Competing to Commit:
Markets with Rational Inattention*

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Abstract

Two homogeneous-good firms compete for a consumer's unitary demand. The consumer is rationally inattentive and pays entropy-based information processing costs to learn about the firms' offers. While trade is always inefficient if firms collude, we obtain efficiency under competition for a range of information costs. Competition puts downward pressure on prices. Additionally, competition increases the demand pointwise, since the consumer's information processing decision depends on the level of competition. For high enough information costs, this effect dominates: Firms' total surplus is larger under competition than under collusion. Our model reveals that markets with common ownership may remain competitive.

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1 Introduction

Consumers' ability to freely process information about prices is naturally at the heart of the idea of competition. Standard models of Bertrand competition assume that a consumer can perfectly spot different sellers' offers and choose the best one. However, in many situations, understanding which offer is the best is a difficult task that requires substantial amounts of effort. For instance, when booking a flight from New York to London with two pieces of luggage, information on the luggage allowance and additional fees has to be processed to understand the final price. Moreover, different websites might offer the same flight at different prices. If processing this information is costly, the Bertrand assumption no longer holds. Importantly, a rational consumer subject to information processing costs has to decide not only which seller's offer to accept, but also how much attention to pay to each offer.

This paper studies the impact of competition in markets with costly information processing. Increasing the level of competition has two effects. First, as in free information processing settings, competition creates downward pressure on prices. Additionally, since the consumer's information strategy depends on the competition level, the demand curve increases pointwise. In our main result, we show that depending on the information processing cost, this second effect dominates: The firms' equilibrium profits may be *higher* when they compete than when they collude.

Following Sims (2003), we introduce a rationally inattentive consumer into an otherwise standard duopoly model to capture consumers' optimal information processing decisions in competitive markets. Two firms sell a good of common stochastic quality. They make take-it-or-leave-it offers which we characterize by their monetary value. A representative consumer has unitary demand for the good and chooses an information structure to learn about the quality of the product and the firms' offers. An information structure is a mapping from quality-offer pairs to signals. The information processing cost or attention cost,[1] is proportional to the expected reduction in entropy that information provides.

Our framework applies to markets where the monetary equivalent of an offer is not easy to understand or compare across firms. Examples of this kind are contracts for health or life insurance, complex loans, the purchase of expensive electronic hardware, or travel arrangements like the one described above. The assumptions we place on demand reflect features of these markets while also simplifying our analysis. Our aim is not to develop the richest possible framework, but a tractable one that is rich enough to capture the main mechanisms at play when competition and costly information processing are combined.

Our model can support any division of surplus as a Bayes Nash Equilibrium. Intuitively, because entropy costs are prior-dependent, Bayes Nash Equilibrium does not put any restrictions on information choices for quality-offer pairs that are not observed in equilibrium.[2] To counteract this multiplicity of equilibria, we focus on consumer's strategies that are *robust to vanishing perturbations* (RVP). This means that, for all possible deviations in the equilibrium price-setting behavior of the firms, the consumer strategy must be optimal against some (possibly correlated)

[1] The two terms will be used interchangeably in our analysis.

[2] This issue is similar to the multiplicity of equilibria in Ravid (2020).

small belief perturbation consistent with such deviations. RVP naturally extends the notion of *credible best response* introduced by Ravid (2020), and is similar in spirit but weaker than Selten's (1975) trembling-hand perfection. Despite its weakness, RVP allows us to obtain sharp predictions regarding the offers accepted by the consumer on-path. These variables are sufficient to characterize both the industry profits and consumer surplus, which are the relevant statistics of our model.

To isolate the equilibrium effect of competition in markets with rational inattention, we first establish a relevant benchmark where the two firms collude. We find that when firms perfectly price-coordinate or perfectly internalize each other's profits, the unique robust *equilibrium outcome* is identical to the credible equilibrium outcome of Ravid's (2020) ultimatum bargaining game.[3] For this reason, we refer to Ravid's model as the monopoly equivalent of our competition model. Applying results from Ravid (2020), we know that trade cannot be efficient if firms collude. Efficiency implies that the consumer always purchases the good from one of the colluding firms, regardless of the offers. However, if this were the case, it would be optimal for the firms to coordinate on overcharging the consumer, making the consumer's strategy sub-optimal.

Our results for the competitive setting are as follows. First, we show that an RVP equilibrium in which both firms trade with positive probability, hereafter *competitive equilibrium*, exists if and only if the parameter k that governs the consumer's unit cost of processing information is not too high, i.e., $k < k^*$. Moreover, whenever a competitive equilibrium exists, it is unique. If attention costs are above k^*, no trade can be sustained: The consumer's optimal level of attention is not enough to deter firms from overcharging. As a result, the effect of competition on prices completely disappears. Interestingly, the threshold k^* is not affected by the level of competition, as it is constant across the two models. If attention is too costly, the consumer is not willing to pay any attention to the market's offers, irrespective of the number of firms in the market and their incentives to compete.

When a competitive equilibrium exists, competition increases trading efficiency. Intuitively, competition acts as a *commitment device* to keep the market's offers in check. Compared to the collusion model, this renders the supply curve less steep, enabling the consumer to engage more often in trade without fearing a drastic increase in prices. Since coordination issues do not arise in the competitive setting because of the firms' undercutting incentives, equilibrium trade efficiency is feasible under competition. In particular, we show that trade is fully efficient if attention costs are below a second, lower threshold $\bar{k} < k^*$. In this region, the optimal attention level implies a strong reaction of demand to differences between offers, creating strong competitive forces that lead to advantageous price offers for the consumer. As a result of this competition effect, the consumer always buys and ignores the quality of the product. If attention costs instead are above $\bar{k}$, prices are higher and there are qualities where buying the product would entail large losses for the consumer. The consumer then pays attention to quality, breaking down the fully efficient equilibrium.

Our main result states that there always exists a *region of attention costs* in which the total

[3]The equilibrium outcome is the probability of trade and the monetary equivalent of the offers accepted by the consumer on-path. We focus on these variables as they determine the most relevant economic predictions, such as the consumer's surplus and the industry profits.

profit generated by competing firms is strictly higher than the one of colluding firms. This is in stark contrast with most of the existing economic literature. Since the firms produce perfect substitutes, competition does not change demand, and firms' surplus in friction-less models is decreasing in competition. In contrast, if the consumer is rationally inattentive, firms benefit from the additional commitment power induced by competition that refrains them from overcharging the consumer.

We apply our results to the literature on common ownership. The common ownership hypothesis states that firms within the same market have fewer incentives to compete if common investors own shares of all of them. Contrary to this hypothesis, our main result shows that common investors may want to create incentives for the firms to compete.

As an illustration, consider the German consumer electronic store chain MediaMarkt. In 1990, MediaMarkt acquired its direct competitor Saturn. Instead of merging the two chains and their management, MediaMarkt decided to keep the management and branding separate: "The MediaMarkt and Saturn brands operate independently on the German market and are in direct competition with one another".[4] Given their position as Germany's market leader, the strategy of MediaMarkt seems counter to standard economic intuition. Horizontal integration should lead to significant savings in management costs. Moreover, a unified position could further increase the market power of the company in a way that could be exploited through higher prices. Our framework can explain the puzzling competition between MediaMarkt and Saturn. When consumers are rationally inattentive about endogenous offers, holdings may benefit from owning actively competing firms. Competition makes the consumer confident that firms will not overcharge. As a result, the consumer is more likely to buy. If attention costs are relatively high, the positive effect of competition on trading efficiency more than offsets the negative effect on prices, leading to higher profits under competition. If firms colluded, they would internalize the positive effect that a high price has on the demand served by their partner. As a result, the commitment power to not overcharge the consumer would vanish.

While the main result in our model fits the example of MediaMarkt and Saturn well, we are not aware of any studies about the actual level of competition between these two firms. However, according to recent studies, some markets with high levels of common ownership display more competition than might be expected. For example, Backus, Conlon, and Sinkinson (2021a) find no significant impact of common ownership on pricing in the cereal industry. Dennis, Gerardi, and Schenone (2021) show that "Common ownership does not have anti-competitive effects in the airline industry". Our result provides a rationale for the joint owner actively designing incentives to promote competition among commonly owned firms in the presence of rational inattention.

Related Literature. This work contributes directly to the literature on rational inattention initiated by Sims (2003). Our theoretical analysis borrows insights and techniques from Ravid (2020). Like us, Ravid studies players that are inattentive about endogenous equilibrium variables. Moreover, we both introduce a refinement in the spirit of, but weaker than, Selten (1975) trembling-hand equilibrium. As entropy-based information cost ignores probability

[4]Translated from `https://www.mediamarktsaturn.com/mediamarktsaturn-deutschland`.

zero events, this refinement is necessary to discipline off-path behavior. Without imposing this refinement, a trivial multiplicity of equilibria would arise where every division of surplus is obtained in equilibrium. In contrast, this refinement selects only the equilibria with the property that consumer's best replies follow the generalized multinomial logit equation formulated by Matějka and McKay (2015) everywhere. This formula characterizes the solution of the single-agent decision problem with rational inattention. We modify the refinement in Ravid (2020) to accommodate for the presence of two firms. We allow for the possibility that belief perturbations about firms' offers are correlated as long as such correlation vanishes in the limit. This further weakens our equilibrium notion.

Bertrand competition with rationally inattentive consumers is not entirely new as it is present in Matějka and McKay (2012). Differently from Matějka and McKay (2012), we do not assume a specific functional form for the consumer's best response, but we derive it as an equilibrium property. This allows us to formulate a novel analysis of the equilibrium effects of competition, which ultimately leads to the statement of our main result. Inattention about equilibrium variables is also present in Matějka (2015). This paper studies a monopoly model where consumers are rationally inattentive about prices and shows that the optimal pricing strategy implies price rigidity. Other works that study rationally inattentive buyers about non-equilibrium features of the markets are: Martin (2017), Roesler and Szentes (2017), Boyacı and Akçay (2018).

Our work has a clear connection with the literature on search costs initiated by Varian (1980), which explains price dispersion as a result of the consumer searching for the best alternative. This literature assumes that prices are not perfectly observable by the consumers. However, the assumption of an entropy-based attention cost with arbitrary information structure makes the two approaches substantially different at the conceptual level. We parametrize the consumer's unit cost of information processing, while this literature mainly focuses on the proportion of informed/ uninformed consumers in the market. The seminal work of Diamond (1971) shows that firms exploit the presence of small search costs to charge monopoly prices. In our setting, as the unit cost of attention approaches zero, the competition effect dominates the attention effect, and prices converge to the ones of competition. Burdett and Judd (1983) is the first work that studies noisy sequential search, and Stahl (1989) introduces endogenous information acquisition by allowing the consumer to pay fixed search costs to observe additional firms' offers. Similarly to Armstrong and Chen (2009), firms' profits are non-monotonic when the parameter that captures the attention level changes. In our setting, however, combined profits under competition may exceed the ones under collusion. De Clippel, Eliaz, and Rozen (2014) model attention as the number of markets that are perfectly observable by the consumer. They find that consumers' welfare is higher if the expected level of attention is lower.

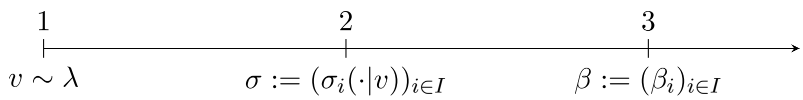
Our model is also related to the behavioral industrial organization literature about inattentive and irrational consumers. In this literature, the unobservability of prices is justified in terms of bounded rationality. Spiegler (2006) studies firms' pricing strategies that the consumer does not perfectly observe but evaluates using an exogenous sampling procedure. In this setting, an increase in competition induces firms to obfuscate prices, which is modeled as the variance of equilibrium prices. The number of competitors does not affect expected equilibrium profits but generates expected surplus losses that are borne by the consumer. This contrasts our main result which states that firms benefit from competition when the positive effect on trade efficiency dominates the negative one on prices. Gabaix and Laibson (2006) study markets where firms exploit non-sophisticated consumers by shrouding add-on prices. Bordalo, Gennaioli, and Shleifer (2016) model consumer's attention as captured by the product's most salient attributes. The key difference with our approach is that the rationally inattentive consumer allocates attention *ex-ante*, generating commitment power. In their approach, attention depends on which attribute is salient *ex-post*. Hefti (2018) studies a competitive setting where firms can manipulate consumers' attention.

Outline. The remainder of the paper proceeds as follows. In Section 2, we introduce the model. Section 3 discusses the equilibrium notion we use for the analysis and studies its implications for the best response of the consumer. In Section 4, we study the benchmark case of colluding firms and show that it is equilibrium outcome-equivalent to Ravid (2020). Section 5 analyzes the competition model. It characterizes the unique equilibrium where both firms trade. In Section 6, we study the equilibrium effects of competition and state our main results. A discussion follows in Section 7.

2 Duopoly with Rational Inattention

Two identical firms compete for a consumer with unitary demand. Marginal costs of production are normalized to zero. The quality of the product is common, stochastic, and perfectly observed by the firms. After observing quality realization, each firm makes a simultaneous offer to the consumer. The consumer does not observe product quality and firms' offers directly but gathers information about them by paying a cost. We interpret the consumer's lack of information as an attention problem, and we use the terms attention and information processing interchangeably in our analysis. The consumer allocates attention optimally, taking into account the trade-off between good decision-making and costly information processing. Following the literature on rational inattention initiated by Sims (2003), we call the consumer *rationally inattentive*, and we assume that the attention cost is entropy-based.

Game structure. The following timeline formalizes the game structure of the model. The description of each element of the timeline is provided below.



1. Product quality v is drawn according to a probability measure $\lambda \in \Delta(\mathbb{R})$. We assume that λ has strictly positive finite support, i.e. $\operatorname{supp}\lambda =: V \subseteq (0,\infty)$ is finite.

2. After observing the realization of product quality v, each firm $i \in I := \{1,2\}$ makes a simultaneous offer to the consumer. Denote firm i's strategy by $\sigma_i : V \to \Delta(X)$, where $X := \mathbb{R}_+$. We interpret every $x \in X$ as the *monetary value* associated with an offer.

3. The consumer holds a prior belief on the exogenous product quality and endogenous firms' offers. We refer to this pair as the *state*. Consumer pays attention by selecting an *information structure* to learn about the realization of the state. The related attention cost is

proportional to Shannon's mutual information between states and signals. This equals the expected entropy reduction between the consumer's prior belief over states and posterior beliefs obtained via Bayesian updating after each signal realization.

Without loss of generality, we restrict the consumer's strategy space to *recommendation* or *attention strategies*.[5] A recommendation strategy β is a profile (β_1, β_2) such that, for every $i \in I$, $\beta_i : V \times X^2 \to [0,1]$ denotes the *conditional* probability of accepting the offer of firm i. That is, $\beta_i(v, x_1, x_2)$ is the probability of receiving the recommendation "accept i's offer" given the state $(v, x_1, x_2) \in V \times X^2$. For every $(v, x_1, x_2) \in V \times X^2$, it holds that $\sum_{i \in I} \beta_i(v, x_1, x_2) \leq 1$.

Denote consumer's prior belief over states by $\mu \in \Delta(V \times X^2)$.[6] Exploiting the restriction on recommendation strategies, we can write mutual information as

$$I(\beta, \mu) := H(\mathbb{E}_\mu[\beta]) - \mathbb{E}_\mu[H(\beta)], \tag{1}$$

where $H(p) = -p_1 \log(p_1) - p_2 \log(p_2) - (1 - p_1 - p_2) \log(1 - p_1 - p_2)$ is the Shannon entropy associated with the probability measure $p = (p_1, p_2, 1 - p_1 - p_2)$. Thus, mutual information as formalized in equation (1) says that attention costs are proportional to the difference in entropy between the conditional and the unconditional distribution of playing each action. It captures how much information reduces prior uncertainty about the optimal plan of action.

Payoffs. Once firms make offers and the consumer selects a recommendation strategy, payoffs are obtained. The payoff of the consumer is given by

$$U := \sum_{i \in I} (v - x_i) \beta_i(v, x_1, v_2) - k \cdot I(\beta, \mu),$$

where $k > 0$ is the unit cost of information processing. That is consumer's payoff is given by the expected gains from trade net of the costs of processing information. If no offer is accepted, the consumer does not receive any payoff from trading but still incurs attention costs equal to $k \cdot I(\beta, \mu)$. The payoff obtained by each firm $i \in I$ instead equals

$$\Pi_i^B := \beta_i(v, x_1, x_2) \cdot x_i,$$

where the superscript B stands for "Bertrand." Competing firms in our model adopt the standard profit-maximizing behavior.

[5] If the consumer did not have access to an ex-post randomization device, this would be a direct consequence of Matějka and McKay (2015). However, even in the presence of an ex-post randomization device, action recommendation strategies are without loss. Intuitively, if the consumer found optimal to strictly randomize between two actions a_1 and a_2 upon receiving a signal $\bar{s}$, he could split $\bar{s}$ into two signals s_1 and s_2 so that (i) action a_i is played upon receiving message s_i, and (ii) the overall probability of playing each action a_i is unchanged. Since the resulting new information structure is neither more nor less informative than the original one, the consumer would be completely indifferent. Moreover, any play of the game between the consumer and the firms would unfold exactly in the same way. Therefore, it is without loss for our equilibrium analysis to consider an "as if" situation where the consumer does not have access to ex-post randomization devices.

[6] We endow $V \times X^2$ with the product σ-algebra between the discrete σ-algebra on V and the standard Borel σ-algebra on X^2. We also endow both $\Delta(X^2)$ and $\Delta(V \times X^2)$ with the topology of strong convergence.

3 Equilibrium Refinement

We introduce Bayes Nash Equilibrium (BNE) as a solution concept for our duopoly model with rational inattention. The *assessment* (μ, σ, β) is a BNE if (i) μ is *consistent* with σ,[7] (ii) β is a best response to μ, and (iii) for every $i \in I$, σ_i is a best response to σ_{-i} given β.

