Selling with Product Recommendation: Below-Cost Pricing and Price Reversal *

Wenji Xu† Shuoguang Yang‡

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Abstract

A long-lived seller sells a product by setting prices and offering product recommendations to short-lived consumers arriving in continuous time. The seller receives consumer feedback about the product value, with an arrival rate increasing in the instantaneous sales volume. We characterize the optimal selling mechanism under general value distributions and feedback technologies and show that the seller may set the price below the production cost for an initial or interim period and may switch the price between low prices and high prices multiple times. The optimal mechanism generates a Pareto improvement compared to the optimal fixed-price mechanism which features an above-cost price and inefficient spamming.

KEYWORDS: Dynamic pricing, information design, product recommendation, consumer feedback, below-cost pricing, spamming, Pareto-improvement.

JEL CLASSIFICATION: D61, D82, D83, L15, L41

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†Department of Economics and Finance, City University of Hong Kong, Email: wenjixu@cityu.edu.hk.

‡Department of Industrial Engineering & Decision Analytics, The Hong Kong University of Science and Technology, Email: yangsg@ust.hk.

1 Introduction

In many markets, sellers offer product recommendations to consumers before they make purchase decisions, and learn about the market demand of the products through consumer feedback. For example, many restaurants offer "Chef's Recommendations" on their menus, and invite the customers to fill in surveys of the food when settling the bill. Similarly, online clothing stores often display their recommended items in a "Featured Collections" panel on their websites and ask consumers to write reviews after purchasing the items. With the rapid development of digital technology, whether it is for selling food, cloth, books, computers, or cars, sellers can easily collect consumer feedback via digital means and adjust the prices and product recommendations on a near real-time basis.[1] In these markets, the possibility of collecting consumer feedback and offering product recommendations alters the seller's considerations in setting prices: lowering the price allows the seller to credibly recommend and sell the product to more consumers, which not only generates additional sales but also accelerates the speed at which the seller collects consumer feedback and learn about the market demand. In contrast, raising the price limits the seller's ability to recommend and sell the product to a large number of consumers and slows down the seller's learning about consumer demand. Pricing and production recommendation are therefore integral parts of the seller's selling strategy to maximize profit.

This paper studies how the seller should dynamically design prices and product recommendations in the presence of consumer feedback. Specifically, since the price affects the seller's ability to recommend the product to consumers and hence learn about market demand, several natural questions follow: How should the seller coordinate her pricing and recommendation strategies? Whether and when should the seller lower the price to speed up her learning about the demand? What are the welfare consequences of the seller's practices? In this paper, we show that, when the seller can flexibly design the prices and product recommendations, not only she may lower the price to speed up her learning, she might even set the price *below* the production cost. Furthermore, prices can alternate between low and high levels multiple times. Below-cost pricing can occur during either an initial or an interim learning period and is accompanied by high prices and authentic recommendations outside the learning period, which together lead to a Pareto improvement in the market, as compared to the outcome when the seller chooses a constant (above-cost) price.

[1] Consumer feedback can be obtained from consumers' reviews, ratings, or survey responses, which can be collected online, through emails, or via phone calls. The sellers may also post specific products on social media and obtain valuable information from the comment section and the Like/Dislike ratio of these posts. These real-time feedback can be digitized, and sellers can use dynamic pricing algorithms and carefully-designed recommender systems to automatically update their prices and product recommendations based on the consumer feedback.

Specifically, we consider a model in continuous time with a long-lived seller (she) who sets prices for a product and offers product recommendations to consumers (he) to maximize the discounted profit. The seller incurs a constant cost for producing each unit of the product. At each instance of time, a unit mass of short-lived consumers arrive. The consumers have unit demands and share a common valuation of the product which is initially unknown to the seller and the consumers. Upon arrival, each consumer observes the current posted price and the seller's product recommendation and then decides whether to purchase the product. Over time, the seller receives consumer feedback, which reveals the product value to her, at a Poisson rate that is increasing in the instantaneous sales volume. The seller sends product recommendations to a fraction of randomly selected consumers at each time, based on the consumer feedback. Each consumer then infers the product value from the seller's recommendation and make a purchase decision accordingly.

The main result of this paper provides a general characterization of the optimal selling mechanism (Theorem 1). Under the optimal mechanism, at each time, if the seller has learned the product value through past consumer feedback, she recommends the product to a consumer if and only if the product value is above the production cost. Alternatively, if the seller has not yet learned the product value, her optimal recommendation strategy can be computed recursively using a function defined in terms of the model primitives. Finally, given these recommendation strategies, at each time, the optimal price is set at the consumers' posterior expected value of the product when they receive a recommendation from the seller.

We then illustrate the main economic implications by considering two special cases of the product value distribution: distributions with binary values or with three values.

First, with binary product values, the optimal mechanism depends on the relative speed of consumer feedback about a high-value product (positive feedback) and about a low-value product (negative feedback). When consumers give positive feedback faster than they give negative feedback, the seller sets a low price for an initial period, during which she gradually raises the price over time and recommends the product to consumers if she has not received a negative feedback; after the initial period, the seller charges the highest possible price and only recommends the product if she has received a positive feedback. On the other hand, when consumers give positive feedback slower than they give negative feedback, the seller initially sets the highest possible price and only recommends the product if she has received a positive feedback; after an initial period, the seller lowers the price and recommend the product to consumers if she has not received a negative feedback. In both cases, the seller could set prices below the production cost for a period time. The timing of below-cost pricing, however, differs across the two cases: in the former case, below-cost pricing can only occur at the beginning, while in the latter case, below-cost pricing can only occur in an interim period. Furthermore, in both cases, the seller never spams the consumers with non-authentic recommendations, i.e., she never recommends a product which has been revealed to have low value through a negative consumer feedback.

Next, we consider a setting with three possible product values. In this case, a novel phenomenon of price reversal may occur under the optimal mechanism. There can be two types of price reversals. A type-1 reversal happens when the price starts at a low level, jumps to a high level for an interim period, and then jumps back to a low level. A type-2 reversal happens when the price starts at a high level, jumps to a low level for an interim period, and then jumps back to a high level. Price reversals occur because, over time, the relative cost-benefit trade-off for the seller in determining whether to lower the price and speed up consumer feedback fluctuates. We provide the necessary and sufficient conditions for both types of reversals to occur and show that a type-1 (or type-2) reversal occurs only if exactly two out of the three product values are below (or above) the production cost.

We then discuss the welfare implication of our results by comparing our baseline model to an alternative setting in which the seller is forced to choose a fixed price over time. In this fixed-price setting, the seller optimally sets a price above the production cost and adjusts the recommendation strategy so that consumers who receive a recommendation are indifferent between purchasing the product or not. Furthermore, after an initial period of time, the seller may start to recommend the product to consumers even if she already knows it is a bad product through consumer feedback. Comparison of the market outcomes under the optimal (flexible-price) mechanism and the optimal fixed-price mechanism shows that the outcome under flexible pricing is Pareto superior: the sales revenue is higher and the consumer surplus is zero in both cases. Hence, our results demonstrate that below-cost pricing can be efficient in the presence of product recommendation and consumer feedback and warns about the danger of blindly viewing below-cost pricing as harmful to efficiency: if a seller is expected to have commitment power in how she recommends products to consumers, below-cost pricing can emerge as an integral part of her optimal selling strategy, which accelerates her learning about the product and eliminates inefficient spamming in product recommendations. Finally, we extend the model in some natural directions, by considering markets in which prices and recommendations are determined separately by upstream and downstream firms, and markets in which the seller lacks the dynamic commitment power in setting prices. Our main results are robust in these directions.

The rest of this paper proceeds in the usual order. Section 2 reviews the related literature. We introduce the model in Section 3. Section 4 presents our main characterization results. Section 5 and Section 6 illustrate the economic implications with binary or three product values. Section 7 discusses the fixed-price mechanisms, welfare implications and the extensions. Section 8 concludes.

2 Related Literature

The paper contributes to the literature of optimal pricing in the standard model of sequential social learning (e.g., Bikhchandani et al. (1992), Banerjee (1992), and Smith and Sørensen (2000)). In particular, Welch (1992) considers how an uninformed monopolist should set a fixed price and sell a product of unknown value to consumers who arrive sequentially, observe a private signal about the product, and see the purchase decisions of previous consumers. Bose et al. (2008) consider a similar setting in which the monopolist, instead of choosing a fixed price, can adjust the prices over time, and show that the seller starts by charging a high price so that the consumers' decisions reveal their private signals and eventually switch to a low price at which consumers always purchase the product regardless of their private signal, resulting in a buying cascade.[2]

In the aforementioned papers, consumers' information about the product and about past purchases is exogenous. In contrast, in our model, the seller can endogenously design the information flow among consumers through designing product recommendations. Our paper is therefore also closely related to the recent literature on information design in social learning environments. The closest to our paper is a recent paper by Che and Hörner (2018) who study the socially optimal design of a recommender system in which users receive product recommendations before making their purchase decisions. Following their approach, our model posits a stochastic Poisson technology for consumer feedback and assumes the seller can recommend the product to a fraction of (randomly selected) consumers based on the feedback received. Kremer et al. (2014) considers the optimal information design problem for a benevolent social planner who discloses information to incentivize sequentially arrived agents to experiment and generate new information about the qualities of two products. In their setting, consumer feedback is instantaneous and perfect and the product quality can be drawn from a continuum of values, giving rise to different incentive considerations and exploration dynamics. Glazer et al. (2021) characterizes the socially optimal disclosure mechanism in social learning when acquiring information is costly for the agents. Our paper differs from these papers in two dimensions. First, in addition to information design, we also consider the optimal (fixed and flexible) pricing by the seller. Therefore, pricing and information must be jointly designed to influence consumer behavior. Moreover, since the designer in our model is the seller, the objective of the design problem is to maximize the profit, rather than the social welfare.

From a methodological perspective, this paper is related to the Bayesian persuasion and

[2]See also: Caminal and Vives (1996) and Caminal and Vives (1999) for social learning in duopoly markets; Glosten and Milgrom (1985), Avery and Zemsky (1998) and Park and Sabourian (2011) for social learning in markets with competitive pricing; and Newberry (2016), who studies empirically how different pricing regimes affect consumer social learning and seller profit when the seller optimally sets prices.

information design literature.[3] In our model, given the seller's pricing strategy, the design of product recommendation is equivalent to a Bayesian persuasion problem. Our paper thus joins the recent literature that studies Bayesian persuasion in dynamic settings (e.g., Ely et al. (2015), Hörner and Skrzypacz (2016), Ely (2017), Orlov et al. (2020) and Ely and Szydlowski (2020)). Into the Bayesian persuasion setting, we introduce optimal pricing by the seller. Thus, the paper also adds to the burgeoning literature on information design and pricing (e.g., Bergemann et al. (2015), Roesler and Szentes (2017), Ravid et al. (forthcoming), Libgober and Mu (2021), Boleslavsky et al. (2019), Armstrong and Zhou (forthcoming), Elliot et al. (2021), Yang (2019), Yang (forthcoming), and Xu and Yang (2022)).

Regarding the applications of our theory, the paper is also related to the literature on rating and recommender systems. For instance, Avery et al. (1999) and Miller et al. (2005) present mechanisms that elicit efficient product evaluations in markets. Bergemann and Ozmen (2006) consider a model in which a firm with a recommender system competes with a competitive fringe without such a system in a market with horizontally differentiated products.[4] Miklos-Thal and Schumacher (2013) analyze a model in which a recommender imperfectly observes a seller's hidden effort in producing a product and focus on how to mitigate the moral hazard problem. Hörner and Lambert (2021) studies a career concern model in which a rating system provides incentives for a worker to exert efforts to boost performance because it conveys an estimate of the worker's productivity to the market. Vellodi (2021) examines the impact of rating systems on firms' entry decisions and characterizes the socially optimal rating systems. In our paper, we abstract away from the issues considered in these papers (endogenous provision of feedback, moral hazard, competition, and entry) and focuses on the profit-maximizing design of dynamic pricing and recommendation for the seller.

3 Model

3.1 Primitives

Time is continuous and indexed by $t \in \mathbb{R}_+$. At $t = 0$, a long-lived seller with discount rate $r \in (0, 1)$ introduces a new product to the market and sells it to consumers. At each time $t \in \mathbb{R}_+$, a unit mass of short-lived consumers arrive and decide whether to purchase the product. The consumers are ex ante homogeneous with unit demands. The seller values the product at zero and incurs a cost of $c > 0$ to supply the product to each unit mass of consumers. Consumers share a common valuation of the product $v \in V = \{v_0, v_1, ..., v_n\} \subseteq \mathbb{R}_+$, which is a priori unknown and has a prior distribution $\tau \in \triangle(V)$. Without loss, we

[3] See Kamenica (2019) and Bergemann and Morris (2019) for surveys of the literature.

