A Three-State Rational Greater-Fool Bubble Model With Intertemporal Consumption Smoothing*

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Abstract

We construct a simple greater-fool bubble model with rational agents, where the motive for trade is intertemporal consumption smoothing. This yields an easy-to-understand bubble model with only three states of the world, instead of the five required in previous research, and may therefore provide a convenient point of departure for future work on greater-fool bubbles. Our model suggests that such bubbles are more likely when there are asset sellers (e.g., innovators) with profitable investment opportunities, but little wealth, so they sell shares in those opportunities to wealthier investors, in order to fund current spending. "Bad sellers" then pretend to sell similar investment opportunities, creating potential bubble assets. We show that bubbles continue to be possible, even if alternative means of consumption smoothing, such as storage (and, in related work, riskless assets) are available. Finally, we show that an anti-bubble policy can reduce the welfare of even the greater fools it's supposed to protect, if it interferes with consumption smoothing by those agents in earlier periods of their lives.

JEL *classification*: D53; D82; E21; G12; G14

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1 Introduction

Journalists, policy makers, and historians have long been interested in "greater-fool bubbles," i.e., bubbles where investors hold overpriced assets in hopes of selling them to greater fools before the bubble bursts. However, only recently has economic theory been able to contribute to an understanding of such greater-fool bubbles, using standard models involving rational agents and asymmetric information (e.g., Allen, et al., 1993, Abreu and Brunnermeier, 2003).[1]

Greater-fool bubbles involve "swindlers," in Kindleberger's (2000) terminology, or "bad sellers," where these sellers are "bad" in the sense that they are trying to sell assets they know are overpriced. This, however, raises the question of why anyone would *buy* from these bad sellers. The answer usually stressed by observers like Kindleberger is that buyers are simply irrational, though Kindleberger also emphasizes monetary expansion and financial liberalization in helping to fund these buyers. It is also often argued that people buy overpriced assets in hopes of reselling them to others before the price crashes (Kindleberger, 2000, p. 31). However, for a reasonable person to buy an overpriced asset for resale, they must think that someone thinks that someone thinks ... that some ultimate buyers think the asset might really be valuable.

Irrationality may certainly play an important role in greater-fool bubbles. However, recent research in this area has tried to take the intelligence of the ultimate buyers more seriously. When one does this, though, one realizes that these buyers will only be willing to buy assets if they think sellers might be "good sellers," with genuinely profitable investment opportunities to offer.[2]

Of course, buyers in greater-fool bubbles will, if at all rational, realize that they face a lemons problem, since they risk buying from bad sellers, and so, risk becoming the greater fools in a bubble. Those buyers will therefore not be willing to pay much for the assets. Good sellers will then get low prices for their assets, so they will not want to sell them unless there are some sorts of gains from trade to compensate them for these low prices.

Previous work building on Allen et al. (1993) assumed these gains from trade came from agents who wanted to share risks with each other. For example, good sellers might be insiders in a startup,

[1] A search of JSTOR found only nine uses of the phrase in economics journals before Temin and Voth (2004) and Conlon (2004) used it in connection with previous greater-fool bubble models.

[2] Kindleberger (2000, p. 38) briefly mentions "profit opportunities" as one of a long list of possible "displacements" that can initiate bubbles. Scheinkman (2014) and others also argue that the arrival of new technologies may encourage bubble formation. Caballero and Krishnamurthy (2006), and others building on their work, consider agents who hold bubble assets as liquidity when young, so they can sell those assets to finance production when middle-aged or old.

willing to sell low-priced shares so they can diversify their portfolios.[3]

This paper, by contrast, assumes that good sellers are willing to sell low-priced assets because they want to smooth their consumption through time. For example, they might be liquidity-constrained insiders in a startup, doing an IPO to obtain funds to invest in a profitable project. In this paper, however, we simply assume good sellers want to sell assets to fund current consumption.

Intertemporal Consumption smoothing as a motive for trade gives a simple, compelling story about how greater-fool bubbles might form. Greater-fool bubbles are more likely to form when there are asset sellers, such as innovators, who have hard-to-evaluate, but potentially profitable investment opportunities, but who also have relatively little wealth. Thus, they are willing to sell shares in those opportunities to wealthier investors. This, however, opens the door for swindlers, who pretend to have similarly profitable investment opportunities, making bubbles possible.

Studies of bubble episodes should therefore not just focus on the presence of swindlers and irrationally optimistic investors. They should also ask whether there were relatively illiquid innovators present in these episodes, willing to sell shares in profitable investment opportunities at bargain prices, and whether it was reasonable for investors during these episodes to buy assets from these innovators, even if they risked becoming greater fools, buying from swindlers.[4]

Intertemporal consumption smoothing as a motive for trade, however, is also important for another reason: it yields models of greater-fool bubbles which are in many ways simpler and easier to understand than models in previous research.

A major goal of recent research has been to develop simple models of "strong bubbles," i.e., simple models in which there is a state of the world where everyone knows that the asset is overpriced. Such strong bubbles are possible because, even though each agent knows that the asset is overpriced, agents may not know that other agents know this, due to asymmetric information. The original such strong bubble model, developed by Allen et al. (1993), was simplified by Conlon (2004, 2015) and Zheng (2011, 2013), until Liu and Conlon (2018) showed that a strong bubble can

[3]Risk sharing as a motive for trade in greater-fool models is most fully explored in Holt (2018), who carefully analyzes the case of risk-averse agents, i.e., agents with concave utility functions. Holt's model provides a major point of departure for the model developed here.

[4]Thus, consider the argument in Kindleberger (2000), p. 31, that participants in a bubble suffer from a "fallacy of composition," because they act as if "the whole" of available gains can exceed "the sum of its parts." This argument implicitly assumes that financial markets are zero-sum games. However, if there are gains from trade due, say, to the presence of innovators in need of funds, then financial markets become *positive* sum games, so bubbles are possible, even if everyone is rational. Of course, even if financial markets involve gains from trade, making bubbles possible, it is not clear whether those bubbles will facilitate or interfere with those gains from trade.

exist in a model with two agents, three periods, and five states of the world.[5]

These earlier strong bubble models assumed that consumption occurred only once, in the last period. Until now, however, no one seems to have noticed the importance of this assumption. The current paper shows that, if agents are allowed to consume in each period, then the number of states needed to model a strong greater-fool bubble can be reduced, from the five states required in Liu and Conlon (2018), to just three states, since we can introduce intertemporal consumption smoothing as a motive for trade (see footnote 19, below, for a detailed comparison of our model with Liu and Conlon's).

Each of the states in our three-state model is easy to understand. One of these states is the bubble state, b, itself, where everyone knows the dividend is zero. The second state, which we call L (for "low"), is a state in which some agents, who know the asset's worthless, can "ride the bubble" and then sell the asset to other agents who don't know it's worthless. The third state, which we call H (for "high"), is the dividend paying state. This simple three-state model will hopefully illuminate the absolutely minimal information structure of a greater-fool bubble.

Intertemporal consumption smoothing has been used as a motive for trade in bubble models dating back to the infinite-horizon overlapping generations (OLG) models of Samuelson (1958), Tirole (1985), and others, where agents buy overpriced assets when young to save for old age.[6] However, unlike OLG models, our model has a finite horizon, so our bubble exists, not because it grows forever in expected value, but because of a greater-fool dynamic, involving asymmetric information.[7] It is also worth noting that our model more closely resembles models in Bewley (1980) and Townsend (1980) than OLG models, since our agents' lifetimes coincide, so agents can trade back and forth with each other to smooth their consumption over time.

Our model is also related to bubble-riding models building on Abreu and Brunnermeier (2003), such as Doblas-Madrid (2012, 2016) and Araujo and Doblas-Madrid (2019). First, our model, like

[5] This result was also independently obtained by Lien et al. (2015).

[6] OLG-based bubble models are sometimes referred to as "rational bubble" models. However, most of the greater-fool bubble models listed above – including the current model – also assume perfectly rational agents. See Barlevy (2015, 2018) for recent surveys of bubble models. Of course, infinite-horizon models have been modified in microeconomically interesting ways to better match features of real world bubbles (see, e.g., Caballero and Krishnamurthy, 2006, Martin and Ventura, 2012, and Farhi and Tirole, 2012).

[7] Greater-fool bubble models therefore also allow us to study the implications of differences in information and/or beliefs. Some additional benefits of greater-fool-style bubble models are underscored by Farhi and Tirole (2012), footnote 6, who argue that these models "typically reach more precise predictions ... regarding which assets are more likely to feature bubbles" and also "have a rich array of implications for volume, turnover, etc."

theirs, has buyers who cannot distinguish sellers needing liquidity from sellers who wish to exploit buyers.[8] In addition, our model also has a bubble-riding state (i.e., the state L mentioned above). Our model may therefore help to bridge the gap between OLG models, the above bubble-riding models, and models based on Allen et al. (1993).[9]

In summary, we present a strong greater-fool bubble model with two rational agents, three periods, and three states of the world, in which agents consume in each period. Our model is therefore simpler than Liu and Conlon's (2018) model in the sense that it has a smaller state space, though our consumption structure is richer than theirs. Of course, our richer consumption structure allows us to consider the implications of bubbles for intertemporal consumption smoothing (savings) behavior, as well as for risk-sharing behavior. For example, we illustrate the model by showing that a policy of deflating overpriced assets may hurt the welfare of even the greater-fool buyers the policy is supposed to protect, by interfering with their prior intertemporal consumption smoothing.[10]

We also consider bubble models where an alternative means of consumption smoothing is available, i.e., risk-free *storage*. One might think that bubbles may cease to exist if agents can use riskless storage to smooth their consumption, but we show that bubble assets can still be held in equilibrium, even in this case. We also briefly discuss risk-free *assets* in Subsection 5.4 and Technical Appendix B. However, in order to model a bubble when such risk-free assets are traded in period 1, a *four*-state model is needed (see Conlon et al., 2021). Nevertheless, while the risk-free asset competes with the bubble asset, this is not the reason why four states are needed. Rather the problem is that the prices of the two assets reveal too much information in a three-state model.

Section 2 describes the model's setup, while Section 3 presents the bubble equilibrium and explains its intuition. Section 4 briefly looks at asset-deflation policy in the numerical example from Section 3. Section 5 considers a model with storage, and Section 6 concludes. Some longer proofs are presented in Appendices A and B, with additional material in technical appendices.

[8] Their models, like ours, therefore build on the need for liquidity as a motive for trade. On the other hand, our motive for trade differs from the motive for trade used in the risk-shifting (borrower-lender) models of Allen and Gale (2000) and Barlevy (2014), as well as from the motive used in the Awaya et al. (2020) middleman model; see footnote 13 below. It also contrasts with the motive for trade in Harrison and Kreps (1978) and Scheinkman and Xiong (2003), where trade and overpricing occur because some agents are overconfident.

[9] However, we believe that an important contribution of these other bubble-riding models may be that they capture *momentum* in a way that the current simple version of our model does not. Thus, while one of our goals is to construct a greater-fool bubble model which is as simple as possible, an important topic for future research is to determine what must be added to our simple model in order to capture momentum, like these other bubble-riding models.

[10] Again, this mirrors earlier results of Holt (2018). See footnote 27 below.

2 Preliminaries

This paper uses a simple asset market structure with asymmetric information, as in Allen et al. (1993) and others, but introduces consumption in every period. Markets are Walrasian, and the model lasts for three periods, so $t = 1, 2, 3$. There are three possible states of the world, with the actual state, ω, randomly selected from the state space $\Omega = \{b, L, H\}$. Here b will be a "strong bubble" state, where every agent knows the asset is overpriced, and L will be a bubble-riding "semi-bubble" state, where some agents know the asset is overpriced, but others do not. State H will be the only dividend-paying state.

There are two rational agents, Ellen and Frank, indexed by $j = E, F$, who have a common prior probability distribution $\pi(\omega)$ over Ω. Agent j's lifetime utility is $\sum_{t=1}^{3} \beta^{t-1} U_j(C_t^j)$, where $\beta > 0$ is the agent's discount factor on utility, C_t^j is Agent j's period-t consumption, and $U_j(\cdot)$ is her smooth concave utility function, so $U_j'(\cdot) > 0$ and $U_j''(\cdot) < 0$.[11] To ensure positive consumption in each state, we also assume $U_j'(0) = \infty$.

There is a single consumption good, which is perishable, and a durable asset, which is risky. There is no riskless asset in our model.[12] The risky asset pays a dividend, in units of the consumption good, only in period 3, state H, so $d(H) = d > 0$, but $d(b) = d(L) = 0$. Agents have equal access to the market, are able to trade the asset in each period, and receive the same dividend for each share they hold in period 3.[13] Agent j is initially endowed with a number of shares of the risky asset, $s_0^j(\omega) \geq 0$, which we assume is constant in ω, so $s_0^j(\omega) = s_0^j$ for $j = E$, F. Let $X_t^j(\omega)$ be Agent j's net sales in period t, state ω, so Agent j owns $s_t^j(\omega) = s_{t-1}^j(\omega) - X_t^j(\omega)$ shares of the risky asset at the end of period t. Agents cannot sell the asset short, so $X_t^j(\omega) \leq s_{t-1}^j(\omega)$.[14]

Agent j also receives an endowment of $e_t^j(\omega)$ units of the consumption good at the beginning of

[11] Our model thus builds on Holt (2018), who first introduced concave utility as a major issue for greater-fool bubble models.

[12] We introduce storage in Section 5. We also briefly consider a risk-free asset and analyze its effects on bubbles in Subsection 5.4, Technical Appendix B, and at greater length in Conlon et al. (2021).

[13] Ellen and Frank therefore have access to the same investment opportunities, removing the motive for trade used in risk-shifting models such as Allen and Gale (2000) and Barlevy (2014). In those models, gains from trade are generated by a standard capital market imperfection. Specifically, safe borrowers' investments earn higher returns than lenders' investments, so lenders are willing to take the chance of potentially lending to risky borrowers, in their attempt to lend to safe borrowers. Similarly, Awaya et al. (2019) assume a sequence of middlemen who cannot, themselves, use the asset, but instead sell it on to a final buyer to whom the asset may be valuable.

[14] Since our markets are competitive, we must assume short-sale constraints. However, suppose sellers have some market power. Then short-sale constraints may not be needed to induce those sellers to limit asset sales, even if they know that prices will crash *next* period, because they will not want to push prices down too much *this* period.

each period t. For simplicity, assume $e_t^j(\omega)$ is strictly positive, so $e_t^j(\omega) > 0$. Again, this, together with $U_j'(0) = \infty$ above, will ensure positive equilibrium consumption in each state. Let $p_t(\omega)$ be the price of the asset in units of the consumption good. Agent j's consumption is then given by

$$C_t^j(\omega) = \begin{cases} e_t^j(\omega) + p_t(\omega) X_t^j(\omega) & \text{for } t = 1, 2, \\ e_3^j(\omega) + p_3(\omega) X_3^j(\omega) + d(\omega) s_3^j(\omega) & \text{for } t = 3. \end{cases} \tag{1}$$

Define $M_t^j(\omega) = \beta^{t-1}\pi(\omega)U_j'(C_t^j(\omega))$ to be Agent j's discounted expected marginal utility of consumption in state ω, period t. In other words, $M_t^j(\omega)$ is the *ex ante* shadow price that Agent j attaches to an additional unit of consumption in state ω, period t. Similarly, $M_t^j(\{\omega_1, ..., \omega_k\})$ is her *ex ante* shadow price of one more unit of consumption in the collection of states $\{\omega_1, ..., \omega_k\}$, so $M_t^j(\{\omega_1, ..., \omega_k\}) = M_t^j(\omega_1) + M_t^j(\omega_2) + ... + M_t^j(\omega_k)$. Note that these shadow prices depend on final consumption, and so, depend endogenously on the equilibrium outcomes.

Our model uses information sets and partitions to describe agents' information structures. An information set, or cell, consists of states that are indistinguishable to an agent, while an information partition is a collection of information sets that are disjoint but cover the state space Ω. Let $I_{t,i}^j$ be the i^{th} information set of Agent j in period t. Then Agent j's period-t information partition is $\mathbb{P}_t^j = \{I_{t,1}^j, I_{t,2}^j, \ldots\}$, where $I_{t,i}^j \cap I_{t,k}^j = \emptyset$, for $i \neq k$, and $I_{t,1}^j \cup I_{t,2}^j \cup \ldots = \Omega$.

Figure 1 illustrates our agents' information partitions. In period 1, Ellen's information partition is $\mathbb{P}_1^E = \{\{b, L\}, \{H\}\}$, so $I_{1,1}^E = \{b, L\}$ and $I_{1,2}^E = \{H\}$. That is, Ellen can distinguish state H from states b and L, but she cannot distinguish between b and L. Frank's period-1 information partition is $\mathbb{P}_1^F = \{\{b\}, \{L, H\}\}$, so Frank can distinguish b from L and H, but not between L and H.

