Bargaining in Non–Stationary Networks

Peter Pusztai*

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Abstract

Dealers in over–the–counter markets bargain over the profit from executing an investor's order with other dealers to whom they are connected via the inter–dealer network. Investor's orders arrive randomly. I study a model of bargaining in continuous time in networks where profitable opportunities arise randomly to agents who contact a neighbor to split the surplus. If a pair of agents fail to reach an agreement, their link is eliminated from the network. This leads to non–stationarity of the network. I prove the existence of a Markov perfect equilibrium using an inductive argument. Players' bargaining power in an equilibrium depends on their continuation values in sub–networks reached when some of their links are eliminated. In particular, the relative bargaining power between a pair of connected agents depends on the difference in the change in their continuation values in the current network and the sub–network without their link.

Under certain conditions, agreement in all bargaining meetings is an equilibrium. These cases are important because the network remains unchanged despite the threat of severance. I prove that agreement in all meetings is an equilibrium if and only if the cost of maintaining a connection is lower than a network specific threshold. Comparison of thresholds across different networks provides insight to their relative stability. I show that star networks are more stable than lines and polygons. Inter–dealer networks in OTC markets exhibit a core–periphery structure which include star networks.

*University of Minnesota, puszt001@umn.edu

1 Introduction

Networks play an important role in many social and economic environments. Among others, friendship, crime, local buyer–seller, and inter–dealer networks in over–the–counter financial markets are of significance. In all of the above, profitable opportunities present themselves to agents. An individual might profit from calling in a favor from an acquaintance, a buyer may profit from a match with a seller and vice versa, and so can a dealer who does not have the inventory to execute an investor's order by contacting another dealer and acquiring the necessary assets. Such opportunities arise at uncertain times, although with some regularity. For example, a dealer cannot perfectly anticipate when an investor will demand an asset. Their meeting might also be delayed by search frictions, adding more uncertainty. The investor's liquidity and hedging needs, however, change over time, which leads to possible future meetings with the dealer. In addition to potential meetings with other investors, this leads to regular opportunities for the dealer. Due to the lack of a centralized inter–dealer market and the uncertainty over the timing, size and the exact assets involved in these transactions, dealers cannot rely on market prices or prespecified contracts to establish terms of their trades. As a result, compensations and possibly other details are determined through bilateral negotiations. If successful, the dealer can execute the investor's order and profit from it. A disagreement between the two dealers, however, can affect the frequency with which they meet in the future. In an extreme case, it could result in no future meetings at all.

Li & Schürhoff (2019) document that the inter–dealer network in the municipal bonds market exhibits certain features inconsistent with random matching. One of them is the *degree distribution*, that is, the distribution of the number of agents' connections. Another is the *persistence of trading relationships*. In particular, if two dealers trade in a given month then, with a 66% chance, they trade the following month as well. These observations suggest that an environment with an explicit network structure and regular opportunities for interaction between pairs of connected agents is well–suited to the analysis of decentralized financial markets. My paper focuses on bilateral bargaining in such environments, where players' decisions also affect their future opportunities.

In the model, players face numerous opportunities to bargain and profit over time. A disagreement between two players leads to the elimination of their link from the network. As a result, the two players will never meet again. How does the threat of no future meetings affect bargaining power? Players, when bargaining, need to consider the present value of their future opportunities, i.e. their continuation value, both under agreement and disagreement. Whether the outcome of previous bargaining situations affects how often one player contacts another in the future has a significant impact on the solution. Consider the following two cases.

Suppose first, that, unlike in the model, the network remains unchanged and players use the same contacting rule regardless of what happened in the past. In this case, an agent's continuation value does not depend on how the current bargaining encounter is resolved and therefore does not play a role. Thus the surplus from a meeting between the two players is the value of the opportunity. Under a symmetric bargaining protocol, which is assumed in the paper, the players split the instantaneous gain equally and, provided the gain is positive, all bargaining situations end in agreement.

Assume now, as in the model, that if two players failed to reach an agreement, they would never contact each other again. When bargaining, continuation values depend on the outcome. In case of agreement, both players remain in the same network and have the same future opportunities as before. In the event of disagreement, however, they continue in a smaller network, with fewer future opportunities. Surplus is now a function of changes in players' continuation values from the loss of their connection. If the two players are affected differently, the one with less to lose has an advantage. Bargaining power arises from the asymmetry in changes in continuation value. This implies that sub–network values directly affect the outcomes and, therefore, the analysis of bargaining in any given network relies on that of all of its sub–networks.

The example of *star networks*[1], shown in Figure 1, illustrates the resulting bargaining power of players. A star network has two types of players — a "star" (player 1) and "periphery players" $(2, \ldots, n)$. The star is connected to all periphery players, who do not have any other connections.

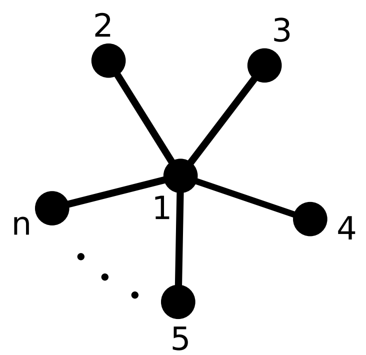


Figure 1: Star network with n players

Bargaining encounters only happen between connected players. Therefore, the only type of bargaining situation is one between the star and a periphery player. As argued later, the

[1] Stars belong to a class of networks which is of special interest. Li & Schürhoff (2019) document that the inter–dealer network in the over–the–counter municipal bonds market exhibits a *core–periphery* structure, that is, it consists of a small number of inter–connected agents (core) and a large number of agents with few connections (periphery). Beth & Atalay (2010) document a similar structure of financial institutions in the federal funds market.

outcome does not depend on which periphery player the star is bargaining with. If a player has less to lose from the elimination of the link with the current bargaining partner, she has relative bargaining power. Intuitively, we might think the star loses less than the periphery player, since she has other connections left. This intuition turns out to be correct. We might also think that the more connections a star has, the stronger her bargaining power. This, however, is false. Figure 2 shows the share of the value of the opportunity the star player receives in an equilibrium where all bargaining situations end in agreement.

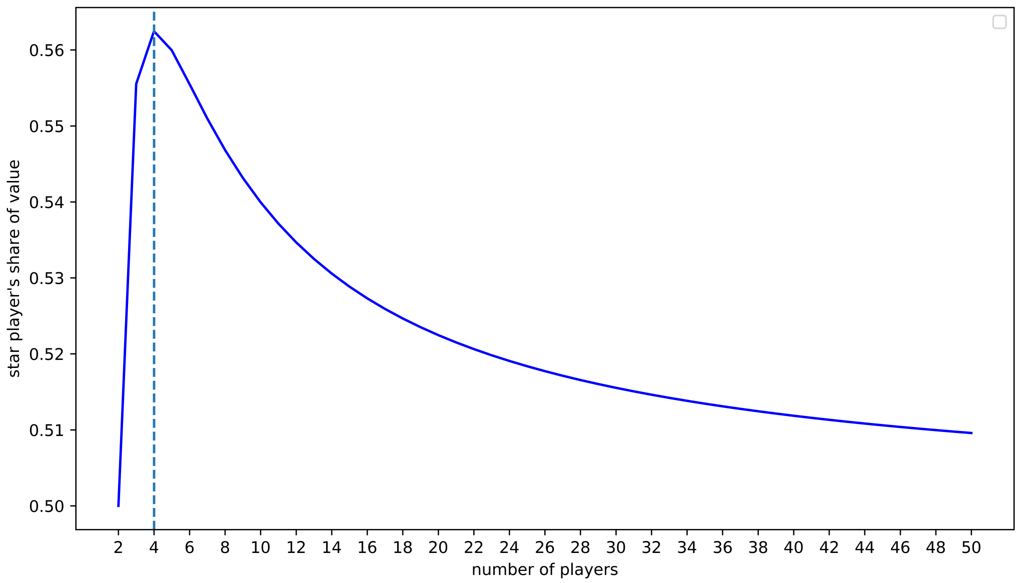


Figure 2: Star player's share of value of the opportunity

Above a certain number of connections, additional links hurt the star's bargaining power. The reason is the change in continuation value depends strongly on the expected loss of future bargaining opportunities. The periphery player can only contact the star as she has no other connections. This has two implications. First, the star's loss of future opportunities equals those that arise to the periphery player in question. Second, without the connection to the star, the periphery player would be isolated, which means she could not possibly profit from her own opportunities. The periphery player, however, also loses out on the opportunities the star would approach her with. The latter is the difference in expected loss of future opportunities for the players. It is, however, decreasing in the number of the star's connections as long as a higher number of potential bargaining partners leads to a lower probability of contacting any particular periphery player. As a result, the star's advantage decreases. As the example of star networks illustrates, differences in the players' expected loss of future bargaining opportunities are crucial in determining relative bargaining power.

I develop a continuous time non–cooperative bargaining game. Profitable opportunities arrive for players according to separate (pairwise independent) Poisson processes. If a player has an opportunity, she randomly contacts one of her neighbors — each with the same probability. Of the two players in the bargaining encounter, one is chosen as the proposer and the other as the responder, determined by a fair coin flip. The proposer makes a take–it–or–leave–it offer to the responder. If the responder accepts the offer, the players split the value of the opportunity as agreed upon and the game continues in the same network. On the other hand, if the responder rejects the offer, both players get an instantaneous payoff of zero and the game continues in the network with their link eliminated. Continuation values are determined as solutions of a system of Hamilton–Jacobi–Bellman equations. Players have perfect information of all past events. Given an initial network, information on the outcome of all past bargaining encounters is sufficient to determine the current network. I focus on Markov strategies, where players can only condition their actions on the current network as opposed to the entire history of the game. The solution concept is Markov perfect equilibrium, that is, sub–game perfect equilibrium in Markov strategies. Using an inductive argument, I show that, in any network, a Markov perfect equilibrium exists.

Under what conditions does the network remain intact in equilibrium? Since disagreement changes the network, it only remains unchanged if all bargaining situations end in agreement. Under what conditions does this constitute an equilibrium? I find that agreement is a Markov perfect equilibrium if and only if the cost of a connection is lower than a network specific cutoff. This cutoff cost can be viewed as a measure of the stability of the network. It shows us how high the cost of maintaining links should be to incentivize some player to break a link through disagreement. If a network has a higher cutoff cost than another, the first network is likelier to remain unchanged. The question arises, what networks are less susceptible to breaking. I find that star networks are more stable than lines or polygons regardless of parameter values, in particular, the arrival rate of profitable opportunities. There is also no other network structure that is more stable than a star for *all* possible arrival rates.

The organization of the paper is as follows. Section 2 reviews the related literature. Section 3 gives a formal description of the game. Section 4 discusses players' values and the equilibrium of the game. Section 5 derives certain properties of players' values from agreement in *all* bargaining situations. Section 6 explores the source of bargaining power under the assumption of agreement. Section 7 shows additional features of certain network structures. Section 8 concludes. The Appendix contains all proofs.

2 Related Literature

The seminal work of Stahl (1972), Rubinstein (1982) and Binmore (1987) used the approach of non–cooperative game theory to model the process of bargaining. Rubinstein and Wolinsky (1985, 1990), Gale (1986a,b, 1987), and McLennan & Sonnenschein (1991) applied the tools of non–cooperative bargaining to study bilateral trade in markets with a large number of agents in order to provide foundations for general equilibrium theory.

Manea (2011) develops a tractable non–cooperative bargaining game to analyze the bargaining power of players in an exogenously given network structure. The paper studies an environment where every period a pair of connected players is chosen randomly to bargain over a unit surplus. If the two players reach agreement, they get the share of surplus determined by the accepted offer, leave the network, and are *replaced by identical copies*. If bargaining ends in disagreement, they remain in the network and wait for new opportunities to bargain. Replacement of players makes the network *stationary* which leads to unique equilibrium payoffs and essentially unique equilibria. Polanski & Vega-Redondo (2017) consider a similar matching and bargaining process in pairwise stable buyer–seller (bipartite) networks with buyers as well as sellers having idiosyncratic valuations of the good traded. The paper shows that player heterogeneity as well as strategic network formation lead to inefficiencies in bargaining even as players become infinitely patient.

Corominas-Bosch (2004), Polanski (2007), and Abreu & Manea (2012a,b) study bargaining games in non–stationary networks. Corominas-Bosch (2004) examines a game where buyers and sellers make alternating public offers. In particular, in even periods, remaining sellers simultaneously post price offers and buyers post their willingness to pay. For any given price, a *maximum cardinality match* is chosen among sellers and buyers willing to trade and the players leave the network after the transaction. In graph theory, a match is a set of edges without common vertices and a maximum cardinality match is such that no other match contains a higher number of edges. Polanski (2007) assumes a maximum cardinality match is chosen among connected buyers and sellers where every pair in the match bargain bilaterally over the division of the surplus. If an agreement is reached, the two players leave the network. The assumption of a maximal match leads to efficient stationary (Markov perfect) equilibria. Abreu & Manea (2012a,b), on the other hand, relax the assumption of a maximum cardinality match chosen every period. As a result, there exist networks with no efficient stationary (Markov perfect) equilibria even as players become infinitely patient. Efficient *sub–game perfect* equilibria can be achieved, however, using a complex incentive system. My paper is closest to this brand of the literature on bargaining in networks as it considers a stochastic non–cooperative game with an exogenously given initial network. However, I analyze a game where players remain in the network after successful trades and *edges* are removed from the network as a result of *disagreement*. The first question of the paper is how bargaining power is derived from the threat of severance of connections between players aiming to carry out regular profitable trades.

