Informed Principal and Screening Problem

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Abstract

This paper studies an informed mechanism designer problem in which the principal's private information is a number of agents. We define mechanical equivalence such that it holds if each agent's and the principal's perspectives are consistent in the sense that a conversion problem for a grand mechanism is resolved – each agent's expected payment taking into account the principal's private information can be incorporated into the principal's revenue. With mechanical equivalence and, additionally, the principal's expected payoff linearity, there is a single threshold for the optimal grand mechanism if a sub-mechanism cannot depend on the principal's private information. Interestingly, the main result shows that if a sub-mechanism can also depend on his private information, the optimal grand mechanism is characterized by double thresholds such that the principal does not announce the number of agents if it is in the middle range. We further extend the signal structure to include rich signal sets.

Keywords and Phrases: informed principal, mechanism, population uncertainty, mechanical equivalence

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1 Introduction

Promoting information sharing and transparency has long been considered as the government's duty. Not just in terms of legitimacy – the ethical point of view – there are numerous supporting arguments for the view with respect to "efficiency" in a large sense. Indeed, during the pandemic (or any other impactful event) by now, it is a familiar scene that everyday, following a government official's announcement of relevant information, the news networks report it, *e.g.*, a number of cases, frantically, and then new policies and subsequent actions for all individuals to take are required with the updated information. Yet, an important question seems missing around the whole discussion. Given the non-negligible cost of such an announcement, what is the *strategic aspect of revealing the information*?

The strategic aspect of information transmission is a main theme in information economics. Signaling, cheaptalk, and recent information design literature all touch on different important aspects of the informed party's strategic choice on transmitting his or her information to an uninformed party (or parties). Little attention, however, has been paid to the case where the sender *owns an organization as a mechanism designer*. On the other hand, in mechanism design literature, it is often the case that the designer himself is *not* informed. That is, the principal's focus is typically on how to elicit private information from agents truthfully in his capacity as one facilitating information extraction, but not on whether and how to transmit his or her own private information to them.

This paper studies an *informed mechanism designer* problem in which the *principal's private information* is a number of agents. In the model, the principal has an object to allocate to agents, each of whom has private information about his or her valuation of it. This paper departs from the standard setting such that agents *randomly* arrive, which is only known to the principal. The principal can announce the number or not. This means that a *grand mechanism* consists of two components: (i) a *signal structure* mapping the principal's private information to public signals and (ii) *sub-mechanisms* such that execution of a sub-mechanism depends on the principal's private information as well as the publicly observable announcement. A collection of sub-mechanisms is designed to satisfy not only every agent's incentive compatibility but the principal's, in light of the revelation principle. Then, we ask whether announcing the information is desirable considering its impact on the strategic behavior of agents, given the cost.

The specific choice of the principal's private information – a number of agents – can be justified since it produces relatively large literature following Myerson (1998a) and fits well for the present paper's applications.[1] More importantly, population uncertainty is only a category of the principal's private information that can resolve a complex problem *innate* to the informed principal problem, with no additional assumptions. From each agent's point of view, given a public signal (that is, announcement of the number or no announcement), each agent must form an updated belief about the principal's type – the number of agents – which yields a corresponding expected transfer. Now, from the principal's point of view, he must construct an ex-ante revenue such that, first, there is the expectation of each agent's type – the agent's valuations – *conditional* on the number of agents, and then the revenue is the expectation of such conditional expectations over his own type, the number of agents. However, without any way of *interchanging* the expectation of the agent's private

[1] On the other hand, a number of agents may pose analytical difficulty as it is an infinite set.

information conditional on the principal's private information and the expectation of conditional expectations over the principal's, it is impossible for the former to be incorporated into the latter. As well known, the procedure is necessary in solving the mechanism design problem by Myerson (1981). We term the problem from such a conflict between the two perspectives a *conversion problem.*

Environmental equivalence is defined by Myerson (1998a) such that it holds if each agent's updated belief about a number of arrived agents is "consistent" with the principal's perspective. We define a novel concept, *mechanical equivalence*, in a parallel manner such that it holds if the two perspectives are consistent in the sense that the conversion problem is resolved. With mechanical equivalence and, additionally, the principal's *expected payoff linearity*, we make the informed principal problem tractable and solve it as in Myerson (1983). That is, the principal commits to a grand mechanism, and the principal's incentive compatibility is satisfied to secure his truth-telling.[2] When the principal is informed, his offering a mechanism can reflect his private information, and thus by observing the mechanism offered, agents can update their beliefs about the principal's private information. This interesting and yet complex problem was first addressed by Myerson (1983), who proposes the principle of inscrutability. The principal can instead construct an equivalent mechanism independent of his type and provide his incentive compatibility condition, extending the revelation principle to *himself.* In this way, the principal solves the informed principal mechanism, without changing beliefs of agents. While the principle of inscrutability is applied to a grand mechanism in this model, our approach marks a departure from the seminal paper in that the principal can still find it optimal to change beliefs of agents in the *interim* stage. Such belief changes can be designed by adopting a signal structure, which procedure is related to Kamenica and Gentzkow (2011) and Bergemann and Morris (2016). Yet, we depart from them in two important ways: The sender in this model is informed, and the sender *is* the principal whose incentive compatibility must be satisfied.

Our strategy to show the principal's truth-telling is to find a link between it and his *interim optimality*, as shown in a section about the generic informed principal problem. This does not mean that the principal can choose a mechanism differently from the committed one; the interim optimality means that the ex-ante committed mechanism is also optimal in the interim point of view. That is, even if the principal *were to* be given a chance to move in the interim stage, he would not do so. We propose the interim optimality since it serves as a vehicle to connect two different concepts, the optimal grand mechanism and the principal's incetnive compatibility.[3] We start with a simple signal structure such that the principal either announces the number with a probability or not but extend it to include rich signal sets in a section later. One important remark, though, is that having the simple signal structure does not undermine the nature of the informed

[2]For concreteness, consider an example with only two numbers of agents, $n = 1$ and $n = 2$, and two signals, a and b. Then, in total, there are four sub-mechanisms and two signal distributions – the probabilities of realizing a and b conditional on n – over a and b, which constitute a grand mechanism. If the principal commits to a grand mechanism, he must execute a specific sub-mechanism contingent on a realized public signal and his report of the private information, together with a signal distribution given his report. For the latter, if the principal commits to $(\Pr(a|n), \Pr(b|n)) = (0.5, 0.5)$ given $n = 1$ and $(0.7, 0.3)$ given $n = 2$, his report of $n = 1$ must carry over the former signal distribution. Yet, the principal can still misreport $n = 2$ even if he observes $n = 1$.

[3]See Subsection 3.4. for a formal definition and discussion.

principal problem. The principal's incentive compatibility is still required: If a number of agents is not announced, the agents must anticipate the number of agents *provided that* the principal reports it truthfully to a grand mechanism.

If the principal is not allowed to make a sub-mechanism contingent on his private information, there is a single threshold for the optimal grand mechanism such that he announces the number if an announcement cost is lower than the threshold; not otherwise. The main result of this paper is for the general case where a sub-mechanism can depend on the private information of the principal as well. Interestingly, it shows that the optimal grand mechanism is characterized by *double thresholds* for a relatively small announcement cost. We first solve the optimal sub-mechanisms when a number of arrived agents is announced and the optimal sub-mechanisms when it is not. Then, the principal finds it optimal to announce the number if it is lower than a lower threshold or higher than a higher one; on the other hand, he finds it optimal not to announce the number if it is between the two thresholds.[4] For intuition behind the results, suppose that the optimal reserve price under population uncertainty *happens to* be identical to the optimal reserve price when a number of agents is announced. Then, given *that* number, the two payoffs yield the same revenue. For any number greater than the zero-difference value, as the number becomes larger, announcing it mitigates the negative effects from intense competition with more agents, so announcement yields a higher revenue. On the other hand, for any number smaller than the zero-difference value, as the number becomes larger – so it gets closer to the zero-difference – the mitigating effect decreases. Overall, the net zero value is in fact the *minimum* point for the revenue difference, which with a positive announcement cost yields the double thresholds. The result has intriguing implications for implementing a grand mechanism. For instance, if an indirect mechanism is an all-pay auction where each agent chooses an effort level, then the main result implies that if the principal wishes to maximize the *expected aggregate effort* level – which is corresponding to the revenue – from agents, given the cost, it is optimal for him *not* to announce the number if it is in the middle range.[5]

We extend the signal structure into two directions. First, we allow the principal to not truthfully announce a number of arrived agents; that is, he could lie about it.[6] A single-peaked revenue function with respect to the choice of a no-assignment rule – which is a reserve price or an entry fee in its implementation – in each sub-mechanism makes this otherwise complex problem simple. That is, the choice of a reserve price given the true number of agents dominates other reserve prices, so the principal reveals it truthfully. Note that such a revelation through a reserve price is based on

[4]Note that the single threshold in the former restricted case is an announcement cost level, whereas the double thresholds in the latter general case are two different levels of number of agents, precisely since the number is the principal's private information

[5]This paper does not model a pandemic case, specifically, but it still sheds light on whether it is *always* desirable to announce a number of cases in order to encourage the expected aggregate effort level on precaution, hygiene or social distancing.

[6]One may argue, in light of the leading example in the introduction, that the principal's information is often verifiable (for instance, a government official cannot lie about it by its nature) in the following sense. For example, the set of number of agents is $\{0, 1, 2, 3\}$. Then, it is unacceptable for the principal to announce, *e.g.*, $n = 2$ when he observed $n = 3$. However, the principal can construct a set of signals or messages with the same cardinality preserving the order such as $S = \{\text{None, Low, Medium, High}\}$, and announce Medium even when he observed $n = 3$. Another related remark is that if S is not finite, we have an existence problem as found in Subsection 3.2, in general, but we extend it to the case with more signals in Section 7, nevertheless.

a grand mechanism that the principal commits to in the first place, not on signaling as in Maskin and Tirole (1990, 1992). The second and more interesting extension is to adopt multiple *vague* language terms such that the principal can choose G or B instead of no announcement, where G stands for a "good" situation and B for a "bad" one. For example, for the former, the principal can say that the number is hopeful or we are watching it with more hope, whereas for the latter, he can say that the number is concerning or we are watching it with some concern. We show that the double thresholds characterization expands such that there are now two thresholds for *each* vague language term.

This paper is related to several literatures, but its relationship is best captured as merging information design into, coincidentally – or inevitably – the three different pillars of Myerson's contribution, Myerson (1981), Myerson (1983), and Myerson (1998a). First, we study the informed principal problem in the area of screening or adverse selection as in Myerson (1981), not moral hazard. This means that the critical aspect of its analysis for direct mechanisms is finding *honest or truthful incentive compatibility* that elicits the truthful reporting of the type of each agent as well as that of the principal, while *obedience incentive compatibility* for moral hazard is simply irrelevant.[7] Therefore, any opportunity of principal "manipulating" agents cannot come from both constraints, dealing with the combinatorial deviation – that is, deviation in action following a misreport of type; it can only come from one's beliefs about the others' types in type revelation.

We also approach the informed principal problem in such a way that the principal *commits* to a grand mechanism as in Myerson (1983). Our focus, however, is more on the role of a signal structure to change the beliefs of agents following the realization of a signal in the interim stage. The commitment means that we do not use the signaling approach as in Maskin and Tirole (1990, 1992). For the latter approach, given auction settings, Jullien and Mariotti (2006) and Cai, Riley and Ye (2007) study a reserve price as a signaling device for an informed seller, without any separate information design technology. In the same vein, this also differentiates the present paper from the signaling approach in the Bayesian persuasion for the informed sender (see Perez-Richet (2014) among others).[8] Our mechanism design approach makes it possible for us to provide the *full characterization* of the optimal grand mechanism, without any refinement.