We call state b the *strong bubble state*, since this is the state where a strong bubble may form. In period 1, state b, Frank knows the true state is b, so he knows that the asset will pay nothing, while Ellen knows the true state is either b or L, so she also knows the asset will pay nothing. Thus, if $p_1(b) > 0$, the asset will be in a strong bubble. We call L the *semi-bubble state*, since in that state Ellen knows the asset is worthless, but Frank thinks the asset might pay a dividend. Lastly, we call H the dividend-paying state, since the asset pays a positive dividend in that state.

Information partitions are exogenously refined at the beginning of each period as new information arrives. In period 2, Ellen learns the true state with certainty, so her period-2 information

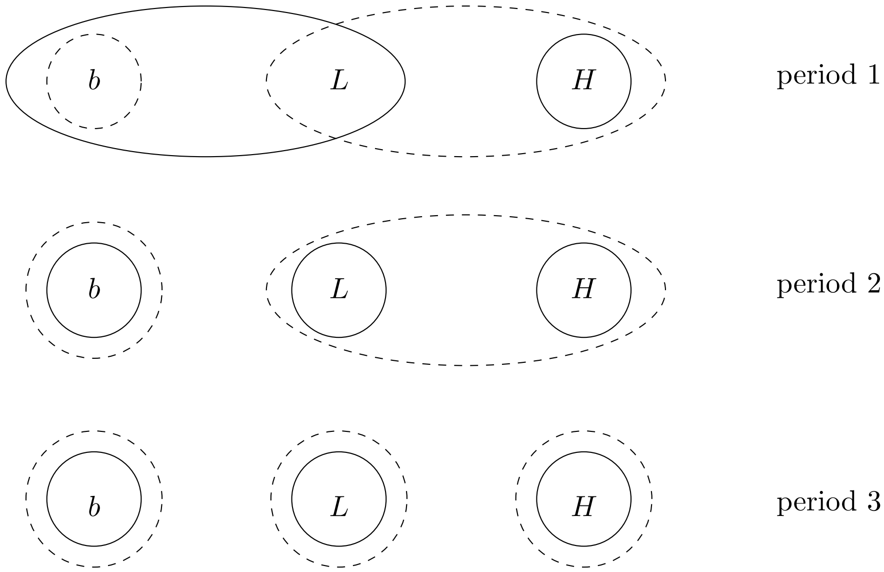

Figure 1: Ellen's and Frank's information partitions. Ellen's information sets are denoted by solid ovals, and Frank's information sets are denoted by dashed ovals.

partition becomes $\mathbb{P}_2^E = \{\{b\}, \{L\}, \{H\}\}$. Frank, however, is assumed to learn nothing between periods 1 and 2, so his period-2 information partition remains $\mathbb{P}_2^F = \mathbb{P}_1^F = \{\{b\}, \{L, H\}\}$. In period 3, Frank also learns the true state, so $\mathbb{P}_3^F = \mathbb{P}_3^E = \{\{b\}, \{L\}, \{H\}\}$. Since the true state becomes common knowledge by period 3, the period-3 asset price will simply equal the dividend, so $p_3(\omega) = d(\omega)$. As a result, there is no motive for trade in period 3, so we can assume that $X_3^j(\omega) = 0$, though this does not affect our results.

Finally, endowments of the consumption good and the risky asset must conform to information structures. For instance, $e_2^F(L)$ must equal $e_2^F(H)$. Otherwise, Frank could refine his period-2 information set $\{L, H\}$. On the other hand, agents can potentially refine their information partitions endogenously by observing market trades and prices (see, e.g., Zheng, 2011). However, as we show below, this will not happen in our bubble equilibrium, so we do not need to discuss it.

As in Liu and Conlon (2018), a *competitive equilibrium* in our model consists of state-and-time dependent asset prices, $p_t(\omega)$, and net sales, $X_t^j(\omega)$, $j = E, F$, such that

(i) the $X_t^j(\omega)$ are feasible and optimal, given Agent j's information and the prices, $p_t(\omega)$;

(ii) $p_t(\omega)$, $X_t^E(\omega)$, and $X_t^F(\omega)$ depend only on the join (coarsest common refinement) of the exogenous information partitions, $\mathbb{P}_t^E$ and $\mathbb{P}_t^F$;

(iii) $X_t^j(\omega)$ depends only on Agent j's (possibly endogenously refined) information partitions; and

(iv) the asset market clears, so $X_t^E(\omega) + X_t^F(\omega) = 0$.

Consider, now, Agent j's choice in her period-t information set, $I_{t,i}^j$, $t \leq 2$. In equilibrium, $C_t^j(\omega)$, $X_t^j(\omega)$, $s_{t-1}^j(\omega)$, and $p_t(\omega)$ will be constant on $I_{t,i}^j$, so we can write these quantities as $C_t^j(I_{t,i}^j)$, $X_t^j(I_{t,i}^j)$, $s_{t-1}^j(I_{t,i}^j)$, and $p_t(I_{t,i}^j)$, by a slight abuse of notation. Also, let $\pi(I_{t,i}^j) = \Sigma_{\omega \in I_{t,i}^j} \pi(\omega)$. Then Agent j's first order condition (FOC) is given by the usual Euler equation,

$$\pi(I_{t,i}^j) U_j'(C_t^j(I_{t,i}^j)) p_t(I_{t,i}^j) \geq \beta \sum_{\omega \in I_{t,i}^j} \pi(\omega) U_j'(C_{t+1}^j(\omega)) p_{t+1}(\omega), \tag{2}$$

or equivalently,

$$p_t(I_{t,i}^j) \geq \frac{1}{M_t^j(I_{t,i}^j)} \sum_{\omega \in I_{t,i}^j} M_{t+1}^j(\omega) p_{t+1}(\omega), \tag{3}$$

where this uses $\beta^{t-1}\pi(I_{t,i}^j) U_j'(C_t^j(I_{t,i}^j)) = M_t^j(I_{t,i}^j)$, which follows from $C_t^j(\omega) = C_t^j(I_{t,i}^j)$ for $\omega \in I_{t,i}^j$. If Agent j holds a positive amount of the risky asset at the end of period t, then (2) and (3) must hold as equalities. If they hold as strict inequalities, then Agent j must strictly prefer to sell any shares she owns, and so, must be short-sale constrained. Thus, each agent's holdings of the risky asset must satisfy (2), or equivalently, (3), together with its complementary slackness conditions.

Note that conditions (2) and (3) can be written more intuitively as

$$p_t(I_{t,i}^j) \geq WTP_t^j(I_{t,i}^j), \tag{4}$$

where $WTP_t^j(I_{t,i}^j)$ is Agent j's willingness-to-pay (WTP) in her period-t cell $I_{t,i}^j$, defined by

$$WTP_t^j(I_{t,i}^j) \equiv \frac{1}{M_t^j(I_{t,i}^j)} \sum_{\omega \in I_{t,i}^j} M_{t+1}^j(\omega) p_{t+1}(\omega) = \beta E\left[\frac{U_j'(C_{t+1}^j(\omega))}{U_j'(C_t^j(I_{t,i}^j))} p_{t+1}(\omega) \,\middle|\, \omega \in I_{t,i}^j \right].$$

To simplify the equilibrium, we choose parameters such that asset sellers will always be at least weakly short-sale constrained. Thus, if Agent j is a seller in $I_{t,i}^j$, she will sell all her shares, so $X_t^j(I_{t,i}^j) = s_{t-1}^j(I_{t,i}^j)$, and she is willing to do this as long as the price at least weakly exceeds her WTP at that point. Similarly, if Agent j is a buyer in $I_{t,i}^j$, then her FOC will hold as an equality, so $p_t(I_{t,i}^j) = WTP_t^j(I_{t,i}^j)$. Thus, her WTP determines the equilibrium price, $p_t(\omega)$. We also choose parameters such that there is no endogenous information refinement. These parameter restrictions turn out to be given by our assumption that initial asset endowments, $s_0^j(\omega)$, are constant in ω for $j = E$, F, together with the three conditions in Proposition 3.1 below. These restrictions vastly simplify the calculation of our equilibria.

Finally, for the purpose of this paper, we say that a strong bubble exists in state ω, period t if the asset has a positive price even though every agent knows with certainty, at that point, that the asset's dividend is *zero*. This is more restrictive than the definition of a strong bubble in Allen et al. (1993), where the asset need not be worthless, but instead must only have a price higher than its largest possible dividend.[15]

3 The Bubble Equilibrium

Given the above setup, the following proposition presents the conditions under which an equilibrium with a strong bubble exists in our model.

Proposition 3.1. *Define $\bar{p}_1$ and $\bar{p}_2$ as solutions to*

$$\bar{p}_2 = \frac{\beta\pi(H)U_F'\big(e_3^F(H) + (s_0^E + s_0^F)d\big)}{[\pi(L) + \pi(H)]U_F'\big(e_2^F(\{L,H\}) - (s_0^E + s_0^F)\bar{p}_2\big)}d, \tag{5}$$

and

$$\bar{p}_1 = \frac{\beta U_E'\big(e_2^E(H) + (s_0^E + s_0^F)\bar{p}_2\big)}{U_E'\big(e_1^E(H) - s_0^F\bar{p}_1\big)}\bar{p}_2. \tag{6}$$

Then, in a three-state, three-period economy, with agents' information structures given in Figure 1, there exists a strong bubble equilibrium if and only if the three conditions in (7) *below are met, with shadow prices $M_t^j(\cdot)$ evaluated at the consumption levels implied by* (8) *below.*

[15]Our definition of a strong bubble needs to be more restrictive than that in Allen et al. (1993) because, in our model, an asset's price may be higher than its largest possible dividend simply because agents greatly value future dividends in states where their future endowment is relatively low, so the discount factors for those states may be greater than one. For instance, consider a two-period model with no uncertainty and Walrasian asset markets, where an agent's utility is $U(C_1, C_2) = \ln C_1 + \ln C_2$. The agent is endowed with 10 units of the consumption good in period 1 and only one unit in period 2. If there exists an asset which pays a dividend of 2 in period 2, then the agent will be willing to pay a price of $p = 4$ for a share in period 1, since her marginal discount factor on consumption would then be $\beta U'(C_2)/U'(C_1) = (1)(C_1/C_2) = (10-4)/(1+2) = 2$. That is, $p = 4$ is a fundamental value to her, even though the dividend is 2. This buyer is willing to pay such a high price because her current endowment is much larger than her future endowment. Thus, this asset is not overpriced by any reasonable definition of overpricing.

$$
\begin{aligned}
&\text{Condition 1}: \quad \frac{M_2^E(L)}{M_1^E(\{b, L\})} = \frac{M_2^E(H)}{M_1^E(H)}, \\
&\text{Condition 2}: \quad \frac{M_2^E(H)}{M_1^E(H)} \geq \frac{M_2^F(\{L, H\})}{M_1^F(\{L, H\})}, \\
&\text{Condition 3}: \quad \frac{M_3^F(H)}{M_2^F(\{L, H\})} \geq \frac{M_3^E(H)}{M_2^E(H)}.
\end{aligned} \tag{7}
$$

The strong bubble equilibrium is essentially unique,[16] *and its prices and net sales are given by*

$$
\begin{aligned}
&p_1(\omega) = \bar{p}_1 \text{ for all } \omega \in \Omega, && X_1^F(\omega) = -X_1^E(\omega) = s_0^F \text{ for all } \omega \in \Omega, \\
&p_2(b) = 0, && X_2^E(b) = X_2^F(b) = 0, \\
&p_2(L) = p_2(H) = \bar{p}_2, && X_2^E(\omega) = -X_2^F(\omega) = s_0^E + s_0^F \text{ for } \omega = L, H, \\
&p_3(b) = p_3(L) = 0,\ p_3(H) = d, && X_3^E(\omega) = X_3^F(\omega) = 0 \text{ for all } \omega \in \Omega.
\end{aligned} \tag{8}
$$

Proof. See Appendix A. □

Note that equation (5) has a unique solution for $\bar{p}_2$, and (6) has a unique solution for $\bar{p}_1$, given $\bar{p}_2$.[17] Condition 2 in (7) then implies that Ellen's period-1 WTP for the risky asset in $\{H\}$ is at least Frank's period-1 WTP in $\{L, H\}$. Also, Condition 1 implies that Ellen's period-1 WTP in $\{b, L\}$ is the same as her period-1 WTP in $\{H\}$. Thus, Ellen will be willing to pay at least as much as Frank in state L, as well as in state H, and so in every state, since Frank's period-1 WTP in $\{b\}$ is zero, as he knows the asset price will be zero in the next period. Thus, Frank will sell the risky asset in period 1 in every state, and at the price determined by Ellen, so his behavior will not reveal his information about the true state to Ellen. Also, by Condition 1 again, Ellen will bid the price of the risky asset up to the same level in every state in period 1, so Ellen's period-1 behavior will not reveal her information to Frank, either. Condition 3 then implies that Frank's period-2 WTP for the risky asset in $\{L, H\}$ is at least Ellen's period-2 WTP in $\{H\}$. Thus, Ellen will sell

[16] We say "essentially" unique because net sales are actually undetermined in state b, period 2, and in all three states in period 3, since agents are indifferent to trade in those situations. Note that, in addition to the strong bubble equilibrium, there is also a no-bubble equilibrium, where the price of the risky asset is only positive in state H.

[17] To see this, observe that the right-hand-side (RHS) of (5) is decreasing in $\bar{p}_2$, with the RHS > 0 for $\bar{p}_2 = 0$ and the RHS $\to 0$ as $\bar{p}_2 \to e_2^F(\{L, H\})/(s_0^E + s_0^F)$, since we've assumed that $U_F'(0) = \infty$. Since the left-hand-side (LHS) of (5) is a 45-Degree line, the LHS and RHS must cross in the first quadrant, and only cross once, which thus gives a unique positive $\bar{p}_2$. A similar argument applies to $\bar{p}_1$ in (6).

the asset in period 2, state H, at the price offered by Frank. Ellen will also prefer to sell in state L, at the same price, since she knows the asset price will crash to zero in period 3. Thus, Ellen will sell and Frank will buy the asset in states L and H in period 2, and at the same price in both states. Therefore, neither Ellen nor Frank learn new information from the market in period 2, so neither will refine their information partitions in period 2, just as neither does in period 1.

Thus, in a strong bubble equilibrium, Ellen buys in $\{b, L\}$, as well as in $\{H\}$, in period 1. If the state turns out to be b, then Ellen winds up being a fool, who buys a worthless asset. If the state turns out to be L, then Ellen, acting like a fool, buys the worthless asset in period 1, but she is able to ride the bubble and sell the asset to Frank in period 2, so Frank is the greater fool. Finally, in state H, there is no fool. Ellen buys the valuable asset in period 1 to smooth her consumption between periods 1 and 2. Frank then risks buying the asset in $\{L, H\}$ in period 2 and, if the state turns out to be H, the asset smooths his consumption between periods 2 and 3, as he was hoping.

3.1 Example 1

We now illustrate the model using a numerical example. Assume the agents' common prior distribution is $\pi(b) = \pi(L) = \pi(H) = 1/3$. Also, assume both agents have logarithmic utility, so $U_j(C) = \ln C$, $j = E, F$, and let $\beta = 1$, so there is no discounting of utility. Initially each agent possesses one share of the risky asset, so $s_0^E = s_0^F = 1$. The asset pays a dividend of four units of the consumption good in period 3, state H, and zero otherwise, so $d(H) = d = 4$ and $d(b) = d(L) = 0$. Agents' endowments of the consumption good, $e_t^E(\omega)$ and $e_t^F(\omega)$, are given in Table 1.[18]

Table 1: Agents' Endowments of the Consumption Good

	Ellen			Frank		
	b	L	H	b	L	H
Period 1	10	10	10	3	3	3
Period 2	5	5	14	36	36	36
Period 3	36	36	36	24	24	24

We can now use Proposition 3.1 to find the bubble equilibrium. Thus, (5) becomes

[18]Only the values in the unshaded areas of Table 1 affect our bubble equilibrium. The values in the shaded areas can be any positive numbers without influencing anyone's behavior, because the asset prices and dividends will be zero in those states and periods. Note also that, in most cases, one can choose agents endowments to be constant across states in each period. The exception is that one of Ellen's endowments in state H must differ from her endowment in states b and L, to ensure that Condition 1 in (7) above holds. Finally note that the parameters in the current example are chosen to make the calculations easy, not to mimic the quantitative features of a real-world economy.

$$\bar{p}_2 = \frac{(1/3) \cdot [24 + (1+1)(4)]^{-1}}{(1/3 + 1/3) \cdot [36 - (1+1)\bar{p}_2]^{-1}} \cdot 4 = \frac{36 - 2\bar{p}_2}{16},$$

which can be easily solved, giving $\bar{p}_2 = 2$. Using $\bar{p}_2 = 2$, (6) becomes

$$\bar{p}_1 = \frac{[14 + (1+1)(2)]^{-1}}{[10 - \bar{p}_1]^{-1}} \cdot 2 = \frac{10 - \bar{p}_1}{9},$$

so $\bar{p}_1 = 1$. Table 2 shows the equilibrium prices, $p_t(\omega)$, and Ellen's net sales, $X_t^E(\omega)$. The asset market clears, so Frank's net sales are $X_t^F(\omega) = -X_t^E(\omega)$.