As discussed by Ravid (2020), standard BNE can be too weak to make sharp predictions about equilibrium outcomes in games with rational inattention, especially when attention is directed towards endogenous variables. In particular, any division of surplus between firms and the consumer can be obtained. The reason lies in the fact that entropy costs are prior-dependent and agents' beliefs are endogenous in equilibrium. The following example illustrates.

Example 1. *We show that, for any $\alpha \in [0,1]$, there exists a BNE such that each firm obtains profits equal to $\alpha \cdot \mathbb{E}_\lambda[v]/2$, while the consumer's payoff is $(1-\alpha)\cdot \mathbb{E}_\lambda[v]$. Consider the following assessment $(\mu^\alpha, \sigma^\alpha, \beta^\alpha)$: μ^α is consistent with σ^α, and for all $i \in I$ and $v \in V$, $\sigma_i^\alpha(\alpha v|v) = 1$ and $\beta_i^\alpha(v, x_1, x_2) = 1/2 \cdot \mathbf{1}_{\{x_1 = x_2 = \alpha v\}}$. Firms have no incentives to deviate since any deviation would imply zero profits. We now argue that the consumer does not want to deviate either. First, the consumer is indifferent between offers and optimally chooses to buy with probability 1 overall. At the same time, the consumer displays deterministic beliefs about firms' behavior conditional on each $v \in V$. Therefore, as the attention strategy β^α implies uniform acceptance of the offers μ^α-almost surely, the consumer incurs no information processing costs. This shows that β^α is optimal given μ^α. As a result, the assessment $(\mu^\alpha, \sigma^\alpha, \beta^\alpha)$ is a BNE. It is immediate to verify that it satisfies the desired surplus allocation.*

Example 1 shows that standard BNE yields a multiplicity of equilibria, each associated with a different division of the surplus between firms and the consumer. Intuitively, costly information processing has no bite when the consumer's strategy induces a deterministic play from the firms. In the above example, this is achieved because the consumer and the firms perfectly coordinate on some arbitrary surplus allocation.

Perfect strategic coordination is hard to achieve in practice. What if we allow firms to make arbitrarily small mistakes on the equilibrium path? As pointed out by Ravid (2020), assessments like the ones described above are fragile to such small mistakes.[8] To address this issue, we impose an additional property on the consumer's best response which we call *robustness to vanishing perturbations* (RVP). RVP naturally extends the notion of *credible best response* introduced by Ravid (2020) to a multi-firm setting. It requires the consumer's strategy to be justified under some arbitrarily small belief perturbations both on and off the conjectured path of play.

Definition 1. *Let μ be consistent with the profile σ and β a best response to μ. We say that*

[7] The belief $\mu \in \Delta(V \times X^2)$ is consistent with the profile σ if for every $v \in V$, and for every Borel measurable set $E \subseteq X^2$, we have

$$\mu(v, E) = \lambda(v) \cdot \int_E 1 \mathrm{d}\sigma_1(\cdot|v) \otimes \sigma_2(\cdot|v).$$

[8] Equivalently, they are not robust to ε-perturbations of the consumer's equilibrium belief. To illustrate, consider the following prior $\mu^{\alpha,\varepsilon} = (1-\varepsilon)\mu^\alpha + \varepsilon\mu^*$ where $\varepsilon \in (0,1)$, μ^a is defined as before for $\alpha \in [0,1]$, and μ^* is consistent with λ and assigns probability 1 to firms' offers being different from $\alpha \cdot v$. An argument similar to Ravid's (2020) shows that, for small enough values of ε, the consumer prefers attention strategy $\hat{\beta}$ to β^α, where $\hat{\beta}$ accepts all offers for sure randomizing uniformly between firms: That is, $\hat{\beta}_i(v, x_1, x_2) = 1/2$ for each (v, x_1, x_2) and $i \in I$.

β is robust to vanishing perturbations (RVP) if for every $v^* \in V$ and $x_1^*, x_2^* \geq 0$, there exists a sequence $(\mu^n, \tilde{\sigma}^n)$ such that

- *$\tilde{\sigma}^n(\cdot|\cdot) \in \Delta(X^2)^V$ is a vector of (possibly correlated) probability measures on X^2,*
- *$\tilde{\sigma}^n(x_1^*, x_2^*|v^*) > 0$ for every $n \in \mathbb{N}$,*
- *$\tilde{\sigma}^n(\cdot|v) \to \sigma_1(\cdot|v) \otimes \sigma_2(\cdot|v)$ strongly for all $v \in V$,*
- *μ^n is consistent with $\tilde{\sigma}^n$,*
- *β is a best reply to μ^n for every $n \in \mathbb{N}$.*

Definition 2. *We say that (μ, σ, β) is an RVP equilibrium if it is a BNE and β is RVP.*

Like Ravid's (2020) credible equilibrium, RVP is weaker than Selten's (1975) trembling-hand perfection. RVP allows for belief perturbations to vary with off-path deviations while trembling hand perfection does not.[9] Moreover, RVP allows for correlated belief perturbations.

We solve our duopoly model using RVP equilibrium (hereafter, just *equilibrium*). Despite its weakness, this refinement is strong enough to obtain sharp predictions regarding the offers accepted by the consumer on-path and the overall trade probability. These variables are sufficient to characterize the most important economic statistics of our model: The industry profits and the consumer surplus.

3.1 Consumer's Best Response

We study the implications of RVP for the best response of the consumer.

Necessity. We show that if β is an RVP best response to μ, then it displays *everywhere* a multinomial logit formula adjusted for the consumer's prior belief. Multinomial logit characterizes optimal behavior in decision problems with rational inattention. However, as Matějka and McKay (2015) show, this property needs to hold only μ-almost surely. Since beliefs are endogenous in a BNE, this characterization is too weak to avoid the trivial multiplicity of equilibria we have previously discussed. The RVP refinement addresses this issue: Robustness uniquely identifies the consumer's best response β for both conjectured and μ-null states.

Lemma 1. *Let μ be consistent with strategy profile σ and β an RVP best response to μ. Then, for every $v \in V$ and $x_1, x_2 \geq 0$ we have*

$$\beta_i(v, x_1, x_2) = \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2}, \ (i \in I) \tag{2}$$

where (π_1, π_2) solves

$$\max_{p_1, p_2 \geq 0} \mathbb{E}_\mu \left[\log \left(p_1 \cdot e^{\frac{v-x_1}{k}} + p_2 \cdot e^{\frac{v-x_2}{k}} + (1 - p_1 - p_2) \right) \right] \quad \textit{subject to} \quad p_1 + p_2 \leq 1. \tag{3}$$

[9] Formally, RVP requires that for every state, a sequence of vanishing belief perturbations exists such that (i) the sequence put a positive probability on that state, and (ii) β is a best reply to every element of the sequence. Instead, trembling hand perfection requires that a sequence of vanishing *full-support* belief perturbations exists such that β is a best reply to every element of the sequence.

Moreover, if $\pi_i > 0$ for some $i \in I$, then

$$\pi_i = \mathbb{E}_\mu[\beta_i]. \tag{4}$$

Intuition for the lemma is as follows. By definition, β is a best response to μ, thereby displaying the multinomial logit formula adjusted for (π_1, π_2) μ-almost everywhere. Each π_i describes the consumer's *trade engagement level* with firm $i \in I$. Problem (3) and equation (4) state that, for every $i \in I$, the trade engagement level π_i is chosen optimally on-path and equals the average probability of purchasing the good from firm $i \in I$. The crucial feature of Lemma 1 is the fact that the consumer's best response is identified everywhere. This follows from the fact that for each state (v, x_1, x_2), robustness requires that β is a best response to some vanishing perturbation that places strictly positive probability on that state.

Symmetry. Consider the following definition.

Definition 3. *We say that*

1. *$\sigma = (\sigma_1, \sigma_2)$ is symmetric if $\sigma_1 = \sigma_2$.*
2. *β is symmetric if for each $v \in V$ and $x_1, x_2 \geq 0$, we have $\beta_i(v, x_1, x_2) = \beta_{-i}(v, x_2, x_1)$.*

We say that (μ, σ, β) is a symmetric assessment if μ is consistent with σ, and both σ and β are symmetric.

As the Appendix shows, it is without loss to restrict the equilibrium analysis of competition to symmetric assessments. The argument is as follows. First, competition plays a role in equilibrium only when both firms *actively trade* with the consumer. Otherwise, the unique *active firm* behaves like a monopolist, making de-facto inconsequential the presence of the competitor. Furthermore, as firms are ex-ante identical, the consumer cannot trade with them asymmetrically. Intuitively, if the consumer traded with firm 1 *more often*, i.e., $\pi_1 > \pi_2 > 0$, firm 1 would charge higher equilibrium prices. However, this would induce the consumer to trade *less often* with firm 1. Otherwise, the trade engagement level would not be optimal for the consumer, and equation (4) would not be satisfied. Thus, any equilibrium assessment where firms actively compete must feature a consumer's recommendation strategy that is symmetric. In turn, this implies a symmetric and pure equilibrium play from the firms.[10]

Since symmetry is without loss whenever both firms are active, we call *equilibrium outcome* the pair $(\psi, \xi) \in [0,1] \times X^V$, where $\psi = \pi_1 + \pi_2$ is the sum of the consumer's trade engagement levels,[11] and $(\xi(v))_{v \in V}$ are the equilibrium offers accepted on-path by the consumer. We say that two assessments are *outcome equivalent* if they imply the same equilibrium outcome. Abusing notation, the equilibrium outcome associated with a no-trade equilibrium is $(0, \emptyset)$.

Full characterization. The above discussion follows from the fact that the consumer's best response satisfies the multinomial logit formula of equation (2). Any application of Lemma 1, however, relies on the existence of vanishing belief perturbations that justify β everywhere. The

[10] This follows from Milgrom and Roberts (1990) and the fact that the two firms face a symmetric logit demand. See the Appendix for formal proofs and discussion.

[11] That is, the ex-ante overall probability of trade in equilibrium.

next result shows that such perturbations exist for the symmetric case. This fully characterizes the consumer's symmetric RVP best responses to beliefs that are consistent with symmetric strategy profiles of the sellers.

Lemma 2. *Let μ be consistent with symmetric strategy profile σ, and β be symmetric. Then, β is a RVP best response if and only if for every $v \in V$ and $x_1, x_2 \geq 0$, we have*

$$\beta_i(v, x_1, x_2) = \frac{\pi \cdot e^{\frac{v-x_i}{k}}}{\pi \cdot \sum_{j=1,2} e^{\frac{v-x_j}{k}} + 1 - 2\pi}, \ (i \in I) \tag{5}$$

where

(i) $\pi = \mathbb{E}_\mu[\beta_i] \in [0, 1/2]$ *for every* $i \in I$.

Moreover, exactly one of the following statements is true.

(ii) $\pi = 0$, *and* $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \leq 1$ *for every* $i \in I$.

(iii) $\pi = 1/2$, *and* $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right] \leq 1/2$.

(iv) $\pi \in (0, 1/2)$, $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \geq 1$ *for every* $i \in I$, *and* $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right] \geq 1/2$.

The only if direction of lemma 2 is a specialization of Lemma 1 to the symmetric case. Equation (5) is the symmetric counterpart of equation (2), while points (i), (ii), (iii) and (iv) characterize the optimal symmetric trade engagement levels when μ is consistent with a symmetric strategy profile of the firms.[12] The proof for the other direction proceeds by finding vanishing perturbations that justify β as a best response for the consumer everywhere. Finding such a sequence of beliefs is possible since the equilibrium refinement we use is relatively weak and allows for potentially correlated deviations of the beliefs about firms' offers.

4 Collusion

To understand the equilibrium effects of competition, we formulate a benchmark case that allows firms to collude. This model is identical to the one of Section 2, except for firms' incentives. As it is standard in industrial organization, we model collusion by letting firms internalize each others' profits. For every $i \in I$, the payoff function is

$$\Pi_i^C = \Pi^C := \sum_{j \in I} \beta_j(v, x_1, x_2) \cdot x_j,$$

where C stands for "Collusion." We refer to this new model as the *collusion model*, and to the one presented in Section 2 as the *competition model.* In the collusion model, firms perfectly internalize each other's profits, leading to identical payoff functions ($\Pi_i^C = \Pi^C$ for every $i \in I$). In the competition model, the competing firms adopt standard profit-maximizing behavior. Demand is identical in both models. Since the analysis of collusion is similar to the one of

[12]There always exists a symmetric solution to (3) as the products offered by the two firms are *a priori homogeneous* when μ is symmetric. See Section IIB in Matějka and McKay (2015) for a relevant discussion.

competition described in Section 5, our discussion here is relatively informal. See the Appendix and Ravid (2020) for more details.

The collusion model is *outcome equivalent* to the monopoly model of Ravid (2020).[13] Intuition is as follows. If the consumer trades with only one of the two firms in equilibrium, the collusion model is *de facto* equivalent to a monopoly model. As the demand for the good produced by one of the two firms is null, the internalization effect plays no role, and the unique active firm has the same incentives as the monopolist. At the same time, the inactive firms' price is as high as possible to increase the active partner's demand, reinforcing the consumer's decision not to engage in trade. On the other hand, if both firms are active in equilibrium and the consumer's best response is robust, then her attention strategy satisfies the adjusted multinomial logit functional form of equation (2) where $\pi_1, \pi_2 > 0$. Under this condition, in the unique BNE, sellers' offers are symmetric and, more importantly, equal to the monopolist's offer of Ravid's (2020) model when facing *aggregate demand*. Intuitively, when firms perfectly internalize each other's profits, they have no incentive to charge different prices. At the same time, when the two colluding firms charge the same price they face the same aggregate demand as the monopolist. As a result, they act as if they were serving the consumer in a monopoly market. This implies that the analysis of Ravid (2020) applies verbatim to our collusion model.

Let $k^* > 0$ be the unique solution to the following equation

$$\mathbb{E}_\lambda \left[e^{v/k-1} \right] = 1. \tag{6}$$

The following result characterizes the main equilibrium predictions of the collusion model.

Theorem 1. *In the collusion model, a unique trading equilibrium outcome exists if and only if $k < k^*$. Moreover, whenever it occurs, equilibrium trade is inefficient. That is, $\pi_1 + \pi_2 < 1$.*

Theorem 1 emphasizes two main features of the collusion model's trade equilibrium outcome. First, trade cannot be sustained as an equilibrium outcome if attention costs are too high. Second, equilibrium trade is never efficient: The probability that the consumer buys the product is always smaller than 1. Both features follow from the fact that the RVP equilibrium is outcome equivalent to the monopoly model of Ravid (2020). The elemental force at play is that the consumer does not process enough information to sustain trade. If information costs are too high, the consumer does not pay enough attention to prevent firms from overcharging. On the other hand, suppose the consumer trades with probability one. Always accepting either offer is equivalent to committing to never using the no-trade *outside option*. In a robust equilibrium, this would mean that the consumer disregards learning about prices in absolute terms. As a result, colluding firms would coordinate a rapid price increase, making equilibrium offers too unappealing to sustain trade.

Two remarks are in order. First, while the equivalence between the models of colluding firms and monopoly is straightforward without rational inattention, it is not immediate in the presence of information processing costs. Indeed, colluding firms may use two different prices to influence

[13]It turns out that firms submit the same offer if both are active. Therefore, the equilibrium outcome in the collusion model can still be described by the overall probability of trade and the offers accepted by the consumer on-path.

the consumer's attention. However, firms' optimal behavior excludes charging different prices in case firms are both active. This implies that the strategic manipulation of attention using two prices does not bite in equilibrium. Second, the results of this section do not depend on the particular interpretation of collusion we employ. As we show in the Appendix, our results hold whether the two firms set the prices together or independently as long as they internalize each other's profits.

5 Competition

Under our equilibrium refinement, competition, like collusion, delivers a finite multiplicity of equilibrium outcomes. First, there always is the uninteresting no-trade equilibrium outcome. In this case, the firms overcharge the consumer, who has no incentive to trade. Furthermore, there is a class of equilibria where the consumer only trades with one firm. Not surprisingly, this class is outcome equivalent to the unique trading equilibrium of the monopoly model of Ravid (2020) and, therefore, to all trading equilibria of the collusion model. Finally, there is a robust equilibrium where the consumer trades with both firms. Since firms actively compete on-path, we call this equilibrium the *competitive equilibrium*. In the remainder of this section, we characterize its properties.

5.1 Firms' Competitive Behavior

As argued in Section 3, every competitive equilibrium must be symmetric. Given the consumer's best reply of Lemma 2, firms behave as if they are facing a symmetric downward sloping multinomial logit demand. The following lemma characterizes the firms' equilibrium strategies.

Lemma 3. *Suppose (μ, σ, β) is a competitive equilibrium. Then, given $v \in V$, each seller $i \in I$ plays a symmetric pure strategy $\sigma_i(\cdot|v) = \delta_{x(v)}$ given by*

$$x(v) = k \cdot (1 + \phi(v)) \tag{7}$$

where $\phi(v)$ is the unique solution to

$$\left(1 + e^{\phi} \cdot \frac{1 - 2\pi}{\pi e^{\frac{v-k}{k}}}\right) \phi = 1. \tag{8}$$

Lemma 3 follows from the fact that firms are not facing a perfectly elastic demand, even though their products are perfectly homogeneous. Intuitively, it is too costly for the consumer to process information about ex-post trade gains perfectly. As a result, the consumer does not know with certainty which offers are best and does not always accept the best option. Interestingly, because the information is noisy, the consumer behaves as if the products were not perfect substitutes. This explains why, according to equation (7), the firms can charge positive prices in equilibrium.

The relationship between the consumer's trade engagement level and firms' offers is crucial. The function ϕ captures the price-setting incentives of the firms. Note that for fixed v and k, equation (8) shows that $\phi(v)$ is increasing in the consumer's trade engagement level π. This shows that π is at the heart of the competitive equilibrium analysis of our model: If firms submit appealing offers to the consumer, the consumer chooses a high trade engagement level, in line with equation (4). At the same time, if the consumer engages more in trade, demand expands and the elasticity of demand goes down. As a result, the firms submit worse equilibrium offers.

5.2 Existence and Uniqueness

The next theorem characterizes the conditions under which trade can be sustained in a competitive equilibrium. It identifies a region of attention costs for which there exists a trade engagement level that is optimal given the optimal offers it induces.

