Arena Games

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Abstract

In an arena game two teams strategize on the order in which their members participate in pairwise matches. An underlying bitournament determines the corresponding winners with the first match being between the two players on the top of the two orders. After each match, the loser is removed and the winner stays in the arena as to play against the successor of the loser. The procedure continues until one of the teams has no player left. Local pure Nash equilibria are shown to always exists and a full characterization of the set of pure Nash equilibria when each team consists of at most three players is provided. In general, the absence of a hamiltonian cycle in the bitournament is necessary and its acyclicity is sufficient for the existence of pure Nash equilibria.

Keywords: acyclicity, arena game, bitournament, hamiltonian cycle, pure Nash equilibrium
JEL Classification: C72, D74

1 Introduction

An arena game describes the competition between two teams of players and possesses the following main features: (1) An underlying bitournament contains the information about who will win the match between every two players from the rival teams; (2) Teams strategize on the order in which their members participate in

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pairwise matches; (3) The sequence of matches starts with the two players on the top of the two orders with the winner playing against the successor of the loser; the procedure continues until one of the teams has no player left in its order; (4) A team's payoff equals the number of its players who, after the last match, have not been beaten by some player from the opposite team.

After introducing our formal setup in Section 2, we focus on the existence of pure Nash equilibria in arena games. Section 3 points out a necessary condition (the absence of a hamiltonian cycle in the underlying bitournament) as well as a sufficient condition (acyclicity of the bitournament) for such an existence. In Section 4 we fully characterize the set of pure Nash equilibria when each team consists of either two or three players. Extending the corresponding characterization turns out to be impossible as we exemplify in Section 5 for the case of each team containing four players. We also show there that each arena game has a pure local Nash equilibrium (where the focus is only on prudent movements in the corresponding team's strategy).

Our model differs from related models in multi-battle contests (cf. Fu et al. 2015a, Fu et al. 2015b) as it is shaped by an underlying bitournament rather than depending on the corresponding players' efforts. The impact of this specific issue is that a team's strategy is a permutation of its members which then results in the specific way in which teams' payoffs are defined. Clearly then, the effect of winning a match on the winning the next match (cf. Chapsal and Vilain 2019) is directly incorporated in the design of a team's strategy. Notice also the fact that we allow, given the structure of the corresponding bitournament, a player to participate in several sequential matches. This is in sharp contrast for instance to Fu and Lu (2020), where the focus is on the equilibrium player ordering in contests with each team consisting of two players who participate in battles in different periods.

2 The model

Our setup includes the following basic ingredients.

Teams and strategies There are two teams denoted by T_1 and T_2. Each team consists of n players. For $i \in \{1, 2\}$, the set of strategies of team T_i coincides with the set of all possible permutations (linear orders) over its n players.

Bitournaments A bipartite tournament (bitournament) is a complete directed bipartite graph $G = (T_1 \cup T_2, A)$ such that there is exactly one arc between any two vertices $p \in T_1$ and $q \in T_2$. The vertex partition (T_1, T_2) of G corresponds to the two teams of players, while A stands for the set of arcs of G. An arc from a player to another player means that the former beats the latter in a battle/match between them. For a graph G, we use $V(G)$ to denote its vertex set.

Arena battles Given an order $\lhd_1$ of players in team T_1, an order $\lhd_2$ of players in team T_2, and a bipartite tournament $G = (T_1 \cup T_2, A)$, the arena game between the two teams goes as follows. There is only one arena where at each time only one battle/match between two players from rival teams is conducted. The players take part in the game between the two teams one by one according to the orders $\lhd_1$ and $\lhd_2$. The first battle is between the two players on the top of the two orders and the winner is determined according to the information contained in G. After each battle, the winner stays in the arena as to play against the successor of the loser. The loser is removed from the corresponding order. The procedure continues until one of the teams has no player left in its order.

Payoffs Given two orders $\lhd_1$ and $\lhd_2$, and a bitournament G as above, we use $w(\lhd_1, \lhd_2, G)$ to denote the player who wins the final match. That is, a team wins if and only if $w(\lhd_1, \lhd_2, G)$ belongs to that team. The payoff of team T_i, $i \in \{1, 2\}$, is denoted by $\pi_i(\lhd_1, \lhd_2, G)$ and equals the number of T_i's players who, after the last arena battle as described above, have not been beaten by some player from the opposite team. Notice that $\pi_i(\lhd_1, \lhd_2, G) > 0$ implies $\pi_{3-i}(\lhd_1, \lhd_2, G) = 0$ for each $i \in \{1, 2\}$.

