Choosing Sides in a Two-sided Matching Market

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Abstract

I model a competitive labor market in which agents of different skill levels decide whether to enter the market as a manager or as a worker. After roles are chosen, a two-sided matching market is realized and a cooperative assignment game occurs. There exists a unique rational expectations equilibrium that induces a stable many-to-one matching and wage structure. Positive assortative matching occurs if and only if the production function exhibits a condition that I call *role supermodularity*, which is stronger than the strict supermodularity condition commonly used in the matching literature because the role(s) that high-skilled agents are willing to enter the market as and the degree of complementarity between roles together determine the equilibrium matching pattern. The wage structure in equilibrium is consistent with empirical evidence that the wage gap is driven both by increased within-firm positive sorting as well as between-firm segregation.

Keywords: Assignment Problems, Two-Sided Matching, Wage Inequality
JEL: C780, J31

1 Introduction

A common assumption in the two-sided matching literature is that if the surplus generated by matching satisfies strict supermodularity, then the resulting stable matching will be positive assortative. This condition is often used in marriage market models as well as several labor matching models, including Kremer (1993)'s O-Ring theory. However, these models do not accurately capture labor markets with two distinct roles in which agents may be able to choose which role they prefer. For example, some doctors open their own practices, some financial analysts will enter their firm's management track, some professors will chair their department, and some entrepreneurs will start their own small business—but all such people likely have qualified peers in their field who prefer to follow the lead of others. In this paper, I model many-to-one labor markets that have a "lead" role and "support" role (which are filled by a manager and worker(s) respectively) that together generate output, in which agents have a pre-matching strategic choice over role.

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The main contribution this paper makes to the literature is adding a role choice to a many-to-one labor market with two sides—agents decide if they prefer to lead or support before the matching market is realized. I find that there exists a unique rational expectations equilibrium that induces a stable matching, and that the matching pattern is socially efficient. In equilibrium, both the matching pattern *and* the wage structure are unique. The latter is generally not the case when the solution concept requires only stability; the unique wage structure is driven by the pre-matching role choice. A condition stronger than strict supermodularity that I call *role supermodularity* determines the equilibrium matching pattern and wage structure, as positive assortative matching occurs if and only if the production function satisfies role supermodularity. A stronger condition is necessary because a high skilled agent is only willing to enter the market as a worker if she is certain that she can profitably cluster with other high skilled agents. The model predicts that there two kinds of wage differentials—differences in wages between agents of the same type in different roles, and differences in wages between agents of different types.

After the literature review in Section 2, I set up the model in Section 3 and solve for equilibrium outcomes. In Section 4, I provide comparative statics and discuss how wage differentials change in response to changes in underlying productivity. I show how the wage structure relates to observed trends in U.S. wage inequality, and discuss possible policy implications. I conclude in Section 5 by discussing directions for future work.

2 Literature Review

This paper combines elements from the two-sided assignment model and the role assignment model.

Shapley and Shubik (1971) is the seminal paper on the standard two-sided assignment model. In a two-sided assignment problem, agents are divided into two disjoint sets and match surplus is generated if agents belonging to different sets match with each other (e.g., men and women in the marriage market, firms and employees in the labor market, managers and workers in this paper). I assume that match surplus is transferable via wages. See Chiappori (2020) for a full review of matching models with transfers, and Chapter 6 of Roth and Sotomayor (1992) for an overview of the literature on many-to-one matching. The paper reassesses the assumption that strict supermodularity in inputs induces positive assortative matching, as introduced by Becker (1973) and often used in the two-sided matching literature.

The pre-matching role choice parallels the decisions agents face in investment and matching models (Chiappori, Iyigun, and Weiss (2009), Nöldeke and Samuelson (2015), and Zhang (2021)). Both are two-sided matching models with transferable utility and share structural similarities: a decision is made before matching (choice of role vs. an investment choice), interim outcomes are realized (each agent's role vs. changes in skill type), and then matching occurs. The first stage is non-cooperative, as agents strategize in anticipation of the matching market that they will face, while the second stage is a cooperative assignment problem. In the investment and matching framework, though, sides of the matching market are fixed and investment decisions change agents' skill levels; in contrast, I allow strategic choices to change the supply and demand of agents on both sides of the market while skill levels are fixed.