[4] They study a two-period model in which a recommender system always perfectly reveals the firm's private signal to consumers and abstract away from the optimal design of the recommender system.

assume $v_0 < v_1 < ... < v_k \leq c < v_{k+1} < ... < v_{n-1} < v_n$.

3.2 Consumer Feedback

Over time, the seller learns about the product value v from consumer feedback. Specifically, if the true product value is v_i and a mass $m_t \geq 0$ of consumers purchase the product over the time interval $[t, t+\mathrm{d}t]$, the seller receives a feedback during this time interval with probability $\lambda_i(\rho + m_t)\,\mathrm{d}t$, which perfectly reveals the value v_i to the seller. Here, $\lambda_i > 0$ governs the rates at which consumers give feedback after consuming the product and observing the product value to be v_i, and $\rho > 0$ measures the extent to which the seller exogenously learns about the product value, through, for example, market research conducted by the seller herself or third-party agencies. We refer to $\Lambda = (\{\lambda_i\}_i, \rho)$ as the *feedback technology*. The feedback is privately observed by the seller. Future consumers do not know whether the seller has received any feedback but can infer about it from the seller's product recommendations.

3.3 Pricing and Recommendation

The seller designs prices and product recommendations to maximize her discounted profit. Since consumers' purchase decisions are binary, it is without loss of generality to assume that the seller simply decides whether to recommend the product.

Specifically, the seller commits to the following selling mechanism: at time t, she offers a price $p_t \geq 0$ to all consumers arriving at t.[5] If she has not received any feedback by time t, she recommends the product to a fraction $a_t \in [0, 1]$ of randomly selected consumers at time t. If the seller has received a feedback by time t and learns that the product value is v_i, she recommends the product to a fraction $b_t^i \in [0, 1]$ of randomly selected consumers among those arriving at time t.[6] A selling mechanism is therefore a tuple of (measurable) functions (p, a, b), where $p := \{p_t\}_t$, $a := \{a_t\}_t$ and $b := \{b_t^i\}_{i,t}$.[7]

[5]We assume that the price is the same for all consumers. If prices can depend on recommendations, then we will get full surplus extraction. Alternatively, if prices can be randomly generated given the feedback but not depend on recommendations, we still get full surplus extraction (in this case, the price itself plays a dual role of determining the monetary transfer and offering recommendations to consumers.

[6]The seller's commitment to the recommendations can arise from long-term relationships between consumers and the seller. For example, if consumers repeatedly purchase different products from the same seller, they can punish the seller if the seller is detected to deviate from its announced recommendation strategy. The commitment assumption can hence be viewed as a reduced form representation of the seller's reputation. Given that the seller is able to commit to a recommendation strategy, it does not matter whether or not she can commit to the prices (see Section 7.4).

[7]An implicit assumption in this formulation is that the seller can send a product recommendation to any consumer. In reality, some consumers may make purchase decisions without paying attention to product recommendations. The existence of these consumers would not affect our results, as long as resale opportunities among consumers are limited: the seller could adopt different strategies to sell the product to the two groups

Product recommendations are private, so that each consumer only knows whether or not the seller recommends the product to himself but does not know whether the seller recommends the product to other consumers.

3.4 Timing and Payoffs

Given the price schedule and the seller's recommendation strategy, events at time $t \geq 0$ unfold as follows: (i) A unit mass of consumers with unit demands arrive, (ii) the seller offers the price p_t to all consumers arriving at time t, (iii) product recommendations are sent to each consumer in private, according to the seller's recommendation strategy $(a_t, \{b_t^i\}_i)$, (iv) each consumer observes the price and his product recommendation and decides whether to purchase the product, (v) payoffs are realized and consumers observe the product value, (vi) the seller receives consumer feedback according to the feedback technology Λ.

Given the outcome at each time t, a consumer's payoff is $v - p_t$ if he buys the product, and his payoff is 0 otherwise; the seller's (flow) payoff is the profit she earns from selling the product to consumers at time t, i.e., $(p_t - c) \cdot m_t$.

3.5 Feasible Mechanisms

By standard arguments (see Kamenica and Gentzkow (2011)), we can without loss focus on *straightforward* mechanisms, i.e., mechanisms in which the consumers are always incentivized to follow the seller's recommendations. We describe the set of feasible straightforward mechanisms below.

Under a straightforward mechanism, the mass of consumers who purchase the product at time t is always equal to the mass of consumers who receive a recommendation from the seller. In particular, in the event that the seller has not received a feedback from the consumers by time t, she recommends the product to a mass a_t of consumers who will then purchase the product. Thus, a_t determines the rate at which the seller learns about the product value.

Let q_t^i denote the ex ante probability that the true product value is v_i and the seller receives no feedback by time t. In particular, since the seller does not have any consumer feedback at $t = 0$, we have $q_0^i = \tau(v_i)$ and $\sum_v q_0^i = 1$. Let g_t^i denote the ex ante probability that the seller receives a feedback by time t which reveals the product value to be v_i. For all i and t, we have

$$\dot{q}_t^i = -\lambda_i(\rho + a_t)q_t^i \quad \text{and} \quad \dot{g}_t^i = \lambda_i(\rho + a_t)q_t^i. \tag{1}$$

where $\dot{q}_t^i$ and $\dot{g}_t^i$ denote $\partial q_t^i/\partial t$ and $\partial g_t^i/\partial t$ respectively. Notice that $\dot{q}_t^i + \dot{g}_t^i = 0$. Hence,

of consumers: those who pay attention to product recommendations and those who do not.

$g_t^i + q_t^i = q_0^i$ and $g_0^i = 0$ for all i and t. Also, at any time $t \geq 0$ and for any $v_i, v_j \in V$, we have

$$q_t^i = q_0^i \Big(\frac{q_t^j}{q_0^j}\Big)^{\lambda_i/\lambda_j} \quad \text{and} \quad g_t^i = q_0^i - q_0^i \Big(\frac{q_t^j}{q_0^j}\Big)^{\lambda_i/\lambda_j}.^{8} \tag{2}$$

If a consumer at time t receives a recommendation from the seller, according to Bayes's rule, his posterior belief about the product value is

$$\mu_t := \frac{\sum_i (q_t^i a_t + g_t^i b_t^i) v_i}{\sum_i q_t^i a_t + g_t^i b_t^i}. \tag{3}$$

Therefore, the consumer wants to follow the seller's recommendation and purchase the product if and only if $\mu_t \geq p_t$. If a consumer at time t does not receive a recommendation from the seller, his belief about the product value is

$$\nu_t := \frac{\sum_i [q_t^i(1-a_t)] v_i + g_t^i(1-b_t^i)}{\sum_i q_t^i(1-a_t) + g_t^i(1-b_t^i)}. \tag{4}$$

In this case, the consumer wants to purchase the product if and only if $\nu_t \geq p_t$.

Formally, a selling mechanism (p, b, a) is *feasible* if it satisfies $\mu_t \geq p_t$ and $\nu_t \leq p_t$ for all t, where $\{q_t^i, g_t^i\}_i$ evolves over time according to (1) with the initial conditions $q_0^i = \tau(v_i)$ and $g_0^i = 0$ for all i. The seller chooses a mechanism from the set of feasible mechanisms Γ to maximize her discounted expected profit. Hence, the seller's problem is

$$\max_{(p,a,b)\in\Gamma} \int_{t=0}^{\infty} e^{-rt} \sum_{i=1}^{n} (q_t^i a_t + g_t^i b_t^i)(p_t - c)\, \mathrm{d}t \tag{5}$$

We say that a selling mechanism is *optimal* if it solves (5).

[8] According to (1), we have

$$\frac{\dot{q}_t^i}{\dot{q}_t^j} = \frac{\lambda_i}{\lambda_j} \frac{q_t^i}{q_t^j},$$

which implies that

$$\ln(q_t^i) - \ln(q_0^i) = \frac{\lambda_i}{\lambda_j}(\ln q_t^j - \ln q_0^j).$$

That is,

$$q_t^i = q_0^i \Big(\frac{q_t^j}{q_0^j}\Big)^{\lambda_i/\lambda_j}.$$

Then, since $g_t^i + q_t^i = q_0^i$, we have

$$g_t^i = q_0^i - q_0^i \Big(\frac{q_t^j}{q_0^j}\Big)^{\lambda_i/\lambda_j}.$$

4 General Characterization

Let $m = \min(\text{argmin}_i \lambda_i)$, i.e., v_m is a product value about which consumer feedback is slowest. Then, define

$$\Phi(q,a) := q^{\frac{r}{\lambda_m(\rho+a)}}\Big[\sum_{i=k+1}^{n}\frac{\lambda_i q_0^i}{r+\lambda_i(\rho+a)}\Big(\frac{q}{q_0^m}\Big)^{\frac{\lambda_i}{\lambda_m}}(c-v_i)-\sum_{i=1}^{n}\frac{(\rho\lambda_i+r)q_0^i}{r+\lambda_i(\rho+a)}\Big(\frac{q}{q_0^m}\Big)^{\frac{\lambda_i}{\lambda_m}}(v_i-c)\Big]. \tag{6}$$

Notice that the sign of $\partial\Phi/\partial q$ is equal to the sign of

$$\Psi(q) := \sum_{i=k+1}^{n}\lambda_i q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m-1}(c-v_i)-\sum_{i=1}^{n}(v_i-c)(\rho\lambda_i+r)q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m-1}, \tag{7}$$

which is independent of a. The optimal mechanism features some switching points $\{\hat{q}_j\}$, which are defined recursively as follows:

1. Let $\hat{q}_0 = 0$. Set $j = 0$.
2. Given $\hat{q}_j$, if $\lim_{x\to\hat{q}_j^+}\text{sign}(\Psi(x)) = 1$, let $\hat{a}_j = 0$; otherwise, let $\hat{a}_j = 1$. Let $\hat{q}_{j+1} = \min\{q_0^m, \min\{q \mid \Phi(q,\hat{a}_j) - \Phi(\hat{q}_j,\hat{a}_j) = 0, q > \hat{q}_j\}\}$.
3. Stop if $\hat{q}_{j+1} = q_0^m$; otherwise, raise j by 1 and go back to Step 2.

Our main result characterizes the optimal selling mechanism using $\{\hat{q}_j\}$ and $\{\hat{a}_j\}$.

Theorem 1. *Under an optimal mechanism (p,a,b), for all $t \geq 0$ and all $i \in \{1,2,..,n\}$, $b_t^i = 0$ if $v_i < c$, $b_t^i = 1$ if $v_i > c$, $p_t = \mu_t$, and $a_t = \hat{a}_j$ if $q_t^m \in (\hat{q}_j, \hat{q}_{j+1}]$.*

5 Binary Values and Below-Cost Pricing

In this section, we illustrate the first price implication of the model in a setting with binary values $V = \{v_0, v_1\}$, where we normalize the values to be $v_0 = 0$ and $v_1 = 1$. As we will show, the optimal mechanism features an aggressive pricing strategy: the seller may set the price *below* the production cost or to the *highest possible* level in different time periods.

In this binary setting, the product is either good ($v = 1$) or bad ($v = 0$). If $\lambda_1 > \lambda_0$, consumers give feedback about good products faster than they give feedback about bad products, which is a case of "feedback through praises". If $\lambda_1 < \lambda_0$, consumers give feedback about bad products faster than they give feedback about good products, which is a case of "feedback through complaints". We consider the two cases separately in this section.