3 Existence of pure Nash equilibria

Given a bitournament G over the two teams T_1 and T_2, a strategy profile $(\lhd_1, \lhd_2)$ is a pure Nash equilibrium of the arena game induced by G if (1) $\pi_1(\lhd_1, \lhd_2, G) \geq \pi_1(\lhd_1', \lhd_2, G)$ for all permutations $\lhd_1'$ of T_1, and (2) $\pi_2(\lhd_1, \lhd_2, G) \geq \pi_2(\lhd_1, \lhd_2', G)$ for all permutations $\lhd_2'$ of T_2.

In what follows in this section we show that the existence of pure Nash equilibria in arena games is closely related to the absence of special cycles in the underlying bitournament. The absence of a *hamiltonian cycle* (i.e., a cycle containing every vertex in the bitournament) is a necessary condition, while the *acyclicity* of the bitournament (the absence of a cycle of any length in it) turns out to be sufficient for the existence of a pure Nash equilibrium of the corresponding arena game.

Proposition 1. *Let G be a bitournament over the two teams. If the corresponding arena game has a pure Nash equilibrium, then G does not contain a hamiltonian* *cycle.*

Proof. We show that if G with $V(G) = 2n$ contains a hamiltonian cycle, then the corresponding arena game has no pure Nash equilibria.

Suppose that, to the contrary, $(\lhd_1, \lhd_2)$ were a pure Nash equilibrium with $\lhd_1 = (p_1, p_2, \dots, p_n)$ and, without loss of generality, assume that $\pi_1(\lhd_1, \lhd_2, G) > 0$. We show that there is a profitable deviation for T_2 from $(\lhd_1, \lhd_2)$.

Fix a hamiltonian cycle and let $q_1, q_2, \dots, q_n \in T_2$ be the players along that cycle who correspondingly defeat $p_1, p_2, \dots, p_n \in T_1$. Consider the strategy $\lhd_2' = (q_1, q_2, \dots, q_n)$ of T_2. We will prove that $\pi_2(\lhd_1, \lhd_2', G) > 0$ (implying that there is a profitable deviation for T_2 due to $\pi_2(\lhd_1, \lhd_2, G) = 0$ following from $\pi_1(\lhd_1, \lhd_2, G) > 0$).

Suppose $\pi_2(\lhd_1, \lhd_2', G) = 0$ would hold. The latter implies that, in the arena game played according to $(\lhd_1, \lhd_2')$, q_n has been defeated and for sure not by p_n (as q_n beats p_n in G). Moreover, q_n has not been defeated by p_1 in the game, either; the reason here is that, by construction, q_1 beats p_1 (i.e., p_1 loses the very first battle in the arena game played according to $(\lhd_1, \lhd_2')$). We are then left with the possibility that some p_i with $i \in \{2, 3, \dots, n-1\}$ is defeating q_n in the game.

Notice first that p_2 defeating q_n would imply, by q_1 winning the battle against p_1, that p_2 is defeating all players from T_2. Thus, we would have a contradiction to G containing a hamiltonian cycle.

Suppose finally that p_i with $i \in \{3, \dots, n-1\}$ is defeating q_n. The fact that p_i has entered the arena implies that p_{i-1} has lost a battle — either against q_k for some $k \in \{1, 2, \dots, i-2\}$ or (and this for sure due to our construction) against q_{i-1}, i.e., there was no battle between p_{i-1} and q_i. The latter fact implies that there has been a battle between p_i and q_i and the winner of this battle is q_i (due to our construction). Thus, it is impossible that p_i participates in a battle against q_n (and defeats that player). $\square$

Before presenting our next result, let us note that acyclicity of a bitournament (the absence of a cycle of any length in it) is closely related to its bitransitivity. A bitournament G is called *bitransitive* if for any $p, p', q, q' \in T_1 \cup T_2$: $p \to q$, $q \to p'$, and $p' \to q'$ imply $p \to q'$. By Theorem 2.5 in Das et al. (2021), a bitournament is bitransitive if and only if it is acyclic. A player is a *dominant player* in G if it beats every player from the opposite team. As it can be easily seen, an acyclic digraph admits a topological order over its vertices so that arcs are only forwards and thus, a dominant player in it always exists. When using the battle procedures in an arena game, the corresponding proof argument runs as follows.