Lastly, the principal's private information is population uncertainty. This has generated a sizable literature since Myerson (1998a) (see, *e.g.*, Kim and Yoo (2021) for related papers).[9] Unlike the nature of information uncertainty in other papers studying cases where each agent may be unsure about his or her valuation, we study settings in which each agent knows his or her own valuation but does not know the distribution of the others' valuations via an uncertain number of participants in a mechanism.

In recent developments, Yamashita (2018) studies information design with an informed principal's commitment on a mechanism for screening, in particular with correlation between information of agents and that of principal; and Koessler and Skreta (2021) focus on the role of obedience incentive compatibility for the moral hazard problem in information design with a principal's private

[7]See Myerson (1982) to know more about the difference in the two incentive compatibility conditions. In particular, obedience incentive compatibility is based on correlated equilibrium by Aumann (1974, 1987).

[8]The literature is relatively large to review them all in this paper.

[9]For example, see Myerson (1998b).

information. While the principal's private information is restricted to a number of agents in the present paper, the key difference is that they study environments where the principal's information is either verifiable or public, with relevant applications, but we study environments in which an informed principal problem needs to be solved with truthful or honest incentive compatibility to induce truth-telling by the principal and the agents; and further, the principal's grand mechanism has sub-mechanisms that depend on his private information, which is closely *tied* to the conversion problem.

In the Bayesian persuasion by Kamenica and Gentzkow (2011) and the information design by Bergemann and Morris (2016), a principal controls a signal structure to influence the beliefs of an agent; or agents.[10] The present paper departs from the literature in that the sender (the principal) has private information and as such, the sender's choice of a signal structure is *constrained* by his incentive compatibility condition. That is, the principal's signal structure is integrated into his information collection process from the principal himself, as well as multiple agents, through truthful incentive compatibility. The key feature of this model – the principal's private information and truthful incentive compatibility – not only expands incomplete information to the principal, but it also has two other ramifications: First, the principal has each sub-mechanism contingent on his private information, as well as a public signal, and second, this is the main culprit of the conversion problem.

The paper is organized as follows. We introduce the model in the next section. We elaborate on the nature of our informed principal problem, including the conversion problem in general, in Section 3, before first discussing a benchmark with no principal's private information in Section 4. Mechanical equivalence extending environment equivalence to resolve the conversion problem is studied in depth in Section 5. We present the main results for the optimal grand mechanism given both the restricted domain – in which sub-mechanisms only depend on public signals – and especially the general case in Section 6. Section 7 extends the information design to include rich signal sets, and Section 8 concludes. All the proofs are collected in an appendix.

2 Model

A principal designs a mechanism to allocate an indivisible item to N agents. Unlike the standard environment with a fixed number of agents, the agents of this model *randomly* arrive according to a Poisson distribution with a parameter $\lambda > 0$, given by[11]

$$p(n|\lambda) \equiv \frac{e^{-\lambda}\lambda^n}{n!} \text{ for } n = 0, 1, 2, ... \tag{1}$$

After the random arrival of agents, the principal only observes the realized number of agents $N = n$; no individual agent observes it. Hence, the realized n is the principal's *private information*, which together with the random arrival makes the mechanism design the *informed principal problem.* The arrived agents, on the other hand, participate in a standard allocation game. Each agent i's value

[10] For earlier, well established, contributions on information transmission, see also Spence (1973) for signaling and Crawford and Sobel (1982) for cheaptalk.

[11] The powerful justification for a Poisson distribution is provided by Myerson (1998a), with the term, environmental equivalence, capturing the state of its concept. See Subsection 3.3 for more details.

from the item is $v_i \in [\underline{v}, \overline{v}]$ with $\overline{v} > \underline{v} \geq 0$; then, agent i obtains payoff $q_i v_i - t_i$ if the principal assigns the item to him with probability q_i with transfer t_i. Like n being the principal's private information, valuation v_i is agent i's private information, but the latter is his *private value*, whereas the former is a *common value* in the sense that it affects each agent's expected payoff as well as the principal's.

The principal *commits* to a *grand mechanism* for any realized n, which is designed to elicit not only v from the agents but n from the principal himself, satisfying the *principle of inscrutability* – in short, a mechanism independent of the principal's type and the revelation principle, as well as individual rationality, applied to the principal himself – following Myerson (1983).[12] Specifically, after Nature chooses $v \equiv (v_1, ..., v_n)$ and n, each agent i observes v_i, and then the principal offers a grand mechanism he commits to.[13] With the principal observing the arrival of agents, the principal and the agents report their types. Following the realization of a public signal, a committed sub-mechanism contingent on the signal and his private information is executed, and all parties receive their payoffs accordingly.

A grand mechanism consists of two components: a collection of sub-mechanisms and a signal structure. The signal structure is a mapping from the set of the principal's types $\mathbb{N}$ to a set of probability distributions over *publicly* observable signals. The set of signals consists of two "effective" signals n, NA and a null signal $\emptyset$ such that $S_n = \{n, \text{NA}, \emptyset\}$, where NA for No Announcement and $\emptyset$ means no sub-mechanism offer. That is, for $s = \emptyset$, the principal can attain at least zero payoff, and for $s = n$, the principal cannot lie about what he observes. We explore two different general sets of signals, $S = \mathbb{N} \cup \{\text{NA}, \emptyset\}$ and $S_n = \{n, \text{G}, \text{B}, \emptyset\}$, where G (Good) and B (Bad) denote two different "language terms" for no announcement in Section 7. Then, the informed principal chooses a signal structure π such that

$$\pi : \mathbb{N} \to \Delta(S). \tag{2}$$

We assume that each arrived agent incurs cost $c > 0$ to participate in each sub-mechanism and the principal incurs cost $d > 0$ to announce it.[14] In addition, we consider a *symmetric framework* for the sub-allocation (or auction) game such that each agent's value v_i is independently and identically drawn from a cumulative distribution F with density $f > 0$ for all $v_i \in (\underline{v}, \overline{v})$. Finally, the principal and agents are risk neutral and the reservation payoff of each agent and the principal is normalized to zero.

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[12] As one way to commit to it, consider a computer program, called "mechanism." This program is a mapping from the principal's report n as well as all agents' type reports v. If the principal reports n, then this program executes an allocation & transfer mechanism contingent on the principal's private information as well as a public signal.

[13] In Myerson (1983), the principal offers a mechanism after observing his type, but does so to make it independent of his type, "... is empirically indistinguishable from saying that "the principal will implement μ^*, no matter what his type is."... " (p.1774) Hence, offering a grand mechanism before observing n and offering it after observing n but in the same way as it should have been done without observing it are analytically identical.

[14] As common in the related literature, this cost c can be interpreted as a cost of reporting a type in a mechanism, which can be translated into any tangible or opportunity cost to participate in an auction. On the other hand, the principal incurs announcement cost d from any subsequent costly action to take following the announcement.

3 Informed principal problem

The informed mechanism designer problem contains several key elements, so in this section, we divert from the main model by taking a general environment in which the principal observes $\theta \in \Theta$ and a conditional distribution of v given θ is $F(v|\theta)$. The generalization in fact makes it easier for one to see not only where the contribution of this paper lies but the nature of complications of the problem.

3.1 Sub-mechanisms and conversion

The principal offers a sub-mechanism depending on his private information as well as a public signal. We start with a single agent case with his valuation $v \in [\underline{v}, \overline{v}]$ (also ignore the participation cost for this section), which in fact suffices to illustrate the main point. We further suppose that a set of the principal types Θ and a set of signals S are discrete and finite.[15] Then, for the future anticipated realized signal s, we consider that the principal solves the optimal sub-mechanism problem in the ex-ante timeline. A signal structure $\pi : \Theta \to \Delta(S)$ together with a realized signal s updates the agent's beliefs about θ, following Bayes rule such that $\mu(\theta|s) = \frac{p(\theta)\pi(s|\theta)}{\sum_{\theta' \in \Theta} p(\theta')\pi(s|\theta')}$, where $p(\theta)$ denotes a prior probability distribution for $\theta \in \Theta$, a general case of (1), with $p(\theta) > 0$ for all $\theta \in \Theta$.

To focus on a mechanism contingent on θ, we proceed for a fixed s; that is, $\Gamma^{(\theta)}$, as well as $q^{(\theta)}, t^{(\theta)}$, depends on s, but we do not consider it explicitly in this subsection. Then, a (direct) sub-mechanism $\Gamma^{(\theta)}$ consists of an allocation rule $q^{(\theta)} : [\underline{v}, \overline{v}] \to [0, 1]$ and a payment rule $t^{(\theta)} : [\underline{v}, \overline{v}] \to \mathbb{R}$. The superscript (θ) is employed to indicate that $\Gamma^{(\theta)}$ is a sub-mechanism that the principal commits to if he observes θ – regardless of whether the agent is informed of θ or not. Anticipating $\Gamma^{(\theta)}$ with μ, the agent obtains his payoff $u(v|\mu, s) = \sum_{\theta \in \Theta} \mu(\theta|s)\left[q^{(\theta)}(v)v - t^{(\theta)}(v)\right]$ for a truthful report, so his incentive compatibility is satisfied if for each $v, v' \in [\underline{v}, \overline{v}]$, $u(v|\mu, s) \geq \sum_{\theta \in \Theta} \mu(\theta|s)\left[q^{(\theta)}(v')v - t^{(\theta)}(v')\right]$, and his individual rationality is satisfied if for each $v \in [\underline{v}, \overline{v}]$, $\sum_{\theta \in \Theta} \mu(\theta|s)\left[q^{(\theta)}(v)v - t^{(\theta)}(v)\right] \geq 0$. Then, from the *agent's point of view*, a standard procedure yields – except for $\sum_{\theta \in \Theta} \mu(\theta|s)$ – for any revenue maximizing sub-mechanism satisfying the two said conditions with an increasing $\sum_{\theta \in \Theta} \mu(\theta|s)q^{(\theta)}(v)$, the agent's expected payment of

$$\sum_{\theta \in \Theta} \mu(\theta|s)t^{(\theta)}(v) = \sum_{\theta \in \Theta} \mu(\theta|s) \left[q^{(\theta)}(v)v - \int_{\underline{v}}^{v} q^{(\theta)}(x)dx\right]. \tag{3}$$

Now, from the *principal's point of view*, the (ex-ante) expected revenue for a realized s is

$$\Pr(s)\mathbb{E}_{\theta,v}\left[t^{(\theta)}(v) \mid s\right] = \sum_{\theta \in \Theta} p(\theta)\pi(s|\theta) \int_{\underline{v}}^{\overline{v}} t^{(\theta)}(v)f(v|\theta)dv, \tag{4}$$

where the probability of generating signal s is given by $\Pr(s) = \sum_{\theta' \in \Theta} p(\theta')\pi(s|\theta')$. Before embarking on discussing the conversion problem further, it is instructive to provide remarks on the principal's commitment to see its role. Note that μ in (3) is based on $\mu(\theta|s) = \frac{p(\theta)\pi(s|\theta)}{\sum_{\theta' \in \Theta} p(\theta')\pi(s|\theta')}$, where $\pi(s|\theta)$ is from a grand mechanism the principal offers in the *ex-ante stage*. On the other hand, $\pi(s|\theta)$ in (4) is from the *interim stage*, after θ is realized, which governs the probability of the principal reaching at that sub-mechanism. In other words, despite the same notation, they are

[15] As below, the same argument applies to continuous sets.

from two different timelines; the commitment does not allow the principal to actually change it.[16] Still, however, in the interim stage, the principal can *misreport* θ. That is, among $\big(\pi(s|\cdot)\big)_{\theta\in\Theta}$ the principal commits to, he can report $\theta' \neq \theta$ to have a signal mapping $\pi(s|\theta')$ instead of $\pi(s|\theta)$ even when he observes θ. We provide the principal's truth-telling condition subsequently.