Theorem 2. *Let k^* be defined as in equation (6). A competitive equilibrium exists if and only if $k < k^*$. If a competitive equilibrium exists, it is unique.*

If attention costs are too high trade cannot be sustained. The consumer is not willing to process any information, and therefore demand does not change with firms' offers. As a result, the firms overcharge the consumer, leading to a breakdown of trade. Conversely, if attention costs are moderately low, a trade equilibrium exists. The consumer is willing to process enough information to find the best offer, implying that the firms face a steep demand curve. This leads firms to make more appealing offers to the consumer. For such low attention costs, there is a unique competitive equilibrium. For instance, suppose there is a second equilibrium in which the consumer's overall trade engagement level is higher. Due to this expansion in demand, firms' marginal revenue is higher everywhere. As a result, firms make less appealing offers when compared to the original equilibrium. This induces the consumer to *reduce* the overall trade engagement level, a contradiction.

As discussed in Section 4, when firms collude there exists an equilibrium with at least one active firm if and only if $k < k^*$. An immediate and perhaps surprising implication of Theorem 2 is that competition does not imply that trade can be sustained in equilibrium if, with collusion, it could not. This is because the consumer processes less information as k grows. In particular, for unit attention costs above k^*, it is not optimal for the consumer to process any information. This includes information about which firm makes the better offer. Therefore, when attention costs are too high, competition has no effect and does not help overcome cases in which trade breaks down.

6 Equilibrium Effects of Competition

This section explains the effects of competition in the presence of information processing costs. To this goal, we compare the equilibrium outcomes of the competitive and the collusion equilibrium (superscript B and C, respectively).

Competition can alleviate the efficiency losses from costly information processing. Recall that trade is always inefficient in the collusion model. The following proposition shows that for any $k \in (0, k^*)$, the total trading probability in a competitive equilibrium is higher than under collusion. For each firm $i \in I$, denote the aggregate trade engagement level under collusion and competition respectively by $\pi_1^C + \pi_2^C = \pi^M$ and $\pi_1^B + \pi_2^B = 2\pi^B$.

Proposition 1. *For any $k \in (0, k^*)$, $0 < \pi^M < 2\pi^B \leq 1$.*

Intuition for the result is as follows. If each of the two competing firms faced half the demand faced by the monopolist, the resulting offers would be more favorable to the consumer due to the effect of competition: Since they do not internalize each other's profits anymore, firms have increased incentives to charge a lower price. But at these lower prices, the consumer engages in trade more often. As a result, the equilibrium trade probabilities have to satisfy $\pi^M < 2\pi^B$.

It is natural to ask whether competition can completely resolve the inefficiency observed in the collusion model. To this end, we investigate whether an equilibrium with efficient trade exists. We refer to any equilibrium with this property as *sure-trade equilibrium*. By Lemma 3, we deduce that such an equilibrium exists. First, as $\pi > 0$, we have that $\frac{1-2\pi}{\pi} = \frac{1}{\pi} - 2$ is strictly decreasing in π. Thus, $\phi(\pi, v)$ is strictly increasing in π and satisfies

$$\phi(\pi, v) \in \left(\lim_{\pi \downarrow 0} \phi(\pi, v), \phi(1/2, v)\right] = (0, 1].$$

Therefore, firms' symmetric offers satisfy $x(v) \in (k, 2k]$ for each $v \in V$. Suppose $x(\cdot)$ is constant and equals $2k$. From Lemma 2, we know that this configuration of prices is consistent with a symmetric equilibrium in which trade occurs with probability 1 if and only if $\mathbb{E}_\lambda \left[e^{2-v/k}\right] \leq 1$. Thus, let $\bar{k} > 0$ be the unique solution to

$$\mathbb{E}_\lambda \left[e^{2-v/k}\right] = 1.$$

Proposition 2. *A (unique) sure-trade competitive equilibrium exists if and only if* $k \leq \bar{k}$.

Competition can restore efficiency if the consumer's unit attention cost is relatively low. Notice, if the consumer trades with probability 1, the consumer's best reply described by equation (5) implies that the consumer does not pay any attention to offers in absolute terms but only to price differences.[14] This attention strategy is not sufficient to prevent firms from submitting unappealing offers under collusion. Firms could coordinate on a simultaneous price increase without impacting consumer's demand. As a result, trade cannot be efficient, as the consumer would be better off never purchasing the product. In contrast, coordination cannot occur under competition because each firm has an incentive to undercut the competitor's offer. This implies that a sure-trade equilibrium is feasible. In particular, as each firm faces the same demand regardless of the product's quality, offers are also constant in the quality level. Moreover, offers strictly increase with $k > 0$. Intuitively, the higher the unit attention cost k is the smaller the consumer's reaction to price changes. This leads firms to charge a higher equilibrium price and explains the existence of the threshold $\bar{k} > 0$ that characterizes equilibrium trade efficiency. If consumer's attention costs exceed $\bar{k}$, the constant price firms would charge becomes too high relative to the expected quality of the good. For this reason, the symmetric sure-trade equilibrium does not exist for large k.

[14] For all $i \in I$, if $\pi = 1/2$, the consumer's symmetric best response β can be written as

$$\beta_i(v, x_1, x_2) = \frac{1}{1 + e^{\frac{x_i - x_j}{k}}}.$$

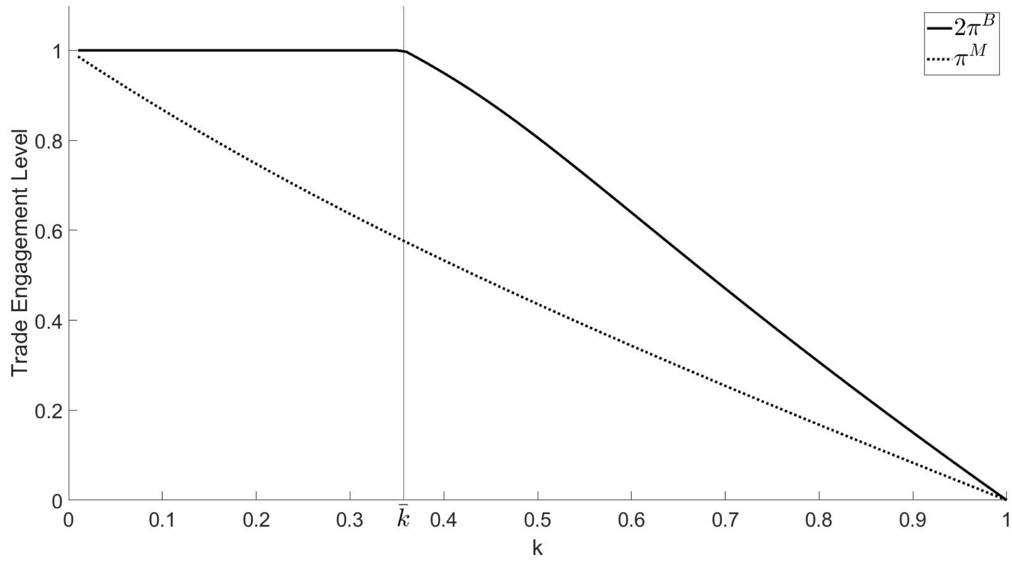


Figure 1: Aggregate trade engagement level in the competitive and collusion equilibrium. The quality of the good v is binary with $v_L = 0.5$ and $v_H = 1.33$, each occurring with equal probability. This normalizes $k^* = 1$.

The threshold $\bar{k}$ is lower than the one characterizing equilibrium trade k^*.[15] Therefore, Theorem 2 implies that whenever a sure-trade competitive equilibrium exists, it is also unique. This underlines the fact that competition is a powerful tool to restore efficiency in markets with rationally inattentive consumers.

Figure 1 shows the equilibrium trade engagement level for the competitive and the collusion equilibrium as a function of $k \in (0, k^*)$ for a binary valuation. The unique competitive equilibrium features sure trade, i.e., $2\pi^B = 1$ for values of $k \leq \bar{k}$. For $k > \bar{k}$, $2\pi^B$ is decreasing in k, but as is shown in Proposition 1, always stays strictly above π^M.

Proposition 2 provides another effective display of the two economic forces at play in the unique competitive equilibrium: The *competition effect* and the *attention effect*. To see this directly, note that the consumer's recommendation strategy in the competitive equilibrium is given by[16]

$$\beta_i(v, x(v), x(v)) = \begin{cases} 1/2 & \text{if } k \leq \bar{k} \\ 1 - k/x(v) & \text{if } k \in (\bar{k}, k^*). \end{cases}$$

When the cost of processing information k is small, equilibrium resembles the standard Bertrand competition outcome. In particular, if $k = 0$ the model is equivalent to the canonical model where price equals marginal cost. For $k \in (0, \bar{k})$, the competition effect is still prevalent: Firms charge a price that does not vary with quality and the consumer buys with certainty. However,

[15]To see that $\bar{k} < k^*$ notice that

$$\mathbb{E}_\lambda\left[e^{2-v/k^*}\right] > \mathbb{E}_\lambda\left[e^{1-v/k^*}\right] = \mathbb{E}_\lambda\left[\frac{1}{e^{v/k^*-1}}\right] \geq \frac{1}{\mathbb{E}_\lambda\left[e^{v/k^*-1}\right]} = 1.$$

[16]Recall that for each $v \in V$, $x(v)$ denotes the symmetric equilibrium offer. The recommendation strategy then follows from $\beta_i(v, x(v), x(v)) = \frac{\phi(v)}{1+\phi(v)}$ for all $i \in I$, and the equilibrium price setting equation (7).

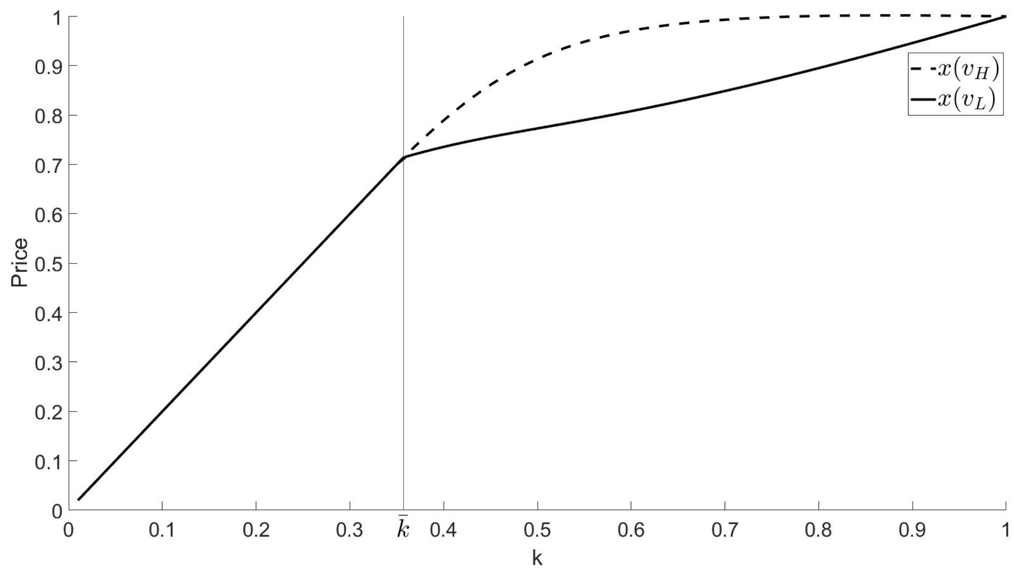


Figure 2: Equilibrium prices in the competitive equilibrium, under the same parameter specification of Figure 1.

the consumer's information processing costs result in the firms having some market power. Therefore, they successfully submit an offer above marginal cost. When k further increases above $\bar{k}$, the attention effect is prevalent and we observe an equilibrium outcome resembling Ravid's (2020) monopoly analysis: Trade with probability 1 cannot be sustained, and firms' offers depend on the quality (see Figure 2). At $k = k^*$, the competition effect on equilibrium price setting disappears completely. At this level of cost, it is not only too costly for the consumer to learn about the product's quality but also to differentiate between prices. As a result, the threshold above which no trade can be sustained remains constant at k^*, regardless of the competition level.

The above discussion also explains why Proposition 2 shows that trade efficiency can be restored, whereas Theorem 2 shows that competition does not extend the range of information costs that support trade in equilibrium compared to collusion. In the limit, i.e., as $k \uparrow k^*$, the industry's behavior is identical in both settings because the competition effect disappears.[17] This implies that equilibrium trade existence is characterized by the same threshold k^*.

Propositions 1 and 2 show that trade efficiency increases with competition. This implies that the sum of industry and consumer surplus, disregarding attention costs, increases. The remainder of this section studies which entity benefits from competition. Without information processing costs, i.e., at $k = 0$, competition benefits the consumer at the expense of the firms. This follows from the standard Bertrand competition model. Since the equilibrium outcomes are continuous in k, it also holds for small information processing costs.

[17] The intuition for which firms' behavior in both frameworks converges as k grows large is as follows. The force that drives prices down under competition, as opposed to collusion, is that firms receive a positive payoff only when they sell their product to the consumer. Thus, each firm's offer must be deemed by the consumer better than both the no-trade outside option and the competitor's offer. However, as the overall trade engagement level decreases, incentives to prevail over the competitor's offer decrease as well since firms focus more on prevailing over the outside option. Therefore, each firm's objectives approximate those of a colluding firm when k increases, implying the result.

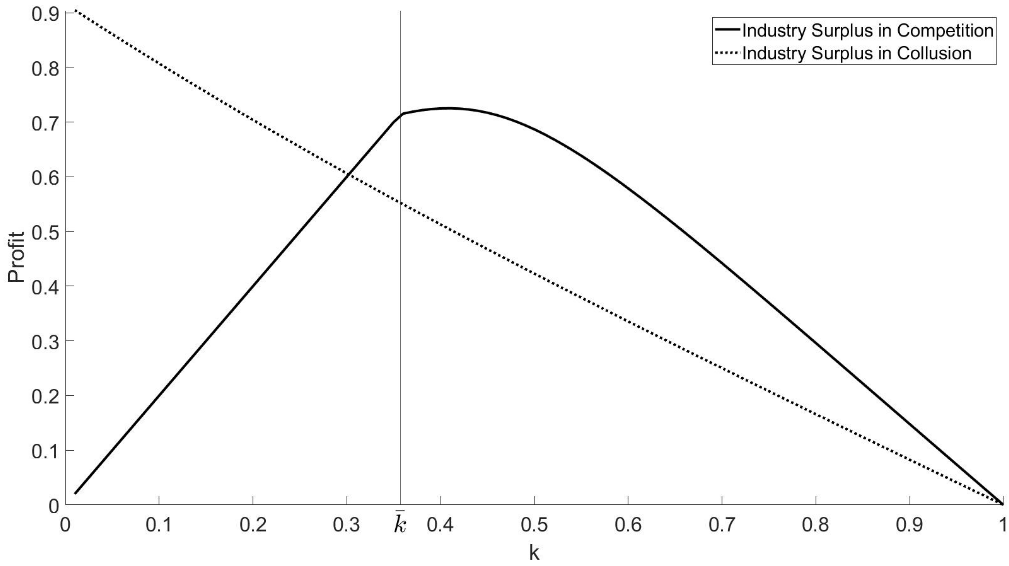


Figure 3: Industry surplus in the competitive and collusion equilibrium, under the same parameter specification of Figure 1.

Our main result shows that for large k the aggregate industry surplus under competition is *higher* than under collusion. Proposition 1 shows that competition increases demand. Below, Theorem 3 adds that when attention costs are high enough, the positive effect that competition has on demand *dominates* the negative effect it has on prices, leading to a higher industry surplus under competition.

For every attention cost $k \in (0, k^*)$, let $\Pi^C(k)$ and $\Pi_1^B(k) + \Pi_2^B(k) = 2\Pi^B(k)$ be the aggregate industry surplus in the collusion and the competitive equilibrium respectively.

Theorem 3. *There exists $\hat{k} \in (0, k^*)$ such that $2\Pi^B(k) > \Pi^C(k) > 0$ for all $k \in (\hat{k}, k^*)$.*

Figure 3 depicts the content of the theorem for one specific distribution of v. In this case, $\hat{k}$ is approximately 0.3. Therefore, the sum of the competitors' profits is larger than the collusive profits for information costs in the interval $(\hat{k}, k^*) = (\approx 0.3, 1)$. Theorem 3 states that such a region can be found for any distribution of v.

The proof of Theorem 3 revolves around the use of *de L'Hopital rule* to prove that

$$\lim_{k \uparrow k^*} \frac{\pi^B(k)}{\pi^M(k)} > \frac{1}{2},$$

even if trading probabilities in both models are converging to zero as $k \uparrow k^*$. This implies that, as k grows large, (i) the behavior of each firm in the competitive equilibrium approximates the equilibrium behavior of a monopolist and, at the same time, (ii) the consumer's equilibrium demand for every single product is strictly more than half the equilibrium demand faced by the monopolist. Given points (i) and (ii), we conclude that for k large enough, the industry surplus under competition is strictly larger than the equilibrium profits under collusion, proving the result.

The driver of this result is that the consumer chooses rationally the amount of information to process and with that her optimal trade engagement level π. If the consumer chose the same attention strategy in both the collusive and competitive settings, the equilibrium prices under competition would be strictly lower. As in any model of competition with fixed demand, the firms would be worse off under competition. However, in equilibrium, the consumer anticipates that competition leads to more favorable terms of trade, and therefore, rationally decides to engage in trade more often, leading to an increase in demand *and* prices. Intuitively, competition serves as a commitment device to avoid overcharging the consumer: It allows supporting higher equilibrium prices through an expansion in demand. For large $k \in (\hat{k}, k^*)$, this positive effect on demand dominates: The firms are better off under competition than under collusion.

Consumer surplus.[18] The increase in industry profits is not necessarily at the expense of the consumer. In the Appendix, we show that consumer surplus is higher under competition whenever the expected prices are lower than in the collusion equilibrium. We show that this is the case for both low ($k \downarrow 0$) and high ($k \uparrow k^*$) levels of information processing costs. We conjecture that consumer surplus is higher under competition for all values of k.

