Optimal menu of tests

Nathan Hancart*

July, 2022

WORK-IN-PROGRESS — PRELIMINARY AND INCOMPLETE

Abstract

I study the optimal design of menus of tests. Prior to taking a binary decision, accept or reject a privately informed agent, a decision-maker (DM) can perform one test from a restricted set. For example, the restriction can come from information processing or technological constraints. The DM wants to accept a subset of types whereas the agent always wants to be accepted. Instead of choosing the test himself, the DM let the agent choose a test from a menu. The choice itself then serves as an additional dimension for information revelation. I characterise when a menu is optimal and show that the DM does not benefit from committing to an action. Using these results, I show conditions under which the DM wants or does not want to include strictly less informative test in the menu. I also define an order on tests that characterises which tests are part of an optimal menu.

*Nathan Hancart: Department of Economics, University College London, 30 Gordon St, London WC1H 0AN, UK, nathan.hancart.16@ucl.ac.uk. I am grateful to Ran Spiegler and Vasiliki Skreta for their guidance and support. I also thank Deniz Kattwinkel for his availability and help in this project. Finally, I would also like to thank Anna Becker, Alex Clyde, Martin Cripps, Duarte Gonçalves, Amir Habibi, Philippe Jehiel and the theory group at UCL for their helpful comments.

1 Introduction

In many economic settings, decision-makers rely on tests. Universities use standardised tests when they select their students; firms interview job candidates before they hire them; regulators test products before they can be authorised. In all these situations, the decision-maker's preferences depend on an agent's private information. To take a more informed decision, he chooses a test the agent must pass. For example, universities care about the student's ability and choose which tests they require in their admission process; firms care about the candidate's productivity and decide on their selection process; regulators care about a product's quality and decide what test it must pass before authorising it. In addition, the decision-maker's testing capacity is usually constrained. Technological or resource constraints can both set a limit to testing.

In this paper, I study the optimal testing strategy for the decision-maker when his testing capacity is constrained. A natural strategy for the decision-maker could be to require the agent to take the best test available. Instead, I show that the decision-maker can do better by choosing a *menu* of tests and letting the agent choose which test to take. By offering a menu, the decision-maker can use the agent's choice as an additional source of information. Furthermore, agents can select into tests that are more suited to detect their type. Therefore, it allows a more efficient allocation of tests to agents. I develop tools to characterise the optimal menu of tests and give conditions to determine whether offering a menu does better or not than offering a single test.

Specifically, I consider a decision-maker who has to make a binary decision, accept or reject. His payoffs depend on an agent's private type. While the DM wants to accept a subset of agents (the A-types) and reject the others (the R-types), the agent always wants to be accepted. Prior to taking the decision, the DM can require the agent to take one test modelled as a Blackwell experiment. The test is chosen from an exogenous set. This set can capture various constraints such as information acquisition, institutional or technological constraints.

The first step in the analysis is to provide a characterisation of the optimal menu for arbitrary type structures and tests available. In Theorem 1, I show that the optimal menu and strategies are the outcome of an auxiliary zero-sum game. The strategies in that auxiliary game determine the optimal menu as well the agent's and DM's strategies. This result greatly simplifies the analysis. Indeed, it is not necessary to compare equilibria from different menus to determine the optimal one. It is enough to find an equilibrium in one auxiliary game to find the optimum across many games. Moreover, I show that if the DM could commit, he would optimally choose a best-response to the information revealed by the test choice and signal realisation. This implies that commitment does not have any value for the DM. Finally, I show that A-types play a pure strategy in an equilibrium of the optimal menu. This means that optimal number of tests is bounded by the number of A-types. In particular, if there is only one A-type, like in binary types models, there is always an optimal menu with only one test. This is true without making any assumptions on the set of available tests.

Using Theorem 1, I analyse properties of the optimal menu. In Section 3.3, I study the optimal menu when there is a Blackwell most informative test in the set of test.[1] A first important result is that the most informative test is always part of an optimal menu. This is a direct consequence of the fact that the DM does not benefit from commitment. Indeed, the DM can always commit to a strategy that has lower probability of both Type-I and Type-II error in the most informative test than any other test in the menu, making it optimal to include it. Thus the incentives in this game do not create distortions in what constitutes "more information".

We know that a most informative test is included in the optimal menu, but would the DM also want to include a strictly less informative test? I provide conditions under which the DM would want to do so. I show that if the A-types are accepted with lower probability than all R-types on a subset of signals, then it is optimal to include a coarsened test that pools this subset of signals on one signal. Pooling can take many forms. One example is taking a test from a $0-100$ scale to a $0-5$ scale. Another extreme example is a completely uninformative test: all signals have been pooled on one signal, making the test completely uninformative. All these

[1] Explanation of Blackwell informativeness?

coarsened tests are strictly less informative. Subsequently, I apply this insight to the question of whether universities should require students to pass a standardised test for admission when the standardised test is not entirely reflective of their ability. I show that allowing students to opt out from the standardised test sometimes leads to strictly more information revelation. That is, it is sometimes optimal to include a completely uninformative test in the menu.

However, I show that in some natural environments only the most informative test is used in the optimal menu. This is true when there is pure vertical differentiation between types, the DM wants to accept all types above a threshold and the tests satisfy the monotone likelihood ratio property.[2] The key intuition is that it is impossible to create a non-singleton menu where both A-types and R-types pool on all tests. If a test is only chosen by R-types, the DM would reject after all signals from that test, making it unattractive to all types and thus never chosen. If it is chosen only by A-types, the DM would accept after any signals, making it attractive to all types. This result points to a fundamental feature of this model that differentiates it from classic signalling models: it is impossible to have separation of good and bad types in equilibrium, unless the tests themselves can separate them. Thus any menu using more than one test must provide incentives such that both A- and R-types pool on each test.

In Section 3.4, I illustrate this point further in an environment where tests are not ordered by their informativeness but by their difficulty. Again, I assume that types are vertically differentiated. The testing technology is a set of pass-fail tests and tests can be ordered by how difficult it is to pass them. Unlike the previous result, I show that it is possible to sustain an equilibrium with more than one test. However, the DM's strategy needed to maintain that equilibrium is such that he is better off offering only one test.

In Section 3.5, I give conditions on the DM's preferences and tests available that guarantee that a test is part of an optimal menu. To this end, I define an order on tests which I call the θ-order. Roughly speaking, this order describes how good a test is at differentiating types restricting attention to one A-type (type θ) and all the R-types. If a test is dominant in that

[2]This result is in terms of Blackwell informativeness (Blackwell, 1953) but also works with weaker notion of informativeness like Lehmann (1988)'s or some weakening of it.

sense, then it is included in the optimal menu. Intuitively, what matters for a test is its informativeness about one A-type versus all the R-types and not its "absolute" informativeness.

Two key properties of the θ-order are worth pointing out. First, it is a relaxation of Blackwell (1953)'s informativeness order as it focuses on a subset of types and is thus easier to satisfy. Second, it is an order that depends on the preferences of the DM. Therefore, it captures both that the properties of the tests and the DM's preferences matter for the inclusion of a test in the optimal menu. The notion of θ-order also captures the idea that a menu allows for a more efficient allocation of tests to agents. Indeed, for the A-type θ, the dominant test in the θ-order is the one that is tailored to differentiate him from all the R-types.

I relate this result to when non-trivial menus can be optimal with horizontal differentiation. For illustration, suppose that types are two-dimensional, $\theta = (\theta_0, \theta_1)$, and there are two tests, one testing dimension θ_0 and the other testing dimension θ_1. Suppose now that the DM cares about each dimension individually, that is he wants to accept if θ_i is high enough. Then the test testing dimension i is dominant in the θ-order where θ is maximal in dimension i. Thus both tests are included in the menu. However, if the DM cares about both dimensions at the same time and wants to reject a type that is good in only one dimension, then no test is dominant in the θ-order anymore and a singleton menu can be optimal. This shows how both horizontal differentiation and the DM's preferences are key elements to determine which tests to include in the menu.

I also show that a necessary and sufficient condition for the DM to never make a mistake is to have for each A-type θ the most dominant test in the θ-order. In this test, the signal support of θ and the support of all the R-types does not intersect. Then, an unravelling argument à la Milgrom (1981) and Grossman (1981) can be implemented by the DM and guarantees he can always differentiate A-types from R-types.

Finally, in Section 4, I show that the model can be easily extended to allow for communication. I model communication as an additional cheap-talk message on top of the test choice. A similar characterisation as in Theorem 1 holds. I show that in this case, each A-type announces his type while the R-types pretend to be some A-types. Furthermore, I show that when there is communication, it is irrelevant for the outcome of the game who chooses the test, the DM or the agents. I relate these results to those of Glazer and Rubinstein (2004) and Carroll and Egorov (2019).

Relation to the literature

This paper is most related to the literature on communication with verification (e.g., Glazer and Rubinstein, 2004; Carroll and Egorov, 2019; Dziuda and Salas, 2018). This project differs in at least two ways from these papers. First, they consider verification problems, that is the problem of finding the optimal test to use after an observable message from the agent. Glazer and Rubinstein (2004) and Carroll and Egorov (2019) consider the optimal verification mechanism for a fixed testing technology. In this paper, I do not allow for communication and the one choosing the test is the agent and not the DM unlike in these papers. The only endogenous information that can be transmitted is through the choice of test.

The second main difference is that I consider an arbitrary type structure and testing technology. On the other hand, Glazer and Rubinstein (2004) and Carroll and Egorov (2019) consider the verification of one dimension of a multidimensional private type. Beyond the added generality, it allows me, for example, to study properties of tests to be included in the optimal mechanism. Glazer and Rubinstein (2004) also show that the DM does not benefit from commitment in their model. I comment more on the connection in Section 4 when I extend the model to allow for communication and shows the connections between the two types of problems. Dziuda and Salas (2018) characterise the structure of communication when lies can be detected with some probability but does not consider any design element.

This paper has also some link to the mechanism design with evidence literature (Green and Laffont, 1986; Bull and Watson, 2007; Deneckere and Severinov, 2008). In its modern form, this literature assumes that a mechanism maps a type report and a piece of hard evidence to an outcome. The variant of this model with commitment to an action can be seen as a mechanism design problem with probabilistic evidence where the tests available are controlled by the designer, not the agent.[3] Related to the no value of commitment result, Ben-Porath et al. (2019) show that in a class of mechanism design problems with a "normal" evidence structure (see Bull and Watson, 2007; Lipman and Seppi, 1995), the designer does not need his commitment power.

The fact that the set of test is constrained in my model can be motivated by the literature on costly information processing and acquisition. Various cost functions have been proposed in the literature, for example entropy costs and its generalisations (Sims, 2003; Caplin et al., Forthcoming) or log-likelihood ratio costs (Pomatto et al., 2018). The set of information structures available can reflect a capacity constraint on the DM's information processing or acquisition abilities. As outlined in the introduction, there are other reasons the set of tests could be limited such as institutional constraints.

There are also links with the sender-receiver literature with manipulation of input or output, see e.g., Frankel and Kartik (2019*a*,*b*); Ball (2019); Nguyen and Tan (2021); Perez-Richet and Skreta (Forthcoming). In this paper however, the agent cannot manipulate the tests but can only mimic another type's test choice, thereby manipulating the inference based on the test choice but not on the test results itself.

Finally, some papers consider the optimal choice of tests for different agents. Motivated by the COVID-19 pandemic, Ely et al. (2021) study the optimal allocation of tests from a restricted set to agents with observable characteristics. Deb and Stewart (2018) study the dynamic choice of test in the presence of asymmetric information and moral hazard, i.e., where both effort and the agent's type affect the outcome of the test. They derive conditions for using the most informative test available.

[3] See also Ball and Kattwinkel (2019) for a different modelling approach to probabilistic evidence.

2 Model

There is a DM and an agent. The agent has a type $\theta \in \Theta = \{1, 2, ...n\}$ with Pr$[\theta] = \mu(\theta)$. The set of types is partitioned in two: $\Theta = A \cup R,\ A \cap R = \emptyset$. The type is private information of the agent. The DM must take an action $a \in \{0, 1\}$. The utilities of the DM and the agent are $v(a, \theta) = a\big(\mathbb{1}[\theta \in A] - \mathbb{1}[\theta \in R]\big)$[4] and $u(a, \theta) = a$. That is the DM wants to accept agents in A and reject agents in R. The agent always wants to be accepted.

There is an exogenous set of test $T \subseteq \Pi \equiv \{\pi : \pi : \Theta \to \Delta X\}$, where X is some finite signal space. I assume that $|T| < \infty$. The conditional probabilities of test t are $\pi_t(\cdot|\theta)$. The exogenous set of tests can capture different constraints on the set of tests. For example, it could come from a capacity constraints in the information processing/acquisition abilities of the DM, $T \subset \{\pi : c \geq C(\pi)\}$ for some cost function C. The constraint can also be on some properties of the tests that can be used $T \subset \{\pi : \pi$ has the MLRP$\}$. Finally, it could be a purely technological constraint, e.g., when choosing amongst standardised test, there is only a limited set of existing tests $T = \{$SAT$, \emptyset\}$.

The DM chooses to post a menu of tests, that is a subset $T' \subseteq T$. A strategy for the agent is a choice of test, $\sigma : \Theta \to \Delta T'$. A strategy for the DM is a probability of accepting after observing the test chosen and the signal realisation, $\alpha : T' \times X \to [0, 1]$. Beliefs are $\tilde{\mu} : T' \times X \to \Delta\Theta$.

The solution concept is DM-preferred weak Perfect Bayesian Equilibrium.

I write $(\alpha, \sigma) \in$ wPBE(T') if there is a belief $\tilde{\mu}$ where $(\alpha, \sigma, \tilde{\mu})$ is a weak PBE when the menu is T'.