Now, we rewrite (4) to have:

$$\sum_{\theta\in\Theta} p(\theta)\pi(s|\theta)\int_{\underline{v}}^{\overline{v}} t^{(\theta)}(v)f(v|\theta)dv = \Pr(s)\sum_{\theta\in\Theta}\mu(\theta|s)\int_{\underline{v}}^{\overline{v}} t^{(\theta)}(v)f(v|\theta)dv, \tag{5}$$

where the equality above follows from Bayes rule. Such an attempt, of course, was made in order to connect (3) and (4) as in the typical analysis; that is, to incorporate (3) into (4). As one can easily find out, it is *impossible* to incorporate the expected payment of the agent over θ into the expected revenue over both the agent's type v conditional on θ and the principal's type θ. For the category of the principal's private information that we consider in this model, that is, $\theta = n$, a single agent's valuation distribution does not depend on n. However, for $n \geq 2$, a distribution for valuations of the other agents changes with n, which involves the same problem.[17]

If a mechanism does not depend on θ, that is, $\Gamma^{(\theta)} = \Gamma$ for all θ, then such a complication does not arise. In other words, the nature of this *conversion problem* is closely tied with two *different perspectives* in (3) and (4) by the agent and the principal through a signal structure in the *presence of the principal's private information*: the principal's optimal sub-mechanism contingent on θ given the agent's private information (and reporting it in a mechanism). We later resolve it with what we term "mechanical equivalence."

3.2 Linearity and existence

Suppose that we resolve the conversion problem for each signal s and denote a sub-mechanism given (θ, s) by $\Gamma^{(\theta)}(s)$. The principal chooses both a collection of sub-mechanisms $\mathbf{\Gamma} \equiv \big(\Gamma^{(\theta)}(s)\big)_{(\theta,s)\in\Theta\times S}$ and a signal structure π to maximize his ex-ante payoff. Choosing the former and the latter optimally are interwoven, which can make the informed principal problem intractable: Unlike a standard decision making under uncertainty, in this case, the principal can choose a "set of prizes" as well, so choosing a set of sub-mechanisms affects the optimal signal structure and vice versa. In this paper, we can achieve a type of linearity for the principal's expected payoff, as a *result of* our setup to tackle it.

Together, a grand mechanism is defined as $\big(\mathbf{\Gamma}, \pi\big)$. Then, we denote by U_0 the principal's *interim* expected payoff from $\big(\mathbf{\Gamma}, \pi\big)$, and if the principal reports θ truthfully, it is given by

$$U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi(\cdot|\theta), \theta\Big). \tag{6}$$

The formula requires further detailed remarks. First, even for the interim stage, that is, after the principal observes θ, his interim payoff can still depend on the entire collection $\mathbf{\Gamma} \equiv$

[16] If π is not committed, then the model becomes signaling problem, and it raises a host of new questions: We need to find out an equilibrium π, which often comes with multiple equilibria and requires proper equilibrium refinement to handle them. We do not pursue that approach in this paper.

[17] That is, we first denote by v_{-i} a vector of all agents' valuations except for agent i's. Then, by defining agent i's expected winning probability and transfer as $Q_i^{(\theta)}(v_i) = \int_{[\underline{v},\overline{v}]^{N-1}} q_i(v_i, v_{-i})f(v_{-i}|\theta)dv_{-i}$ and $T_i^{(\theta)}(v_i) = \int_{[\underline{v},\overline{v}]^{N-1}} t_i(v_i, v_{-i})f(v_{-i}|\theta)dv_{-i}$, we can essentially have the same formula.

$\left(\Gamma^{(\theta)}(s)\right)_{(\theta,s)\in\Theta\times S}$. In other words, U_0 depends not only on the particular sub-mechanism given θ, $\Gamma^{(\theta)}(s)$, but also $\Gamma^{(\theta')}(s)$ for $\theta' \neq \theta$ since if θ is not revealed to the agents, each agent's expected payment is derived via the whole mechanism with their beliefs about θ. Likewise, even for the interim payoff, the whole signal mapping π matters: Both the probability $\pi(\cdot|\theta)$ of the sub-mechanism $\Gamma^{(\theta)}(s)$ being reached, and the other probabilities of $\left(\Gamma^{(\theta')}(s)\right)$ being reached for $\theta' \neq \theta$ – *i.e.*, $\pi(\cdot|\theta')$ – affect each agent's payment and thus the principal's interim payoff. Note that we have π for a grand mechanism $\left(\mathbf{\Gamma}, \pi\right)$ and, separately, $\pi(\cdot|\theta)$ as well. This is precisely because $\left(\mathbf{\Gamma}, \pi\right)$ affects the principal's interim payoff *through* each agent's payment, whereas the second argument, $\pi(\cdot|\theta)$, governs the probabilities with which the principal can reach at each sub-mechanism for $s \in S$.

Then, the informed principal's (expected) payoff is said to satisfy *linearity* if

$$U_0\Big(\left(\mathbf{\Gamma}, \pi\right), \pi(\cdot|\theta), \theta\Big) = \sum_{s\in S} \pi(s|\theta) U_0\Big(\left(\mathbf{\Gamma}, \pi\right), e_s, \theta\Big), \tag{7}$$

where e_s denotes a *degenerate distribution* that assigns probability 1 to signal s. As will be shown, finding the principal's ex-ante optimal mechanism is related to solving the optimal interim optimal mechanism. For the optimal interim optimal mechanism given θ, one can find that even with linearity, to solve the problem requires a *finite* set S to simply make $\Delta(S)$ compact for $\pi(\cdot|\theta) \in \Delta(S)$; that is, the finiteness is *necessary*.[18] Hence, if S is not finite, we have an existence problem in general, but in this paper, the model can be extended to $S = \mathbb{N} \cup \{\mathrm{NA}, \emptyset\}$ in Section 7. For $S_n = \{n, \mathrm{NA}, \emptyset\}$, to simplify notations, we denote by $\Gamma^{(\theta)}$ a sub-mechanism if the principal observes θ and announces it; and $\widetilde{\Gamma}^{(\theta)}$ a sub-mechanism if the principal observes θ but does not announce it such that

$$\Gamma^{(\theta)}(s) = \begin{cases} \Gamma^{(\theta)} & \text{if } \theta \text{ and } s = \mathrm{A}, \\ \widetilde{\Gamma}^{(\theta)} & \text{if } \theta \text{ and } s = \mathrm{NA}. \end{cases} \tag{8}$$

3.3 Population uncertainty

Lastly, even without a signal structure, for the category of the principal's private information we consider – population uncertainty with $\theta = n$ – applying the typical Bayes rule is not a straightforward problem. That is, even when $\pi(\mathrm{NA}|n) = 1$ for all $n \in \mathbb{N}$, each agent can use the information that he or she has already arrived. In other words, Bayes rule applies to *both* π and the arrival of oneself.

Yet, thanks to Myerson (1998a), with a Poisson distribution, we can achieve a simple updating for the latter, which is called *environmental equivalence*: Each agent's beliefs about his facing $(n-1)$ other agents – *i.e.* the number of agents *including himself* is n – are the same as those that the external game theorist, the principal in this model, would assess, which is $p(n-1|\lambda)$. Now, for the principal's private information $\theta = n$ in this paper, given π coupled with the equivalence, each agent's beliefs are given by

$$\mu(n|s) = \frac{p(n-1|\lambda)\pi(s|n)}{\sum_{m=1}^{\infty} p(m-1|\lambda)\pi(s|m)}. \tag{9}$$

[18]The sufficiency, on the other hand, requires that a collection of sub-mechanisms yields a continuous U_0.

3.4 Definition

We provide a formal definition for the optimal mechanism of the informed principal problem in this paper. The principal commits to a grand mechanism, but he must provide his incentive compatibility as well, to make the agents in his mechanism believe that he reports it truthfully, following the principle of inscrutability of Myerson (1983).

Especially, the principal's incentive compatibility is not a simple extension of an agent's, since his feasible set is constrained by his committed grand mechanism. To see it closely, first, we define a set of probability distributions that the principal commits to such that

$$\Pi(\pi) \equiv \{\pi(\cdot|\theta)\}_{\theta\in\Theta}. \tag{10}$$

Then, the principal's incentive compatibility (PIC) is said to be satisfied if for each $\theta \in \Theta$,

$$U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi(\cdot|\theta), \theta\Big) \geq U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi(\cdot|\theta'), \theta\Big) \text{ for all } \pi(\cdot|\theta') \in \Pi(\pi). \tag{11}$$

In addition, his (interim) individual rationality is said to be satisfied if for each $\theta \in \Theta$,

$$U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi(\cdot|\theta), \theta\Big) \geq 0. \tag{12}$$

The informed principal chooses the optimal signal structure π given a set of sub-mechanisms $\mathbf{\Gamma} \equiv \big(\Gamma^{(\theta)}(s)\big)_{(\theta,s)\in\Theta\times S}$ depending on both θ and s, and the grand mechanism includes the principal's truthful report of θ, *i.e.*, *the principal's incentive compatibility* (11) as well as each agent's incentive compatibility. The second equally intriguing point that makes this paper depart from the existing literature is *the principal's individual rationality* (12). The principal should not offer a sub-mechanism if its payoff is negative – that is, lower than his reservation payoff, zero. This is not an empty case. For instance, if the principal's information is not announced at all for all θ, the optimal sub-mechanisms cannot vary depending on θ, and such a "fixed mechanism" can result in a negative maximum interim payoff; it is optimal not to offer a sub-mechanism in that case. In the main model, we consider the environment in which the principal reports truthfully if he announces his private information. It is worthwhile commenting on PIC of such a case in which $\pi(\cdot|\theta) \in \Delta(S_\theta)$, where $S_\theta = \{\theta, \mathrm{NA}, \emptyset\}$ for all $\theta \in \Theta$.

Now, we are ready to provide a formal definition of the optimal grand mechanism.