Proposition 2. *If the underlying bitournament of an arena game is acyclic, then the game has a pure Nash equilibrium. Moreover, all such equilibria are Pareto-optimal.*

Proof. Let G be the underlying acyclic bitournament. It suffices to show that one of the teams contains a dominant player in G. Notice then that every strategy profile, where the dominant player is at the first place in the ordering of the corresponding team, constitutes a Pareto-optimal pure Nash equilibrium.

Take a strategy profile $(\lhd_1, \lhd_2)$ and suppose that, w.l.o.g., team T_1 is winning in the arena game played according to this profile. Denote by $p^* \in T_1$ the last player in the ordering $\lhd_1$ who participates in a battle; we show that p^* is a dominant player in G.

Clearly, $\pi_1(\lhd_1, \lhd_2, G) = n$ would imply that p^* is the first player in $\lhd_1$ and thus, he would be definitely a dominant player in G.

Suppose next that $\pi_1(\lhd_1, \lhd_2, G) < n$ holds. That is, there is at least one other player before p^* in $\lhd_1$ (who, of course, has been defeated in the game). Denote by $T_2^* \subset T_2$ the set of players from T_2 who lost a battle against p^* in the arena game. Clearly, by T_1 being the winning team and the definition of p^*, T_2^* is non-empty and it contains the last $|T_2^*|$ players from T_2 in the ordering $\lhd_2$. Let q^* be the first player (according to $\lhd_2$) from T_2^* who plays (and loses) against p^*. Clearly, q^* has defeated the player before p^* in the ordering $\lhd_1$ (otherwise, q^* would not be on turn). Two cases are possible:

(1) q^* defeats all players before p^* in $\lhd_1$. This implies that q^* is the first player from T_2 in $\lhd_2$ and thus, p^* beats all players from T_2 implying that he is a dominant player in G.

(2) q^* does not defeat all players before p^* in $\lhd_1$. Let p^{**} be the first player from T_1 in $\lhd_1$ who loses a battle against q^*. Notice that q^* is not the first player from T_2 in $\lhd_2$ (otherwise, he should have defeated all players from T_1 before p^* in $\lhd_1$ in contradiction to the fact that we are in case (2)). Let q^{**} be the player from T_2 just before q^* in $\lhd_2$, and notice that q^{**} should have been defeated by p^{**}. So, we have $p^* \to q^*$, $q^* \to p^{**}$, and $p^{**} \to q^{**}$. By acyclicity (and thus, by bitransitivity due to Theorem 2.5 in Das et al. 2021), we have that $p^* \to q^{**}$ holds. By the same type of argument, p^* beats any player from T_2 who is beaten by p^{**} (and, in particular, all players before q^{**} in $\lhd_2$ who lost a battle against p^{**}). In other words, p^* can win all the battles p^{**} was participating in (and these battles are consecutive with respect to $\lhd_2$). One can then move all players from T_1 who, according to $\lhd_1$, are between p^{**} and p^* (including p^{**} but not including p^*) just after p^* (no matter in which order). Denote the new order by $\lhd_1'$ and notice that,

when the strategy profile is $(\lhd_1', \lhd_2)$, the winning team is again T_1. So, one can repeat the above arguments as many times as necessary in order to conclude that $p^* \in T_1$ is a dominant player in G. $\square$

4 Teams with at most three players

This section is devoted to the analysis of arena games with each participating team having either two or three players. For the case of two-players teams, the necessary condition and the sufficient condition discussed in the previous section do coincide and thus, each pure Nash equilibrium is Pareto-optimal as well. However, when each team consists of three players, the absence of a hamiltonian cycle is not any more sufficient for the existence of a pure Nash equilibrium. In this case we are able to fully describe the structure of the bitournaments inducing arena games with a pure Nash equilibrium as well as the structure of the corresponding equilibrium sets.