Role assignment models (Kremer and Maskin (1997), Li and Suen (2001), McCann and Trokhimtchouk (2008), Anderson (2020)) are one-sided matching frameworks that analyze changes in worker sorting and their downstream effects on wage dispersion.[1] In role assignment models, the firm (acting as a social planner) first determines who is matched with whom, and then assigns roles within each match; in this paper, the timing is reversed. Kremer and Maskin (1997) show that in the role assignment model, matching patterns become positive assortative as skill levels become dispersed, and mean skill level correlates positively with wage inequality. Li and Suen (2001) show that for sufficiently dispersed skill distributions, segregation by type and wage inequality depends on how the social planner chooses to sort the agents who are indifferent between managing a lower-skilled worker or working for a higher-skilled manager. My model captures a similar tension without requiring a social planner; the equilibrium matching pattern depends on whether high-skilled agents can profitably become a worker and match to a high-skilled manager, or whether high-skilled agents always prefer to become a manager.

In role assignment, there are two standard assumptions imposed that pull the matching pattern in opposite directions: (1) managers and workers are complements (i.e., a highly-qualified accountant should work in a junior position at a top firm), and (2) output is more sensitive to managerial skill (i.e., a highly-qualified accountant should work in a senior position at a lower-tier firm). I similarly impose assumptions that imply that the manager role is more sensitive to skill type, but unlike the standard role assignment environment, I do *not* restrict attention to supermodular production functions. Empirical justification for these assumptions is unclear. Due to data limitations and difficulty in quantifying productivity, empirical research on managerial impact is modest. There are multiple data issues: it is not clear how to choose the best measure of productivity, some types of productivity may be unobservable, and one would need rich data across many firms. However, existing papers validate the main assumptions. Lazear, Shaw, and Stanton (2015) show that in a technology-based service workplace, the average manager contributes more to output than the average worker. Bertrand and Schoar (2003) find that differences in corporate managerial practices are systematically and significantly related to differences in performance. Finally, Bloom and Reenen (2007) find that better managerial practices are significantly and positively related to higher productivity in manufacturing firms.

That said, assumptions on the production function are not innocuous and I do not claim results generalize to all labor markets. The model I present most closely models small business ownership (e.g., a single owner employs a small number of workers and all agents perform a variety of tasks) and larger firms in which internal distribution of human capital is important, such as technology-based service and/or innovation sectors, fields in which a high level of qualification is necessary to enter the market (e.g., law or academia), start-up companies, or sectors with a significant freelance presence.

3 Model

The labor market is competitive with two employment roles, $r \in \{m, w\}$ such that a manager m must match with exactly $n \in \mathbb{N}$ workers for production to occur.[2] Worker skill is additive. There is a unit mass of agents, all risk neutral, who are of a skill type $\theta \in \{H, L\}$ such that $H, L \in \mathbb{R}_{++}$ and $H > L$. The measure of H-type agents is $M_H \in (0, \frac{1}{n+1})$. Let θ_r denote a θ-type agent in role r. The production technology is $f(\theta_m, \sum_{i=1}^{n} \theta_w^i) : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$. f is monotone in both arguments: (1) $f(H, n\theta_w) > f(L, n\theta_w)$ for all θ_w, and

[1] I call it the role assignment model, following Anderson (2020). It has also been described as a two-step assignment problem (Gavilan 2012).

[2] I consider a more specific model with endogenized $n \in \{1, 2\}$ in Appendix D.

(2) $f(\theta_m, H + \sum_{i=1}^{n-1} \theta_w^i) > f(\theta_m, L + \sum_{i=1}^{n-1} \theta_w^i)$ for all $\theta_m, \sum_{i=1}^{n-1} \theta_w^i$.

In this paper, I solely concentrate on the choice to become a manager, so I make two marginal productivity assumptions: (MP1) I rule out the possibility that any mixed coworker groups are socially efficient by assuming that the marginal productivity of H-type workers is weakly increasing in the number of H-type workers, and (MP2) I assume that the marginal productivity of changing a manager from an L-type agent to an H-type agent is always greater than adding one H-type worker for all worker compositions.[3]

The game takes place over two stages.