Table 2: The Bubble Equilibrium

	Asset Prices			Ellen's Net Sales		
	b	L	H	b	L	H
Period 1	1	1	1	−1	−1	−1
Period 2	0	2	2	0	+2	+2
Period 3	0	0	4	0	0	0

This equilibrium involves a strong bubble in period 1, state b, since $p_1(b) = 1$, even though Ellen and Frank both know the asset is worthless there. Also, a semi-bubble is present in state L, periods 1 and 2, since $p_1(L) = 1$ and $p_2(L) = 2$, even though Ellen knows the asset is worthless.

To check that this is an equilibrium, first note that the shadow prices for all relevant information sets are

$$M_1^E(\{b, L\}) = \frac{2}{3} \cdot \frac{1}{10 - (1)(1)} = \frac{2}{27}, \quad M_1^E(H) = \frac{1}{3} \cdot \frac{1}{10 - (1)(1)} = \frac{1}{27}, \quad M_2^E(L) = \frac{1}{3} \cdot \frac{1}{5 + (2)(2)} = \frac{1}{27},$$

$$M_2^E(H) = \frac{1}{3} \cdot \frac{1}{14 + (2)(2)} = \frac{1}{54}, \quad M_3^E(H) = \frac{1}{3} \cdot \frac{1}{36} = \frac{1}{108},$$

$$M_1^F(\{L, H\}) = \frac{2}{3} \cdot \frac{1}{3 + (1)(1)} = \frac{1}{6}, \quad M_2^F(\{L, H\}) = \frac{2}{3} \cdot \frac{1}{36 - (2)(2)} = \frac{1}{48}, \quad M_3^F(H) = \frac{1}{3} \cdot \frac{1}{24 + (2)(4)} = \frac{1}{96}.$$

Thus, Condition 1 holds since

$$\frac{M_2^E(L)}{M_1^E(\{b, L\})} = \frac{1}{2} = \frac{M_2^E(H)}{M_1^E(H)},$$

and Condition 2 holds since

$$\frac{M_2^E(H)}{M_1^E(H)} = \frac{1}{2} \geq \frac{1}{8} = \frac{M_2^F(\{L, H\})}{M_1^F(\{L, H\})}.$$

Finally, Condition 3 holds since

$$\frac{M_3^F(H)}{M_2^F(\{L,H\})} = \frac{1}{2} \geq \frac{1}{2} = \frac{M_3^E(H)}{M_2^E(H)}.$$

While the above used Proposition 3.1 to find the equilibrium, it is also easy to obtain the equilibrium directly, by considering each information set, one at a time.

3.2 The Economic Intuition for the Bubble Equilibrium

To understand the model, consider Example 1. In period 1, Frank sells his share in $\{L, H\}$ because he is relatively illiquid: his period-1 endowment (3 units in Table 1) is only one twelfth of his period-2 endowment (36 units). He therefore assumes the role of the relatively illiquid "good seller" in the introduction, selling his asset, even if the price is low, to fund current consumption. This then opens the door for "bad seller" Frank, in state b, to pretend that he is selling a similarly valuable asset, even though he knows the asset is worthless. Ellen, in her information set $\{b, L\}$, then faces a lemons problem. She does not know whether Frank is a bad seller in state b, or a good (but illiquid) seller in state L. In spite of this, Ellen is willing to purchase Frank's share in $\{b, L\}$, since she has a relatively large endowment in that cell in period 1 ($e_1^E(b) = e_1^E(L) = 10$ but $e_2^E(L) = 5$), so her WTP is large. Ellen therefore buys in period 1, in $\{b, L\}$, in hopes that the true state is L, so she can ride the bubble and resell the asset to Frank in period 2, before the price crashes, and thus smooth her state-L consumption between periods 1 and 2. In period 1, state H, Ellen is less willing to supply liquidity compared to $\{b, L\}$ (her endowments are $e_1^E(H) = 10$ and $e_2^E(H) = 14$), but she knows for sure that the asset price will rise. She is therefore willing to pay the same price for the asset in $\{H\}$ as in $\{b, L\}$. This supplies the coincidence required by Condition 1 in (7) (see Subsection 3.3 below).

In period 2, state b, agents learn that the true state is b, so the asset price crashes to zero. If the true state is not b, Ellen sells her two shares to Frank in state L because she knows they are worthless. Ellen is also willing to sell in H, at a discount, because of her large liquidity demand there ($e_2^E(H) = 14$, but $e_3^E(H) = 36$). Now, Frank, as a buyer, faces a lemons problem in $\{L, H\}$. He does not know whether Ellen is selling him the asset because she knows it is worthless or because she has a large liquidity demand. However, the gains from trade in state H are large enough to overcome the lemons problem created by state L, so Frank is willing to buy in $\{L, H\}$.

Thus, the model's state space is as simple as possible, and easy to understand. We clearly need the bubble state, b. We also need a bubble-riding state like state L, where agents who know the asset's worthless are willing to hold it in order to ride the bubble and then sell it in the future to other agents who don't know it's worthless. Finally, we need a state like H, where the asset is valuable, so the ultimate buyers actually have a reason to buy. Thus, since we are using intertemporal consumption smoothing to motivate trade, we are able to avoid the extra states, ω^1 and e^2, which Liu and Conlon (2018) use to motivate risk sharing in their model.[19]

This three-element state space yields an extremely simple "skeletal structure." Also, while allowing consumption in each period adds some complexity to our model, the basic idea is very simple: people sell assets when their endowment of consumption is low, and buy assets when this endowment is high. The only wrinkle is that Ellen must have the same WTP in state H and in her cell $\{b, L\}$, as in Condition 1 in (7). However, a similar condition is also needed in Liu and Conlon's model, and other models building on Allen et al. (1993) (see also Subsection 3.3 below).

The current model was chosen to be as simple as possible, but it could easily be elaborated to better match real-world bubbles. For example, a more realistic model might involve additional agents. Thus, the buyers in period 2 might be different people from the sellers in period 1. Alternatively, "good sellers" might represent different people from "bad sellers," rather than the same agents in different states – as long as buyers do not know who is good or bad. The number of periods could also be increased, which may allow us to model momentum in bubbles, as in Araujo and Doblas-Madrid (2019). One could also allow the bubble asset to be a produced asset, as in Conlon (2015), or give agents market power, to eliminate the need for short-sale constraints, as suggested in footnote 14 above. Finally, real-world bubble episodes may actually involve many different potentially bubbly assets, sold by many different sellers, some good, and some bad. Once one understands the simplest possible model, richer models are easy to develop.

[19] In Liu and Conlon (2018), the pattern of trade is similar to here: Frank sells to Ellen in period 1 and then, in certain states, Ellen sells back to Frank in period 2. However, their model is in some ways more complicated than the current model. Thus, their model needs their state ω^1 for two reasons: (i) by having the price crash in period 2 in state ω^1, this gives "good Frank" a motive to sell the asset to Ellen in period 1, before the crash, assuming he cares relatively more about consumption in state ω^1 than Ellen does, and (ii) the possibility of this crash lowers the willingness to pay of "good Ellen" in period 1 to the point where Frank cannot distinguish between the price offered by good Ellen and bad Ellen. Liu and Conlon's model also needs their state e^2, to motivate good Ellen to sell to Frank in period 2: if the price crashes in state e^2 in period 3, and good Ellen cares relatively more about consumption in state e^2 than Frank does, this gives good Ellen a motive to sell, to avoid the possibility of this crash.

However, while the above extensions should be interesting, one early question remains central: the role of bubbles and anti-bubble policy in business cycles (Barlevy, 2018). While this issue may be more challenging than some of the issues raised just above, White (2019) and White and Conlon (2019) have done some preliminary work on this, based on the current approach.[20]

3.3 The Lack of Robustness of the Equilibrium, and Possible Solutions

In the above equilibrium, $p_1(b) = p_1(L)$ because Ellen is the period-1 buyer in $\{b, L\}$, whose WTP determines these two prices, and she cannot distinguish state b from state L. However, the equilibrium also requires the coincidence $p_1(L) = p_1(H)$ because, if $p_1(L) \neq p_1(H)$, then Frank would be able to distinguish L from H. He would therefore know, in state L, that the asset is worthless, and so, would not buy in period 2. Thus, Ellen would not buy in $\{b, L\}$ in period 1, so the bubble equilibrium would unravel. The equilibrium thus requires the coincidence, $WTP_1^E(\{b, L\}) = WTP_1^E(\{H\})$, between Ellen's WTPs. This coincidence is implied by Condition 1 in (7).

Since Ellen must have the same period-1 WTP in $\{b, L\}$ as in $\{H\}$, our model is not robust to changes in parameters. However, this lack of robustness is just a consequence of our discrete state-space simplification. In more realistic continuous state-space models, bubble equilibria can easily be made robust. Thus, suppose we replace the cell $\{b, L\}$, belonging to "bad Ellen," with a continuum of cells, $\{b_\alpha, L_\alpha\}$, involving a range of different WTPs, and we also replace the cell $\{H\}$, belonging to "good Ellen," with a continuum of cells $\{H_\beta\}$. Then the coincidence in Condition 1 is replaced by a matching function, $\beta = \beta(\alpha)$, between bad and good types of Ellen, as in Zheng and Conlon (2019). Changes in parameters would then simply change this matching function. See also Doblas-Madrid (2012) and, for a different approach, Zhang and Zheng (2017).

Our model's lack of robustness, however, is inconvenient from a practical point of view, since it is difficult to do comparative statics analysis in simple models with discrete state spaces. Thus, in order to analyze the effects of some policy, we may want to use a continuous state-space version of the model, rather than a discrete state-space version. Otherwise the policy may break the coincidence required in Condition 1, and the breaking of this coincidence may dominate the more

[20] Thus, they consider a model of a "semi-bubble," where some agents know whether the asset is overpriced, while others do not, and incorporate this into a very simple New Keynesian model. A major role for bubble-bursting policy in their context is then to allow information, held by the better informed agents in asset markets, to be revealed to price setters in goods markets, so those price setters are better able to anticipate shocks to aggregate demand.

relevant effects, which could only be captured in models with richer state spaces.[21]

It is also worth noting that the above lack of robustness is inconvenient primarily because it requires the equality of two *endogenous* variables in equilibrium, $WTP_1^E(\{b,L\})$ and $WTP_1^E(\{H\})$. Therefore, a change in any parameter can potentially disrupt the equilibrium. By contrast, the model also lacks robustness in terms of the *exogenous* initial endowments, $s_0^j(\omega)$, of the risky asset, but this is less of a problem. Thus, consider Frank in period 1. We clearly require $s_0^F(L) = s_0^F(H)$, since states L and H are in the same information set, $\{L, H\}$, of Frank's. However suppose, as in our equilibria, that Frank is short-sale constrained in $\{L, H\}$, so he sells all of his holdings of the risky asset in that cell. Then the bubble equilibrium also requires the coincidence $s_0^F(b) = s_0^F(L)$. This is because Frank also sells all of his shares in state b, since he knows there that the price will crash next period. Thus, if his initial shares differed between states b and L, his sales would also differ, so Ellen could tell whether the state was b or L, and she would never buy in state b. Nevertheless, this lack of robustness is less of a problem than that implied by Condition 1 above, since it is less likely to interfere with comparative statics analysis.[22]

4 Consumption Smoothing and Welfare in Example 1

A complete welfare analysis of a bubble in the current model is beyond the scope of this paper. We nevertheless make some brief comments about the effects of an anti-bubble policy on intertemporal consumption smoothing and welfare in Example 1 above, to illustrate some of the workings of the model. For a more general analysis, see Liu (2020).

Consider an asset-deflation policy, for example, where the central bank follows a policy rule of revealing to all agents, before period 1, whether the bubble or semi-bubble state, b or L, occurred.[23]

[21] However, it is relatively easy to extend discrete state-space models to accommodate comparative statics analysis, without resorting to fully continuous state spaces. For example, Conlon (2015) uses a discrete state-space model to analyze a policy rule of announcing whether an asset is in a strong bubble. In that model, bad sellers behave like *low* confidence good sellers in the absence of the policy rule, but they switch to behaving like *high* confidence good sellers if the policy rule is in place, but no actual bubble announcement is made, since the economy is not in the bubble state, but instead is in a semi-bubble state. For a similar example, see footnote 1 in Technical Appendix A.

[22] Thus, the lack of robustness of this second variety has been present in models going back at least to Allen et al. (1993), but as far as we know, no one has ever considered it worth mentioning.

[23] The asset-deflation policy we discuss here – simply announcing whether states b or L occurred – thus resembles the policies in Asako and Ueda (2014), Conlon (2015), Holt (2018), and Awaya et al. (2020), but it is not the only sort of anti-bubble policy a central bank might pursue. Instead of announcing the true state of the world – what we might call a "bubble bursting" policy – central banks often use open market operations to counteract bubbles. That is, they sell bonds to raise interest rates – what we might call a "bubble suppression" policy. Presumably such bond sales compete with overpriced assets for investors' savings, and so, push down the prices of these overpriced assets

We first find the equilibrium under this policy for Example 1. The policy disrupts the bubble equilibrium, so the price of the asset is always zero in states b and L. In state H, however, the policy eliminates the lemons problem. Agents thus know for sure that the asset will pay a dividend, so they are willing to pay more for it.[24] The prices in state H then rise to $p_1(H) = 45/31 \approx 1.45$ in period 1 and to $p_2(H) = 3.60$ in period 2. Given these new prices, Ellen buys Frank's one share in state H, period 1, Frank buys Ellen's two shares in state H, period 2, and no trade occurs in other states and periods. Table 3 presents the resulting consumption levels, for both the bubble and the policy equilibria.

Table 3: Equilibrium Consumption Levels in Example 1

	Bubble						Policy					
	Ellen			Frank			Ellen			Frank		
	b	L	H	b	L	H	b	L	H	b	L	H
Period 1	9	9	9	4	4	4	10	10	8.55	3	3	4.45
Period 2	5	9	18	36	32	32	5	5	21.20	36	36	28.80
Period 3	36	36	36	24	24	32	36	36	36	24	24	32

We next use the consumption levels from Table 3 to compute agents' welfare with and without the asset-deflation policy. Table 4 provides agents' lifetime utilities state-by-state for each equilibrium. E.g., Frank's state-L utility under the policy is given by $\ln 3 + \ln 36 + \ln 24 = \ln 2{,}592$.

Table 4: State-by-State Utility in Example 1

	Ellen			Frank		
	b	L	H	b	L	H
Bubble Equilibrium	ln 1,620	ln 2,916	ln 5,832	ln 3,456	ln 3,072	ln 4,096
Policy Equilibrium	ln 1,800	ln 1,800	ln 6,524.13	ln 2,592	ln 2,592	ln 4,102.61

We can now use Tables 3 and 4 to find the effect of the policy on consumption smoothing and welfare. We first discuss these effects state-by-state, beginning with state b. The asset is essentially useless for consumption smoothing in state b, because its price drops to zero in period 2. However, the policy prevents Ellen from buying this asset from Frank in period 1, so Ellen's consumption rises and Frank's falls in period 1, but neither's consumption changes in periods 2 or 3. Ellen's

(for a related effect, see, e.g., Caballero and Krishnamurthy, 2006). The welfare effects of such a policy, in a model related to the current one, are studied in Conlon et al. (2021). Here we consider a much simpler policy, since our major goal is to illustrate the role of intertemporal consumption smoothing in our model.

[24] Typically when a lemons problem is eliminated, people will trade more. However, since sellers are always short-sale constrained in our equilibria, elimination of the lemons problem increases prices, not the volume of trade.

lifetime welfare therefore rises and Frank's falls in that state.

State L is more interesting. Comparing Table 1 to the left-hand panels of Table 3, we can see how trading the bubble asset back and forth allows agents to smooth their consumption in state L, so their consumption paths in that state more closely follow the path of the aggregate supply of the consumption good. For, as seen from Table 1, the aggregate endowment in state L rises from $10 + 3 = 13$ to $5 + 36 = 41$ between periods 1 and 2, or 215%, while Ellen's endowment falls from 10 to 5, or 50%, and Frank's rises from 3 to 36, or 1,100%. That is, the growth rates of Ellen's and Frank's endowments are very different from the growth rate of the aggregate endowment. By contrast, in the bubble equilibrium in Table 3, Ellen's consumption stays constant at 9 units, instead of falling, while Frank's consumption grows 700% instead of 1,100%. Thus, even though consumption growth rates are still not equal to the growth rate of the aggregate endowment, as they would be with perfect consumption smoothing, they are quite a bit closer. That is, trading the bubble asset back and forth allows both agents to smooth their consumption intertemporally.[25]

Since the asset-deflation policy causes the price of the asset to collapse immediately to zero in state L, agents no longer have any useful asset they can trade to smooth their consumption in that state. Thus, they can only consume their endowments. Asset-deflation policy can therefore be welfare reducing in that state. In fact, Ellen's utility necessarily falls in state L because, when Ellen purchases Frank's share in $\{b, L\}$ in period 1, she is effectively choosing to participate in the bubble equilibrium, in spite of the risk that the true state might be b. Thus, by a standard revealed preference argument, Ellen must benefit in state L from the bubble equilibrium. Intuitively, the policy prevents Ellen from saving, and it also interferes with her exploitation of Frank.