[4] The analysis is virtually unchanged by allowing for DM's utility functions of the form $v(a, \theta) = a\nu(\theta)$ for some $\nu : \Theta \to \mathbb{R}$.

The optimal design of menu solves

$$V = \max_{T' \subseteq T} \max_{\sigma,\alpha} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T'} \sigma(t|\theta) \sum_{x} \alpha(t,x)\pi_t(x|\theta) - \sum_{\theta \in R} \mu(\theta) \sum_{t \in T'} \sigma(t|\theta) \sum_{x} \alpha(t,x)\pi_t(x|\theta)$$

$$\text{s.t. } (\alpha, \sigma) \in \text{wPBE}(T')$$

Note that without loss of generality, we can set $T' = T$ using off-path beliefs to deter any deviation. The implicit off-path beliefs used throughout is $\tilde{\mu}(\theta \in R|x,t) = 1$, i.e., any deviation from equilibrium strategies is attributed to an R-type.

Notation: For any α, I write $\pi_t(a|\theta) = \sum_x \alpha(t,x)\pi_t(x|\theta)$.

3 Results

3.1 Characterisation of the optimal menu

In this subsection, I show that the value of the optimal menu is characterised by an equilibrium of a zero-sum game.

Let $s : A \to \Delta T$ and $s' : R \to \Delta A$ and abusing notation, let

$$v(\alpha, s, s') \equiv \sum_{\theta \in A} \sum_{t} s(t|\theta) \Big[\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta') s'(\theta|\theta') \pi_t(a|\theta') \Big].$$

The function $\alpha : T \times X \to [0,1]$ keeps the same definition. The function s can be interpreted as A-types choosing a test, s' as R-types choosing an A-type to mimic and v as the DM's expected payoffs from a distribution over tests induced by s, s'. I explain these objects in more detail in the discussion of Theorem 1.

Theorem 1. *The value of an optimal menu is*

$$V = \max_{s,\alpha} \min_{s'} v(\alpha, s, s') = \min_{s'} \max_{s,\alpha} v(\alpha, s, s')$$

A saddle point (α, s, s') of v such that $s(\cdot|\theta)$ is in pure strategies for all $\theta \in A$ exists and characterises an optimal menu and strategies:

- *for* $\theta \in A : \sigma(t|\theta) = s(t|\theta)$
- *for* $\theta' \in R : \sigma(t|\theta') = \sum_{\theta \in A} s'(\theta|\theta') s(t|\theta)$
- *and the DM's strategy is* α*.*

Moreover, the DM does not benefit from committing over α*.*

Proof. See appendix. □

There are two main take-aways from this proposition. First, the DM does not benefit from committing over α.

Second, the fact that an optimal menu is an equilibrium of *a* game gives us a powerful tool to test equilibria. Indeed, it is now not necessary to compare equilibria to establish that a menu is not optimal. It is enough to find that $(\tilde{\alpha}, \tilde{s})$ such that

$$\min_{s'} v(\alpha, s, s') < \min_{s'} v(\tilde{\alpha}, \tilde{s}, s')$$

to show that (α, s, s') does not constitute an optimal menu without having to care whether $(\tilde{\alpha}, \tilde{s})$ are optimal.

To understand the structure of this game better, consider the following auxiliary game. Fix a strategy α.

Definition 1 (Auxiliary game)**.** *An auxiliary game is a game* $< \Theta, S, \tilde{u} >$ *where*

$$\textit{for } \theta \in A,\ S_\theta = T \textit{ and for } \theta' \in R,\ S_{\theta'} = A$$

and

$$\tilde{u}_\theta(t, s'(\cdot|\theta')) = \mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta') s'(\theta|\theta') \pi_t(a|\theta')$$
$$\tilde{u}_{\theta'}(\theta, s(\cdot|\theta)) = \sum_t s(t|\theta) \pi_t(a|\theta')$$

The auxiliary game is the problem faced by A-types and R-types of the zero-sum game for a fixed α, presented at the interim stage.

In the auxiliary game, each A-type is choosing a test and each R-type is choosing an A-type to mimic. The R-types are maximising their probability of being accepted like in the original problem while the A-type maximise a modified version of their utility where they maximise their probability of being accepted while being "punished" every time a R-type mimics them and is accepted. In a sense, the A-types' utility is modified to align it with the DM's payoffs. An equilibrium of the auxiliary game is also an equilibrium of the zero sum-game for a fixed α.

The strategies of the auxiliary game induce a distribution over tests for each type. The A-types get the distribution over test they choose and the R-types the distribution of the A-types they choose to mimic. Theorem 1 shows that this distributions are actually the equilibrium strategies of the optimal menu game. Moreover, the A-types play a pure strategy.

To understand why choosing test t for type $\theta \in A$ in the auxiliary game delivers the right equilibrium behaviour in the original game, consider the following interpretation of the game. The payoffs of a type $\theta \in A$ can be understood as a gross payoff

$$\mu(\theta)\pi_t(a|\theta),$$

corresponding to the payoffs in the original game and a net payoff

$$\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(a|\theta').$$

The equilibrium behaviour of R-types means that the test chosen in equilibrium carries the largest negative term. Indeed, assuming for simplicity a pure strategy from the A-types, if $s'(\theta|\theta') > 0$, then $\pi_t(a|\theta') \geq \pi_{t'}(a|\theta')$ for any t' chosen by some other A-type. Let's consider a deviation of that A-type θ to test t' when the equilibrium is (s, s'). Equilibrium behaviour gives us

$$\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(a|\theta') \geq \mu(\theta)\pi_{t'}(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{t'}(a|\theta')$$

$$\Rightarrow \mu(\theta)\big(\pi_t(a|\theta) - \pi_{t'}(a|\theta)\big) \geq \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\big(\pi_t(a|\theta') - \pi_{t'}(a|\theta')\big) \geq 0$$

where the last inequality comes from the equilibrium behaviour of the R-types. This delivers the desired equilibrium behaviour in the original game. Intuitively, the test chosen in equilibrium is "the most expensive" amongst all the tests. This means that the gross payoffs from it must be the largest.

Theorem 1 gives an upper bound on the number of tests needed in an optimal menu. If A-types are playing a pure strategy and R-types only use tests A-types use, then the number of tests used is at most $|A|$.

Corollary 1. *The number of tests used in the optimal menu is at most* $|A|$.

An immediate corollary is also that if there is only one type the DM would like to accept an optimal menu is to use only one test. In particular, this results shows that in a binary state environment, the optimal mechanism uses only one test, no matter what the available set of test is.

Corollary 2. *Suppose* $|A| = 1$*. Then for any* T*, there is an optimal menu that uses only one test.*

3.2 Sketch of proof Theorem 1

I here provide a sketch of the proof of Theorem 1. To prove it I will first need to introduce test allocation *mechanism*. A (direct) mechanism is a mapping $\tilde{\sigma} : \Theta \to \Delta T$, a mapping that maps types to distribution over tests. Suppose there is a designer that could design $\tilde{\sigma}$ to maximise the DM payoffs. The DM only observes the realised test and signal, thus the definition of his strategy is unchanged. The agent's strategy is now to report a type into the mechanism. The solution concept is still DM-preferred wPBE. Standard arguments show that without loss of generality we can restrict attention to direct truthful mechanism. The designer's problem is

$$
\begin{aligned}
\tilde{V} = \max_{\tilde{\sigma},\alpha} \sum_{\theta\in A}\mu(\theta)\sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) - \sum_{\theta\in R}\mu(\theta)\sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) \\
\text{s.t. } \sum_t (\tilde{\sigma}(t|\theta) - \tilde{\sigma}(t|\theta'))\pi_t(a|\theta) \geq 0 \text{ for all } \theta, \theta' \\
\sum_t \tilde{\sigma}(t|\theta) = 1 \text{ for all } \theta \\
\alpha \in BR(\tilde{\sigma})
\end{aligned}
$$

where the first constraint is the agent's incentive compatibility constraint, the second is a feasibility constraint and $\alpha \in BR(\tilde{\sigma})$ means that the α is a best-response to some beliefs consistent with the mechanism. We have $\tilde{V} \geq V$, i.e., the value of the optimal mechanism is larger than the value of the optimal menu, as imposing a menu is simply restricting the class of mechanism the designer could use.

The first part of the proof shows that $\tilde{V} = \max_{\alpha,s}\min_{s'} v(\alpha, s, s')$. The second part shows that the optimal mechanism can be implemented by posting a menu of tests.

To show that $\tilde{V} = \max_{\alpha,s}\min_{s'} v(\alpha, s, s')$, I characterise the optimal mechanism when the DM commits to α. The DM problem is then

$$
\begin{aligned}
\tilde{V}(\alpha) = \max_{\tilde{\sigma}} \sum_{\theta\in A}\mu(\theta)\sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) - \sum_{\theta\in R}\mu(\theta)\sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) \\
\text{s.t. } \sum_t (\tilde{\sigma}(t|\theta) - \tilde{\sigma}(t|\theta'))\pi_t(a|\theta) \geq 0 \text{ for all } \theta, \theta' \\
\sum_t \tilde{\sigma}(t|\theta) = 1 \text{ for all } \theta
\end{aligned}
$$

This is a linear program and verifying complementary slackness conditions shows that $\tilde{V}(\alpha) = \max_s \min_{s'} v(\alpha, s, s')$. As in the statement of the theorem, (s, s') characterise an optimal mechanism $\tilde{\sigma}$ by setting $\tilde{\sigma}(t|\theta) = s(t|\theta)$ for $\theta \in A$ and $\tilde{\sigma}(t|\theta') = \sum_{\theta\in A} s(t|\theta)s'(\theta|\theta')$.

The value of the DM if he could commit to α is $\max_\alpha \tilde{V}(\alpha)$. Now notice that, $\max_\alpha \tilde{V}(\alpha) = \max_\alpha \max_s \min_{s'} v(\alpha, s, s') = \max_{\alpha,s} \min_{s'} v(\alpha, s, s')$. Using a result from Baye et al. (1993) on the existence of Nash equilibrium in non-quasiconcave games, $\max_\alpha \tilde{V}(\alpha)$ is attained by (α^*, s^*, s'^*) such that $v(\alpha, s, s'^*) \leq v(\alpha^*, s^*, s'^*) \leq v(\alpha^*, s^*, s')$ for all α, s, s'. This in turn implies that α^* is a best-response to the mechanism implied by (s^*, s'^*) as $v(\alpha^*, s^*, s'^*) \geq v(\alpha, s^*, s'^*)$ for all α. Therefore $\max_\alpha \tilde{V}(\alpha) = \tilde{V} = \max_{\alpha,s} \min_{s'} v(\alpha, s, s')$. This proves that the DM would not benefit from committing to α if he could offer a mechanism.

The second part shows that the optimal mechanism can be implemented by posting a menu of tests. The way the proof proceeds is by showing that there is $(\alpha^*, s^*) \in \arg\max \min_{s'} v(\alpha, s, s')$, where s^* is in pure strategy for all $\theta \in A$. If this is the case, then we can take the menu of tests as the support of tests in the optimal mechanism. Each type $\theta \in A$ is better off choosing "his" test as choosing another one would violate the incentive compatibility constraints. Types $\theta' \in R$ possibly have a randomised allocation but they are indifferent between any tests they are allocated to. Indeed, their randomised allocation corresponds to a mixed strategy in the auxiliary game where they are maximising their probability of being accepted, just like in the menu-game.

To understand why s^* must be in pure strategy, note that given s'^*, the DM and types in $\theta \in A$ must choose α and s to maximise $v(\alpha, s, s'^*)$. If the A-types are willing to mix, they must be indifferent between all the tests in the support for a fixed α^*. This α^* is itself a best-response to (s^*, s'^*). Choosing a pure strategy in the support of s^* allows then the DM to re-optimise and get a higher payoff for both the DM and the A-types.

3.3 Optimal menu with Blackwell dominant test

In this subsection I explore the design of the optimal menu when there is a Blackwell dominant test in the set of available tests T. First, I give sufficient conditions under which a strictly less Blackwell informative test is used in the optimal menu. Then, I give conditions under which having a Blackwell dominant test implies that it is the only test used in the optimal menu.

Definition 2 (Blackwell (1953))**.** *A test t Blackwell-dominates t', $t \succeq t'$, if there is function $\beta : X \times X \to [0,1]$ such that for all $x' \in X$, $\sum_x \beta(x,x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x,x') = 1$.*

If a test Blackwell-dominates another then we know that for any decision problem, i.e., a pair of utility function and a prior, using the Blackwell dominant test yields higher expected utility. A first important fact we will record here is that if there is a Blackwell dominant test, then it is part of an optimal menu.

Lemma 1. *If there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$, then there is an optimal menu with t.*

Proof. See appendix. □

This lemma follows from the zero-sum characterisation of Theorem 1 and the properties of Blackwell-dominant test. Indeed, if we find a menu where the dominant test t is not included, we can modify the DM's strategy such that one A-type is accepted with higher probability than the test he is choosing, say t', and all R-types are accepted with lower probability than in t'. Then this A-type has a profitable deviation to t.

Having established that a Blackwell dominant test is always included in the optimal menu, a natural question is whether this should be the only test in the menu or is it sometimes optimal to add a strictly less informative test?

I will focus on a special kind of Blackwell dominated test which are coarsening of a test.

Definition 3. *A test t is a coarsening of test t' if there is a partition of X, $\{X_i\}$, such that for* *all* $\theta \in \Theta$,

$$\pi_t(x_i|\theta) = \sum_{x \in X_i} \pi_{t'}(x|\theta) \quad \textit{for some } x \in X_i$$

$$\pi_t(x'|\theta) = 0 \quad \textit{for all } x' \in X_i,\, x' \neq x_i$$

The idea of a coarsening is that it pools all the signal in one element of the partition X_i on one signal x_i. The test t' is more informative than t as any strategy under t can be implemented under t'. I say that a test pools signals in X' if the partition is $\{X', \{x\} : x \notin X'\}$.