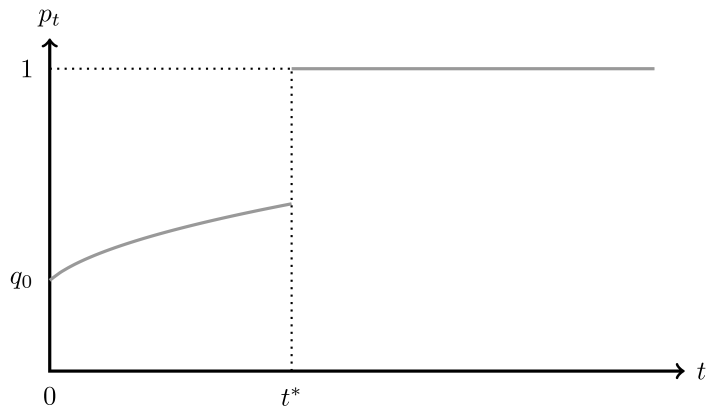

Figure 1: Price dynamics when $\lambda_1 > \lambda_0$

Proposition 1. *Suppose $\lambda_1 > \lambda_0$. Under an optimal mechanism, $b_t^1 = 1$ and $b_t^0 = 0$ for all $t \geq 0$, and*

$$a_t = \begin{cases} 1, & \text{if } t < t^* \\ 0, & \text{if } t > t^* \end{cases} ; \quad p_t = \begin{cases} q_0^1\left(1 - q_0^0(1 - e^{-\lambda_0(\rho+1)t})\right)^{-1}, & \text{if } t < t^* \\ 1, & \text{if } t > t^* \end{cases} ;$$

where

$$t^* := \left[\frac{\ln\left(\frac{q_0^1(1-c)}{q_0^0 c}\left(1 + \frac{\lambda_1}{r+\lambda_1\rho}\right)\right)}{(\lambda_1 - \lambda_0)(\rho+1)}\right]^+ .$$

According to Proposition 1, when $t < t^*$, the seller starts by setting a low price and gradually raises the price over time; during this period, the seller recommends the product to all consumers as long as she has not received a negative feedback: $a_t = b_t^1 = 1$ and $b_t^0 = 0$. When $t > t^*$, the seller switches to charge the highest possible price $p_t = 1$, and only recommends the product if she has confirmed it to be a good product through consumer feedback: $b_t^1 = 1$ and $a_t = b_t^0 = 0$. The price dynamics is depicted in Figure 1.

To gain some intuition, we first notice that upon receiving either a positive feedback ($v = 1$) or a negative feedback ($v = 0$), the seller has learned the product quality perfectly. Thus, the choice of b_t^1 and b_t^0 does not have direct effect on the seller's learning about the product value. However, the choice of b_t^1 and b_t^0 indirectly affects the seller's ability to credibly recommend the product to consumers when she has not received any feedback, i.e., the largest a_t for which the consumer's incentive compatibility constraint is satisfied: Raising the value of b_t^1 increases the instantaneous sales volume and allows the seller to increase a_t while satisfying the consumers' incentive compatibility constraint. This generates more sales and induces faster learning. Thus, setting $b_t^1 = 1$ is optimal for the seller. Increasing b_t^0, however, has two countervailing effects, it raises sales in the event of a negative feedback, but it tightens the consumer's incentive compatability constraint and reduces sales in the event of no feedback (given the same price). Proposition 1 demonstrates that the first effect always dominates and it is optimal to set $b_t^0 = 0$ for all $t \geq 0$.

Next, consider the trade-off the seller faces when maximizing her *flow* profit at time t. Given $b_t^1 = 1$ and $b_t^0 = 0$, the highest price that will be accepted by consumers who receive a product recommendation is

$$p_t = \frac{g_t^1 + q_t^1 a_t}{g_t^1 + (q_t^1 + q_t^0)a_t} \tag{8}$$

The expected sales volume at time t is

$$Q_t := g_t^1 + (q_t^1 + q_t^0)a_t \in [g_t^1, 1 - g_t^0].$$

Hence, (8) is equivalent to

$$p_t = \frac{q_t^1}{q_t^1 + q_t^0} + \frac{g_t q_t^0}{(q_t^1 + q_t^0)} \cdot \frac{1}{Q_t}, \tag{9}$$

which can be interpreted as the seller's demand function at t, as depicted in Figure 2. As in standard monopoly pricing, the demand function and the production cost determine the seller's flow trade-off. If the seller sells more products to consumers, the *flow* marginal revenue is

$$\frac{\partial p_t Q_t}{\partial Q_t} = \frac{q_t^1}{q_t^1 + q_t^0}$$

and the marginal cost is c. Unique to our setting, the marginal revenue is constant at $\frac{q_t^1}{q_t^1+q_t^0}$, which is the seller's posterior expectation of the product value when she has not received a feedback by time t. Therefore, ignoring the dynamic benefit of learning from consumer feedback, the seller would sell the highest quantity $Q_t = 1 - g_t^0$ by setting $a_t = 1$ if $\frac{q_t^1}{q_t^1+q_t^0} > c$, and sell the lowest quantity $Q_t = g_t^1$ by setting $a_t = 0$ if $\frac{q_t^1}{q_t^1+q_t^0} < c$. In other words, when the seller does not have conclusive information about the product value, she should recommend and sell the product to consumers if and only if the gains from trade $(\frac{q_t^1}{q_t^1+q_t^0} - c)$ is positive.

Now, we consider the *dynamic* benefit to the seller for learning about the product value from consumer feedback. If the seller recommends the product to more consumers by increasing a_t, she would learn about the product at a faster pace, which gives her greater flexibility when recommending and selling the product to future consumers. Suppose the seller induces a higher rate of learning by increasing a_t. With probability $(\frac{q_t^1}{q_t^1+q_t^0})$, the product is good and consumer feedback reveals this information to the seller at rate λ_1. As shown in Proposition 1, the seller uses the acquired information to decide when to switch to a high price at $p_t = 1$, recommend only good products, and stop acquiring information through consumer feedback. Hence, a faster learning rate allows the seller to sell a larger quantity of good products to consumers after she switches to the high price at $p_t = 1$. The marginal benefit from the additional information is therefore $\frac{1-c}{r}$, which should also be discounted at the rate

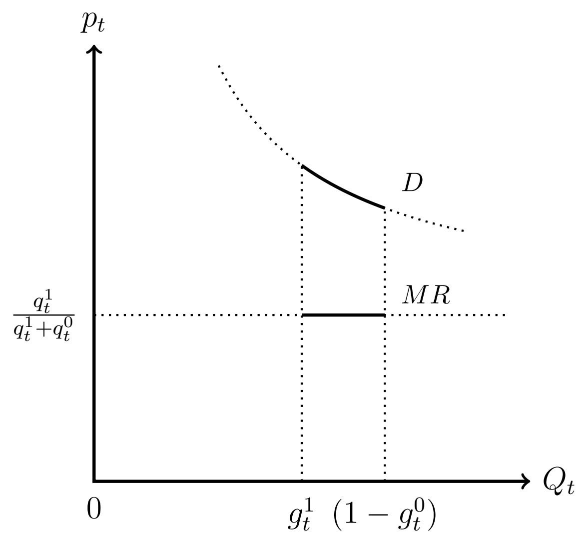

Figure 2: Seller's flow demand (D) and marginal revenue (MR) curves

$\left(\frac{1}{\frac{\lambda_1 \rho}{r}+1}\right)$ because the seller can learn about the product exogenously at rate ρ. Thus, the present value of this additional dynamic benefit of learning about the product is

$$\lambda_1 \left(\frac{q_t^1}{q_t^1+q_t^0}\right)\left(\frac{1-c}{r}\right)\left(\frac{1}{\frac{\lambda_1 \rho}{r}+1}\right).$$

The additional benefit of learning implies that the seller may sometimes sell the product even if the expected gains from trade is negative, i.e., $\frac{q_t^1}{q_t^1+q_t^0} < c$. Specifically, taking into account the flow marginal revenue, the flow marginal cost, and the dynamic marginal benefit, if the seller sells more products at t, her discounted profit at t increases by

$$\xi_t := \frac{q_t^1}{q_t^1+q_t^0} - c + \lambda_p \left(\frac{q_t^1}{q_t^1+q_t^0}\right)\left(\frac{1-c}{r}\right)\left(\frac{1}{\frac{\lambda_1 \rho}{r}+1}\right). \tag{10}$$

Since ξ_t does not depend on the quantity Q_t, the seller would sell the largest quantity $Q_t = 1-g_t^0$ if $\xi_t > 0$ and sell the lowest quantity $Q_t = g_t^1$ if $\xi_t < 0$. Since $\lambda_1 > \lambda_0$, consumers reveal a good product to the seller through positive feedback faster than they reveal a bad product through negative feedback. In this case, if the seller continues to not receive a feedback, she tends to believe the product is increasingly likely to be a bad one: $\left(\frac{q_t^1}{q_t^1+q_t^0}\right)$ decreases over time. Therefore, the seller's marginal profit ξ_t also decreases over time. The cutoff time t^* in Proposition 1 is precisely the time at which ξ_t equals zero under the optimal mechanism.

As already discussed, under the optimal selling mechanism, the seller never knowingly recommend a bad product to consumers, i.e., $b_t^0 = 0$ for all t. That is, the seller never spams the consumers with non-authentic recommendations. Another immediate implication of Proposition 1 is that the seller may charge a price below the production cost for an initial period of time, as demonstrated in Corollary 1.

Corollary 1 (Below-Cost Pricing)**.** *Suppose $\lambda_1 > \lambda_0$. There exists $\bar{t} > 0$ such that $p_t < c$ for $t \in [0, \bar{t})$ under the optimal mechanism, if and only if*

$$\frac{q_0^1}{q_0^0} < \frac{c}{1-c} < \frac{q_0^1}{q_0^0} \cdot \left(1 + \frac{\lambda_1}{r + \lambda_1 \rho}\right).$$

Proof. According to Theorem 1, p_t is increasing on $[0, t^*)$ and $p_t = 1$ on (t^*, ∞). Thus, the necessary and sufficient condition for below-cost pricing ($p_t < c$) to occur during an initial period $t \in [0, \bar{t})$ is that $t^* > 0$ and $p_0 < c$. Note that $t^* > 0$ if and only if

$$\frac{c}{1-c} < \frac{q_0^1}{q_0^0} \cdot \left(1 + \frac{\lambda_1}{r + \lambda_1 \rho}\right).$$

When $t^* > 0$, $p_0 < c$ is equivalent to

$$\frac{q_0^1}{q_0^0} < \frac{c}{1-c}.$$

Combining the two inequalities, we obtain the desired result. ■

Now, we consider the optimal mechanism in the alternative scenario when $\lambda_1 < \lambda_0$.

Proposition 2. *When $\lambda_1 < \lambda_0$, under the optimal mechanism, $b_t^1 = 1$ and $b_t^0 = 0$ for all $t \geq 0$, and*

$$a_t = \begin{cases} 0, & \text{if } t < t^{**} \\ 1, & \text{if } t > t^{**} \end{cases}; \quad p_t = \begin{cases} 1, & \text{if } t < t^{**} \\ q_0^1 \left(1 - q_0^0 \left(1 - e^{-\lambda_0((\rho+1)t - t^{**})}\right)\right)^{-1}, & \text{if } t > t^{**} \end{cases};$$

where

$$t^{**} := \left[\frac{\ln\left(\frac{q_0^0 c}{q_0^1(1-c)}\left(1 - \frac{\lambda_0}{r + (1+\rho)\lambda_0}\right)\right)}{(\lambda_0 - \lambda_1)\rho}\right]^+.$$

According to Proposition 2, when $\lambda_1 < \lambda_0$, the seller again does not knowingly recommend a bad product, i.e., $b_t^0 = 0$ for all t, and always recommends a good product, i.e., $b_t^1 = 1$ for all t. However, she adopts a different pricing strategy: She initially charges a high price at $p_t = 1$. At time $t = t^{**}$, the price drops discontinuously to a lower level and gradually increases afterwards. The price dynamics is depicted in Figure 3.

Intuitively, according to arguments similar to those in Section 5, at any time t, the seller's marginal benefit from increasing a_t to sell an additional unit of product is proportional to the posterior product value $\left(\frac{q_t^1}{q_t^1 + q_t^0}\right)$ given she received no consumer feedback until time t. When $\lambda_1 < \lambda_0$, the posterior value $\left(\frac{q_t^1}{q_t^1 + q_t^0}\right)$ increases over time, and therefore the seller becomes increasingly optimistic about the product value over time if she continues to not receive any consumer feedback. Hence, the seller finds it beneficial to lower the price to sell an additional

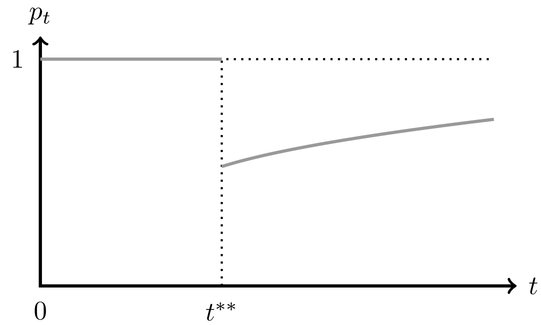

Figure 3: Price dynamics when $\lambda_1 < \lambda_0$

unit of product when she has not received a consumer feedback only when $\left(\frac{q_t^1}{q_t^1+q_t^0}\right)$ becomes large enough, which happens after time t^{**}.