7 Discussion

We conclude by discussing the implication of our main result on the common ownership hypothesis and by extending our analysis to more than two firms.

Common ownership. As shown by Backus, Conlon, and Sinkinson (2021b), the increased concentration in the investment-fund industry and the diffusion of new financial instruments facilitating portfolio diversification have implied a dramatic growth in common ownership in the US stock market over the last forty years. Some economists have regarded this phenomenon as a threat to competition, liking common ownership to increased concentration and lower competition in important American industries.[19] Others, inspired by these early contributions, have challenged the anti-competitive effects of common ownership previously found.[20] These mixed results sparked a fruitful scientific and political debate about the implications of common ownership on prices, quantities, investments, and market entry strategies. Refer to Backus, Conlon, and Sinkinson (2019) for a complete survey of the academic literature on common ownership in economics.

At the heart of such debate is the *common ownership hypothesis*, a theory first advanced by Rotemberg (1984) stating that firms within the same market have fewer incentives to compete if they have common owners. The basic intuition behind Rotemberg's theory is as follows: If managers act to maximize their value to investors, the presence of common owners may induce firms to internalize competitors' profits. Under this circumstance, firms would not act competitively anymore, thus generating possible scenarios of tacit collusion.

Skeptics of the common ownership hypothesis criticize two aspects of the theory. First, the mechanism that makes the theory work. The common ownership hypothesis requires institutional

[18]This subsection is still work in progress.

[19]See the seminal works of Azar (2011) and Azar, M. C. Schmalz, and Tecu (2018), and more recently, Azar, Raina, and M. Schmalz (2022).

[20]Kennedy et al. (2017) and Dennis, Gerardi, and Schenone (2021), among others.

investors to have perfect control over the firms' business strategies, an assumption considered not verified empirically.[21] Second, and more interestingly, the fact that the theory focuses on just one industry. The common ownership hypothesis "fails to account for the effects that weaker competition in one industry may have on other industries and investors who hold diversified portfolios of stocks." As a result, one "cannot simply assume that common shareholders would benefit if competing companies practice anti-competitive pricing.[22]"

Our framework enriches the theoretical debate around the common ownership hypothesis by providing a third reason why anti-competitive concerns associated with common ownership may not be well-grounded. In the presence of rational inattentive consumers, we show that internalizing competitors' profits may produce counterproductive effects on value-oriented common investors with perfect control, even disregarding general equilibrium effects to other industries. To illustrate this point, suppose the same investor (e.g., a holding) owns both firms in a duopoly market. The investor has perfect control over the managers' objectives via designing different managerial payment schemes. For instance, paying bonuses based on individual profits would generate an incentive to compete, while offering remuneration indexed to the industry performance would generate an incentive to collude. If attention costs are high enough, our theory predicts that the value of the investor's portfolio is maximized when the two firms compete. Therefore, a value maximizing investor would design the managers' incentives as if the firms were owned separately. The empirical evidence against the common ownership hypothesis documented by Kennedy et al. (2017), Dennis, Gerardi, and Schenone (2021), and Backus, Conlon, and Sinkinson (2021a) can be explained as the industry's optimal response to the consumers' rational inattention. We leave the empirical verification of the relation between attention and competition to future research.

More than two firms. It is natural to ask how an increase in the number of firms affects the economic outcomes of the model. Since collusion is outcome-equivalent to monopoly, predictions concerning the collusion model are robust to such extension. This is not the case when firms compete. Call *competitive* an equilibrium where *all* firms are active. One can verify that, if there are $N \geq 2$ firms, Theorem 2 and Lemma 3 holds *verbatim* with the exception that (8) must be replaced by

$$1 = \phi \cdot \left((N-1) + e^{\phi} \cdot \frac{1 - N\pi}{\pi e^{\frac{v-k}{k}}} \right). \tag{9}$$

Let $\bar{k}(N) > 0$ be the unique solution to

$$\mathbb{E}_{\lambda} \left[e^{\frac{N}{N-1} - v/k} \right] = 1. \tag{10}$$

The following result characterizes how the consumer's overall engagement level interacts with the number of active firms.

Proposition 3. *In the competitive equilibrium, the consumer's overall trade engagement level increases with N. Moreover, a sure-trade competitive equilibrium exists if and only if $k \leq \bar{k}(N)$.*

Proposition 3 states that equilibrium efficiency expands as the number of active firms increases.

[21] See the March 2017 viewpoint of BlackRock: *Index Investing and Common Ownership Theories.*
[22] *Common Sense Doesn't Support Common Ownership Hypothesis*, January 2022 ICI viewpoint.

Since $\bar{k}(N)$ strictly increases with N, it also affirms that full-efficiency becomes easier to sustain when N grows large. The latter statement is of special importance because it implies that, under some parametric restrictions, the presence of an extra competing firm could restore efficiency in equilibrium. This would be the case if, for example, $N = 2$ and $\bar{k}(2) < k < \bar{k}(3)$.

The intuition behind Proposition 3 is similar to the one given for Propositions 1 and 2. As N grows large, firms have stronger incentives to undercut their competitors' offers, i.e., the competition effect gets more powerful. Anticipating this, the consumer wants to engage in trade more often, implying that, in equilibrium, the overall trade engagement level increases with N.

It is interesting to study what happens in the limit as $N \uparrow \infty$. Let $\bar{k}(\infty) := \lim_{N\uparrow\infty} \bar{k}(N)$. In general, $\bar{k}(\infty) < k^*$ unless v is a degenerate random variable. In other words, an infinite number of firms does not imply efficient trade when the quality of the product is unknown. This is because, even with a large number of firms, market's offers still linearly increase with the information cost parameter $k > 0$. As a result, when information costs are large, market's offers will be too bad on average to sustain a sure-trade equilibrium.

How does Theorem 3 extend to the environment with N-firms? Let $\Pi^B(N)$ be the expected profit earned by each firm in the competitive equilibrium with N firms. One can use the same proof techniques developed earlier to show the following proposition.[23]

Proposition 4. *Let $N > M \geq 2$. There exists $\hat{k} \in (\bar{k}(N), k^*)$ such that $N \cdot \Pi^B(N) > M \cdot \Pi^B(M)$ for all $k \in (\hat{k}, k^*)$.*

In Section 6 we discussed how competition can serve as a commitment device that allows supporting higher equilibrium prices through an expansion in demand. Proposition 4 adds to this discussion that as $k \uparrow k^*$, such positive effect on demand is stronger the larger the number of active firms. This implies that for any number of firms N, there is a region of parameters k where the addition of a new competitor strictly increase total industry surplus.

[23]The competitive equilibrium must be symmetric even with $N \geq 2$ firms. Thus, $N \cdot \Pi^B(N)$ represents the industry surplus in the competitive equilibrium with N active firms.

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Appendix

The Appendix is structured as follows: Appendix A includes all the omitted proofs from the main text. Appendices B and C pertain to our *competition model.* Appendix B shows that the restriction to pure strategies of firms is without loss in any competitive equilibrium, whereas Appendix C shows that competitive equilibria must be in symmetric assessments only. Appendix D pertains to the *collusion model.* There, we prove the equilibrium outcome-equivalence between Ravid (2020)'s monopoly model, perfect price coordination, and perfect profit internalization, i.e., the collusion model discussed in Section 4. Appendix E discusses changes in consumer surplus due to competition. Finally, proofs of the results presented in the section 7 are in Appendix F.

A Omitted proofs

Proof of Lemma 1

Proof. Let μ be consistent with a strategy profile σ of the sellers. Suppose that β is a best response to μ and that is robust to vanishing perturbations. Since β is a best response to μ, we know from Matějka and McKay (2015) that

$$\beta_i(w, y_1, y_2) = \frac{\pi_i \cdot e^{\frac{w-y_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{w-y_j}{k}} + 1 - \pi_1 - \pi_2}, \ (i \in I)$$

μ-a.s., where each (π_1, π_2) is a solution to problem (3). Now, fix $v \in V$ and $x_1, x_2 \geq 0$ arbitrarily. Then, there exists a sequence $(\mu^n, \tilde{\sigma}^n)$ with the desired properties such that β is a best response to μ^n for every $n \in \mathbb{N}$. Again from Matějka and McKay (2015), we know that β must take the following logit functional form

$$\beta_i(w, y_1, y_2) = \frac{\pi_i^n \cdot e^{\frac{w-y_i}{k}}}{\sum_{j=1,2} \pi_j^n \cdot e^{\frac{w-y_j}{k}} + 1 - \pi_1^n - \pi_2^n}, \ (i \in I) \tag{11}$$

μ^n-a.s. for every $n \in \mathbb{N}$, where each $\pi^n = (\pi_1^n, \pi_2^n)$ is a solution to

$$\max_{p_1, p_2 \geq 0} \mathbb{E}_{\mu^n}\left[\log\left(p_1 \cdot e^{\frac{v-x_1}{k}} + p_2 \cdot e^{\frac{v-x_2}{k}} + (1 - p_1 - p_2)\right)\right] \ \text{subject to} \ \ p_1 + p_2 \leq 1. \tag{12}$$

Since β is a best reply to all μ^n, and $\mu^n(v, x_1, x_2) > 0$ for all n, we know that $(\pi_1^n, \pi_2^n) = (\bar{\pi}_1, \bar{\pi}_2)$ for some $\bar{\pi}_1, \bar{\pi}_2 \in [0, 1]$.

Now, let (v', x_1', x_2') be a generic element in the support of μ. Since $\tilde{\sigma}^n \to \sigma$ strongly implies that $\mu^n \to \mu$ strongly, we have that $\mu^n(\text{Supp}(\mu)) > 0$ for large n. Once again, because β has to be a best response at all n, we know that

$$\beta_i(v', x_1', x_2') = \frac{\bar{\pi}_i \cdot e^{\frac{v'-x_i'}{k}}}{\sum_{j=1,2} \bar{\pi}_j \cdot e^{\frac{v'-x_j'}{k}} + 1 - \bar{\pi}_1 - \bar{\pi}_2} = \frac{\pi_i \cdot e^{\frac{v'-x_i'}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v'-x_j'}{k}} + 1 - \pi_1 - \pi_2}, \ (i \in I).$$

Therefore, $\bar{\pi}_i = \pi_i$ for all $i \in I$. Finally, equation (4) directly follows from Corollary 2 in Matějka and McKay (2015). This concludes the proof of the Lemma. □

Consider the following definition.

Definition 4. *We say that μ is symmetric if for every measurable $A \subseteq V \times X^2$, we have $\mu(A) = \mu(A^{sym})$, where*

$$A^{sym} := \bigcup \{(v, x_1, x_2) : (v, x_2, x_1) \in A\}$$

is the symmetric conjugate of A.

Lemma 4 (Proof omitted.)**.** *Let σ be symmetric, and suppose that μ is consistent with σ. Then, μ is symmetric.*

Proof of Lemma 2

Proof. ("Only if" direction.) Since σ is symmetric, Lemma 4 implies the belief μ is symmetric. As a result, given the concavity of the objective function, it is without loss to focus on solutions (π_1, π_2) to (3) such that $\pi_1 = \pi_2$.[24] Points (i), (ii), (iii) and (iv) follows directly from the use of Lagrangian methods. (See also the results in Matějka and McKay (2015).) We omit the details.

("If" direction.) Let $\beta = (\beta_1, \beta_2)$ be given by (5). Clearly, β is symmetric. Furthermore, given the symmetry of μ, we know from Matějka and McKay (2015) that β is *a* best response to μ.[25] To prove that β is indeed robust to vanishing perturbations we need to distinguish between three cases.

Case 1. Suppose $\pi = \mathbb{E}_\mu[\beta_i] \in (0, 1/2)$. At the end of this proof, Lemma 5 shows that the other conditions displayed in statement (iv) of Lemma 2 are redundant. Fix $v \in V$ and $x_1, x_2 \geq 0$ arbitrarily. Let

$$A := \frac{1}{2}\beta_1(v, x_1, x_2) + \frac{1}{2}\beta_1(v, x_2, x_1) > 0.$$

Note that $A = \frac{1}{2}\beta_2(v, x_1, x_2) + \frac{1}{2}\beta_2(v, x_2, x_1)$ due to symmetry. Also, observe that $\beta_i(v, x, x)$ is strictly decreasing in $x \geq 0$ for each $v \in V$ and $i \in I$, and that $\beta_i(v, v, v) = \pi$.

- Suppose $A < \pi$. Then, there exists $\alpha \in (0, 1)$ and $\varepsilon > 0$ such that (i) $v - \varepsilon \geq 0$ and (ii) $\alpha A + (1 - \alpha)\beta_i(v, v - \varepsilon, v - \varepsilon) = \pi$ for each $i \in I$. Let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v', v')}$ if $v' \neq v$, and

$$\tilde{\sigma}'(\cdot|v') = \alpha\left(\frac{1}{2}\delta_{(x_1, x_2)} + \frac{1}{2}\delta_{(x_2, x_1)}\right) + (1 - \alpha)\delta_{(v-\varepsilon, v-\varepsilon)}$$

  when $v' = v$. Then, for each $n \in \mathbb{N}$, let $\tilde{\sigma}^n = \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$. By construction, $\tilde{\sigma}^n \to \sigma$ strongly and $\tilde{\sigma}^n(x_1, x_2|v) > 0$ for every $n \in \mathbb{N}$. Let μ^n be consistent with $\tilde{\sigma}^n$. We now show that β is a best response to μ^n for each $n \in \mathbb{N}$. Each μ^n is symmetric, implying that

[24] If (π_1, π_2) is a solution to (3) with $\pi_1 \neq \pi_2$, the symmetry of μ implies that (π_2, π_1) is a distinct solution. Because of concavity, $\frac{1}{2}(\pi_1, \pi_2) + \frac{1}{2}(\pi_2, \pi_1)$ is then another (symmetric) solution to (3), proving the assertion.

[25] More precisely, β is the unique best response to μ in the class given by (1) if $\sigma_i(\cdot|v) = \sigma_0(\cdot|v) \in \Delta(X)$ is a non-degenerate probability measure for at least one $v \in V$. Otherwise, the problem of the consumer has a continuum of solutions, and the β given by (5) represents the unique solution in the class given by (1) satisfying symmetry.

it is without loss of generality to focus on symmetric best responses. Moreover, we have $\mathbb{E}_{\mu^n}[\beta_i] = \pi$ for each $i \in I$. The result now follows from Lemma 1, Lemma 5 and Corollary 2 of Matějka and McKay (2015).

- Suppose $A > \pi$. Then, there exists $\alpha \in (0,1)$ and $\varepsilon > 0$ such that $\alpha A + (1-\alpha)\beta_i(v, v+\varepsilon, v+\varepsilon) = \pi$ for each $i \in I$. Let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v',v')}$ if $v' \neq v$, and
$$\tilde{\sigma}'(\cdot|v') = \alpha\left(\frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(x_2,x_1)}\right) + (1-\alpha)\delta_{(v+\varepsilon,v+\varepsilon)}$$
when $v' = v$. For each $n \in \mathbb{N}$, let $\tilde{\sigma}^n = \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$. Once again, $\tilde{\sigma}^n \to \sigma$ strongly and $\tilde{\sigma}^n(x_1, x_2|v) > 0$ for every $n \in \mathbb{N}$. Moreover, if we let μ^n be the belief consistent with $\tilde{\sigma}^n$, we obtain that μ^n is symmetric, and satisfies $\mathbb{E}_{\mu^n}[\beta_i] = \pi$ for each $i \in I$ and $n \in \mathbb{N}$. Like before, this implies that β is a best response to μ^n for each $n \in \mathbb{N}$ as required.

- Suppose $A = \pi$. There exist $\alpha \in (0,1)$ and $\varepsilon > 0$ such that (i) $v - \varepsilon \geq 0$ and (ii) $\alpha\beta_i(v, v+\varepsilon, v+\varepsilon) + (1-\alpha)\beta_i(v, v-\varepsilon, v-\varepsilon) = \pi$. Now, let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v',v')}$ if $v' \neq v$, and
$$\tilde{\sigma}'(\cdot|v') = \frac{1}{2}\left(\frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(x_2,x_1)}\right) + \frac{1}{2}(\alpha\delta_{(v+\varepsilon,v+\varepsilon)}(1-\alpha)\delta_{(v-\varepsilon,v-\varepsilon)})$$
when $v' = v$. For each $n \in \mathbb{N}$, let $\tilde{\sigma}^n = \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$. One can easily verify that $\tilde{\sigma}^n \to \sigma$ strongly and $\tilde{\sigma}^n(x_1, x_2|v) > 0$ for every $n \in \mathbb{N}$. Moreover, letting μ^n be the belief consistent with $\tilde{\sigma}^n$, once again we obtain that μ^n is symmetric, and it holds that $\mathbb{E}_{\mu^n}[\beta_i] = \pi$ for each $i \in I$ and $n \in \mathbb{N}$. This implies that β is a best response to μ^n for each $n \in \mathbb{N}$ as required.

Case 2. Suppose $\pi = 0$, so that $\beta_1 = \beta_2 = 0$. We must have $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \leq 1$ for every $i \in I$. Fix $v \in V$ and $x_1, x_2 \geq 0$ arbitrarily. Let
$$A := \frac{1}{2}e^{\frac{v-x_1}{k}} + \frac{1}{2}e^{\frac{v-x_2}{k}} > 0.$$

- If $A \leq 1$, let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v',v')}$ if $v' \neq v$, and
$$\tilde{\sigma}'(\cdot|v') = \frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(x_2,x_1)}$$
when $v' = v$, and define $\tilde{\sigma}^n = \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$ for each $n \in \mathbb{N}$. We get that $\tilde{\sigma}^n \to \sigma$ strongly and $\tilde{\sigma}^n(x_1, x_2|v) > 0$ for every $n \in \mathbb{N}$. Let μ^n be consistent with $\tilde{\sigma}^n$. Each μ^n is symmetric, implying that it is without loss of generality to focus on symmetric best responses. By construction, for every $n \in \mathbb{N}$, we have $\mathbb{E}_{\mu^n}\left[e^{\frac{v-x_i}{k}}\right] \leq 1$ for each $i \in I$. This implies that β is a best reply to μ^n for every $n \in \mathbb{N}$ as required.