Models of intermediation or, more specifically, over–the–counter financial markets naturally have agents who trade regularly in a decentralized (network) setting. The literature mostly consists of papers which use one of two approaches. *Random search* models as in Duffie, Garleanu, Pederson (2005), Üslü (2019), Hugonnier, Lester & Weill (2020), Farboodi, Jarosch & Shimer (2021), among others, have a continuum of agents, therefore the consideration of future bargaining encounters with the same partner do not affect the outcome of any present meetings. The *static network* approach assumes connections between agents are determined by an exogenously given, fixed network. Malamud & Rostek (2017) consider the effect of market decentralization on price impact and welfare. Babus & Kondor (2018) study a model of agents trading an information good and the resulting properties of information diffusion. Chang & Zhang (2021) analyze endogenous network formation with agents optimally choosing their trading partners. Agents are heterogeneous with respect to their exposure to risk. Unsuccessful encounters, however, do not rule out any potential future bargaining partners as disagreement does in my paper. The impact of threat of breaking connections in bargaining among regularly trading agents has, to the best of my knowledge, not been studied.

The second main question of my paper is which networks are less susceptible to breaking due to disagreement in bargaining. This notion of *relative stability* of networks is markedly different from stability concepts derived from that of *pairwise stability* introduced by Jackson and Wolinsky (1996). A network is pairwise stable if there is neither a pair of connected agents who wish to break their link, nor a pair of non–connected agents who wish to form a connection. In the paper, I do not consider players adding links to the network. Therefore, a network is considered stable if no pair of connected agents wish to break their link strongly enough to forgo the value of the profitable opportunity they are bargaining over. Thus, for any network, there exists a sufficiently low cost (or sufficiently high subsidy) of connections such that the network remains intact in an equilibrium of the game. A network is considered more stable than another if it remains unchanged for even higher costs (lower subsidies). To the best of my knowledge, such measures of (relative) stability of networks have not been studied in the literature.

3 The Model

Time is continuous and infinite, indexed by $t \in \mathbb{R}_+$. A *network* is given by an undirected, unweighted, connected[2], finite graph $G = (I^G, E^G)$. Every vertex in I^G corresponds to a

[2]The reason we can restrict our focus to connected graphs is that two players can only interact with each other if they are connected by an edge, as detailed later.

player. Two players k and k' are connected if there is an edge between them, that is, if $\{k,k'\} \in E^G$.[3] Let $I_k^G \equiv \left\{k' \in I^G | \{k,k'\} \in E^G\right\}$. I_k^G denotes the set of players connected to k, i.e. her *connections* or *neighbors*, in network G. All players have to pay a flow cost c per connection.[4]

Fix a network G. Consider the following bargaining game. Let $G_0 = G$. For any player $l \in I^G$, let P_l denote a Poisson process with arrival rate π.[5] By definition, none of the players' Poisson processes have an arrival at $t = 0$.

Players only make moves when a Poisson process has an arrival. Changes in the network, as detailed later, can only happen as a result of players' actions. Therefore the network remains unchanged at any point in time without a Poisson jump. The time between two jumps of a Poisson process is exponentially distributed, therefore, almost surely, there is a time interval without any arrivals. That is, the notion of *current network*, the network *just before* a Poisson jump, is, almost surely, well defined. At any possible decision point, the current network determines each player's potential bargaining partners. Formally, for any $t > 0$, almost surely, there exists $\theta \in \mathbb{R}_+$ such that the network does not change during (θ, t). Let G_{t-} denote this network. A player $k \in I^{G_{t-}}$ gets a profitable opportunity at time t if P_k has a jump at t. She cannot take advantage of this opportunity alone, so she randomly contacts one of her neighbors in order to bargain. By assumption, k contacts all of her connections in G_{t-} with equal probability. Suppose $k' \in I_k^{G_{t-}}$ is chosen.

The bargaining protocol is as follows. One of the two players, each with probability one half, is chosen as the *proposer* and the other as the *responder*. The proposer makes an offer $x \in \mathbb{R}$ to the responder, which the responder can accept or reject. If the responder accepts offer x, she gets instantaneous payoff x, the proposer gets instantaneous payoff $q-x$ and both wait for new opportunities in network $G_t \equiv G_{t-}$. If the responder rejects the offer, both of them receive an instantaneous payoff of zero and wait for new opportunities in network $G_t \equiv G_{t-} \setminus \{\{k,k'\}\} = \left(I^{G_{t-}\setminus\{\{k,k'\}\}}, E^{G_{t-}\setminus\{\{k,k'\}\}}\right)$, where $E^{G_{t-}\setminus\{\{k,k'\}\}}) = E^{G_{t-}} \setminus \{\{k,k'\}\}$, that is, the link between the two players is eliminated; and $I^{G_{t-}\setminus\{\{k,k'\}\}} = I^{G_{t-}}$, unless either k or k' (or both) become isolated from the rest of the network, in which case the isolated agents are dropped to preserve connectedness. Contacting and bargaining happen instantaneously.

Arrival rate π, value of an opportunity q, cost of a connection c, and discount rate ρ are common across all agents. Poisson processes $\{P_l\}_{l\in I^G}$ are independent. Agents have common knowledge of the game as well as perfect information of past events.

[3] The notation uses the convention $\{k,k'\} = \{k',k\}$, which means edges are undirected.

[4] A negative cost is interpreted as a subsidy or intrinsic value to connections.

[5] A Poisson process P with arrival rate π is a non–negative, non–decreasing, integer–valued stochastic process (or counting process) defined by the following three conditions. $P(0) = 0$, P has independent increments, and the number of arrivals (or "jumps") during any time period with length Δ follows a Poisson distribution with parameter $\pi\Delta$.

Let h_t denote the full history up to time $t \in \mathbb{R}_+$, that is, a collection of bargaining opportunities of all agents up to and including t, as well as the resulting bargaining situations and their outcomes up to but *not including* t. Given initial network G, past rejections fully determine the current network. Hence, h_t includes the network in place before any time t bargaining situations are resolved. Let $G(h_t)$ denote this network. Let $t_k \in \mathbb{R}_+$ be a time when player $k \in I^{G(h_{t_k})}$ has a profitable opportunity. The following events all happen at t_k. Nature chooses one of k's connections in network $G(h_{t_k})$. Suppose k' is chosen. Nature chooses the roles of proposer and responder. Let $(h_{t_k}, k \to k')$ denote the history where k is chosen as the proposer and $(h_{t_k}, k' \to k)$ when k'.

A *strategy* s_k of player k specifies — for all $t \in \mathbb{R}_+$, $k' \in I_k^G$, and histories h_t where either k or k' has an opportunity at t and $k, k' \in I^{G(h_t)}$ — an *offer* $s_k(h_t, k \to k') \in \Delta(\mathbb{R})$ and, given offer $x \in \mathbb{R}$, an *acceptance decision* $s_k(h_t, k' \to k, x) \in \Delta(\{Accept, Reject\})$, where $\Delta(Z)$ denotes the set of probability distributions with a finite support over a set Z.

A player's payoff after bargaining is resolved is the sum of the instantaneous payoff and the continuation value. The instantaneous payoff is zero if bargaining ends in disagreement (the offer is rejected) and is determined by the accepted offer otherwise. A player's continuation value is the net present value of her future cost and instantaneous payoff stream in the current network if bargaining ended in agreement, or in the network without the two players' link if bargaining ended in disagreement.

As solutions of the game, I consider sub–game perfect equilibria among Markov strategy profiles where the state is the current network $H \subseteq G$. That is, strategy profiles constituting a Nash equilibrium in all sub–games of the game and where any two histories leading to the same current network also lead to the same actions by all players. Formally, let histories h_t and $h'_{t'}$, for some $t, t' \in \mathbb{R}_+$, lead to the same network $H \subseteq G$. Then Markov strategy s_k satisfies $s_k(h_t, k \to k') = s_k(h'_{t'}, k \to k')$ and $s_k(h_t, k' \to k, x) = s_k(h'_{t'}, k' \to k, x)$ for all $k' \in I_k^H$ and $x \in \mathbb{R}$. With a slight abuse of notation, the resulting networks will be used in place of histories.

4 Values and Equilibrium

Section 3 provided a description of the game, as well as strategies, payoffs and equilibrium concept. In particular, I restrict attention to the analysis of Markov strategies, where players can only condition their actions on the current network instead of the entire full history. In this section, first I discuss players' values from playing Markov strategies. Afterwards, I characterize Markov perfect equilibrium strategy profiles. In subsequent sections, a point of emphasis is conditions for the existence of a MPE without any rejections and, therefore, elimination of links. I derive conditions for the existence of a MPE inducing any fixed vector of rejection probabilities. The section concludes by showing the existence of Markov perfect Equilibria of the game.

Consider a Markov strategy profile s, a sub–network $H \subseteq G$ and a pair of connected players $k \in I^H$, $k' \in I^H_k$. Let us denote by $s_k(H, k \to k')(x)$ the probability of offer x from k to k' in H and $s_{k'}(H, k \to k', x)$ the probability of acceptance of x by k'. Let $u((H, k' \to k, x), \hat{k}, s)$ denote the expected payoff of $\hat{k} \in \{k, k'\}$ at partial history $(H, k' \to k, x)$ under strategy profile s. Then we can write

$$\begin{aligned} u((H, k' \to k, x), k, s) &= s_k(H, k' \to k, x)\,(x + V(H, k, s)) \\ &\quad + \big(1 - s_k(H, k' \to k, x)\big)\, V(H \setminus \{\{k, k'\}\}, k, s) \\ u((H, k' \to k, x), k', s) &= s_k(H, k' \to k, x)\,\big(q - x + V(H, k', s)\big) \\ &\quad + \big(1 - s_k(H, k' \to k, x)\big)\, V(H \setminus \{\{k, k'\}\}, k', s) \end{aligned}$$

where $V(\cdot)$ denotes continuation value, as discussed later.

A player's value is potentially affected by the elimination of another player's link, even if it does not involve the first player, as it changes the network and therefore the set of possible future bargaining encounters. The exact offer leading to such an event, however has no impact on the player's value as she is not involved in the bargaining meeting. What matters is the *probability* of two other players' meeting ending in disagreement, which is captured by the concept of *induced rejection probabilities*. At partial history $(H, k \to k')$, the probability of k' *rejecting* an offer (any offer) by k under s is given by

$$\delta(H, k, k', s) \equiv \sum_{x \in supp(s_k(H, k \to k'))} s_k(H, k \to k')(x)(1 - s_{k'}(H, k \to k', x))$$

Let $\delta(s)$ denote the rejection probability vector induced by strategy profile s, i.e. $\delta(s) = \Big((\delta(H, k, k', s))_{k \in I^H, k' \in I^H_k}\Big)_{H \subseteq G}$.

Let $V(H, k, s)$ denote player k's value under strategy profile s in sub–network H, that is, her expected net present value of cost and payoff stream in H provided that players follow s. The value is given by the following Hamilton–Jacobi–Bellman equation.

$$\underbrace{\rho}_{\substack{\text{discount}\\ \text{rate}}} V(H, k, s) = \underbrace{-c|I^H_k|}_{\substack{\text{flow}\\ \text{cost}}} + \underbrace{\pi}_{\substack{\text{arrival rate}\\ \text{of opportunities}}} \sum_{k' \in I^H_k} \underbrace{\left(\frac{1}{|I^H_k|} + \frac{1}{|I^H_{k'}|}\right)}_{\text{sum of contacting probabilities}} \times$$

$$\times \left\{ \underbrace{\frac{1}{2} \sum_{x \in supp(s_{k'}(H, k' \to k))} s_{k'}(H, k' \to k)(x)\,\big(u((H, k' \to k, x), k, s) - V(H, k, s)\big)}_{\substack{\text{expected change in value}\\ \text{when } k \text{ is the responder}}} \right.$$

$$
\left.+\frac{1}{2}\underbrace{\sum_{y\in supp(s_k(H,k\to k'))} s_k(H,k\to k')(y)\left(u((H,k\to k',y),k,s)-V(H,k,s)\right)}_{\substack{\text{expected change in value}\\ \text{when } k \text{ is the proposer}}}\right\}
$$

$$
+\,\pi \sum_{\substack{\{\bar{k},\hat{k}\}\in E^H\\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}} \left(\frac{1}{|I^H_{\bar{k}}|}+\frac{1}{|I^H_{\hat{k}}|}\right) \underbrace{\frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s))}_{\text{probability of a rejected offer in a meeting between } \hat{k} \text{ and } \bar{k}}
$$

$$
\times \underbrace{\left(V(H\setminus\{\{\bar{k},\hat{k}\}\},k,s)-V(H,k,s)\right)}_{\substack{\text{change in value}\\ \text{from elimination of edge } \{\bar{k},\hat{k}\}}} \tag{1}
$$

The left–hand side of equation (1) is the discount rate times the value. The right–hand side consists of three parts. First, the flow cost of maintaining connections which is linear in the number of k's neighbors. Second, expected changes in value from k being in a bargaining encounter, multiplied by the Poisson arrival rate of such events. The arrival rate of a bargaining meeting between k and a specific neighbor is the arrival rate of opportunities π times the sum of contacting probabilities — the inverses of the number of neighbors of k and the neighbor in question, respectively. The expected change from a meeting with a neighbor (in braces) is the expected value of the difference between the payoff resulting from an offer, as discussed in the previous section, and the current value. Player k can have the role of the proposer or the responder, each with probability one half. The first summation shows the expected change in value from an offer if k is the responder, whereas the second pair if k is the proposer. The third term is expected changes from the elimination of a link not involving k times the Poisson arrival rate of such events — the product of the arrival rate of opportunities, the sum of contacting probabilities, and the probability of a rejected offer conditional on the pair being in a bargaining situation. The Appendix contains the derivation of (1) using discrete time approximation.

Let $S(H,k,k',s)$ denote the *surplus* from a meeting between neighbors k and k' in sub–network H. The surplus is the difference in total value of the two players in case they come to an agreement as opposed to a disagreement. If the meeting results in an agreement, the two players split the value of the opportunity q and network remains intact. If the meeting ends in disagreement, the players wait for opportunities in the network without their link. Therefore, we can write

$$
S(H,k,k',s)=q+V(H,k,s)+V(H,k',s)-V(H\setminus\{\{k,k'\}\},k,s)-V(H\setminus\{\{k,k'\}\},k',s)
$$

The concept of surplus from a meeting appears in the characterization of Markov perfect Equilibria as shown in Lemma 1.