In this case, there could be an interim period immediately after t^{**}, during which the seller sets a price below the production cost and incurs a loss. Corollary 2 provides a necessary and sufficient condition for below-cost pricing to occur.

Corollary 2. *Suppose $\lambda_1 < \lambda_0$. There exists $\bar{t} > t^{**}$ such that $p_t < c$ for $t \in (t^{**}, \bar{t})$ under the optimal mechanism, if and only if*

$$\frac{q_0^1}{q_0^0} < \frac{c}{1-c} < \frac{q_0^1}{q_0^0} \cdot \left(1 + \frac{\lambda_0}{r + \lambda_0 \rho}\right)^{\lambda_0/\lambda_1}.$$

Proof. According to Theorem 2, $p_t = 1$ on $[0, t^{**})$, and p_t is increasing on (t^{**}, ∞). Thus, below-cost pricing ($p_t < c$) can occur during a non-empty time period $t \in [t^{**}, \bar{t})$ if either $t^{**} > 0$ and $p_{t^{**}} < c$, or $t^{**} = 0$ and $p_0 < c$.

Note that $t^{**} > 0$ if and only if

$$\frac{c}{1-c} > \frac{q_0^1}{q_0^0} \cdot \left(1 + \frac{\lambda_0}{r + \lambda_0 \rho}\right). \tag{11}$$

When $t^{**} > 0$, $p_{t^{**}} < c$ is equivalent to

$$\frac{c}{1-c} < \frac{q_0^1}{q_0^0} \cdot \left(1 + \frac{\lambda_0}{r + \lambda_0 \rho}\right)^{\lambda_0/\lambda_1}.$$

Therefore, below-cost pricing can occur if

$$\frac{q_0^1}{q_0^0} \cdot \left(1 + \frac{\lambda_0}{r + \lambda_0 \rho}\right) < \frac{c}{1-c} < \frac{q_0^1}{q_0^0} \cdot \left(1 + \frac{\lambda_0}{r + \lambda_0 \rho}\right)^{\lambda_0/\lambda_1}. \tag{12}$$

On the other hand, when $t^{**} = 0$, i.e., when (11) is violated, below-cost pricing occurs if $p_0 = q_0^1 < c$. Thus, below-cost pricing can also occur if

$$\frac{q_0^1}{q_0^0} < \frac{c}{1-c} \leq \frac{q_0^1}{q_0^0} \cdot \left(1 + \frac{\lambda_0}{r + \lambda_0 \rho}\right). \tag{13}$$

Combining (12) and (13), we obtain the desired result. ∎

Therefore, below-cost pricing is featured in the optimal mechanism, regardless of whether $\lambda_1 > \lambda_0$ or $\lambda_1 < \lambda_0$. But the timing of below-cost pricing differs across the two cases: it could occur either in an initial period or in an interim period. In Section 7, we illustrate the welfare and policy implications by comparing the optimal mechanism in this section with the optimal mechanism under an alternative setting with constant pricing, which, as we will show, features above-cost pricing and active spamming and induces a Pareto inferior outcome.

6 Three Values and Price Reversal

In this section, we illustrate another important phenomenon of "price reversal" by considering a setting with three possible product values, i.e., $V = \{v_0, v_1, v_2\}$. We normalize the values to $v_0 = 0$, $v_1 = x$ and $v_2 = 1$, where $x \in (0, 1)$.

Unlike the binary-value case which features a one-time price switch between high prices and low prices, the price dynamics in a three value may feature multiple switches between high prices and low prices. Formally, we say that a *type-1 reversal* occurs if there exist two distinct times $t_1, t_2 \geq 0$ where $t_1 < t_2$ such that, under the optimal mechanism $p_t = 1$ for $t \in (0, t_1) \cup (t_2, \infty)$ and $p_t = \mu_t < 1$ for $t \in (t_1, t_2)$, and a *type-2 reversal* occurs if there exist two distinct times $t_1, t_2 \geq 0$ where $t_1 < t_2$ such that, under the optimal mechanism $p_t = 1$ for $t \in (t_1, t_2)$ and $p_t = \mu_t < 1$ for $t \in (0, t_1) \cup (t_2, \infty)$. Proposition 3 provides the necessary and sufficient condition for the two types of price reversals to occur.

Proposition 3. *Suppose $V = \{0, x, 1\}$, where $x \in (0, 1)$. Then,*

- *a type-1 reversal occurs if and only if $x < c$, the smallest positive root of*

$$\frac{(\lambda_2(\rho+1)+r)q_0^2}{r+\lambda_2\rho}\left(\frac{q}{q_0^m}\right)^{\lambda_2/\lambda_m}(c-1) - q_0^1\left(\frac{q}{q_0^m}\right)^{\lambda_1/\lambda_m}(x-c) + qc\left(\frac{q}{q_0^m}\right)^{\lambda_0/\lambda_m} = 0, \quad (14)$$

  denoted by $\hat{q}_1$, is less than q_0^m, $\Psi(\hat{q}_1) < 0$, and $\Phi(q_0^m, 1) - \Phi(\hat{q}_1, 1) > 0$;

- *a type-2 reversal occurs if and only if $x > c$, the smallest positive root of*

$$q_0^2\left(\frac{q}{q_0^m}\right)^{\lambda_2/\lambda_m}(c-1) - q_0^1\left(\frac{q}{q_0^m}\right)^{\lambda_1/\lambda_m}(x-c) + \frac{(\rho\lambda_0+r)c}{r+\lambda_0(\rho+1)}\left(\frac{q}{q_0^m}\right)^{\lambda_0/\lambda_m} = 0, \quad (15)$$

  denoted by $\hat{q}_1$, is less than q_0^m, $\Psi(\hat{q}_1) > 0$, and $\Phi(q_0^m, 0) - \Phi(\hat{q}_1, 0) > 0$.

7 Discussions

In this section, we discuss the economic implications of our results and extend the model in several natural directions.

7.1 Fixed-Price Mechanism

In this section, we analyze the market outcome under an alternative setting with a constant price, which, we will later compare with the market outcome in our baseline model and derive policy implications for market efficiency and price regulation.

While sellers may flexibly set prices and offer recommendations in some markets, as captured in our baseline model, in other markets with high menu costs (Sheshinski and Weiss, 1977) or where a fixed price is mandated by law (e.g., U.S. law mandates a fixed price per share when a firm sells its equity shares in an Initial Public Offering), the sellers may not be able to adjust the price over time. We refer to the former case (our baseline model) as the "flexible pricing" regime, and the latter as the "constant pricing" regime.

In what follows, we focus on the "constant pricing" regime and assume that the seller can only choose a constant price $p^* \in \mathbb{R}$ that is offered to all consumers arriving at any time, i.e., $p_t = p^*$ for all t. In this environment, the seller's problem becomes:

$$\max_{(p,a,b)\in\Gamma:\frac{dp_t}{dt}=0} \int_{t=0}^{\infty} e^{-rt} \sum_i (q_t^i a_t + g_t^i b_t^i)(p_t - c)\,\mathrm{d}t$$

Proposition 4. *Under the optimal constant-price mechanism, the price p^* is such that $p^* > c$ and $\mu_t = p^*$ for all $t \geq 0$. If $V = \{0, 1\}$, there exists $\bar{t} \geq 0$ such that $b_t^0 > 0$ for $t > \bar{t}$.*

As is demonstrated in Proposition 4, when the seller can only set a constant price for all consumers, she always chooses a price strictly above her production cost. Furthermore, in the binary value case, the seller may spam the consumers with non-authentic product recommendations ($b_t^0 > 0$) when a sufficiently long period of time has elapsed. These phenomena are observationally opposite to those under our baseline model. Such differences have important welfare and policy implications, which we discuss in the next section.

Remark 1. The above-cost pricing and spamming phenomena under constant pricing are reminiscent of some classic results in economics:

- In monopoly pricing, given a downward-slopping demand curve, the seller always sets prices above her marginal cost of production. In our setting, the consumer demand is ex ante degenerate at the prior. However, through endogenous product recommendation, the seller effectively generates a downward-sloping demand curve for herself: with a higher price, the seller recommends the product less aggressively, inducing the consumer to demand less of the product but to accept the higher price. Hence, given the optimal recommendation strategy under different prices, the seller effectively solves a standard monopoly pricing problem and thus chooses an above-cost price.

- In Bayesian persuasion, when a sender designs a signal to influence a receiver's action, the optimal signal structure often involves "garbling of the state", which aims at persuading the receiver to take the sender-preferred action(s) more often than the receiver

> would do by himself. In our setting, the seller's product recommendation problem is equivalent to the design of a binary signal to influence consumer behavior. Hence, the phenomenon of "spamming" in our setting is an example of garbling.

Hence, our main results in the flexible pricing case show that when prices and production recommendations can be jointly designed, classic intuitions in monopoly pricing and Bayesian persuasion no longer apply.

7.2 Efficient Below-Cost Pricing

Since it is often easier for regulators to collect data about prices than about product recommendations, it is likely that regulatory decisions will rely more on sellers' pricing decisions and less on her product recommendation strategy.[9]

Conventional price regulation frameworks often treat below-cost prices as alarms for predatory pricing: incumbent firms set prices below the production cost to drive out competitors, which hurts market efficiency and consumer welfare.[10] Therefore, if policy makers adopt a conventional policy framework to regulate pricing behavior of a seller in our setting, it may appear that the outcome under constant pricing (with above-cost prices) is seemingly more efficient than the outcome under flexible pricing (which can involve below-cost prices). However, as Proposition 5 demonstrates, flexible pricing generates efficiency gains in the Pareto sense, compared to a constant above-cost pricing strategy.

Proposition 5. *The outcome under the optimal flexible-price mechanism Pareto dominates the outcome under the optimal constant-price mechanism.*

Proof. According to Theorem 1 and Proposition 4, under both the optimal constant-price mechanism and the optimal flexible-price mechanism, we have $p_t = \mu_t$ for all $t \geq 0$. That is, the consumers' posterior expected product value upon receiving a recommendation equals the price. Hence, consumers always get zero surplus. On the other hand, the seller gets strictly higher surplus under the optimal flexible-price mechanism, compared to her surplus under the constant-price mechanism, because the set of feasible mechanisms is strictly larger under flexible pricing and the optimal flexible-price mechanism features a non-constant price. ■

[9]Even if product recommendations can be tracked by the regulators, it is difficult to interpret and regulate the recommendations: In our model, to detect spamming, a regulator needs to not only observe the seller's product recommendations but also the consumer feedback that the seller has observed.

[10]For example, many courts adopted a rule proposed by Areeda and Turner (1975) which proposes that "a firm's pricing is predatory if its price is less than its short-run marginal cost". Areeda and Turner also suggest using averable varibale cost as a proxy for short-run marginal cost if short-run marginal cost is hard to determine due to data limitation. In our model, the short-run marginal cost and average variable cost are both c. Hence, prices below c will be deemed as predatory in nature according to the Areeda-Turner test.

In fact, as demonstrated in Section 5, below-cost pricing can be an integral part of a comprehensive price-recommendation strategy for the seller and may not be driven by anti-competitive motives.[11] There are two sources of efficiency gains from flexible pricing. First, under constant pricing, the seller sets a price above the production cost, which in turn forces herself to spam the consumers after the product has been released for some time. This type of spamming is socially wasteful: the seller can generate higher profit (without hurting consumers) by adjusting the price over time and stop spamming. Second, flexible pricing allows the seller to fully optimize her learning about the product: she lowers the price when learning is beneficial and raises the price when learning is not beneficial, which cannot be achieved with a constant price. Hence, in markets where sellers also offer product recommendations to consumers, one should be careful in judging the welfare consequences of sellers' using below-cost prices.

