Splitting Games over finite sets

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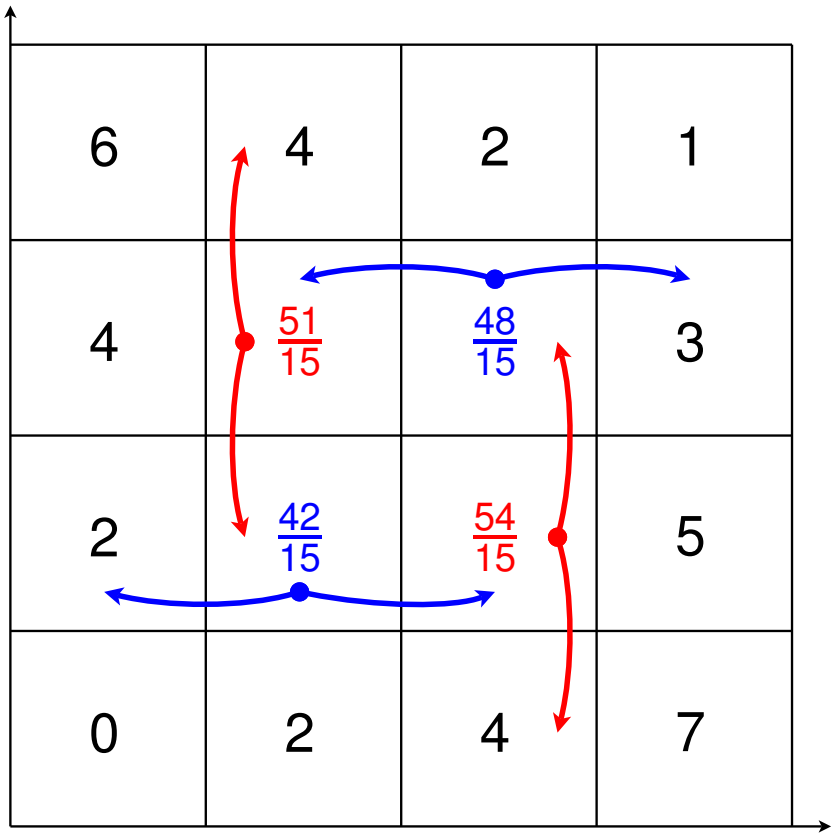

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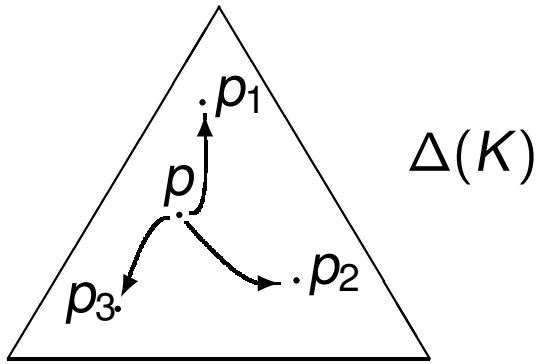
Introduction: Splittings

K being finite, a splitting of a proba p on K is a proba on $\Delta(K)$ with mean p.



Often interpreted as information revelation.

Example $K = \{0, 1\}$, the player (pharmaceutical firm) can run one or several experiments, each experiment returns the true state with proba $2/3$.
One experiment from $p = (1/2, 1/2)$ induces the splitting

$$\frac{1}{2}\delta_{2/3} + \frac{1}{2}\delta_{1/3}.$$

Comes from repeated games with incomplete information, has become very popular in economic theory under the name “information design” or “Bayesian persuasion". (Aumann-Maschler, Mertens-Zamir, Sorin Laraki, Oliu-Barton, Kamenica Gentzkow ...)

Here we consider the zero-sum case with independent states, and finite possible sets of posteriors.

Outline:

1. The Mertens-Zamir transform of a matrix
2. Approximation scheme for the original continuous case
3. Splitting on a finite grid: general case.

This paper: Math Programming 2022.
Companion paper in the continuous case : “Long information design”, Theoretical Economics 2022.

Row-convex and column-concave matrices

Definitions: Let $U = (u_{i,j})$ be in $\mathbb{R}^{I \times J}$, with $I = \{0, ..., m\}$ and $J = \{0, ..., n\}$.