In our numerical example, Frank is also hurt by the policy in state L, even though he is the ultimate greater fool in that state, who the policy is supposed to protect. Specifically, Frank's state-L lifetime utility falls from ln 3,072 in the bubble equilibrium to ln 2,592 under the policy.

It might not be surprising that Frank is hurt by the policy, since he is the initial holder of the asset whose price is deflated by the policy, costing him one unit of the consumption good in period 1. However, the policy also *saves* Frank *four* units of the consumption good in period 2, since it prevents him from becoming the greater fool, buying a worthless asset. His period-2 gain

[25] While the consumption smoothing calculations here are only meant to be illustrative, it is easy to show that the corresponding results hold generally in our model (Liu, 2020). Note, however, that the bubble asset does not necessarily smooth consumption between periods 2 and 3 in state L, since its price falls to zero in period 3.

from the policy, in units of the consumption good, is therefore four times bigger than his period-1 loss. The fall in his welfare thus comes down to consumption smoothing. Since Frank is poor in period 1, he is hurt more by his one-unit loss in that period than he benefits from his four-unit gain in period 2, when he is wealthy. Therefore, since the policy makes his consumption stream less smooth intertemporally, his overall welfare falls in state L in this example.[26]

While the policy might not *always* hurt Frank (see footnote 29 below), this result is reminiscent of results in Holt (2018), who first showed that this sort of asset-deflation policy tends to reduce welfare in greater-fool bubble models, since it interferes with trade. However, in Holt's case the policy interferes with risk sharing, not intertemporal consumption smoothing.[27]

Our results are also consistent with the results in infinite-horizon bubble models like Samuelson (1958). Bubble assets tend to improve welfare in those models, by allowing agents to smooth consumption from youth to old age, so anti-bubble policies tend to be harmful.

However, unlike in OLG models, the reason Frank buys in period 2, here, is not because he wants to smooth his consumption in state L – he knows the price is about to crash if the state is L. Instead, he buys in state L because he can't tell states L and H apart, and he wants to smooth his consumption in state H. Thus, Frank's benefit from the bubble equilibrium in state L of our model may be more surprising than his benefit in OLG models, since, in our model, he is the greater fool in state L, holding the asset when the price crashes. The reason why Frank can benefit here is because, though he is hurt in period 2 by the bubble equilibrium, the bubble asset facilitates his consumption smoothing between period 2 and the *earlier* period, period 1.

Note, though, that our welfare result in state L requires the greater-fool buyers, in period 2 of our model, to be the same agents as the period-1 sellers. This allows us to avoid interpersonal comparisons of utility, so the greater fools' period-1 loss from the policy can be subtracted from their own period-2 gain from it.

The plausibility of our state-L result presumably also requires that the greater fools, who buy

[26] Similar results continue to hold even if alternative means of consumption smoothing, such as riskless *storage*, are available; see Technical Appendix A. For the case of a bubble in the presence of a riskless *asset*, see Subsection 5.4 below and Technical Appendix B, as well as Conlon et al. (2021).

[27] Thus, as Holt clearly explains, his results depend on the "Hirshleifer effect" (Hirshleifer, 1971), where information revelation interferes with agents' ability to share risks. By contrast, in our model, the policy hurts welfare in state L because it eliminates an asset which would have been useful for intertemporal consumption smoothing. Other papers that consider announcement policies include Conlon (2015), who discusses the effects of asset-deflating announcements on the *allocation of production*, Asako and Ueda (2014), who discuss the effects of the *timing* of asset-deflating announcements on prices, and Awaya et al. (2020) who consider Bayesian persuasion in their middleman model.

the overpriced asset in period 2, are wealthy enough to bear the risk. If the greater fools are poor and naive investors just entering the market (the "greengrocers and servant girls" in Kindleberger, 2000, p. 29 – or recent GameStop buyers), then the welfare costs to them from mistakenly buying overpriced assets may very well be higher. Thus, in considering the advisability of anti-bubble policy, policymakers may want to consider whether buyers of potentially bubbly assets are wealthy, sophisticated investors, or lower-income, inexperienced newcomers.[28]

Next consider state H. Even though the bubble equilibrium facilitates intertemporal consumption smoothing between periods 1 and 2 in state L, it also creates a lemons problem which distorts consumption smoothing in state H. An asset-deflation policy, which eliminates this lemons problem, therefore improves consumption smoothing in state H, as can be easily checked for our example. This result can also be shown to hold more generally (Liu, 2020).

Intuitively, when the policy eliminates the lemons problem, everyone learns the state is H, so buyers bid up asset prices. Also, since agents tend to sell the asset in periods of their lives when they are relatively poor, and buy it when they are relatively wealthy, the higher prices raise agents' consumption when they are poor sellers and reduce their consumption when they are rich buyers. Their consumption therefore becomes smoother under the policy than in the bubble equilibrium.

In the current example, the improved consumption smoothing under the policy is large enough to increase the welfare of both Frank and Ellen in state H, as shown in Table 4. It turns out that Ellen always benefits from the policy in state H in our three-state model; see Liu (2020). However, Frank could either benefit or be hurt by the policy in state H.[29]

Besides examining agents' state-by-state welfare, we can also look at their *ex ante* welfare, as well as their interim welfare in different information sets. In Example 1, the prior probabilities are $\pi(b) = \pi(L) = \pi(H) = 1/3$. Thus, Frank's *ex ante* welfare falls from $(\ln 3{,}456 + \ln 3{,}072 + \ln 4{,}096)/3 = \ln 3{,}516.56$ in the bubble equilibrium to $(\ln 2{,}592 + \ln 2{,}592 + \ln 4{,}102.61)/3 = \ln 3{,}020.72$ under the policy, and Ellen's falls from $\ln 3{,}020.23$ to $\ln 2{,}764.96$. Both are therefore

[28] Of course, this raises the issue of "paternalistic" policies. On the other hand, if we extend our model to allow the greater fools to be fragile financial intermediaries, who are able to pass their financial difficulties on to others as negative externalities, then this might create an additional argument for an asset-deflation policy.

[29] In general, the policy does not always hurt Frank in state L, nor does it always benefit him in state H. For example, if we change Frank's period-1 endowment $e_1^F(\{L, H\})$ from 3 to 10, then the bubble and the policy equilibria would stay the same. However, the policy would now raise Frank's state-L utility from $\ln 11 + \ln 32 + \ln 24 = \ln 8{,}448$ to $\ln 10 + \ln 36 + \ln 24 = \ln 8{,}640$, while it would lower his state-H utility from $\ln 11 + \ln 32 + \ln 32 = \ln 11{,}264$ to $\ln 11.45 + \ln 28.8 + \ln 32 = \ln 10{,}553.81$. That is, when Frank's endowment, $e_1^F(\{L, H\})$, becomes bigger, he is hurt less in state L by his period-1 loss, and benefits less in state H from his period-1 gain under the policy.

hurt by the asset-deflation policy *ex ante*. In addition, Frank's interim welfare in $\{L, H\}$ falls from $\ln 3{,}547.24$ to $\ln 3{,}260.97$, and Ellen's interim welfare in $\{b, L\}$ falls from $\ln 2{,}173.46$ to $\ln 1{,}800$.

Of course, the above is only intended to illustrate the range of issues that naturally arise in this model. Asset-deflation policies will obviously have additional effects in richer bubble models, so the effects found here only illustrate a subset of the different possibilities. For example, asset-deflation policies may be more beneficial in models involving produced assets, as in Conlon (2015), since those policies can shift production to states where the asset is valuable. Such policies might also mitigate business cycles (see footnotes 20 and 28 above).

Thus, while even a simple model like ours is capable of raising interesting issues, it is clear that the full positive and negative welfare impacts of anti-bubble policies cannot begin to be assessed in such a simple model. These impacts are therefore an important topic for future research.

5 Bubbles with a Durable Consumption Good

In the previous sections, we assumed the consumption good was completely perishable, and no riskless asset existed. There were therefore no alternative means available for agents to save portions of their endowment, other than through trades of the bubble asset, and so, no alternative way for agents to smooth their consumption over time. One might therefore conjecture that, if we introduce an alternative means of consumption smoothing, then bubbles will cease to exist in our model. In this section, however, we assume that the consumption good is durable (or storable), so an alternative means of consumption smoothing *is* available, and show that investors can continue to hold overpriced assets in equilibrium, so strong bubbles are still possible. Thus, our bubble equilibria do not depend on the absence of alternative means of consumption smoothing.

We first derive agents' utility-maximizing savings and storage conditions. We then show that the strong bubble equilibrium in Example 1 above remains unchanged, even if the consumption good is partially durable, as long as the storage technology has a sufficiently high depreciation rate. In fact, we show that, for *any* pre-existing three-state bubble model, if the consumption level is positive in each state and period, then a storage technology can be incorporated into the model without changing its strong-bubble equilibrium, as long as the return to storage is sufficiently low.

More interestingly, we show that a strong bubble can coexist with a durable good more generally.

Specifically, Example 3 below presents a strong bubble with a perfectly durable good and positive levels of storage in equilibrium, and Example 4 presents a very similar model, but with a storage technology that involves quantitative *appreciation* instead of depreciation. Technical Appendix A shows that the welfare effects in Example 3 resemble those in Example 1 above, even though the consumption good is perfectly durable in Example 3, and storage is used in equilibrium. Similar results follow for Example 4 where storage involves appreciation (see footnote 1 in that appendix).

Finally we show that, in any strong bubble equilibrium where storage is available, Ellen never stores in state H in periods 1 or 2, and Frank never stores in period 1 in states L or H. These results dramatically simplify the calculation of strong bubble equilibria in the durable goods case.

5.1 Utility-Maximizing Storage Conditions

Storage is private, so others cannot observe an agent's storage decisions. Let $\delta < 1$ be the one-period depreciation rate, so if Agent j places $v_t^j(\omega)$ units of consumption good into storage in period t, she will receive $(1-\delta)v_t^j(\omega)$ units of consumption good in period $t+1$. Note that, if $\delta < 0$, then the quantity of the good placed in storage does not depreciate, but appreciates, so agents can use quantities of the consumption good today to privately produce larger quantities in the future. Also, no one uses storage in period 3 because agents do not consume after period 3, so $v_3^j(\omega) = 0$ is optimal for all $\omega \in \Omega$ and $j = E, F$ (of course, a fourth period, after the dividend is paid, could easily be added). With this setup, Agent j's state-ω consumption is

$$\begin{aligned} C_1^j(\omega) &= e_1^j(\omega) + p_1(\omega)X_1^j(\omega) - v_1^j(\omega), \\ C_2^j(\omega) &= e_2^j(\omega) + p_2(\omega)X_2^j(\omega) + (1-\delta)v_1^j(\omega) - v_2^j(\omega), \\ C_3^j(\omega) &= e_3^j(\omega) + d(\omega)s_3^j(\omega) + (1-\delta)v_2^j(\omega), \end{aligned} \tag{9}$$

where, without loss of generality, let $X_3^j(\omega) = 0$, since the period-3 asset price equals the dividend.

Agents are free to choose any nonnegative portion of their endowment to store, but they cannot store a negative amount, i.e., they cannot borrow through the storage technology. Thus, Agent j's level of storage, $v_t^j(\omega)$, for $\omega \in I_{t,i}^j$, is determined by the FOC in her cell $I_{t,i}^j$,

$$M_t^j(I_{t,i}^j) \geq M_{t+1}^j(I_{t,i}^j)\,(1-\delta)\,, \tag{10}$$

where (10) holds as an equality if the agent stores a positive amount in period t, while if (10) holds as a strict inequality, then the agent must place none of her consumption good in storage. Storage levels must therefore satisfy (10) and its complementary slackness conditions. Agent j also cannot store more than her available resources, and she will never store all of her resources, since $U_j'(0) = \infty$, $j = E, F$ (this can be seen using a modification of the argument in footnote 30 below). Finally, because storage is private, a storage decision by one agent will not cause endogenous refinement of other agents' information partitions. This means, for instance, that Ellen can store different amounts of the consumption good in $\{b, L\}$ versus $\{H\}$ in period 1, without revealing to Frank whether the true state is L or H.

Proposition 5.1. *In our three-state, three-period economy, with a durable consumption good, an allocation as in* (8) *above is a strong bubble equilibrium if and only if the three conditions in* (7) *are met, together with the FOCs for storage in* (10)*, though with shadow prices evaluated at consumption levels from* (9)*, not* (1)*.*

Note that the previous FOCs, given in (18) through (22) of Appendix A below, will still hold, though again, with the consumption levels from (9). The proof of Proposition 5.1 is similar to the proof of Proposition 3.1 above, and so is omitted. We illustrate equilibria with a storage technology in Examples 2, 3 and 4 below.

5.2 Numerical Examples

We first consider the case where stored goods depreciate so quickly that no one stores in equilibrium, before turning to examples where storage is more attractive.

Example 2 (Partially Durable Good): Consider again Example 1 above. In that example, we assumed the consumption good was completely perishable, so $\delta = 1$. Now let $\delta = 0.5$, so the consumption good is partially durable. It is easy to check that the strong bubble equilibrium in that example still applies, with zero storage in all states and periods.

The change in the parameter δ does not change the equilibrium because storage is so unattractive. In fact, (10) turns out to hold as a strict inequality for all cells, as can be seen using Ellen's and Frank's shadow prices from Subsection 3.1 above, together with the shadow prices for state b in period 2, and for states b and L in period 3. Thus, neither agent will use storage, so the equilibrium results are not affected. We generalize this example in Proposition 5.2.

Proposition 5.2. *Assume that each agent's endowment of the consumption good is positive in each state and period of a three-state, three-period bubble model (as we did just above equation* (1) *in Section 2). Then a storage technology can be incorporated into the model without changing the pre-existing no-storage equilibrium, provided that $\delta < 1$ is sufficiently close to one.*

Proof. The shadow prices $M_t^j(\cdot)$ are all positive and finite, since consumption is always positive, so $0 < U_j'(C_t^j(\omega)) < \infty$ for all t, j, and ω.[30] Let $a = \min_{i,j,t} \left\{ M_t^j(I_{t,i}^j) \right\}$ and $b = \max_{i,j,t} \left\{ M_{t+1}^j(I_{t,i}^j) \right\}$. Then a and b are both positive and finite because they are extremes over a finite number of positive finite cases. Also, (10) is a strict inequality for all cells, for any $\delta \in (1 - a/b, 1)$. Thus, no one stores, so the strong bubble equilibrium is unchanged. □

In the more general case, where storage might be used in equilibrium, one can use the following algorithm to find the strong bubble equilibria. Note that Propositions 5.3, 5.4, and 5.5 from Subsection 5.3 below allow us to simplify the calculations (i.e., Ellen does not store in state H in periods 1 and 2, and Frank does not store in period 1 in $\{L, H\}$). We thus use these results first, before proving them in Subsection 5.3. Of course, none of the results in Subsection 5.3 depend on anything in this subsection. Also, Algorithm 1 might yield multiple strong bubble equilibria.

Note also that the examples below are easy to check, without using Algorithm 1, so the reader can skip the algorithm entirely, without loss of continuity.

Algorithm 1:

1. Find $X_t^j(\omega)$ as in (8). Also, $v_1^E(H) = v_2^E(H) = v_1^F(\{L, H\}) = 0$, by Propositions 5.3 through 5.5 below.
2. Find $v_2^F(\{L, H\})$, which we abbreviate v_2^F, and $\bar{p}_2$, using the FOC (18) from Appendix A below, together with Frank's version of (10) in his period-2 cell $\{L, H\}$.[31] That is, jointly

[30] To show this, start by noting that there is no storage in the pre-existing equilibrium, by assumption. Thus, consumption is given by the formulas in (1). Consider period 3 first. Period-3 consumption equals the positive endowment plus any dividends. Thus, $C_3^j(\omega) > 0$ for each agent in each state.

Next consider the period-2 cell $\{b\}$. Since the price of the asset is zero, $C_2^j(b)$ simply equals the positive endowment, so $C_2^j(b) > 0$ for each agent. Also, Ellen sells in states L and H in period 2, so $C_2^E(L) > 0$ and $C_2^E(H) > 0$. Turn, therefore, to Frank's period-2 cell $\{L, H\}$. If $C_2^F(\{L, H\}) = 0$, then $U_F'(C_2^F(\{L, H\})) = \infty$. Using Frank's FOC (18) in Appendix A, it follows that $\bar{p}_2 = 0$. However, $\bar{p}_2 = 0$ implies $C_2^F(\{L, H\}) = e_2^F(\{L, H\}) > 0$. Thus, $C_2^F(\{L, H\})$ must be positive, so $U'(C_2^F(\{L, H\}))$ is finite. Similar arguments apply for period-1 consumption and shadow prices.