- If $A > 1$, there exists $\alpha \in (0,1)$ and $\varepsilon > 0$ such that $\alpha A + (1-\alpha)e^{-\varepsilon/k} \leq 1$. Let $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v') = \delta_{(v',v')}$ if $v' \neq v$, and
$$\tilde{\sigma}'(\cdot|v') = \alpha\left(\frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(x_2,x_1)}\right) + (1-\alpha)\delta_{(v+\varepsilon,v+\varepsilon)}$$

when $v' = v$, and for each $n \in \mathbb{N}$, let $\tilde{\sigma}^n = \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$. We get that $\tilde{\sigma}^n \to \sigma$ strongly and $\tilde{\sigma}^n(x_1, x_2|v) > 0$ for every $n \in \mathbb{N}$. Let μ^n be consistent with $\tilde{\sigma}^n$. Once again, μ^n is symmetric. Moreover, for every $n \in \mathbb{N}$, we have $\mathbb{E}_{\mu^n}\left[e^{\frac{v-x_i}{k}}\right] \leq 1$ for each $i \in I$. This implies that β is a best reply to μ^n for every $n \in \mathbb{N}$ as required.

Case 3. The proof for the case $\pi = 1/2$ is similar to the one for Case 2. The details are omitted. $\square$

The following Lemma was invoked during the proof of Lemma 2.

Lemma 5. *Let μ be symmetric, and β be given by (5). If $\pi = \mathbb{E}_\mu[\beta_i] \in (0, 1/2)$ for every $i \in I$, then $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \geq 1$ for every $i \in I$, and $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right] \geq 1/2$.*

Proof. For every $y > 0$ and $\gamma \in (0, 1/2)$, let

$$g(y, \gamma) = \frac{1 - 2\gamma}{\gamma y + (1 - 2\gamma)},$$

and

$$h(y, \gamma) = \frac{\gamma}{\gamma + (1 - 2\gamma)y}.$$

Observe that g and h are strictly decreasing and strictly convex in $y > 0$ for every $\gamma \in (0, 1/2)$. Moreover, $g(y, \gamma) = 1 - 2\gamma$ if and only if $y = 2$, and $h(y, \gamma) = 2\gamma$ if and only if $y = 1/2$.

From Jensen's inequality, we have

$$2\pi = \mathbb{E}_\mu[\beta_1 + \beta_2] = \mathbb{E}_\mu\left[h\left(\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}, \pi\right)\right] \geq h\left(\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right], \pi\right),$$

which implies that $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right] \geq 1/2$ because h is strictly decreasing in $y > 0$. Similarly,

$$1 - 2\pi = 1 - \mathbb{E}_\mu[\beta_1 + \beta_2] = \mathbb{E}_\mu\left[g\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}, \pi\right)\right] \geq g\left(\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right], \pi\right),$$

which implies that $\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right] \geq 2$. Since μ is symmetric, $\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}}\right] = \mathbb{E}_\mu\left[e^{\frac{v-x_2}{k}}\right]$. Therefore, $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \geq 1$ for every $i \in I$ as required. $\square$

Proof of Lemma 3

Proof. A profile (μ, β, σ) is a competitive equilibrium if and only if it is symmetric trading equilibrium. That is, (a) μ is consistent with σ, (b) β is given by (5) with $\pi > 0$, and (c) σ is symmetric and σ_i is a best response to σ_{-i} given β.

We focus on the equilibrium behavior of the firms. Fix $v \in V$ arbitrarily. As Appendix B shows, for fixed symmetric logit demand β of the consumer, the unique equilibrium of the pricing game played by the firms is pure and symmetric. To characterize it, suppose (b) holds and let

$\sigma_i(\cdot|v) = \delta_{x_i(v)}$ for all $i \in I$. Then, taking $\pi \in (0, 1/2]$ and $-i$'s offer $x_{-i}(v) = x_{-i}$ as given, firm i maximizes profits given the demand function $\beta_i(v, x_i, x_{-i})$ as follows:

$$\max_{x_i \geq 0} \frac{x_i \cdot \pi e^{\frac{v-x_i}{k}}}{\pi \cdot \left(e^{\frac{v-x_i}{k}} + e^{\frac{v-x_{-i}}{k}}\right) + 1 - 2\pi}$$

The first order condition is given by

$$\frac{d\Pi_i(v, x_1, x_2)}{dx_i} = \frac{\pi e^{\frac{v-x_i}{k}}}{\sum_{j \in 1,2} \pi e^{\frac{v-x_j}{k}} + 1 - 2\pi} + \frac{-x_i \cdot \pi \frac{1}{k} e^{\frac{v-x_i}{k}} (\pi e^{\frac{v-x_{-i}}{k}} + 1 - 2\pi)}{(\sum_{j \in 1,2} \pi e^{\frac{v-x_j}{k}} + 1 - 2\pi)^2} = 0$$

We can rewrite this as

$$\frac{d\Pi_i(v, x_1, x_2)}{dx_i} = \beta_i(v, x_1, x_2) \left(1 - \frac{x_i}{k} \cdot [1 - \beta_i(v, x_1, x_2)]\right) = 0. \tag{13}$$

Note that $\beta_i > 0$ for all $x_i \geq 0$ and $\lim_{x_i \to \infty} \beta_i = 0$. Therefore, $x_i \mapsto \frac{d\Pi_i(v, x_1, x_2)}{dx_i}$ crosses zero exactly once from above. It follows that $x_i \mapsto \Pi_i(v, x_i, x_{-i})$ admits a unique (interior) global maximum characterized by the first order condition. We rearrange (13) and use symmetry to see that, in equilibrium, $x_i(v) = x_{-i}(v) = x(v)$ satisfies:

$$x(v; \pi) = k \cdot \left[1 + \frac{1}{1 + \frac{1-2\pi}{\pi e^{\frac{v-x(v;\pi)}{k}}}}\right].$$

Define $\phi(v; \pi) = \frac{1}{1 + \frac{1-2\pi}{\pi e^{\frac{v-x(v;\pi)}{k}}}}$. The equilibrium firm behavior is then given by

$$x(v; \pi) = k \cdot [1 + \phi(v; \pi)],$$

where optimality requires that

$$\left(1 + e^{\phi(v;\pi)} \frac{1 - 2\pi}{\pi e^{\frac{v-k}{k}}}\right) \phi(v; \pi) = 1.$$

The above equation uniquely pins down $\phi(\pi, v)$ in $\mathbb{R}_+$: The LHS goes to 0 as $\phi \to 0$, and it goes to ∞ as $\phi \to \infty$. Furthermore, the LHS is continuously increasing in ϕ.

Lastly, note that $\phi(v; \pi) = \frac{\beta(v, x(v;\pi), x(v;\pi))}{1 - \beta(v, x(v;\pi), x(v;\pi))}$ for all $v \in V$, where $\beta_i = \beta$ because of symmetry. $\square$

Proof of Theorem 2

Proof. We start with the proof of the if and only if statement. The necessity direction is easy to prove and an argument identical to the one given in Ravid (2020) suffices. As we argued after Lemma 3, in any competitive equilibrium, the sellers charge a price $x_i(v) = x(v)$ strictly above k for each $v \in V$. Now, suppose by way of contradiction that a trading equilibrium exists but

$k \geq k^*$, i.e., $\mathbb{E}_\lambda \left[e^{v/k-1}\right] \leq 1$. In equilibrium, we would have

$$\mathbb{E}_\mu \left[e^{\frac{v-x(v)}{k}}\right] < \mathbb{E}_\lambda \left[e^{v/k-1}\right] \leq 1.$$

This is in contradiction with our hypothesis of on-path equilibrium trade. Indeed, according to Lemma 2, the consumer's trade engagement level with each firm $i \in I$ would be equal to zero. We conclude that $k < k^*$ is necessary for the existence of a competitive equilibrium.

We now turn to the sufficiency direction. We split the proof in two parts. First, we restrict attention to values of k for which trade occurs with probability 1. We then consider the remaining parameter values. To this end, we need to introduce some further notation. Let $\bar{k}$ be the unique solution to $\mathbb{E}_\lambda \left[e^{2-v/k}\right] = 1$. Notice that

$$\mathbb{E}_\lambda \left[e^{2-v/k^*}\right] > \mathbb{E}_\lambda \left[e^{1-v/k^*}\right] = \mathbb{E}_\lambda \left[\frac{1}{e^{v/k^*-1}}\right] \geq \frac{1}{\mathbb{E}_\lambda \left[e^{v/k^*-1}\right]} = 1.$$

Thus, $0 < \bar{k} < k^*$.

Suppose first that $k \leq \bar{k}$. Take $x(v) = 2k$ for all $v \in V$. Observe that this configuration of prices is an equilibrium of the pricing game played by the firms when they face a symmetric logit demand with $\pi = 1/2$. At the same time, from Lemma 2, a symmetric trade engagement level $\pi = 1/2$ is consistent with this configuration of prices if and only if $\mathbb{E}_\lambda \left[e^{2-v/k}\right] \leq 1$, or equivalently, $k \leq \bar{k}$. Therefore, a symmetric sure-trade equilibrium exists. Since a symmetric sure-trade equilibrium is indeed a competitive equilibrium, we are done.

Now, consider the case where $k \in (\bar{k}, k^*)$. We show that if $k \in (\bar{k}, k^*)$ - or equivalently, $\mathbb{E}_\lambda \left[e^{2/v/k}\right] > 1$ and $\mathbb{E}_\lambda \left[e^{v/k-1}\right] > 1$ -, then a symmetric RVP trading equilibrium where trade occurs with probability strictly between 0 and 1 exists. To do so, define the functions $\phi = \phi(p, v)$, $x = x(p, v)$ and $F = F(p)$ as follows. For each $p \in (0, 1/2]$ and $v \in V$, let $\phi = \phi(p, v)$ be the unique solution to

$$1 = \phi \left(1 + e^{\phi} \frac{1 - 2p}{p \cdot e^{(v-k)/k}}\right), \tag{14}$$

let $x = x(p, v)$ be given by $x := k \cdot (1 + \phi(p, v))$, and finally let $F = F(p)$ be defined as

$$F(p) := \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1 - 2p)}\right]. \tag{15}$$

Since the function $F(\cdot)$ satisfies $F(p) = \frac{1}{p}\mathbb{E}_\lambda \left[\beta_i(v, x(p, v), x(p, v))\right]$ for each $i \in I$, it is sufficient to show that $F(p^*) = 1$ for some $p^* \in (0, 1/2)$. We prove this by relying on the Intermediate Value Theorem, hence exploiting the continuity of $F(\cdot)$ in $p \in (0, 1/2]$. In particular, we show that there exists $0 < p_0 < p_1 < 1/2$ such that for all $p \in (0, p_0)$, we have $F(p) > 1$, and for all $p \in (p_1, 1/2)$, we have $F(p) < 1$.

Existence of $0 < p_1 < 1/2$*:* We exploit the fact that $F(\cdot)$ is continuosly differentiable. This follows from the Implicit Function Theorem that guarantees that $\phi(v, p)$ is continuously differentiable in $p \in (0, 1/2]$ for all $v \in V$.[26] Given that V is finite and $\phi(p, v) \uparrow 1$ as $p \uparrow 1/2$, for every $\varepsilon > 0$ there exists a $\bar{p}_1 \in (0, 1/2)$ such that $\phi(p, v) > 1 - \varepsilon$ for all $v \in V$ and $p \in (\bar{p}_1, 1/2)$. Fix $\varepsilon > 0$ and $\delta > 0$ small enough so that $\mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k}\right] - \delta > 1$, and let $\bar{p}_1$ be the p-threshold corresponding to ε.[27] For every $v \in V$, define

$$A(v) := \max_{p \in [\bar{p}_1, 1/2]} e^{1-v/k+\phi(p,v)} \cdot \frac{\partial}{\partial p}\phi(p, v) \cdot \frac{D_{\max}(p)}{D_{\min}(p)}$$

where

$$D_{\max}(p) := \max_{v \in V} \left(2p + (1 - 2p) \cdot e^{\phi(p,v)+1-v/k}\right)^2 > 0$$

and

$$D_{\min}(p) := \min_{v \in V} \left(2p + (1 - 2p) \cdot e^{\phi(p,v)+1-v/k}\right)^2 > 0.$$

We make two observations.

Obs. 1: Each $A(v)$ is a well-defined real number since it is the maximum value of a continuous function on a compact support. Again by the finiteness of V, there exists $\bar{p}_2 \in (0, 1/2)$ such that $(1 - 2p) \cdot A(v) \leq \delta$ for all $v \in V$ and $p \in (\bar{p}_2, 1/2)$.

Obs. 2: Since $D_{\max}(p), D_{\min}(p) \to 1$ as $p \uparrow 1/2$, we have that $D_{\max}(p)/D_{min}(p) \to 1$ as $p \uparrow 1/2$. Therefore, there exists $\bar{p}_3 \in (0, 1/2)$ such that $D_{\max}(p)/D_{min}(p) \leq 1 + \delta/2$ for all $p \in (\bar{p}_3, 1/2)$.

Now, let $\bar{p} = \max\{\bar{p}_1, \bar{p}_2, \bar{p}_3\} < 1/2$. For all $p \in (\bar{p}, 1/2)$, we have:

$$\begin{aligned}
F'(p) &= \mathbb{E}_\lambda \left[\frac{2 \cdot \left(e^{1+\phi(p,v)-v/k} - 1\right) - (1 - 2p) \cdot e^{\phi(p,v)+1-v/k} \cdot \frac{\partial}{\partial p}\phi(p, v)}{\left(2p + (1 - 2p) \cdot e^{\phi(p,v)+1-v/k}\right)^2}\right] \\
&\geq \mathbb{E}_\lambda \left[\frac{2}{D_{\max}(p)} \cdot e^{1+\phi(p,v)-v/k} - \frac{2}{D_{\min}(p)} - (1 - 2p) \cdot \frac{e^{\phi(p,v)+1-v/k}}{D_{\min}(p)} \cdot \frac{\partial}{\partial p}\phi(p, v)\right] \\
&= \frac{1}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[2 \cdot e^{1+\phi(p,v)-v/k} - 2 \cdot \frac{D_{\max}(p)}{D_{\min}(p)} - (1 - 2p) \cdot e^{\phi(p,v)+1-v/k} \cdot \frac{\partial}{\partial p}\phi(p, v) \cdot \frac{D_{\max}(p)}{D_{\min}(p)}\right] \\
&\geq \frac{1}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[2 \cdot e^{1+\phi(p,v)-v/k} - 2 \cdot \frac{D_{\max}(p)}{D_{\min}(p)} - (1 - 2p) \cdot A(v)\right] \\
&\geq \frac{2}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[e^{1+\phi(p,v)-v/k} - 1 - \delta\right] \\
&\geq \frac{2}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k} - 1 - \delta\right] > 0,
\end{aligned}$$

where the first inequality comes from the fact that $\frac{\partial}{\partial p}\phi(p, v) \geq 0$,[28] and the others are obvious. Since $F(1/2) = 1$ and $F'(p) > 0$ for all $p \in (0, 1/2)$ sufficiently close to $1/2$, the existence of p_1 immediately follows.

[26] More formally, for $\phi \in (0, \infty)$, $v \in V$, and $p \in (0, 1/2 + \tau)$, let

$$G(p, \phi, v) := \phi \cdot \left(1 + e^\phi \cdot (1 - 2p) / \left[p \cdot e^{\frac{v-k}{k}}\right]\right) - 1.$$

Given that V is finite, one can show for a $\tau > 0$ small enough, the assumptions of the Implicit Function Theorem are satisfied by G. Thus, there exists a continuously differentiable function $\bar{\phi}(p, v)$ on $(0, 1/2 + \tau) \times V$ such that $G(p, \bar{\phi}(p, v), v) = 0$ for all $v \in V$ and $p \in (0, 1/2 + \tau)$. As a result, $\bar{\phi}(p, v) = \phi(p, v)$ on $(0, 1/2] \times V$.

[27] Such $\varepsilon, \delta > 0$ exist because $\mathbb{E}_\lambda \left[e^{2-v/k}\right] > 1$ by assumption.

[28] See the proof of Lemma 9.

Existence of $0 < p_0 < p_1 < 1/2$*:* Given that V is finite, $\phi(p,v) \downarrow 0$ and $2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p) \to 1$ as $p \downarrow 0$, for every $\varepsilon > 0$ there exists a $\underline{p} \in (0,1/2)$ such that $\phi(p,v) < \varepsilon$ and $2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p) < 1+\varepsilon$ for all $v \in V$ and $p \in (0,\underline{p})$. Let $\varepsilon > 0$ be small enough so that $\mathbb{E}_\lambda \left[e^{v/k-1-\varepsilon}\right]/(1+\varepsilon) > 1$. (Such an $\varepsilon > 0$ exists because $\mathbb{E}_\lambda \left[e^{v/k-1}\right] > 1$.) For all $p \in (0,\underline{p})$, we have:

$$
\begin{aligned}
F(p) &= \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p)} \right] \\
&\geq \frac{\mathbb{E}_\lambda \left[e^{\frac{v-k}{k}} \cdot e^{-\phi}\right]}{1+\varepsilon} \\
&\geq \frac{\mathbb{E}_\lambda \left[e^{v/k-1-\varepsilon}\right]}{1+\varepsilon} > 1.
\end{aligned}
$$

Thus, a $p_0 \in (0,p_1)$ with the desired properties exists. This concludes the proof of existence of a trading equilibrium.