U is row-convex if for each i the linear interpolation of $(j \mapsto u_{i,j})$ is convex: $\forall i \in I, \forall 1 \leq j \leq n-1, \;\; u_{i,j} \leq \frac{1}{2}(u_{i,j-1} + u_{i,j+1})$.

U is column-concave if for each j the linear interpolation of $(i \mapsto u_{i,j})$ is concave: $\forall j \in J, \forall 1 \leq i \leq m-1, \;\; u_{i,j} \geq \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$.

$\mathrm{vex}(U)$, the row-convex envelope of U, is the greatest row-convex matrix V such that $V \leq U$.

$\mathrm{cav}(U)$, the column-concave envelope of U, is the smallest column-concave matrix V such that $V \geq U$.

Example:

$$U = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 2 & 4 & 3 \\ 2 & 4 & 2 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}, \mathrm{vex}U = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 2 & 5/2 & 3 \\ 2 & 2 & 2 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}, \mathrm{cav}U = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 4 & 4 & 3 \\ 2 & 4 & 4 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}.$$

1. The MZ Transform of a matrix

Theorem 1: Given U in $\mathbb{R}^{I\times J}$, there exists a unique matrix V in $\mathbb{R}^{I\times J}$ that is column-concave, row-convex and such that for all $(i,j)\in I\times J$:

(C1): if $v_{i,j} > u_{i,j}$, then $0 < i < m$ and $v_{i,j} = \frac{1}{2}(v_{i-1,j} + v_{i+1,j})$,

and (C2): if $v_{i,j} < u_{i,j}$, then $0 < j < n$ and $v_{i,j} = \frac{1}{2}(v_{i,j-1} + v_{i,j+1})$.

Definition: We say that V is the Mertens-Zamir transform of U: $V = MZ(U)$.

Examples: If U is a row vector, $V = \operatorname{vex} U$.
If U is a column vector, $V = \operatorname{cav} U$.

$$\text{If } U = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 2 & 4 & 3 \\ 2 & 4 & 2 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}, \text{ then } V = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 51/15 & 48/15 & 3 \\ 2 & 42/15 & 54/15 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}.$$

Uniqueness: via a splitting game with terminal payoffs

Define the zero-sum game $\Gamma(U)$:

- an initial position (i_1, j_1) in $I \times J$ is given,
- given $(i_t, , j_t)$, player 1 can choose $i_{t+1} = i_t$ (stay) or select i_{t+1} from $\frac{1}{2}\delta_{i_t-1} + \frac{1}{2}\delta_{i_t+1}$ (move) and simultaneously player 2 can stay or select j_{t+1} from $\frac{1}{2}\delta_{j_t-1} + \frac{1}{2}\delta_{j_t+1}$,
- the payoff for player 1 is $\mathbb{E}(u_{i_\infty, j_\infty})$, where (i_∞, j_∞) is the a.s. limit of the martingale $(i_t, j_t)_t$.

Assume $V = MZ(U)$ exists, and consider the following strategy σ for player 1:

in state (i, j), stay at i if $u_{i,j} \geq v_{i,j}$ and move if $u_{i,j} < v_{i,j}$.

Let player 2 play some arbitrary strategy τ.

By def of V, for all t $\mathbb{E}(v_{i_{t+1}, j_t} | (i_t, j_t)) = v_{i_t, j_t}$.

Moreover $\mathbb{E}(v_{i_{t+1}, j_{t+1}} | (i_{t+1}, j_t)) \geq v_{i_{t+1}, j_t}$ since V is row-convex. Hence,

$$\mathbb{E}(v_{i_{t+1}, j_{t+1}}) \geq \mathbb{E}(v_{i_{t+1}, j_t}) = \mathbb{E}(v_{i_t, j_t}) \geq v_{i_1, j_1}.$$

By definition of σ, $u_{i_\infty, j_\infty} \geq v_{i_\infty, i_\infty}$ a.s. So:

$$\mathbb{E}(u_{i_\infty, j_\infty}) \geq \mathbb{E}(v_{i_\infty, j_\infty}) = \lim_t \mathbb{E}(v_{i_t, j_t}) \geq v_{i_1, j_1}.$$

And σ guarantees v_{i_1, j_1} to player 1.