One way to distinguish inefficient below-cost pricing (predatory pricing) from efficient below-cost pricing (our model) is to examine the inter-temporal prices, product recommendations and consumer feedback. The market outcome in our model has some unique features. For example, below-cost price could occur in an initial period (e.g., when consumer feedback features praises more than complaints as in Proposition 1), but it could also appear in an interim period (e.g., when consumer feedback features complaints more than praises as inProposition 2). The price can cycle between high prices and low prices (as discussed in Section 4). Additionally, high prices are often accompanied by authentic product recommendations. These properties can be useful in practice for judging whether the seller is using below-cost pricing for learning (as in our model) or for other motives (such as predation). This observation is consistent with proposals by Easterbrook (2003) and Posner (2003) who advocated observing price patterns over time to determine whether a firms is engaged in predatory conduct. To ensure accurate policy judgements, it would be important to maintain accessibility of data about inter-temporal prices, product recommendations and consumer feedback.

7.3 Intermediated Market

In this paper, we have assumed that the pricing and product recommendation decisions are both made by the seller. In reality, some producers may sell their products to final consumers through an intermediary which in turn offers product recommendations to consumers. Hence, the pricing decision and the recommendation decision are made by separate market partici-

[11]In our model, the seller is a monopolist and do not face any competition. The below-cost pricing is entirely driven by the seller's incentive to acquire information about the product value.

pants.[12] We can incorporate a certain aspect of this feature into the baseline model, which we describe in this section.

In this section, we assume $V = \{0, 1\}$ and consider an alternative setting with a producer who sets prices and an intermediary who offers recommendations to consumers. Specifically, the intermediary commits to a recommendation strategy $\{a_t, b_t^0, b_t^1\}_{t\geq 0}$ at $t = 0$. At each time $t \geq 0$, the timing of events is as follows: the producer sets the price p_t; the intermediary sends product recommendations according to $\{a_t, b_t^0, b_t^1\}$; consumers observe the price and the product recommendation and decide whether to purchase the product. Consumer feedback is generated according to the same Poisson technology as in the baseline model. The intermediary retains a fraction $\phi \in (0, 1)$ of the sales revenue as commission fees and incurs an operating/transaction cost κ in facilitating each unit mass of purchase. The producer incurs a cost χ in producing each unit mass of the product and earns a fraction $(1 - \phi)$ of the sales revenue, i.e., the sales revenue net of the commission fees paid to the intermediary.

Proposition 6. *Suppose $\phi\chi \geq (1 - \phi)\kappa$. There is a unique equilibrium outcome. If $\lambda_1 > \lambda_0$, the equilibrium strategy profile $\{a_t, b_t^0, b_t^1, p_t\}_{t\geq 0}$ takes the same form as that in Proposition 1 with c replaced by κ/ϕ; if $\lambda_1 < \lambda_0$, the equilibrium strategy profile $\{a_t, b_t^0, b_t^1, p_t\}_{t\geq 0}$ takes the same form as that in Proposition 2 with c replaced by κ/ϕ.*

According to Proposition 6, if the transaction cost incurred by the intermediary is small and the production cost incurred by the seller is large, the equilibrium outcome in the intermediated market features similar price dynamics and recommendation strategies as in the baseline model. The only difference is that the relevant "cost" which determines the equilibrium outcome is κ/ϕ.

7.4 Lack of Price Commitment

In the baseline model, it is assumed that the seller commits to the selling mechanism at $t = 0$. In practice, the seller may sometimes lack dynamic commitment power on prices. In this section, we consider an extension of the baseline model that captures this scenario.

Specifically, instead of committing to the entire price schedule $\{p_t\}$ up front, the seller chooses the price p_t at each time $t \geq 0$ right before time-t consumers arrive. We maintain the assumption that the seller can commit at time $t = 0$ to the recommendation strategy $\{a_t, \{b_t^i\}_i\}_t$. With these assumptions, the market outcome remains exactly the same as the outcome under the optimal mechanism in the baseline model. To see this, suppose that

[12] For example, in e-commerce, third-party sellers sell their products through intermediaries such as Amazon and Ebay which operate as recommender systems for consumers; in traditional markets, manufacturers can also fix the prices a priori through Resale Price Maintenance contracts, and retailers take the price as given and make product recommendations to consumers.

the seller commits to the recommendation strategy given by Theorem 1. Then, at each time $t \geq 0$, choosing the prices according to Theorem 1 must be the optimal continuation strategy for the seller because this strategy attains the highest possible continuation profit. Anticipating that prices will be chosen optimally in the future, the seller must also find it optimal to commit at time $t = 0$ to the recommendation strategy in Theorem 1. Hence, lack of price commitment will not alter the market outcome.

8 Conclusion

In this paper, we analyze sellers' optimal dynamic pricing and product recommendation strategy to sell a product in the presence of consumer feedback. We show that the optimal selling mechanism exhibits periods of high prices and low (and even below-cost) prices, and the seller never spams consumers with non-authentic product recommendations. For a general value and feedback structure, we provide a simple procedure to compute the optimal mechanism. We illustrate the main economic implications of our model using the special cases when there are two or three possible product values.

We then discuss the welfare implications of our results by comparing our baseline model to an alternative environment in which the seller can only choose a price that is constant over time. Under this constant pricing environment, the seller always sets the price above the production cost, which inevitably incentives herself to spam the consumers with non-authentic product recommendations after the product has been released for a sufficiently long period of time. Comparison of the optimal mechanisms under the two environments highlight the economic channels through which below-cost prices can be an integral part of a price-recommendation strategy for the seller to learn about the product more efficiently, which raises the seller's profit without hurting consumers. We also extend the baseline model to consider environments in which the seller lacks commitment power on prices and markets in which prices and product recommendation are determined by different market participants.

Several directions are worth exploring as future research questions. For example, while we assumed that the seller and consumers possess no ex ante private information about the product, in practice, the seller can be privately informed about the product quality, and consumers may have private tastes which are not be observable to the seller. It would be valuable to understand how the optimal mechanism changes when there is ex ante information asymmetry between the seller and consumers and what are the welfare implications. In addition, we have abstracted away from how consumers decide whether to give feedback about the product, it would also be useful to endogenize the consumer feedback and analyze how the seller may incentivize consumer feedback through rewards. Lastly, while our model assumes that the seller sells only one product, it would be useful to extend the analysis to a setting with a multi-product seller.

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Appendix

A.1 Proof of Theorem 1

First, in any mechanism, if the incentive compatibility constraint is slack at time t, i.e., $\mu_t > p_t$, then increasing the price to $p_t = \mu_t$ would raise the flow profit for the seller without affecting the evolution of $\{g_t^i, q_t^i\}_{v\in\{0,1\}}$ over time. Thus, in an optimal mechanism, the incentive compatibility constraint $\mu_t \geq p_t$ must be binding for all t. That is, for all $t \geq 0$, given $\{g_t^i, q_t^i\}_{v\in\{0,1\}}$, the price must be

$$p_t = \frac{\sum_i (g_t^i b_t^i + q_t^i a_t) v_i}{\sum_i (g_t^i b_t^i + q_t^i a_t)}. \tag{A.16}$$

Substituting (A.16) into the objective function of (5), the objective becomes

$$\text{Obj} = \int_{t\geq 0}^{\infty} e^{-rt} \sum_i (g_t^i b_t^i + q_t^i a_t)(v_i - c) dt \tag{A.17}$$

The objective is increasing in b_t^i for $i \leq k$ and is decreasing in b_t^i for $i \geq k+1$. Since $\{b_t^i\}_i$ do not affect the evolution of $\{g_t^i, q_t^i\}$ over time, we must have $b_t^i = 1$ for $i \leq k$ and $b_t^i = 0$ for $i \geq k+1$ in an optimal mechanism. Thus, the objective is

$$\text{Obj} = \int_{t\geq 0}^{\infty} e^{-rt} [\sum_{i=k+1}^{n} g_t^i (v_i - c) + a_t \sum_{i=1}^{n} (v_i - c) q_t^i] dt \tag{A.18}$$

By (2), we have

$$q_t^i = q_0^i \Big(\frac{q_t^m}{q_0^m}\Big)^{\lambda_i/\lambda_m} \quad \text{and} \quad g_t^i = q_0^i - q_0^i \Big(\frac{q_t^m}{q_0^m}\Big)^{\lambda_i/\lambda_m}. \tag{A.19}$$

The objective then becomes

$$\begin{aligned} \text{Obj} &= \int_{t\geq 0}^{\infty} e^{-rt} [\sum_{i=k+1}^{n} \Big(q_0^i - q_0^i \Big(\frac{q_t^m}{q_0^m}\Big)^{\lambda_i/\lambda_m}\Big)(v_i - c) + a_t \sum_{i=1}^{n} (v_i - c) q_0^i \Big(\frac{q_t^m}{q_0^m}\Big)^{\lambda_i/\lambda_m}] dt \\ &= C + \int_{t\geq 0}^{\infty} e^{-rt} [-\sum_{i=k+1}^{n} q_0^i \Big(\frac{q_t^m}{q_0^m}\Big)^{\lambda_i/\lambda_m} (v_i - c) + a_t \sum_{i=1}^{n} (v_i - c) q_0^i \Big(\frac{q_t^m}{q_0^m}\Big)^{\lambda_i/\lambda_m}] dt \end{aligned} \tag{A.20}$$

where C is a constant. The value of q_t^m evolves over time according to the law of motion:

$$\dot{q}_t^m = -\lambda_m (\rho + a_t) q_t^m. \tag{A.21}$$

We conduct a change of variable by defining $t(q) := \inf\{t | q_t^m \geq q\}$ and $u(q) := \frac{1}{\rho + a_{t(q)}}$. Then, (A.21) becomes

$$t'(q) = -\frac{u(q)}{\lambda_m q}, \tag{A.22}$$

and the objective (minus C) becomes

$$\text{Obj} - C = \int_0^{q_0^m} \frac{e^{-rt}}{\lambda_m q} \Big[-u(q) \sum_{i=k+1}^{n} q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m} (v_i - c) + (1 - \rho u(q)) \sum_{i=1}^{n} (v_i - c) q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m} \Big] dq \tag{A.23}$$

Hence, the optimal control problem is transformed to a problem with an objective (A.23), where the control variable is u, the state variable is t, and the law of motion is (A.21). Notice that this is a linear optimal control problem, to which a solution exists by the Filippov-Cesari theorem (Cesari, 1983). Let $\mathcal{H}$ denote the Hamiltonian of the optimal control problem:

$$\mathcal{H}(t,u,q,\eta) = \frac{e^{-rt}}{\lambda_m q}\Big[-u\sum_{i=k+1}^{n} q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}(v_i-c) + (1-\rho u)\sum_{i=1}^{n}(v_i-c)q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}\Big] - \eta\frac{u}{\lambda_m q}. \tag{A.24}$$

We will use the necessary conditions for optimality to derive the optimal control, which require:

$$u(q) = \operatorname*{argmax}_{u\in[\frac{1}{\rho+1},\frac{1}{\rho}]} \mathcal{H}(t(q),u,q,\eta(q)), \tag{A.25}$$

$$t'(q) = -\frac{u(q)}{\lambda_m q}, \quad t(q_0^m) = 0, \tag{A.26}$$

$$\eta'(q) = -\frac{\partial\mathcal{H}(t(q),u(q),q,\eta(q))}{\partial t}, \quad \eta(0) = 0. \tag{A.27}$$

Let

$$\phi(q) := \lambda_m q\frac{\partial\mathcal{H}(t(q),u,q,\eta(q))}{\partial u} = e^{-rt(q)}\Big[-\sum_{i=k+1}^{n} q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}(v_i-c) - \rho\sum_{i=1}^{n}(v_i-c)q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}\Big] - \eta(q) \tag{A.28}$$

Meanwhile, (A.27) implies that

$$\eta'(q) = \frac{re^{-rt(q)}}{\lambda_m q}\Big[-u(q)\sum_{i=k+1}^{n} q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}(v_i-c) + (1-\rho u(q))\sum_{i=1}^{n}(v_i-c)q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}\Big] \tag{A.29}$$

Taking derivative of $\phi(q)$ with respect to q, we get

$$\begin{aligned}\phi'(q) &= re^{-rt(q)}\Big[\sum_{i=k+1}^{n} q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}(v_i-c) + \rho\sum_{i=1}^{n}(v_i-c)q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}\Big]t'(q)\\ &\quad+\frac{e^{-rt(q)}}{q}\Big[-\sum_{i=k+1}^{n}\frac{\lambda_i q_0^i}{\lambda_m}\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}(v_i-c) - \rho\sum_{i=1}^{n}(v_i-c)\frac{q_0^i\lambda_i}{\lambda_m}\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}\Big] - \eta'(q)\\ &= \frac{e^{-rt(q)}}{\lambda_m q_0^m}\Big[\sum_{i=k+1}^{n}\lambda_i q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m-1}(c-v_i) - \sum_{i=1}^{n}(v_i-c)(\rho\lambda_i+r)q_0^i\Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m-1}\Big].\end{aligned} \tag{A.30}$$

where the second equality follows from (A.26) and (A.29). Note that the sign of $\phi'(q)$ is independent of $u(q)$, and it cannot be zero over any compact interval with positive measure. Also, since $\lambda_i \geq \lambda_m$ for all i, we have $\phi(0) = 0$. Therefore, $\phi(q) = 0$ has a finite number of roots on $[0, q_0^m]$, which we denote by $0 = \hat{q}_0 < \hat{q}_1 < \hat{q}_2 < ... < \hat{q}_r$. For each integer $0 \leq j \leq r$, $\phi(q)$ has the same sign on $[\hat{q}_j, \hat{q}_{j+1}]$. Hence, according to (A.25), the optimal control $u(q)$ is a constant $\hat{u}_j$ on $[\hat{q}_j, \hat{q}_{j+1}]$, where $\hat{u}_j = 1/\rho$ if $\phi(q) > 0$ on $[\hat{q}_j, \hat{q}_{j+1}]$ and $\hat{u}_j = 1/(\rho+1)$ otherwise. Accordingly, under the optimal mechanism, we have $a_{t(q)} = \hat{a}_j := 1/\hat{u}_j - \rho$ for $q \in [\hat{q}_j, \hat{q}_{j+1}]$.