[31] This uses the result that Frank does not store in his period-1 cell $\{L, H\}$. If $v_1^F(\{L, H\}) \neq 0$ had been possible, $\bar{p}_2$ and v_2^F would have had to be solved jointly with $v_1^F(\{L, H\})$, making the problem much more difficult.

solve

$$\bar{p}_2 = \frac{\beta\pi(H)U'_F\left(e_3^F(H) + (s_0^E + s_0^F)d + (1-\delta)v_2^F\right)}{[\pi(L)+\pi(H)]\,U'_F\left(e_2^F(\{L,H\}) - (s_0^E + s_0^F)\bar{p}_2 - v_2^F\right)}d, \tag{11}$$

and

$$\begin{aligned}&\big[\pi(L)+\pi(H)\big]U'_F\big(e_2^F(\{L,H\}) - (s_0^E+s_0^F)\bar{p}_2 - v_2^F\big)\\ &\quad\geq \beta\big[\pi(L)U'_F\big(e_3^F(L) + (1-\delta)v_2^F\big) + \pi(H)U'_F\left(e_3^F(H) + (s_0^E+s_0^F)d + (1-\delta)v_2^F\right)\big](1-\delta),\end{aligned} \tag{12}$$

along with the complementary slackness condition for (12). These yield at least one nonnegative solution for v_2^F and $\bar{p}_2$ (see Appendix B.1). To obtain all possible solutions, first set $v_2^F = 0$, solve (11) for $\bar{p}_2$ and use it in (12). If (12) holds, keep this solution. Then solve (11) and (12) jointly, with (12) regarded as an equality, and keep all solutions with $v_2^F > 0$.

3. For each solution from Step 2, check whether Condition 3 in (7) holds. If it does not, then that solution cannot give a strong bubble equilibrium, and is discarded.
4. For each remaining $\bar{p}_2$, find $p_1(H) = \bar{p}_1$ using (6). Given $\bar{p}_2$, this is one equation in one unknown, and it yields a unique positive solution for $\bar{p}_1$.[32]
5. For each solution from Step 4, check whether Condition 2 in (7) holds. If it does not, then that solution does not give a strong bubble equilibrium, and is discarded.
6. For any remaining solutions, assume temporarily that $p_1(b) = p_1(L) = \bar{p}_1$, from Step 4. Find $v_1^E(\{b,L\}) \equiv v_1^E$ for short, as well as $v_2^E(b)$ and $v_2^E(L)$, using Ellen's version of (10) in her period-1 cell $\{b,L\}$ and her two period-2 cells $\{b\}$ and $\{L\}$. That is, jointly solve

$$\begin{aligned}[\pi(b)+\pi(L)]\,U'_E\big(e_1^E(\{b,L\}) - s_0^F\bar{p}_1 - v_1^E\big) &\geq \beta\pi(b)U'_E\big(e_2^E(b) + (1-\delta)v_1^E - v_2^E(b)\big)(1-\delta)\\ &\quad+\beta\pi(L)U'_E\left(e_2^E(L) + (s_0^E+s_0^F)\bar{p}_2 + (1-\delta)v_1^E - v_2^E(L)\right)(1-\delta),\end{aligned} \tag{13}$$

$$U'_E\left(e_2^E(b) + (1-\delta)v_1^E - v_2^E(b)\right) \geq \beta U'_E\left(e_3^E(b) + (1-\delta)v_2^E(b)\right)(1-\delta), \tag{14}$$

$$U'_E\left(e_2^E(L) + (s_0^E+s_0^F)\bar{p}_2 + (1-\delta)v_1^E - v_2^E(L)\right) \geq \beta U'_E\left(e_3^E(L) + (1-\delta)v_2^E(L)\right)(1-\delta), \tag{15}$$

along with their complementary slackness conditions. For each possible $(\bar{p}_1, \bar{p}_2)$ pair, these conditions yield unique nonnegative solutions for v_1^E, $v_2^E(b)$, and $v_2^E(L)$ (see Appendix B.2).
7. Check whether Condition 1 in (7) holds, so the assumption $p_1(b) = p_1(L) = \bar{p}_1$ in step 6 is actually valid. If it holds, then that solution gives a strong bubble equilibrium.

[32] Note that (6) applies because Ellen does not store in state H in periods 1 and 2, by Propositions 5.3 and 5.5 below. Also, recall that (6) must yield a unique positive solution for $\bar{p}_1$, given $\bar{p}_2$ (see footnote 17 above).

8. Find $v_1^F(b)$ and $v_2^F(b)$ using Frank's version of (10) in his cell $\{b\}$ in periods 1 and 2, i.e.,

$$U_F'\left(e_1^F(b) + s_0^F\bar{p}_1 - v_1^F(b)\right) \geq \beta U_F'\left(e_2^F(b) + (1-\delta)v_1^F(b) - v_2^F(b)\right)(1-\delta), \tag{16}$$

$$U_F'\left(e_2^F(b) + (1-\delta)v_1^F(b) - v_2^F(b)\right) \geq \beta U_F'\left(e_3^F(b) + (1-\delta)v_2^F(b)\right)(1-\delta), \tag{17}$$

together with their complementary slackness conditions. Again, for each possible $(\bar{p}_1, \bar{p}_2)$ pair, unique nonnegative solutions exist for $v_1^F(b)$ and $v_2^F(b)$.[33]

We now give an example where a strong bubble exists, even though the consumption good is perfectly durable and agents actually use storage in equilibrium. This shows that, even though storage competes with the risky asset, the occurrence of a strong bubble depends on the pattern of trade, which, in turn, depends on agents' endowments and information structures, regardless of the depreciation rate. If the rate of return on storage is very favorable, then the price of the risky asset can simply fall to the point where agents are willing to hold the asset in spite of its risk.

Example 3 (A Perfectly Durable Good and Positive Storage): Suppose $\delta = 0$. Suppose also that $U_j(C) = \ln C$ for $j = E, F$, $\beta = 1$, $s_0^E = s_0^F = 1$, $\pi(\omega) = 1/3$ for $\omega \in \Omega$, and $d = 4$. Ellen's and Frank's endowments of the consumption good are given in Table 5.

Table 5: Agents' Endowments of the Consumption Good in Example 3

	Ellen			Frank		
	b	L	H	b	L	H
Period 1	17	17	7	7	6	6
Period 2	14	10	8	7	40	40
Period 3	12	12	25	6	16	8

Using Algorithm 1, or just checking, one can find a strong bubble equilibrium. Ellen buys Frank's one share of the risky asset in period 1 in all three states, at the price $\bar{p}_1 = 1$, while Frank buys Ellen's two shares of the risky asset in period 2 in states L and H, at the price $\bar{p}_2 = 2$. Agents' levels of storage are shown in the left-hand panels of Table 6, and their final consumption levels are shown in the right-hand panels. Note that there is no storage in period 3, as expected. Also, Ellen does not store in state H in periods 1 or 2, and Frank does not store in $\{L, H\}$ in period 1. We generalize these later zero-storage results in the following subsection.

[33] A proof like that in Appendix B.2 applies here, though the proof is easier here than in Appendix B.2, since it only involves two storage decisions, not three.

Table 6: Storage and Consumption Levels in Example 3

	Storage						Consumption					
	Ellen			Frank			Ellen			Frank		
	b	L	H	b	L	H	b	L	H	b	L	H
Period 1	2	2	0	1	0	0	14	14	6	7	7	7
Period 2	2	2	0	1	10	10	14	14	12	7	26	26
Period 3	0	0	0	0	0	0	14	14	25	7	26	26

Example 4 (Productive Storage): In this example we construct a model with a bubble, even though $\delta = -1$, so the quantity of the consumption good, when stored, appreciates over time, at a net rate of 100% per period, rather than depreciating. We construct this example by modifying Example 3 above, in an extremely simple, mechanical way. Thus, suppose again that $U_j(C) = \ln C$ for $j = E, F$, $\beta = 1$, $s_0^E = s_0^F = 1$, and $\pi(\omega) = 1/3$ for $\omega \in \Omega$. However, let $d = 16$, rather than 4. Also, let Ellen's and Frank's endowments be as given in Table 7.

Comparing Table 7 to Table 5 above, note that we've chosen agents' endowments to be the same in period 1 for the two examples, while their period-2 endowments are twice as high in the current example, and their period-3 endowments are four times higher here.

Table 7: Agents' Endowments of the Consumption Good in Example 4

	Ellen			Frank		
	b	L	H	b	L	H
Period 1	17	17	7	7	6	6
Period 2	28	20	16	14	80	80
Period 3	48	48	100	24	64	32

The equilibria follow a similar pattern. Thus, in the current example, the equilibrium price of the asset turns out to rise from 1 to 4 to 16 before crashing, so prices are twice as high in period 2 and four times higher in period 3 here. A similar pattern holds for equilibrium storage and consumption levels, given in Table 8. Again, they're the same in period 1, and twice as high in period 2 in the current example, while agents' consumption levels are four times higher in period 3 here (of course, their storage levels are also four times higher, since $0 = 4 \times 0$).

Intuitively, if we double all quantities in period 2, and quadruple them in period 3, then we reduce all one-period stochastic discount factors by a factor of two, since utility is logarithmic. This allows a doubling of the gross rate of return on the risky asset, which matches the doubling

Table 8: Storage and Consumption Levels in Example 4

	Storage						Consumption					
	Ellen			Frank			Ellen			Frank		
	b	L	H	b	L	H	b	L	H	b	L	H
Period 1	2	2	0	1	0	0	14	14	6	7	7	7
Period 2	4	4	0	2	20	20	28	28	24	14	52	52
Period 3	0	0	0	0	0	0	56	56	100	28	104	104

of the gross return to storage. Also, it is clear, from this example's structure, that one can easily construct examples where the production/storage technology has any finite level of productivity.

This example shows that agents continue to hold bubble assets, even if, when goods are put into storage, they *increase* in quantity over time, so storage is productive. Intuitively, agents will still hold the risky asset as long as it appreciates in price sufficiently rapidly before crashing. Another way to put this is that agents will hold the asset if the initial price is sufficiently low compared to what agents think that others might think that others might think ... the asset is worth.

In addition, note that the agents who are most likely to hold a risky bubble asset are those who can safely bear the risk. For example, Frank, in our examples, buys the bubble asset in period 2 in large part because he is relatively wealthy in that period.

5.3 General Results on Storage Decisions

We now obtain some general results about storage for any strong bubble equilibrium in our model. Specifically, we show that, in any such equilibrium, Ellen never stores in her cell $\{H\}$ in periods 1 or 2, and Frank never stores in $\{L, H\}$ in period 1. That is, in a strong-bubble equilibrium for our model, an agent never stores the consumption good in those of her information sets for which the risky asset happens to have a *positive riskless* return since, in those information sets, the risky asset is actually riskless, and its return turns out to exceed the return on storage. Of course, agents also never store in period 3. Example 3 above shows that storage *can* occur in all other situations, so our results are best possible.

Proposition 5.3. *In any strong bubble equilibrium of a three-state, three-period economy where storage is available, the return to the risky asset in period 1, state H, is greater than the return to storage, so $\bar{p}_2/\bar{p}_1 > 1 - \delta$. Thus, Ellen never uses storage in period 1, state H, so $v_1^E(H) = 0$.*

Proof. Using $M_2^E(b) > 0$, Ellen's period-1 FOC for the risky asset in $\{b, L\}$, given in equation (21) in Appendix A below, implies that

$$\frac{\bar{p}_2}{\bar{p}_1} = \frac{M_1^E(\{b, L\})}{M_2^E(L)} > \frac{M_1^E(\{b, L\})}{M_2^E(\{b, L\})} \geq 1 - \delta,$$

where the last inequality uses Ellen's FOC for storage, from (10), in her period-1 cell $\{b, L\}$. Thus, the price rise between periods 1 and 2 in state L is $\bar{p}_2/\bar{p}_1 > 1 - \delta$. The price rise is therefore also $\bar{p}_2/\bar{p}_1 > 1 - \delta$ in Ellen's period-1 cell $\{H\}$, by Condition 1 of (7), and is riskless in that cell. Thus, in period 1, state H, Ellen knows that the risky asset is actually riskless and has a higher return than storage. Ellen therefore allocates all of her savings to the risky asset there, and none to storage, so $v_1^E(H) = 0$. □

The result $\bar{p}_2/\bar{p}_1 > 1 - \delta$ is intuitive. In her period-1 cell $\{b, L\}$, Ellen knows that she can sell the risky asset for a positive price in period 2 in state L, but not in state b, while the storage technology pays off in *both* states. Thus, Ellen is only willing to buy the risky asset in $\{b, L\}$ if it provides a sufficiently large return in state L, to compensate for her loss in state b. That is, the asset's gross return in state L, i.e. $\bar{p}_2/\bar{p}_1$, must be larger than $1 - \delta$. Ellen receives this same gross return on the risky asset in her cell $\{H\}$, where it is riskless. Thus, the risky asset strictly dominates storage in cell $\{H\}$, period 1, since both are riskless there.

Proposition 5.4. *In any strong bubble equilibrium of a three-state, three-period economy where storage is available, Frank never uses storage in his cell $\{L, H\}$ in period 1, so $v_1^F(\{L, H\}) = 0$.*

Proof. In Frank's period-1 information set $\{L, H\}$, the risky asset is actually riskless and has a larger return, $\bar{p}_2/\bar{p}_1$, than storage (see the proof of Proposition 5.3). Frank would therefore prefer buying the risky asset to using storage, so $v_1^F(\{L, H\}) = 0$. □

In any strong bubble equilibrium, Frank actually sells all his shares of the risky asset in $\{L, H\}$ in period 1. Given that he is unwilling to hold the risky asset to earn the larger riskless gross return of $\bar{p}_2/\bar{p}_1$, he must also be unwilling to store to earn the smaller gross return of $1 - \delta$.

Proposition 5.5. *In any strong bubble equilibrium of a three-state, three-period economy where storage is available, the return to the risky asset in period 2, state H, is greater than the return to storage, so $d/\bar{p}_2 > 1 - \delta$. Thus, Ellen never uses storage in period 2, state H, so $v_2^E(H) = 0$.*

Proof. Using $M_3^F(L) > 0$, Frank's period-2 FOC for the risky asset in $\{L, H\}$, given in equation (18) in Appendix A below, implies that

$$\frac{d}{\bar{p}_2} = \frac{M_2^F(\{L,H\})}{M_3^F(H)} > \frac{M_2^F(\{L,H\})}{M_3^F(\{L,H\})} \geq 1 - \delta,$$

where the last inequality uses Frank's FOC for storage, from (10), in his period-2 cell $\{L, H\}$. Thus, $d/\bar{p}_2 > 1 - \delta$. In period 2, in her cell $\{H\}$, Ellen thus knows that the risky asset actually has a larger, riskless return than storage. The risky asset is therefore more appealing to her than storage, so $v_2^E(H) = 0$. □

The reasoning for $d/\bar{p}_2 > 1 - \delta$ is similar to the reasoning for $\bar{p}_2/\bar{p}_1 > 1 - \delta$. In his period-2 cell $\{L, H\}$, Frank knows that the risky asset only pays off in state H, not in state L. However, storage pays off in both states. Thus, Frank is only willing to buy the risky asset in $\{L, H\}$ in period 2 if it provides a sufficiently large return in state H, so $d/\bar{p}_2 > 1 - \delta$. Ellen therefore prefers the risky asset in state H. Since Ellen does not even hold the risky asset, she also does not store.

5.4 Brief Comments on Risk-free Assets

First, note that there is a subtle difference between the rates of return on risk-free assets versus storage. While the depreciation rate on storage is given exogenously, the rate of return on a risk-free asset is determined endogenously by its market price, given its risk-free final payoff. This also implies that, while a long-term risk-free asset has no long-run buy-and-hold risk, it does have a short-run buy-and-sell risk, since the asset's price can fluctuate randomly before reaching maturity.

More importantly, it is reasonable to assume that agents can store *privately*, without revealing their information to other agents, while the market for risk-free assets is presumably public, so agents can see the prices that are determined in these markets. Thus, as Technical Appendix B shows, bubbles cannot exist in three-state models if a risk-free asset is traded in period 1. Notably, though, while the risk-free asset does compete with the bubble asset, this is not the reason why the bubble cannot exist in a three-state model. Instead, the bubble cannot exist because the prices of the two assets reveal too much information in a three-state model. Thus, bubbles and risk-free assets *can* coexist in four-state models, as shown in Conlon et al. (2021).