The following propositions give two criteria under which including a coarsened version of a test in the menu yields higher expected payoff to the DM.

Proposition 1. *Let* $\alpha(x,t)$ *be the optimal strategy when only test* t *is used. If there is* $\theta \in A$ *and* $X' \subseteq X$ *such that*

$$\frac{\sum_{x \in X'} \alpha(x,t)\pi_t(x|\theta)}{\sum_{x \in X'} \pi_t(x|\theta)} \leq \frac{\sum_{x \in X'} \alpha(x,t)\pi_t(x|\theta')}{\sum_{x \in X'} \pi_t(x|\theta')} \quad \textit{for all } \theta' \in R$$

then it is optimal to include a coarsened version of t *that pools signals in* X'.

Proof. See appendix. □

The proposition gives us one criterion to include a coarsened version of a test. If one A-type, say θ, is doing worse than all R-types on a subset of signals, it is beneficial for the DM to offer a test where these signals are coarsened. The type θ can then choose the coarsened test where the superior performance of the R-types is comparatively less important than in the original test. The proof works by considering a deviation from the zero-sum game of Theorem 1 where θ is given exactly the same probability of being accepted in the coarsened test and in t. The inequality in Proposition 1 implies that all R-types do worse and thus this forms a deviation or a new equilibrium in the zero-sum game.

The proposition above gives us a useful criterion, but is quite demanding. In particular, it requires a form of absolute dominance: the one A-type is doing worse than all the R-types on that subset. We can relax that requirement by trading off how many A-types and R-types would benefit from introducing the new test. Let $a^+ = \max\{0, a\}$.

Proposition 2. *Let $\alpha(x,t)$ be the optimal strategy when only test t is used. If there is $\tilde{\alpha} \in [0,1]$ and $X' \subseteq X$ such that*

$$\sum_{\theta \in A} \sum_{x \in X'} \mu(\theta) \left[(\tilde{\alpha} - \alpha(x,t)) \pi_t(x|\theta) \right]^+ \geq \sum_{\theta' \in R} \sum_{x \in X'} \mu(\theta') \left[(\tilde{\alpha} - \alpha(x,t)) \pi_t(x|\theta') \right]^+$$

then it is optimal to include a coarsened version of t that pools signals in X'.

Proof. See appendix. □

Proposition 2 is less demanding than Proposition 1. If the condition of Proposition 1 is satisfied and we set $\tilde{\alpha} = \frac{\sum_{x \in X'} \alpha(x,t) \pi_t(x|\theta)}{\sum_{x \in X'} \pi_t(x|\theta)}$ then the condition of Proposition 2 holds.

This result is a direct application of the zero-sum game of Theorem 1. It considers using the same strategy as in test t for the coarsened test but for the coarsened signal in X' where it uses $\tilde{\alpha}$. The condition then boils down to checking for a profitable deviation.

Example (Test with gaming concerns)**.** Suppose a university depends on some standardised test for university admission. Suppose there are three types of students: $A = \{A1, A2\}$ and $R = \{R1\}$. Consider the testing set $T = \{t, \emptyset\}$ where $\emptyset$ is an uninformative test. The test t is described by $X = \{1, 2, 3\}$ and

$$\pi_t(x|A1) = \begin{cases} 1/2 & \text{if } x = 1 \\ 1/2 & \text{if } x = 2 \end{cases} \qquad \pi_t(x|R1) = \begin{cases} 1/4 & \text{if } x = 1 \\ 1/2 & \text{if } x = 2 \\ 1/4 & \text{if } x = 3 \end{cases}$$

$$\pi_t(x|A2) = \begin{cases} 1/2 & \text{if } x = 2 \\ 1/2 & \text{if } x = 3 \end{cases}$$

Furthermore, suppose that $2\mu(A1) < \mu(R1) < \frac{1}{2}\mu(A2)$. Clearly, the test $\emptyset$ is a coarsening of t.

This example can be interpreted in the following way. The test t is a standardised test the university uses to get information about students. However, the test is not very good at identifying some type of good student, $A1$. One reason could be that $A1$ did not learn how to do well on the test or that it does not test a dimension he is good on. Another reason could be that $R1$ got some special training that allows him to get good grades on the test despite not being "intrinsically" a good student. Offering $\emptyset$ allows students to opt out from the standardised test.

This information structure and prior delivers the following best response when only t is offered,

$$\alpha(x,t) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x \geq 2 \end{cases}$$

The acceptance probabilites of each types is then

$$\pi_t(a|R1) = 3/4 \qquad \pi_t(a|A1) = 1/2 \qquad \pi_t(a|A2) = 1$$

Clearly, $\pi_t(a|A1), \pi_t(R1)$ and thus we can apply Proposition 1 with $X' = X$ to conclude that offering $\emptyset$ would benefit the DM. We can also easily compute the resulting equilibrium and acceptance probabilities.

$$\sigma(\emptyset|R1) = \mu(R1) - \mu(A1) \qquad \sigma(\emptyset|A1) = 1 \qquad \sigma(t|A2) = 1$$

After t, the strategy remains the same as before. When the DM observes $\emptyset$, he is indifferent between accepting and rejecting. He then mixes in a way to make $R1$ indifferent between $\emptyset$ and t: $\alpha(x, \emptyset) = 3/4$. The resulting acceptance probabilities are

$$\mathbb{E}[\pi(a|R1)] = 3/4 \qquad \pi_\emptyset(a|A1) = 3/4 \qquad \pi_t(a|A2) = 1$$

Therefore allowing to opt out strictly increases the DM's payoffs.

I conclude this subsection by showing that in a natural environment, it is however optimal to use the Blackwell most informative test, if there exists one.

Definition 4 (MLRP environment)**.** *An environment is MLRP if* $\Theta, X \subset \mathbb{R}$, $A = \{\theta : \theta > \overline{\theta}\}$ *for some* $\overline{\theta}$ *and all tests in* T *satisfy the monotone likelihood ratio property: for* $\theta > \theta'$,

$$\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \text{ is increasing in } x$$

This is a natural environment to capture that the agent's types are ordered and that a higher signal is indicative of a higher type. In this case, if there is a test Blackwell dominating all the others, then an optimal menu is to use only that test.

Proposition 3. *Suppose we are in an MLRP environment. If there is* $t \in T$ *such that* $t \succeq t'$ *for all* $t' \in T$*, then an optimal menu is to only offer* t*.*

Being in an MLRP environment has two consequences. First, the DM follows a cutoff strategy: there is $\overline{x} \in X$ such that $\alpha(x,t) = 1$ for $x > \overline{x}$ and $\alpha(x,t) = 0$ for $x < \overline{x}$. Second, Blackwell dominance implies a single-crossing condition on the acceptance decisions: if $t \succeq t'$, then $\pi_t(a|\theta') \geq \pi_{t'}(a|\theta')$ for $\theta' \in R$ implies $\pi_t(a|\theta) \geq \pi_{t'}(a|\theta)$ for $\theta \in A$. Intuitively, the reason is that more informative tests send relatively higher signals for higher types. So if a low type chooses the most informative test, the higher types must also choose that one. This prevents any pooling of A-types and R-types on two different tests. Combined with Lemma 1 that guarantees the inclusion of the Blackwell dominant test, we get our result. Note also that this result would hold using weaker information order like Lehmann (1988) or some weakening of it. The key property delivering the result is the single-crossing condition described above.

3.4 Tests ranked by difficulty

So far, we have looked at test that can be compared by how informative they are. Another natural way of ordering experiment when thinking about testing is to rank them by their difficulty. We are going to focus on types ordered on the real line and the available tests are binary and ordered by how difficult it is to get a high grade.

Definition 5 (Difficulty environment)**.** *An environment is a Difficulty environment if it is an MLRP environmnent, $X = \{0, 1\}$, $T \subset \mathbb{R}$ and for all $t > t'$, and $\theta > \theta'$,*

$$\frac{\pi_t(x=1|\theta)}{\pi_t(x=1|\theta')} \geq \frac{\pi_{t'}(x=1|\theta)}{\pi_{t'}(x=1|\theta')} \quad \textit{and} \quad \frac{\pi_t(x=0|\theta)}{\pi_t(x=0|\theta')} \geq \frac{\pi_{t'}(x=0|\theta)}{\pi_{t'}(x=0|\theta')}$$

If $t > t'$, I will say that t is harder than t'. To understand the last condition better, let $\mu(\cdot|x, t)$ be a posterior belief after observing signal x in test t. The monotone likelihood ratio property implies $\mu(\cdot|t, x = 1) \succeq_{FOSD} \mu(\cdot|t, x = 0)$, a higher signal is "good news" about the type (Milgrom, 1981). The last property in the definition further implies $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|t', x)$. That means that a pass grade shifts more beliefs towards higher type in a harder test and a fail grade shifts more beliefs towards lower types in an easy test. Or put differently, the harder a test the more informative it is about a high type when there is a pass-grade whereas an easier test is informative about the low types when the test is failed. As an example, if $\Theta \subset (0, 1)$ and $\pi_t(x = 1|\theta) = \theta^t$ we are in a Difficulty environment.

Proposition 4. *In a Difficulty environment, a singleton menu is optimal.*

Proof. See appendix. □

The proof proceeds in two steps. First, I show that there are at most two tests in the optimal menu and if there are two tests, the harder test must be more lenient that the easy test. In particular, I show that after the hard test, the DM must accept with some probability after a fail signal and in the easy test, reject with positive probability after a pass grade.

This means in particular that to maintain incentives to select both tests, the DM only reacts to the least informative signal from the test: in the hard test after a fail grade, in the easy test after a pass grade. This in turn implies that it would be better for the DM to use only one test and reject after a fail grade and accept after a pass grade.

3.5 Sufficient conditions for test inclusion

In this subsection, I explore in more details the notion of efficient allocation of tests to agents. I show that a sufficient condition to include a test in the optimal menu is if it is good at differentiating one A-type from all the R types. This captures a notion of a test tailored for the A-type.

Definition 6. *Fix $\theta \in A$. Test t θ-dominates t', $t \succeq_\theta t'$, if there is $\beta : X \times X \to [0,1]$ such that for all $x' \in X$*

$$\begin{aligned} & \sum_x \beta(x,x')\pi_t(x|\theta) \leq \pi_{t'}(x'|\theta) \\ \textit{for all } \theta' \in R, \quad & \sum_x \beta(x,x')\pi_t(x|\theta') \geq \pi_{t'}(x'|\theta') \\ \textit{for all } x \in X, \quad & \sum_{x'} \beta(x,x') = 1 \end{aligned}$$

To understand this definition better, compare it to Blackwell (1953)'s informativeness order. It requires the existence of a function β such that for all $x' \in X$, $\sum_x \beta(x,x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x,x') = 1$. There are two main differences. First, Definition 6 restricts attention to one A-type and all the R-types. Second, it is less demanding in the sense that the Blackwell order requires equality whereas inequalities are sufficient for our purpose. This is because we are fixing the utility function we are interested in, unlike in Blackwell (1953).

If a type $\theta \in A$ has a $\succeq_\theta$-dominant test, then this test is used in an optimal menu. This shows that an important property of tests is not so much how good they are at differentiating types, but how good they are at differentiating one type the DM wants to accept from all the types he wants to reject.

Proposition 5. *Suppose there is $t \in T$ and $\theta \in A$ such that $t \succeq_\theta t'$ for all $t' \in T$, then t is part of an optimal menu.*

Proof. See appendix. $\square$

Note that if there is a test for $\theta \in A$ such that $\operatorname{supp} \pi_t(\cdot|\theta) \cap \Big(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \Big) = \emptyset$, then this test is θ-dominates any other test. If each type in A has such a test, then the principal never makes a mistake. This condition is also necessary.

Proposition 6. *The principal's expected payoff is $\sum_{\theta \in A} \mu(\theta)$ if and only if for all $\theta \in A$, there exists $t \in T$ such that*

$$\operatorname{supp} \pi_t(\cdot|\theta) \cap \Big(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \Big) = \emptyset$$

Proof. ($\Leftarrow$) For each $\theta \in A$, let t_θ such that

$$\operatorname{supp} \pi_t(\cdot|\theta) \cap \Big(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \Big) = \emptyset$$

Then posting a menu $(t_\theta)_{\theta \in A}$ is optimal (eliminating duplicates if there are some). Each $\theta \in A$ chooses t_θ. For any strategy of $\theta' \in R$, the DM accepts after any $(x, t) \in \cup_{\theta : \sigma(t|\theta)=1} \operatorname{supp} \pi_t(\cdot|\theta)$ and rejects otherwise. This gives the DM and the A-types maximal payoffs and the R-types get rejected for any strategy they follow.

($\Rightarrow$) Suppose the DM's payoffs are maximal and there is $\theta \in A$ and for all $t \in T$ there is $\theta' \in R$ and $x \in X$ such that $\pi_t(x|\theta), \pi_t(x|\theta') > 0$. Then when θ chooses t out of the menu of tests, if θ' chooses t as well, at x, either the DM accepts θ' or rejects θ. Therefore, payoffs cannot be maximal. $\square$

Here, the principal just needs for each type he wants to accept a test where he can discriminate between that type and the R-types. Then he can offer a menu of tests where each A-type self selects into the test that discriminates him from the R-types. The actual learning only happens by observing the test selected and the testing technology serves as a detriment to deviations from R-types. The argument is then similar to an unravelling argument à la Milgrom (1981) and Grossman (1981). These are not fully revealing tests but tests that allow to perfectly discriminate *one* A-type from all the R-types. But it could be a very noisy tests for the other A-types.

There are different ways we can relax the conditions for inclusion of a test in the optimal menu. One way one can do that is by considering simplified environment like binary signals $X = \{x_0, x_1\}$.