Definition 1 *The informed principal obtains the optimal mechanism* $\big(\mathbf{\Gamma}, \pi\big)$ *if it maximizes*

$$\sum_{\theta\in\Theta} p(\theta) U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi(\cdot|\theta), \theta\Big) \textit{ given PIC and PIR.}$$

The informed principal *commits* to a grand mechanism as in Myerson (1983), but differently, he chooses π to influence beliefs of agents as in Kamenica and Gentzkow (2011) and Bergemann and Morris (2016). In other words, we explore further how the principal can choose a signal structure to influence the beliefs of agents in the interim stage. While the principle of inscrutability of the former applies to a grand mechanism, in the interim stage, our signal structure can change the beliefs of agents following the realization of a signal. On the other hand, this model advances the information design literature further such that the sender is *informed and a principal*.

$$\text{Ex-ante Optimality} \Longrightarrow \text{P-Interim Optimality} \Longrightarrow \text{Principal's IC \& IR}$$

Figure 1: P-Interim optimality

The principal's interim optimality is closely related to PIC. We say that the signal structure π is *P-interim optimal* for $\big(\mathbf{\Gamma}, \pi\big)$ if for each $\theta \in \Theta$, $\pi(\cdot|\theta)$ solves

$$\max_{\nu \in \Delta(S)} U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \nu, \theta\Big) \text{ and } U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi(\cdot|\theta), \theta\Big) \geq 0. \tag{13}$$

We need to clarify the interim optimality (13) before proceeding further, since one may argue what we mean by that given the ex-ante committed mechanism. This does *not* mean that the principal can choose some $\widehat{\pi}(\cdot|\theta)$ differently from a committed $\pi(\cdot|\theta)$ in the interim stage – again, he can misreport θ; rather, P-interim optimality means that the ex-ante committed $\big(\mathbf{\Gamma}, \pi\big)$ is optimal from the interim point of view – in the hypothetical sense. Put differently, we use P-interim optimality as a means to check the principal's incentive compatibility because it implies the latter, as illustrated in Figure 1. The second important remark is that the interim optimality depends on the ex-ante offered π, which entails that there can be more than one interim optimal grand mechanism $\big(\mathbf{\Gamma}, \pi\big)$.[19]

We show the relationship between the optimality of a grand mechanism and P-interim optimality and that between P-interim optimality and PIC. First, it is relatively straightforward to show that if π is P-interim optimal for $\big(\mathbf{\Gamma}, \pi\big)$, the principal's incentive compatibility PIC and individual rationality PIR are satisfied. Now, to show that any (ex-ante) optimal grand mechanism is P-interim optimal, by supposing that it is not P-interim optimal, one can construct another grand mechanism satisfying PIC and PIR that can yield a higher ex-ante payoff to the principal. Note that without the principal's PIC and PIR, optimality of this model is incomplete, because the optimality is based on the payments by the agents in the mechanism believing in the principal's truthful report – just like one agent's beliefs about the others' truth-telling in the mechanism.

Proposition 1 *Suppose that the informed principal's payoff linearity is satisfied and given $\big(\mathbf{\Gamma}, \pi\big)$, P-interim optimal $\pi(\cdot|\theta)$ exists for all θ. Then a grand mechanism satisfies the following properties.*

(i) If π is P-interim optimal for $\big(\mathbf{\Gamma}, \pi\big)$, it satisfies the principal's PIC and PIR.

(ii) If the grand mechanism $\big(\mathbf{\Gamma}, \pi\big)$ is optimal, then (a) for each $\theta \in \Theta$ and every $s \in supp(\pi(s|\theta))$, $\Gamma^{(\theta)}(s)$ is optimal for π, and (b) π is P-interim optimal for $\big(\mathbf{\Gamma}, \pi\big)$.

The significance of the proposition is not just limited to the problem of whether there is a divergence between the ex-ante optimal mechanism and the P-interim optimal mechanism, but more importantly, it shows that to find out the optimal mechanism, we just need to focus on the two conditions (a) and (b) in (ii) of Proposition 1, as will be analyzed in Sections 5 and 6. Note that the P-interim optimality of π is contingent on $\big(\Gamma^{(\theta)}(s)\big)_{(\theta,s)\in\Theta\times S}$, so without the principal's payoff linearity, solving it becomes complicated; that is, due to the linearity, we are allowed to *pin down* the optimal sub-mechanism for each $(\theta, s) \in \Theta \times S$.

[19]That is, both $\big(\mathbf{\Gamma}, \pi\big)$ and $\big(\mathbf{\Gamma}, \widehat{\pi}\big)$ are interim optimal if for each $\theta \in \Theta$,
$U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi(\cdot|\theta), \theta\Big) \geq U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi'(\cdot|\theta), \theta\Big) \,\forall\, \pi'(\cdot|\theta)$ and $U_0\Big(\big(\mathbf{\Gamma}, \widehat{\pi}\big), \widehat{\pi}(\cdot|\theta), \theta\Big) \geq U_0\Big(\big(\mathbf{\Gamma}, \widehat{\pi}\big), \pi'(\cdot|\theta), \theta\Big) \,\forall\, \pi'(\cdot|\theta)$.

4 Benchmark: no private information of the principal

We first examine the benchmark in which the arrived number of agents $N = n$ is *complete information*, while v is still incomplete information. The benchmark is, as shown later, equally applicable to the incomplete information n case where the realized n is announced by the principal, if it is *optimal for him to do so.* Additionally, it is useful for introducing some key notations along with participation costs that will appear in the subsequent sections.

The standard analysis for entry with participation costs is well known, but we provide the following procedure for the benchmark to make our analysis self-contained – to make it applicable even to the case where n is unknown to agents. Using the *semirevelation mechanism* proposed by Stegeman (1996), without loss of generality, we consider a direct mechanism $\Gamma^{(n)}$ consisting of functions $q^{(n)}$, $t_i^{(n)}$ and $\sigma_i^{(n)}$ for $i \in \{1, ..., n\}$ such that $q^{(n)} : [\underline{v}, \overline{v}]^n \to \Delta^{(n)}$, $t_i^{(n)} : [\underline{v}, \overline{v}]^n \to \mathbb{R}$ and $\sigma_i^{(n)} : [\underline{v}, \overline{v}] \to \{1, 0\}$, where denote $\Delta^{(n)} \equiv \{(q_1^{(n)}, ..., q_n^{(n)}) \in \mathbb{R}_+^n : \sum_{i=1}^n q_i \leq 1\}$. With the semirevelation for such costs, the principal allocates the item to agent i with a positive probability only if $\sigma_i^{(n)}(v_i) = 1$, where 1 stands for entry.[20]

Let v_{-i} be a vector of all agents' types except for agent i's and denote their CDF by $F^{(n)}(v_{-i}) \equiv \times_{j\in\{1,...,n\}\setminus\{i\}} F(v_j)$. Then, the expected probability that agent i obtains the item is defined as $Q_i^{(n)}(v_i) \equiv \int_{[\underline{v},\overline{v}]^{n-1}} q_i^{(n)}(v_i, v_{-i}) dF^{(n)}(v_{-i})$ and the expected payment that agent i makes to the principal is $T_i^{(n)}(v_i) \equiv \int_{[\underline{v},\overline{v}]^{n-1}} t_i^{(n)}(v_i, v_{-i}) dF^{(n)}(v_{-i})$.[21] With them, agent i's payoff when he reports truthfully is

$$U_i^{(n)}(v_i) = v_i Q_i^{(n)}(v_i) - T_i^{(n)}(v_i) - c\sigma_i^{(n)}(v_i). \tag{14}$$

A mechanism $\Gamma^{(n)}$ is said to be *incentive compatible* (IC) if each $i \in \{1, ..., n\}$ and for every $v_i, v_i' \in [\underline{v}, \overline{v}]$,

$$U_i^{(n)}(v_i) \geq v_i Q_i^{(n)}(v_i') - T_i^{(n)}(v_i') - c\sigma_i^{(n)}(v_i'), \tag{15}$$

and a mechanism is said to be *individually rational* (IR) if for each $i \in \{1, ..., n\}$, $U_i^{(n)}(v_i) \geq 0$ for all $v_i \in [\underline{v}, \overline{v}]$. Then, a mechanism is IC if and only if for each $i \in \{1, ..., n\}$, $Q_i^{(n)}(v_i)$ is increasing and $U_i(v_i) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} Q_i^{(n)}(x)dx$, and since any revenue maximizing mechanism must have $U_i^{(n)}(\underline{v}) = 0$, as well known, under revenue maximization, we have

$$T_i^{(n)}(v_i) = v_i Q_i^{(n)}(v_i) - \int_{\underline{v}}^{v_i} Q_i^{(n)}(x)dx - c\sigma_i^{(n)}(v_i). \tag{16}$$

Then, the principal's expected payoff is given as

$$\int_{[\underline{v},\overline{v}]^n} \sum_{i=1}^n t_i(v) dF(v_1) \times \cdots \times F(v_N) = \sum_{i=1}^n \int_{\underline{v}}^{\overline{v}} T_i^{(n)}(v_i) f(v_i) dv_i. \tag{17}$$

A sub-mechanism $\Gamma^{(n)}$ is optimal if it maximizes (17) given IC and IR. Now, we characterize each agent's entry assignment, which will also be applied to the main analysis where n is only

[20] That is, if agent i reports his type v_i for all $i \in \{1, ..., n\}$, then the principal commits to allocating the item to agent i with probability $q_i^{(n)}(v)$ for a transfer $t_i^{(n)}(v)$ from agent i if $\sigma_i^{(n)}(v_i) = 1$, whereas $q_i^{(n)}(v) = t_i^{(n)}(v) = 0$ if $\sigma_i^{(n)}(v_i) = 0$. As shown in the proof of Proposition 2, formally, $\sigma_i^{(n)}(v_i)$ is a *composite* function combining the principal's entry assignment and agent i's entry strategy.

[21] Hence, if $n = 1$, the principal assigns the item to agent number 1 with probability $q_1^{(1)}(v_1)$ with a transfer $t_1^{(1)}(v_1)$.

known to the principal. Suppose that a sub-mechanism satisfies IC and IR. Then, if the principal adopts no-assignment rule such that $Q_i^{(n)}(v_i) = 0$ for some $v_i \in [\underline{v}, \overline{v}]$, it must satisfy a *threshold* property, like a reserve price (or an entry fee) in an auction: that is, there exists $k \in [\underline{v}, \overline{v}]$ such that $Q_i^{(n)}(v_i) > 0$ if $v_i \geq k$, and $Q_i^{(n)}(v_i) = 0$ if $v_i < k$.[22] Then, by substituting (16), the standard analysis yields

$$\sum_{i=1}^{n} \left[\int_{k}^{\overline{v}} \big(Q_i^{(n)}(v_i)\psi(v_i) - c \big) f(v_i) dv_i \right], \tag{18}$$

where we denote the *virtual valuation* by $\psi(v_i) \equiv v_i - \frac{1-F(v_i)}{f(v_i)}$ and $G^{(n)}(x) \equiv [F(x)]^{n-1}$.

We define the optimal revenue for participating types *given* no-assignment rule k by

$$R(k,n) \equiv nM(k,n), \tag{19}$$

where the ex-ante payment of a buyer is given as

$$M(k,n) \equiv \int_{k}^{\overline{v}} \big(G^{(n)}(x)\psi(x) - c \big) f(x) dx. \tag{20}$$

Both of the notations will keep recurring as key concepts throughout our analysis. It can be readily shown that $R(k,n)$ satisfies the *single-peaked property with respect to* k; there exists $k' \in [\overline{v}, \underline{v}]$ such that it is strictly increasing if $k < k'$ and strictly decreasing if $k > k'$. We say that F is regular if ψ is strictly increasing. The optimal allocation rule for participating types assigns the item to the highest value with probability 1 if F is regular. Hence, it resembles the optimal mechanism of Myerson (1981) except for no-assignment rule $r(n)$, as in Menezes and Monteiro (2000) and Lu (2009), which depends on the number of agents n.