In addition to showing that bubbles cannot occur in three-state models if risk-free assets are traded in period 1, Technical Appendix B also considers a three-state bubble model in which risk-free assets are available for trade starting in period 2, and shows that the behavior of this model is very similar to the behavior of the models in the main paper.

6 Conclusion

This paper shows that, if intertemporal consumption smoothing is introduced as a motive for trade, then a strong bubble can exist in a very simple, easy-to-understand model, with only three states of the world, instead of five. This may therefore provide a useful workhorse model of greater-fool bubbles, and may be simple enough to make applied work on greater-fool bubble models extremely straightforward (see, e.g., the list of possible extensions in the last two paragraphs of Subsection 3.2 above).[34] Currently we know very little about greater-fool bubbles and their implications. The range of different greater-fool bubble models now available, however, both ours and others in the literature, will hopefully accelerate future progress on this important but controversial topic.

Appendix A Proof of Proposition 3.1

We first prove uniqueness, then sufficiency, then necessity.

Proof of Uniqueness: Suppose that a strong bubble equilibrium exists in our three-state, three-period economy. Then this equilibrium is essentially unique and given by (8).

First, when it becomes common knowledge that the dividend is zero, the price of the asset will also be zero. Thus, from the information structure given in Figure 1, we get $p_2(b) = p_3(b) = p_3(L) = 0$. Similarly, $p_3(H) = d$. Given these prices, neither Ellen nor Frank has a motive to trade the asset, so $X_2^E(b) = X_2^F(b) = 0$ and $X_3^E(\omega) = X_3^F(\omega) = 0$ for all $\omega \in \Omega$ are optimal.

Next, any strong bubble must occur in state b, because this is the only state where everyone knows the asset is worthless (L cannot be a strong bubble state since, if Frank learns, in that state, that the asset is worthless, then the price will collapse there). Also, the bubble must occur in period 1 because $p_2(b) = p_3(b) = 0$. Thus, the existence of a strong bubble implies $p_1(b) > 0$.

[34] Bubble models based on asymmetric information may also complement other approaches. Thus, Perez and Santos (2018) argue that asymmetric information can reduce a bubble model's dependence on behavioral traders. E.g., Matsushima (2013) considers a bubble model with a small probability of behavioral traders where, the longer rational agents are able to ride a bubble, the more sure they become that other investors are irrationally optimistic.

However, in state b, Frank knows for certain that the state is b, so he knows that the asset price will fall to zero in period 2. Thus, he strictly prefers to sell all his shares in period 1 in that state, so $X_1^F(b) = s_0^F$. Frank must therefore also sell all his shares in state L in period 1, so $X_1^F(L) = s_0^F$ (this uses $s_0^F(L) = s_0^F(b) = s_0^F$). Otherwise Ellen could distinguish state b from state L and would not buy in state b at a positive price, so there would be no strong bubble.

Of course, Ellen knows that, if the true state is b, then the price of the asset will fall to zero in period 2, while if the true state is L, then the price will fall to zero in period 3. Thus, she will only buy the asset in her cell $\{b, L\}$ in period 1 if she knows that she can sell it to Frank in state L in period 2, before the price crashes. This, in turn, means that Frank must not be able to distinguish the zero-dividend state L from the dividend-paying state H in period 2, since otherwise he would not be willing to buy from Ellen in state L in period 2, and the bubble equilibrium would collapse.

Ellen must therefore behave the same way in state L as in state H in period 1, and also in period 2. Thus, in period 1, Ellen must buy the same amount of shares in state H as she does in her cell $\{b, L\}$, and at the same price, so Frank doesn't refine his cell $\{L, H\}$. Thus, $p_1(b) = p_1(L)$ $= p_1(H)$ and $X_1^E(b) = X_1^E(L) = X_1^E(H) = -s_0^F$. Similarly, in period 2, since Ellen sells all her shares to Frank in state L, she must also be willing to sell all her shares at Frank's WTP in state H, so $p_2(L) = p_2(H)$ and $X_2^E(L) = X_2^E(H) = s_0^E + s_0^F$.[35]

Finally, since Frank buys in $\{L, H\}$ in period 2, his FOC, from (3), must hold as an equality. Given $p_3(L) = 0$ and $p_3(H) = d$, this implies that $p_2(L) = p_2(H) = \bar{p}_2$, with $\bar{p}_2$ from (5). Similarly, Ellen buys in her cell $\{H\}$ in period 1, so her FOC from (3) must hold as an equality. Thus, $p_1(H)$ $= \bar{p}_1$, with $\bar{p}_1$ from (6). It therefore also follows that $p_1(b) = p_1(L) = \bar{p}_1$, since otherwise Frank could distinguish state L from state H. Thus, if a strong bubble equilibrium exists, it is essentially unique, and its prices and net sales must take the form in (8).

Proof of Sufficiency: Given the conditions in (7), we show that the allocation in (8) is a competitive equilibrium, so (7) is sufficient for a strong bubble equilibrium. To show this, we first check optimality. Specifically, we show that the prices and net sales in (8) are solutions to the FOCs from (3) and their complementary slackness conditions. This implies that the behavior in (8) is optimal, since the global second-order conditions hold automatically, as the market is competitive

[35]Note that this all implies that, in a strong bubble equilibrium in our model, there must be no actual price-or-trade refinement of the information structures given in Figure 1.

and utility functions are concave. We then check the other conditions for a competitive equilibrium.

First, as argued at the beginning of the uniqueness proof just above, $p_2(b) = p_3(b) = p_3(L) = 0$, and $p_3(H) = d$, while $X_2^j(b) = X_3^j(b) = X_3^j(L) = X_3^j(H) = 0$ are optimal for $j = E,\ F$. These all agree with (8). Also, given $p_1(b) > 0$ and $p_2(b) = 0$, Frank's FOC in his period-1 cell $\{b\}$ holds as a strict inequality, so Frank must sell any shares he owns, as in (8). Similarly, given $p_2(L) > 0$ and $p_3(L) = 0$, Ellen's FOC in her period-2 cell $\{L\}$ holds as a strict inequality, so Ellen must sell any shares she owns in that situation. Note that none of this uses the conditions in (7) yet.

Next consider Frank's period-2 cell $\{L, H\}$, where (8) sets the asset price equal to $\bar{p}_2$ from (5). Note that (5) is actually

$$\bar{p}_2 = \frac{\beta\pi(H)U_F'\left(C_3^F(H)\right)}{\left[\pi(L) + \pi(H)\right]U_F'\left(C_2^F(\{L, H\})\right)}d = \frac{M_3^F(H)}{M_2^F(\{L, H\})}d. \tag{18}$$

Thus, given the prices $p_2(H) = p_2(L) = \bar{p}_2$, $p_3(H) = d$, and $p_3(L) = 0$ from (8), it follows that Frank's FOC, (3), in his period-2 cell $\{L, H\}$, holds as an equality, as it should since he buys the asset there, according to (8). Further, (18), along with Condition 3 in (7), implies that

$$\bar{p}_2 \geq \frac{M_3^E(H)}{M_2^E(H)}d, \tag{19}$$

so Ellen's FOC, (3), holds in her period-2 cell $\{H\}$, where (19) need not be an equality since, according to (8), Ellen sells all her shares of the asset in period 2 in $\{H\}$.

Next consider Ellen's period-1 cell $\{H\}$, where (8) sets the asset price equal to $\bar{p}_1$ from (6). Note that (6) is equivalent to

$$\bar{p}_1 = \frac{\beta U_E'\left(C_2^E(H)\right)}{U_E'\left(C_1^E(H)\right)}\bar{p}_2 = \frac{M_2^E(H)}{M_1^E(H)}\bar{p}_2. \tag{20}$$

Also, combining (20) and Condition 1 in (7) gives

$$\bar{p}_1 = \frac{M_2^E(L)}{M_1^E(\{b, L\})}\bar{p}_2. \tag{21}$$

Thus, given $p_1(H) = p_1(b) = p_1(L) = \bar{p}_1$, $p_2(H) = p_2(L) = \bar{p}_2$, and $p_2(b) = 0$ from (8), it follows that Ellen's FOCs, from (3), hold as equalities in both of her period-1 cells $\{H\}$ and $\{b, L\}$, as they should since Ellen buys the asset in both $\{H\}$ and $\{b, L\}$ in period 1, as in (8). Further, (20) together with Condition 2 in (7) yields

$$\bar{p}_1 \geq \frac{M_2^F(\{L,H\})}{M_1^F(\{L,H\})}\bar{p}_2, \tag{22}$$

so Frank's FOC, (3), also holds in his period-1 cell $\{L, H\}$, where again, (22) need not be an equality, since according to (8), Frank sells all his shares in that cell.

Therefore, given the prices and net sales in (8), agents' FOCs, together with their complementary slackness conditions, are satisfied, so the behavior in (8) is optimal. We next check the other conditions of a competitive equilibrium. First, the prices and net sales in (8) depend only on the join of the exogenous information partitions in Figure 1. It is also easy to check that each agent's net sales depend only on her own information partitions, since there is no refinement of those partitions. In addition, $X_t^E(\omega) = -X_t^F(\omega)$ in all states and periods, so markets always clear. Thus, all conditions of a competitive equilibrium are met, so (8) is a competitive equilibrium.

Finally, note that (8) has a strong bubble in state b, period 1, since $p_1(b) = \bar{p}_1 > 0$. The conditions in (7) are therefore sufficient for a strong bubble equilibrium to exist.

Proof of Necessity: We have shown that, if there is a strong bubble equilibrium, it is essentially unique and given by (8). Thus, if (8) implies the conditions in (7), then these conditions are necessary for a strong bubble equilibrium.

First, (8) gives $X_1^E(H) = -s_0^F$, so Ellen buys the risky asset in period 1 in $\{H\}$. Ellen's FOC in her period-1 cell $\{H\}$ must therefore hold as an equality, as in (20), using $p_1(H) = \bar{p}_1$ and $p_2(H) = \bar{p}_2$ from (8). Similarly, $X_1^E(b) = X_1^E(L) = -s_0^F$, so Ellen's FOC in her period-1 cell $\{b, L\}$ also holds as an equality, as in (21), using $p_1(\{b, L\}) = \bar{p}_1$, $p_2(b) = 0$, and $p_2(L) = \bar{p}_2$ from (8). Equalities (20) and (21) together give Condition 1.

Next, (8) gives $X_1^F(L) = X_1^F(H) = s_0^F$, so Frank sells all his shares of the risky asset in period 1 in $\{L, H\}$. Thus, Frank's FOC in his period-1 cell $\{L, H\}$ holds as an inequality, as in (22), using $p_1(\{L, H\}) = \bar{p}_1$ and $p_2(\{L, H\}) = \bar{p}_2$. Inequality (22), together with (20), gives Condition 2.

Finally, (8) gives $X_2^F(L) = X_2^F(H) = -\left(s_0^E + s_0^F\right)$, so Frank's FOC in his period-2 cell $\{L, H\}$ must hold as an equality, as in (18), using $p_2(\{L, H\}) = \bar{p}_2$, $p_3(L) = 0$, and $p_3(H) = d$. Correspondingly, $X_2^E(H) = s_0^E + s_0^F$, so Ellen's FOC in her period-2 cell $\{H\}$ holds as an inequality, as in (19), using $p_2(H) = \bar{p}_2$ and $p_3(H) = d$. Combining (18) and (19) gives Condition 3.

Appendix B Proofs for Algorithm 1, Subsection 5.2

B.1 Proof that Step 2 Has At Least One Solution

To see that Step 2 of Algorithm 1 has a solution, consider a graph with $\bar{p}_2$ on the vertical axis and v_2^F on the horizontal axis. Note that $\bar{p}_2$ is decreasing in v_2^F both in (11) and in (12) regarded as an equality. Thus, a little care is needed to show that the corresponding curves cross. Consider the $\bar{p}_2$-intercept of (11) first, and call this $\bar{p}_2^0$ temporarily. If (12) holds with $\bar{p}_2 = \bar{p}_2^0$ and $v_2^F = 0$, then this is a solution. If it does not hold then, at $\bar{p}_2 = \bar{p}_2^0$ and $v_2^F = 0$, the LHS of (12) must be small relative to the RHS. Thus, $\bar{p}_2$ must be raised above $\bar{p}_2^0$ to increase the LHS of (12), and so to make (12) hold. That is, the $\bar{p}_2$-intercept of (12), regarded as an equality, must be above $\bar{p}_2^0$.

If this $\bar{p}_2$-intercept of (12) is above $\bar{p}_2^0$, however, then the two curves must cross in the first quadrant, since the horizontal intercept of (12) regarded as an equality is to the left of the horizontal intercept of (11). To see this note that, as $\bar{p}_2 \to 0$, we get $v_2^F \to e_2^F(\{L, H\})$ in (11) (because these give $C_2^F(\{L, H\}) \to 0$ in (11), so the RHS of (11), i.e., Frank's WTP, approaches zero, matching $\bar{p}_2 \to 0$ on the LHS). However, in (12) regarded as an equality, $v_2^F < e_2^F(\{L, H\})$ as $\bar{p}_2 \to 0$ (since (12) regarded as an equality requires the LHS of that equation to remain finite as $\bar{p}_2 \to 0$). The intersections of the two curves then give nonnegative solutions for v_2^F and $\bar{p}_2$.

Note also that the two curves might cross more than once, and the two curves might also cross in the interior of the first quadrant, even if zero storage, $v_2^F = 0$, is also a solution. Thus, multiple nonnegative solutions might exist.

B.2 Proof that Step 6 Yields a Unique Solution

First, given v_1^E, solve $v_2^E(b)$ from (14). The LHS of (14) is increasing in $v_2^E(b)$ and the RHS is decreasing in $v_2^E(b)$. Thus, with the LHS and RHS on the vertical axis of a graph, and $v_2^E(b)$ on the horizontal axis, (14) must yield a unique, positive $v_2^E(b)$ if the LHS and RHS curves cross in the first quadrant. However, if they do not cross in the first quadrant, then the vertical intercept of the LHS curve must be above that of the RHS curve. This is because, as $v_2^E(b) \to e_2^E(b) + (1 - \delta)v_1^E$, the LHS $\to \infty$ and the RHS remains finite and positive. Thus, in this case, (14) holds at $v_2^E(b) = 0$

(though possibly as a strict inequality), which yields a corner solution. In both cases, therefore, (14) yields a unique nonnegative value for $v_2^E(b)$ as a function of v_1^E, say $v_2^E(b) \equiv g(v_1^E)$.

Next, we show $0 \leq g'(v_1^E) < 1-\delta$. First, if $v_2^E(b) = g(v_1^E) > 0$, then (14) holds as an equality by complementary slackness. Thus, since $U_E''(C) < 0$ is finite for $C > 0$, it follows that $g(\cdot)$ is differentiable. Differentiating (14), regarded as an equality, with respect to v_1^E, gives

$$U_E''\big(e_2^E(b) + (1-\delta)v_1^E - v_2^E(b)\big)\big[(1-\delta) - g'(v_1^E)\big] = \beta U_E''\big(e_3^E(b) + (1-\delta)v_2^E(b)\big)(1-\delta)^2 g'(v_1^E).$$

Since $U_E'' < 0$, $(1-\delta) - g'(v_1^E)$ must have the same sign as $g'(v_1^E)$, so $0 < g'(v_1^E) < 1-\delta$. On the other hand, if $v_2^E(b) = g(v_1^E) = 0$, then $g'(v_1^E) = 0$. Thus, in both cases, $0 \leq g'(v_1^E) < 1-\delta$.

Similarly, given v_1^E, solve $v_2^E(L)$ from (15). Equation (15) again yields a unique positive $v_2^E(L)$ if the LHS and RHS curves cross in the first quadrant, or $v_2^E(L) = 0$ if they do not cross. Letting $v_2^E(L) \equiv f(v_1^E)$, we again obtain $0 \leq f'(v_1^E) < 1-\delta$.

Finally, consider v_1^E. Substitute $v_2^E(b)$ and $v_2^E(L)$ into (13) using $g(v_1^E)$ and $f(v_1^E)$, respectively. The LHS of (13) is increasing in v_1^E, while the RHS is decreasing in v_1^E since $g'(v_1^E) < 1-\delta$ and $f'(v_1^E) < 1-\delta$. Thus, as with (14) and (15), equation (13) must yield a unique, positive v_1^E if the LHS and RHS curves cross in the first quadrant. Otherwise, since the LHS $\to \infty$ and the RHS is positive and finite as $v_1^E \to e_1^E(\{b, L\}) - s_0^F \bar{p}_1$, the vertical intercept of the LHS must be above that of the RHS, so $v_1^E = 0$ is the solution. Thus, (13), (14) and (15), together with their complementary slackness conditions, yield a unique nonnegative solution for v_1^E, $v_2^E(b)$ and $v_2^E(L)$, given $\bar{p}_1$ and $\bar{p}_2$.

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Technical Appendices

Feng Liu, Joseph S. White and John R. Conlon

These technical appendices briefly consider some simple extensions of the topics covered in the main paper. None of the results in the main paper depend on anything in these appendices.