Definition 7. *Suppose $X = \{x_0, x_1\}$ and let $\theta \in A$. A test t θ-dominates t', $t \succeq_\theta^b t'$, if for all $\theta' \in R$,*

$$\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} \geq \max\left\{\frac{\pi_{t'}(x_1|\theta)}{\pi_{t'}(x_1|\theta')}, \frac{\pi_{t'}(x_0|\theta)}{\pi_{t'}(x_0|\theta')}\right\} \geq \min\left\{\frac{\pi_{t'}(x_1|\theta)}{\pi_{t'}(x_1|\theta')}, \frac{\pi_{t'}(x_0|\theta)}{\pi_{t'}(x_0|\theta')}\right\} \geq \frac{\pi_t(x_0|\theta)}{\pi_t(x_0|\theta')}$$

The advantage of this much easier to check if it holds. It builds on a recent result of Di Tillio et al. (2022) on dominance of Blackwell experiment in decision problem satisfying the interval dominance order (**?**), a generalisation of single-crossing preferences (Milgrom and Shannon, 1994). We then get the following result:

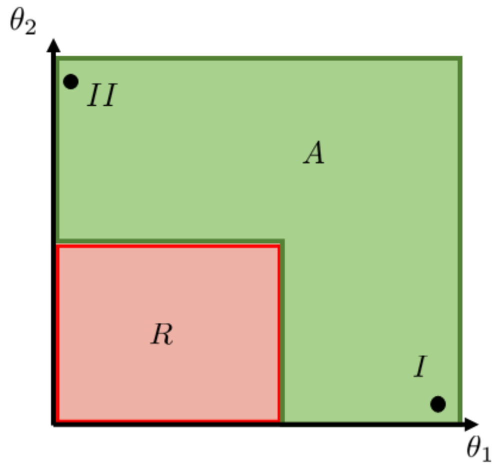
Proposition 7. *Let $\theta \in A$. If there is $t \in T$ such that $t \succeq_\theta^b t'$ for all $t' \in T$, then t is part of an optimal menu.*

Proof. See appendix. □

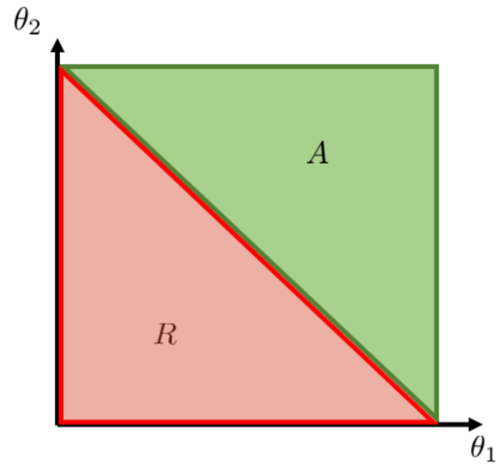
The following example illustrates how I can use Proposition 7 to get a better sense of which test is included in the optimal menu.

Example (Hiring a worker with bi-dimensional skills - a bit sketchy)**.** A firm (the DM) must hire a worker (the agent) whose type is bi-dimensional, $\Theta \subset (0,1)^2$ where each dimension represents a different skill. For simplicity suppose that Θ is ϵ-dense in $[0,1]^2$ for some $\epsilon > 0$, i.e., for any $a \in [0,1]^2$, there is $\theta \in \Theta$ such that $\theta \in B_\epsilon(a)$. Suppose that there are three different tests, $T = \{0, 1\}$ where $\pi_t(x = 1|(\theta_1, \theta_2)) = t\theta_1 + (1-t)\theta_2$ for $t = 0, 1$.

Figure 1a and Figure 1b represent two possibilities for the set A and R. It is relatively easy to show that for the case depicted in Figure 1a, $\theta = II$ (see Figure 1a), $t = 0 \succeq_{II}^b t'$ and for $\theta = I$, $t = 1 \succeq_I^b t'$ for all $t' \in T$. Therefore the optimal menu contains the two tests. On the other hand, for the case depicted in Figure 1b, for some prior μ, the optimal menu consists of only one test.



(a) $R = \{\theta : \max\{\theta_1, \theta_2\} \leq \overline{\theta}\}$



(b) $R = \{\theta : \theta_1 + \theta_2 \leq 1\}$

4 Extension: Communication

I consider here the possibility of adding a communication channel on top of the test choice. I will also relate my results to those of Glazer and Rubinstein (2004) and Carroll and Egorov (2019). There is now a finite set M of output messages with $|M| \geq |A|$ and a strategy is a mapping $\sigma : \Theta \to \Delta(T \times M)$. Note that all the results from the previous sections go through as from any finite set T one can create another T' that duplicate each test $|M|$ times. I call this variant of the model the menu game with communication.

In line with Theorem 1, each A-type chooses a message-test pair deterministically and each R-type mixes over some A-types message-test pair. A parallel theorem to Theorem 1 is thus obtained. Moreover, I show that when communication is added, each type in A announces his type, thus maximally differentiating himself, and each R-type pretends to be an A-type.

Theorem 2. *If communication is allowed, the same construction as Theorem 1 holds. Moreover, there is a DM-preferred equilibrium where each A-type reports his own type.*

Proof. The only thing we need prove is that it is optimal to have a different message for each type $\theta \in A$, the rest follows from Theorem 1. Suppose it is not the case and take a saddle-point (α, s, s') of the zero-sum game.

There is $\theta_1, \theta_2 \in A$ and $(t, m) \in T \times M$ such that $s(t, m|\theta_1) = s(t, m|\theta_2) = 1$ (if they use a different test then we can also change the message and nothing is changed). Then consider the alternative strategy α' where, for some unused (t, m') in the original mechanism, $\alpha'(t, m', x) = \alpha(t, m, x)$ for all $x \in X$ and $\alpha'(t'', m'', x) = \alpha'(t'', m'', x)$ for all other $(t'', m'') \in T \times M$ and all $x \in X$ otherwise. The new strategy α' is thus the same as α but makes sure that if the pair (t, m') is chosen, it uses the same actions as (t, m). Now consider the following strategy $\tilde{s}(\cdot|\theta)$ for $\theta \in A$ in the auxiliary game, $\tilde{s}(\cdot|\theta) = s(\cdot|\theta)$ for $\theta \neq \theta_1$ and $\tilde{s}(t, m'|\theta_1) = 1$. In the zero-sum game under the strategy α', the payoffs are the same than under (α, s, s') for all types. Moreover, any deviations under α' gives the same payoff than under α. Therefore, $(\alpha', \tilde{s}, s')$ is an equilibrium of the zero-sum game and $v(\alpha, s, s') = v(\alpha', \tilde{s}, s')$. Either α' is a best response to $(\tilde{s}, s')$, $(\alpha', \tilde{s}, s')$ is saddle-point of v and characterises an optimal menu. Or, α' is not a best-response and there is $\tilde{\alpha}$ such that $v(\tilde{\alpha}, \tilde{s}, s') > v(\alpha', \tilde{s}, s') = v(\alpha, s, s')$. This would contradict that (α, s, s') is a saddle point of v. □

Theorem 2 shows that the results extend naturally to an environment where communication is allowed. Because the DM could commit to a strategy, he can always guarantee each A-type at least as much as he would have if he would pool with another A-type. This guarantees that there is an equilibrium where he separates from the other A-types.

Note that because each A-type uses a different message and does not mix over tests, the test chosen does not contain any information: $\mu(\theta|m, t) = \mu(\theta|m)$. Thus all the information revealed by the test is through the signal realisations and not the test choice.

In the remainder of this section, I will connect the results developed in this model to the existing literature, and in particular to Glazer and Rubinstein (2004). Consider the following model generalising the one of Glazer and Rubinstein (2004). They consider a model of persuasion and verification where the agent sends a message and the DM chooses a test and a decision. Formally, the DM designs a mechanism defined by $\tau : M \to \Delta(T \times [0, 1]^X)$, that is a mechanism commits to a test and a decision for each test and signal realisation for each message. A strategy for the agent is $\delta : \Theta \rightarrow \Delta M$. The solution concept is weak Perfect Bayesian Equilibrium. In Glazer and Rubinstein (2004), the state space is some multidimensional set and each test in T perfectly reveals one dimension. I will call the mechanism τ a GR-mechanism.

One of the results of Glazer and Rubinstein (2004) is that the outcome of the optimal mechanism τ can be implemented without commitment in a wPBE of the following game: the agent chooses a message in M, based on the message, the DM chooses a test and based on the signal realisation and test, the DM accepts or rejects the agent. If the outcome of the optimal Glazer and Rubinstein (2004)-mechanism is the same as the one of the game above, I will say that it is credible. I will call that game a GR-game.

The fundamental difference between the Glazer and Rubinstein (2004) model and the one we have studied so far is that it is now the DM that chooses the test and not the agent. But as we will see, if we allow for communication in the menu test model, this distinction does not matter anymore.

Proposition 8. *The outcome of a GR-mechanism is credible for any T. Moreover, its outcome coincides with the menu game with communication.*

Proof. See appendix. □

This proposition generalises the commitment result of Glazer and Rubinstein (2004) to an arbitrary testing technology and type structure. Moreover, it shows that when there is communication, who chooses the test is not important. To understand this better, let us first note the dual role of test choice in the model without communication. In this case, the test is used both to communicate to the DM and to provide evidence which type the agent is. When we add communication on top of the menu of test, all the communication is through the cheap-talk message and the test is only used to provide evidence about the type.

Now consider the zero-sum game characterisation of the optimal menu, and in particular the payoffs of the A-types. Remember that in the zero-sum game, the A-types were maximising the DM's payoffs. Combined with the fact that the test choice does not carry additional information, we can let the DM choose it. If it was optimal for A-types to choose test t after message m, it will also be for the DM.

Carroll and Egorov (2019) study a similar model as Glazer and Rubinstein (2004), multi-dimensional types with the testing technology revealing one dimension, but with a different agent payoff function. Their objective is to study under which condition on the agent's payoffs there is full information revelation. They show that when there is full information revelation and some technical conditions are satisfied, the mechanism can be implemented by having the agent choosing the test, a parallel result to Proposition 8. Thus I show that the equivalence result they have also applies to other environments and is not a feature of full information revelation and their testing technology.

5 Concluding remarks

I study the design of optimal menus of tests. Menus allow the DM to have an additional dimension for information revelation as well as allow for a more efficient allocation of tests to agents. I provide a characterisation of the optimal menu in terms of an auxiliary zero-sum game. While proving this result, I also show that the characterisation holds for a general class of mechanisms allocating agent to tests. I use this result to characterise the optimal menu in different economic environments.

When the set of tests contains a Blackwell dominant test, that test is included in the optimal menu. When the acceptance probability of some A-types is lower than that of R-types on a subset of signals, it is optimal to include a test that pools all signals in that subset. However, when types are purely vertically differentiated and tests satisfy the monotone likelihood ratio property, the only possible menu with the Blackwell most informative test only contains that test.

It is interesting to think about this last result allowing for DM's commitment. Standard properties of Blackwell more informative test allows to show that they should be included in the menu with commitment. But to show that in the MLRP case, only the most informative test is part of the optimal menu, I used that the DM must best-reply, in particular that his best-reply takes the form of a cutoff function. It is exactly that form that prevents any other test to be included in the menu. With other strategies of the DM, it would be possible to sustain a non-singleton menu. Perhaps speculatively, this observation points to a hierarchy in the learning sources. It seems that the DM should always prioritise learning from the test result rather than from the test choice as *even with commitment* it is better to best-reply to the information rather than distorting the strategy to sustain a menu.

In an environment where tests are ranked by their difficulty, a singleton test is also optimal. This was not caused by the impossibility of sustaining a menu, but by the fact that the strategies needed to sustain a menu made it unprofitable to do so.

Turning to sufficient conditions for the inclusion of a test, I show that it if a test is more informative when restricting attention to one A-type and all the R-types, then it is included in the menu. I apply this insight to multidimensional type structures.

The sufficient conditions I derive in Proposition 1 and Proposition 5 are comparing properties of a test or acceptance probability of one A-type to those of all the R-types. This asymmetry between the R-types and the A-types is due to their different incentives to separate as reflected by their strategy in the auxiliary zero-sum game. While an A-type wants to be singled-out by choosing a different test, the R-types want to "hide behind" A-types and only choose to mimic them.

Finally, I show that adding a communication channel links the current model to existing models in the literature and generalises their results. Interestingly, adding the communication helps highlight the role of tests when there is no communication. Indeed, when communication is allowed, only Blackwell dominant tests are used which is not true without communication. Without communication, the role of tests are to serve as a communication channel and less informative tests are useful despite the learning loss from using them. When communication is allowed, this channel is channel is redundant with the cheap talk message and it is thus better to use a Blackwell more informative test.

References

Ball, I. (2019), ‘Scoring strategic agents’, *arXiv preprint arXiv:1909.01888* .

Ball, I. and Kattwinkel, D. (2019), Probabilistic verification in mechanism design, *in* ‘Proceedings of the 2019 ACM Conference on Economics and Computation’, EC ’19, Association for Computing Machinery, New York, NY, USA, p. 389–390.
URL: *https://doi.org/10.1145/3328526.3329657*

Baye, M., Tian, G. and Zhou, J. (1993), ‘Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs’, *Review of Economic Studies* **60**(4), 935–948.

Ben-Porath, E., Dekel, E. and Lipman, B. L. (2019), ‘Mechanisms with evidence: Commitment and robustness’, *Econometrica* **87**(2), 529–566.
URL: *https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA14991*

Bertsimas, D. and Tsitsiklis, J. N. (1997), *Introduction to linear optimization*, Vol. 6, Athena Scientific Belmont, MA.