Uniqueness: Once again, we distinguish between two cases. First, suppose $k \leq \bar{k}$, or equivalently, $\mathbb{E}\left[e^{2-v/k}\right] \leq 1$. From the proof of existence, we know that a sure-trade competitive equilibrium exists. We want to show that no other symmetric trading equilibrium can exist. For each $p \in (0,1/2]$ and $v \in V$, let $\phi = \phi(p,v)$, $x = x(p,v)$, and $F = F(p)$ be defined as above. Note that $F(1/2) = 1$. This again confirms that a sure-trade competitive equilibrium exists because $x(1/2,v) = 2k$ for every $v \in V$ and $\mathbb{E}_\lambda \left[e^{2-v/k}\right] \leq 1$ holds. To prove that no other symmetric trading equilibrium exists, it is sufficient to show that $F(p) \neq 1$ for all $p \in (0,1/2)$. With this goal in mind, first note that $\phi(p,v)$ is strictly increasing in $p \in (0,1/2]$ for every $v \in V$, and that $\phi(1/2,v) = 1$. Thus, given that V is finite, when p is strictly below $1/2$, there exists $\varepsilon > 0$ small enough such that $\phi(p,v) < 1-\varepsilon$ for all $v \in V$. Second, observe that $\mathbb{E}\left[e^{2-c-v/k}\right] < 1$ for any constant $c > 0$. Now, fix $p \in (0,1/2)$ and its corresponding $\varepsilon > 0$. We have

$$
\begin{aligned}
F(p) &= \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p)} \right] \\
&= \mathbb{E}_\lambda \left[\frac{1}{2p + (1-2p) \cdot e^{\phi+1-v/k}} \right] \\
&> \mathbb{E}_\lambda \left[\frac{1}{2p + (1-2p) \cdot e^{2-\varepsilon-v/k}} \right] \\
&\geq \frac{1}{2p + (1-2p) \cdot \mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k}\right]} > 1.
\end{aligned}
$$

where the first strictly inequality comes from $\phi = \phi(p,v) < 1-\varepsilon$ for all $v \in V$, the second is an application of Jensen's inequality, and the last strict inequality is implied by $\mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k}\right] < 1$. Hence, $F(p) \neq 1$ for all $p < 1/2$ as required.

Now, consider the case where $k \in (\bar{k}, k^*)$. Suppose towards a contradiction that there exist $0 < p^* < p^{**} < 1/2$ such that $F(p^*) = F(p^{**}) = 1$. Define $\gamma \in (0,1)$ implicitly by $p^{**} =$

$\gamma p^* + (1-\gamma)1/2$. We have

$$
\begin{aligned}
F(p^{**}) &= \mathbb{E}_\lambda\left[\frac{1}{2p^{**} + (1-2p^{**})\cdot e^{\phi(p^{**},v)+1-v/k}}\right] \\
&= \mathbb{E}_\lambda\left[\frac{1}{2\gamma p^* + 1 - \gamma + \gamma(1-2p^*)\cdot e^{\phi(p^{**},v)+1-v/k}}\right] \\
&< \mathbb{E}_\lambda\left[\frac{1}{1-\gamma+\gamma\left(2p^* + (1-2p^*)\cdot e^{\phi(p^*,v)+1-v/k}\right)}\right] \leq 1.
\end{aligned}
$$

The first inequality follows from the fact that $p^{**} > p^*$ and that $\phi(p, v)$ is strictly increasing in $p \in (0, 1/2)$ for all $v \in V$. In order to prove the second inequality, we define

$$
g(\gamma) := \mathbb{E}_\lambda\left[\frac{1}{1-\gamma+\gamma\left(2p^* + (1-2p^*)\cdot e^{\phi(p^*,v)+1-v/k}\right)}\right].
$$

Note that $g(0) = 1$ and $g(1) = F(p^*) = 1$. It remains to show that $g(\gamma)$ is convex for all $\gamma \in [0, 1]$. Taking the second derivative, we get

$$
g''(\gamma) = \mathbb{E}_\lambda\left[\frac{2\left(2p^* + (1-2p^*)\cdot e^{\phi(p^*,v)+1-v/k} - 1\right)^2}{\left(1-\gamma+\gamma\left(2p^* + (1-2p^*)\cdot e^{\phi(p^*,v)+1-v/k}\right)\right)^3}\right] \geq 0.
$$

Thus, we reached the contradiction that $F(p^{**}) < 1$. We conclude that there is at most one $\hat{p} \in (0, 1/2)$ such that $F(\hat{p}) = 1$. This concludes the proof of uniqueness. □

Proof of Proposition 1

Proof. Fix $k \in (0, k^*)$, and let $(\mu^C, \sigma^C, \beta^C)$ and $(\mu^B, \beta^B, \sigma^B)$ be the unique symmetric equilibrium of the collusion and competition model respectively associated with the cost parameter k. Set $\pi^M = \mathbb{E}_{\mu^C}[\beta^C_1 + \beta^C_2]$ and $\pi^B = \mathbb{E}_{\mu^B}[\beta^B_i]$ for each $i \in I$.

If $k \leq \bar{k}$, the result is an immediate consequence of Proposition 2 and Corollary 1 of Ravid (2020). In words, while a sure-trade equilibrium cannot exist in the collusion model, it is the only competitive equilibrium outcome. Hence, $0 < \pi^M < 1 = 2\pi^B$, as required.

Now assume that $k \in (\bar{k}, k^*)$. In the collusion model,[29] for every $v \in V$, each active firm plays a strategy $\sigma^M(\cdot|v) = \delta_{x^M(v)}$ such that

$$
x^M(v) = k \cdot \left(1 + W\left(\frac{\pi^M}{1-\pi^M}e^{v/k-1}\right)\right), \tag{16}
$$

and $W(\cdot)$ is the Lambert's function.[30] Compared to the equilibrium price formula of the competition model displayed in equation (7), we note that the only difference is in the second multiplicative component, where $W\left(\frac{\pi^M}{1-\pi^M}e^{v/k-1}\right)$ is replaced by $\phi(\pi^B, v)$. As a first step, we

[29] See also Proposition 2 in Ravid (2020).

[30] The Lambert's function is the inverse of the function $z \in \mathbb{R}_+ \mapsto ze^z$.

show that if the *overall* equilibrium trade engagement level was identical in both models, the equilibrium prices under collusion would be strictly higher than the competitive ones.

Lemma 6. *For all $p \in (0, 1/2)$ and $v \in V$, we have $W\left(\frac{2p}{1-2p}e^{v/k-1}\right) > \phi(p, v)$.*

Proof. Fix $p \in (0, 1/2)$ arbitrarily. According to Lemma 3, $\phi = \phi(p, v)$ is the unique solution to equation (8) where π is replaced by p. Note that (8) is equivalent to

$$\frac{p}{1-2p}e^{v/k-1} = \phi \cdot \frac{p}{1-2p}e^{v/k-1} + \phi e^{\phi}.$$

Therefore

$$\frac{2p}{1-2p}e^{v/k-1} > \frac{p}{1-2p}e^{v/k-1} = \phi \cdot \frac{p}{1-2p}e^{v/k-1} + \phi e^{\phi} > \phi e^{\phi}.$$

Applying the Lambert's function on both sides of the above inequality yields the desired result. □

Let $W(2p, v) := W\left(\frac{2p}{1-2p}e^{v/k-1}\right)$ for all $v \in V$ and $p \in (0, 1/2)$. Following the proof of Theorem 1 in Ravid (2020), the overall equilibrium engagement level in the collusion model is given by $\pi^M = 2p^M$, where p^M the unique solution in $(0, 1/2)$ to the equation:

$$G(2p) := \mathbb{E}_\lambda\left[\frac{e^{\frac{v-k}{k}} \cdot e^{-W(2p,v)}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-W(2p,v)} + (1-2p)}\right] = 1. \tag{17}$$

Let F be defined as in the proof of Theorem 2. We have

$$\begin{aligned} 1 &= G(2p^M) \\ &= \mathbb{E}_\lambda\left[\frac{1}{2p^M + (1-2p^M) \cdot e^{W(2p^M,v)+1-v/k}}\right] \\ &< \mathbb{E}_\lambda\left[\frac{1}{2p^M + (1-2p^M) \cdot e^{\phi(p^M,v)+1-v/k}}\right] = F(p^M), \end{aligned}$$

where the strict inequality follows from Lemma 6. From the proof of Theorem 2, we conclude that $p^M < \pi^B$. This is equivalent to say that $\pi^M < 2\pi^B$. □

Proof of Proposition 2

Proof. Follows directly from the proof of Theorem 2. □

Proof of Theorem 3

*Preliminary analysis for the monopoly/collusion model

For each $k \in (0, k^*]$, let $F_k^M : [0, 1) \to \mathbb{R}_+$ be defined as

$$F_k^M(p) := \mathbb{E}_\lambda\left[\frac{1}{p + (1-p) \cdot e^{W(p,v,k)+1-v/k}}\right].$$

As in the previous section, we abuse notation and write $W(p,v,k)$ for $W\left(\frac{p}{1-p}e^{v/k-1}\right)$ and let $W(\cdot)$ be the Lambert's function. We are interested in the solution $p^M(k)$ to the equation $F_k^M(p)=1$. By the Implicit Function Theorem,[31] we know that whenever this solution exists it is continuously differentiable. In his Theorem 1, Ravid (2020) shows that $p^M(k)$ exists uniquely in $(0,1)$ whenever $k\in(0,k^*)$. The following Lemma characterizes additional properties that the solution $p^M(k)$ satisfies as k ranges in $(0,k^*]$.

Lemma 7. *We have:*

(i) $\lim_{k\uparrow k^*} p^M(k)=0$.

(ii) $\lim_{k\uparrow k^*}\frac{\partial}{\partial k}p^M(k)=-\mathbb{E}_\lambda\left[\frac{v}{(k^*)^2}\cdot e^{v/k^*-1}\right]/\mathbb{E}_\lambda\left[\frac{2-e^{1-v/k^*}}{e^{2\cdot(1-v/k^*)}}\right]$.

Proof. (i): Recall from Ravid (2020) that the $F_k^M(\cdot)$ function crosses the line $y=1$ only once from above.[32] Therefore, it is sufficient to show that (#): for every $p\in(0,1/2)$, there exists $k_p\in(0,k^*)$ such that for all k strictly between k_p and k^*, $F_k^M(p)<1$.

Since the Lambert function $W(\cdot)$ is strcitly increasing $W(p,v,k)$ is strictly decreasing in k for every $p\in(0,1/2)$ and $v\in V$. It further satisfies $W(p,v,k)>0$ for all $p\in(0,1/2)$, $v\in V$ and $k>0$. Fix $p\in(0,1/2)$ arbitrarily. Given the finiteness of V, there exists $c_p>0$ such that $W(p,v,k)>c_p$ for all $v\in V$ and $k\in(0,k^*]$. Since $\mathbb{E}_\lambda\left[e^{v/k^*-1}\right]=1$, we have $\mathbb{E}_\lambda\left[e^{v/k^*-1-c_p}\right]<1$. Therefore, continuity implies that there exists k_p strictly between 0 and k^* so that $\mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right]<1$ for all $k\in(k_p,k^*)$. Fix any such k. We have:

$$\begin{aligned}F_k^M(p)&=\mathbb{E}_\lambda\left[\frac{1}{2p+(1-2p)\cdot e^{W(p,v,k)+1-v/k}}\right]\\&\le\mathbb{E}_\lambda\left[\frac{1}{2p+(1-2p)\cdot e^{c_p+1-v/k}}\right]\\&\le 2p+(1-2p)\cdot\mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right]<1.\end{aligned}$$

Thus, (#) holds.

(ii): For each $k\in(0,k^*)$, we totally differentiate the equation $F_k^M(p(k))=1$ to obtain:[33]

$$\frac{\partial}{\partial k}p^M(k)=-\frac{A_M}{B_M}\tag{18}$$

where

$$A_M=\mathbb{E}_\lambda\left[\frac{v\cdot(1-p^M(k))\cdot e^{W(p^M(k),v,k)+1-v/k}}{k^2\cdot D_M^2\cdot(1+W(p^M(k),v,k))}\right],$$

$$B_M=\mathbb{E}_\lambda\left[\frac{1}{D_M^2}\cdot\left(1-e^{W(p^M(k),v,k)+1-v/k}+(1-p^M(k))\cdot\frac{e^{W(p^M(k),v,k)}\cdot W'\left(\frac{p^M(k)}{1-p^M(k)}e^{v/k-1}\right)}{(1-p^M(k))^2}\right)\right],$$

[31] More precisely, one can show that there exists $\tau_k,\tau_p>0$ small enough so that the assumptions of the Implicit Function Theorem are satisfied once the domain of $F_k^M(\cdot)$ is extended to let k range in $(0,k^*+\tau_k)$ and p range in $(-\tau_p,1)$.

[32] This is shown by Ravid (2020) in the proof of Theorem 1.

[33] To derive equation (21), we used the fact that $W'(x)=\frac{W(x)}{x\cdot(1+W(x))}$ for all $x>0$.

and

$$D_M = p^M(k) + (1 - p^M(k)) \cdot e^{W(p^M(k),v,k)+1-v/k}.$$

As $k \uparrow k^*$, we know from (i) that $p^M(k) \to 0$. Therefore, $A_M \to \mathbb{E}_\lambda\left[\frac{v}{(k^*)^2} \cdot e^{v/k^*-1}\right]$ and $B_M \to \mathbb{E}_\lambda\left[\frac{2-e^{1-v/k^*}}{e^{2\cdot(1-v/k^*)}}\right]$.[34] This concludes the proof of Lemma 7. $\square$

*Preliminary analysis for the competition model

We first state the following result:

Lemma 8. *Assume $k < k^*$. Let $(\mu^B, \sigma^B, \beta^B)$ be the unique competitive equilibrium. For each $v \in V$, denote by $x(v)$ the symmetric equilibrium offer made by both firms after observing v. When V has at least two elements, the consumer's ex-post gains from trade $v - x(v)$ strictly increase with v. Additionally, if $k \in (\bar{k}, k^*)$, we have $v_{\min} - x(v_{\min}) < 0 < v_{\max} - x(v_{\max})$ where $v_{\min} := \min V$ and $v_{\max} := \max V$.*

Proof. In the symmetric RVP trading equilibrium, $\beta_1 = \beta_2 = \beta$ because $\pi_1 = \pi_2 = \pi$. Suppose first that $k \leq \bar{k}$. In equilibrium trade is efficient, i.e., $\pi = 1/2$, implying immediately that $\beta(v, x(v), x(v)) = 1/2$ for every $v \in V$. Moreover, from Lemma 3, we know that $x(v) = 2k$ for every $v \in V$. Therefore, the consumer's ex-post gains from trade strictly increase with v.

Suppose now that $k \in (\bar{k}, k^*)$, which implies that $\pi \in (0, 1/2)$. From the proof of Lemma 3, we know that (i) $\beta(v, x(v), x(v)) = \frac{\phi(\pi,v)}{1+\phi(\pi,v)}$, and (ii) $x(v) = k(1 + \phi(\pi, v))$. Therefore,

$$\beta(v, x(v), x(v)) = \frac{\phi(\pi, v)}{1 + \phi(\pi, v)} = \frac{k \cdot (\phi(\pi, v) + 1 - 1)}{k \cdot (1 + \phi(\pi, v))} = 1 - k/x(v).$$

Finally, from equation (8), we know that $\phi(v, \pi)$ increases with $v \in V$. Therefore, $x(v)$ strictly increases with v as well, implying that $\beta(v, x(v), x(v)) = 1 - k/x(v)$ also strictly increases with v. Since $\beta_i(v, x, x)$ strictly co-increases with $v - x$ for every $i \in I$, we conclude that $v - x(v)$ is strictly increasing in v.

Lastly, since $\beta_i(v, x, x) = \pi$ if and only if $v - x = 0$ when $\pi \in (0, 1/2)$, we immediately deduce that $v_{\min} - x(v_{\min}) < 0 < v_{\max} - x(v_{\max})$ as required. $\square$

For each $k \in (\bar{k}, k^*]$, we define $F_k^B : [0, 1/2) \to \mathbb{R}_+$ as

$$F_k^B(p) := \mathbb{E}_\lambda\left[\frac{1}{2p + (1 - 2p) \cdot e^{\phi(p,v,k)+1-v/k}}\right],$$

where for $p > 0$, we let $\phi(p, v, k)$ be defined as the unique solution to equation (8), and we set $\phi(0, v, k) := 0$ for all $v \in V$ and $k \in (\bar{k}, k^*]$. Let $p^B(k)$ be a solution to $F_k^B(p) = 1$. From Theorem 2, we know that $p^B(k)$ exists and is unique for all $k \in (\bar{k}, k^*)$. Again, by the Implicit Function theorem we know that $p^B(k)$ is continuously differentiable on $(\bar{k}, k^*)$. The next Lemma provides additional properties that $p^B(k)$ satisfies.

Lemma 9. *We have:*

[34] Here, we used the fact that $W'(x) = 1$ as $x \downarrow 0$.

(i) $\lim_{k\uparrow k^*} p^B(k) = 0$.

(ii) $\lim_{k\uparrow k^*} \frac{\partial}{\partial k} p^B(k) = -\mathbb{E}_\lambda\left[\frac{v}{(k^*)^2}\cdot e^{v/k^*-1}\right]/\mathbb{E}_\lambda\left[\frac{2(1-e^{1-v/k^*})+1}{e^{2\cdot(1-v/k^*)}}\right]$.

Proof. (i): We show that (#): for every $p \in (0, 1/2)$, there exists $k_p \in (\bar{k}, k^*)$ such that for all k strictly between k_p and k^*, $F_k^B(p) < 1$. Given our proof of Theorem 2, (#) implies that for all k sufficiently close to k^*, $p^B(k) < p$, proving the statement.

From equation (8), it is not hard to see that $\phi(p, v, k)$ is strictly decreasing in k for every $p \in (0, 1/2)$ and $v \in V$, and satisfies $\phi(p, v, k) > 0$ for all $p \in (0, 1/2)$, $v \in V$ and $k > 0$. Fix $p \in (0, 1/2)$ arbitrarily. Given the finiteness of V, there exists $c_p > 0$ such that $\phi(p, v, k) > c_p$ for all $v \in V$ and $k \in (\bar{k}, k^*]$. Since $\mathbb{E}_\lambda\left[e^{v/k^*-1}\right] = 1$, we have $\mathbb{E}_\lambda\left[e^{v/k^*-1-c_p}\right] < 1$. Therefore, continuity implies that there exists k_p strictly between $\bar{k}$ and k^* so that $\mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1$ for all $k \in (k_p, k^*)$. Fix any such k. We have:

$$\begin{aligned}
F_k^B(p) &= \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{\phi(p,v,k)+1-v/k}}\right] \\
&\leq \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{c_p+1-v/k}}\right] \\
&\leq 2p + (1-2p)\cdot \mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1.
\end{aligned}$$

Thus, (#) holds.