The *change in continuation value* of k from the elimination of the link under s is given by

$$V(H \setminus \{\{k,k'\}\}, k, s) - V(H, k, s)$$

A player's change in value from the elimination of one of her connections is used to characterize the offers she receives, as a responder, from her neighbor in any MPE.

Lemma 1. *A strategy profile s is a MPE iff for any sub–network $H \subseteq G$ and any pair of neighbors $k \in I^H$ and $k' \in I^H_k$, the following conditions are satisfied.*

(i) At partial history $(H, k' \to k, x)$,

(a) k accepts (rejects) offer x if it is higher (lower) than her change in continuation value $V(H \setminus \{\{k,k'\}\}, k, s) - V(H, k, s)$

(b) k accepts an offer equal to her change in continuation value if the surplus from her meeting with k' is positive

(ii) At partial history $(H, k' \to k)$,

(a) the offer of k' never exceeds the change in continuation value of k

(b) the offer of k' is equal to (lower than) the change in continuation value of k if the surplus from their meeting is positive (negative)

A rejected offer leads to zero instantaneous payoff for both the proposer and responder, therefore a player's payoff at such a partial history is her change in continuation value from losing the link. The offer itself does not affect either player's value in the current network, as determined by equation (1). Lemma 1 tells us that, in any Markov perfect equilibrium, an offer that is made and accepted with positive probability equals the responder's change in continuation value from losing the link with the proposer. As a result, *rejection probabilities* are sufficient to determine players' values.

Moreover, the responder's change in value is the same after any offer the proposer makes with positive probability. The change in value from a rejected offer equals the responder's change in continuation value from losing the link; and so does the change in value from accepting the offer.

Formally, let $\delta = \left((\delta(H,k,k'))_{k \in I^H, k' \in I^H_k}\right)_{H \subseteq G}$ be a rejection probability vector. That is, a vector specifying rejection probabilities for all possible proposer–responder pairs in all sub–networks. The change in value for k from a meeting with k' can be written as

$$\frac{1}{2} \underbrace{\left(V(H \setminus \{\{k,k'\}\}, k, \delta) - V(H, k, \delta)\right)}_{k \text{ is the responder}}$$

$$+\frac{1}{2}\Bigg[\delta(H,k,k')\underbrace{\big(V(H\setminus\{\{k,k'\}\},k,\delta)-V(H,k,\delta)\big)}_{k\text{ is the proposer and }k'\text{ rejects the offer}}$$

$$+(1-\delta(H,k,k'))\underbrace{\big(q-V(H\setminus\{\{k,k'\}\},k',\delta)+V(H,k',\delta)-V(H,k,\delta)\big)}_{k\text{ is the proposer and }k'\text{ accepts the offer}}\Bigg]$$

This allows us to write the Hamilton–Jacobi–Bellman equation for a player's value given a rejection probability vector as follows.

$$\rho V(H,k,\delta)=-c|I_k^H|+\pi\sum_{k'\in I_k^H}\left(\frac{1}{|I_k^H|}+\frac{1}{|I_{k'}^H|}\right)\times$$
$$\times\frac{1}{2}\Big[\big(V(H\setminus\{\{k,k'\}\},k,\delta)-V(H,k,\delta)\big)+\delta(H,k,k')\big(V(H\setminus\{\{k,k'\}\},k,\delta)-V(H,k,\delta)\big)$$
$$+(1-\delta(H,k,k'))\big(q-V(H\setminus\{\{k,k'\}\},k',\delta)+V(H,k',\delta)\big)\Big]$$
$$+\pi\sum_{\substack{\{\bar{k},\hat{k}\}\in E^H\\ \text{s.t. }\ k\notin\{\bar{k},\hat{k}\}}}\left(\frac{1}{|I_{\bar{k}}^H|}+\frac{1}{|I_{\hat{k}}^H|}\right)\frac{1}{2}(\delta(H,\hat{k},\bar{k})+\delta(H,\bar{k},\hat{k}))\Big(V(H\setminus\{\{\bar{k},\hat{k}\}\},k,\delta)-V(H,k,\delta)\Big) \tag{2}$$

Proposition 1. *Values are well–defined for any rejection probability vector.*

Proposition 1 has two implications. First, it is straightforward to see that equation (2) given the rejection probability vector induced by strategy profile s gives the same solution as equation (1), provided that s satisfies parts (i)/(a) and (ii)/(a) of Lemma 1.[6] The two conditions are satisfied by all MPE strategy profiles, therefore, MPE values are well–defined.

Second, surpluses given a rejection probability are well–defined. Therefore we can write

$$S(H,k,k',\delta)=q+V(H,k,\delta)+V(H,k',\delta)-V(H\setminus\{\{k,k'\}\},k,\delta)-V(H\setminus\{\{k,k'\}\},k',\delta)$$

which, in turn, allows us to rewrite values in equation (2) as

$$\rho V(H,k,\delta)=-c|I_k^H|$$
$$+\pi\sum_{k'\in I_k^H}\left(\frac{1}{|I_k^H|}+\frac{1}{|I_{k'}^H|}\right)\left[V(H\setminus\{\{k,k'\}\},k,\delta)-V(H,k,\delta)+\frac{1}{2}(1-\delta(H,k,k'))S(H,k,k',\delta)\right]$$
$$+\pi\sum_{\substack{\{\bar{k},\hat{k}\}\in E^H\\ \text{s.t. }\ k\notin\{\bar{k},\hat{k}\}}}\left(\frac{1}{|I_{\bar{k}}^H|}+\frac{1}{|I_{\hat{k}}^H|}\right)\frac{1}{2}(\delta(H,\hat{k},\bar{k})+\delta(H,\bar{k},\hat{k}))\Big(V(H\setminus\{\{\bar{k},\hat{k}\}\},k,\delta)-V(H,k,\delta)\Big) \tag{3}$$

[6] That is, $V(H,k,s)=V(H,k,\delta(s))$.

Equation (3) highlights the importance of a neighbor's rejection probability. If surplus $S(H, k, k', \delta)$ is positive and rejection probability of k', $\delta(H, k, k')$, is positive as well, a decrease in the rejection probability would increase k's value. Similarly, if surplus is negative and rejection probability is smaller than one, an increase would result in a higher value for k. The concept of *individual rationality* captures this insight.

Definition 1. *A rejection probability vector δ is* individually rational *if for all $H \subseteq G$, $k \in I^H$ and $k' \in I_k^H$, we have*

(i) Rejection probability $\delta(H, k, k') = 0$ whenever surplus $S(H, k, k', \delta) > 0$

(ii) Rejection probability $\delta(H, k, k') = 1$ whenever surplus $S(H, k, k', \delta) < 0$

Proposition 2. *If s is a MPE strategy profile, then its induced rejection probability vector $\delta(s)$ is individually rational. If a rejection probability vector δ is individually rational, then there exists a MPE strategy profile $\tilde{s}$ such that its induced rejection probability vector $\delta(\tilde{s})$ equals δ.*

In subsequent sections, a key question is whether the network remaining intact through a lack of disagreement is consistent with rational players' behavior. Proposition 2 says that provided this rejection probability vector satisfies individual rationality, there exists a MPE with agreement in all bargaining encounters and therefore an unchanged network.

Theorem 1. *For all networks and parameter values, a Markov perfect equilibrium exists.*

Proofs of Proposition 1 and Theorem 1 proceed by induction on the number of edges. The appendix contains all proofs as well as a detailed discussion on the induction approach.

5 Equilibrium with Agreement

In this section, I consider a specific rejection probability vector, namely that of agreement in any bargaining situation, in any sub–network, i.e. δ equal to the zero vector. This rejection probability vector will be referred to as *agreement*.

A desirable feature of *agreement* is that agents do not miss out on valuable opportunities due to an inability to split the surplus. Analyzing *agreement* also allows us to derive conditions under which the network remains unchanged. If *agreement* is consistent with a MPE strategy profile, then a lack of changes to the network structure is consistent with rational players' behavior. Studying the case of *agreement* constituting an equilibrium enables us to compare different networks based on their susceptibility to breaking — a possible notion of *relative stability*.

Values under *agreement* are defined as a special case of values under a rejection probability vector, hence Proposition 1 implies they are well defined. Moreover, they are additively separable in costs and gains from bargaining. This leads to the observation that surplus can be written as the sum of surplus under free connections and a term involving costs. As a result, for any network, there exists a cutoff cost such that *agreement* is an equilibrium in the network for any cost lower than or equal to the cutoff, but not for any higher cost. The cutoff cost of a network can be viewed as a measure of how susceptible the network is to breaking. If the network remains intact in equilibrium, we call it *stable*. Comparison of cutoff costs among networks thus allows us to discuss relative stability of different network structures.

For any sub–network $H \subseteq G$ and player $k \in I^H$, let $W(H,k)$ denote player k's value in network H given *agreement*. For ease of notation, σ will be dropped during the discussion of values and bargaining outcomes under *agreement*. Proposition 1 implies that $W(H,k)$ is well–defined.

Equation (2) implies for *agreement*

$$\begin{aligned}\rho W(H,k) = -c|I_k^H| + \frac{\pi}{2}\sum_{k' \in I_k^H}\left(\frac{1}{|I_k^H|} + \frac{1}{|I_{k'}^H|}\right)\big(q + W(H,k') - W(H \setminus \{\{k,k'\}\}, k') \\ + W(H \setminus \{\{k,k'\}\}, k) - W(H,k)\big) \qquad (4)\end{aligned}$$

Let $\gamma(H,k,k')$ denote the expected gain of k from bargaining with k' before the roles of proposer and responder are drawn and $\beta(H,k,k')$ the expected gain as a fraction of the value of the opportunity. We can write

$$\gamma(H,k,k') = \frac{1}{2}\left(q + W(H,k') - W(H \setminus \{\{k,k'\}\}, k') + W(H \setminus \{\{k,k'\}\}, k) - W(H,k)\right) \qquad (5)$$

$$\beta(H,k,k') = \frac{\gamma(H,k,k')}{q}$$

and

$$\rho W(H,k) = -c|I_k^H| + \pi \sum_{k' \in I_k^H}\left(\frac{1}{|I_k^H|} + \frac{1}{|I_{k'}^H|}\right)\gamma(H,k,k') \qquad (6)$$

Lemma 2 shows that, in any sub–network H, values under *agreement* $(W(H,k))_{k \in I^H}$ are additively separable in cost and gains from bargaining.

Lemma 2. *For any sub–network $H \subseteq G$ and any $k \in I^H$, there exists a unique $\alpha(H,k)$ such that*

$$W(H,k) = -\frac{c|I_k^H|}{\rho} + \frac{\pi q}{\rho}\alpha(H,k) \qquad (7)$$

and neither $\alpha(H,k)$, nor $\beta(H,k,k') \equiv \frac{\gamma(H,k,k')}{q}$ depend on c or q, for any $k,k' \in I^H$ s.t. $k' \in I_k^H$.

One interpretation of $\alpha(H,k)$ is the following. If connections were free, i.e. $c = 0$, player k would be just as well off waiting for opportunities in sub–network H and bargaining or in a scenario where opportunities had value $\alpha(H,k)q$ for her and she could take the gains from these opportunities for herself without having to bargain. Therefore $\alpha(H,k)$ can be viewed as an *implied gain* of profitable opportunities without bargaining.[7]

Equation (7) has implications for equilibria with *agreement*. In particular, Proposition 2 shows that whether *agreement* is an equilibrium in a network depends on the relationship between flow cost c and a network specific cutoff cost.

Proposition 3. *Fix an arbitrary network G. There exists* cutoff cost $\bar{c}^G$ *such that* agreement *is an equilibrium in G if and only if $c \leq \bar{c}^G$.*

In the proof of Proposition 3 I show that the cutoff cost $\bar{c}^G$ is given by

$$\bar{c}^G = \min\left\{ \frac{\rho}{2} S_0(H,k,k') \middle| H \subseteq G, \{k,k'\} \in E^H \right\} \tag{8}$$

where $S_0(H,k,k')$ denotes the surplus from a meeting between k and k' in sub–network H *under the assumption of free connections.*

The cutoff–cost property of equilibrium with *agreement* allows us to develop a notion of relative stability of networks.

Definition 2. *Consider two networks G_1 and G_2. Network G_1 is more stable than G_2 if and only if $\bar{c}^{G_1} \geq \bar{c}^{G_2}$*

Definition 2 can be interpreted as follows. A network G_1 is more stable than another network G_2 if, for any cost level c such that *agreement* with the corresponding values is an equilibrium in G_2, it is also an equilibrium in G_1. That is, for any cost level, for which it is possible to not see any elimination of links in equilibrium in G_1, it is also possible in G_2.

In Section 7, relative stability of different networks will be discussed in detail. In particular, I study the relative stability of star networks in comparison to lines and polygons.

Proposition 4 shows another property $(\alpha(H,k))_{k\in I^H}$. The result is used in Section 7 to simplify the derivation of a first order difference equation characterizing agents' values in a star network.

[7]Suppose player k can get the value of $\alpha(H,k)$ many opportunities for herself without bargaining whenever her Poisson process has a jump. The present value of her payoff stream, $U(H,k)$ satisfies

$$\underbrace{\rho}_{\text{discount rate}} U(H,k) = \underbrace{\pi}_{\text{Poisson arrival rate}} \Big(\underbrace{\alpha(H,k)q}_{\text{instantaneous gain}} + \underbrace{U(H,k) - U(H,k)}_{\text{change in continuation value}} \Big)$$

$$U(H,k) = \frac{\pi q}{\rho}\alpha(H,k)$$

which is the same as her present value of bargaining (without cost) in H under *agreement.*

Proposition 4. *For any sub–network $H \subseteq G$,*

$$\sum_{k \in I^H} \alpha(H,k) = |I^H|. \tag{9}$$

Intuitively, (9) is the result of the nature of Poisson processes and *agreement*. Since Poisson processes $(P_k)_{k \in I^H}$ are independent and have the same arrival rate π, at an arrival rate of $\pi|I^H|$, *some* player in H has an opportunity. Also, *agreement* implies that no profitable opportunities are wasted by disagreement.