Proposition 2 *Suppose that n is complete information and F is regular. Then, among all incentive compatible and individually rational mechanisms $\Gamma^{(n)}$, the optimal mechanism is given by*

$$q_i^{(n)}(v) = \begin{cases} 1 & \text{if } v_i > r(n) \text{ and } v_i > \max_{j \neq i} v_j, \\ 0 & \text{otherwise,} \end{cases}$$

and $t_i^{(n)}(v)$ that is any function satisfying $T_i^{(n)} = v_i Q_i^{(n)}(v_i) - \int_{\underline{v}}^{v_i} Q_i^{(n)}(x)dx - c$ for $v_i > r(n)$ and 0 for almost all $v_i < r(n)$; and furthermore, the optimal revenue is

$$R(r(n), n),$$

where $r(n)$ satisfies $G^{(n)}(r(n))\psi(r(n)) = c$.

One can find that as n increases, $r(n)$ increases from the monotonicity of ψ. As the number of bidders increase in an auction, a corresponding optimal reserve price must increase in order to account for the expected revenue loss from a larger set of participants.

[22] If not, obviously, the monotonicity of $Q_i^{(n)}(v_i)$ is violated. Formally, there exists $k \in [\underline{v}, \overline{v}]$ such that $q^{(n)}(v_i, v_{-i}) > 0$ for $v_{-i} = (\underline{v}, ..., \underline{v})$ if $v_i \geq k$, and $q^{(n)}(v_i, v_{-i}) = 0$ almost all $v_{-i} \in [\underline{v}, \overline{v}]^{n-1}$ if $v_i < k$.

5 Environmental and mechanical equivalence

In this section, we consider the privately informed principal as in the main model of Section 2, and provide important intermediate steps to establish our main results in the next section. We start by noting that under population uncertainty given $s = \mathrm{NA}$, one needs to be careful about each agent's name. Each agent has his or her "generic" name, say number 70, but upon arrival of some subset of agents, say five agents $n = 5$, assigning a new number is necessary in order to extend the idea of environmental equivalence to mechanism design. Specifically, first, to extend the idea of environmental equivalence to a mechanism design, the principal must also hide each agent's name i that he assigns so that the title i is only known to the principal; otherwise, for instance, if an agent is informed that he is agent number 7, he knows that there are at least 7 agents. The second requirement to fulfill the additional equivalence in a mechanism is to make it *anonymous* such that for each $n \in \mathbb{N}$ and every $i \neq j \in \{1, ..., n\}$,

$$\widetilde{q}_i^{(n)}(v_i, v_{-i}) = \widetilde{q}_j^{(n)}(v_j, v_{-j}) \text{ and } \widetilde{t}_i^{(n)}(v_i, v_{-i}) = \widetilde{t}_j^{(n)}(v_j, v_{-j}) \text{ for all } v_i = v_j \text{ and } v_{-i} = v_{-j}. \tag{21}$$

The property simply requires that what matters is only the valuations of the agents, not their names. Any non-anonymous way of allocating the item or making a transfer necessarily reveals information about name i. We say that a grand mechanism is under full anonymity if it satisfies secrecy and anonymity above, and full anonymity turns out to be essential in resolving the conversion problem as discussed in Subsection 3.1.

Now, to find the optimal mechanism, we rely on Proposition 1, which lays the foundations of the characterization of the optimal grand mechanism under the payoff linearity. That is, we can proceed with two steps: First, for each π, we find the optimal $(\Gamma^{(n)})_{n\in\mathbb{N}}$ and $(\widetilde{\Gamma}^{(n)})_{n\in\mathbb{N}}$ in (8), and then we find P-interim optimal π. Since the optimal $(\Gamma^{(n)})_{n\in\mathbb{N}}$ for $s = \mathrm{A}$ can be found in Proposition 2, for an arbitrary π, it remains to determine the optimal $\left(\widetilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$ for $s = \mathrm{NA}$. Despite the two effective signals $S_n = \{n, \mathrm{NA}, \emptyset\}$ in the model, we find that it is useful to retain $\pi(s|n)$ to keep track of the role of a signal structure, in general, and to extend it to a model with more signals in Section 7. For each n, a direct mechanism $\widetilde{\Gamma}^{(n)}$ consists of functions $\widetilde{q}^{(n)}$, $\widetilde{t}_i^{(n)}$ and $\widetilde{\sigma}_i^{(n)}$ for $i \in \{1, ..., n\}$, where $\widetilde{q}^{(n)} : [\underline{v}, \overline{v}]^n \to \Delta^{(n)}$, $\widetilde{t}_i^{(n)} : [\underline{v}, \overline{v}]^n \to \mathbb{R}$ and $\widetilde{\sigma}_i^{(n)} \to \{1, 0\}$. We adopt the notations with tilde to indicate that the principal offers a sub-mechanism without announcing n, as in Subsection 3.2.

Upon arrival, under population uncertainty, each agent reports his type, anticipating the number of competitors. By incorporating beliefs about n, $\mu(n|s)$ in (9), from the arbitrary agent i's point of view, the expected "winning" probability $\widetilde{Q}_i(v_i|\mu, s)$ and the expected transfer $\widetilde{T}_i(v_i|\mu, s)$ are derived, respectively, as

$$\widetilde{Q}_i(v_i|\mu, s) = \sum_{n=1}^{\infty} \mu(n|s) \widetilde{Q}_i^{(n)}(v_i) \text{ and } \widetilde{T}_i(v_i|\mu, s) = \sum_{n=1}^{\infty} \mu(n|s) \widetilde{T}_i^{(n)}(v_i), \tag{22}$$

where $\widetilde{Q}_i^{(n)}(v_i) = \int_{[\underline{v},\overline{v}]^{n-1}} \widetilde{q}_i^{(n)}(v_i, v_{-i}) dF^{(n)}(v_{-i})$ and $\widetilde{T}_i^{(n)}(v_i) = \int_{[\underline{v},\overline{v}]^{n-1}} \widetilde{t}_i^{(n)}(v_i, v_{-i}) dF^{(n)}(v_{-i})$ denote the expected probability and transfer *conditional* on n. Then, agent i obtains the expected payoff if he reports truthfully such that

$$\widetilde{U}_i(v_i|\mu, s) = v_i \widetilde{Q}_i(v_i|\mu, s) - \widetilde{T}_i(v_i|\mu, s) - c\widetilde{\sigma}_i(v_i|\mu, s),$$

where $\widetilde{\sigma}_i$ is agent i's entry strategy. A collection of sub-mechanisms $\big(\widetilde{\Gamma}^{(n)}\big)_{n\in\mathbb{N}}$ is said to be *incentive compatible under population uncertainty* (IC-U) if for each $i \in \{1, ..., n\}$ and for every $v_i, v_i' \in [\underline{v}, \overline{v}]$,

$$\widetilde{U}_i(v_i|\mu, s) \geq v_i\widetilde{Q}_i(v_i'|\mu, s) - \widetilde{T}_i(v_i'|\mu, s) - c\widetilde{\sigma}_i(v_i'|\mu, s), \tag{23}$$

and a collection of sub-mechanisms $\big(\widetilde{\Gamma}^{(n)}\big)_{n\in\mathbb{N}}$ is said to be *individually rational under population uncertainty* (IR-U) if for each $i \in \{1, ..., n\}$, $\widetilde{U}_i(v_i|\mu, s) \geq 0$ for all $v_i \in [\underline{v}, \overline{v}]$. Then, it is IC-U in (23) if and only if for each $i \in \{1, ..., n\}$, $\widetilde{Q}_i(v_i|\mu, s)$ is increasing and $\widetilde{U}(v_i|\mu, s) = \widetilde{U}(\underline{v}|\mu, s) + \int_{\underline{v}}^{\overline{v}} \widetilde{Q}_i(x|\mu, s)dx$. As in the complete information case, a mechanism comes with a threshold no-assignment rule k. Hence, under revenue maximization, for $v_i \in [k, \overline{v}]$, we have

$$\widetilde{T}_i(v_i|\mu, s) = v_i\widetilde{Q}_i(v_i|\mu, s) - \int_k^{v_i} \widetilde{Q}_i(x|\mu, s)dx - c. \tag{24}$$

On the other hand, for a collection of sub-mechanisms $\big(\Gamma^{(n)}(s)\big)_{n\in\mathbb{N}}$, the principal obtains the revenue such that

$$R = \sum_{n=1}^{\infty} p(n|\lambda)\pi(s|n) \sum_{i=1}^{n} \int_{[\underline{v},\overline{v}]} \int_{[\underline{v},\overline{v}]^{n-1}} \widetilde{t}_i^{(n)}(v_i, v_{-i})dF^{(n)}(v_{-i})dF(v_i). \tag{25}$$

We denote an arbitrary agent i's value by X, since the auction model is symmetric, and a vector of valuations except for an arbitrary agent i's by Y. Then, by the anonymity (21), the revenue (25) is rewritten as

$$R = \int_{[\underline{v},\overline{v}]} \Pr(s)\mathbb{E}\Big[\sum_{i=1}^{N} \widetilde{t}_i^{(N)}(X, Y) \mid X = x, s\Big]dF(x),$$

where the part inside the integral is given by

$$\Pr(s)\mathbb{E}\Big[\sum_{i=1}^{N} \widetilde{t}_i^{(N)}(X, Y) \mid X = x, s\Big] = \sum_{n=1}^{\infty} p(n|\lambda)\pi(s|n) \sum_{i=1}^{n} \int_{[\underline{v},\overline{v}]^{n-1}} \widetilde{t}_i^{(n)}(x, y)dF^{(n)}(y). \tag{26}$$

This single step is also non-trivial, apart from mechanical equivalence below, so one need to appreciate it; that is, for the alternative formulation, we condition the sum of the expected payments by n agents (26) on the same value x for all of them.[23] Nonetheless, with the symmetry and anonymity, the two formulations are equivalent in terms of generating their ex-ante expected values.

Now, a critical step is how to relate (24) with the above expected payment (26) so that the principal's ex-ante revenue can be reformulated with the allocation rule to find out the optimal mechanism, as in the standard analysis – for instance, the equality we use in (17) without population uncertainty. That is, we need to resolve the conversion problem in Subsection 3.1: (26) is the ex-ante sum of expected transfer from *the principal's point of view*, knowing $N = n$ as well as each agent's title i, whereas (24) is the ex-ante expected transfer from *each arrived agent's point of view* under population uncertainty. If the two are related in such a way that for each s, the sum of expected payments the principal receives is the same as a *representative agent's allocation & transfer* times the expected number of n, we say that a mechanism is *mechanically equivalent* under environmental equivalence.

[23] To do that, let us consider the case that $n = 2$ realizes. From (25), we have $\sum_{i=1}^{2} \int_{[\underline{v},\overline{v}]} \widetilde{t}_i^{(2)}(v_i, v_{-i})dF^{(2)}(v_{-i}) = \int_{[\underline{v},\overline{v}]} \widetilde{t}_1^{(2)}(v_1, v_2)dF^{(2)}(v_2) + \int_{[\underline{v},\overline{v}]} \widetilde{t}_2^{(2)}(v_2, v_1)dF^{(2)}(v_1)$, whereas from (26), $\sum_{i=1}^{2} \int_{[\underline{v},\overline{v}]} \widetilde{t}_i^{(2)}(x, y)dF^{(2)}(y) = \int_{[\underline{v},\overline{v}]} \widetilde{t}_1^{(2)}(x, y)dF^{(2)}(y) + \int_{[\underline{v},\overline{v}]} \widetilde{t}_2^{(2)}(x, y)dF^{(2)}(y)$ conditional on x.