For $q \in (\hat{q}_j, \hat{q}_{j+1})$, the law of motion (A.26) is

$$t'(q) = -\frac{\hat{u}_j}{\lambda_m q} \iff t(q) = -\frac{\hat{u}_j}{\lambda_m}\ln q + t(\hat{q}_j) \iff e^{-rt(q)} = q^{\frac{r\hat{u}_j}{\lambda_m}}e^{-rt(\hat{q}_j)}. \tag{A.31}$$

Substituting the above expression into $\phi'(q)$, we know that, for $q \in (\hat{q}_j, \hat{q}_{j+1})$,

$$\phi'(q) = \frac{e^{-rt(\hat{q}_j)}}{\lambda_m q_0^m} q^{\frac{r\hat{u}_j}{\lambda_m}} \Big[\sum_{i=k+1}^{n} \lambda_i q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m - 1} (c - v_i) - \sum_{i=1}^{n} (v_i - c)(\rho\lambda_i + r) q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m - 1} \Big]. \tag{A.32}$$

Therefore, for $q \in [\hat{q}_j, \hat{q}_{j+1}]$, we have

$$\begin{aligned}
\phi(q) &= \int_{\hat{q}_j}^{q} \phi'(q) dq + \phi(\hat{q}_j) \\
&= \frac{e^{-rt(\hat{q}_j)}}{\lambda_m q_0^m} \int_{\hat{q}_j}^{q} q^{\frac{r\hat{u}_j}{\lambda_m}} \Big[\sum_{i=k+1}^{n} \lambda_i q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m - 1} (c - v_i) - \sum_{i=1}^{n} (v_i - c)(\rho\lambda_i + r) q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m - 1} \Big] dq \\
&= e^{-rt(\hat{q}_j)} (\rho + \hat{a}_j) \Big[\sum_{i=k+1}^{n} \frac{\lambda_i q_0^i}{\lambda_i + r\hat{u}_j} \Big(\frac{q}{q_0^m}\Big)^{\frac{\lambda_i}{\lambda_m}} q^{\frac{r\hat{u}_j}{\lambda_m}} (c - v_i) - \sum_{i=1}^{n} (v_i - c) \frac{(\rho\lambda_i + r) q_0^i}{\lambda_i + r\hat{u}_j} \Big(\frac{q}{q_0^m}\Big)^{\frac{\lambda_i}{\lambda_m}} q^{\frac{r\hat{u}_j}{\lambda_m}} \Big] \Big|_{\hat{q}_j}^{q} \\
&= e^{-rt(\hat{q}_j)} (\rho + \hat{a}_j) \Big[\Phi(q, \hat{a}_j) - \Phi(\hat{q}_j, \hat{a}_j) \Big].
\end{aligned} \tag{A.33}$$

Hence, $\Phi(q, \hat{a}_j) - \Phi(\hat{q}_j, \hat{a}_j)$ has the same sign as that of $\partial\mathcal{H}(t(q), u, q, \eta(q))/\partial u$. (A.25) and (A.33) then imply that $a_t = \hat{a}_j$ for $q_t^m \in (\hat{q}_j, \hat{q}_{j+1}]$ is indeed optimal. It remains to verify that the solution also satisfies the sufficient condition for the optimal control, which, according to Arrow sufficiency theorem (see, e.g., Seierstad and Sydsaeter (1977)), requires that the maximized Hamiltonian is concave in the state variable. In our problem, this is requires

$$\begin{aligned}
0 &\geq -u(q) \sum_{i=k+1}^{n} q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m} (v_i - c) + (1 - \rho u(q)) \sum_{i=1}^{n} (v_i - c) q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m} \\
&= (1 - (\rho + 1) u(q)) \sum_{i=k+1}^{n} q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m} (v_i - c) + (1 - \rho u(q)) \sum_{i=1}^{k} (v_i - c) q_0^i \Big(\frac{q}{q_0^m}\Big)^{\lambda_i/\lambda_m}.
\end{aligned}$$

The inequality is satisfied for all $u(q) \in [\frac{1}{\rho+1}, \frac{1}{\rho}]$. ■

A.2 Proof of Proposition 1

Since $\lambda_1 > \lambda_0$, we have $m = 0$. Hence,

$$\Phi(q, a) = q^{\frac{r}{\lambda_0(\rho+a)}} \Big[\frac{\lambda_1 q_0^1}{r + \lambda_1(\rho + a)} \Big(\frac{q}{q_0^0}\Big)^{\frac{\lambda_1}{\lambda_0}} (c - 1) - \frac{(\rho\lambda_1 + r) q_0^1}{r + \lambda_1(\rho + a)} \Big(\frac{q}{q_0^0}\Big)^{\frac{\lambda_1}{\lambda_0}} (1 - c) + \frac{\rho\lambda_0 + r}{r + \lambda_0(\rho + a)} qc \Big].$$

The sign of $\partial\Phi/\partial q$ is equal to

$$\Psi(q) = -(1 - c)((\rho + 1)\lambda_1 + r) q_0^1 \Big(\frac{q}{q_0^0}\Big)^{\lambda_1/\lambda_0 - 1} + c(\rho\lambda_0 + r) q_0^0. \tag{A.34}$$

Let $\hat{q}_0 = 0$. Note that $\Psi(0) > 0$, i.e., $\lim_{x \to \hat{q}_0^+} \mathrm{sign}(\Psi(x)) = 1$. Thus, $\hat{a}_0 = 0$. Next, we determine $\hat{q}_1 = \min\{q_0^0, \min\{q \mid \Phi(q, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = 0, q > \hat{q}_0\}\}$.

Notice that $\Phi(\hat{q}_0, \hat{a}_0) = 0$. Since $\Psi(q)$ is decreasing in q when $\lambda_1 > \lambda_0$, we know that $\Phi(q, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = \Phi(q, \hat{a}_0)$ is concave in q. Thus, to find $\hat{q}_1$, there are only two cases to consider, according to whether $\Phi(q_0^0, \hat{a}_0) \geq 0$. First, note that $\Phi(q_0^0, \hat{a}_0) \geq 0$ if and only if

$$\frac{q_0^1(1 - c)}{q_0^0 c} \left(1 + \frac{\lambda_1}{r + \lambda_1\rho}\right) \leq 1. \tag{A.35}$$

In this case, we have $\hat{q}_1 = q_0^0$, and the optimal control is $u(q) = \frac{1}{\rho}$ for all $q \in (0, q_0^0]$, or equivalently, $a_t = 0$ for all $t \geq 0$. On the other hand, if (A.35) is violated, $\hat{q}_1$ will be in the interior of $[0, q_0^0]$, and must satisfy $\Phi(\hat{q}_1, \hat{a}_0) = 0$, i.e.,

$$\left[(\rho+1)(1-c) - \frac{r(1-c)}{r+\lambda_1\rho}\right] \frac{q_0^1}{q_0^0} \left(\frac{\hat{q}_1}{q_0^0}\right)^{\lambda_1/\lambda_0 - 1} = \rho c \tag{A.36}$$

When $q \in [\hat{q}_1, q_0^0]$, the optimal mechanism features $a_t = 1$. Thus, the law of motion for $q \geq \hat{q}_1$ is

$$\dot{q}_t^0 = -\lambda_0(\rho+1)q_t^0 \tag{A.37}$$

which implies

$$q_t^0 = q_0^0 \exp(-\lambda_0(\rho+1)t). \tag{A.38}$$

Thus, at the cutoff time $t^* := \inf\{t : q_t^m < \hat{q}_1\}$, we have

$$\hat{q}_1 = q_0^0 \exp(-\lambda_0(\rho+1)t^*). \tag{A.39}$$

Together, (A.36) and (A.39) imply

$$\exp(-(\rho+1)t^*(\lambda_1 - \lambda_0)) = \rho c \left((\rho+1)(1-c)\frac{q_0^1}{q_0^0} - \frac{r(1-c)}{r+\lambda_1\rho}\frac{q_0^1}{q_0^0}\right)^{-1}, \tag{A.40}$$

which is equivalent to

$$t^* = \frac{\ln\left(\frac{q_0^1(1-c)}{q_0^0 c}\left(1 + \frac{\lambda_1}{r+\lambda_1\rho}\right)\right)}{(\lambda_1-\lambda_0)(\rho+1)}. \tag{A.41}$$

Combining the two cases, we therefore have

$$t^* = \left[\frac{\ln\left(\frac{q_0^1(1-c)}{q_0^0 c}\left(1 + \frac{\lambda_1}{r+\lambda_1\rho}\right)\right)}{(\lambda_1-\lambda_0)(\rho+1)}\right]^+. \tag{A.42}$$

Therefore, under the optimal mechanism $a_t = 1$ for $t < t^*$ and $a_t = 0$ for $t > t^*$.

Finally, since $p_t = \mu_t$ under the optimal mechanism, by (A.38), we have

$$p_t = q_0^1\left(1 - q_0^0(1 - e^{-\lambda_0(\rho+1)t})\right)^{-1} \quad \text{for } t < t^*, \tag{A.43}$$

and $p_t = 1$ for $t > t^*$. This completes the proof. ■

A.3 Proof of Proposition 2

Since $\lambda_1 < \lambda_0$, we have $m = 1$. Hence,

$$\Phi(q,a) = q^{\frac{r}{\lambda_1(\rho+a)}}\left[\frac{\lambda_1 q}{r+\lambda_1(\rho+a)}(c-1) - \frac{(\rho\lambda_1+r)q}{r+\lambda_1(\rho+a)}(1-c) + \frac{(\rho\lambda_0+r)q_0^0}{r+\lambda_0(\rho+a)}\left(\frac{q}{q_0^1}\right)^{\lambda_0/\lambda_1} c\right].$$

The sign of $\partial\Phi/\partial q$ is equal to

$$\Psi(q) = (c-1)((\rho+1)\lambda_1 + r)q_0^1 + c(\rho\lambda_0 + r)q_0^0\left(\frac{q}{q_0^1}\right)^{\lambda_0/\lambda_1 - 1}. \tag{A.44}$$

Let $\hat{q}_0 = 0$. Note that $\Psi(0) < 0$, i.e., $\lim_{x \to \hat{q}_0^+} \text{sign}(\Psi(x)) = -1$. Thus, $\hat{a}_0 = 1$. Next, we determine $\hat{q}_1 = \min\{q_0^1, \min\{q \mid \Phi(q, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = 0, q > \hat{q}_0\}\}$.