Proposition 3. *Let G be a bitournament over the two teams with $|V(G)| = 4$. Then the corresponding arena game has a pure Nash equilibrium if and only if G does not contain a hamiltonian cycle. Moreover, each Nash equilibrium is Pareto-optimal.*

Proof. Due to $V(G) = 4$ (i.e., $|T_1| = |T_2| = 2$), the bitournament G is acyclic if and only if it does not contain a hamiltonian cycle. By combining Propositions 1 and 2, the assertion follows. $\square$

In order to fully describe the set of pure Nash equilibria when each team consists of three players, we introduce the following additional notation. Given a bitournament G over the two teams, we denote by $Nash^k(G)$, $k \in \{1, 2, \ldots, n\}$, the set of all pure Nash equilibria of the corresponding arena game with equilibrium payoff of k for the winning team. The set of all pure Nash equilibria is denoted by $Nash(G)$. For $i \in \{1, 2\}$, we call a team T_i *completely diverse* in G, if for each $\ell \in \{1, 2, \ldots, n\}$ there is exactly one player in T_i who is beaten in G by ℓ players from T_{3-i}. Moreover, the structure of the bitournament G^* as displayed in Figure 1 will be helpful for our characterization.

Finally, we say that two directed graphs H and H' with the same number of vertices are *isomorphic* if there is a one-to-one mapping $f : V(H) \to V(H')$ such that the following holds: There is an arc from a to b in H if and only if there is an arc from $f(a)$ to $f(b)$ in H'.

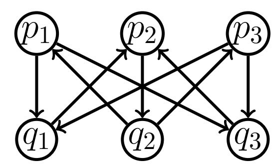

Figure 1: Bitournament G^*

Proposition 4. *Let G be a bitournament over the two teams with $|V(G)| = 6$. Then,*

(1) $Nash(G) = Nash^1(G) \cup Nash^2(G)$ with $Nash^1(G) \neq \emptyset \neq Nash^2(G)$ if and only if G is isomorphic to G^*;

(2) $Nash(G) = Nash^2(G) \neq \emptyset$ if and only if there is no dominant player in G and one of the teams is completely diverse in G;

(3) $Nash(G) = Nash^3(G) \neq \emptyset$ if and only if there is a dominant player in G.

Proof. (1) Given the structure of the bitournament G^*, it is easy to check that the strategy profile $(\lhd_1, \lhd_2)$ with $\lhd_1 = (p_1, p_2, p_3)$ and $\lhd_2 = (q_1, q_2, q_3)$ is a pure Nash equilibrium with $\pi_1(\lhd_1, \lhd_2, G^*) = 1$, and that the strategy profile $(\lhd_1, \lhd_2')$ with $\lhd_2' = (q_1, q_3, q_2)$ is a pure Nash equilibrium with $\pi_1(\lhd_1, \lhd_2', G^*) = 2$.

We show now that if a bitournament G induces an arena game having a pure Nash equilibrium $(\lhd_1, \lhd_2)$ with $\pi_1(\lhd_1, \lhd_2, G) + \pi_2(\lhd_1, \lhd_2, G) = 1$, then G has to be isomorphic to G^*. Suppose the latter does not hold and let the two teams be $T_1 = \{p_1', p_2', p_3'\}$ and $T_2 = \{q_1', q_2', q_3'\}$. Without loss of generality, let $\lhd_1 = (p_1', p_2', p_3')$, $\lhd_2 = (q_1', q_2', q_3')$, $\pi_1(\lhd_1, \lhd_2, G) = 1$, and $\pi_2(\lhd_1, \lhd_2, G) = 0$. As $(\lhd_1, \lhd_2)$ is a Nash equilibrium, it must be that p_3' beats q_3' in G. We enumerate all the four possibilities of the arcs between $\{p_1', p_2'\}$ and q_1' and show that each case results in a contradiction.

(1.1) $q_1' \to p_1'$ and $q_1' \to p_2'$. Since $\pi_1(\lhd_1, \lhd_2, G) = 1$, it must be that p_3' beats all players in T_2. In this case, changing the order $\lhd_1$ to (p_3', p_1', p_2') makes team T_1 better off, a contradiction to $(\lhd_1, \lhd_2)$ being a Nash equilibrium.