→ Lemma: The game $\Gamma(U)$ has value v_{i_1,j_1}, where $V = (v_{i,j}) = MZ(U)$. And players have pure optimal strategies.

Shows uniqueness of V. Interpretation for (C1), (C2).

Existence (as in Laraki-R 2020)

Consider the following variant of $\Gamma(U)$ where the payoff for player 1 is $\sum_{t=1}^{\infty} \lambda^{t-1} u_{i_t,j_t}$, with λ a fixed discount factor in $(0,1]$:

This game has a value $v_\lambda(i_1, j_1)$.
Possible to show that a limit point of $(v_\lambda)_\lambda$ must satisfy the conditions for $MZ(U)$.

Rem: Also uniform value

(Proof of Theorem 1 without games ?)

Some properties: For U, U' in $\mathbb{R}^{I\times J}$,

1) if $U \geq U'$ then $MZ(U) \geq MZ(U')$,

2) $\|MZ(U) - MZ(U')\|_\infty \leq \|U - U'\|_\infty$,

3) $\operatorname{vex}(\operatorname{cav} U) \geq MZ(U) \geq \operatorname{cav}(\operatorname{vex} U)$,

4) $MZ(MZ(U)) = MZ(U)$,

5) $MZ(U - MZ(U)) = 0$.

Computation of $V = MZ(U)$, for $U = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 2 & 4 & 3 \\ 2 & 4 & 2 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}$.

First check first and last rows and columns: OK, $V = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & a & b & 3 \\ 2 & c & d & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}$

Guess $a > 2$, $b < 4$, $c < 4$ and $d > 2$ (v>u, v<u, v=u).
Solve $a = (4 + c)/2$, $b = (a + 3)/2$, $c = (2 + d)/2$ and $d = (b + 4)/2$.
Then

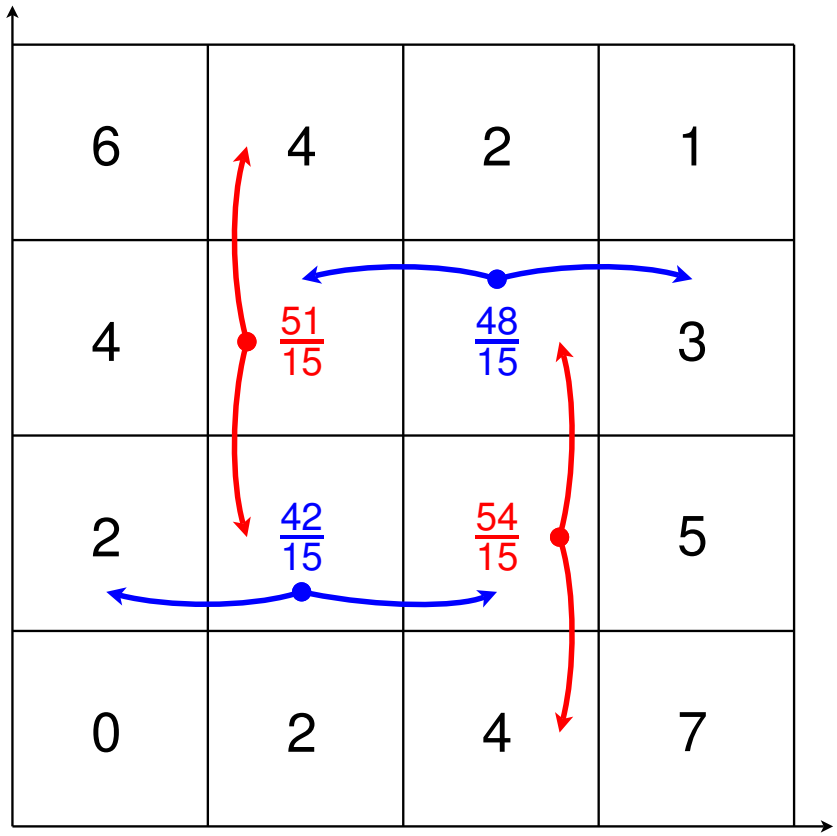
$$V = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 51/15 & 48/15 & 3 \\ 2 & 42/15 & 54/15 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}$$

$\rightarrow$ finite algorithm

Here

$$\text{vex cav}U = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 11/3 & 10/3 & 3 \\ 2 & 9/3 & 12/3 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}, \text{cav vex}U = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 4 & 10/3 & 8/3 & 3 \\ 2 & 8/3 & 10/3 & 5 \\ 0 & 2 & 4 & 7 \end{pmatrix}$$

Optimal strategies:



The number of disclosure periods is unbounded, but disclosure stops with probability 1 in finite time (reminiscent of the 4 frogs in Forges,1990).