Blackwell, D. (1953), ‘Equivalent comparisons of experiments’, *The Annals of Mathematical Statistics* **24**(2), 265–272.

Bull, J. and Watson, J. (2007), ‘Hard evidence and mechanism design’, *Games and Economic Behavior* **58**(1), 75–93.

Caplin, A., Dean, M. and Leahy, J. (Forthcoming), ‘Rationally inattentive behavior: Characterizing and generalizing shannon entropy’, *Journal of Political Economy* .
URL: *https://doi.org/10.1086/719276*

Carroll, G. and Egorov, G. (2019), ‘Strategic communication with minimal verification’, *Econometrica* **87**(6), 1867–1892.

Deb, R. and Stewart, C. (2018), 'Optimal adaptive testing: informativeness and incentives', *Theoretical Economics* **13**(3), 1233–1274.
URL: *https://econtheory.org/ojs/index.php/te/article/view/2914/0*

Deneckere, R. and Severinov, S. (2008), 'Mechanism design with partial state verifiability', *Games and Economic Behavior* **64**(2), 487–513.

Di Tillio, A., Ottaviani, M. and Sørensen, P. N. (2022), 'Comparison of experiments in mononone problems'.

Dziuda, W. and Salas, C. (2018), 'Communication with detectable deceit', *Available at SSRN 3234695* .

Ely, J., Galeotti, A., Jann, O. and Steiner, J. (2021), 'Optimal test allocation', *Journal of Economic Theory* **193**, 105236.

Frankel, A. and Kartik, N. (2019*a*), 'Improving information from manipulable data', *Journal of the European Economic Association* .

Frankel, A. and Kartik, N. (2019*b*), 'Muddled information', *Journal of Political Economy* **127**(4), 1739–1776.

Glazer, J. and Rubinstein, A. (2004), 'On optimal rules of persuasion', *Econometrica* **72**(6), 1715–1736.

Green, J. R. and Laffont, J.-J. (1986), 'Partially verifiable information and mechanism design', *The Review of Economic Studies* **53**(3), 447–456.

Grossman, S. J. (1981), 'The informational role of warranties and private disclosure about product quality', *The Journal of Law and Economics* **24**(3), 461–483.

Lehmann, E. L. (1988), 'Comparing location experiments', *The Annals of Statistics* **16**(2), 521–533.

Lipman, B. L. and Seppi, D. J. (1995), ‘Robust inference in communication games with partial provability’, *Journal of Economic Theory* **66**(2), 370–405.

Milgrom, P. R. (1981), ‘Good news and bad news: Representation theorems and applications’, *The Bell Journal of Economics* pp. 380–391.

Milgrom, P. and Shannon, C. (1994), ‘Monotone comparative statics’, *Econometrica* **62**(1), 157–180.

Nguyen, A. and Tan, T. Y. (2021), ‘Bayesian persuasion with costly messages’, *Journal of Economic Theory* **193**, 105212.

Perez-Richet, E. and Skreta, V. (Forthcoming), ‘Test design under falsification’, *Econometrica* .

Persico, N. (2000), ‘Information acquisition in auctions’, *Econometrica* **68**(1), 135–148.

Pomatto, L., Strack, P. and Tamuz, O. (2018), ‘The cost of information’.
URL: *https://arxiv.org/abs/1812.04211*

Rockafellar, R. T. (2015), *Convex Analysis*, Princeton University Press.
URL: *https://doi.org/10.1515/9781400873173*

Sims, C. A. (2003), ‘Implications of rational inattention’, *Journal of Monetary Economics* **50**(3), 665–690. Swiss National Bank/Study Center Gerzensee Conference on Monetary Policy under Incomplete Information.
URL: *https://www.sciencedirect.com/science/article/pii/S0304393203000291*

A Omitted proofs

A.1 Proof of Theorem 1

The plan of the proof is the following. First, I will characterise the optimal mechanism allocating types to test. Unlike menus, mechanisms allow the designer to randomise over tests allocation. Menus are a special type of mechanisms, where any randomisation has to be over tests where agents are indifferent. Therefore profits from the optimal mechanism are weakly greater than profits from the optimal menu. Then I show that the optimal mechanism can be implemented by posting a menu.

A direct mechanism is a mapping $\tilde{\sigma} : \Theta \to \Delta T$. The designer's problem is

$$\begin{aligned}
\tilde{V} = \max_{\tilde{\sigma},\alpha} \; & \sum_{\theta \in A} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) \\
\text{s.t.} \; & \sum_t (\tilde{\sigma}(t|\theta) - \tilde{\sigma}(t|\theta'))\pi_t(a|\theta) \geq 0 \text{ for all } \theta, \theta' \\
& \sum_t \tilde{\sigma}(t|\theta) = 1 \text{ for all } \theta \\
& \alpha \in BR(\tilde{\sigma})
\end{aligned}$$

If the DM could commit over a strategy α, his problem would be

$$\begin{aligned}
\tilde{V}(\alpha) = \max_{\tilde{\sigma}} \; & \sum_{\theta \in A} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) \\
\text{s.t.} \; & \sum_t (\tilde{\sigma}(t|\theta) - \tilde{\sigma}(t|\theta'))\pi_t(a|\theta) \geq 0 \text{ for all } \theta, \theta' \\
& \sum_t \tilde{\sigma}(t|\theta) = 1 \text{ for all } \theta
\end{aligned}$$

We have that $\max_\alpha V\tilde{(\alpha)} \geq \tilde{V}$ as the DM could always commit to the strategy used to get $\tilde{V}$.

Show that $\tilde{V}(\alpha) = \max_s \min_{s'} v(\alpha, s, s')$.

The dual problem of $V\tilde{(\alpha)}$ is

$$
\begin{aligned}
&\min_{y_{\theta,\theta'}, z_\theta} \sum_\theta z_\theta \\
\text{s.t. } &\text{for } \theta \in A, t: \ -\pi_t(a|\theta)\sum_{\theta'} y_{\theta,\theta'} + \sum_{\theta'} \pi_t(a_t|\theta') y_{\theta',\theta} + z_\theta \geq \mu(\theta)\pi_t(a|\theta) \\
&\text{for } \theta \in R, t: \ -\pi_t(a|\theta)\sum_{\theta'} y_{\theta,\theta'} + \sum_{\theta'} \pi_t(a_t|\theta') y_{\theta',\theta} + z_\theta \geq -\mu(\theta)\pi_t(a|\theta) \\
&y_{\theta,\theta'} \geq 0, \ z_\theta \in \mathbb{R}
\end{aligned}
$$

Note that $y_{\theta,\theta'}$ is the dual variable associated to the IC constraint of type θ deviating to θ' and z_θ the dual variable associated with the feasibility constraint of type θ.

I will show that the optimal mechanism can be characterised by an equilibrium of the zero-sum game for a fixed α by verifying complementary slackness conditions. To this end I will

1. Guess values for $\tilde{\sigma}, y, z$
2. Verify that the guessed variables are feasible in their respective problem
3. Verify complementary slackness conditions

If variables are feasible and satisfy complementary slackness then they are optimal (see e.g., Bertsimas and Tsitsiklis, 1997, Theorem 4.5).

Take an equilibrium of the zero-sum game fixing α, (s, s'), i.e., $s \in \arg\max \min_{s'} v(\alpha, \tilde{s}, s')$ and $s' \in \arg\min \max_s v(\alpha, s, \tilde{s}')$.

Guess

- $y_{\theta,\theta'} = 0$ for $\theta \in A$
- $y_{\theta',\theta} = 0$ for $\theta', \theta \in R$
- $y_{\theta',\theta} = \mu(\theta')s'(\theta|\theta')$ for $\theta' \in R$, $\theta \in A$

- $z_{\theta'} = 0$ for $\theta' \in R$
- $z_\theta = \mu(\theta)\pi_{t^\theta}(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{t^\theta}(a|\theta')$ for some $t^\theta \in \operatorname{supp} s(\cdot|\theta)$ for $\theta \in A$
- $\tilde{\sigma}(t|\theta) = s(t|\theta)$ for $\theta \in A$
- $\tilde{\sigma}(t|\theta') = \sum_{\theta \in A} s'(\theta|\theta')s(t|\theta)$ for $\theta' \in R$

Plugging in these guessed values in the constraints of the dual problem, we get for the constraints $(\theta \in R, t)$,

$$-\pi_t(a|\theta) \sum_{\theta' \in A} \mu(\theta)s'(\theta'|\theta) \geq -\mu(\theta)\pi_t(a|\theta)$$

which holds with equality because $\sum_{\theta' \in A} s'(\theta'|\theta) = 1$.

For the constraints $(\theta \in A, t)$, plugging in the guessed values gives

$$\mu(\theta)\pi_{t^\theta}(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{t^\theta}(a|\theta') \geq \mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(a|\theta')$$

which holds because (s, s') is an equilibrium of the zero-sum game and thus t^θ maximises this expression.

I will now verify that the induced $\tilde{\sigma}$ is feasible. Note that any allocation is either an A-type allocation or a convex combination of A-type allocation.

For the A-types, if (s, s') is an equilibrium of the auxiliary game, $\theta \in A$ must be weakly worse off mimicking another A-type, $\tilde{\theta}$, in the auxiliary game:

$$\sum_t s(t|\theta)\Big[\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(a|\theta')\Big] \geq \sum_t s(t|\tilde{\theta})\Big[\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(a|\theta')\Big]$$

$$\Leftrightarrow \mu(\theta) \sum_t (s(t|\theta) - s(t|\tilde{\theta}))\pi_t(a|\theta) \geq \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta') \sum_t (s(t|\theta) - s(t|\tilde{\theta}))\pi_t(a|\theta')$$

Note that the LHS is the IC constraint of θ deviating to $\tilde{\theta}$ and the RHS is positive. Indeed, whenever $\sum_t (s(t|\theta) - s(t|\tilde{\theta}))\pi_t(a|\theta') < 0$, we have $s'(\theta|\theta') = 0$. Therefore the IC constraints of an A-type deviating to an A-type are satisfied. Because the $\tilde{\sigma}(t|\theta')$ for $\theta' \in R$ is a convex combination of A-type allocation, all the IC constraints of A-types are satisfied.

For the R-types, any R-type is indifferent between reporting his type and an A-type he is mimicking in the zero-sum game. He also weakly prefers reporting his own type over an A-type he is not mimicking. Thus there are no deviations to A-types. Because any other R-type allocation is a convex combination of A-types allocation, no R-type is willing to report another R-type.

We are left with checking that all complementary slackness conditions are satisfied. That is, if a variable in the primal or dual problem is strictly positive, then the corresponding constraint must be binding. The dual variables y is strictly positive if and only if $\theta \in A$, $\theta' \in R$ and $s'(\theta|\theta') > 0$. The corresponding IC constraint is θ' deviating to θ. But in that case the IC constraint binds as mimicking θ maximises the probability of being accepted in the zero-sum game and thus θ' gets the same expected probability of being as if he would get θ's distribution.

On the other hand the dual constraints are only slack for $(\theta \in A, t)$ such that

$$\mu(\theta)\pi_{t^\theta}(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{t^\theta}(a|\theta') > \mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(a|\theta')$$

In this case $\tilde{\sigma}(t|\theta) = 0$ as $s(t|\theta) = 0$. Therefore, the complementary slackness conditions are satisfied and we have characterised an optimal mechanism when the DM commits to α.

Remember that $v(\alpha, s, s') = \sum_t \sum_{\theta \in A} s(t|\theta)\big[\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(a|\theta')\big]$ and note that it is the DM's payoff in the induced mechanism. Thus, we can express the value of an optimal mechanism with commitment to α, $\tilde{V}(\alpha) = \max_s \min_{s'} v(\alpha, s, s')$. The optimal value of the DM, when he can commit is therefore $\max_\alpha \max_s \min_{s'} v(\alpha, s, s') = \max_{\alpha,s} \min_{s'} v(\alpha, s, s')$.

Show that the DM does not benefit from commitment.

Consider the two-players game where player one chooses (s, α) to maximise v and player two chooses s' to maximise $-v$. This game satisfies the condition for the existence of a NE in Baye et al. (1993). Indeed, a sufficient condition for the existence of NE is that (1) strategy spaces are a subset of $\mathbb{R}^m$, (2) v is continuous in all arguments, (3) v is linear in one player's strategy and (4) there are two players. Condition (2) guarantees diagonal transfer continuity (see Proposition 2 in Baye et al., 1993), conditions (2) and (3) guarantee diagonal transfer quasi-concavity (see Proposition 1(e) in Baye et al., 1993). Together this implies the conditions stated in Theorem 1 in Baye et al. (1993). (For complete definitions see the paper.)

Therefore there is (α^*, s^*, s'^*) such that

$$v(\alpha, s, s'^*) \leq v(\alpha^*, s^*, s'^*) \leq v(\alpha^*, s^*, s')$$

for all α, s, s' and $v(\alpha^*, s^*, s'^*) = \max_{\alpha,s} \min_{s'} v(\alpha, s, s')$.

Notice that $v(\alpha, s^*, s'^*) \leq v(\alpha^*, s^*, s'^*)$ for all α. Because v is the DM's expected utility and s^*, s'^* induce the optimal mechanism, $\alpha^* \in BR(\tilde{\sigma}^*)$ where $\tilde{\sigma}^*$ is the mechanism induced by (s^*, s'^*).

Show that an optimal mechanism can be implemented by posting a menu.