1. **Strategic Stage:** Agents simultaneously make pre-matching strategic decisions over what role to enter into the market as. Becoming a manager has a known cost $c(\theta) \in \mathbb{R}_+$, and costs are relatively small compared to productivity. I assume type symmetric strategies and represent the cumulative actions of θ-type agents as a function $\sigma_\theta : \Theta \to [0, 1]$ such that σ_θ is the probability that a θ-type agent chooses to become a manager, and denote $\sigma = (\sigma_H, \sigma_L)$ an arbitrary strategy profile.

2. **Outcome Stage:** σ induces a matching market $P \in \mathbb{R}_+^4$. Denote an arbitrary component $P_{r\theta}$ as the measure of θ-type agents in role r (e.g., the measure of L-type workers is $P_{wL} = (1-M_H)(1-\sigma_L)$). Once P has formed, a cooperative, non-strategic assignment game occurs. A *market outcome* is a matching μ along with a wage vector V. Since worker composition can be expressed as a linear combination of skill types, I denote a matching μ as a function $\mu : \Theta \times \{0, 1, ..., n\} \to [0, 1]$ such that for each (θ, ℓ), $\mu(\theta, \ell)$ is the measure of θ-type managers matched with ℓ H-type workers and $(n - \ell)$ L-type workers. Denote V as a payoff vector of up to four components, such that $v_{r\theta} \in \mathbb{R}_+$ is the wage of a θ-type agent in the role r whenever $\mu_{r\theta} > 0$.

Solutions to the assignment game that occurs in the outcome stage are stable outcomes.

Definition 1. *A **stable market outcome** for a matching market P is a market outcome (μ, V) such that:*

1. *μ is such that all unmatched agents share the same role and satisfies feasibility constraints:*
   - *(a) $\mu(\theta, \ell) \geq 0$ for all θ,*
   - *(b) $\sum_{\ell=0}^{n} \mu(\theta, \ell) \leq P_{m\theta}$ for all θ,*
   - *(c) $\sum_{\ell=0}^{n} \ell(\mu(H, \ell) + \mu(L, \ell)) \leq P_{wH}$, and*
   - *(d) $\sum_{\ell=0}^{n}(n - \ell)(\mu(H, \ell) + \mu(L, \ell)) \leq P_{wL}$.*
2. *The payoff vector V satisfies:*
   - *(a) Individual rationality: $v_{r\theta} \geq 0$ for all r and θ.*
   - *(b) Pairwise efficiency: If $\mu(\theta, \ell) > 0$, then $f(\theta, \ell H + (n - \ell)L) = v_{m\theta} + \ell v_{wH} + (n - \ell)v_{wL}$.*
   - *(c) Market efficiency: $v_{m\theta} + \ell v_{wH} + (n - \ell)v_{wL} \geq f(\theta, \ell H + (n - \ell)L)$ for all θ, ℓ.*

Stability imposes two features: V is feasible and compatible with μ, and there do not exist any managers and groups of workers who all prefer to be matched with each other over their current assignment. Stable

[3] I discuss how to loosen MP1 in Appendix C.

outcomes to the assignment game exist; while the stable matching is generally unique, it can be supported by a continuum of wage vectors (see Chiappori, Pass, and McCann (2016) and Chiappori (2020)). If f satisfies strict supermodularity, then the unique stable matching is positive assortative; otherwise, it is negative assortative. Because wage determination isn't unique, stability alone cannot generate unique predictions on what stable outcome(s) will occur in a competitive market setting. However, in this setting, P isn't fixed until agents have made their role choice; I show later in this section that this pre-matching role choice gives more structure to the potential wage vectors that can emerge.

The solution concept for the game follows.

Definition 2. *A* ***rational expectations equilibrium*** *is a list* $(\sigma^*, (\mu^*, V^*))$ *that satisfies the following:*

1. (μ^*, V^*) *is a stable outcome in the matching market induced by* σ^**.*

2. σ^*_θ *maximizes* θ*-type agents' expected wages minus costs incurred for all* θ*. If* $P_{r\theta} > 0$ *in the matching market induced by* σ^**, then* V^* *explicitly defines the expected wage in the labor market,* $v^*_{r\theta}$*.*

   If $P_{m\theta} = 0$*, then the wage that the* θ*-type agent expects to receive when individually deviating to becoming a manager is*

$$v_{m\theta} = \max_{\sum_{i=1}^{n} \theta_w^i : P_{r\theta_w} > 0 \forall \theta_w^i} [f(\theta, \sum_{i=1}^{n} \theta_w^i) - \sum_{i=1}^{n} v^i_{w\theta_i}]. \tag{1}$$