2. The original MZ system

Given K, L finite and $u : \Delta(K) \times \Delta(L) \to \mathbb{R}$ continuous there exists a unique continuous solution $v = \mathrm{MZ}(u) : \Delta(K) \times \Delta(L) \to \mathbb{R}$ of the system:

$$v(p,q) = \operatorname{cav} \min\{u(p,q), v(p,q)\} = \operatorname{vex} \max\{u(p,q), v(p,q)\}.$$

Equivalently, $\mathrm{MZ}(u)$ is the unique continuous v which is concave in p, convex in q and such that ($\sim$ Heuer 1992):
(C1') if $v(p,q) > u(p,q)$ $\exists\, s \in SPL(p)$ s.t. $v(p,q) = v(s,q) = u(s,q)$,
(C2') if $v(p,q) < u(p,q)$ $\exists\, r \in SPL(q)$ s.t. $v(p,q) = v(p,r) = u(p,r)$.

(u is linearly extended to distributions).

Computing $MZ(u)$ is in general difficult.

Theorem 2: An approximation scheme in dimension 1

Let $u : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and $v = \mathrm{MZ}(u)$. For each $n \geq 1$, write $I_n = \{0, \ldots, n\}$, $P_n = \{i/n, i \in I_n\}$, and let $U^n = (U^n_{i,j})_{i,j} \in \mathbb{R}^{I_n \times I_n}$ be the matrix which coincides with u at each point $(\frac{i}{n}, \frac{j}{n})$ in the grid $P_n \times P_n$.

Let $V^n = (V^n_{i,j})_{i,j} = MZ(U^n)$ be the MZ transform of U^n and $v_n : [0,1] \times [0,1] \to \mathbb{R}$ be the piecewise biaffine extension of V^n, using the grid $P_n \times P_n$, .

Then, $(v_n)_n$ uniformly converges to $\mathrm{MZ}(u)$. Moreover, if u is C-Lipschitz, then for each $n \geq 1$,

$$\|v - v_n\|_\infty \leq \frac{4C}{n}.$$

3. Splitting on finite sets: general case

Finite set of states: $P \subset \mathbb{R}^K$ for Player 1 and $Q \subset \mathbb{R}^L$ for Player 2,
For each p in P, a set $S(p)$ of probas on P with mean p, containing δ_p and similarly for each q in Q, a set $R(q)$ of probas on Q with mean q, containing δ_q. A payoff function $u : P \times Q \to \mathbb{R}$.

Game $\Gamma(P, Q, S, R, u)$:

- There is a given initial state $(p_1, q_1) \in P \times Q$.
- At each stage $t \geq 1$, if the current state is $(p_t, q_t) \in P \times Q$, Player 1 chooses a splitting $s \in S(p_t)$ and Player 2 chooses a splitting $r \in R(q_t)$. The next state (p_{t+1}, q_{t+1}) is drawn from $s \otimes r$, and observed by both players before playing stage $t + 1$.
- Given a strategy profile, the payoff is the expectation of $u(p_\infty, q_\infty)$ where (p_∞, q_∞) is the almost-sure limit of the martingale $(p_t, q_t)_t$.

We show that this game has a value which admits a Mertens-Zamir-like characterization.