Notice that $\Phi(\hat{q}_0, \hat{a}_0) = 0$. Since $\Psi(q)$ is increasing in q when $\lambda_1 < \lambda_0$, we know that $\Phi(q, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = \Phi(q, \hat{a}_0)$ is convex in q. Thus, to find $\hat{q}_1$, there are only two cases to consider, according to whether $\Phi(q_0^1, \hat{a}_0) \leq 0$. First, note that $\Phi(q_0^1, \hat{a}_0) \leq 0$ if and only if

$$\frac{q_0^0 c}{q_0^1(1-c)} \left(1 - \frac{\lambda_0}{r + (1+\rho)\lambda_0}\right) \leq 1. \tag{A.45}$$

In this case, we have $\hat{q}_1 = q_0^1$, and the optimal control is $u(q) = \frac{1}{\rho+1}$ for all $q \in (0, q_0^1]$, or equivalently, $a_t = 1$ for all $t \geq 0$. On the other hand, if (A.45) is violated, $\hat{q}_1$ will be in the interior of $[0, q_0^1]$, and must satisfy $\Phi(\hat{q}_1, \hat{a}_0) = 0$, i.e.,

$$\left(\frac{\hat{q}_1}{q_0^1}\right)^{\lambda_0/\lambda_1 - 1} = \frac{(1-c)q_0^1}{cq_0^0} \left(\frac{r + \lambda_0(\rho+1)}{r + \lambda_0 \rho}\right) \tag{A.46}$$

When $q \in [\hat{q}_1, q_0^1]$, the optimal mechanism features $a_t = 0$. Thus, the law of motion for $q \geq \hat{q}_1$ is

$$\dot{q}_t^1 = -\lambda_1 \rho q_t^1 \tag{A.47}$$

which implies

$$q_t^1 = q_0^1 \exp(-\lambda_1 \rho t) \tag{A.48}$$

Thus, at the cutoff time $t^{**} := \inf\{t : q_t^1 < \hat{q}_1\}$, we have

$$\exp(-\lambda_1 \rho t^{**}) = \frac{\hat{q}_1}{q_0^1} \tag{A.49}$$

Together, (A.46) and (A.49) imply that

$$t^{**} = \frac{1}{\rho(\lambda_0 - \lambda_1)} \ln\left(\frac{cq_0^0}{(1-c)q_0^1}\left(1 - \frac{\lambda_0}{r + (\rho+1)\lambda_0}\right)\right). \tag{A.50}$$

Combining the two cases, we conclude that

$$t^{**} := \left[\frac{\ln\left(\frac{q_0^0 c}{q_0^1(1-c)}\left(1 - \frac{\lambda_0}{r+(1+\rho)\lambda_0}\right)\right)}{(\lambda_0 - \lambda_1)\rho}\right]^+.$$

Therefore, under the optimal mechanism $a_t = 0$ for $t < t^{**}$ and $a_t = 1$ for $t > t^{**}$.

Under the optimal mechanism, for $t > t^{**}$, $a_t = 1$. Thus, the law of motion is

$$\dot{q}_t^1 = -\lambda_1(\rho+1)q_t^1,$$

which implies that

$$q_t^1 = q_0^1 \exp(-\lambda_1(\rho+1)t + \lambda_1 t^{**}). \tag{A.51}$$

Hence, $p_t = \mu_t$ and (A.51) imply that, under the optimal mechanism, $p_t = 1$ for $t < t^{**}$, and

$$p_t = q_0^1 \left(1 - q_0^0 \left[1 - \exp\left(-\lambda_0((\rho+1)t - t^{**})\right)\right]\right)^{-1}.$$

for $t > t^{**}$. This completes the proof. ■

A.4 Proof of Proposition 3

With three values, if $x = c$, Φ and Ψ take the same form as the case when there are two product values with $V = \{0, 1\}$. According to Theorem 1, Proposition 1 and Proposition 2, price reversals do not occur in this case. We consider the remaining cases: $x < c$ and $x > c$.

Case 1: $x < c$.
In this case,

$$\Psi(q) = (c-1)((\rho+1)\lambda_2+r)q_0^2\left(\frac{q}{q_0^m}\right)^{\lambda_2/\lambda_m-1} + (\rho\lambda_1+r)q_0^1\left(\frac{q}{q_0^m}\right)^{\lambda_1/\lambda_m-1}(c-x) + c(\rho\lambda_0+r)q_0^0\left(\frac{q}{q_0^m}\right)^{\lambda_2/\lambda_m-1}.$$

First, suppose $m = 2$, i.e., λ_2 is the smallest among all the feedback rates. We apply Theorem 1 to this case. Let $\hat{q}_0 = 0$. Note that $\Psi(0) = (c-1)((\rho+1)\lambda_2 + r)q_0^2 < 0$, i.e., $\lim_{x\to\hat{q}_0^+} \text{sign}(\Psi(x)) = -1$. Thus, $\hat{a}_0 = 1$. Next, let $\hat{q}_1 = \min\{q_0^2, \min\{q \mid \Phi(q, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = 0, q > \hat{q}_0\}\}$. If $\hat{q}_1 = q_0^2$, price reversals obviously do not occur. Alternatively, if $\hat{q}_1 < q_0^2$, we must have

$$\Phi(\hat{q}_1, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = \Phi(\hat{q}_1, \hat{a}_0) = 0.$$

Notice that $\Psi(q)$ is increasing in q on $[0, q_0^2]$. Since the sign of $\partial\Phi/\partial q$ is equal to $\Psi(q)$ and $\Phi(\hat{q}_0, \hat{a}_0) = \Phi(\hat{q}_1, \hat{a}_0) = 0$, we must have $\Psi(q) > 0$ for all $q > \hat{q}_1$, which in turn implies that $\hat{q}_2 = q_0^2$. Hence, again, price reversals do not occur.

Now, suppose $m \neq 2$. In this case, $\Psi(0)$ is either $(\rho\lambda_1 + r)q_0^1$ (if $m = 1$) or $c(\rho\lambda_0 + r)q_0^0$ (if $m = 0$). Thus, $\Psi(0) > 0$ and $\hat{a}_0 = 0$. So, $p_t = 1$ for t large enough. Hence, only a type-1 reversal may occur. Theorem 1 then implies that the necessary and sufficient condition for type-1 reversal to occur is $0 < \hat{q}_1 < \hat{q}_2 < q_0^m$. To have $\hat{q}_1 < q_0^m$, we need

$$\min\{q \mid \Phi(q, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = 0, q > \hat{q}_0\} < q_0^m, \tag{A.52}$$

Since $\Phi(\hat{q}_0, \hat{a}_0) = 0$, (A.53) is equivalent to requiring the smallest positive root of (14) to be less than q_0^m. Given that $\hat{q}_1 < q_0^m$, for a price reversal to occur, we need $\Psi(\hat{q}_1) < 0$, otherwise, $\hat{a}_1 = 0$ and $\hat{q}_2 = q_0^m$ (recall that $\Psi(q)$ can have at most two positive roots). Suppose $\Psi(\hat{q}_1) < 0$, we have $\hat{a}_1 = 1$. The remaining condition for price reversal to occurs is $\hat{q}_2 < q_0^m$, which is equivalent to $\Phi(q_0^m, 1) - \Phi(\hat{q}_1, 1) > 0$.

Case 2: $x > c$.
In this case,

$$\Psi(q) = (c-1)((\rho+1)\lambda_2+r)q_0^2\left(\frac{q}{q_0^m}\right)^{\lambda_2/\lambda_m-1} + ((\rho+1)\lambda_1+r)q_0^1\left(\frac{q}{q_0^m}\right)^{\lambda_1/\lambda_m-1}(c-x) + c(\rho\lambda_0+r)q_0^0\left(\frac{q}{q_0^m}\right)^{\lambda_2/\lambda_m-1}.$$

First, suppose $m = 0$, i.e., λ_0 is the smallest among all the feedback rates. We apply Theorem 1 to this case. Let $\hat{q}_0 = 0$. Note that $\Psi(0) = c(\rho\lambda_0 + r)q_0^0 > 0$, i.e., $\lim_{x\to\hat{q}_0^+} \text{sign}(\Psi(x)) = 1$. Thus, $\hat{a}_0 = 0$. Next, we consider $\hat{q}_1$. If $\hat{q}_1 = q_0^0$, price reversals obviously do not occur. Alternatively, if $\hat{q}_1 < q_0^0$, we must have

$$\Phi(\hat{q}_1, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = \Phi(\hat{q}_1, \hat{a}_0) = 0.$$

Notice that $\Psi(q)$ is decreasing in q on $[0, q_0^0]$. Since the sign of $\partial\Phi/\partial q$ is equal to $\Psi(q)$ and $\Phi(\hat{q}_0, \hat{a}_0) = \Phi(\hat{q}_1, \hat{a}_0) = 0$, we must have $\Psi(q) < 0$ for all $q > \hat{q}_1$, which in turn implies that $\hat{q}_2 = q_0^2$. Again, price reversals do not occur.

Now, suppose $m \neq 0$. In this case, $\Psi(0)$ is either $((\rho+1)\lambda_1 + r)q_0^1$ (if $m = 1$) or $(c-1)((\rho+1)\lambda_2 + r)$ (if $m = 2$). Thus, $\Psi(0) < 0$ and $\hat{a}_0 = 1$. So, $p_t = \mu_t$ for t large enough. Hence, only a type-2 reversal may occur. Theorem 1 then implies that the necessary and sufficient condition for type-2 reversal to occur is $0 < \hat{q}_1 < \hat{q}_2 < q_0^m$. To have $\hat{q}_1 < q_0^m$, we need

$$\min\{q \mid \Phi(q, \hat{a}_0) - \Phi(\hat{q}_0, \hat{a}_0) = 0, q > \hat{q}_0\} < q_0^m, \tag{A.53}$$

Since $\Phi(\hat{q}_0, \hat{a}_0) = 0$, (A.53) is equivalent to requiring the smallest positive root of (15) to be less than q_0^m. Given that $\hat{q}_1 < q_0^m$, for a price reversal to occur, we need $\Psi(\hat{q}_1) > 0$ because, otherwise, $\hat{a}_1 = 1$ and $\hat{q}_2 = q_0^m$. Suppose $\Psi(\hat{q}_1) > 0$, we have $\hat{a}_1 = 0$. The remaining condition for price reversal to occurs is $\hat{q}_2 < q_0^m$, which is equivalent to $\Phi(q_0^m, 0) - \Phi(\hat{q}_1, 0) > 0$.

Combining the two cases, we obtain the results in the proposition. ∎

A.5 Proof of Proposition 4

First, notice that if the seller chooses a constant price weakly below c, she will get a non-positive discounted profit under any constant-price mechanism. On the other hand, with a price p strictly above c, a feasible (not necessarily optimal) constant-price mechanism is such that $a_t = 0$, $b_t^i = 1$ for all $v_i > p$, and $b_t^i = 0$ for all $v_i \leq p$. This mechanism generates strictly positive discounted profit. Hence, it must be that $p^* > c$ under the optimal constant-price mechanism.

Next, we argue that $\mu_t = p^*$ for all $t \geq 0$ under the optimal constant-price mechanism. First, suppose p^* is lower than the prior expectation $\mathbb{E}_\tau(v)$, then the mechanism is dominated by a mechanism with $a_t = b_t^i = 1$ for all i and $p^* = \mathbb{E}_\tau(v)$. Hence, $p^* \geq \mathbb{E}_\tau(v)$ in the optimal constant-price mechanism. In this case, if $p^* < \mu_t$ for some t, we can always increase a_t or b_t^i, which increases the flow profit and potentially also increases the speed of learning, making the seller strictly better off. Hence, under the optimal constant-price mechanism, we must have $p^* = \mu_t$ for all $t \geq 0$.

Lastly, we show that $b_t^0 > 0$ for t large enough when $V = \{0, 1\}$. To see this, first note that if $p^* \geq 1$, we must have $a_t = 0$ in any feasible mechanism so that the incentive compatibility constraint $\mu_t \geq p^*$ can be satisfied. This in turn implies that the seller never acquires any information about the product values and will therefore get zero profit when $p^* \geq 1$. This is clearly not optimal for the seller. Thus, under the optimal mechanism, we have $p^* < 1$. Next, note that since $\rho > 0$, the seller must eventually gets fully informed about the product values as time passes, i.e.,

$$g_t^i \to q_0^i \text{ and } q_t^i \to 0, \text{ as } t \to \infty.$$

From the discussion above, we also know that, for all $t \geq 0$, we have $p^* = \mu_t$. Thus,

$$p^* = \lim_{t\to\infty} \mu_t = \lim_{t\to\infty} \frac{q_0^1 b_t^1}{q_0^1 b_t^1 + q_0^0 b_t^0}, \tag{A.54}$$

Since $p^* < 1$, (A.54) implies that $\lim_{t\to\infty} b_t^0 > 0$. ∎

A.6 Proof of Proposition 6

To simplify the exposition, we introduce notations that represent the mechanisms and cutoff times in Section 5 as functions of the production cost c. Let $(p(c), a(c), b(c))$ denote the optimal mechanism in Proposition 1 when the production cost is c, and $(\hat{p}(c), \hat{a}(c), \hat{b}(c))$ denote the optimal mechanism in Proposition 2 when the production cost is c. Let $t^*(c)$ denote the cutoff time in Proposition 1 when the production cost is c, and $t^{**}(c)$ denote the cutoff time in Proposition 2 when the production cost is c.