The idea of the proof is the following. Take a saddle point (α, s, s') of v and suppose that s mixes over two tests for some $\theta \in A$. Then for $\tilde{s}$ that puts probability one on one test, we must have $v(\alpha, s, s') = v(\alpha, \tilde{s}, s')$. The idea is that under the new mechanism induced by $(\tilde{s}, s')$, there might be $\tilde{\alpha}$ such that $v(\alpha, \tilde{s}, s') < v(\tilde{\alpha}, \tilde{s}, s')$, contradicting that (α, s, s') is a saddle point. Intuitively, putting probability one on one test for $\theta \in A$ does not change his payoffs but allows the DM to reoptimise his strategy and increase payoffs. If all A-types have a "pure" allocation in the optimal mechanism, then a menu implementation is feasible: IC constraints imply that A-types prefer their tests over another A-type's test and any randomisation from R-types can be implemented in equilibrium as they maximise their probability of being accepted in the auxiliary game.

The bulk all the proof is devoted to show that generically, there is such $\tilde{\alpha}$ that *strictly* increase payoffs. This is done by showing that generically $\theta \in A$ is only willing to mix if the DM also mixes after some signal. The $\tilde{\alpha}$ that increases payoffs is one that only uses pure strategies. Then there is a pure strategy of A-types that increases v as they were only willing to mix if the DM also mixes after some signal. Thus we would have found a $(\tilde{\alpha}, \tilde{s})$ such that $v(\alpha, \tilde{s}, s') <$

$v(\tilde{\alpha}, \tilde{s}, s')$. A convergence argument shows that A-types' strategies are also pure in the non-generic case.

We first are going to record properties of equilibrium strategies when mixed strategies are played. Their aim is to characterise under which condition an A-type mixing implies that the DM is using a mixed strategy after some signal.

Lemma 2. *For any fixed α, if there is an equilibrium of the zero-sum game where some type $\theta \in A$ plays a mixed strategy, then there is also an equilibrium of the the zero-sum game (s, s') where*

- *either for all $t \in \operatorname{supp} s(\cdot|\theta)$, there is $x, x' \in \operatorname{supp} \cup_\theta \pi_t(\cdot|\theta)$ such that $\alpha(t, x) < 1$ and $\alpha(t, x') > 0$, i.e., some signals are accepted, some signals are not.*
- *or θ plays a pure strategy on a test t with $\pi_t(a|\tilde{\theta}) = 0$ for all $\tilde{\theta} \in \Theta$.*

Proof. Fix α and take some $s^* \in \arg\max \min_{s'} v(\alpha, s, s')$, i.e., it is an equilibrium strategy of the zero-sum game. Suppose s^* is in mixed strategy for some $\theta \in A$.

If the DM accepts after all signals in all tests in the support of θ's strategy, then playing a pure strategy on one test for θ does not change the payoffs of any type on- and off-path and thus constitutes a new equilibrium. Similarly if the DM rejects after any signal in the support of θ's strategy.

Suppose there are two tests, t, t' in the support of $s^*(\cdot|\theta)$ where the DM accepts after all signals in t but not in the other test t'. Then I can show that θ deviating to a strategy $s(\cdot|\theta)$ putting probability zero on t can only increase v. Indeed, type $\theta \in A$ is indifferent between all test in the support of his strategy, if the behaviour of types in R does not change. On the other hand, the overall payoff of the R types can only decrease as they can only choose strategies that either were previously available or have a smaller probability of being accepted if they keep the same strategy. Therefore, $\min_{s'} v(\alpha, s^*, s') \leq \min_{s'} v(\alpha, s, s')$ and the new strategy s is also in $\arg\max \min_{s'} v(\alpha, s, s')$. Thus we can always find another equilibrium strategy where the tests that accept after all signals is not used.

If there is a test in the support of θ's strategy where the DM rejects after all signals, then a similar argument as above shows that playing that test with probability one also constitutes an optimal strategy for θ and therefore he can play a pure strategy in equilibrium. $\square$

We will now focus on equilibria where when a type $\theta \in A$ mixes, for all t in the support, some signals are accepted and some are rejected.

Continue to fix α and consider an equilibrium of the auxiliary game (s^*, s'^*) such that for some $\theta \in A$ and $t, t' \in T$, $s^*(t|\theta)$ and $s^*(t'|\theta) > 0$. Note that $s^* \in \arg\max_s \min_{s'} v(\alpha, s, s')$.

The following lemma shows us that if there is an equilibrium in mixed strategy for some $\theta \in A$, it is possible to perturb the strategy of that A-type and still being an equilibrium strategy.

Lemma 3. *Fix α. If there is an equilibrium of the zero-sum game (s^*, s'^*) such that for some $\theta \in A$ and $t, t' \in T$, $s^*(t|\theta)$ and $s^*(t'|\theta) > 0$, then there is some $\epsilon > 0$ such that the strategy $\tilde{s}$ that increases type θ's probability on t by ϵ and decreases it on t' by ϵ and keeps all the others' fixed is also an equilibrium strategy.*

Proof. Note that the type $\theta \in A$ is indifferent between $s^*, \tilde{s}$, i.e., $v(\alpha, s^*, s'^*) = v(\alpha, \tilde{s}, s'^*)$ and $s^* \in \arg\max_s \min_{s'} v(\alpha, s, s')$. We want to show that $\tilde{s} \in \arg\max_s \min_{s'} v(\alpha, s, s')$. For that it will be enough to show that $\min_{s'} v(\alpha, \tilde{s}, s') \geq \min_{s'} v(\alpha, s^*, s')$.

When evaluating $\max \min v(\alpha, s, s')$, take a selection of $\arg\min_{s'} v(\alpha, s, s')$ such that if $\sum_t s(t|\theta)\pi_t(a|\theta') \geq \sum_t s(t|\tilde{\theta})\pi_t(a|\theta')$ for all $\tilde{\theta} \in A$, then $s'(\theta|\theta') = 1$ (by linearity of v in s', this is without loss of generality).

Consider all the types $\theta' \in R$ such that $s'(\theta|\theta') \neq 1$ under the selection described above. That is all R-types that strictly prefer another $\tilde{\theta} \in A$ over θ. In that case, modifying θ's strategy by a small enough ϵ will not change the strategy of these R-types.

For the types $\theta' \in R$ such that $s'(\theta|\theta') = 1$, either they now strictly prefer another type or θ is still their preferred type. In any case, the payoffs v is weakly increasing, as either s' does not change or some $\theta' \in R$ choose to mimic another $\tilde{\theta} \in A$, i.e., choose a strategy that was previously available. Thus the probability of being accepted must be weakly lower for these R-types. Therefore, $\tilde{s} \in \arg\max_s \min_{s'} v(\alpha, s, s')$. $\square$

We use Lemma 3 to show some properties of equilibrium strategies whenever $s^*(\cdot|\theta)$ is in mixed strategy.

Lemma 4. *Fix α. If there is an equilibrium of the zero-sum game (s^*, s'^*) such that for some $\theta \in A$ and $t, t' \in T$, $s^*(t|\theta)$ and $s^*(t'|\theta) > 0$ then for all $\theta' \in R$ such that $s'^*(\theta|\theta') > 0$, either $\pi_t(a|\theta') = \pi_{t'}(a|\theta')$ or they strictly prefer θ, $\sum_t s^*(t|\theta)\pi_t(a|\theta') > \sum_t s^*(t|\tilde{\theta})\pi_t(a|\theta')$ for all $\tilde{\theta} \in A$.*

Proof. As the set of saddle points is a product set, s'^* must be a BR to $\tilde{s}$ as defined in Lemma 3. One possibility is that the payoffs when facing s^* and $\tilde{s}$ do not change for some R-types, i.e., $\pi_t(a|\theta') = \pi_{t'}(a|\theta')$ and so the same strategy is best-response for them.

To show the other possibility, suppose that $\sum_t s^*(t|\theta)\pi_t(a|\theta') = \sum_t s^*(t|\tilde{\theta})\pi_t(a|\theta')$ for some $\tilde{\theta} \in A$ and $\pi_t(a|\theta') \neq \pi_{t'}(a|\theta')$. Then to have s'^* to be a best response $\tilde{s}$, it must be that the change to $\tilde{s}$ strictly increases the payoffs of θ', i.e., $\pi_t(a|\theta') > \pi_{t'}(a|\theta')$. But then consider the other strategy $\tilde{s}_1$ that puts additional probability ϵ_1 on t' and decreases the probability on t by ϵ_1. Using the same argument as in Lemma 3, $\tilde{s}_1 \in \arg\max_s \min_{s'} v(\alpha, s, s')$. The strategy s'^* must also be a best-response to $\tilde{s}_1$, which cannot be true if $\sum_t s^*(t|\theta)\pi_t(a|\theta') = \sum_t s^*(t|\tilde{\theta})\pi_t(a|\theta')$ for some $\tilde{\theta} \in A$ and $\pi_t(a|\theta') > \pi_{t'}(a|\theta')$. Therefore, $\sum_t s^*(t|\theta)\pi_t(a|\theta') > \sum_t s^*(t|\tilde{\theta})\pi_t(a|\theta')$, i.e., θ' plays a pure strategy. $\square$

This means that if there is a mixed strategy over t, t' for some $\theta \in A$ in the zero-sum game, at least one of these two assertions are true (1) for some $\theta' \in R$, $\pi_t(a|\theta') = \pi_{t'}(a|\theta')$ or (2) there exists a subset $Z \subseteq R$ such that $\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_t(a|\theta') = \mu(\theta)\pi_{t'}(a|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_{t'}(a|\theta')$. Condition (1) was explicitly shown. Condition (2) is the indifference condition of type $\theta \in A$ when a subset $Z \subseteq R$ mimicks him. Lemma 4 tells us that if condition (1) does not hold then all R-types mimicking θ must play a pure strategy which here are the types in Z.

Consider the following condition:

Condition 1 (M)**.** *For any test $t \in T$, for all $\tilde{\theta} \in \Theta$, $\pi_t(\cdot|\tilde{\theta})$ has full support on X.*

For any two tests $t, t' \in T$, there are no subsets of signals $\emptyset \neq X_1, X_2 \subset X$ such that

- $\sum_{x \in X_1} \pi_t(x|\theta') - \sum_{x \in X_2} \pi_{t'}(x|\theta') = 0$ *for some $\theta' \in R$*
- *or for some $\theta \in A$ and $Z \subseteq R$, $\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_t(a|\theta') = \mu(\theta)\pi_{t'}(a|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_{t'}(a|\theta')$.*

If condition (M) is satisfied then we cannot have $s^*(\cdot|\theta)$ in mixed strategy and $\alpha(t, x) \in \{0, 1\}$ for all on-path (t, x). Indeed, if we do have a mixed strategy, the full support assumption guarantees that if one type is accepted or rejected with probability one, then all types are. Then from Lemma 2, we can assume that $\pi_t(a|\theta') \in (0, 1)$ for all $\theta' \in R$ in a mixed strategy equilibrium.

But in Lemma 4, we have proven that to have a mixed strategy, either $\pi_t(a|\theta') = \pi_{t'}(a|\theta')$or each $\theta' \in R$ mimicking θ plays a pure strategy. The first possibility is not possible as $\pi_t(a|\theta') \in (0, 1)$ and thus that would contradict the first condition in (M).

The second possibility from Lemma 4 is that there is a subset $Z \subseteq R$ that mimics θ and from the indifference condition, we get $\mu(\theta)\pi_t(a|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_t(a|\theta') = \mu(\theta)\pi_{t'}(a|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_{t'}(a|\theta')$. Again, if $\alpha(t, x) \in \{0, 1\}$ for all on-path (t, x) and condition (M) holds, then we have a contradiction.

Now take a saddle point of v, $\big((\alpha^*, s^*), s'^*\big)$. Suppose that T satisfies condition (M). By linearity of v in α, it must be that $v(\alpha^*, s^*, s'^*) = v(\tilde{\alpha}, s^*, s'^*)$, where $\tilde{\alpha}$ sets $\tilde{\alpha}(t, x) = 0$ whenever $\alpha^*(t, x) \in (0, 1)$ and such (t, x) exists. From condition (M), no $\theta \in A$ is willing to mix under $\tilde{\alpha}$, so there is $\tilde{s}$ that is a BR to $(\tilde{\alpha}, s'^*)$ such that $v(\tilde{\alpha}, \tilde{s}, s'^*) > v(\tilde{\alpha}, s^*, s'^*) =$

$v(\alpha^*, s^*, s'^*)$, a contradiction. Therefore, if condition (M) is satisfied, $s^*(\cdot|\theta)$ is a pure strategy for all $\theta \in A$.

I now show that condition (M) is generic in the following sense. First, endow Π, the set of tests with signal space X, with the following metric, $d_2(\pi, \pi') \equiv \max_\theta \sqrt{\sum_x \left(\pi(x|\theta) - \pi'(x|\theta)\right)^2}$.[5]

Lemma 5. *For any T, either T satisfies (M) or for each $t \in T$, there is a sequence of tests $\pi_t^n \to \pi_t$ such that $T^n \equiv \{\pi_t^n\}_{t \in T}$ satisfy (M).*[6]

Proof. Remember that there is a finite number of types, tests and signals, thus there is a finite number of equalities to check to make sure that condition (M) is satisfied.

Suppose T does not satisfy (M). The first possibility is that there is $t \in T$ that does not satisfy full-support. In that case, for each $\tilde{\theta} \in \Theta$, take some $x \in X$ with $\pi_t(x|\tilde{\theta}) > 0$ and modify it by setting $\pi'_t(x|\tilde{\theta}) = \pi_t(x|\theta) - \epsilon$ and $\pi'_t(x'|\tilde{\theta}) = \pi_t(x'|\theta) + \frac{\epsilon}{|X|-1}$ for $x' \neq x$ for some $\epsilon > 0$ small enough.