(1.2) $q_1' \to p_1'$ and $p_2' \to q_1'$. In this case, it must be that q_2' beats p_2', since otherwise T_1 can increase its payoff by changing $\lhd_1$ to (p_2', p_3', p_1'), contradicting that $(\lhd_1, \lhd_2)$ is a Nash equilibrium. Then, it follows that p_3' beats q_2'. However, in this case, T_1 can again be made better off by changing $\lhd_1$ to (p_2', p_3', p_1'), again contradiction to $(\lhd_1, \lhd_2)$ being a Nash equilibrium.

(1.3) $p_1' \to q_1'$ and $p_2' \to q_1'$. In this case, it must be that q_2' beats p_1', since otherwise T_1 can be made better off by changing $\lhd_1$ to (p_1', p_3', p_2'), a contradiction. Similarly, we can assume that q_2' beats p_3'. Then, we can further assume that p_2' beats q_2', since otherwise q_2' has to beat all players in T_1 in contradiction to $\pi_1(\lhd_1, \lhd_2, G) > 0$. However, in this case T_1 can increase its payoff by changing $\lhd_1$ to (p_2', p_3', p_1').

(1.4) $p_1' \to q_1'$, and $q_1' \to p_2'$. In this case, it must be that q_2' beats p_1', since otherwise T_1 can increase its payoff by changing its strategy to (p_1', p_3', p_2'), contradicting that $(\lhd_1, \lhd_2)$ is a Nash equilibrium. Moreover, it must be that q_2' beats p_3', since otherwise T_1 can again increase its payoff by changing the order to (p_1', p_3', p_2'). It follows from $\pi_1(\lhd_1, \lhd_2, G) > 0$ that p_2' beats q_2'. If q_1' beats p_3', then T_2 can be made better off by changing its strategy to (q_2', q_1', q_3'). So, let us assume that p_3' beats q_1'. Furthermore, as $\pi_1(\lhd_1, \lhd_2, G) = 1$, it must be that q_3' beats p_2'. Finally, as the graph G was assumed not isomorphic to G^*, it must be that q_3' beats p_1'. In this case, T_2 can increase it payoff by changing its strategy to (q_3', q_2', q_1'), a contradiction. We conclude that G has to be isomorphic to G^*.

(2) First we prove the "if" part. Assume without loss of generality that team T_2 is completely diverse with q_1 being beaten by everyone from T_1, q_2 being beaten by p_1 and p_3, and q_3 being beaten by p_2. It is then straightforward to verify that $(\lhd_1, \lhd_2) = ((p_1, p_2, p_3), (q_1, q_2, q_3))$ constitutes a pure Nash equilibrium with $\pi_1(\lhd_1, \lhd_2, G) = 2$ if and only if there is no dominant player in G.

Next we prove the "only if" part. Suppose the game has a pure Nash equilibrium $(\lhd_1, \lhd_2) = ((p_1, p_2, p_3), (q_1, q_2, q_3))$ with $\pi_1(\lhd_1, \lhd_2, G) = 2$; the latter in particular implies that there is no dominant player in G. Notice that $\pi_1(\lhd_1, \lhd_2, G) = 2$ implies $p_2 \to q_3$ and, as there is no dominant player, $p_1 \to q_1$. There are four possible cases to be considered:

(2.1) $q_2 \to p_1$ and $q_2 \to p_2$. This case is not possible as T_2 can deviate to $\lhd_2' = (q_2, q_1, q_3)$ and win the game (as there is no dominant player).

(2.2) $q_2 \to p_1$ and $p_2 \to q_2$. As there is no dominant player, $q_1 \to p_2$. We then have $p_3 \to q_1$ and $p_3 \to q_3$, for otherwise $\lhd_2' = (q_2, q_1, q_3)$ would be a profitable deviation for T_2. As there is no dominant player, $q_2 \to p_3$. Furthermore, $p_1 \to q_3$ must hold so T_2 has no incentive to deviate to $\lhd_2'' = (q_3, q_1, q_2)$. This completes the graph G, and it can be readily verified that T_2 is completely diverse in G.