We want to show that, if $\lambda_1 > \lambda_0$, the intermediary's equilibrium recommendation strategy is $(a(\frac{\kappa}{\phi}), b(\frac{\kappa}{\phi}))$ and the producer's pricing strategy is $p(\frac{\kappa}{\phi})$; if $\lambda_1 < \lambda_0$, the intermediary's equilibrium recommendation strategy is $(\hat{a}(\frac{\kappa}{\phi}), \hat{b}(\frac{\kappa}{\phi}))$ and the producer's pricing strategy is $\hat{p}(\frac{\kappa}{\phi})$. Note that the equilibrium strategy profile corresponds to the optimal mechanism for the intermediary if the intermediary were able to design both prices and recommendations. Hence, to establish the equilibrium, it suffices to verify that, given that intermediary strategy, it is optimal for the producer choose $p(\frac{\kappa}{\phi})$ (if $\lambda_1 > \lambda_0$) or $\hat{p}(\frac{\kappa}{\phi})$ (if $\lambda_1 < \lambda_0$).

Case 1: $\lambda_1 > \lambda_0$.
Suppose the intermediary chooses $(a(\frac{\kappa}{\phi}), b(\frac{\kappa}{\phi}))$. We need to show that at any time $\hat{t} \geq 0$, the producer finds that the continuation price schedule $\{p_t(\frac{\kappa}{\phi})\}_{t \geq \hat{t}}$ is optimal among all continuation price schedules. There are two sub-cases to consider: $\hat{t} \geq t^*(\frac{\kappa}{\phi})$ or $\hat{t} < t^*(\frac{\kappa}{\phi})$.

Sub-case 1: $\hat{t} \geq t^*(\frac{\kappa}{\phi})$.
By the definition of $t^*(\frac{\kappa}{\phi})$, for $t \geq \hat{t}$, the intermediary's strategy $(a(\frac{\kappa}{\phi}), b(\frac{\kappa}{\phi}))$ specifies $a_t = 0$, $b_t^0 = 0$, and $b_t^1 = 1$. In this case, the on-path equilibrium price schedule $p(\frac{\kappa}{\phi})$ specifies $p_t = 1$ for all $t \geq \hat{t}$. To show that the producer does not want to deviate to another continuation price schedule, it suffices to show that, even if the producer were able to jointly design the continuation price schedule and the continuation recommendation strategy, she finds it optimal to choose $p_t = 1$, $a_t = 0$, $b_t^0 = 0$, and $b_t^1 = 1$ for all $t \geq \hat{t}$. This hypothetical problem is exactly the problem in the baseline model when the production cost is $\chi/(1-\phi)$, whose solution is $(p(\frac{\chi}{1-\phi}), a(\frac{\chi}{1-\phi}), b(\frac{\chi}{1-\phi}))$. Note that

$$\frac{\kappa/\phi}{1-\kappa/\phi}\left(1 - \frac{\lambda_1}{r + (1+\rho)\lambda_1}\right) \geq \frac{\chi/(1-\phi)}{1-\chi/(1-\phi)}\left(1 - \frac{\lambda_1}{r + (1+\rho)\lambda_1}\right), \tag{A.55}$$

where the inequality follows from the fact that $\phi\chi > (1-\phi)\kappa$. Thus, by the definition of $t^*(\cdot)$, we have $t^*(\frac{\kappa}{\phi}) > t^*(\frac{\chi}{1-\phi})$, which in turn implies that $\hat{t} > t^*(\frac{\chi}{1-\phi})$. Hence, for all $t \geq \hat{t}$, the mechanism $(p(\frac{\chi}{1-\phi}), a(\frac{\chi}{1-\phi}), b(\frac{\chi}{1-\phi}))$ specifies $p_t = 1$, $a_t = 0$, $b_t^0 = 0$, and $b_t^1 = 1$. Therefore, given that the intermediary has already chosen $a_t = 0$, $b_t^0 = 0$, and $b_t^1 = 1$ for $t \geq \hat{t}$, it is indeed optimal for the producer to choose $p_t = 1$ for $t \geq \hat{t}$.

Sub-case 2: $\hat{t} < t^*(\frac{\kappa}{\phi})$.
Consider the time interval between $\hat{t}$ and $t^*(\frac{\kappa}{\phi})$. In this time interval, the intermediary's strategy $(a(\frac{\kappa}{\phi}), b(\frac{\kappa}{\phi}))$ specifies $a_t = 1$, $b_t^0 = 0$, and $b_t^1 = 1$. Hence, for each $t \in (\hat{t}, t^*(\frac{\kappa}{\phi}))$, the consumers' posterior belief is either $\mu_t = q_0^1/(1-g_t^0)$ (upon receiving a recommendation) or $\nu_t = 0$ (upon not receiving a recommendation). Given the intermediary's strategy, the producer therefore finds it optimal to set the price at $p_t = \mu_t$ for all $t \in (\hat{t}, t^*(\frac{\kappa}{\phi}))$ because setting p_t above μ_t induces no trade at time t, and setting p_t below μ_t will not induce more consumers to purchase the product unless $p_t = 0$, which generates a negative flow profit without affecting the rate of consumer feedback at time t. Hence, the optimal continuation price schedule for $t \geq \hat{t}$ can differ from the on-path prices only for $t \geq t^*(\frac{\kappa}{\phi})$. Such continuation price schedules are those considered in Sub-case 1 above. Hence, given the intermediary's recommendation strategy, the producer finds it optimal to choose prices according to $p(\frac{\kappa}{\phi})$.

Case 2: $\lambda_1 < \lambda_0$.
Suppose the intermediary chooses $(\hat{a}(\frac{\kappa}{\phi}), \hat{b}(\frac{\kappa}{\phi}))$. We need to show that at any time $\tilde{t} \geq 0$, the producer finds that the continuation price schedule $\{\hat{p}_t(\frac{\kappa}{\phi})\}_{t\geq\tilde{t}}$ is optimal among all continuation price schedules. There are two sub-cases to consider: $\tilde{t} \geq t^{**}(\frac{\kappa}{\phi})$ or $\tilde{t} < t^{**}(\frac{\kappa}{\phi})$.

Sub-case 1: $\tilde{t} \geq t^{**}(\frac{\kappa}{\phi})$.
In this case, for any $t \geq \tilde{t}$, the intermediary's strategy is $a_t = 1$, $b_t^0 = 0$, and $b_t^1 = 1$. The same arguments as those in Sub-case 2 of Case 1 above shows that, given the intermediary's recommendation strategy, the producer does not want to deviate from the on-path price $p_t = \mu_t$ for $t > \tilde{t}$.

Sub-case 2: $\tilde{t} < t^{**}(\frac{\kappa}{\phi})$.
In this case, the intermediary's strategy is $a_t = 0$, $b_t = 0$ and $b_t^1 = 1$ for $t < t^{**}(\frac{\kappa}{\phi})$, and $a_t = 1$, $b_t^0 = 0$ and $b_t^1 = 1$ for $t > t^{**}(\frac{\kappa}{\phi})$. Consider any time $\tilde{t}$ continuation price schedule for the producer. First, notice that it is always optimal for the producer to price at $p_t = \mu_t$ for $t > t^{**}(\frac{\kappa}{\phi})$ (for the same reasons as in Sub-case 2 of Case 1 and Sub-case 1 of Case 2). Hence, we only need to consider continuation price schedules $\{p_t(\frac{\kappa}{\phi})\}_{t\geq\tilde{t}}$ in which the prices differ from $\{\hat{p}_t(\frac{\kappa}{\phi})\}_{t\geq\tilde{t}}$ only when $t < t^{**}(\frac{\kappa}{\phi})$. To rule out the possibility that the producer may benefit from deviating to such a continuation price schedule, it suffices to show that the producer does not want to deviate even if she can choose both prices and recommendation strategy for $t \in (\tilde{t}, t^{**}(\frac{\kappa}{\phi}))$. Similar arguments as those in the proof of Theorem 1 implies that $p_t = \mu_t$ under the optimal strategy. Substituting $p_t = \mu_t$ into the producer's objective function, we obtain that the discounted profit for the producer at time $t = \tilde{t}$ is

$$\int_{t\geq\tilde{t}}^{\infty} e^{-rt}\left[(g_t^1 b_t^1 + q_t^1 a_t)((1-\phi)-\chi) - (g_t^0 b_t^0 + q_t^0 a_t)\chi\right] \mathrm{d}t$$

Hence, it suffices to show that the solution to the the optimal control problem below is $a_t = 0$ for $t \in (\tilde{t}, t^{**}(\frac{\kappa}{\phi}))$ and $a_t = 1$ for $t > t^{**}(\frac{\kappa}{\phi})$.

$$\begin{aligned}
\max_{\{a_t, b_t^0, b_t^1\}} & \int_{t\geq\tilde{t}}^{\infty} e^{-rt}\left[(g_t^1 b_t^1 + q_t^1 a_t)(1-\frac{\chi}{1-\phi}) - (g_t^0 b_t^0 + q_t^0 a_t)\frac{\chi}{1-\phi}\right] \mathrm{d}t \\
\text{s.t. } & \dot{q}_t^1 = -\lambda_1(\rho + a_t)q_t^1, \\
& a_t, b_t^0, b_t^1 \in [0,1] \text{ for all } t \geq 0, \\
& a_t = 1 \text{ if } t \geq t^{**}(\frac{\kappa}{\phi}),
\end{aligned} \tag{A.56}$$

This is the same problem considered in Theorem 1 in the special case when there are binary values with $\lambda_1 < \lambda_0$ (i.e., the special case of Proposition 2), the production cost is $\frac{\chi}{1-\phi}$, and there is an additional constraint: $a_t = 1$ if $t \geq t^{**}(\frac{\kappa}{\phi})$. Define $\hat{q}_1(c)$, $\Psi(\cdot; c)$, and $\Phi(\cdot,\cdot; c)$ as the cutoff $\hat{q}_1$ and functions Φ, Ψ in Section 4 when the production cost is c. By similar arguments as those in the proofs of Theorem 1 and Proposition 2, we know that the optimal solution to (A.56) is $b_t^0 = 0$, $b_t^1 = 1$, and $a_t = \tilde{a}_j$ if $q_t^1 \in (\tilde{q}_j, \tilde{q}_{j+1}]$, where $\{\tilde{q}_j\}$ and $\{\tilde{a}_j\}$ are obtained through the following procedure:

1. Let $\tilde{q}_0 = 0$, $\tilde{a}_0 = 1$, and $\tilde{q}_1 = \hat{q}_1(\frac{\kappa}{\phi})$. Set $j = 1$.

2. Stop if $\tilde{q}_j = q_0^1$.

3. If $\lim_{x \to \tilde{q}_j^+} \text{sign}(\Psi(x; \frac{\chi}{1-\phi})) = 1$, let $\tilde{a}_{j+1} = 0$; otherwise, let $\tilde{a}_{j+1} = 1$. Let $\tilde{q}_{j+1} = \min\{q_0^1, \min\{q \mid \Phi(q, \tilde{a}_j; \frac{\chi}{1-\phi}) - \Phi(\tilde{q}_j, \tilde{a}_j; \frac{\chi}{1-\phi}) = 0, q > \tilde{q}_j\}\}$.

4. Raise j by 1 and go back to Step 2.

According to the proof of Proposition 2, $\hat{q}_1(c)$ is characterized by (A.46) and is therefore decreasing in c. Since $\phi\chi > (1-\phi)\kappa$, we have $\hat{q}_1(\frac{\chi}{1-\phi}) < \hat{q}_1(\frac{\kappa}{\phi})$. By definition, $t^{**}(\frac{\kappa}{\phi})$ satisfies $q^1_{t^{**}(\frac{\kappa}{\phi})} = \hat{q}_1(\frac{\kappa}{\phi})$ when $a_t = 0$ for $t \in (0, t^{**}(\frac{\kappa}{\phi}))$. Thus, the procedure above implies that the solution to (A.56) is indeed $a_t = 0$ for $t \in (\tilde{t}, t^{**}(\frac{\kappa}{\phi}))$ and $a_t = 1$ for $t > t^{**}(\frac{\kappa}{\phi})$.

■