Equation (9) also implies that the average of $(\alpha(H,k))_{k \in I^H}$ equals one. In an environment, where a player never gets to bargain, but can seize the entire value q from her own opportunities, her value could be written in the form of (7) and her α would be equal to one. Therefore, on average, agents are just as well off under the two scenarios.

Given the number of k's connections in H, $\alpha(H,k)$ and $W(H,k)$ contain the same information. Calculations and proofs of subsequent sections use implied gains extensively, in particular equations (10), (11), and (12).

Equations (4) – (7), and the definition of $\beta(H,k,k')$ imply

$$\alpha(H,k) = \sum_{k' \in I_k^H} \left(\frac{1}{|I_k^H|} + \frac{1}{|I_{k'}^H|} \right) \beta(H,k',k) \tag{10}$$

$$\beta(H,k,k') = \frac{1}{2} \left(1 + \alpha(H,k') - \alpha(H \setminus \{\{k,k'\}\}, k') + \alpha(H \setminus \{\{k,k'\}\}, k) - \alpha(H,k)\right) \tag{11}$$

$$\begin{aligned} \alpha(H,k) = \frac{1}{2} \sum_{k' \in I_k^H} \left(\frac{1}{|I_k^H|} + \frac{1}{|I_{k'}^H|} \right) \Big(1 + \frac{\pi}{\rho} \big(\alpha(H,k') - \alpha(H \setminus \{\{k,k'\}\}, k') \\ + \alpha(H \setminus \{\{k,k'\}\}, k) - \alpha(H,k)\big)\Big) \end{aligned} \tag{12}$$

6 Bargaining Power and Values under Agreement

What causes bargaining power in an environment where connected agents meet regularly and come to an agreement on path, bargaining outcomes are determined by the threat of the severance of the link?

The bargaining protocol is symmetric in the sense that both players have the same chance of becoming a proposer. When bargaining, however, two agents need not split the value of the opportunity equally between them. In this section, I derive sufficient as well as necessary conditions on when two agents split the gain equally, and when they have the same values of waiting under *agreement*. Moreover, I derive necessary conditions for a player to have a higher implied gain than another as well as for a player to have a relative bargaining power over a connection, for any parameter values. Next, I discuss necessary conditions for surplus along an edge to be higher for all parameter values than surplus along another edge — in possibly a different network. The latter can be used to obtain results on relative stability of networks.

First, I introduce a notion of two agents being *in identical positions*. Definition 3 provides this notion.

Definition 3. *Player k is* in an identical position *to k' in network G (notation: $k \sim^G k'$) if $k, k' \in I^G$ and there exists an automorphism mapping k to k' in G, that is a function $p : I^G \to I^G$ such that*

(i) p is a bijection

(ii) $p(k) = k'$

(iii) For all $l, l' \in I^G$, $\{l, l'\} \in E^G$ if and only if $\{p(l), p(l')\} \in E^G$

It is straightforward to verify that $\sim^G$ is an equivalence relation for any network G.[8]

Definition 3 provides a notion of symmetry of a pair of agents. A notion of symmetry of the network itself is *vertex transitivity*. We call a network vertex transitive if there is an automorphism between any two agents, i.e. all agents are pairwise in identical positions. A simple example of a vertex transitive network is a polygon, which is discussed in detail in Section 7.

Corollary 1. *If agents k and k' are in identical positions in G, i.e. $k \sim^G k'$, then $\alpha(G, k) = \alpha(G, k')$.*

Corollary 1 provides a sufficient condition for two agents to have the same value under *agreement*. Note, if two agents are in identical positions, they necessarily have the same number of connections. Hence not only does $k \sim^G k'$ imply $\alpha(G, k) = \alpha(G, k')$, but also, by Lemma 1, $W(G, k) = W(G, k')$.

In what networks do *all* agents have the same value under agreement? Under what conditions do two connected agents split the value of the opportunity equally? To answer the first question, I introduce notion of *equitable* networks.

Agents' implied gains $(\alpha(G, k))_{k \in I^G}$ are functions of parameters. Lemma 2 tells us that they do not depend on c or q. It can be shown that they are only functions of the fraction $\frac{\pi}{\rho}$. In earlier sections, this dependence was suppressed for notational ease, however, in the remainder of the paper it will be of significance. In what follows, when necessary to prevent confusion, I will use the notation $\alpha(G, k, \frac{\pi}{\rho})$.

[8] For any network G, $\sim^G$ is reflexive since the identity satisfies conditions (i),(ii) and (iii); $\sim^G$ is symmetric since if $k \sim^G k'$ then there is a function p with $p(k) = k'$ that satisfies conditions (i) and (iii) and then p^{-1} satisfies (i) and (iii) as well and $p^{-1}(k') = k$ (the inverse is well–defined since p is a bijection); $\sim^G$ is transitive since if $k \sim^G k'$ and $k' \sim^G k''$ then there exist p_1 and p_2 satisfying (i) and (iii) with $p_1(k) = k'$ and $p_2(k') = k''$ and then $p_1 \circ p_2$ maps k to k'' and also satisfies (i) and (iii).

Definition 4. *A network G is called* equitable *at $\frac{\pi}{\rho}$, if $\forall k, k' \in I^G$, $\alpha(G, k, \frac{\pi}{\rho}) = \alpha(G, k', \frac{\pi}{\rho})$.*

Proposition 5 provides sufficient conditions for a network to be equitable and also for two connected agents to each get a half of the value of the opportunity.

Proposition 5. *Fix an arbitrary network G and agents $k, k' \in G$ such that $k' \in I_k^G$.*

(i) If a network is vertex transitive, then it is equitable for all values of $\frac{\pi}{\rho}$.

(ii) If k and k' are connected as well as in identical positions in both G and $G \setminus \{\{k, k'\}\}$, then $\beta(G, k, k') = \frac{1}{2}$ for all values of $\frac{\pi}{\rho}$.

The first statement of Proposition 5 follows directly from Corollary 1 and the definition of vertex transitivity. The second statement gives a partial answer to the question, "How does the network structure create differences in bargaining payments?" It tells us that if two agents are in identical positions in the network and also in the sub–network without their link, they will split the value of the opportunity q in half. The result follows from the fact that if both agents lose the same in continuation value from the loss of their link, then neither has a relative strength in bargaining.

So far we have established (sufficient) conditions on two agents having the same implied gain as well as an equal division of the value of the opportunity, if connected. This tells us when, in a pair of connected agents, neither player can have relative bargaining power over their partner. To gain more insight to the source of bargaining power, we consider necessary conditions for a player having relative bargaining power.

Let us call the limiting case of $\frac{\pi}{\rho}$ going to zero from above *infrequent opportunities*. The infrequent opportunities limit means that either profitable opportunities are arbitrarily infrequent, or players are infinitely impatient. Neither case is of economic significance. However, the limit can be obtained in closed form and used to derive necessary conditions on relative bargaining power and relative stability.

As argued above, the infrequent opportunities limit of $\alpha(G, k)$ can be derived in closed form for any network G and player $k \in I^G$.

Note, (10) implies

$$\beta(G, k, k') = \frac{1}{2}\left(1 + \frac{\pi}{\rho}\left(\alpha(G, k) - \alpha^{G\setminus\{(k,k')\}}(k)\right) - \frac{\pi}{\rho}\left(\alpha(G, k') - \alpha^{G\setminus\{(k,k')\}}(k')\right)\right)$$

and with infrequent opportunities, the weight on changes in continuation values goes to zero. As a result, under *agreement*, all bargaining situations end in value of the opportunity q split in half, i.e. $\beta(G, k, k') = \frac{1}{2}$ for any network G and edge $\{k, k'\} \in E^G$.

This leads to

$$\alpha(G, k) = \sum_{k' \in I_k^G} \left(\frac{1}{|I_{k'}^G|} + \frac{1}{|I_k^G|}\right) \beta(G, k', k) = \frac{1}{2} \sum_{k' \in I_k^G} \left(\frac{1}{|I_{k'}^G|} + \frac{1}{|I_k^G|}\right) \qquad (13)$$

where the parentheses contain the fractions of k's and her connections future opportunities k gets to bargain over. All of her own as well as the inverse degree of all of her connections, since they all contact their neighbors with equal probability.

As a side remark, note that if we assumed that in case of disagreement only the profitable opportunity was lost but not the link between the agents, for *any* parameter values, we would get the same α's as those in equation (13). This is because all we used to derive equation (13) is that all bargaining situations under *agreement* end in the two agents getting an equal share of the value of the opportunity. This is true if the link remains in case of disagreement, as the continuation value of a player would be the same regardless of the outcome of bargaining. Therefore continuation values would cancel out and surplus in any bargaining situation would be equal to q. This also means that agreement would be an equilibrium in this alternative model for any parameter values.

Equation (13) captures the direct effect that differences in the frequency of being contacted have on values of agents. For general values of $\frac{\pi}{\rho}$, we need to take into account differences in β's as well, which, in turn, come from the direct effect of differences in frequency of being contacted in sub–networks. As in the infrequent opportunities case there are no differences in β's, the above direct effect alone gives us necessary conditions on bargaining power as well as relative shares and stability.

Lemma 3. *For networks G and H, and agents $k \in G$, $k' \in H$, the following hold.*

(i) If $\alpha(G,k,\frac{\pi}{\rho}) \geq \alpha(H,k',\frac{\pi}{\rho})$ for all $\frac{\pi}{\rho} > 0$, then $\sum_{k'' \in I_k^G} \frac{1}{|I_{k''}^G|} \geq \sum_{k''' \in I_{k'}^H} \frac{1}{|I_{k'''}^H|}$.

(ii) Consider $G = H$ and $k' \in I_k^G$. If $\beta(G,k,k',\frac{\pi}{\rho}) \geq \frac{1}{2}$ for all $\frac{\pi}{\rho} > 0$, then either $|I_k^G| = 1$ or $|I_k^G| \geq |I_{k'}^G| \geq 2$.

(iii) Let $\hat{k} \in I_k^G$ and $\bar{k} \in I_{k'}^H$. If $S(G,k,\hat{k},\frac{\pi}{\rho}) \geq S(H,k',\bar{k},\frac{\pi}{\rho})$ for all $\frac{\pi}{\rho} > 0$, then

$$\frac{1}{|I_k^G|} + \frac{1}{|I_{\hat{k}}^G|} + \chi\left(|I_k^G| = 1\right) + \chi\left(|I_{\hat{k}}^G| = 1\right) \geq \frac{1}{|I_{k'}^H|} + \frac{1}{|I_{\bar{k}}^H|} + \chi\left(|I_{k'}^H| = 1\right) + \chi\left(|I_{\bar{k}}^H| = 1\right)$$

Part (i) of Lemma 3 tells us that if a player has a higher gain from bargaining regardless of the value of $\frac{\pi}{\rho}$, then the sum of inverse degrees of her neighbors is higher. The inverse degree of a neighbor is the probability of her contacting the player when the neighbor has a profitable opportunity.

Consider networks with a fixed number of agents n. Among these networks, the highest possible value of a player's sum of neighbors' inverse degrees is achieved in a star network, by the star player. This implies no player in a class of networks of fixed size can have an implied gain from bargaining always larger than that of the star player in the star network of appropriate size.

Part (ii) of Lemma 3 says that if a player gets a larger fraction of the value of the opportunity than her bargaining partner for all values of $\frac{\pi}{\rho}$ then either her partner has no other connections, or her partner has more connections than the player. From equation (11), if the fraction of the value of the opportunity a player gets when bargaining with a partner is larger than a half then the player's loss of continuation value is smaller than that of her partner. From equation (13), this holds in the infrequent opportunities limit only if either her partner has one connection, as then her partner loses access to her own profitable opportunities as well, or if both have at least two connections and her partner has at least many as the player. In the latter case, the agents lose out on the expected bargaining opportunities that arise from contacting each other, the probability of which is higher for the player's partner as the player has fewer connections and as a result contacts her partner with a higher probability.

The above result can be illustrated by the Martini glass in Figure (3).

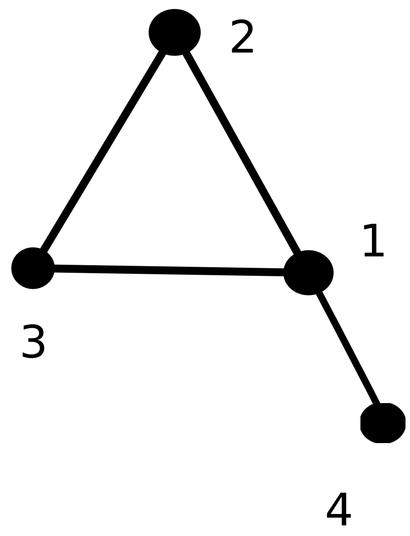


Figure 3: Martini glass

Player 1 has three connections, whereas player 2 has only two. Intuitively, one might think player 1 gets a higher share of the value of the opportunity when bargaining with player 2. However, player 1's share equals

$$\frac{1}{2}-\frac{12\frac{\pi}{\rho}+5\left(\frac{\pi}{\rho}\right)^2}{288+1392\frac{\pi}{\rho}+1994\left(\frac{\pi}{\rho}\right)^2+720\left(\frac{\pi}{\rho}\right)^3}$$

which is lower than a half. This, in turn, implies that player 2's share is higher than a half and therefore larger than that of player 1. Player 2 has greater relative bargaining power than player 1 in their meeting despite having fewer connections.

Lemma 3 also yields a result on relative stability of networks.