Definition 2 *A mechanism is mechanically equivalent under environmental equivalence if for each $i \in \{1,...,n\}$ and every $s \in S$,*

$$\begin{aligned}\Pr(s)\mathbb{E}\Big[\sum_{i=1}^{N} \widetilde{q}_i^{(N)}(X,Y) \mid X = x, s\Big] &= \lambda \Pr(s)\widetilde{Q}_i(x|\mu, s),\\ \Pr(s)\mathbb{E}\Big[\sum_{i=1}^{N} \widetilde{t}_i^{(N)}(X,Y) \mid X = x, s\Big] &= \lambda \Pr(s)\widetilde{T}_i(x|\mu, s),\end{aligned} \tag{27}$$

where $\Pr(s) = \sum_{m=1}^{\infty} p(m-1|\lambda)\pi(s|m)$.

To understand the intuition behind the formula above, we only examine the allocation rule further, which, by moving $\lambda \Pr(s)$ in the right-hand side to the left-hand side, can be rewritten as

$$\sum_{n=1}^{\infty} \mu(n|s)\frac{1}{n}\sum_{i=1}^{n}\int_{[\underline{v},\overline{v}]^{n-1}} \widetilde{q}_i^{(n)}(x,y)dF^{(n)}(y) = \widetilde{Q}_i(x|\mu, s),$$

where we use $p(n|\lambda) = \frac{e^{-\lambda}\lambda^n}{n!} = \frac{\lambda}{n}p(n-1|\lambda)$ and Bayes rule in (9) in Subsection 3.3. Then, by (22), the equality holds if and only if $\widetilde{q}_i^{(n)}(x,y) = \frac{1}{n}\sum_{i=1}^{n}\widetilde{q}_i^{(n)}(x,y)$. As shown in the following lemma, this can be accomplished if the principal assigns names of arrived agents randomly, which we term *random assignment of names*. Formally, random assignment of names holds if for each $i \in \{1,...,n\}$ and every $x \in [\underline{v},\overline{v}]$, $y \in [\underline{v},\overline{v}]^{n-1}$,

$$\widetilde{q}_i^{(n)}(x,y) = \frac{1}{n}\sum_{j=1}^{n}\widetilde{q}_j^{(n)}(x,y) \text{ and } \widetilde{t}_i^{(n)}(x,y) = \frac{1}{n}\sum_{j=1}^{n}\widetilde{t}_j^{(n)}(x,y). \tag{28}$$

As the concept is not standard in mechanism design, we explain it in detail. If the principal commits to a grand mechanism satisfying random assignment of names, upon arrival of n agents whose names only he knows, he assigns each title $i \in \{1,...,n\}$ to every agent randomly, *i.e.* with probability $\frac{1}{n}$. Then, agent i's allocation and transfer are identical to the *expected* allocation and transfer, respectively, of all j agents under random number generation.

In fact, while random assignment of names is conceptually new to the literature, it is essentially nothing more than full anonymity; that is, it is equivalent to full anonymity in (21) together with secrecy. We already required them in order to extend environment equivalence to our mechanism design problem in the beginning of this section. The following lemma also shows that mechanical equivalence holds under random assignment of names.

Lemma 1 *A (grand) mechanism satisfies the following properties.*

(i) A mechanism satisfies random assignment of names if and only if it satisfies full anonymity.

(ii) Under random assignment of names, a mechanism is mechanically equivalent.

By Lemma 1, from (26), the principal's expected revenue is derived as

$$\sum_{n=1}^{\infty} p(n|\lambda)\pi(s|n)\sum_{i=1}^{n}\int_{k}^{\overline{v}} \widetilde{T}_i^{(n)}(v_i)f(v_i)dv_i. \tag{29}$$

Then, a collection of sub-mechanisms $\left(\widetilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$ for the signal $s = \text{NA}$ is optimal if it maximizes (29) given IC-U and IR-U. The optimal allocation rule for participating types assigns the item to the highest value with probability 1, like the complete information benchmark, but the principal chooses the optimal no-assignment rule such that it now exploits the population uncertainty that each agent faces.

Proposition 3 *Suppose that F is regular and signal $s = NA$ is generated.*[24] *Then, under population uncertainty $\left(\pi(A|1), \pi(A|2), ...\right) \neq (1, 1, ...)$, among all incentive compatible and individually rational mechanisms $\left(\widetilde{\Gamma}^{(n)}\right)_{n\in\mathbb{N}}$, for each $\widetilde{\Gamma}^{(n)}$, the optimal mechanism for signal $s = NA$ is given by*

$$\widetilde{q}_i^{(n)}(v) = \begin{cases} 1 & \text{if } v_i > \widetilde{r}(\pi, s) \text{ and } v_i > \max_{j\neq i} v_j, \\ 0 & \text{otherwise,} \end{cases}$$

and $\widetilde{t}_i^{(n)}(v)$ that is any function satisfying (24) for $v_i > \widetilde{r}(\pi, s)$ and 0 for almost all $v_i < \widetilde{r}(\pi, s)$; and furthermore, the optimal revenue given $s = NA$ is

$$\sum_{n=1}^{\infty} p(n|\lambda)\pi(s|n) \max\{R(\widetilde{r}(\pi, s), n), 0\},$$

where $\widetilde{r}(\pi, s)$ satisfies $\frac{\sum_{n=1}^{\infty} p(n|\lambda)\pi(s|n)nG^{(n)}(\widetilde{r}(\pi,s))\psi(\widetilde{r}(\pi,s))}{\sum_{m=1}^{\infty} p(m|\lambda)\pi(s|m)m} = c.$

To satisfy the principal's individual rationality (12), the principal should not offer a sub-mechanism if it is negative, which leads to $\max\{R(\widetilde{r}(\pi, s), n), 0\}$. With the revenue (19) and the ex-ante payment of a buyer (20), under population uncertainty, the corresponding revenue and payment given $\widetilde{r}(\pi, s)$ are derived as $R(\widetilde{r}(\pi, s), n) = nM(\widetilde{r}(\pi, s), n)$, and $M(\widetilde{r}(\pi, s), n) = \int_{\widetilde{r}(\pi,s)}^{\overline{v}} \left(G^{(n)}(x)\psi(x) - c\right) f(x)dx$, respectively. It can be readily shown that given the optimal $\widetilde{r}(\pi, s)$, the payment $M(\widetilde{r}(\pi, s), n)$ is a strictly decreasing function of n, and further, the positive term $G^{(n)}(x)$ keeps decreasing. This entails that there exists a unique $\widetilde{n} > 0$ such that

$$R(\widetilde{r}(\pi, s), \widetilde{n}) = 0, \tag{30}$$

and $R(\widetilde{r}(\pi, s), n) > 0$ for $n < \widetilde{n}$, whereas $R(\widetilde{r}(\pi, s), n) < 0$ for $n > \widetilde{n}$ for which the null signal $\emptyset$ can be used to have no mechanism offer. We discuss it in depth and more explicitly in the next section.

As an extreme case, if $\pi(\text{NA}|n) = 1$ for all $n \in \mathbb{N}$, the principal offers the mechanism, without announcing n for *all n with probability* 1, such that for any number of arrived agents, each agent's allocation and transfer will be chosen according to $\widetilde{\Gamma}^{(n)}$. In this case, the optimal reserve price is derived neatly as below.

Corollary 1 *Suppose that $\pi(NA|n) = 1$ for all $n \in \mathbb{N}$ and F is regular. Then, the optimal revenue is*

$$\sum_{n=1}^{\infty} p(n|\lambda) \max\{R(\widetilde{r}(\pi, s), n), 0\},$$

where $\widetilde{r}$ satisfies $e^{-\lambda(1-F(\widetilde{r}))}\psi(\widetilde{r}) = c$.

[24]Note that to generate signal $s = \text{NA}$ with positive probability requires that $\pi(\text{A}|n) \neq 1$ for *at least* one n, that is, $\left(\pi(\text{A}|1), \pi(\text{A}|2), ...\right) \neq (1, 1, ...)$, so writing both conditions in the statement is redundant. Despite that, we do so to emphasize population uncertainty.

One can find that this optimal mechanism is identical to that from Proposition 3 if $\pi(\mathrm{NA}|n) = 1$ for all n such that

$$\frac{\sum_{n=1}^{\infty} p(n|\lambda) n G^{(n)}(\widetilde{r}(\pi,s))\psi(\widetilde{r}(\pi,s))}{\lambda} = c \Rightarrow \widetilde{r}(\pi,s) = \widetilde{r}.$$

6 Optimal mechanism under the two equivalences

We first study the optimal mechanism under a restricted domain such that a mechanism does not depend on θ, that is, $\Gamma^{(\theta)}(s) = \Gamma(s)$ for all $\theta \in \Theta$ in terms of a general notation from Subsection 3.1. This is translated into either $\pi(\mathrm{NA}|n) = 1$ for all n or $\pi(\mathrm{A}|n) = 1$ for all n in this model. The former is to conceal all n, whereas the latter is to reveal all of them. The same type of the revealing and concealing policies with auctions are studied for risk-averse bidders by McAfee and McMillan (1987) and for ambiguity aversion by Levin and Ozdenoren (2004), *without information design and participation costs*. Note that this is the case where the mechanism designer commits to a policy, either the revealing or the concealing policy. In that sense, we call it the optimal policy rather than the optimal mechanism.

For the revealing policy, it is straightforward to find that $\left(\Gamma^{(n)}\right)_{n\in\mathbb{N}}$ is incentive compatible and individually rational if and only if $\Gamma^{(n)}$ is IC and IR for all n. The incentive compatibility IC-U and individual rationality IR-U for the concealing policy follow from Proposition 3. Then, by combining Proposition 2 and Corollary 1 for Proposition 3 together with announcement cost $d > 0$, the net benefit of the ex-ante committed revealing policy is given by

$$\sum_{n=1}^{\infty} p(n|\lambda)[R(r(n),n) - nd] - \sum_{n=1}^{\infty} p(n|\lambda) R(\widetilde{r}, n) = \mathbb{E}[R(r(N),N)|\lambda] - \lambda d - \mathbb{E}[R(\widetilde{r},N)|\lambda].$$

The first main result shows that for each $\lambda > 0$, there exists a unique announcement cost $\widehat{d}(\lambda)$ such that the revealing policy dominates if d is lower than $\widehat{d}(\lambda)$; the concealing policy dominates otherwise. The ex-ante committed disclosure policy has a *single threshold* with respect to d.

Theorem 1 *For each $\lambda > 0$, there exists a unique $\widehat{d}(\lambda) > 0$ such that*

$$\begin{cases} \mathbb{E}[R(r(N),N)|\lambda] - \lambda d > \mathbb{E}[R(\widetilde{r},N)|\lambda] & \text{if } d < \widehat{d}(\lambda), \\ \mathbb{E}[R(r(N),N)|\lambda] - \lambda d < \mathbb{E}[R(\widetilde{r},N)|\lambda] & \text{if } d > \widehat{d}(\lambda). \end{cases}$$

The result appears rather immediate. However, without participation cost $c > 0$ of agents – as in the basic case – the optimal reserve price r does not depend on n, so unlike $\mathbb{E}[R(r(N),N)|\lambda] > \mathbb{E}[R(\widetilde{r},N)|\lambda]$ above, the two expected revenues are the same; *i.e.*, $\mathbb{E}[R(r(N),N)|\lambda] = \mathbb{E}[R(\widetilde{r},N)|\lambda]$ (see Kim and Yoo (2021) among others). Then, we have a quite different characterization: For any positive announcement cost, the concealing policy is *always* better.