(2.3) $p_1 \to q_2$ and $p_2 \to q_2$. As there is no dominant player, $q_3 \to p_1$ and $q_1 \to p_2$. As T_2 has no incentive to deviate to $\lhd_2' = (q_3, q_1, q_2)$, $p_3 \to q_1$ and $p_3 \to q_2$. As there is no dominant player, $q_3 \to p_3$, which makes T_2 completely diverse in G.

(2.4) $p_1 \to q_2$ and $q_2 \to p_2$. As there is no dominant player, $q_3 \to p_1$. To ensure that T_2 has no incentive to deviate to $\lhd_2' = (q_3, q_2, q_1)$, we must have $p_3 \to q_1$ and $p_3 \to q_2$. As there is no dominant player, $q_3 \to p_3$. There are only two possibilities to complete the graph G:

(2.4.1) $p_2 \to q_1$. This results in T_2 being completely diverse in G.

(2.4.2) $q_1 \to p_2$. This would make G isomorphic to G^*. By (1), $Nash^1(G) \neq \emptyset$ follows and thus, we have a contradiction to $Nash(G) = Nash^2(G)$. Therefore this case is not possible.

(3) The proof is trivial. It is also obvious that, provided that a dominant player exists, the set of pure Nash equilibria can be fully characterized. $\square$

The reader might get the impression that Proposition 4 covers all non-hamiltonian bitournamens for the case when each team consists of three players. Such an impression is wrong as we show next. For this, consider the bitournament G^{**} shown in Figure 2.

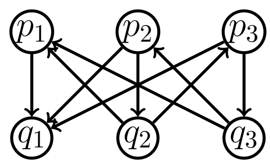

Figure 2: Bitournament G^{**}

Notice that G^{**} is non-hamiltonian as q_1 is beaten by every player from T_1.

Proposition 5. *The arena game induced by G^{**} has no pure Nash equilibrium.*

Proof. As it can be easily checked, there is no dominant player in G^{**}. Moreover, neither is G^{**} isomorphic to G^* nor is there a team which is completely diverse in G^{**}. The assertion then follows from Proposition 4. $\square$

5 Local Nash equilibria

In view of Proposition 4(2), the reader may then wonder whether games where each team consists of more than three players and one of the teams is completely diverse do have a pure Nash equilibrium. The answer is into the negative as we show next.

For this, let each team consist of four players and consider the bitournament G^{***} shown in Figure 3.

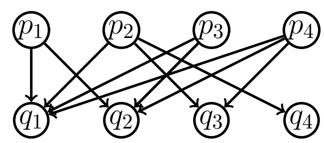

Figure 3: Bitournament G^{***}. Missing arcs are upwards (from q_i to p_j where $i, j \in [4]$).

Notice that G^{***} is non-hamiltonian as q_1 is beaten by every player from T_1. Moreover, T_2 is completely diverse in G^{***}.

Proposition 6. *The arena game induced by G^{***} has no pure Nash equilibrium.*

Proof. The assertion follows from the following two claims.

Claim 1. *There is no pure Nash equilibrium where T_2 is winning.*

Proof of Claim 1. Since $q_1 \in T_2$ is beaten in G^{***} by every player from T_1, it is easy to see that if $(\lhd_1, \lhd_2)$ is a pure Nash equilibrium with T_2 winning, then q_1 should be the last player in $\lhd_2$ (otherwise, there would be a profitable deviation for T_2 which keeps the order of the other players and places q_1 at the end of the corresponding order).

Notice first that $\pi_2(\lhd_1, \lhd_2, G^{***}) > 0$ would imply that $\{q_2, q_3, q_4\}$ is a dominating set of T_2. Moreover, this set is embedded into the following cycle in G^{***}: $p_1 \to q_2 \to p_2 \to q_4 \to p_4 \to q_3 \to p_1$. So, we can accordingly apply the proof idea from Proposition 1 and re-order the players p_1, p_2, and p_4 in such a way by following this cycle that, given the way in which q_2, q_3, and q_4 are ordered in $\lhd_2$, each player in $\{q_2, q_3, q_4\}$ is beaten by some player in $\{p_1, p_2, p_4\}$. Since q_1 is beaten by every player from T_1, we conclude that there is a profitable deviation for T_1 (as $\pi_1(\lhd_1, \lhd_2, G^{***}) = 0$) in contradiction to $(\lhd_1, \lhd_2)$ being a Nash equilibrium.