(ii): We first totally differentiate equation (8) to find the partial derivatives of ϕ with respect to p and k. That is, $\phi_p(p, v, k) := \frac{\partial}{\partial p}\phi(p, v, k)$ and $\phi_k(p, v, k) := \frac{\partial}{\partial k}\phi(p, v, k)$. After some algebra, one can show that

$$\phi_p(p, v, k) = \frac{1-\phi(p, v, k)}{(1-2p)\cdot\left(p + e^{\phi(p,v,k)}\cdot(1+\phi(p, v, k))\frac{1-2p}{e^{v/k-1}}\right)} \geq 0, \tag{19}$$

and

$$\phi_k(p, v, k) = -\frac{v}{k^2}\cdot\frac{\phi(p, v, k)e^{\phi(p,v,k)}}{\frac{p}{1-2p}e^{v/k-1} + e^{\phi(p,v,k)}(1+\phi(p, v, k))} \leq 0. \tag{20}$$

Note that, as $k \uparrow k^*$ and, therefore, $p \to 0$, we have $\phi \to 0$. Therefore, $\phi_p \to e^{v/k^*-1}$ and $\phi_k \to 0$ as $k \uparrow k^*$.

Next, we totally differentiate the equation $F_k^B(p^B(k)) = 1$ with respect to $k > \bar{k}$. One can show that

$$\frac{\partial}{\partial k}p^B(k) = -\frac{A_B}{B_B} \tag{21}$$

where

$$A_B = \mathbb{E}_\lambda \left[\frac{1}{D_B^2} \cdot \left((1 - 2p^B(k)) e^{\phi(p^B(k),v,k)+1-v/k} \cdot \left(\frac{v}{k^2} + \phi_k(p^B(k), v, k) \right) \right) \right],$$

$$B_B = \mathbb{E}_\lambda \left[\frac{1}{D_B^2} \cdot \left(2(1 - e^{\phi(p^B(k),v,k)+1-v/k}) + (1 - 2p^B(k)) \cdot e^{\phi(p^B(k),v,k)+1-v/k} \cdot \phi_p(p^B(k), v, k) \right) \right],$$

and

$$D_B = 2p^B(k) + (1 - 2p^B(k)) \cdot e^{\phi(p^B(k),v,k)+1-v/k}.$$

Letting $k \uparrow k^*$, we conclude that

$$\frac{\partial}{\partial k} p^B(k) \to -\mathbb{E}_\lambda \left[\frac{v}{(k^*)^2} \cdot e^{v/k^*-1} \right] / \mathbb{E}_\lambda \left[\frac{2(1 - e^{1-v/k^*}) + 1}{e^{2\cdot(1-v/k^*)}} \right]$$

as required. □

*Concluding the proof of Theorem 3

We now use *de L'Hopital rule* to show that as $k \uparrow k^*$, the ratio $p^B(k)/p^M(k)$ is bounded above $1/2$ *strictly.* Formally:

Lemma 10. *There exists $\Theta > 0$ such that*

$$\lim_{k \uparrow k^*} \frac{p^B(k)}{p^M(k)} > \frac{1}{2} + \Theta.$$

Proof. Note that $\lim_{k\uparrow k^*} \frac{\partial}{\partial k} p^M(k)$ exists and is different from 0. Therefore, by *de L'Hopital rule*

$$\lim_{k \uparrow k^*} \frac{p^B(k)}{p^M(k)} = \lim_{k \uparrow k^*} \frac{\frac{\partial}{\partial k} p^B(k)}{\frac{\partial}{\partial k} p^M(k)} = \frac{\mathbb{E}_\lambda \left[\frac{2 - e^{1-v/k^*}}{e^{2\cdot(1-v/k^*)}} \right]}{\mathbb{E}_\lambda \left[\frac{2(1-e^{1-v/k^*})+1}{e^{2\cdot(1-v/k^*)}} \right]} = \frac{1}{2 - \frac{\mathbb{E}_\lambda \left[e^{2(v/k^*-1)} \right]}{2\mathbb{E}_\lambda \left[e^{2(v/k^*-1)} \right] - 1}}.$$

Since

$$2\mathbb{E}_\lambda \left[e^{2(v/k^*-1)} \right] - 1 > \mathbb{E}_\lambda \left[e^{2(v/k^*-1)} \right] - 1 \geq 0$$

because of Jensen inequality, the conclusion of the lemma follows. □

As the last step, note that as $k \uparrow k^*$, $p^M(k), p^B(k) \to 0$. It follows that $x_k^M(v), x_k^O(v) \to k^*$ for all $v \in V$. Now, fix $\varepsilon > 0$ so small that

$$1 + 2(\Theta - \varepsilon) > \frac{k^* + \varepsilon}{k^* - \varepsilon},$$

and let $\hat{k} \in (\bar{k}, k^*)$ be such that $p^B(k)/p^M(k) > 1/2 + \Theta - \varepsilon$ and $x_k^m(v) \in (k^* - \varepsilon, k^* + \varepsilon)$ for all $k > \hat{k}$, $v \in V$, and $m \in \{B, C\}$. For all $k > 0$, we have that $2p^B(k)(k^* - \varepsilon) > p^M(k)(k^* + \varepsilon)$ if and only if

$$2 \cdot \frac{p^B(k)}{p^M(k)} > \frac{k^* + \varepsilon}{k^* - \varepsilon}. \tag{22}$$

Notice that (22) holds by assumption as long as $k \in (\hat{k}, k^*)$. Since by construction we have $\Pi^B(k) \geq p^B(k)(k^* - \varepsilon)$ and $p^M(k)(k + \varepsilon) \geq \Pi^C(k)$, we conclude that $2\Pi^B(k) > \Pi^C(k)$ for all $k \in (\hat{k}, k^*)$ as required.

Q.E.D.

B The restriction to pure strategy equilibria is without loss

In this section, we show that focusing on pure and symmetric strategies for the sellers is without loss of generality in any trading equilibrium of the competitive model where the demand function is given by (5). To do so, we apply the results in Milgrom and Roberts (1990) (henceforth MR90). Fix $v \in V$ arbitrarily and let β be given by (5). For fixed $\pi \in (0, 1/2]$, the payoff function of seller $i \in I$ is:

$$u_i(x_i, x_{-i}) = \left[\frac{\pi \cdot e^{\frac{v-x_i}{k}}}{\pi \cdot \sum_{j \in I} e^{\frac{v-x_j}{k}} + 1 - 2\pi} \right] \cdot x_i.$$

Let $w_i = \log(u_i)$ be the log-transformed payoff function of seller $i \in I$. One can verify that each w_i is twice continuously differentiable in $\mathbb{R}^2_+$ and satisfies the following conditions:

1. $\frac{\partial^2}{\partial x_i \partial x_j} w_i(x_i, x_j) \geq 0.$
2. $\frac{\partial^2}{\partial x_i \partial x_j} w_i(x_i, x_j) + \frac{\partial^2}{\partial x_i^2} w_i(x_i, x_j) < 0.$

Condition (i) guarantees that the log-transformed game played by the two sellers is *smooth supermodular*. As a result, it admits at least one pure strategy NE.[35] As argued in section 4 of MR90, condition (ii) guarantees that such an equilibrium is unique and corresponds to the unique profile of rationalizable actions. Thus, no other equilibrium (either pure or mixed) can exist. Denote the unique pure equilibrium profile by $x^* = (x_1^*, x_2^*)$. That x^* is symmetric now follows from the fact that the profile $(u_i)_i$ (hence, $(w_i)_i$) satisfies $u_i(x_i, x_j) = u_j(x_j, x_i)$ for $i \neq j$. For, if $x_1^* \neq x_2^*$, then $x^{**} = (x_2^*, x_1^*)$ would be another pure strategy NE, contradicting the uniqueness of x^*.

Remark 1. *The results of MR90 can be used even when the number of firms $N := |I| > 2$. This can help us later in this project, if want to do comparative statics regarding the parameter N.*

[35] See Theorem 5 in MR90.

C The restriction to symmetric equilibria is without loss

We proceed by showing that any competitive equilibrium must be symmetric. For this, we use the finding that all best responses by the consumer have to be of the form described in Lemma 1.

More formally, suppose a competitive equilibrium exists, and let the assessment (μ, σ, β) be one of them. From Matějka and McKay (2015) we know that since both firms are active, we must have $\pi_i = \mathbb{E}_\mu[\beta_i] > 0$ for all $i \in I$. Taking $v \in V$, and β_1 and β_2 as given, it can be shown that the log-transformation of the game played by the two firms is *smooth supermodular*. Furthermore, from Milgrom and Roberts (1990) (henceforth, MR90) it follows that the game played between the two firms admits a unique equilibrium in pure strategies. For each $v \in V$, denote such unique equilibrium by $(\hat{x}_1(v), \hat{x}_2(v))$. In Lemma 12 below, we show that if $\pi_i > \pi_j$ then $\hat{x}_i(v) > \hat{x}_j(v)$. We now use this fact to show that in case a competitive equilibrium exists, it must be symmetric.

Lemma 11. *Let (μ, σ, β) be a competitive equilibrium, i.e., $\pi_i = \mathbb{E}_\mu[\beta_i] > 0$ for every $i \in I$. Then, (μ, σ, β) is symmetric.*

The intuition behind the result is simple. Suppose that $\pi_1 > \pi_2$. Then firm 1 faces a higher demand than firm 2 at any fixed price. In equilibrium, firm 1 would therefore optimally charge a higher price than firm 2. However, this implies that the consumer should buy from firm 1 *less often* than firm 2, i.e., $\pi_1 < \pi_2$, which leads to a contradiction.

Lemmas 1, 11 and 2 taken together imply the following important corollary that we used extensively in Section 5.

Corollary 1. *(μ, σ, β) is a symmetric trading equilibrium if and only if it is competitive equilibrium.*

Proof of Lemma 11

Proof. First, notice that if $\pi_1 = \pi_2$, then $\hat{x}_1(v) = \hat{x}_2(v)$ for all $v \in V$, implying the symmetry of (μ, σ, β). Thus, it is sufficient to show that $\pi_1 = \pi_2 = \pi \in (0, 1/2]$ is a necessary condition for an equilibrium where both firms are active. To this goal, we first state and prove Lemma 12, which says that if $\pi_1 > \pi_2 > 0$, for each $v \in V$, the unique (pure strategy) NE between the two firms $\hat{x}(v) = (\hat{x}_1(v), \hat{x}_2(v))$ is such that $\hat{x}_1(v) > \hat{x}_2(v)$.

Lemma 12. *If $\pi_1 > \pi_2 > 0$ then $\hat{x}_1(v) > \hat{x}_2(v)$ in equilibrium.*

Proof. Consider the following two player game where the action sets are $A_i = [0, \infty)$ for every $i \in I$, and the payoffs are given by

$$w_i(x_1, x_2) = \log\left(\frac{B^i \cdot e^{\frac{v - x_i}{k}}}{\sum_{j=1,2} B^j \cdot e^{\frac{v - x_j}{k}} + C}\right) + \log(x_i), \ \ (i \in I)$$

where B^1, B^2, C are parameters such that $B^1, B^2 > 0$, and $C \geq 0$. Again, applying the same arguments of MR90, we know that this game is smooth supermodular and admits a unique NE equilibrium which is pure. Denote it by $\hat{x} = (\hat{x}_1, \hat{x}_2)$. The equilibrium must be interior and is characterized by the following system of FOC:

$$\frac{k}{\hat{x}_i} = \frac{B^{-i} \cdot e^{\frac{v-\hat{x}_{-i}}{k}} + C}{\sum_{j=1,2} B^j \cdot e^{\frac{v-\hat{x}_j}{k}} + C}, \ (\forall i \in I).$$

If $B^1 = B^2 = B > 0$, the unique equilibrium $\hat{x}$ must be symmetric. In what follows we show that (a) the best response function of firm $i \in I$ moves up as B^i grows, while the best response function of firm $j \neq i$ moves downwards. Moreover, we prove that (b) each best response function is upward-sloping with a slope always strictly below unity. Since such best responses cross only once, from (a) and (b) we immediately conclude that

$$B^1 > B^2 \implies \hat{x}_1 > \hat{x}_2.$$

The best response function of firm 1, say $x_1^* = x_1^*(x_2) \in [0, \infty)$, is given implicitly as the unique solution to the equation

$$k \cdot \left(\sum_{j=1,2} B^j \cdot e^{\frac{v-x_j}{k}} + C \right) = \left(B^2 \cdot e^{\frac{v-x_2}{k}} + C \right) \cdot x_1. \tag{23}$$

First, notice that since $\sum_{j=1,2} B^j \cdot e^{\frac{v-x_j}{k}} > B^2 \cdot e^{\frac{v-x_2}{k}} > 0$, we immediately infer that $x_1^*(x_2) > k$ for all $x_2 \geq 0$. Totally differentiating equation (23) for B^1 we further obtain

$$\frac{\partial x_1^*}{\partial B^1}(x_2) = \frac{e^{\frac{v-x_1^*}{k}}}{B^1 \cdot e^{\frac{v-x_1^*}{k}} + B^2 \cdot e^{\frac{v-x_2}{k}} + C}.$$

This shows that $\frac{\partial x_1^*}{\partial B^1}(x_2) > 0$ for all $x_2 \geq 0$. We now show that the opposite strict inequality obtains for the BR function $x_2^* = x_2^*(x_1)$ of firm 2 over the (relevant) domain for x_1 given by $(k, \bar{x}]$. The best response function x_2^* of firm 2 is given as the unique solution to the equation

$$k \cdot \left(\sum_{j=1,2} B^j \cdot e^{\frac{v-x_j}{k}} + C \right) = \left(B^1 \cdot e^{\frac{v-x_1}{k}} + C \right) \cdot x_2. \tag{24}$$

Totally differentiating for B^1 equation (24), we get

$$\frac{\partial x_2^*}{\partial B^1}(x_1) = \frac{e^{\frac{v-x_1}{k}}}{B^1 \cdot e^{\frac{v-x_2^*}{k}} + B^2 \cdot e^{\frac{v-x_1}{k}} + C} \cdot (k - x_2^*)$$

which is < 0 because $x_2^* > k$ always. Finally, totally differentiating equation (23) with respect to x_2 (respectively, equation (24) with respect to x_1), we get

$$\frac{\partial}{\partial x_j} x_i^*(x_j) = \frac{B^i \cdot e^{\frac{v-x_i^*}{k}}}{B^i \cdot e^{\frac{v-x_i^*}{k}} + B^j \cdot e^{\frac{v-x_j}{k}} + C} \cdot \frac{B^j \cdot e^{\frac{v-x_j}{k}}}{B^j \cdot e^{\frac{v-x_j}{k}} + C} < 1,$$

for all $i \in I$, $j \neq i$, and $x_j \geq 0$. We conclude that $B^1 > B^2$ implies that $\hat{x}_1 > \hat{x}_2$. Finally, notice that when $\pi_i = B^i > 0$ for each $i \in I$, and $C = 1 - \pi_1 - \pi_2 \geq 0$, we get back our model of Bertrand competition with a rationally inattentive consumer. Therefore, $\pi_1 > \pi_2$ implies that $\hat{x}_1(v) > \hat{x}_2(v)$ for every $v \in V$, as required. □

Suppose towards a contradiction that (μ, σ, β) is an equilibrium with $\pi_1 > \pi_2 > 0$. From Lemma 12, we know that $\hat{x}_1(v) > \hat{x}_2(v)$ for every $v \in V$. This is in contradiction with the consumer trading with firm 1 strictly more often. To see this more formally, notice that $\hat{x}_1(\cdot) > \hat{x}_2(\cdot)$ immediately implies that for each $v \in V$, we have

$$\frac{\beta_1(v, \hat{x}_1(v), \hat{x}_2(v))}{\pi_1} < \frac{\beta_2(v, \hat{x}_1(v), \hat{x}_2(v))}{\pi_2}.$$

(Recall that $\pi_1 > \pi_2 > 0$.) But then, given that V is finite and in equilibrium $\mathbb{E}_\mu[\beta_i/\pi_i] = 1$ for every $i \in I$, we would reach the conclusion that $1 = \mathbb{E}_\mu[\beta_1/\pi_1] < \mathbb{E}_\mu[\beta_2/\pi_2] = 1$, an absurd. □

D Ravid's (2020) monopoly model is equilibrium outcome-equivalent to perfect price coordination and perfect profit internalization

First, we show that a model with two firms and one manager setting two offers simultaneously to maximize joint profits is equilibrium-outcome equivalent to the monopoly model of Ravid (2020). We do so by showing that the offers *accepted* by the consumer in any equilibrium with trade of this game will be equal to

$$\hat{x}(v) = k\left[1 + W\left(\frac{\pi_1 + \pi_2}{1 - \pi_1 - \pi_2} \cdot e^{v/k-1}\right)\right] \tag{25}$$

for every $v \in V$, where W denotes the Lambert's function, and each π_i is the consumer's equilibrium trade engagement level with firm $i \in I$. In other words, we will show that on-path, accepted equilibrium offers must take the *same* functional form derived by Ravid (2020) in his monopoly model when the monopolist produces a good of quality v and faces the aggregate demand

$$Q(x) = \frac{(\pi_1 + \pi_2) \cdot e^{\frac{v-x}{k}}}{(\pi_1 + \pi_2) \cdot e^{\frac{v-x}{k}} + 1 - \pi_1 - \pi_2}.$$

Since the shape of the monopolist's best response determines all the properties satisfied by the unique trading equilibrium in Ravid's (2020) model, equation (25) suffices to prove the equilibrium outcome-equivalance of the two models.