Now, given the results from Propositions 2 and 3, we find the optimal grand mechanism. First, by Proposition 3 and mechanical equivalence, the principal's expected payoff for concealed n is

$$\sum_{n=1}^{\infty} p(n|\lambda)\pi(\mathrm{NA}|n) \max\{R(\widetilde{r}(\pi,s),n), 0\}. \tag{31}$$

Second, with the probability $\pi(\mathrm{A}|n)$ of announcing n for $n \in \mathbb{N}$, by Proposition 2 coupled with the announcement cost $d > 0$, the principal's expected payoff for announced n is

$$\sum_{n=1}^{\infty} p(n|\lambda)\pi(\mathrm{A}|n) \max\{R\big(r(n), n\big) - dn, 0\}, \tag{32}$$

where the max is introduced in order to satisfy the principal's individual rationality (12). Like $\widetilde{n}$ from (30), we can find a cut-off value for the zero interim payoff of the principal when n is announced. With the revenue (19) and the ex-ante payment of a buyer (20), for the case of announced n, the corresponding revenue and payment given $r(n)$ are derived as $R(r(n), n) = nM(r(n), n)$, and $M(r(n), n) = \int_{r(n)}^{\overline{v}} \big(G^{(n)}(x)\psi(x) - c\big) f(x)dx$, respectively. Since $r(n)$ is strictly increasing and $G^{(n)}(x)$ is strictly decreasing in n, $M(r(n), n)$ is a strictly decreasing function of n, and further, finding that for a sufficiently large n, $M(r(n), n) - d$ has a negative value is straightforward. Hence, there exists a unique $\widehat{n} > 0$ such that

$$R(r(\widehat{n}), \widehat{n}) - dn = n[M(r(\widehat{n}), \widehat{n}) - d] = 0, \tag{33}$$

and $R(r(\widehat{n}), \widehat{n}) - dn > 0$ for $n < \widehat{n}$, whereas $R(r(\widehat{n}), \widehat{n}) - dn < 0$ for $n > \widehat{n}$.

The benefit of introducing max in (31) and (32) is to incorporate the null signal $\emptyset$ in a set of signals S into the payoff functions *directly*, so that we can focus on the two effective signals. To simplify notations, we use $\xi_n \in [0, 1] \cup \{\emptyset\}$ for $\pi(s|n)$ such that

$$\pi(s|n) = \begin{cases} \xi_n & \text{if } s = \mathrm{A}, \\ 1 - \xi_n & \text{if } s = \mathrm{NA}, \end{cases} \tag{34}$$

and $\xi_n = \emptyset$ if $s = \emptyset$; and further, denote $\widetilde{r}(\xi) \equiv \widetilde{r}(\pi, \mathrm{NA})$ with a slight abuse of notations. In other words, by having max, it is not necessary to consider the probability of generating the null signal $\emptyset$ separately. Then, by combining it and the result of Proposition 3, the principal's expected payoff given ξ from a grand mechanism $\mathbf{\Gamma}$ is

$$\begin{aligned} \mathbb{E}\Big[U_0\Big(\big(\mathbf{\Gamma}, \xi\big), \xi, n\Big)\Big] &= \sum_{n=1}^{\infty} p(n|\lambda) U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \xi, n\Big) \\ &= \sum_{n=1}^{\infty} p(n|\lambda) \left[\xi_n \max\{R\big(r(n), n\big) - dn, 0\} + (1 - \xi_n) \max\{R\big(\widetilde{r}(\xi), n\big), 0\}\right], \end{aligned} \tag{35}$$

where note that in terms of the notations from Definition 1 in Subsection 3.4.,

$$U_0\Big(\big(\mathbf{\Gamma}, \xi\big), e_s, n\Big) = \begin{cases} \max\{R\big(r(n), n\big) - dn, 0\} & \text{if } s = \mathrm{A}, \\ \max\{R\big(\widetilde{r}(\xi), n\big), 0\} & \text{if } s = \mathrm{NA}, \end{cases}$$

where the max is to incorporate the principal's individual rationality (12). Importantly, this procedure shows that we can achieve the informed principal's payoff linearity as in Subsection 3.2. Finally, with the payoff and $\Xi \equiv [0, 1] \cup \{\emptyset\} \times [0, 1] \cup \{\emptyset\} \times \cdots$, the principal solves

$$\max_{\xi} \mathbb{E}\Big[U_0\Big(\big(\mathbf{\Gamma}, \xi\big), \xi, n\Big)\Big] \text{ subject to } \xi \in \Xi. \tag{36}$$

We consider $n < \max\{\widehat{n}.\widetilde{n}\}$, and it can be readily shown that $\widehat{r}$ is differentiable with respect to ξ_n for each $n < \widetilde{n}$ from the inverse function theorem since both $G^{(n)}(x)$ and $\psi(x)$ are strictly increasing in x. The derivative of the objective function in (35) with respect to each ξ_n, by the envelope theorem, yields

$$\begin{aligned}\frac{\partial \mathbb{E}\Big[U_0\Big(\big(\mathbf{\Gamma}, \xi\big), \xi, n\Big)\Big]}{\partial \xi_n} &= p(n|\lambda)\left\{\Big[R\big(r(n), n\big) - nd\Big] - R\big(\widetilde{r}(\xi), n\big)\right\} \\ &= p(n|\lambda)n[\Delta M(n, \xi) - d],\end{aligned} \tag{37}$$

where we define the difference in the expected payment of a single buyer between the two signals – $s = \text{A}$ and $s = \text{NA}$ – as

$$\Delta M(n, \xi) \equiv M\big(r(n), n\big) - M\big(\widetilde{r}(\xi), n\big). \tag{38}$$

To appreciate the simple formulation (37) further, we delve into an intermediate step. The partial derivative of the ex-ante payoff function (36) of the principal with respect to ξ_n yields

$$p(n|\lambda)\left\{\Big[R\big(r(n), n\big) - nd\Big] - R\big(\widetilde{r}(\xi), n\big)\right\} + \underbrace{\Big[\sum_{n'=1}^{\infty} p(n'|\lambda)(1 - \xi_{n'})\frac{\partial R\big(\widetilde{r}(\xi), n'\big)}{\partial r}\Big]}_{=0 \text{ from Proposition 3}}\frac{\partial \widetilde{r}(\xi)}{\partial \xi_n}.$$

The second term disappears by the choice of the optimal $\widetilde{r}(\xi)$ from Proposition 3 – the envelope theorem. On the other hand, consider the interim stage; that is, suppose that n realizes. Then, the principal's payoff from the corresponding sub-mechanism is an element from the ex-ante payoff (36) such that $\big[R\big(r(n), n\big) - nd\big] - R\big(\widetilde{r}(\xi), n\big)$. Hence, the partial derivative of this interim payoff with respect to ξ_n yields

$$\Big[R\big(r(n), n\big) - nd\Big] - R\big(\widetilde{r}(\xi), n\big) + (1 - \xi_n) \times \underbrace{\frac{\partial R\big(\widetilde{r}(\xi), n\big)}{\partial r}}_{=0 \text{ from the commitment}} \times \frac{\partial \widetilde{r}(\xi)}{\partial \xi_n}.$$

Note that from the principal's ex-ante point of view, $\frac{\partial R(\widetilde{r}(\xi), n)}{\partial r}$ is not necessarily zero; only the *sum* of them, $\sum_{n'=1}^{\infty} p(n'|\lambda)(1 - \xi_{n'})\frac{\partial R(\widetilde{r}(\xi), n')}{\partial r}$, is zero from Proposition 3. Yet, in the beginning, the principal commits to a grand mechanism consisting of sub-mechanisms in such a way that he implements a specific sub-mechanism depending on a realized public signal as well as his private information. This means that given the realization of the signal $s = \text{NA}$, the principal must implement $\widetilde{r}(\xi)$; that is, he is not allowed to make such a change in the interim stage.

We thus strengthen Proposition 1: There is the *equivalence relationship* between the ex-ante optimal mechanism and the P-interim optimal mechanism if the private information of the principal is given as the number of agents. This is summarized as the first result in the following lemma (see the Appendix for the second result).

Lemma 2 *The optimal (grand) mechanism satisfies the following properties.*

(i) A mechanism $\big(\mathbf{\Gamma}, \pi\big)$ is ex-ante optimal if and only if it is P-interim optimal. Suppose the interim payoff is positive. Then, the condition for the P-interim optimality is satisfied if for each $n < \min\{\widetilde{n}, \widehat{n}\}$,

$$\xi_n = \begin{cases} 1 & \text{if } \Delta M(n, \xi) - d > 0, \\ 0 & \text{if } \Delta M(n, \xi) - d < 0. \end{cases}$$

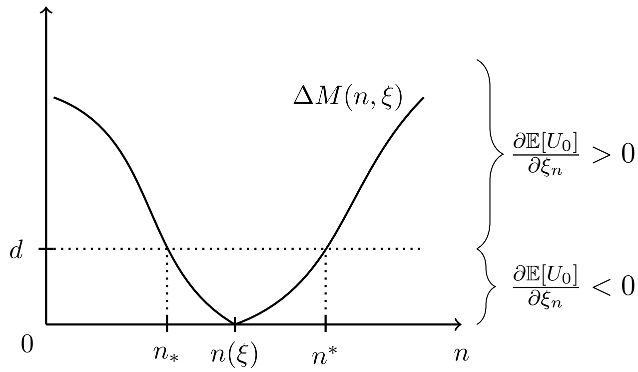

Figure 2: $\Delta M(n,\xi)$

(ii) $M\big(r(n_*), n_*\big) - d > 0$ *for* $n_* \equiv \arg\min_n\{\Delta M(n,\xi) - d = 0\}$.

By the first result of the lemma, the P-interim optimality is now sufficient for the optimal grand mechanism. To satisfy his individual rationality, we use, as discussed in the previous sections, the fact that both $M\big(r(n), n\big) - d$ and $M\big(\widetilde{r}(\xi), n\big)$ are strictly decreasing in n, making them negative for a sufficiently large n, and in addition, two cut-off values; the one for the latter is denoted by $\widetilde{n}$ in (30) and that for the former is $\widehat{n}$ in (33). In addition, the second result shows that we have "effective" sub-mechanisms; that is, the principal offers them for small n since $\Delta M(n,\xi) - d > 0$ and $M\big(\widetilde{r}(\xi), n\big) > 0$ for $n < n_*$, where n_* is defined in (ii) of the lemma above.

Now, we are ready to establish the main results of this paper. The optimal mechanism is characterized by a single threshold for ξ_n if d is large, and, interestingly, it has *double thresholds* if d is relatively small. That is, for the double thresholds, the principal discloses n with probability 1 if n is sufficiently small or sufficiently large, whereas he conceals n with probability 1 if n is in the middle range. In addition, for relatively large n values, to satisfy the principal's individual rationality, it is optimal for him not to offer a sub-mechanism, and $\xi_n = \emptyset$ stands for no mechanism offer.