Proposition 6. *No other network structure is more stable for all values of $\frac{\pi}{\rho}$ than a star.*

Proposition 6 follows directly from part (iii) of Lemma 3, which relies on equation (13). If the surplus along an edge is higher then the two agents' total loss from the elimination of the link is higher. In the infrequent opportunities limit, this means the total expected loss of bargaining opportunities is higher. Note, if both agents have another connection, then the total expected loss is bounded from above by one. If one of the agents has no other connection, however, than it is strictly larger than one. If a network G is more stable than another network H for all values of $\frac{\pi}{\rho}$, then the smallest surplus in G is larger than the smallest surplus in H. If a network only has edges where one player has no other connection, then the smallest surplus is larger than one. If not, it is at most one. Star networks are the only ones satisfying this condition, therefore a network can only be more stable than a star network if it is also a (smaller) star network.

7 Lines, Polygons, and Stars

In this section I analyze specific network structures, namely lines, polygons and star networks. I discuss properties of values, and bargaining payments as well as the relative stability of these network structures.

Star networks are extreme cases of core–periphery networks, which arise in over–the–counter financial markets as well as other environments. Polygons are simple examples of vertex transitive, and therefore also regular networks. By eliminating a link from a polygon, the network becomes a line. The solution and several qualitative properties of polygons, as a result, depend on those of lines. Lines and star networks are also closed under elimination of links, which greatly simplifies their analysis.

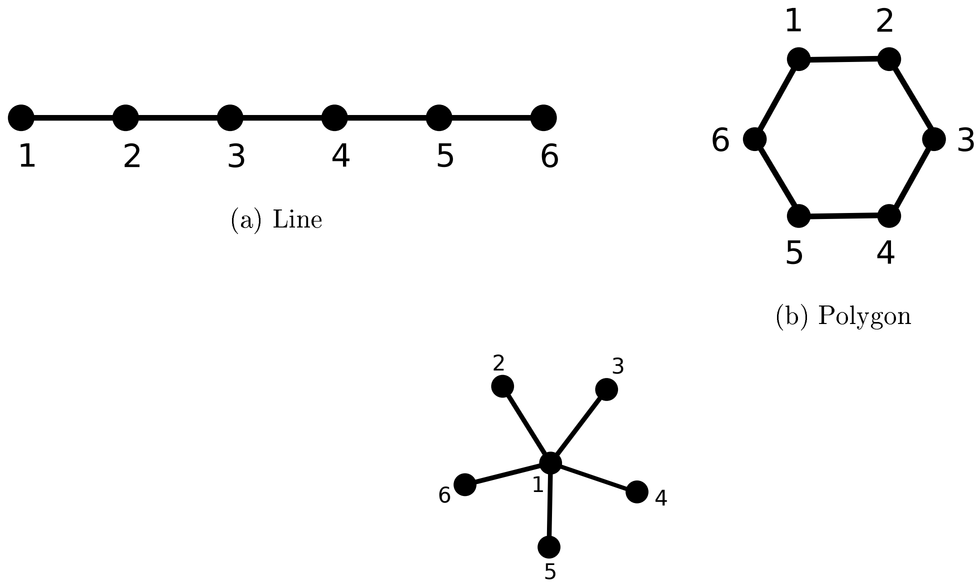
Figure 4 illustrates the three network structures with 6 agents and the labeling convention.

7.1 Lines

Labeling of agents starts at one end of the line and proceeds towards the other end. This means, if $|I^G| = n$, then $I^G = \{1, \ldots, n\}$ and $E^G = \{\{l, l+1\}\,|l \in \{1, \ldots, n-1\}\}$.

Solutions to (12), (11), and (4) are given by

$$\alpha(G,1) = \alpha(G,n) = \begin{cases} 1 & \text{if } n = 2 \\ \frac{3+6\frac{\pi}{\rho}}{4+9\frac{\pi}{\rho}} & \text{otherwise} \end{cases}$$



(a) Line

(b) Polygon

(c) Star

Figure 4: The Three Network Structures with 6 Agents

$$
\alpha(G,2) = \alpha(G,n-1) = \begin{cases} 1 & \text{if } n=2 \\ 3 - 2\frac{3+6\frac{\pi}{\rho}}{4+9\frac{\pi}{\rho}} & \text{if } n=3 \\ 2 - \frac{3+6\frac{\pi}{\rho}}{4+9\frac{\pi}{\rho}} & \text{otherwise} \end{cases}
$$

$$
\forall 3 \le l \le n-2, \qquad \alpha(G,l) = 1
$$

$$
\beta(G,1,2) = 1 - \beta(G,2,1) = \begin{cases} \frac{1}{2} & \text{if } n=2 \\ \frac{2+4\frac{\pi}{\rho}}{4+9\frac{\pi}{\rho}} & \text{otherwise} \end{cases}
$$

$$
\forall 2 \le l \le n-2, \qquad \beta(G,l,l+1) = \beta(G,l+1,l) = \frac{1}{2}
$$

$$
S(G,1,2) = \begin{cases} q(1+2\frac{\pi}{\rho}) - \frac{2c}{\rho} & \text{if } n=2 \\ q\frac{4+14\frac{\pi}{\rho}+12\frac{\pi^2}{\rho^2}}{4+9\frac{\pi}{\rho}} - \frac{2c}{\rho} & \text{otherwise} \end{cases}
$$

$$
\forall 2 \le l \le n-2, \qquad S(G,l,l+1) = q\frac{4+11\frac{\pi}{\rho}+6\frac{\pi^2}{\rho^2}}{4+9\frac{\pi}{\rho}} - \frac{2c}{\rho}
$$

Agents k and $n-k+1$ are identical in the line, for any $k \in I^G$, therefore Proposition 2 implies that $\alpha(G,k) = \alpha(G,n-k+1)$. However, such pairs are not the only ones with the same value for sufficiently long lines. In particular, for $n \ge 7$, agents $3,4,\dots,n-2$ all have the same value, even though they are not all pairwise identical. This means, in general, the sufficient condition in Proposition 2, (i) is not necessary.

It can be shown that all surpluses are strictly increasing in π, which means agents lose more from the elimination of the connection if opportunities arise more frequently. It can also be shown that, for $n \geq 4$, the surplus along the edge adjacent to the end of the line is higher than along edges in the middle of the line. Agents at the end of a line face an outside option of value zero, which leads to the higher surplus.

Using (8), we can calculate

$$\bar{c}^G = \begin{cases} \frac{q\rho}{2}(1 + 2\frac{\pi}{\rho}) & \text{if } n = 2 \\ \frac{q\rho}{2}\frac{4+14\frac{\pi}{\rho}+12\frac{\pi^2}{\rho^2}}{4+9\frac{\pi}{\rho}} & \text{if } n = 3 \\ \frac{q\rho}{2}\frac{4+11\frac{\pi}{\rho}+6\frac{\pi^2}{\rho^2}}{4+9\frac{\pi}{\rho}} & \text{if } n \geq 4 \end{cases}$$

Therefore lines with two agents are more stable than lines with three which, in turn, are more stable than lines with four or more agents, but those are then just as stable, regardless of the size of the network.

7.2 Polygons

Assume $n \geq 3$. Labeling convention is from 1 to n, clockwise.

It is straightforward to show that all agents are pairwise identical and also any two connected agents are also identical in the sub–network without their connection — as, in that case, they are on the two ends of a line. Proposition 4 then implies $\alpha(G, k) = 1$ for all $k \in I^G$ and $\beta(G, k, k') = \frac{1}{2}$ for any two connected agents k and k'. Using (4), the solution for lines, and (8), we can calculate

$$\forall l \in I^G, \qquad S(G, l, l+1) = q\frac{4 + 11\frac{\pi}{\rho} + 6\frac{\pi^2}{\rho^2}}{4 + 9\frac{\pi}{\rho}} - \frac{2c}{\rho}$$

$$\bar{c}^G = \frac{q\rho}{2}\frac{4 + 11\frac{\pi}{\rho} + 6\frac{\pi^2}{\rho^2}}{4 + 9\frac{\pi}{\rho}}$$

Which means that polygons with at least three agents, but otherwise regardless of the size of the network, are just as stable as lines with four or more agents.

7.3 Stars

Assume $n \geq 4$ as for $n = 2$ or $n = 3$, there is no distinction between lines and stars. The labeling convention is that the "star" player has label 1 and the other agents are labeled clockwise.

Since all agents other than the star are (pairwise) in identical positions, all terms on the right–hand side of (11) are the same regardless of the identity of the player bargaining with player 1. As a result, $\beta(G, k, 1) = \beta(G, 2, 1)$ for any $k \geq 2$ and (10) implies

$$\alpha(G, 1) = n\beta(G, 2, 1) \tag{14}$$

Let us add the number of agents n as an additional argument in the notation.

Note, since all agents other than the star are pairwise identical, (9) implies $\alpha(G,n,k) = \frac{n-\alpha(G,n,1)}{n-1}$, for any $k \geq 2$. This, together with (14) leads to the difference equation

$$\alpha(G,n,1) = \frac{n(n-1)+n^2\frac{\pi}{\rho}}{2(n-1)+n^2\frac{\pi}{\rho}} + \frac{n(n-1)\frac{\pi}{\rho}}{2(n-1)+n^2\frac{\pi}{\rho}}\alpha(G,n-1,1) \tag{15}$$

with the closed form solution of a line with three agents as the initial condition. Equation (15) allows us to derive certain properties of star networks.

Proposition 7. *The star player's share of value of the opportunity, $\beta(n,2,1)$ is larger than a half for any number of agents $n \geq 3$ and goes to a half as the number of agents n goes to infinity.*

At first glance, Proposition 7 might seem counter–intuitive. It tells us that a star player's relative strength in bargaining compared to a periphery player vanishes as the number of agents grows sufficiently large. This also implies that a star player's relative strength is non–monotone in the number of her connections. The intuition behind the result is that relative bargaining strength is determined by the difference in loss of continuation value through the elimination of the link. As argued in the infrequent opportunities case, the loss of the probability that the current bargaining partner contacts the player in the future has a direct effect. However, the periphery player does not only lose future bargaining opportunities with the star when the star has a profitable opportunity, but also the ability to get any value out of her own opportunities. The difference in direct effect as shown in (13) would be $\frac{1}{2}$ for the star and $\frac{1}{2}+\frac{1}{2(n-1)}$ for the periphery player. In the limit as n goes to infinity, the two direct effects cancel out.

Proposition 8. *A star network with at least four agents is strictly more stable than a line with at least four agents or a polygon with at least three agents.*

Differences in surpluses are determined by differences in the pair of agents' total loss of continuation value from losing the edge. The direct effect is always larger if one of the agents has no other connections, since that player loses the ability to take advantage of her own opportunities as well. The direct effect between two agents with degrees d_1 and d_2, as shown by (13) is $\frac{1}{2d_1}+\frac{1}{2d_1}+\frac{\chi(d_1=1\vee d_2=1)}{2}$, where χ is the characteristic function of an event. If neither of the two degrees equals one, the direct effect is necessarily smaller than or equal to a half. However, if one of the degrees is equal to one, the direct effect is strictly larger than a half. This means if a network only has edges where one of the agents has degree one, the minimal direct effect on surpluses is strictly larger than in a network with an edge where both agents have at least degree two. Start networks fall in the first category, whereas lines and polygons in the latter. This implies that, without indirect effects, stars are always more stable than lines and polygons. Proposition 8 shows that, taking indirect effects into account, star networks are still more stable.

8 Conclusion

Real–world examples include networks with relatively few agents who interact with each other frequently. I analyze a non–cooperative bargaining game in an explicit network structure, with the threat of severed links in case of disagreement. I restrict attention to Markov strategies, where for any two histories that result in the same sub–network, players choose the same action. First I prove the existence of Markov perfect equilibria, then discuss a specific strategy profile — one with no disagreement and therefore the network intact. I derive necessary and sufficient conditions for such a strategy profile to be a MPE, which take the form of a cutoff rule. No disagreement is a MPE if and only if the flow cost of maintaining a connection is lower than the network specific cutoff. Comparison of these cutoff costs across networks provides a measure of relative stability. In a class of networks including lines, polygons and stars, star networks are the most stable for any parameter values. Furthermore, there is no other network that is more stable than a star for all parameter values. In addition to the analysis of relative stability, the main finding of the paper is that relative bargaining power under the equilibrium without breaking is largely determined by the difference in players' loss of future bargaining encounters resulting from the severance of the link.

Appendix

The Induction Approach

Most proofs in the paper follow an induction argument. We can see from (3) that values in any given network depend directly on values attained in its sub–networks, more precisely those reached by eliminating one edge. Definition 2 requires an equilibrium to be an equilibrium in any sub–networks of the original network. As a result, to show existence, we need to show existence in all sub–networks. This requires an induction approach. Induction requires a strict complete order on the (countable) set of objects in question with a well–defined starting point. For our purposes, the order has to satisfy the requirement that any network is preceded by all of its sub–networks. Therefore we need a strict complete order on the set of connected, undirected, unweighted and unlabeled[9] finite graphs (which are indeed contained in a countable set) with a well–defined starting point. More precisely, what we need is, for any given network G, an order which, restricted to the set of sub–networks of the network in question, satisfies that for any $H \subseteq G$, any $\{k, k'\} \in E^H$, $H \setminus \{\{k, k'\}\}$ (strictly) precedes H in the order, and the order has a well–defined starting point. There are several possible ways to construct such an order, I will propose one. First, form equivalence classes of networks based on the number of edges, i.e. $\mathcal{G}(m) = \big\{G \big| |E^G| = m\big\}$. Second,

[9]Since sub–networks inherit the labeling of the original network in question.

order equivalence classes such that $\mathcal{G}(m)$ precedes $\mathcal{G}(m')$ if and only if $m \leq m'$.[10] Third, use *any* order within equivalence classes. This suffices because for any network G, any $H \subseteq G$ and $\{k,k'\} \in E^H$, $H \setminus \{\{k,k'\}\}$ belongs to indifference class $\mathcal{G}(|E^H|-1)$, whereas H belongs to indifference class $\mathcal{G}(|E^H|)$ and the former necessarily (strictly) precedes the latter. An order constructed as above also has a well–defined starting point, as $\mathcal{G}(1)$ is a singleton.[11] Let G_0 denote the unique element of $\mathcal{G}(1)$.