Suppose that after this change, there are two tests $t, t' \in T$ such that for all $\emptyset \neq X_1, X_2 \subset X$ and for some $\theta' \in R$, $\pi_t(X_1|\theta') = \pi_{t'}(X_2|\theta')$. Consider the following modification, take $x \in X_1$, $x' \in X \setminus X_1$ and set $\pi'_t(x|\theta') = \pi_t(x|\theta') + \epsilon$ and $\pi'_t(x'|\theta') = \pi_t(x'|\theta') - \epsilon$ for some $\epsilon > 0$ small enough. We get $\pi_t(X_1|\theta') \neq \pi_{t'}(X_2|\theta')$ and that would be true for all $\epsilon' \in (0, \epsilon]$. Moreover, there is only a finite number of ϵ' that would create new equalities that would upset condition (M). Indeed, suppose that for some ϵ', there is $\emptyset \neq X'_1, X'_2 \subset X$ such that $\pi_t(X'_1|\theta') \neq \pi_{t'}(X'_2|\theta')$ but $\pi'_t(X'_1|\theta') = \pi_{t'}(X'_2|\theta')$, i.e., a new inequality is created. This means that $x \in X'_1$ and $x' \in X \setminus X'_1$ or vice-versa. But then for any $\epsilon'' < \epsilon'$, the equality would be upset again.

A similar modification can be done for any $\theta \in A$ by modifying $\pi_t(\cdot|\theta)$ in a similar way.

[5] Note that under this metric, two information structures inducing the same posterior beliefs can be "far" from each other. However, two information structures "close" to each other under this metric will have posterior beliefs "close" to each other. This does not play any role in the analysis.

[6] Note that by a small abuse of notation, T is here both a set of tests and an index set.

Because for each $\tilde{\theta} \in \Theta$, there is only a finite number of inequalities to check and a modification like the one described above is possible, we can find a sequence of $(\epsilon^n) \to 0$, where for each n the associated set T^n satisfies (M). As each modified test converges to the original t, we have $\pi_t^n \to \pi_t$ for all $t \in T$. $\square$

Suppose T does not satisfy (M). Take a sequence (T^n) converging to T as described in Lemma 5. For each $t \in T$, $\pi_t^n(\cdot|\theta) \to \pi_t(\cdot|\theta)$ where the convergence is in the Euclidian metric. To make the dependence of v with T explicit, I write $v(\alpha, s, s'; T)$. I will now show that the Nash Equilibrium correspondence of the zero-sum game satisfies the closed graph property and that allows to conclude that there is an s in pure strategies in an equilibrium of the zero-sum game at T.

Take a sequence (α^n, s^n, s'^n) of NE for each T^n such that s^n is in pure strategy for each $\theta \in A$.[7] Because (α^n, s^n, s'^n) is a bounded sequence of $\mathbb{R}^m$, it admits a converging subsequence. Let us focus on that converging subsequence to some (α^*, s^*, s'^*). If (s^n) is in pure strategy for all $\theta \in A$ and converges, it must be that it converges to a pure strategy for all $\theta \in A$.

I will now show that (α^*, s^*, s'^*) constitutes a NE at T thus proving there is a pure strategy for A-types in equilibrium at T. The function v is a bounded product and sum of continuous function and so it is jointly continuous in $(\alpha, s, s'; T)$. For each n,

$$\begin{aligned} & v(\alpha^n, s^n, s'^n; T^n) \geq v(\tilde{\alpha}, \tilde{s}, s'^n; T^n), \text{ for all } \tilde{\alpha}, \tilde{s} \\ \Rightarrow\ & v(\alpha^*, s^*, s'^*; T) \geq v(\tilde{\alpha}, \tilde{s}, s'^*; T), \text{ for all } \tilde{\alpha}, \tilde{s}, \text{by continuity of } v \end{aligned}$$

Therefore (α^*, s^*) is a best-response to s'^* at T. Similarly, we can show that s'^* is a best-response to (α^*, s^*) at T. This shows that (α^*, s^*, s'^*) constitutes a NE at T and establishes that there is an equilibrium with pure strategies for A-types when (M) is not satisfied.

In the optimal mechanism characterised in the first part of the proof, the tests used are only those on the support of s^*. Therefore, the incentive compatibility constraints imply that

[7] Note that the definitions of α and s depend on T but only as an index of tests and not on the test set itself. Because T and T^n have the same cardinality, the sequence is well-defined.

$\sum_t \tilde{\sigma}(t|\theta)\pi_t(a|\theta) \geq \sum_t \tilde{\sigma}(t|\tilde{\theta})\pi_t(a|\theta)$ for all $\tilde{\theta} \in A$, where each $\tilde{\sigma}(\cdot|\tilde{\theta})$ puts probability one on one test. Thus when offered with a menu of tests, each $\theta \in A$ is better off choosing "his" test over any other.

Each $\theta' \in R$ is indifferent between any tests in $\operatorname{supp} \tilde{\sigma}(\cdot|\theta')$ and therefore can mix according to $\tilde{\sigma}(\cdot|\theta')$.

This concludes the proof of Theorem 1.

A.2 Proof of Lemma 1

Because $t \succeq t'$ implies $t \succeq_\theta t'$ for any $\theta \in A$, Lemma 1 is a corollary of Proposition 5 proven below.

A.3 Proof of Proposition 2 and Proposition 1

Proof. Suppose the DM only uses t and let t' be the coarsened version of t that pools signals in X'. Let $T = \{t, t'\}$. Let $\pi_{t'}(x'|\theta) = \sum_{x \in X'} \pi_t(x|\theta)$ for some $x' \in X'$.

Consider the deviation, $(\tilde{\alpha}, \tilde{s})$: $\tilde{\alpha}(x', t') = \tilde{\alpha}$ and $\tilde{\alpha}(x, \tilde{t}) = \alpha(x, \tilde{t})$ for $x \neq x'$, $\tilde{t} = t, t'$ and $\tilde{s}(t'|\theta) = 1$ if $\sum_{x \in X'} \tilde{\alpha}\pi_t(x|\theta) > \sum_{x \in X'} \alpha(x, t)\pi_t(x|\theta)$and $\tilde{s}(\cdot|\theta) = s(\cdot|\theta)$ otherwise. We want to show that

$$\min_{s'} v(\tilde{\alpha}, \tilde{s}, s') \geq \min_{s'} v(\alpha, s, s')$$
$$\Leftrightarrow \sum_{\theta \in A} \sum_{x \in X'} \mu(\theta) \big[(\tilde{\alpha} - \alpha(x, t))\pi_t(x|\theta)\big]^+ \geq \sum_{\theta' \in R} \sum_{x \in X'} \mu(\theta') \big[(\tilde{\alpha} - \alpha(x, t))\pi_t(x|\theta')\big]^+$$

which is exactly the condition in Proposition 2. Note that the strategy of the R-types is to mimick a type choosing t' iff $\sum_{x \in X'} \tilde{\alpha}\pi_t(x|\theta') > \sum_{x \in X'} \alpha(x, t)\pi_t(x|\theta')$.

To prove Proposition 1, take $\theta \in A$ and $X' \subseteq X$ such that

$$\frac{\sum_{x \in X'} \alpha(x, t)\pi_t(x|\theta)}{\sum_{x \in X'} \pi_t(x|\theta)} \leq \frac{\sum_{x \in X'} \alpha(x, t)\pi_t(x|\theta')}{\sum_{x \in X'} \pi_t(x|\theta')} \quad \text{for all } \theta' \in R$$

then setting $\tilde{\alpha} = \frac{\sum_{x\in X'} \alpha(x,t)\pi_t(x|\theta)}{\sum_{x\in X'} \pi_t(x|\theta)}$ is enough. Indeed this implies that $\sum_{x\in X'} \big(\tilde{\alpha} - \alpha(x,t)\big)\pi_t(x|\theta) = 0$ and rearranging the expression above, we get

$$\sum_{x\in X'} \pi_t(x|\theta') \cdot \frac{\sum_{x\in X'} \alpha(x,t)\pi_t(x|\theta)}{\sum_{x\in X'} \pi_t(x|\theta)} \leq \sum_{x\in X'} \alpha(x,t)\pi_t(x|\theta') \Leftrightarrow \sum_{x\in X'} \big(\tilde{\alpha} - \alpha(x,t)\big)\pi_t(x|\theta') \leq 0$$

for all $\theta' \in R$ and thus the condition of Proposition 2 is satisfied. $\square$

A.4 Proof of Proposition 3

Proof. Note that in an MLRP environment, the strategy of the DM takes the form of a cutoff strategy. For each test t, there is $x_t \in X$ such that $\alpha(x,t) = 0$ for $x < x_t$, $\alpha(x,t) = 1$ for $x > x_t$ and $\alpha(x_t,t) \in [0,1]$. From Lemma 1, we know that there is an optimal menu containing the Blackwell most informative test. Because all tests are MLRP and the DM's payoffs satisfy single-crossing condition, the Lehmann order is well-defined and the Blackwell order implies the Lehmann order (Lehmann, 1988; Persico, 2000). Let $\succeq^a$ denote the Lehmann order.

The Lehmann order is defined on continuous information structure. But as outlined in Lehmann (1988), we can always make our conditional probabilities continuous by adding independent uniform between each signal. Let's assume, without loss of generality, that $X = \{1, ..., n\}$. The new distribution over signal is $\tilde{y}|\theta = \tilde{x}|\theta - u$ where $u \sim$U$[0,1]$. Denote by F_t the cdf associated with the new information structure.

We have that $t \succeq^a t'$ if $y^*(\theta, y) \equiv F_t(y^*|\theta) = F_{t'}(y|\theta)$ is nondecreasing in θ for all y. In particular, this condition implies that if $F_t(y|\theta') \leq (<) F_{t'}(y'|\theta')$ then $F_t(y|\theta) \leq (<) F_{t'}(y'|\theta)$ for all $\theta > \theta'$.

Let α be the optimal strategy and x_t be the cutoff signal associated to each test. To each $\big(\alpha(\cdot,t), x_t\big)$ we can associate a $y_t \equiv x_t - \alpha(x_t,t)$.

If t is part of an optimal menu, it must be that there is some $\theta' \in R$ such that $\pi_t(a|\theta') \geq \pi_{t'}(a|\theta')$ for all t'. Or put differently, $F_t(y_t|\theta') \leq F_{t'}(y_{t'}|\theta')$ for all t'. But then $F_t(y_t|\theta) \leq F_{t'}(y_{t'}|\theta)$ for all t' and all $\theta > \theta'$, in particular all $\theta \in A$. Therefore all type in A prefer test t

as well and there is an equilibrium of the zero-sum game where all types in $\theta \in A$ choose t. (If there is an A-type that is indifferent between t and t' then all types in R must be indifferent or prefer t' so choosing t is an equilibrium strategy for such A-type.) $\square$

A.5 Proof of Proposition 4

I first show that if $t > t'$, then $\mu(\cdot|t,x) \succeq_{FOSD} \mu(\cdot|t',x)$ where $\succeq_{FOSD}$ denotes first-order stochastic dominance.

Proof. The proof is similar to the one in Milgrom (1981). Denote by $G_t(\cdot|x)$ the cdf of posterior beliefs after signal x in test t. For all $\theta > \theta'$,

$$\mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Take some $\theta^* \geq \theta'$. Summing over θ, we get

$$\sum_{\theta>\theta^*}\mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \sum_{\theta>\theta^*}\mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Inverting and summing over θ', we get

$$\frac{\sum_{\theta^*\geq\theta'}\mu(\theta')\pi_t(x|\theta')}{\sum_{\theta>\theta^*}\mu(\theta)\pi_t(x|\theta)} \leq \frac{\sum_{\theta^*\geq\theta'}\mu(\theta')\pi_{t'}(x|\theta')}{\sum_{\theta>\theta^*}\mu(\theta)\pi_{t'}(x|\theta)}$$

which implies

$$\frac{G_t(\theta^*|x)}{1-G_t(\theta^*|x)} \leq \frac{G_{t'}(\theta^*|x)}{1-G_{t'}(\theta^*|x)} \quad \Rightarrow \quad G_t(\theta^*|x) \leq G_{t'}(\theta^*|x)$$

$\square$

The way this proof proceeds is by fixing a menu and dividing tests in two categories: (1) those for which $\alpha(x=0,\tilde{t}) \in (0,1)$ and $\alpha(x=1,\tilde{t})=1$ and (2) $\alpha(x=0,\tilde{t})=0$ and $\alpha(x=1,\tilde{t}) \in (0,1]$. I exclude the possibility that the DM always accepts or rejects after any signal as it would either be the only test chosen in equilibrium or never chosen. Then, I show that within each category, it is without loss of optimality to have at most one test. It is thus optimal to have at most two tests in the menu. The last part of the proof shows that the resulting menu is dominated by having only one test.

If there are two tests, $t > t'$ such that $\alpha(x = 0, \tilde{t}) = 0$ and $\alpha(x = 1, \tilde{t}) \in (0, 1]$, I will show that,

$$\pi_t(a|\theta') \geq \pi_{t'}(a|\theta') \quad \Rightarrow \quad \pi_t(a|\theta) \geq \pi_{t'}(a|\theta) \text{ for all } \theta > \theta'$$

Take two tests such that $\alpha(x = 0, \tilde{t}) = 0$, $t > t'$. Let α, α' denote their respective probability of accepting after $x = 1$. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x = 1|\theta) - \alpha'\pi_{t'}(x = 1|\theta) = 0$. Clearly, $\alpha(\theta) = \alpha' \frac{\pi_{t'}(x=1|\theta)}{\pi_t(x=1|\theta)}$. From our assumption on the difficulty environment, $\alpha(\theta)$ is decreasing in θ. If $\pi_t(a|\theta') \geq \pi_{t'}(a|\theta')$ for some θ' then $\alpha \geq \alpha(\theta')$. Then $\alpha \geq \alpha(\theta)$ for all $\theta > \theta'$.

In equilibrium, we must have that there is one $\theta' \in R$ that chooses t and thus for all $\theta \in A$, $\pi_t(a|\theta) \geq \pi_{t'}(a|\theta)$. Then there is an equilibrium of the zero-sum game where t' is never chosen.