Analysis. Fix $v \in V$. Suppose each firm $i \in I$ faces a demand for its product given by

$$Q^i(x_1, x_2) := \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2}.$$

Suppose the same manager runs both firms and aims at maximizing joint profits. His payoff is given by

$$\Pi(x_1, x_2) = \sum_{i=1,2} \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2} \cdot x_i.$$

He therefore solves (P): $\max_{x_1,x_2 \geq 0} \Pi(x_1, x_2)$. If $0 = \pi_i < \pi_j$ for some $i \in I$, the problem of the common-manager is identical to the problem faced by the monopolist in Ravid's model. The equilibrium outcome-equivalence is therefore immediate. Thus, from now on suppose that $\pi_i > 0$ for all $i \in I$. Below, we show that (P) admits a unique critical point which coincides with the global optimum. Moreover, we show that the solution to (P) is symmetric.

Let $D = D(x_1, x_2) = \sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2$. The first order conditions associated to problem (P) are:

$$\frac{\pi_i e^{\frac{v-x_i}{k}}}{D} - \frac{\pi_i x_i}{k \cdot D^2} \cdot \left[e^{\frac{v-x_i}{k}}\left(\pi_j e^{\frac{v-x_j}{k}} + 1 - \pi_i - \pi_j\right)\right] + \frac{\pi_j e^{\frac{v-x_j}{k}}}{D^2} x_j \cdot \frac{e^{\frac{v-x_i}{k}}}{k} = 0, \ (i \in I).$$

Multiplying both sides by $D^2 > 0$, dividing both sides by $\pi_i e^{\frac{v-x_i}{k}} > 0$ and re-arrenging, we get

$$D = \frac{\pi_j e^{\frac{v-x_j}{k}}}{k} \cdot (x_i - x_j) + \frac{x_i}{k}(1 - \pi_i - \pi_j), \; (i \in I). \tag{26}$$

Notice that for all $x_j \geq 0$, if $x_i = 0$, then the LHS of (26) is strictly greater than the RHS. This implies that any solution to (P) (if exists) has to be interior. Conversely, there exists a $\bar{x}_i > 0$ such that, for all $x_i \geq \bar{x}_i$, the RHS of (26) is strictly greater than the LHS for all $x_j \geq 0$. This means that $x_i \mapsto \Pi(x_i, x_j)$ is eventually decreasing in x_i for all $x_j \geq 0$, implying that (P) indeed admits a bounded solution.

Now, combing both equations in (26) yields

$$\frac{x_1 - x_2}{k} \cdot D = 0$$

which is true if and only if $x_1 = x_2$. Thus, any critical point of Π must lie on the 45^o-line. Therefore, problem (P) is equivalent to solve

$$\max_{x \geq 0} \frac{(\pi_1 + \pi_2) \cdot e^{\frac{v-x}{k}}}{(\pi_1 + \pi_2) \cdot e^{\frac{v-x}{k}} + 1 - \pi_1 - \pi_2} \cdot x \tag{27}$$

which, from the analysis in Ravid (2020), we know admits a unique critical point

$$\hat{x}(v) = k \left[1 + W \left(\frac{\pi_1 + \pi_2}{1 - \pi_1 - \pi_2} \cdot e^{v/k - 1} \right) \right]$$

that coincides with the global maximum. Therefore, the solution to (P) is symmetric and given by $(x_1^*, x_2^*) = (\hat{x}(v), \hat{x}(v))$ in (25).

Two comments are in order. First, given $v \in V$, the problem in (27) is equivalent to the problem faced by the monopolist in Ravid's model when the logit demand he faces is characterized by $\pi^M = \pi_1 + \pi_2$. Second, given that at the optimum $x_1^* = x_2^* = \hat{x}(v)$ for all $v \in V$, firms' offers are *a priori homogeneous*. Therefore, Lemma 1 implies that the consumer's best response depends on (π_1, π_2) only through $\pi_1 + \pi_2$. The first comment implies that Ravid's (2020) model and our two-firms-one-manager model are equilibrium outcome-equivalent. Indeed, in any equilibrium with trade, the following variables coincide across the two models: (i) Offers accepted on-path, (ii) industry's total profits, (iii) the consumer's surplus, and (iv) trading efficiency. The second comment implies that it is without loss to focus on symmetric assessments to characterize all the equilibrium outcomes of the model.[36] Indeed, the following holds.

Lemma 13. *An equilibrium with total trade engagement level $\pi^M = \pi_1 + \pi_2$ exists if and only if a symmetric equilibrium with total trade engagement level π^M exists.*

[36] To see this more clearly, fix (π_1, π_2) arbitrarily. Let $\sigma_i(\cdot|v) = \delta_{\hat{x}(v)}$ for all $v \in V$, and μ be consistent with $\sigma = (\sigma_1, \sigma_2)$. By definition, if the recommendation strategy β given in (2) characterized by (π_1, π_2) solves the consumer's problem given μ and is RVP, then (μ, σ, β) is an equilibrium. Let $\bar{\pi} = \frac{\pi_1 + \pi_2}{2}$, and consider the assessment (μ, σ, β^*), where β^* is given in (5) and is characterized by $(\bar{\pi}, \bar{\pi})$. Clearly, (μ, σ, β^*) is symmetric. We show that (μ, σ, β^*) is an equilibrium as well. Since μ is symmetric, β^* is still a best response to μ. Moreover, by Lemma 2, it is RVP. On the other hand, because $\hat{x}(v)$ depends on (π_1, π_2) only through $\pi_1 + \pi_2$, each σ_i is a best response to σ_{-i} given β^*.

Relation to the collusion model presented in Section 4: For fixed $v \in V$, given that Π admits a unique critical point $(\hat{x}(v), \hat{x}(v))$, we conclude that $(\hat{x}(v), \hat{x}(v))$ is also the unique NE of the game played by two independent managers that perfectly internalize each other's profits in their payoff function.[37] This is because the FOC of (P) are sufficient to characterize the managers' best responses, and any NE between the two independent managers must by a critical point of Π. Therefore, like the model of perfect price coordination discussed above, the equilibrium predictions of the collusion model presented in Section 4 are equivalent to those obtained by Ravid (2020) in his monopoly analysis. In other words, the monopoly model of Ravid (2020), the model of perfect price coordination presented above, and the model of perfect profits internalization of Section 4 are all equilibrium outcome-equivalent.

The proof of Theorem 1 follows.

Proof of Theorem 1.

The necessity of $k < k^*$ for the existence of any equilibrium with trade follows from the same arguments presented in the proof of Theorem 2. For the sufficiency part, focus on symmetric assessments. (This is without loss in light of Lemma 13.) Given that, for every $v \in V$, firms' equilibrium offers are given by (25), a moment of thought shows that invoking Theorem 1 of Ravid (2020) suffices to prove that a trading equilibrium exists. Finally, the uniqueness of the trading equilibrium *outcome* follows from Theorem 1 of Ravid (2020), and the fact that the offers accepted on path $(\hat{x}(v))_{v \in V}$ only depend on the overall trade engagement level $\pi^M = \pi_1 + \pi_2$.

Q.E.D.

[37]Since $(\hat{x}(v), \hat{x}(v))$ is the unique NE in pure strategies, no equilibrium in mixed strategies exists.

E Consumer Surplus

This section is devoted to studying the surplus the consumer obtains in any trading equilibrium. To avoid trivialities, assume throughout this section that V contains at least two elements, i.e., the quality of the product is random.[38] The following proposition summarizes some key preliminary observations.

Proposition 5. *The following statements hold:*

(i) Whenever trade occurs, the consumer surplus in equilibrium is strictly positive, $\mathbb{E}[U] > 0$. This is true irrespective of the number of active firms and their incentives to compete.

(ii) If only one firm is active, the consumer surplus in the competition model is identical to the one in any trading equilibrium of the collusion model.

We omit the proof of the proposition. Part (i) follows from the strict convexity of the entropy cost function with respect to recommendation strategies, a fact already noted by Ravid (2020). Part (ii), instead, is a direct consequence of the outcome-equivalence between any trading equilibrium of the collusion model and the equilibria of the competition model where only one firm is active.

Part (i) of Proposition 5 confirms the intuition provided in Ravid (2020) that rational inattention helps consumers extract some surplus whenever trade occurs. Not surprisingly, this intuition is robust to the particular market structure the consumer faces, i.e., it is not affected by the number of active firms and their incentives to compete. On the other hand, part (ii) shows once again that competition does not have any bite unless both firms are active.

Proposition 5 clarifies that competition cannot nullify the effect that rational inattention has on consumer surplus. However, it does not fully characterize how these two factors interact. In the remainder of this section, we investigate precisely this question. *Is competition always beneficial for the consumer, even when she is rationally inattentive?* In light of the previous results, the answer to this question is not at all immediate. After all, Theorem 3 shows that the industry surplus can strictly increase with competition in markets with rational inattention. The next proposition provides a sufficient condition for when competition strictly improves the surplus the consumer gets in equilibrium.

Proposition 6. *Fix $0 < k < k^*$, and let $(\mu^B, \sigma^B, \beta^B)$ and $(\mu^C, \sigma^C, \beta^C)$ be the unique symmetric trading equilibria of the competition model and of the collusion model respectively. Suppose that the expected prices in the collusion model are weakly higher than those in the competition model, i.e., $\mathbb{E}_\lambda[x^C(v)] \geq \mathbb{E}_\lambda[x^B(v)]$. Then, the consumer surplus is strictly higher in the equilibrium with competition. That is,*

$$\mathbb{E}[U^B] > \mathbb{E}[U^C].$$

Proof. The proof of Proposition 6 relies on the following lemma.

[38] Like Ravid (2020), the consumer surplus is easy to characterize when v is common-knowledge, i.e., when $V = \{v_o\}$ is a singleton. In this case, equilibrium offers accepted on-path must equal v_o in the collusion model, implying that the consumer is left with no trading surplus. In the competitive equilibrium, offers equal v_o if $k > \bar{k} = v_o/2$, i.e., there is not efficient trade. Otherwise, offers equal to $2k \leq v_o$. Additionally, in both models the consumer does not incur any information processing costs because his ex-post gains from trade in equilibrium are non-random. Therefore, the consumer surplus is always zero in the collusion model. In the competitive model, it is equal to zero if $k \geq \bar{k}$. Else, it is equal to $v_o - 2k > 0$.

Lemma 14. *There exists a threshold $v^* > 0$ such that $x^C(v) > x^B(v)$ if and only if $v \geq v^*$.*

Proof of Lemma 14. Let $2\pi^C$ be the overall equilibrium engagement level of the consumer when the firms collude, and $2\pi^B$ be the overall engagement level of the consumer in the competitive equilibrium. For every $v \in V$, let $W^C(v) = W\left(\frac{2\pi^C}{1-2\pi^C} e^{v/k-1}\right)$ and $\phi^B(v) = \phi(\pi^B, v)$ solving (8). Since $x^C(v) = k(1+W^C(v))$ and $x^B(v) = k(1+\phi^B(v))$, it follows that $x^C(v) > x^B(v)$ if and only if $W^C(v) > \phi^B(v)$. By the definition of the Lambert function, $W^C(v)e^{W^C(v)} = \frac{2\pi^C}{1-2\pi^C} e^{v/k-1}$. Moreover, $x \mapsto xe^x$ is a strictly increasing function of $x > 0$. Therefore, $W^C(v) > \phi^B(v)$ if and only if

$$\frac{2\pi^C}{1-2\pi^C} e^{v/k-1} > \phi^B(v)e^{\phi^B(v)}. \tag{28}$$

From equation (8), we know that $\phi^B(v)e^{\phi^B(v)} = \frac{1-\phi^B(v)}{2} \cdot \frac{2\pi^B}{1-2\pi^B} e^{v/k-1}$. Therefore, (28) is equivalent to

$$\frac{1-\phi^B(v)}{2} < \frac{2\pi^C}{1-2\pi^C} \cdot \frac{1-2\pi^B}{2\pi^B}. \tag{29}$$

Because $\phi^B(v)$ is strictly increasing in v, the conclusion of the lemma follows. $\square$

Lemma 2 in Matějka and McKay (2015) shows that, in equilibrium,

$$\mathbb{E}[U^C] = \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right],$$

and

$$\mathbb{E}[U^B] = \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^B}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right].$$

Consider the random variables Y^B and Y^C defined by $Y^B(v) := v - x^B(v)$ and $Y^C(v) = v - x^C(v)$. Let G^B and G^C be the CDF of Y^B and Y^C respectively, and define $\omega := \mathbb{E}_\lambda[x^C(v)] - \mathbb{E}_\lambda[x^B(v)]$. By assumption, $\omega \geq 0$. Finally, denote with u_1 and u_0 the maximal and minimal element in the support of Y^B respectively. From Lemma 14, we know that $u_0 \leq Y^C \leq u_1$ with probability 1. Furthermore, one can verify that $\omega \geq 0$ together with Lemma 14 imply

$$\int_u^{\bar{u}} G^B(y)dy \leq \int_u^{\bar{u}} G^C(y)dy \quad \text{for all } u \in [u_0, u_1].$$

This means that any expected utility maximizer with an increasing and convex Bernulli utility function $w : [u_0, u_1] \to \mathbb{R}$ would prefer the lottery Y^B over Y^C (see Theorem 4 in Meyer (1977)). Now, observe that for every $\pi \in [0, 1/2]$, we have

$$k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] = k \cdot \mathbb{E}\left[\ln\left(2\pi \cdot e^{\frac{Y^C}{k}} + 1 - 2\pi\right)\right],$$

and

$$k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] = k \cdot \mathbb{E}\left[\ln\left(2\pi \cdot e^{\frac{Y^B}{k}} + 1 - 2\pi\right)\right].$$

Furthermore, the function $y \in (0, +\infty) \mapsto \ln\left(2\pi \cdot e^{y/k} + 1 - 2\pi\right)$ is strictly increasing and strictly convex in $y > 0$ whenever $\pi \in (0, 1/2)$. Therefore,

$$\begin{aligned}
\mathbb{E}[U^C] &= \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] \\
&= k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi^C \cdot e^{\frac{v-x}{k}} + 1 - 2\pi^C\right)\right] \\
&= k \cdot \mathbb{E}\left[\ln\left(2\pi^C \cdot e^{\frac{Y^C}{k}} + 1 - 2\pi^C\right)\right] \\
&\leq \mathbb{E}\left[\ln\left(2\pi^C \cdot e^{\frac{Y^B}{k}} + 1 - 2\pi^C\right)\right] \\
&= k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi^C \cdot e^{\frac{v-x}{k}} + 1 - 2\pi^C\right)\right] \\
&< \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^B}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] = \mathbb{E}[U^B].
\end{aligned}$$

where the first inequality is implied by $\pi^C \in (0, 1/2)$, while the last strict inequality is implied by the fact that the engagement level π^C is not a best response to μ^B. We conclude that $\mathbb{E}[U^B] > \mathbb{E}[U^C]$ as required. □

We have a strong conjecture (confirmed by several simulations) that the condition of Proposition 6, i.e., that on average equilibrium prices go down with competition, must be always met. However, due to the fact that π^B, π^C and ϕ^B are only defined implicitly in our model, we are not able to prove this conjecture generally. However, we are able to prove that $\mathbb{E}_\lambda[x^C(v)] \geq \mathbb{E}_\lambda[x^B(v)]$ whenever k is sufficiently close to either k^* or 0.

Proposition 7. *There exist thresholds $0 < k_1 < k_2 < k^*$ such that for all $k \in (0, k_1) \cup (k_2, k^*)$, we have $\mathbb{E}_\lambda[x^C(v)] \geq \mathbb{E}_\lambda[x^B(v)]$.*

Proof. The existence of a threshold $k_1 > 0$ sufficiently close to 0 follows immediately from the fact that prices in the competition model converge to 0 as $k \downarrow 0$,[39] while prices in the collusion model are bounded below by $v_{\min} > 0$, the lowest possible quality level in V.[40]

To show that a threshold $k_2 > 0$ exists as well, we make use of equation (29) introduced earlier. Specifically, in the Appendix, the proof of Theorem 3 shows that while both π^B and π^C converge to 0 as $k \uparrow k^*$, we have

$$\lim_{k \uparrow k^*} \frac{\pi^B(k)}{\pi^C(k)} = \frac{1}{2 - \frac{\mathbb{E}_\lambda\left[e^{2(v/k^*-1)}\right]}{2\mathbb{E}_\lambda\left[e^{2(v/k^*-1)}\right]-1}}.$$

It is easy to see that such limit is strictly less than 1, because $\mathbb{E}_\lambda\left[e^{2(v/k^*-1)}\right] > \left(\mathbb{E}_\lambda\left[e^{v/k^*-1}\right]\right)^2$ due to Jensen inequality. (Recall that $\mathbb{E}_\lambda\left[e^{v/k^*-1}\right] = 1$ by definition.) Therefore, while the LHS of equation (29) converges to $\frac{1}{2}$ because $\phi^B(v) \downarrow 0$ as $k \uparrow k^*$, the RHS of (29) is converging to a limit strictly greater than 1. As a result, equation (29) is satisfied eventually (i.e., as k approaches k^* from below) for all $v \in V$. The existence of the threshold $k_2 > 0$ follows immediately from this observation. □

[39] Recall that $\lim_{k\downarrow 0} x^B(v) = \lim_{k\downarrow 0} 2k = 0$ for all $v \in V$.

[40] This follows from Corollary 2 in Ravid (2020), and the fact that equilibrium prices in the collusion model are increasing in v.

F Discussion Section

Proof of Proposition 3

A straightforward extension of the proof of Proposition 1 shows that, in the competitive equilibrium, the overall trade engagement level increases with N. To see why the second statement holds as well, notice that with $N \geq 2$ firms, the maximal price that can be ever sustained in a symmetric equilibrium becomes $x(v, N) = k \cdot \left(1 + \frac{1}{N-1}\right) = k \cdot \frac{N}{N-1}$.[41] Therefore, a sure-trade competitive equilibrium exists if and only if $k \leq \bar{k}(N)$, where $\bar{k}(N)$ is the unique solution to $\mathbb{E}_\lambda\left[e^{\frac{N}{N-1} - v/k}\right] = 1$.

Q.E.D.

[41] This price corresponds to $\pi = 1/N$ which generalizes the case $\pi = 1/2$ of the duopoly setting.