Definition: Consider a map $v : P \times Q \to \mathbb{R}$.
1) v is S-concave if: $\forall (p,q) \in P \times Q, \forall s \in S(p), v(p,q) \geq v(s,q)$.
v is R-convex if: $\forall (p,q) \in P \times Q, \forall r \in R(q), v(p,q) \leq v(p,r)$.
2)

$$\forall (p,q) \in P \times Q,\ v_-(p,q) := \sup\left\{v(s_{\neq p}, q), s \in S(p)\backslash\{\delta_p\}\right\} \in \mathbb{R} \cup \{-\infty\}$$

with $s_{\neq p}$ the conditional probability induced by s given that the state is not p, and similarly
$v_+(p,q) = \inf\left\{v(p, r_{\neq q}), r \in Q(q)\backslash\{\delta_q\}\right\} \in \mathbb{R} \cup \{+\infty\}$.

Equivalently,

$$v_-(p,q) = \sup\left\{v(s',q) : s' \in \Delta(P\backslash\{p\}), \exists\, \alpha \in [0,1) \text{ s.t. } \alpha\delta_p + (1-\alpha)s' \in S(p)\right\},$$

$$v_+(p,q) = \inf\left\{v(p,r') : r' \in \Delta(Q\backslash\{q\}), \exists\, \alpha \in [0,1) \text{ s.t. } \alpha\delta_q + (1-\alpha)r' \in R(q)\right\}.$$

Remarks:
If v is S-concave and R-convex, then $v_-(p,q) \leq v(p,q) \leq v_+(p,q)$.
If $S(p) = \{\delta_p\}$, $v_-(p,q) = -\infty$; if $R(q) = \{\delta_q\}$, $v_+(p,q) = +\infty$.

Example: Let $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\} \subset \mathbb{R}$, $S(p) = \{\delta_p\}$ for $p \in \left\{0, \frac{1}{4}, \frac{3}{4}, 1\right\}$, and

$$S\left(\frac{1}{2}\right) = \left\{(1 - 2\varepsilon - 2\varepsilon^2)\delta_{\frac{1}{2}} + \varepsilon\delta_{\frac{1}{4}} + \varepsilon\delta_{\frac{3}{4}} + \varepsilon^2\delta_0 + \varepsilon^2\delta_1 : \varepsilon \in \left[0, \frac{1}{4}\right]\right\}.$$

At $\frac{1}{2}$, Player 1 can split, but the posterior is likely to remain at $\frac{1}{2}$ and is much more likely to be $\frac{1}{4}$ or $\frac{3}{4}$ than 0 or 1. Suppose that for each q in Q, $u(0, q) = u(1/2, q) = u(1, q) = 0$ and $u(1/4, q) = u(3/4, q) = 1$, so that at $p = 1/2$ player 1 would like to reach the distribution $\frac{1}{2}\delta_{\frac{1}{4}} + \frac{1}{2}\delta_{\frac{3}{4}}$.

Then, the value satisfies $v(0, q) = v(1, q) = 0$ and $v(\frac{1}{4}, q) = v(\frac{1}{2}, q) = v(\frac{3}{4}, q) = 1$. It is S-concave and we have $v(\frac{1}{2}, q) = v_-(\frac{1}{2}, q)$. Here, the splitting $\frac{1}{2}\delta_{\frac{1}{4}} + \frac{1}{2}\delta_{\frac{3}{4}}$ is infeasible at $\frac{1}{2}$ but can be approximately achieved with many stages by choosing a very small ε.

Theorem 3 The splitting game $\Gamma(P, Q, S, R, u)$ admits a value, which is the unique function $v : P \times Q \to \mathbb{R}$ which is S-concave, R-convex, and such that for all (p, q) in $P \times Q$,

(C1″) if $v(p,q) > u(p,q)$, then $v(p,q) = v_-(p,q)$, and

(C2″) if $v(p,q) < u(p,q)$, then $v(p,q) = v_+(p,q)$.

Moreover, the value is obtained in pure strategies: using stopping times, we construct pure Markovian ε-optimal strategies .

Questions:

- Extension to countable compact spaces ?
- Extension of Theorem 2 to higher dimension ?
- Without assuming that Dirac measures are available, Theorem 3 does not hold, but does the value exist ?
- $P = Q = [0, 1]$, all splittings available but u is severely discontinuous: $u(p, q) = 1$ if $p - q$ is rational, $= -1$ otherwise. Both $v = 1$ and $v = -1$ are concave/convex and satisfy $(C1')$ and $(C2')$. What about the value of the game with terminal payoffs ?
- Does at least one $MZ(u)$ exist when u is not continuous ?