Claim 2. *There is no pure Nash equilibrium where T_1 is winning.*

Proof of Claim 2 Suppose that T_1 wins at the strategy profile $(\lhd_1, \lhd_2)$ and thus $\pi_2(\lhd_1, \lhd_2, G^{***}) = 0$ holds. We show that there is a profitable deviation for T_2 such that the assertion in Claim 2 would follow.

Notice first that, no matter how $\lhd_1$ looks like, the following observation holds: either p_1 and p_3, or p_1 and p_4, or p_3 and p_4 are next to each other in $\lhd_1$. Moreover: both p_1 and p_3 are beaten by q_3 in G^{***}; both p_1 and p_3 (both p_3 and p_4) are beaten by q_4 in G^{***}. Having in mind that q_2 beats p_2 in G^{***}, we conclude that

$\{q_2, q_3, q_4\}$ is a dominating set of T_2. Clearly then, it is always possible to order the players from $\{q_2, q_3, q_4\}$ in such a way that T_2 wins in the arena game. $\square$

Inspired by Alós-Ferrer and Ania (2001) and given the above insight, we consider now two solutions for arena games which are based on local deviations. As for the first notion (we call LN1 equilibrium), we consider (allow) for (profitable) deviations from a team's strategy only by exchanging the places of exactly two players who are next to each other in the given order of players. The second notion (LN2 equilibrium) is even weaker as it allows only for prudent movements of players in the sense that it puts on two-players places' exchange the following additional condition: the players from the opposite team beaten in the underlying bitournament by the player placed later are also beaten by the player placed earlier. In other words, the exchange here is based on monotonicity by set inclusion of the corresponding outneighborhoods. Clearly, every pure Nash equilibrium is a pure LN1 equilibrium and every pure LN1 equilibrium is also a pure LN2 equilibrium.

When contrasted to the notion of a pure Nash equilibrium, the notion of pure LN1 equilibrium is strictly weaker; recall that, by Proposition 5, the arena game induced by the bitournament G^{**} has no pure Nash equilibria which is not anymore the case when a pure LN1 equilibrium is considered.

Proposition 7. *The arena game induced by G^{**} has a pure LN1 equilibrium.*

Proof. Consider the strategy profile $(\lhd_1, \lhd_2)$ with $\lhd_1 = (p_2, p_3, p_1)$ and $\lhd_2 = (q_2, q_1, q_3)$. We have $\pi_1(\lhd_1, \lhd_2, G^{**}) = 2$ and thus, as T_1 does not contain a dominant player in G^{**}, there is no profitable deviation for this team. On the other hand, exchanging the places of q_2 and q_1 results in exactly the same payoff (zero) for T_2, the reason being that both players are beaten by p_2 in G^{**}. Finally, the reader can easily check that exchanging the places of q_1 and q_3 leads again to a zero payoff for T_2 since both players are beaten by p_3 in G^{**}. We conclude that $(\lhd_1, \lhd_2)$ is a pure LN1 equilibrium. $\square$

The reader can easily check that not every arena game has a pure LN1 equilibrium. For instance, when each team consists of two players and there is a hamiltonian cycle in the underlying bitournament, then the game has no such equilibrium. However, we show next that every arena game has a pure LN2 equilibrium.

A directed graph is *strongly connected* if either it consists of only one vertex or for every pair (v, u) of two distinct vertices there is a directed path from v to u. A *strongly connected component* of a directed graph G is a maximal subgraph of G that is strongly connected. A subset S of vertices in a directed graph *dominates* another subset S' of vertices if there does not exist any arc from S' to S. From the definition of a strongly connected component, it is clear that for every two such components S and S' of a directed graph it holds that either S dominates S', or S' dominates S. It is known that every bitournament has a strongly connected component which dominates all the other strongly connected components of the bitournament.

Proposition 8. *Let G be a bitournament over the two teams. Then the corresponding arena game has a pure LN2 equilibrium.*

Proof. We consider the following two possible cases.