Technical Appendix A: The Welfare Effects of Policy with Storage

This appendix considers the welfare effects of an asset-deflation announcement policy in Example 3 of the main paper. A storage technology is available in this example. Thus, even if the price of the bubble asset collapses in states b and L under this policy, so the asset becomes useless for consumption smoothing, agents can still use storage to save some of their endowments, and so, smooth their consumption. The policy may then interfere less with agents' consumption smoothing, so, unlike in Example 1 above, it may turn out that the policy always benefits those it is supposed to protect. We will see that this is not the case, and that the policy can still hurt both agents in state L.

Note, however, that we again only focus on welfare in this specific example. A general welfare analysis with storage is beyond the scope of this technical appendix, but is an important topic for future research. Also, a detailed welfare comparison of this example with Example 1 is inappropriate, since the endowments of the consumption good in Example 3 are so different from those in Example 1. When we examine the effect of adding a storage technology to a pre-existing bubble equilibrium, however, robustness becomes an issue.[1]

We again first find the policy equilibrium. Like in Example 1, the price of the asset is always zero in states b and L under the policy, but state-H prices rise to $p_1(H) = 1.4$ in period 1 and $p_2(H) = 4$ in period 2. Given these prices, Ellen continues to buy Frank's one share in period 1, state H, while Frank continues to buy Ellen's two shares in period 2, state H. Agents' storage

[1] One way to handle this might be to add a fourth state, i.e., a second dividend-paying state, so we could split H into the two states, H_1 and H_2. Then Ellen, in her information set $\{b, L\}$, might behave like Ellen in state H_1 in the absence of a storage technology, but like Ellen in state H_2 if a storage technology is introduced. Presumably the availability of a storage technology would then reduce the period-1 and period-2 prices of the risky asset.

Also note that, because of the way Example 4 was derived from Example 3, the welfare effects in Example 4 will be qualitatively identical to those in Example 3. While consumption levels in periods 2 and 3 are higher in Example 4, the *proportional changes* in consumption levels induced by policy would be identical to those in Example 3, and this is what matters, given logarithmic utility.

levels are given by (10) and its complementary slackness conditions, but with every $I^j_{t,i}$ a singleton. These storage levels are presented in the left-hand panels of Table A1.

Table A1 shows that Ellen stores more under the policy in period 1, states b and L, than in the bubble equilibrium in the left-hand panels of Table 6 in the main paper. This is because she no longer spends resources on the risky asset in those states. She also stores more in period 1, state L, as well as less in period 2, state L, because she cannot sell the risky asset in period 2, state L, which reduces her holdings of the consumption good in period 2. Ellen stores more in period 2, state b, since she rolls over some of her additional storage from period 1, state b. Frank stores less in state b in both periods 1 and 2, since the policy prevents him from selling the asset in period 1. Frank stores more in period 2, state L because he is no longer spending resources on the asset. Finally, Frank stores less in period 2, state H because the asset costs more, so he has less remaining wealth to store. The resulting consumption levels under the policy are presented in the right-hand panels of Table A1.

Table A1: Storage and Consumption under the Policy in Example 3

	Storage						Consumption					
	Ellen			Frank			Ellen			Frank		
	b	L	H	b	L	H	b	L	H	b	L	H
Period 1	2.67	4	0	0.33	0	0	14.33	13	5.60	6.67	6	7.40
Period 2	2.33	1	0	0.67	12	8	14.33	13	16	6.67	28	24
Period 3	0	0	0	0	0	0	14.33	13	25	6.67	28	24

Table A2: State-by-State Utility in Example 3

	Ellen			Frank		
	b	L	H	b	L	H
Bubble Equilibrium	ln 2,744	ln 2,744	ln 1,800	ln 343	ln 4,732	ln 4,732
Policy Equilibrium	ln 2,944.70	ln 2,197	ln 2,240	ln 296.30	ln 4,704	ln 4,262.40

Next, Table A2 presents the state-by-state welfare effects of the asset-deflation policy. These results are similar to those for Example 1 in Table 4, i.e., Ellen benefits and Frank is hurt in state b, and Ellen and Frank are both hurt in state L. However, only Ellen benefits in state H here.

Thus, even though an alternative mean of consumption smoothing, i.e., storage, is available, Ellen and Frank can still benefit from the bubble equilibrium in state L, and so, can still be hurt by the asset-deflation policy in that state. It is perhaps more surprising that Ellen continues to benefit from the bubble equilibrium, since she can now save through storage, so she doesn't need to use the risky asset to save in state L. However, by the same revealed preference argument used in Section 4 of the main paper, Ellen must benefit from buying the asset in state L of the bubble equilibrium, so she must be hurt by the asset-deflation policy in that state. Intuitively, while the asset is risky in Ellen's cell $\{b, L\}$, since it will crash in state b, the gross return on the asset is twice as high as the gross return on storage in state L, so Ellen is willing to risk buying it.

The policy also again hurts Frank in state L in this example, even though he is the greater-fool buyer the policy is presumably supposed to protect. In state L, Frank's endowment is low in period 1 but high in period 2. Thus, in the bubble equilibrium, selling the asset in period 1 and buying it back in period 2 allows him to shift some of his consumption backwards in time, from period 2 to period 1, just like in Section 4. However, storage can only move consumption *forward* in time, from period 1 to period 2, so it is useless to Frank in state L, period 1. Indeed, he never stores at all in $\{L, H\}$ in period 1 (see Proposition 5.4 of the main paper). It would therefore seem that the availability of storage shouldn't matter to Frank, at least in period 1.

However, the fact that the *other* agent, *Ellen*, *can* store, has an *indirect* effect on Frank in period 1, since Frank now has to compete against Ellen's ability to store when he sells her the risky asset. This can presumably reduce Ellen's WTP for the asset, and so, reduce its price, $p_1(L)$. This would then reduce Frank's gain from selling the asset which, in turn, would reduce the harm to him when the policy interferes with selling the asset.

More directly, of course, Frank can now store in period 2, so this will reduce his dependence on buying the risky asset in his cell $\{L, H\}$. This will presumably reduce the price, $p_2(L)$, he pays for the asset, and so, may reduce the loss he experiences from being the greater fool in state L, period 2. It may therefore reduce his period-2 gain from the policy, which would increase his overall net loss from the policy in this example. On the other hand, the lower $p_2(L)$ will contribute to a lower $p_1(L)$, and so, further reduce Frank's period-1 gain from the bubble equilibrium, and so, reduce his net loss from the policy.

In summary, the presence of a storage facility presumably reduces both $p_1(L)$ and $p_2(L)$. Thus, when the policy prevents Frank from selling the asset in period 1, the loss is smaller, but when it prevents him from buying a worthless asset in period 2, the benefit is smaller. The policy can therefore still potentially hurt Frank, as the current example suggests.

Unlike in Example 1, Frank is hurt by the policy in state H in the current example, since his benefit from the 0.40-unit gain in period 1 is not large enough to compensate for his loss from paying 4 units more for the two shares in period 2 under the policy. This is in part because his period-1 endowment in state H higher here than in Example 1. For example, the welfare effect is reversed if Frank's period-1 endowments in L and H are 1 unit each rather than 6 units each (compare to footnote 29 in the main paper).

Finally, we look at the *ex ante* as well as the interim welfare in this example. Frank's *ex ante* welfare falls, from $(\ln 343 + \ln 4{,}732 + \ln 4{,}732)/3 = \ln 1{,}973.00$ in the bubble equilibrium, to $(\ln 296.30 + \ln 4{,}704 + \ln 4{,}262.40)/3 = \ln 1{,}811.14$ under the policy, and Ellen's *ex ante* welfare rises from $\ln 2{,}384.22$ to $\ln 2{,}438.03$. Thus, Ellen now benefits from the asset-deflation policy *ex ante* in this example, but Frank is still hurt by it. In addition, Frank's interim welfare in $\{L, H\}$ falls from $\ln 4{,}732$ to $\ln 4{,}477.76$, and Ellen's interim welfare in $\{b, L\}$ falls from $\ln 2{,}744$ to $\ln 2{,}543.52$.

Of course, the above is again only meant to briefly illustrate a range of possibilities. A more general welfare analysis in the presence of storage is therefore an important topic for future research.

Technical Appendix B: Bubbles when a Risk-Free Asset is Present

This appendix examines the possibility of strong bubbles when a risk-free asset, in addition to the risky asset, is present in the market. For simplicity, we again assume the consumption good is perishable, although this makes no difference to our results. Similar to storage, which has a constant depreciation rate, a risk-free asset pays a constant dividend in each state at maturity.

However, unlike the level of storage, which we can reasonably assume is private information, the price of the risk-free asset is publicly determined in the market. As a result, when a risk-free asset is present, too much information might be revealed to the market, which can then preclude a strong bubble equilibrium. In fact, we show in Subsection B.1 that, if a risk-free asset is present in period 1, then the prices of the two assets will allow Frank to distinguish state H from state L. Thus, the presence of a risk-free asset in period 1 prevents bubbles in this model, even though a bubble can coexist with storage in period 1.

Since a bubble is possible when risk-free *storage* is available in period 1, but not when a risk-free *asset* is available in that period, it is not the ability of the risk-free asset to *smooth consumption* that prevents the bubble equilibrium, but its tendency to reveal too much information. Thus, a strong bubble can exist in similar models, but with *four* states, even if a risk-free asset is present in period 1, as shown in Conlon et al. (2021). This is because, in four-state models, asset prices do not necessarily reveal enough information to allow Frank to distinguish zero-dividend states from dividend-paying states. Therefore, the presence of a risk-free asset prevents bubbles here, not because it gives investors a safe alternative to the potential bubble asset, but only because it transmits too much information between investors.

Also, if a risk-free asset only becomes available in period 2, then Frank may not be able to tell states L and H apart, so strong bubbles are possible in that case, as Example 5 below shows.

There are a few possible cases of our model with a risk-free asset, depending on when the risk-free asset becomes available and when it pays off. For simplicity of exposition, we assume that each share of the risk-free asset pays one unit of the consumption good in each state, but only in period 3. Let $q_t(\omega)$ be the period-t price of the risk-free asset in state ω, so $q_3(\omega) = 1$. Agent j is endowed with $n_t^j(\omega)$ shares of the risk-free asset in period $t = 1$ or 2, so $n_3^j(\omega) = 0$. Note that this endowment, like the endowment of the consumption good and of the risky asset, must conform to Agent j's information partition. Lastly, Agent j's net sales of the risk-free asset are $Y_t^j(\omega)$, and short-sales are not allowed. Agent j's state-ω consumption is thus

$$\begin{aligned}
C_1^j(\omega) &= e_1^j(\omega) + p_1(\omega)X_1^j(\omega) + q_1(\omega)Y_1^j(\omega), \\
C_2^j(\omega) &= e_2^j(\omega) + p_2(\omega)X_2^j(\omega) + q_2(\omega)Y_2^j(\omega), \\
C_3^j(\omega) &= e_3^j(\omega) + d(\omega)s_3^j(\omega) + \sum_{t=1}^{2}\left[n_t^j(\omega) - Y_t^j(\omega)\right],
\end{aligned} \tag{B.1}$$

where $Y_3^j(\omega) = X_3^j(\omega) = 0$ are optimal since, in period 3, the risk-free asset's price, like the risky asset's price, equals its dividend.

Short-sales are not allowed on either the risky or the risk-free asset, so Agent j chooses her net sales of the risky asset, $X_t^j(\omega)$, according to the FOC, (2) or (3), from the main paper, and chooses net sales of the risk-free asset, $Y_t^j(\omega)$, for $\omega \in I_{t,i}^j$, according to the following FOC in her cell $I_{t,i}^j$:

$$M_t^j(I_{t,i}^j)q_t(I_{t,i}^j) \geq \sum_{\omega \in I_{t,i}^j} M_{t+1}^j(\omega)q_{t+1}(\omega), \text{ for } t = 1, 2 \tag{B.2}$$

(where, by abuse of notation, $q_t(I^j_{t,i}) = q_t(\omega)$ for all $\omega \in I^j_{t,i}$). Condition (B.2) necessarily holds as an equality if Agent j holds a positive amount of the risk-free asset at the end of period t, while if (B.2) holds as a strict inequality, then Agent j must sell all her shares of the risk-free asset, and so, must be short-sale constrained. Thus, the holdings of the risk-free asset must satisfy (B.2) and its complementary slackness conditions. Note that the conditions in (7) of the main paper remain necessary for a strong bubble equilibrium, and the FOCs (18) through (22) from the main paper still hold, though with shadow prices now evaluated at the consumption levels from (B.1).

B.1 A Risk-Free Asset Available in Period 1

This subsection considers the case where agents are endowed with the risk-free asset in period 1, so $n^E_1(\omega) > 0$ or $n^F_1(\omega) > 0$ or both, and agents can observe period-1 risk-free asset prices. Agents may also be endowed with additional shares in period 2, but not necessarily. We now prove that, if a market exists for the risk-free asset in period 1 of our three-state model, then that model cannot sustain a strong bubble.

Proposition B.1. *A strong bubble cannot exist in a three-state, three-period economy, if there is a market for a risk-free asset in period 1.*

Proof. For a strong bubble to exist in our model, the conditions in (7) from the main paper remain necessary, and agents' information partitions in Figure 1 of the main paper must not get further refined (see footnote 35 of the main paper). Thus, the equilibrium prices of the risk-free asset must satisfy $q_1(b) = q_1(L) = q_1(H)$ in period 1 and $q_2(L) = q_2(H)$ in period 2, so these prices do not reveal information. We now show that, if $q_2(L) = q_2(H)$, which we abbreviate as $\bar{q}_2$ for short, then $q_1(L) \neq q_1(H)$, so a strong bubble is not possible.

Consider $q_1(L)$ first. Given $q_2(L) = q_2(H) \equiv \bar{q}_2$ and the information structure in the main paper's Figure 1, Ellen's and Frank's period-1 WTPs for the risk-free asset in state L are respectively

$$WTP^E_1(\{b, L\}) = \frac{M^E_2(b)q_2(b) + M^E_2(L)\bar{q}_2}{M^E_1(\{b, L\})} \quad \text{and} \quad WTP^F_1(\{L, H\}) = \frac{M^F_2(\{L, H\})\bar{q}_2}{M^F_1(\{L, H\})}.$$

Using $M^E_2(b)q_2(b) > 0$, together with Conditions 1 and 2 in (7) from the main paper, we get

$$\frac{M^E_2(b)q_2(b) + M^E_2(L)\bar{q}_2}{M^E_1(\{b, L\})} > \frac{M^E_2(L)\bar{q}_2}{M^E_1(\{b, L\})} = \frac{M^E_2(H)\bar{q}_2}{M^E_1(H)} \geq \frac{M^F_2(\{L, H\})\bar{q}_2}{M^F_1(\{L, H\})}. \tag{B.3}$$

Equation (B.3) implies that Ellen bids the price of the risk-free asset higher than Frank in state L, so her WTP determines the period-1 price of the risk-free asset in that state. That is,

$$q_1(L) = WTP_1^E(\{b, L\}) = \frac{M_2^E(b)q_2(b) + M_2^E(L)\bar{q}_2}{M_1^E(\{b, L\})}. \tag{B.4}$$

Next consider $q_1(H)$. Again, given $q_2(L) = q_2(H) \equiv \bar{q}_2$ and the main paper's information structure, Ellen's and Frank's period-1 WTPs for the risk-free asset in state H are respectively

$$WTP_1^E(H) = \frac{M_2^E(H)\bar{q}_2}{M_1^E(H)} \quad \text{and} \quad WTP_1^F(\{L, H\}) = \frac{M_2^F(\{L, H\})\bar{q}_2}{M_1^F(\{L, H\})}.$$

The last step of (B.3) (i.e., Condition 2 in (7)) implies that Ellen has a (weakly) higher WTP, so her WTP also determines the risk-free asset's period-1 price in state H. That is,

$$q_1(H) = WTP_1^E(H) = \frac{M_2^E(H)\bar{q}_2}{M_1^E(H)}. \tag{B.5}$$

However, the first two steps of (B.3) then imply that $q_1(L) > q_1(H)$. Frank can thus use the period-1 price of the risk-free asset to distinguish state H from L. Ellen therefore knows, in her period-1 cell $\{b, L\}$, that she will not be able to sell the risky asset to Frank in state L in period 2. Thus, Ellen does not buy that asset in $\{b, L\}$ in period 1, and a strong bubble is not possible. $\square$

An argument similar to the above can also be used to show that a strong bubble is impossible in our three-state model when a risk-free asset is available in period 1 and pays off in period 2, since in that case, $q_2(L) = q_2(H)$ is simply given by the risk-free asset's dividend.