Derivation of Equation (1)

I will derive equation (1) using a discrete time approximation. This means I will write the value in network G to player k under s recursively in discrete time. Then look at the limit as the length of time periods goes to zero. Let $\Delta > 0$ denote the length of a time period. Start at time t. $V(H,k,s,t)$ satisfies equation (16)

$$
\begin{aligned}
V(H,k,s,t) = &-c\Delta|I_k^H| + e^{-\rho\Delta}\Bigg[\pi\Delta \sum_{k' \in I_k^H}\left(\frac{1}{|I_k^H|}+\frac{1}{|I_{k'}^H|}\right)\times \\
&\times \Bigg\{\frac{1}{2}\sum_{x \in supp(s_{k'}(H,k'\to k))} s_{k'}(H,k'\to k)(x)u((H,k'\to k,x),k,s,t+\Delta) \\
&+\frac{1}{2}\sum_{y \in supp(s_k(H,k\to k'))} s_k(H,k\to k')(y)u((H,k\to k',y),k,s,t+\Delta)\Bigg\} \\
&+\pi\Delta \sum_{\substack{\{\bar{k},\hat{k}\}\in E^H \\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}}\left(\frac{1}{|I_{\bar{k}}^H|}+\frac{1}{|I_{\hat{k}}^H|}\right)\frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s))\Big(V(H\setminus\{\{\bar{k},\hat{k}\}\},k,s,t+\Delta)\Big) \\
&+\left(1-\pi\Delta\sum_{k'\in I_k^H}\left(\frac{1}{|I_k^H|}+\frac{1}{|I_{k'}^H|}\right)-\pi\Delta\sum_{\substack{\{\bar{k},\hat{k}\}\in E^H \\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}}\left(\frac{1}{|I_{\bar{k}}^H|}+\frac{1}{|I_{\hat{k}}^H|}\right)\frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s))\right) \\
&\times V(H,k,s,t+\Delta)\Bigg]
\end{aligned}
\tag{16}
$$

where the first term is the cost of connections player k has to pay over length of time Δ, $e^{-\rho\Delta}$ is the discount factor which multiplies the expected value of k in the next period, that is at time $t+\Delta$. The probability of a Poisson process P_l (for any $l \in I^H$) having a jump in the time interval in question is $\pi\Delta$ by the definition of Poisson processes. Multiple jumps in the

[10] Note, there cannot be ties between two different equivalence classes. If $m = m'$, then by definition $\mathcal{G}(m) = \mathcal{G}(m')$.

[11] Technically, for many applications, we could view the singleton $\mathcal{G}(0)$, i.e. a network with only an isolated vertex in it, as an (uninteresting) starting point. This network is not a focus of interest, however, and all proofs will start the induction from the one network in $\mathcal{G}(1)$.

same time period, however, happen with probabilities which are higher order terms in Δ. (The same process P_l having $m > 1$ many jumps happens with probability $\frac{(\pi\Delta)^m}{m!}$ because the number of jumps follows a Poisson distribution with parameter $\pi\Delta$. Multiple processes having one jump each also happens with a probability that is a higher order term in Δ because of the independence of the processes.) We can, therefore, ignore the possibility of multiple opportunities arising between time t and $t+\Delta$. These are also excluded in (16). The first summation includes the possibilities of k or one of her connections having a profitable opportunity, meeting each other with k getting her expected payoff, as a responder and a proposer, as given by s. The second summation includes the cases of other players' bargaining encounters ending in disagreement. In this case, an edge is eliminated and the continuation value of player k is that of her value in the new network, $V(H \setminus \left\{\left\{\bar{k},\tilde{k}\right\}\right\}, k, s, t)$. The third term considers the case when no player gets an opportunity, in which case the continuation value of k is that of waiting in the same network.

For small Δ, $e^{-\rho\Delta}$ is approximately equal to $\frac{1}{1+\rho\Delta}$. Hence we can multiply both sides of (16) by $1+\rho\Delta$ and subtract $V(H,k,s,t)$ from both sides to get

$$
\begin{aligned}
\rho\Delta V(H,k,s,t) = -c\Delta(1+\rho\Delta)|I_k^H| + \Bigg[& \pi\Delta \sum_{k'\in I_k^H} \left(\frac{1}{|I_k^H|}+\frac{1}{|I_{k'}^H|}\right) \times \\
\times \Bigg\{ & \frac{1}{2} \sum_{x\in supp(s_{k'}(H,k'\to k))} s_{k'}(H,k'\to k)(x)\left(u((H,k'\to k,x),k,s,t+\Delta) - V(H,k,s,t)\right) \\
& + \frac{1}{2} \sum_{y\in supp(s_k(H,k\to k'))} s_k(H,k\to k')(y)\left(u((H,k\to k',y),k,s,t+\Delta) - V(H,k,s,t)\right) \Bigg\} \\
& + \pi\Delta \sum_{\substack{\{\bar{k},\hat{k}\}\in E^H \\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}} \left(\frac{1}{|I_{\bar{k}}^H|}+\frac{1}{|I_{\hat{k}}^H|}\right) \frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s)) \left(V(H\setminus\{\{\bar{k},\hat{k}\}\},k,s,t+\Delta) - V(H,k,s,t)\right) \\
& + \left(1 - \pi\Delta \sum_{k'\in I_k^H} \left(\frac{1}{|I_k^H|}+\frac{1}{|I_{k'}^H|}\right) - \pi\Delta \sum_{\substack{\{\bar{k},\hat{k}\}\in E^H \\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}} \left(\frac{1}{|I_{\bar{k}}^H|}+\frac{1}{|I_{\hat{k}}^H|}\right) \frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s)) \right) \\
& \times (V(H,k,s,t+\Delta) - V(H,k,s,t)) \Bigg]
\end{aligned} \tag{17}
$$

Dividing through (17) by Δ yields

$$
\begin{aligned}
\rho V(H,k,s,t) = -c(1+\rho\Delta)|I_k^H| + \Bigg[& \pi \sum_{k'\in I_k^H} \left(\frac{1}{|I_k^H|}+\frac{1}{|I_{k'}^H|}\right) \times \\
\times \Bigg\{ & \frac{1}{2} \sum_{x\in supp(s_{k'}(H,k'\to k))} s_{k'}(H,k'\to k)(x)\left(u((H,k'\to k,x),k,s,t+\Delta) - V(H,k,s,t)\right)
\end{aligned}
$$

$$+\frac{1}{2}\sum_{y\in supp(s_k(H,k\to k'))} s_k(H,k\to k')(y)\left(u((H,k\to k',y),k,s,t+\Delta)-V(H,k,s,t)\right)\Bigg\}$$

$$+\pi\sum_{\substack{\{\bar{k},\hat{k}\}\in E^H\\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}}\left(\frac{1}{|I^H_{\bar{k}}|}+\frac{1}{|I^H_{\hat{k}}|}\right)\frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s))\left(V(H\setminus\{\{\bar{k},\hat{k}\}\},k,s,t+\Delta)-V(H,k,s,t)\right)$$

$$+\left(1-\pi\Delta\sum_{k'\in I^H_k}\left(\frac{1}{|I^H_k|}+\frac{1}{|I^H_{k'}|}\right)-\pi\Delta\sum_{\substack{\{\bar{k},\hat{k}\}\in E^H\\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}}\left(\frac{1}{|I^H_{\bar{k}}|}+\frac{1}{|I^H_{\hat{k}}|}\right)\frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s))\right)$$

$$\times\frac{V(H,k,s,t+\Delta)-V(H,k,s,t)}{\Delta}\Bigg] \tag{18}$$

Taking the $\Delta\to 0$ limit then leads to

$$\rho V(H,k,s,t)=-c|I^H_k|+\left[\pi\sum_{k'\in I^H_k}\left(\frac{1}{|I^H_k|}+\frac{1}{|I^H_{k'}|}\right)\times\right.$$

$$\times\Bigg\{\frac{1}{2}\sum_{x\in supp(s_{k'}(H,k'\to k))} s_{k'}(H,k'\to k)(x)\left(u((H,k'\to k,x),k,s,t)-V(H,k,s,t)\right)$$

$$+\frac{1}{2}\sum_{y\in supp(s_k(H,k\to k'))} s_k(H,k\to k')(y)\left(u((H,k\to k',y),k,s,t)-V(H,k,s,t)\right)\Bigg\}$$

$$+\pi\sum_{\substack{\{\bar{k},\hat{k}\}\in E^H\\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}}\left(\frac{1}{|I^H_{\bar{k}}|}+\frac{1}{|I^H_{\hat{k}}|}\right)\frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s))\left(V(H\setminus\{\{\bar{k},\hat{k}\}\},k,s,t)-V(H,k,s,t)\right)$$

$$+\left(1-\pi\Delta\sum_{k'\in I^H_k}\left(\frac{1}{|I^H_k|}+\frac{1}{|I^H_{k'}|}\right)-\pi\Delta\sum_{\substack{\{\bar{k},\hat{k}\}\in E^H\\ \text{s.t. } k\notin\{\bar{k},\hat{k}\}}}\left(\frac{1}{|I^H_{\bar{k}}|}+\frac{1}{|I^H_{\hat{k}}|}\right)\frac{1}{2}(\delta(H,\hat{k},\bar{k},s)+\delta(H,\bar{k},\hat{k},s))\right)$$

$$\times\frac{\partial V(H,k,s,t)}{\partial t}\Bigg]$$

where the last term is the time derivative of $V(H,k,s,t)$. Since we restrict our attention to Markov and hence time–independent values (since the state is time–independent), the time derivative equals 0 and all t subscripts can be dropped, which yields (1).

Proof of Lemma 1

(i) (a) Follows directly from sub–game perfection. Her change in value from the elimination of the link is the offer that makes the responder just indifferent.

(b) Has to hold, otherwise the proposer's problem has no solution.

(ii) (a) Follows directly from sub–game perfection. An offer that makes the responder strictly prefer acceptance can never be optimal.

(b) The proposer is (strictly) better off with the acceptance of an offer equal to the responder's change in value from the elimination of the link if and only if surplus is (positive) non–negative. To see this, note that the proposer's (assumed to be k') payoff conditional on acceptance is $q - (V(H \setminus \{\{k, k'\}\}, k, s) - V(H, k, s)) + V(H, k', s)$ whereas in case of rejection, her payoff is $V(H \setminus \{\{k, k'\}\}, k', s)$. Payoff conditional on acceptance is (strictly) larger if and only if surplus is (positive) non–negative.

This concludes the proof.

Proof of Proposition 1

We want to show that for any network H, and rejection probability vector δ, there exists a unique solution to the system of equations described by (2), taking sub–network values as given, which are also uniquely determined.

The proof follows the induction procedure discussed at the start of the Appendix. For the starting point, we need to solve for the values of agents in G_0. For notational ease, drop G_0 and call rejection probabilities simply $\delta(1,2)$, $\delta(2,1)$ and . Note, in any case of the one edge being eliminated, both agents end up with a continuation value of zero, as they cannot bargain or take advantage of profitable opportunities if isolated. Using this observation, (1) gives us two equations

$$\rho V(1,\delta) = -c - \pi(1 + \delta(1,2))V(1,\delta) + \pi(1 - \delta(1,2))\,(q + V(2,\delta))$$
$$\rho V(2,\delta) = -c - \pi(1 + \delta(2,1))V(1,\delta) + \pi(1 - \delta(2,1))\,(q + V(1,\delta))$$

This system of equations has a unique solution,

$$\begin{aligned} V_\delta(1) &= \frac{-c(\rho + 2\pi + \delta(2,1) - \delta(1,2) + \pi q(\rho + 2\pi)(1 - \delta(1,2))}{\rho^2 + \pi(\rho + 2\pi)(\delta(1,2) + \delta(2,1)} \\ V_\delta(2) &= \frac{-c(\rho + 2\pi + \delta(1,2) - \delta(2,1) + \pi q(\rho + 2\pi)(1 - \delta(2,1))}{\rho^2 + \pi(\rho + 2\pi)(\delta(1,2) + \delta(2,1)} \end{aligned} \tag{19}$$

Hence G_0 satisfies the claim and the induction has a valid starting point. Now suppose that every network satisfies the claim up to a certain index in the order of induction and G is the network in the order. I will refer to this assumption as the induction hypothesis. Let $V(H,\delta) \in \mathbb{R}^{|I^H|}$ denote the vector containing values $(V(H,l,\delta))_{l \in I^H}$. It suffices to show, as the induction step, that there exists a unique solution $V(H,\delta)$ to the system of equations given by (2). Rearranging (2) allows us to write the system of equations in the following matrix form.

$$A(H,\delta)V(H,\delta) = y(H,\delta)$$

where

$$
\begin{aligned}
A(H,\delta)_{ll'} &= 0 \qquad \text{if } \{l,l'\} \notin E^H \\
A(H,\delta)_{ll} &= 1, \qquad \forall l \in I^H \\
A(H,\delta)_{ll'} &= -\tilde{\pi}^{H,\delta}_{ll'}, \qquad \text{if } \{l,l'\} \in E^H \\
y(H,\delta)_l &= -c|I^H_l| + \sum_{l' \in I^H_l} \tilde{\pi}^{H,\delta}_{ll'} \left(q + V(H \setminus \{\{l,l'\}\}, l, \delta) - V(H \setminus \{\{l,l'\}\}, l', \delta)\right) \\
&\quad + \sum_{\{\bar{l},\hat{l}\} \in E^H} \tilde{\pi}^{H,\delta}_{l\bar{l}\hat{l}} V(H \setminus \left\{\left\{\bar{l},\hat{l}\right\}\right\}, l, \delta)
\end{aligned}
$$

where

$$
\tilde{\pi}^{H,\delta}_{ll'} = \frac{\pi\left(1-\delta(H,l,l')\right)\left(\frac{1}{|I^H_l|}+\frac{1}{|I^H_{l'}|}\right)}{2\rho + \pi\sum_{l' \in I^H_l}\left(1-\delta(H,l,l')\right)\left(\frac{1}{|I^H_l|}+\frac{1}{|I^H_{l'}|}\right) + \pi\sum_{\{\bar{l},\hat{l}\} \in E^H}\left(\delta(H,\bar{l},\hat{l})+\delta(H,\hat{l},\bar{l})\right)\left(\frac{1}{|I^H_{\bar{l}}|}+\frac{1}{|I^H_{\hat{l}}|}\right)}
$$

$$
\tilde{\pi}^{H,\delta}_{l\bar{l}\hat{l}} = \frac{\pi(\delta(H,\bar{l},\hat{l})+\delta(H,\hat{l},\bar{l}))\left(\frac{1}{|I^H_{\bar{l}}|}+\frac{1}{|I^H_{\hat{l}}|}\right)}{2\rho + \pi\sum_{l' \in I^H_l}\left(1-\delta(H,l,l')\right)\left(\frac{1}{|I^H_l|}+\frac{1}{|I^H_{l'}|}\right) + \pi\sum_{\{\bar{l},\hat{l}\} \in E^H}\left(\delta(H,\bar{l},\hat{l})+\delta(H,\hat{l},\bar{l})\right)\left(\frac{1}{|I^H_{\bar{l}}|}+\frac{1}{|I^H_{\hat{l}}|}\right)}
$$

By the construction of the order and the induction hypothesis, all sub–network values in right–hand side vector $y(H,\delta)$ are uniquely determined. All coefficients in $y(H,\delta)$ and entries of $A(H,\delta)$ are uniquely determined by rejection probability vector δ and the network structure of H, that is sets I^H and E^H. It suffices to show that $A(H,\delta)$ is invertible.