A similar argument can be made for all tests where $\alpha(x = 0, \tilde{t}) > 0$.

Thus we conclude that it is without loss of optimality that the optimal menu has at most two tests.

Suppose the optimal menu uses two tests, $t > t'$. I will now show that it must be that $\alpha(x = 0, t) \in (0, 1)$ and $\alpha(x = 1, t') \in (0, 1)$, i.e., the DM must accept in the hard test when there is a fail grade and only accept in the easy test if there is a pass grade. Suppose it is not the case and denote by α, α' their respective mixing probabilities. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x = 1|\theta) - \alpha'\pi_{t'}(x = 0|\theta) - \pi_{t'}(x = 1|\theta) = 0$, which is equivalent to $\alpha(\theta) = \alpha' \frac{1}{\pi_t(x=1|\theta)} + (1 - \alpha') \frac{\pi_{t'}(x=1|\theta)}{\pi_t(x=1|\theta)}$. Again from our assumptions, this is decreasing in θ. A type θ chooses t if $\alpha \geq \alpha(\theta)$. Thus if one $\theta \in A$ chooses t all $\theta \in R$ choose t and there is no pooling of A and R-types on t', or it is payoff equivalent to just offering t. Therefore, $\alpha(x = 0, t) \in (0, 1)$ and $\alpha(x = 1, t') \in (0, 1)$ for $t > t'$.

If the DM mixes, he must be indifferent and thus we have

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta)\pi_t(x=0|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')\pi_t(x=0|\theta') = 0$$
$$\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x=1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x=1|\theta') = 0$$

In the easy test, because the DM rejects with positive probability after $x = 1$ and rejects for sure after $x = 0$ (as he uses a cutoff strategy), his payoffs from t' is 0, i.e., he does as well as rejecting for sure.

In the hard test, he accepts with some probability after $x = 0$ and thus his payoffs are

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')$$

that is the payoffs he would get from accepting all types choosing t. Thus the overall payoffs from the menu is $\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')$. Offering a menu is better than a singleton menu if this value is strictly greater than offering t and following the signal

$$\begin{aligned}\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta') &> \sum_{\theta \in A} \mu(\theta)\pi_t(x=1|\theta) - \sum_{\theta' \in R} \mu(\theta')\pi_t(x=1|\theta') \\ &= \sum_{\theta \in A} \sigma(t|\theta)\mu(\theta)\pi_t(x=1|\theta) + \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x=1|\theta) \\ &- \sum_{\theta' \in R} \sigma(t|\theta')\mu(\theta')\pi_t(x=1|\theta') - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x=1|\theta')\end{aligned}$$

We can rearrange and use the indifference condition at $(x = 0, t)$ to get

$$0 > \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x=1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x=1|\theta')$$

Using the indifference condition at $(x = 1, t')$, we can replace 0 on the LHS and get

$$\begin{aligned}&\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x=1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x=1|\theta') \\ &\qquad > \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x=1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x=1|\theta')\end{aligned}$$

But from the definition of the environment, for all $\theta > \theta'$,

$$\frac{\pi_t(x=1|\theta)}{\pi_t(x=1|\theta')} \geq \frac{\pi_{t'}(x=1|\theta)}{\pi_{t'}(x=1|\theta')}$$

which implies that $\mu(\theta|x=1,t) \succeq_{FOSD} \mu(\theta|x=1,t')$. Thus we get a contradiction.

A.6 Proof of Proposition 5

Proof. I will first prove the following lemma. This result already exists in the literature and I provide a proof for completeness.

Lemma 6. *For any $t \succeq t'$ and $\alpha(\cdot, t')$, there is $\alpha(\cdot, t)$ such that*

$$\sum_x \alpha(x,t)\pi_t(x|\theta) \geq \sum_x \alpha(x,t')\pi_{t'}(x|\theta)$$
$$\textit{for all } \theta' \in R, \quad \sum_x \alpha(x,t)\pi_t(x|\theta') \leq \sum_x \alpha(x,t')\pi_{t'}(x|\theta')$$

Proof. We can prove this lemma by using a theorem of the alternative (see e.g., Rockafellar (2015) Section 22). Only one of the following statement is true:

- There exists $\alpha(\cdot, t)$ such that

$$\sum_x \alpha(x,t)\pi_t(x|\theta) \geq \sum_x \alpha(x,t')\pi_{t'}(x|\theta)$$
$$\text{for all } \theta' \in R, \quad \sum_x \alpha(x,t)\pi_t(x|\theta') \leq \sum_x \alpha(x,t')\pi_{t'}(x|\theta')$$
$$\text{for all } x \in X, \quad \alpha(x,t) \leq 1$$
$$\text{for all } x \in X, \quad \alpha(x,t) \geq 0$$

- There exists $z, y \geq 0$ such that

$$\text{for all } x \in X, \quad -z_\theta \pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_t(x|\theta') + y_x \geq 0 \tag{1}$$
$$-z_\theta \sum_{x'} \alpha(x',t')\pi_t(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x',t')\pi_{t'}(x'|\theta') + \sum_{x'} y_{x'} < 0 \tag{2}$$

Take inequality (**??**) from the second alternative and multiply by $\beta(x,x')$ as described in **??** and sum over $x \in X$:

$$-z_\theta \sum_x \beta(x,x')\pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_x \beta(x,x')\pi_t(x|\theta') + \sum_x \beta(x,x') y_x \geq 0$$

Because $t \succeq_\theta t'$, we get for all $x' \in X$,

$$-z_\theta \pi_{t'}(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_{t'}(x'|\theta') + \sum_x \beta(x, x') y_x \geq 0$$

We can then multiply by $\alpha(x', t')$ and sum over $x' \in X$:

$$-z_\theta \sum_{x'} \alpha(x', t') \pi_t(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x', t') \pi_{t'}(x'|\theta') + \sum_{x,x'} \alpha(x', t') \beta(x, x') y_x \geq 0 \quad (3)$$

Because $\sum_{x'} \beta(x, x') = 1$ and $\alpha(x', t') \leq 1$ for all $x' \in X$, we have $\sum_{x,x'} \alpha(x', t') \beta(x, x') y_x \leq \sum_x y_x$. Therefore, the inequality (2) cannot hold and the first alternative holds. $\square$

With this result in hand, we can now prove our result. Suppose that t is not part of the optimal menu. Thus we can find an equilibrium of the zero-sum game of Theorem 1, (α, s, s') with $s(t|\theta) = 0$ for all $\theta \in A$. Fix a test t' used in equilibrium. Then from Lemma 6, we can construct a $\tilde{\alpha}$ such that

$$\pi_t(\tilde{a}|\theta) \geq \pi_{t'}(a|\theta)$$

$$\text{for all } \theta' \in R, \quad \pi_t(\tilde{a}|\theta') \leq \pi_{t'}(a|\theta')$$

If the first inequality is strict or the second such that $s'(\theta'|\theta) > 0$ is strict then we have a strict profitable deviation. Otherwise, we have constructed a new equilibrium of the zero-sum game.

$\square$

A.7 Proof of Proposition 7

The proof shows that if $t \succeq_\theta^b t'$ for all t', then for any $\alpha(\cdot, t')$, we can always find $\alpha(\cdot, t)$ such that

$$\sum_x \alpha(x, t) \pi_t(x|\theta) \geq \sum_x \alpha(x, t') \pi_{t'}(x|\theta)$$

$$\text{for all } \theta' \in R, \quad \sum_x \alpha(x, t) \pi_t(x|\theta') \leq \sum_x \alpha(x, t') \pi_{t'}(x|\theta')$$

$$\text{for } x = 0, 1 \quad \alpha(x, t) \leq 1$$

Using a theorem of the alternative, the other possibility is that There exists $z, y \geq 0$ such that

$$\text{for all } x = 0,1, \quad -z_\theta \pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_t(x|\theta') + y_x \geq 0 \tag{4}$$

$$-z_\theta \sum_{x'} \alpha(x', t') \pi_t(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x', t') \pi_{t'}(x'|\theta') + \sum_{x'} y_{x'} < 0 \tag{5}$$

Suppose the second alternative holds. Clearly we must have at least one $z_{\tilde{\theta}} > 0$. Note that without loss of generality, we can set $\sum_{\theta \cup R} z_{\tilde{\theta}} = 1$ as we can always divide through by that value and the inequalities would still be satisfied.

Define the complete order $\triangleright$ on $\{\theta\} \cup R$ with $\theta \triangleright \theta'$ for any $\theta' \in R$ and the other binary relation be arbitrary. Under that order, the preferences of the DM are single-crossing (Milgrom and Shannon, 1994) and given that $t \succeq_\theta^b t'$, Proposition 1 of Di Tillio et al. (2022), there is $\alpha_1, \alpha_0 \in [0,1]$ such that

$$\begin{aligned}
&\alpha_1 \big(z_\theta \pi_t(x=1|\theta) - \sum_{\theta'} z_{\theta'} \pi_t(x=1|\theta') \big) + \alpha_0 \big(z_\theta \pi_t(x=0|\theta) - \sum_{\theta'} z_{\theta'} \pi_t(x=0|\theta') \big) \\
&\geq \alpha_1' \big(z_\theta \pi_{t'}(x=1|\theta) - \sum_{\theta'} z_{\theta'} \pi_{t'}(x=1|\theta') \big) + \alpha_0' \big(z_\theta \pi_{t'}(x=0|\theta) - \sum_{\theta'} z_{\theta'} \pi_{t'}(x=0|\theta') \big)
\end{aligned}$$

for all $\alpha_1', \alpha_0' \in [0,1]$. That is experiment t dominates experiment t' in the sense that the DM can always get higher payoff for any prior, represented by z here. We can then multiply inequalities (4) for $x = 0,1$ by α_1 and α_0 and sum over them to get

$$\begin{aligned}
&\alpha_1 \big(-z_\theta \pi_t(x=1|\theta) + \sum_{\theta'} z_{\theta'} \pi_t(x=1|\theta') \big) \\
&\qquad + \alpha_0 \big(-z_\theta \pi_t(x=0|\theta) + \sum_{\theta'} z_{\theta'} \pi_t(x=0|\theta') \big) + \alpha_1 y_1 + \alpha_0 y_0 \geq 0
\end{aligned}$$

Because $\alpha_x \in [0,1]$ and $y_x \geq 0$, we get a contradiction with inequality (5).

Once this is established, the rest of the proof follows the lines of Lemma 1.

A.8 Proof of Proposition 8

The way this proof proceed is by first arguing that an optimal mechanism $\tilde{\sigma} : \Theta \to \Delta(T \times M)$ does weakly better than an optimal GR-mechanism, τ. Then I will show that the outcome of the optimal mechanism $\tilde{\sigma}$ can be implemented by a GR-game.

To see the first part, note that a GR-mechanism can be rewritten as a mechanism $\tilde{\tau} : M \to \Delta(T)$ and a DM-strategy $\tilde{\alpha} : M \times T \times X \to [0, 1]$. Then we can implement any equilibrium outcome of $(\tilde{\tau}, \tilde{\alpha}, \delta)$, where δ is the agent's strategy by a mechanism and strategy of the DM, $(\tilde{\sigma}, \alpha)$ by setting $\tilde{\sigma} = \tilde{\tau} \circ \delta$, the composition of the GR-mechanism and the agent's strategy and $\alpha = \tilde{\alpha}$. This does not change the outcome so all the agent's incentives are preserved.

I will now show that the outcome of the menu game with communication can be implemented in a GR-game.

Remember that we have established that in the zero-sum game, all the A-types play a pure strategy and send a different message (Theorem 2). This implies that it is without loss of optimality to decompose the A-types' strategy s in choosing a message $m \in M$, call it $\phi : A \to M$ and a test for each message, call it $\rho : M \to T$.

Abusing notation define

$$\pi_t(a|\theta, m) = \sum_x \alpha(t, x, m)\pi_t(x|\theta)$$

$$v(\alpha, \phi, \rho, s') = \sum_m \mathbb{1}\left[(t, \theta) : t = \rho(m), m = \phi(\theta)\right]\left[\mu(\theta)\pi_t(a|\theta, m) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(a|\theta', m)\right]$$

To understand the new version of v, we sum over all messages and for each message, we select the the test associated with it and the A-type choosing that message.

We get,

$$\min_{s'} \max_{\alpha, s} v(\alpha, s, s') = \max_{\alpha, s} \min_{s'} v(\alpha, s, s') = \max_{\alpha, \phi, \rho} \min_{s'} v(\alpha, \phi, \rho, s') = \min_{s'} \max_{\alpha, \phi, \rho} v(\alpha, \phi, \rho, s')$$

But now observe that we could equivalently interpret ρ as being chosen by the DM as it maximises his payoffs. We are left to check that ϕ and s' generate equilibrium strategies.

As before the R-types select an A-type's strategy. Because they are playing a pure strategy, this is equivalent to choosing an on-path m taking into account that the test will be $t = \phi(m)$ to maximise $\pi_{t=\phi(m)}(a|\theta', m)$. The A-types choose m if

$$\mu(\theta)\pi_{\phi(m)}(a|\theta, m) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{\phi(m)}(a|\theta', m) \geq \mu(\theta)\pi_{\phi(m')}(a|\theta, m') - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{\phi(m')}(a|\theta', m')$$

$$\Leftrightarrow \mu(\theta)\big[\pi_{\phi(m)}(a|\theta, m) - \pi_{\phi(m')}(a|\theta, m')\big] \geq \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\big(\pi_{\phi(m)}(a|\theta', m) - \pi_{\phi(m')}(a|\theta', m')\big) \geq 0$$

where the last line uses the equilibrium behaviour of R-types to get that $s'(\theta|\theta')$ implies $\pi_{\phi(m)}(a|\theta', m) - \pi_{\phi(m')}(a|\theta', m') \geq 0$.