(1) G contains a hamiltonian cycle. Suppose $p_1 \to q_1 \to p_2 \to q_2 \to \cdots \to q_{n-1} \to p_n \to q_n \to p_1$ is a hamiltonian cycle in G, and consider the strategy profile $(\lhd_1, \lhd_2)$ with $\lhd_1 = (p_1, p_2, \ldots, p_{n-1}, p_n)$ and $\lhd_2 = (q_1, q_2, \ldots, q_{n-1}, q_n)$; as a matter of fact, team T_1 wins at this strategy profile. For $i \in \{1, 2, \ldots, n-1\}$ and given the cycle, notice that p_i beats q_i (who is not beaten by p_{i+1}) and that q_i beats p_{i+1} (who is not beaten by q_{i+1}). Clearly then, $(\lhd_1, \lhd_2)$ is an LN2 equilibrium.

(2) G does not contain a hamiltonian cycle. Let G' be the strongly connected component which dominates all the other strongly connected components of G. If G' consists of only one player, then this player is a dominant player in the whole graph G, and in this case a pure Nash equilibrium exists. So, suppose that this is not the case (note that in this case G' contains at least four players.) Let C be a longest cycle in G'. By Corollary 5.7.19 in Bang-Jensen and Gutin (2009), the subgraph of G' restricted to $V(G') \setminus V(C)$ is acyclic and thus, it contains a player who is dominant in that subgraph. In fact, as G' dominates all the other strongly connected components of G, this implies that p_d is dominant in a larger subtournament, i.e., the subtournament of G restricted to $V(G) \setminus V(C)$.

Without loss of generality, let this player belong to T_1 and denote the player by p_d. Let, for some $k \in \{1, 2, \ldots, n\}$, the cycle C be as follows:

$$p_1 \to q_1 \to p_2 \to q_2 \to \cdots \to q_{k-1} \to p_k \to q_k \to p_1.$$

Let $k' \leq k$ be the smallest index such that, for all k'' with $k' \leq k'' \leq k$, the outneighborhood of $p_{k''}$ is included in the outneighborhood of p_d.

Consider then the strategy profile $(\lhd_1, \lhd_2)$ defined as follows ("irr" means that the order from that point on can be set arbitrarily): $\lhd_1 = (p_1, p_2, \ldots, p_{k'-1}, p_d, irr)$ and $\lhd_2 = (q_1, q_2, \ldots, q_{k-1}, q_k, irr)$; if there is no such k' as defined above, we place in $\lhd_1$ player p_d immediately after p_k. As it can be easily checked, team T_1

is winning the arena game at this strategy profile. We now show that $(\lhd_1, \lhd_2)$ is a pure LN2 equilibrium.

Notice first that, by an analogous reasoning as in (1), the definition of LN2 excludes the exchange of the places of any two players who are next to each other in the corresponding ordering, and who are between p_1 and $p_{k'-1}$ and between q_1 and $q_{k'-1}$, respectively. Moreover, exchanging the places of any two players after p_d in $\lhd_1$ is immaterial as they do not participate in any battle of the arena game. We have also that exchanging the places of any two players after $q_{k'}$ in $\lhd_2$ is irrelevant, the reason being that each of these players is beaten in G by p_d.

Notice further that, by construction and the definition of LN2, an exchange of the places of $p_{k'-1}$ and p_d (with $p_{k'-1}$ being just before p_d in $\lhd_1$) is not allowed. Consider finally the players $q_{k'-1}, q_{k'} \in T_2$ and notice that, by construction, $p_{k'-1} \to q_{k'-1}$ and $p_d \to q_{k'}$ holds. If $q_{k'-1} \to p_d$ holds as well, then exchanging the places of $q_{k'-1}$ and $q_{k'}$ is excluded by the definition of LN2 as $q_{k'-1}$ beats a player from T_1 who is not beaten by $q_{k'}$. Finally, if $p_d \to q_{k'-1}$ is the case, then a possible exchange of the places of $q_{k'-1}$ and $q_{k'}$ is irrelevant for the payoff of T_2 due to our construction and the fact that both players are beaten in G by p_d. We conclude that the constructed strategy profile $(\lhd_1, \lhd_2)$ is a pure LN2 equilibrium. □

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