Define $B(H,\delta) \equiv I_{|I^H| \times |I^H|} - A(H,\delta)$ where $I_{|I^H| \times |I^H|}$ denotes the $|I^H| \times |I^H|$ identity matrix. It is straightforward to show that all row–sums of $B(H,\delta)$ are strictly smaller than one and hence so is its spectral radius. This means $B(H,\delta)$ is a convergent matrix and hence $A(H,\delta) = I_{|I^H| \times |I^H|} - B(H,\delta)$ is invertible, which concludes the proof.

Proof of Proposition 2

If s is a MPE, then induced rejection probability vector $\delta(s)$ is individually rational:

Suppose s is a MPE strategy profile. Pick an arbitrary $H \subseteq G$, $k \in I^H$ and $k' \in I^H_k$. Assume, w.l.o.g., that k is the proposer. First, we need to show that if $S(H,k,k',s) > 0$, then $\delta(H,k,k',s) = 0$. Assume $S(H,k,k',s) > 0$. Lemma 1 (i)/(b) and (ii)/(b) imply that the proposer's offer is accepted with probability one. That is, rejection probability $\delta(H,k,k',s)$ equals zero. Second, we need to show that if $S(H,k,k',s) < 0$, then $\delta(H,k,k',s) = 1$. Lemma 1 (i)/(a) and (ii)/(b) imply that the proposer's offer is rejected with probability one, that is $\delta(H,k,k',s) = 1$ is satisfied.

If δ is individually rational, then there exists a MPE s such that $\delta(s) = \delta$:

Suppose δ is an individually rational rejection probability vector. Pick an arbitrary $H \subseteq G$, $k \in I^H$ and $k' \in I^H_k$. We know from Proposition 1 that $S(H,k,k',\delta$ is well defined.

Assume, w.l.o.g., that k is the proposer. If $S(H, k, k', \delta) > 0$ then, by individual rationality, $\delta(H, k, k') = 0$. Let $s_k(H, k \to k') = V(H \setminus \{\{k, k'\}\}, k', \delta) - V(H, k', \delta)$ and let $s_{k'}(H, k \to k', x) = 1$ iff $x \geq V(H \setminus \{\{k, k'\}\}, k', \delta) - V(H, k', \delta)$. If $S(H, k, k', \delta) < 0$ then, by individual rationality, $\delta(H, k, k') = 1$. Let $s_k(H, k \to k') = x$ for some $x < V(H \setminus \{\{k, k'\}\}, k', \delta) - V(H, k', \delta)$ and let $s_{k'}(H, k \to k', x) = 1$ iff $x \geq V(H \setminus \{\{k, k'\}\}, k', \delta) - V(H, k', \delta)$. If $S(H, k, k', \delta) = 0$, let $s_k(H, k \to k') = V(H \setminus \{\{k, k'\}\}, k', \delta) - V(H, k', \delta)$ and let $s_{k'}(H, k \to k', x) = \delta(H, k, k')$. It follows that $\delta(H, k, k', s) = \delta(H, k, k')$, therefore δ and s induce the same values and surpluses. As a result, all conditions of Lemma 1 are satisfied and s is a MPE strategy profile.

This concludes the proof.

Proof of Theorem 1

The proof follows the induction argument outlined at the beginning of the Appendix. By Proposition 2, it suffices to show that there exists an individually rational rejection probability vector for any network. For a starting point, we need to show that there exists an equilibrium in G_0.

(19) and (4) imply that the surplus among the two agents given recommended disagreement probability δ equals

$$S(1, 2, \delta) = \frac{-2c(\rho + 2\pi) + q(\rho^2 + 2\pi(\rho + 2\pi) + 2\rho\pi)}{\rho^2 + \pi(\rho + 2\pi)(\delta(1, 2) + \delta(2, 1)}$$

Clearly the denominator is positive, therefore the sign of the surplus is determined by the sign of the numerator. We get

$$S(1, 2, \delta) \begin{cases} > 0 if c < q\frac{\rho+2\pi}{2} \\ < 0 if c > q\frac{\rho+2\pi}{2} \\ = 0 if c = q\frac{\rho+2\pi}{2} \end{cases}$$

It is straightforward to construct an individually rational rejection probability vector for any parameter values and therefore the induction has a valid starting point.

Now suppose, that for all networks up to a certain index in the order an equilibrium exists and G is the next network in the order. An individually rational rejection probability vector in a network H restricted to some sub–network $H' \subset H$ is also individually rational in H'. By the induction hypothesis, for all sub–networks of G, an individually rational rejection probability vector exists. Thus we can piece together $\hat{\delta}$ which contains rejection probabilities for all proper sub–networks of G and, restricted to any such sub–network, is individually rational. Then, given $\hat{\delta}$, all relevant sub–network values are uniquely determined (by Proposition 1), which allows us to calculate surpluses under $(\hat{\delta}, \delta')$ for any guess $\delta' \in [0, 1]^{|E^G|^2}$. Then for any $\{k, k'\} \in E^G$, $S(G, k, k', (\hat{\delta}, \delta'))$ is well–defined. Define

$$\Psi_k(\delta') \equiv \underset{\left(\tilde{\delta}(G,k,k')\right)_{k' \in I_k^G} \in [0,1]^{|I_k^G|}}{\arg\max} \left\{ -c|I_k^G| + \pi \sum_{k' \in I_k^G} (1 - \tilde{\delta}(G, k, k')) \left(\frac{1}{|I_k^G|} + \frac{1}{|I_{k'}^G|} \right) (V(G \setminus \{\{k, k'\}\}, k, \hat{\sigma}) \right.$$

$$
-V(G,k,(\hat{\sigma},\delta'))(k)+\frac{1}{2}S(G,k,k',(\hat{\sigma},\delta')\Big)+\pi\sum_{k'\in I_k^G}\tilde{\delta}(G,k,k')\left(\frac{1}{|I_k^G|}+\frac{1}{|I_{k'}^G|}\right)(V(G\setminus\{\{k,k'\}\},k,\hat{\sigma})
$$

$$
-V(G,k,(\hat{\sigma},\delta')))
$$

$$
+\pi\sum_{\left\{\bar{k},\tilde{k}\right\}\in E^G|k\neq\bar{k}andk\neq\tilde{k}}\frac{1}{2}(\delta'(G,\bar{k},\tilde{k})+\delta'(G,\bar{k},\tilde{k}))\left(\frac{1}{|I_{\tilde{k}}^G|}+\frac{1}{|I_{\tilde{k}}^G|}\right)\left(V(G\setminus\left\{\left\{\bar{k},\tilde{k}\right\}\right\},k,\hat{\sigma})-V(G,k,(\hat{\sigma},\delta'))\right)\Bigg\} \tag{20}
$$

A few comments on (20). The constraint set correspondence is constant, and therefore continuous, and obviously non–empty valued and compact valued. The objective function is continuous in the choice variables, hence a maximizer exists and Ψ_k is non–empty valued. Moreover, (3) implies that $V(G,k,(\hat{\sigma},\delta')$ as well as $S(G,k,k'(\hat{\sigma},\delta))$ are continuous in δ' (for all $k'\in I_k^G$). Hence, by the Theorem of the Maximum, Ψ_k is upper hemi–continuous and compact valued. As it will be argued later, Ψ_k is also convex valued.

Define $\Psi(\delta')\equiv\times_{k\in I^G}\Psi_k(\delta')$. The sign of surplus $S(G,k,k',(\hat{\sigma},\delta'))$, which determines the optimal choice as the objective function is linear in all choice variables. Ψ hence is an upper hemi–continuous, non–empty valued, compact valued and convex valued correspondence from $[0,1]^{|E^G|}$ to $[0,1]^{|E^G|}$ and, by Kakutani's fixed point theorem, has a fixed point δ^{**} such that $\delta^{**}\in\hat{\Psi}(\delta^{**})$.

Due to the objective function in (20) being linear in the choice variables, we can characterize the maximizers. If $S(G,k,k',(\hat{\sigma},\delta))>0$, only $\tilde{\delta}(k,k')=0$ is optimal, if $S_{\hat{\sigma},\delta}^G(k,k')<0$, only $\tilde{\delta}(k,k')=1$ and if $S_{\hat{\sigma},\delta}^G(k,k')=0$, any $0\le\tilde{\delta}(k,k')\le 1$ is optimal. This observation means Ψ_k, and as a result Ψ and $\hat{\Psi}$ are indeed convex valued. Moreover, it implies that any fixed point δ^{**} of $\hat{\Psi}$ satisfies individual rationality and hence $(\hat{\sigma},\delta^{**})$ is individually rational in G. This concludes the proof.

Proof of Lemma 2

First, for an arbitrary network G, and agents $k,k'\in I^G$ such that $k'\in I_k^G$, let us define

$$
\alpha(G,k)\equiv\left(W(G,k)+\frac{c|I_k^G|}{\rho}\right)\frac{\rho}{\pi q}
$$

$$
\beta(G,k,k')\equiv\frac{\gamma(G,k,k')}{q}
$$

Note, Proposition 1 guarantees that $W(H,l)$ exists, for any $H\subseteq G$ and $l\in I^H$. As a result, $\alpha(G,k)$ is well–defined and so is $\gamma(G,k,k')$ through (5), and therefore $\beta(k,k')$.

(19) implies $\alpha^{G_0}(1)=\alpha^{G_0}(2)=1$, which is indeed a rational function of $\frac{\pi}{\rho}$ and does not depend on c or q. (19) also implies $\beta^{G_0}(1,2)=\beta^{G_0}(2,1)=\frac{1}{2}$, which does not depend on c or q and hence the induction has a valid starting point.

Assume that the statement holds up to a certain index in the order of induction and G is the next network in the order.

Equation (12) allows us to write the system of equations in the following matrix equation form.

$$
A^G\alpha^G=z^G \tag{21}
$$

where A^G ($|I^G|\times|I^G|$ matrix) and z^G (vector of length $|I^G|$) are as follows.

Let $A_{l,l'}^G$ denote the entry of A^G in the (l,l') position and z_l^G denote the lth entry of z^G. Then

$$
A_{l,l'}^G=0,l\neq l'\wedge l'\notin I_l^G
$$

$$
\begin{aligned}
A^G_{l,l} &= 1 \\
A^G_{l,l'} &= -\tilde{\pi}_{l,l'}, l' \in I^G_l \\
z^G_l &= \sum_{l' \in I^G_l} \tilde{\pi}_{l,l'} \left(\frac{\rho}{\pi} + \alpha^{G \setminus \{\{l,l'\}\}}(l) - \alpha^{G \setminus \{(l,l')\}}(l') \right)
\end{aligned}
$$

where

$$
\tilde{\pi}^G_{kk'} \equiv \frac{\pi \left(\frac{1}{|I^G_k|} + \frac{1}{|I^G_{k'}|} \right)}{2\rho + \pi \sum_{k'' \in I^G_k} \left(\frac{1}{|I^G_k|} + \frac{1}{|I^G_{k''}|} \right)}
$$

Note, in (21), neither A^G nor z^G depends on c or q and it suffices to show that A^G is invertible. Note, $I - A^G$ is a convergent matrix (all of its row–sums are smaller than 1) and hence A^G is invertible, which concludes the proof.

Proof of Proposition 3

Let $S(H,k,k')$ denote the surplus between k and k' in sub–network $H \subseteq G$ under *agreement*, i.e.

$$
S(H,k,k') = q + W(H,k) - W(H \setminus \{\{k,k'\}\}, k) + W(H,k') - W(H \setminus \{\{k,k'\}\}, k') \tag{22}
$$

And let $S_0(H,k,k')$ denote the surplus between k and k' in sub–network $H \subseteq G$ under *agreement*, but with the additional assumption that $c = 0$. Obviously, $S_0(H,k,k')$ is not a function of cost any more.