Theorem 2 *Suppose that F is regular. The optimal mechanism $\big(\mathbf{\Gamma}, \xi\big)$ is characterized by a collection of sub-mechanisms $\mathbf{\Gamma}$ and a signal structure ξ. Among all incentive compatible and individually rational mechanisms $\mathbf{\Gamma}$, the optimal sub-mechanism $\Gamma^{(n)}$ for announced each n is given by Proposition 2, and the optimal sub-mechanism $\widetilde{\Gamma}^{(n)}$ for concealed each n is given by Proposition 3. There exists $0 < n_* < n^* < \infty$ such that*

$$\Delta M(n_*,\xi) = \Delta M(n^*,\xi) = d.$$

The optimal signal structure ξ is given as follows. If $d > M\big(r(n^*), n^*\big)$,

$$\xi_n = \begin{cases} 1 & \text{if } n < n_*, \\ 0 & \text{if } n_* < n < \widetilde{n}, \\ \emptyset & \text{if } n > \widetilde{n}. \end{cases}$$

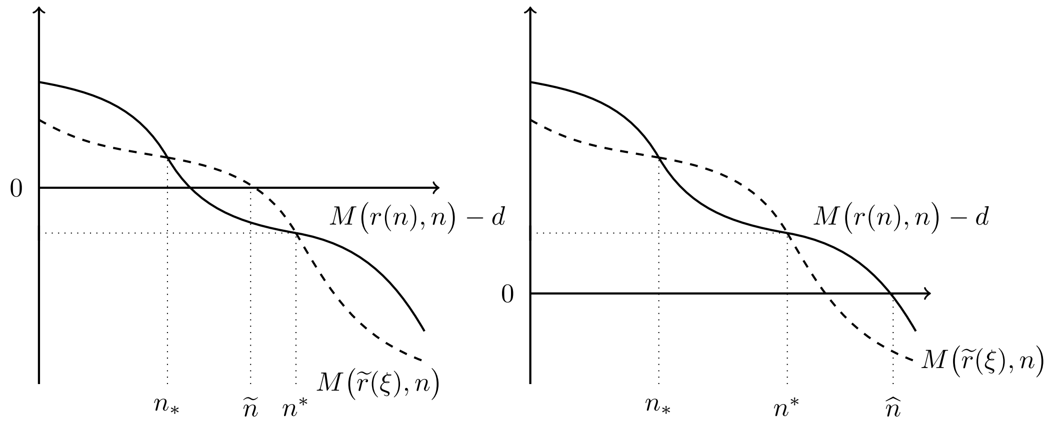

Figure 3: Single threshold ($d > M\big(r(n^*), n^*\big)$) (a); Double thresholds ($d < M\big(r(n^*), n^*\big)$) (b)

and if $d < M\big(r(n^*), n^*\big)$,

$$\xi_n = \begin{cases} 1 & \textit{if } n < n_*, \\ 0 & \textit{if } n_* < n < n^*, \\ 1 & \textit{if } n_* < n < \widehat{n}, \\ \emptyset & \textit{if } n > \widehat{n}, \end{cases}$$

where $\widetilde{n}$ *and* $\widehat{n}$ *are from (30) and (33).*

For intuition behind the results, consider $\Delta M(n,\xi)$ in (38). As shown in the proof of Theorem 2, there exists a unique $n(\xi)$ such that $\Delta M(n,\xi)$ is strictly increasing for $n > n(\xi)$, whereas it is strictly decreasing for $n < n(\xi)$, which is illustrated in Figure 1. Hence, as the number of bidders increases, each bidder's expected winning probability decreases, and so does his expected payment. If n is greater than $n(\xi)$, as n gets far away from $n(\xi)$, *i.e.*, $n' > n > n(\xi)$, by requiring a higher reserve price compared with no announcement case, announcing n can *mitigate* the negative effects from the increase in the number of competitors. On the other hand, if n is smaller than $n(\xi)$, as n gets closer to $n(\xi)$, *i.e.*, $n < n' < n(\xi)$, announcing n has comparatively less mitigating advantage to account for the decrease in the expected payment.

To understand the role of d in determining a single threshold or double ones, recall that both $M\big(r(n),n\big) - d$ and $M\big(\widetilde{r}(\xi),n\big)$ are strictly decreasing in n, but given the aformentioned characterization, they are decreasing in such a way that they *cross exactly at two points* – n_* and n^* – as illustrated in Figure 2. Depending on whether the value of $M\big(r(n),n\big) - d$ or $M\big(\widetilde{r}(\xi),n\big)$ at the larger crossing point n^* is negative or positive, we have a single threshold or double thresholds.

7 Extensions

We restrict the signal set to $S_n = \{n, \mathrm{NA}, \emptyset\}$ in the model and consider a range for n generating effective sub-mechanisms. This section generalizes it to two different directions. First, we allow the principal to lie about n even for announcement, which changes the set to $S = \mathbb{N} \cup \{\text{NA}, \emptyset\}$.[25] To incorporate that, denote by $\rho : \mathbb{N} \to \mathbb{N}$ the principal's reporting function and further by $\mathcal{S}(\pi, \rho(n)) \equiv \{m \in \mathbb{N} : \pi(m|\rho(n)) > 0\}$ a set of signals – number of agents – that arise with positive probability given a signal structure π and report $\rho(n)$. Then, by applying the same procedure in Proposition 3, the principal's payoff linearity is extended such that the ex-ant payoff $\mathbb{E}\big[U_0\big((\mathbf{\Gamma}, \xi), \xi, n\big)\big]$ in (35) changes to

$$\mathbb{E}\Big[U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi, n\Big)\Big] = \sum_{n=1}^{\infty} p(n|\lambda) \left\{ \sum_{m=1}^{\infty} \pi(m|\rho(n)) \Big[R\big(\widetilde{r}(\pi, m), n\big) - nd\Big] + \pi(\text{NA}|\rho(n)) R\big(\widetilde{r}(\pi, \text{NA}), n\big) \right\},$$

where with more than two effective signals, now we use π, not ξ.

Since by single-peaked property, $R(r(n), n) > R(r(n'), n)$ for $n' \neq n$, we obtain that for each ρ and every $\big(\widetilde{r}(\pi, 1), \widetilde{r}(\pi, 2), ...\big) \neq (r(1), r(2), ...)$, choosing $r(n)$ for all n dominates such that

$$\begin{aligned} &\sum_{n=1}^{\infty} p(n|\lambda) \left\{ \sum_{m \in M(\pi, \rho(n))} \pi(m|\rho(n)) \Big[R\big(r(n), n\big) - nd\Big] + \pi(\text{NA}|\rho(n)) R\big(\widetilde{r}(\pi, \text{NA}), n\big) \right\} \\ &> \sum_{n=1}^{\infty} p(n|\lambda) \left\{ \sum_{m \in M(\pi, \rho(n))} \pi(m|\rho(n)) \Big[R\big(\widetilde{r}(\pi, m), n\big) - nd\Big] + \pi(\text{NA}|\rho(n)) R\big(\widetilde{r}(\pi, \text{NA}), n\big) \right\}. \end{aligned}$$

Hence, choosing $r(n)$ dominates the choices of other reserve prices. Further, since committing to a reserve price r is *credible* with its announcement, the true n is revealed through the principal's optimal choice of the reserve price that is a monotonic function of n. As a consequence, restricting $S_n = \{n, \text{NA}, \emptyset\}$ in the model in fact comes without loss of generality. Note that the principal commits to a grand mechanism from the beginning, so we are not solving the signaling problem as in Jullien and Mariotti (2006) and Cai, Riley and Ye (2007).

The second and more interesting extension is to expand the language for no announcement into two kinds such as G and B, where G is for a "good" situation and B for a "bad" one. Since the nature of such language is *vague*, we assume that there is no announcement cost from it like No Announcement in the model of Section 2. Then, a set of signals is given as $S_n = \{n, \text{G}, \text{B}, \emptyset\}$ and we further denote by a_n the probability of announcing n whereas g_n and b_n denote the probabilities for the two signals G and B, respectively, such that for each n, $(a_n, g_n, b_n) \in [0, 1]^3$ and $a_n + g_n + b_n = 1$. Then, by applying the same procedure in Proposition 3, the principal's payoff linearity is extended such that the ex-ant payoff $\mathbb{E}\big[U_0\big((\mathbf{\Gamma}, \xi), \xi, n\big)\big]$ in (35) changes to

$$\mathbb{E}\Big[U_0\Big(\big(\mathbf{\Gamma}, \pi\big), \pi, n\Big)\Big] = \sum_{n=1}^{\infty} p(n|\lambda) \Big\{ a_n \Big[R\big(r(n), n\big) - nd\Big] + g_n R\big(\widetilde{r}(\pi, \text{G}), n\big) + b_n R\big(\widetilde{r}(\pi, \text{B}), n\big) \Big\}.$$

The linearity combined with Theorem 2 shows that the expanded language for No Announcement enriches Theorem 2: Now there are double thresholds for *each vague language term*, that is for each $s \in S = \{\text{G}, \text{B}\}$.

[25] As discussed in Section 2, a set of signals does not have to be the same as $\mathbb{N}$; any S with the same cardinality suffices.

Theorem 3 *Suppose that F is regular and d is sufficiently small. Then, among all incentive compatible and individually rational mechanisms $\mathbf{\Gamma}$, the optimal sub-mechanism $\Gamma^{(n)}(s)$ for $s = A$ is given by Proposition 2, and the optimal sub-mechanism $\Gamma^{(n)}(s)$ for each n and every $s \in \{G, B\}$ is given by Proposition 3. Furthermore, the optimal signal structure π is given by two thresholds $\underline{n}_s, \overline{n}_s \in \mathbb{R}$ for each $s \in \{G, B\}$ satisfying $0 < \underline{n}_B < \overline{n}_B < \underline{n}_G < \overline{n}_G < \infty$ such that*

$$\pi(s|n) = \begin{cases} 1 & \text{if } n < \underline{n}_B \text{ and } s = A, \\ 1 & \text{if } \underline{n}_B < n < \overline{n}_B \text{ and } s = B, \\ 1 & \text{if } \overline{n}_B < n < \underline{n}_G \text{ and } s = A, \\ 1 & \text{if } \underline{n}_G < n < \overline{n}_G \text{ and } s = G, \\ 1 & \text{if } n > \overline{n}_G \text{ and } s = A. \end{cases}$$

The endogenous choice of value language is beyond the scope of our analysis, but it is certainly of interest for the future research.[26]

8 Concluding remarks

This paper provides the full characterization of the optimal grand mechanism in which the principal's private information is a number of randomly arrived agents. The main result shows that we have a quite different characterization of the optimal grand mechanism if a signal structure not only depends on the principal's private information but each sub-mechanism also depends on it. The auction setting with the single-peaked revenue function makes it possible for us to easily extend it to the cases with rich signal sets.

As articulated in Section 3 and subsequent analysis in Section 5, the essential complication arises from the conversion problem. This means that to have an analysis tractable for different environments – different private information of the principal as well as basic games agents play, *e.g.*, public good or voting – we need additional strong assumptions on the probability distributions. Finding reasonable conditions under which the conversion problem can be resolved for general environments and thereby establishing the full characterization for each such environment can be further explored. Nonetheless, in spite of the inherently complex nature of the informed principal problem, the category of private information – a number of agents – that we consider in this paper enables us to obtain a clear and full characterization of the optimal grand mechanism.

Another possible extension is that even with this type of private information, it would be interesting to investigate how further the endogenous choice vague language terms can change the nature of the optimal grand mechanism, thereby extending Theorem 3 in Section 7.

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[26]An additional modelling choice is about the difference between the levels of lower cost for vague language. That is, here, we assume that for both NA and B & G, there is no annoucement cost, but we could have a slightly higher cost for the latter compared with the former, and so on, as the number of vague language terms increases.

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