Proposition B.1's no-bubble result is intuitive. Note first that the risky asset, as well as the risk-free asset, are both riskless to Ellen in her cell $\{H\}$ in period 1. Thus, since Ellen is the state-H asset buyer whose WTP determines the prices, she requires the same rate of return on these two assets. However, in Ellen's period-1 cell $\{b, L\}$, the period-2 price of the risky asset will only be positive in state L, but zero in state b, while the period-2 price of the risk-free asset will be positive in both states. Ellen then requires a higher return on the risky asset in state L, to compensate her for the asset's state-b loss. As a result, the risk-free asset must have the same return as the risky asset in state H, but a lower return in state L. Thus, assuming $p_1(L) = p_1(H)$, it must be the case that $q_1(L) > q_1(H)$, since a strong bubble equilibrium requires $p_2(L) = p_2(H)$ and $q_2(L) = q_2(H)$. Frank would therefore be able to tell states H and L apart, so a strong bubble is not possible.

Note, incidentally, that the argument for $q_1(L) > q_1(H)$, here, is similar to the argument in the main paper's Proposition 5.3. Thus, Ellen's lower WTP for the risk-free asset in state H, period 1, here is analogous to her choice not to store in state H, period 1, in Proposition 5.3.

B.2 A Risk-Free Asset Only Becomes Available in Period 2

Suppose now that agents are endowed with the risk-free asset only in period 2, but not in period 1. That is, assume that $n_1^E(\omega) = n_1^F(\omega) = 0$, and that no market for the risk-free asset exists in period 1, while $n_2^E(\omega) > 0$, or $n_2^F(\omega) > 0$, or both. For simplicity, assume $n_2^j(\omega)$ is independent of ω and henceforth write it as n_2^j. In the present case, the FOCs remain (2) or (3) from the main paper, together with (B.2) here, with the shadow prices evaluated at the consumption levels from (B.1) here. We now determine a further condition, in addition to those in (7) from the main paper, which must be met for a strong bubble to exist in this case.

Again, for a strong bubble to exist in our three-state model, it must be the case that neither agent's information partitions from the main paper get refined. Also, it turns out that Frank necessarily has a higher WTP than Ellen for the risk-free asset in state H, period 2. He must therefore also have a higher WTP than Ellen in state L, period 2, since otherwise he would be able to distinguish state H from state L, and a strong bubble would not be possible. This, in turn, implies that $q_2(L) = q_2(H)$ will be determined by Frank's WTP in his cell $\{L, H\}$.

Consider state H first. Given $q_3(\omega) = 1$, Ellen's and Frank's period-2 WTPs for the risk-free asset in state H are respectively

$$WTP_2^E(H) = \frac{M_3^E(H)}{M_2^E(H)} \quad \text{and} \quad WTP_2^F(\{L, H\}) = \frac{M_3^F(\{L, H\})}{M_2^F(\{L, H\})}.$$

Using $M_3^F(L) > 0$, together with Condition 3 in (7) from the main paper, we get

$$\frac{M_3^F(\{L, H\})}{M_2^F(\{L, H\})} > \frac{M_3^F(H)}{M_2^F(\{L, H\})} \geq \frac{M_3^E(H)}{M_2^E(H)}.$$

Thus, Frank bids the price of the risk-free asset strictly higher than Ellen in state H. Ellen must therefore be short-sale constrained in the risk-free asset in state H, period 2, and Frank's WTP determines that asset's price in that situation. That is,

$$q_2(H) = WTP_2^F(\{L, H\}) = \frac{M_3^F(\{L, H\})}{M_2^F(\{L, H\})}. \tag{B.6}$$

Again, note that the argument that Ellen has a lower WTP, here, is similar to the argument, from Proposition 5.5 in the main paper, that Ellen does not store in period 2, state H.

Next, consider $q_2(L)$. Ellen's and Frank's period-2 WTPs for the risk-free asset in state L are respectively

$$WTP_2^E(L) = \frac{M_3^E(L)}{M_2^E(L)} \quad \text{and} \quad WTP_2^F(\{L, H\}) = \frac{M_3^F(\{L, H\})}{M_2^F(\{L, H\})}.$$

Recall that a strong bubble equilibrium requires $q_2(L) = q_2(H)$. Given $q_2(H)$ in (B.6), this requires that Frank's WTP is at least as high as Ellen's in state L, just like we showed it is in state H, i.e.,

$$\frac{M_3^F(\{L, H\})}{M_2^F(\{L, H\})} \geq \frac{M_3^E(L)}{M_2^E(L)}. \tag{B.7}$$

Thus, (B.7) gives another condition, in addition to those in (7) from the main paper, for a strong bubble to exist in our model, when a risk-free asset is available in period 2. The period-2 price of the risk-free asset in the cell $\{L, H\}$ is then given by Frank's WTP in $\{L, H\}$, so

$$q_2(L) = q_2(H) = \frac{M_3^F(\{L, H\})}{M_2^F(\{L, H\})} \equiv \bar{q}_2. \tag{B.8}$$

At this price, (B.7) implies that Ellen is willing to sell all her shares of the risk-free asset in period 2, states L and H, and Frank is willing to buy in those states, so the equilibrium net sales are

$$Y_2^E(\omega) = -Y_2^F(\omega) = n_2^E, \text{ for } \omega = L, H.$$

In general, in a three-state, three-period economy where a risk-free asset only becomes available to trade in period 2, one can use Algorithm 2 below to find a unique strong bubble equilibrium.

Algorithm 2:

1. Take $X_t^j(\omega)$ from (8) in the main paper. Also, $Y_2^E(\omega) = -Y_2^F(\omega) = n_2^E$ for $\omega = L, H$, and $Y_1^j(\omega) = n_1^j = 0$ for all ω.

2. Find $\bar{p}_2$ and $\bar{q}_2$ by solving the FOC (18) from the main paper jointly with (B.8), i.e.,

$$\bar{p}_2 = \frac{\beta\pi(H)U_F'\left(e_3^F(H)+(s_0^E+s_0^F)d+n_2^E+n_2^F\right)}{[\pi(L)+\pi(H)]\,U_F'\left(e_2^F(\{L,H\})-(s_0^E+s_0^F)\bar{p}_2-n_2^E\bar{q}_2\right)}d, \tag{B.9}$$

$$\bar{q}_2 = \frac{\beta\pi(L)U_F'\left(e_3^F(L)+n_2^E+n_2^F\right)+\beta\pi(H)U_F'\left(e_3^F(H)+(s_0^E+s_0^F)d+n_2^E+n_2^F\right)}{[\pi(L)+\pi(H)]\,U_F'\left(e_2^F(\{L,H\})-(s_0^E+s_0^F)\bar{p}_2-n_2^E\bar{q}_2\right)}. \tag{B.10}$$

These are two equations in two unknowns, and yield unique positive solutions for $\bar{p}_2$ and $\bar{q}_2$.[2]

3. Check whether (B.7) and Condition 3 in (7) from the main paper hold. If not, there is no strong bubble equilibrium.

4. Find $\bar{p}_1$ using the FOC (20) from the main paper, i.e.,

$$\bar{p}_1 = \frac{\beta U_E'\left(e_2^E(H)+(s_0^E+s_0^F)\bar{p}_2+n_2^E\bar{q}_2\right)}{U_E'\left(e_1^E(H)-s_0^F\bar{p}_1\right)}\bar{p}_2. \tag{B.11}$$

Given $\bar{p}_2$ and $\bar{q}_2$, this is one equation in one unknown, and it yields a unique positive solution for $\bar{p}_1$, by an argument similar to that in footnote 17 from the main paper.

5. Check whether Conditions 1 and 2 in (7) hold. If these conditions hold, then we have a unique strong bubble equilibrium.

6. Finally, find $q_2(b)$, $Y_2^E(b)$, and $Y_2^F(b)$ using Ellen's and Frank's versions of (B.2) in their period-2 cell $\{b\}$, together with the associated complementary slackness conditions. Substituting $Y_2^E(b) = -Y_2^F(b)$, and provisionally assuming an interior solution, this requires us to jointly solve

$$q_2(b) = \frac{\beta U_E'\left(e_3^E(b)+n_2^E+Y_2^F(b)\right)}{U_E'\left(e_2^E(b)-Y_2^F(b)q_2(b)\right)}, \tag{B.12}$$

$$q_2(b) = \frac{\beta U_F'\left(e_3^F(b)+n_2^F-Y_2^F(b)\right)}{U_F'\left(e_2^F(b)+Y_2^F(b)q_2(b)\right)}. \tag{B.13}$$

These are two equations in the two unknowns, $q_2(b)$ and $Y_2^F(b)$, and they yield a unique

[2]To see this, divide (B.9) by (B.10) to obtain

$$\bar{p}_2 = \frac{\pi(H)U_F'\left(e_3^F(H)+(s_0^E+s_0^F)d+n_2^E+n_2^F\right)d}{\pi(L)U_F'\left(e_3^F(L)+n_2^E+n_2^F\right)+\pi(H)U_F'\left(e_3^F(H)+(s_0^E+s_0^F)d+n_2^E+n_2^F\right)}\bar{q}_2,$$

so $\bar{p}_2 = A\bar{q}_2$, for A a positive constant, independent of $\bar{p}_2$ and $\bar{q}_2$. Using this for $\bar{p}_2$ in (B.10), the RHS of (B.10) is now decreasing in $\bar{q}_2$, with the RHS > 0 for $\bar{q}_2 = 0$ and the RHS $\to 0$ as $\bar{q}_2 \to e_2^F(\{L,H\})/\left[(s_0^E+s_0^F)A+n_2^E\right]$. Since the LHS of (B.10) is a 45-Degree line, the LHS and the RHS must cross in the first quadrant and must only cross once, which thus gives a unique positive solution for $\bar{q}_2$, and so for $\bar{p}_2$.

solution.[3] Keep the solution if $-n_2^E \leq Y_2^F(b) \leq n_2^F$. Otherwise, if $Y_2^F(b) > n_2^F$, set $Y_2^F(b) = n_2^F$ and solve (B.12) for $q_2(b)$. If $Y_2^F(b) < -n_2^E$, set $Y_2^F(b) = -n_2^E$ and solve (B.13) for $q_2(b)$.

Example 5: Suppose $U_j(C) = \ln C$ for $j = E, F$, $\beta = 1$, and $\pi(\omega) = 1/3$ for $\omega \in \Omega$. There are two assets in the market, with one risky and the other risk-free. A share of the risky asset pays four units of the consumption good only in period 3, state H, and the endowments are $s_0^E = s_0^F = 1$. By contrast, a share of the risk-free asset pays one unit of the consumption good in each state in period 3, and the endowments are $n_1^E = 0$, $n_2^E = 8$ for Ellen and $n_1^F = n_2^F = 0$ for Frank. The market for the risk-free asset only opens in period 2. Finally, the endowments of the consumption good are shown in Table B1.

Table B1: Agents' Endowments of the Consumption Good in Example 5

	Ellen			Frank		
	b	L	H	b	L	H
Period 1	17	17	9	10	8	8
Period 2	8	4	4	36	40	40
Period 3	64	64	64	20	20	12

Using Algorithm 2, or just checking, one can find a strong bubble equilibrium. Ellen buys Frank's one share of the risky asset in period 1 in each state, at the price $\bar{p}_1 = 1$, while Frank buys Ellen's two shares of the risky asset in period 2 in states L and H, at the price $\bar{p}_2 = 2$. Frank also buys Ellen's eight shares of the risk-free asset in period 2 in each state, at the price $\bar{q}_2 = q_2(b) = 1$. Agents' consumption levels in the bubble equilibrium are shown in the left-hand panels of Table B2. Note that $p_1(b) = 1$, so a strong bubble occurs in state b, period 1.

Table B2: Equilibrium Consumption Levels in Example 5

	Bubble						Policy					
	Ellen			Frank			Ellen			Frank		
	b	L	H	b	L	H	b	L	H	b	L	H
Period 1	16	16	8	11	9	9	17	17	7.52	10	8	9.48
Period 2	16	16	16	28	28	28	16	12.89	18.55	28	31.11	25.45
Period 3	64	64	64	28	28	28	64	64	64	28	28	28

Since a risk-free asset is available in period 2 in the current example, we reconsider the welfare

[3] Put $q_2(b)$ on the vertical axis and $Y_2^F(b)$ on the horizontal axis of a graph. Consider $q_2(b)$ as a function of $Y_2^F(b)$ in (B.12) and in (B.13). As $Y_2^F(b)$ rises from $-e_3^E(b) - n_2^E$ to $e_3^F(b) + n_2^F$ in (B.12), $q_2(b)$ declines from $+\infty$ to a positive finite value, but in (B.13), $q_2(b)$ rises from a positive finite value to $+\infty$. Thus, the two curves must across exactly once, so (B.12) and (B.13) must jointly yield a unique solution with $q_2(b) > 0$.

effect of an asset-deflation announcement policy. As before, we first find the policy equilibrium. The prices of the risky asset fall to zero in states b and L, but rise in state H to $p_1(H) = 90/61 \approx 1.48$ in period 1, and to $p_2(H) = 40/11 \approx 3.64$ in period 2. The prices of the risk-free asset become $q_2(b) = 1$ in state b, $q_2(L) = 10/9 \approx 1.11$ in state L, and $q_2(H) = 10/11 \approx 0.91$ in state H. Thus, Frank bids $q_2(L)$ higher in state L, since he doesn't waste resources on the risky asset, but bids $q_2(H)$ lower in state H, since he devotes more resources to the risky asset in that state. Agents' trading behavior is the same as in the presence of a bubble, except that agents no longer trade the risky asset in states b and L. The resulting consumption levels under the policy are shown in the right-hand panels of Table B2.

Table B3: State-by-State Utility in Example 5

	Ellen			Frank		
	b	L	H	b	L	H
Bubble Equilibrium	ln 16,384	ln 16,384	ln 8,192	ln 8,624	ln 7,056	ln 7,056
Policy Equilibrium	ln 17,408	ln 14,023.11	ln 8,931.00	ln 7,840	ln 6,968.89	ln 6,753.38

Table B3 shows agents' state-by-state lifetime utilities in the two equilibria. As in Table A2 from Technical Appendix A, Ellen benefits but Frank is hurt in state b, Ellen and Frank are both hurt in state L, and Ellen benefits but Frank is hurt in state H.

Again, in state L, Frank is so poor in period 1 and so wealthy in period 2 that he cares more about the one-unit gain from selling the risky asset in period 1 than about the four-unit loss from buying a worthless asset in period 2. Since he is prevented from making these transactions under the policy, Frank is hurt in state L. In addition, the state-L price of the risk-free asset rises under the policy, so Frank has to pay more in period 2, which makes his utility fall even more.

Ellen is also hurt by the policy in state L, even though the fall in her period-2 consumption is cushioned somewhat by receiving a higher price when she sells the risk-free asset. Note however that, since the policy affects the price of the risk-free asset, the revealed preference argument, used in Section 4 of the main paper and in Technical Appendix A above, to show that the policy *necessarily* hurts Ellen in state L, no longer applies. Also, Frank is hurt in state H, even though the fall in his period-2 consumption is cushioned by paying a lower price for the risk-free asset.

Finally, we can again look at the *ex ante* and interim welfare in this example. Frank's *ex ante* welfare falls from $(\ln 8{,}624 + \ln 7{,}056 + \ln 7{,}056)/3 = \ln 7{,}544.12$ in the bubble equilibrium to $\ln 7{,}172.44$ under the policy, and Ellen's *ex ante* welfare falls from $\ln 13{,}003.99$ to $\ln 12{,}966.75$. Thus, in this example, both agents are hurt, *ex ante*, by the asset-deflation policy. In addition, Frank's interim welfare in $\{L, H\}$ falls from $\ln 7{,}056$ to $\ln 6{,}860.29$, and Ellen's interim welfare in $\{b, L\}$ falls from $\ln 16{,}384$ to $\ln 15{,}624.16$. Of course, an analysis of the general case would also be interesting.

Since there is no risk-free asset in period 1 in this example, we cannot perform a fully satisfying analysis of an asset-deflation policy here. As mentioned above, if a risk-free asset is available in period 1, we need four-states to model a greater-fool bubble (see Conlon et al., 2021).

B.3 Summary

We have shown that, if there is a market for a risk-free asset in period 1, then a strong bubble cannot exist in a three-state, three-period economy. This is because the prices of the assets reveal to Frank whether the state is H or L. Since Ellen knows this, she does not buy the risky asset in $\{b, L\}$ in period 1, which then precludes a strong bubble.

This result suggests that, when a market is prone to asset bubbles, policymakers may reduce the likelihood of such bubbles by introducing new assets, to encourage information transmission in the market. However, there are some caveats. First, as argued both in the main paper and here, anti-bubble policies might hurt welfare, even for the greater fools who ultimately hold the overpriced assets, since the policy can interfere with agents' prior intertemporal consumption smoothing. Second, the ability of new assets to prevent bubbles depends on the structure of the model. For example, strong bubbles may still exist in our three-state, three-period model if a risk-free asset only becomes available in period 2. Similarly, Conlon et al. (2021) shows, in related models with four states, that strong bubbles may still exist, even if a risk-free asset is present in period 1.