(7) and (22) allow us to write

$$
S_0(H,k,k') = q\left(1 + \frac{\pi}{\rho}\left(\alpha(H,k) - \alpha(H \setminus \{\{k,k'\}\}, k) + \alpha(H,k') - \alpha(H \setminus \{\{k,k'\}\}, k')\right)\right) \tag{23}
$$

$$
S(H,k,k') = -\frac{2c}{\rho} + q\left(1 + \frac{\pi}{\rho}\left(\alpha(H,k) - \alpha(H \setminus \{\{k,k'\}\}, k) + \alpha(H,k') - \alpha(H \setminus \{\{k,k'\}\}, k')\right)\right) \tag{24}
$$

where the term in parentheses only depends on $\frac{\pi}{\rho}$, by Lemma 1. Equations (23) and (24) imply

$$
S(H,k,k') = -\frac{2c}{\rho} + S^H_0(k,k') \tag{25}
$$

And hence $S(H,k,k')$ is additively separable in cost and gain from bargaining. Definition 2 says that for a fixed network G, the *agreement* recommendation together with the resulting $\left((W(H,l))_{l \in I^H}\right)_{H \subseteq G}$ is an equilibrium in G, if and only if $S(H,k,k') \geq 0$ for all $H \subseteq G$ and $\{k,k'\} \in E^H$. Equation (25) then implies that if *agreement* with the corresponding values is an equilibrium for some level of c, it would also be an equilibrium for $c' < c$. Moreover, (25) implies that for any network G, there is a maximal level of cost, $\bar{c}^G$ such that *agreement* with the corresponding values is an equilibrium in G for any $c \leq \bar{c}^G$, but it is not an equilibrium for any $c > \bar{c}^G$.

For *agreement* to be an equilibrium, all surpluses in all sub–networks have to be non–negative, in particular, the minimal surplus. For *agreement* to cease to be an equilibrium, it is enough for one – the minimal – surplus to be negative. Thus the cutoff cost is such, that it makes the minimal surplus equal to zero. (14) then allows us to construct

$$
\bar{c}^G = \min\left(\left(S_0(H,k,k')\right)_{\{k,k'\} \in E^H}\right)_{H \subseteq G} \frac{\rho}{2}
$$

which concludes the proof.

Proof of Proposition 4

Equation (11) implies $\beta(H,k,k') = 1 - \beta(H,k',k)$. Theni f we sum up the right–hand side of (10) over all $k \in I^G$, every edge will be counted twice — once as $\{k,k'\}$ and once as $\{k',k\}$ — and hence the sum will be a sum of zeros plus $|I^G|$. This concludes the proof.

Proof of Corollary 1

Fix a network G and two agents k,k' such that $k \sim^G k'$. This means there exists an automorphism p such that $p(k) = k'$. We want to show that $\alpha(G,k) = \alpha(G,k')$. First, I define a permuted network $p(G)$, then argue, in two steps, that $\alpha^G_{p^{-1}(l)} = \alpha^{p(G)}_l = \alpha^G_l$, for all $l \in I^G$, which yields the desired result.

For any permutation f, define network $f(G)$ as follows. $I^{f(G)} \equiv I^G$ and $E^{f(G)} \equiv \{\{f(l),f(l')\} \,|\, \{l,l'\} \in E^G\}$. As a result, $I^{f(G)}_l = I^G_{f^{-1}(l)}$ for any $l \in I^G = I^{f(G)}$. The relabeled solutions to the system of equations described by (12) in G obviously solve the analogous system of equations in $f(G)$, i.e. for any $l \in I^G$ we have $\alpha^G_{f^{-1}(l)} = \alpha^{f(G)}_l$.

Automorphisms are permutations, hence the above definition can be used for p as well. This leads to $\alpha^G_{p^{-1}(l)} = \alpha^{p(G)}_l$ for any $l \in I^G$. Since p is an automorphism, however, it satisfies the property that $\{l,l'\} \in E^G$ if and only if $\{p(l),p(l')\} \in E^G$, i.e. the set of edges does not change by the relabeling. Thus $E^G = E^{p(G)}$ and this implies $G = p(G)$, i.e. the two networks are the same. Since for any network G, the vector α^G is determined uniquely, by Proposition 1, $\alpha^G = \alpha^{p(G)}$, which concludes the proof.

Proof of Proposition 5

Both statements follow directly from Corollary 1.

Proof of Lemma 3

(i) If $\alpha(G,k,\frac{\pi}{\rho}) \geq \alpha(H,k',\frac{\pi}{\rho})$ for all $\frac{\pi}{\rho} > 0$ then it also holds for the $\frac{\pi}{\rho} \to 0$ limit. Equation (13) then implies

$$\frac{1}{2}\sum_{\hat{k}\in I^G_k}\left(\frac{1}{|I^G_{\hat{k}}|} + \frac{1}{|I^G_k|}\right) \geq \frac{1}{2}\sum_{\hat{k}\in I^G_k}\left(\frac{1}{|I^G_{\hat{k}}|} + \frac{1}{|I^G_k|}\right)$$

which yields the statement.

(ii) From equation (22), $\beta(G,k,k',\frac{\pi}{\rho}) \geq \frac{1}{2}$ implies

$$\alpha(G,k',\frac{\pi}{\rho}) - \alpha(G \setminus \{\{k,k'\}\},k',\frac{\pi}{\rho}) \geq \alpha(G,k,\frac{\pi}{\rho}) - \alpha(G \setminus \{\{k,k'\}\},k,\frac{\pi}{\rho})$$

Since the above holds for all $\frac{\pi}{\rho} > 0$, it also holds for the $\frac{\pi}{\rho} \to 0$ limit as well. Equation (13) then implies

$$\frac{1}{|I^G_{k'}|} + \chi(|I^G_k| = 1) \geq \frac{1}{|I^G_k|} + \chi(|I^G_{k'}| = 1)$$

Since we only consider connected networks, $|I^G_k| = |I^G_{k'}| = 1$ can only hold in the line with two agents. G is a different network. Then either $|I^G_k| = 1$ and the LHS is larger than one while the RHS is smaller or both agents have at least two connections. In the latter case we have

$$\frac{1}{|I_{k'}^G|} \geq \frac{1}{|I_k^G|}$$

which yields the statement.

(iii) From equations (4) and (7), $S(G,k,\hat{k},\frac{\pi}{\rho}) \geq S(H,k',\bar{k},\frac{\pi}{\rho})$ implies

$$\begin{aligned}
&\alpha(G,k,\frac{\pi}{\rho}) - \alpha^{G\setminus\{\{k,\hat{k}\}\}}(k,\frac{\pi}{\rho}) + \alpha(G,\hat{k},\frac{\pi}{\rho}) - \alpha^{G\setminus\{\{k,\hat{k}\}\}}(\hat{k},\frac{\pi}{\rho}) \\
&\geq \alpha(H,k',\frac{\pi}{\rho}) - \alpha^{H\setminus\{\{k',\bar{k}\}\}}(k',\frac{\pi}{\rho}) + \alpha(H,\bar{k},\frac{\pi}{\rho}) - \alpha^{H\setminus\{\{k',\bar{k}\}\}}(\bar{k},\frac{\pi}{\rho})
\end{aligned}$$

Since the above holds for all $\frac{\pi}{\rho} > 0$, it also holds for the $\frac{\pi}{\rho} \to 0$ limit.

Equation (13) then yields the result.

Proof of Proposition 7

First, I show, by induction on the number of agents n, that $\alpha(n,1) < \frac{n+1}{2}$ holds for all $n \geq 2$. Note, this implies $\beta(n,2,1) = \frac{\alpha(n,1)}{n} < \frac{1}{2} + \frac{1}{2n}$. A star network with two agents is isomorphic to a line with two agents and hence we have $\alpha(2,1) = 1 < \frac{3}{2}$. Therefore $n = 2$ is a valid starting point. Suppose that $\alpha(n,1) < \frac{n+1}{2}$ holds for all $2 \leq n \leq m-1$ for some $m \geq 3$. It suffices to show that $\alpha(m,1) < \frac{m+1}{2}$. Equation (15) then implies

$$\begin{aligned}
\alpha(G,m,1) &= \frac{m(m-1) + m^2\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}} + \frac{m(m-1)\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}}\alpha(G,m-1,1) \\
&< \frac{m(m-1) + m^2\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}} + \frac{m(m-1)\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}}\frac{m}{2} \\
&= \frac{1}{2}\frac{2m(m-1) + 2m^2\frac{\pi}{\rho} + m^2(m-1)\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}} \\
&= \frac{1}{2}\left(m\frac{2(m-1)}{2(m-1) + m^2\frac{\pi}{\rho}} + (m+1)\frac{m^2\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}}\right) < \frac{m+1}{2}
\end{aligned}$$

therefore we have established that $\alpha(n,1) < \frac{n+1}{2}$ holds for all $n \geq 2$.

Second, I show $\beta(n,2,1) > \frac{1}{2}$ for all $n \geq 3$. It suffices to show $\alpha(n,1) > \frac{n}{2}$ for all $n \geq 3$. Suppose that $\alpha(n,1) > \frac{n}{2}$ holds for all $3 \leq n \leq m-1$ for some $m \geq 4$. It suffices to show that $\alpha(m,1) > \frac{m}{2}$. Equation (15) then implies

$$\begin{aligned}
\alpha(G,m,1) &= \frac{m(m-1) + m^2\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}} + \frac{m(m-1)\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}}\alpha(G,m-1,1) \\
&> \frac{m(m-1) + m^2\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}} + \frac{m(m-1)\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}}\frac{m-1}{2} \\
&= \frac{1}{2}\frac{2m(m-1) + 2m^2\frac{\pi}{\rho} + m(m-1)^2\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}} \\
&= \frac{1}{2}\frac{2m(m-1) + (m^3+m)\frac{\pi}{\rho}}{2(m-1) + m^2\frac{\pi}{\rho}} > \frac{m}{2}
\end{aligned}$$

therefore we have established that $\alpha(n,1) > \frac{n}{2}$ holds for all $n \geq 3$.

As $\frac{1}{2} < \beta(n,2,1) < \frac{1}{2} + \frac{1}{2n}$ for all $n \geq 3$, the squeeze theorem implies $\beta(n,2,1) \to \frac{1}{2}$, which concludes the proof.

Proof of Proposition 8

Define $\phi(n) = \frac{S_0^G(n,1,2)}{q}$. First, I show that $\phi(n) \geq 1+\frac{\pi}{\rho}$ for all $n \geq 2$. By equation (8) and the closed form solutions for lines and polygons, this implies $\bar{c}^{star}(n) \geq \frac{\rho q}{2}\left(1+\frac{\pi}{\rho}\right) > \bar{c}^{line}(n) = \bar{c}^{polygon}(m)$ for any $n \geq 4$ and $m \geq 3$.

Equations (15) and (12) imply

$$\phi(n+1) = \frac{n(n-1)}{2(n-1)+2n\frac{\pi}{\rho}} \frac{2(n+1)\frac{\pi^2}{\rho^2}+2n\frac{\pi}{\rho}}{(n+1)^2\frac{\pi}{\rho}+2n}\phi(n) + \left(1+\frac{\pi}{\rho}\right)\frac{\frac{\pi}{\rho}+2n}{(n+1)^2\frac{\pi}{\rho}+2n} \tag{26}$$

Let us define $\tilde{\phi}(n) \equiv \frac{\phi(n)-1}{\frac{\pi}{\rho}}$, i.e. we have $\phi(n) = 1+\tilde{\phi}(n)\frac{\pi}{\rho}$. It suffices to show that $\tilde{\phi}(n) \geq 1$ for all $n \geq 4$. We can show $\tilde{\phi}(n) \geq 1$ for all $n \geq 2$. A star with two agents is isomorphic to a line with two agents and the surplus in the latter equals $q\left(1+\frac{2\pi}{\rho}\right)$, i.e. $\tilde{\phi}(2) = 2$. Therefore $n = 2$ is a valid starting point for induction. Using (26), we can write

$$\tilde{\phi}(n+1) = \frac{2n^2-n-1+(4n^2-2)\frac{\pi}{\rho}+(2n^2+2n)\left(\frac{\pi}{\rho}\right)^2}{\left(\frac{\pi}{\rho}(n+1)^2+2n\right)\left(n-1+n\frac{\pi}{\rho}\right)} + \frac{n(n-1)\left(n\frac{\pi}{\rho}+(n+1)\left(\frac{\pi}{\rho}\right)^2\right)}{\left(\frac{\pi}{\rho}(n+1)^2+2n\right)\left(n-1+n\frac{\pi}{\rho}\right)}\tilde{\phi}(n) \tag{27}$$

Suppose $\tilde{\phi}(n) \geq 1$ for all $2 \leq n \leq m$ for some $m \geq 2$. It suffices to show $\tilde{\phi}(m+1) \geq 1$. Equation (27) implies

$$\begin{aligned}\tilde{\phi}(n+1) &= \frac{2n^2-n-1+(4n^2-2)\frac{\pi}{\rho}+(2n^2+2n)\left(\frac{\pi}{\rho}\right)^2}{\left(\frac{\pi}{\rho}(n+1)^2+2n\right)\left(n-1+n\frac{\pi}{\rho}\right)} + \frac{n(n-1)\left(n\frac{\pi}{\rho}+(n+1)\left(\frac{\pi}{\rho}\right)^2\right)}{\left(\frac{\pi}{\rho}(n+1)^2+2n\right)\left(n-1+n\frac{\pi}{\rho}\right)} \\ &= \frac{n^3\left(\frac{\pi}{\rho}\right)^2+n^3\left(\frac{\pi}{\rho}\right)+2n^2\left(\frac{\pi}{\rho}\right)^2+3n^2\left(\frac{\pi}{\rho}\right)+2n^2+n\left(\frac{\pi}{\rho}\right)^2-n-2\left(\frac{\pi}{\rho}\right)-1}{n^3\left(\frac{\pi}{\rho}\right)^2+n^3\left(\frac{\pi}{\rho}\right)+2n^2\left(\frac{\pi}{\rho}\right)^2+3n^2\left(\frac{\pi}{\rho}\right)+2n^2+n\left(\frac{\pi}{\rho}\right)^2-n\left(\frac{\pi}{\rho}\right)-2n-\left(\frac{\pi}{\rho}\right)} \geq 1\end{aligned}$$

which establishes that $\tilde{\phi}(n) \geq 1$ for all $n \geq 2$ and therefore $\bar{c}^{star}(n) \geq \frac{\rho q}{2}\left(1+\frac{\pi}{\rho}\right) > \bar{c}^{line}(n) = \bar{c}^{polygon}(m)$ for any $n \geq 4$ and $m \geq 3$, which concludes the proof.

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