   If $P_{w\theta} = 0$*, then all workers are of the other type* $\theta' \neq \theta$*, so wage that the* θ*-type agent expects to receive when individually deviating to becoming a worker is*

$$v_{w\theta} = \max_{\theta_m : P_{r\theta_m} > 0} [f(\theta_m, \theta + (n-1)\theta') - v_{m\theta_m} - (n-1)v_{w\theta'}]. \tag{2}$$

The first part of the definition is a consistency condition. If agents optimally play σ^* because they expect to face the stable market outcome (μ^*, V^*), then σ^* induces a matching market that not only can, but actually *does* sustain (μ^*, V^*) as a stable market outcome. This prevents cases where, for example, $\sigma_H = \sigma_L = 1$, but agents incorrectly expect to be matched in the second period. The second part is a utility maximizing condition. V^* along with Equations 1 and 2 together allow agents to have rational expectations on all wages they could possibly face, even if some type-role combinations are not in the market induced by σ^*.

Before solving for the competitive equilibrium, consider the social planner's problem of assigning roles and matches. $f(H, nH)$ is the most productive arrangement, but the opportunity cost of grouping H-type agents together is that the n H-type agents who are workers could have instead been managers to groups of L-type workers. Hence, after taking into account the cost of becoming a manager, the social planner groups agents of the same type together if and only if

$$f(H, nH) + n \cdot f(L, nL) + n[c(H) - c(L)] > (n+1)f(H, nL). \tag{3}$$

I call this matching pattern *clustering*. Otherwise, the social planner has one H-type manager oversee n L-type workers while excess L-type agents form clusters, a matching pattern that I call *specialization*. In equilibrium, the same condition determines the competitive market outcome.

Theorem 1. *Outside of knife-edge cases, a unique equilibrium always exists.*[4]

1. *If Equation 3 holds, then there exists a unique* ***clustering equilibrium****. The equilibrium strategies are* $\sigma^*_{CE} = (\frac{1}{n+1}, \frac{1}{n+1})$*. In the unique stable market outcome, all agents match with their own type, and wages are*

$$v^*_{m\theta} = \frac{1}{n+1}(f(\theta, n\theta) + n \cdot c(\theta)),$$
$$v^*_{w\theta} = \frac{1}{n+1}(f(\theta, n\theta) - c(\theta)).$$

2. *If not, then there exists a unique* ***specialization equilibrium****. The equilibrium strategies are* $\sigma^*_{SE} = (1, \frac{1-(n+1)M_H}{(n+1)(1+M_H)})$*. In the unique stable market outcome, all H-type agents become managers, L-type agents mix such that the market clears, and the wages are*

$$v^*_{mH} = f(H, nL) - \frac{n}{n+1}(f(L, nL) - c(L)),$$
$$v^*_{mL} = \frac{1}{n+1}(f(L, nL) + n \cdot c(L)),$$
$$v^*_{wL} = \frac{1}{n+1}(f(L, nL) - c(L)).$$

Equilibrium matching patterns are socially efficient.

Appendix A has the full proof; I outline the argument here. Existence can be proven by construction. Uniqueness follows by ruling out all other possibilities: intuitively, an equilibrium shouldn't induce an unbalanced market. Given that the market clears, θ-type agents mix strategies if and only if expected wages net of costs on both sides of the market are the same. At least some L-type agents must always cluster for the market to clear, so $v_{mL} - c(L) = v_{wL}$ regardless of the matching pattern. Given that, utility-maximizing H-type agents consider whether to manage L-type agents (who demand lower wages) or to cluster with other H-type agents (a more productive arrangement). This trade-off makes the H-type agents' optimization problem equivalent to the social planner's problem, hence Equation 3 determines both the unique, stable matching pattern as well as the accompanying wage vector in equilibrium.

To compare the equilibrium to the setting without a role choice, set costs to 0 and let $n = 1$. Then Equation 3 simplifies to

$$f(H, H) + f(L, L) > 2f(H, L) \qquad \text{(Role supermodularity)}$$

and positive assortative matching occurs if and only if role supermodularity is satisfied. I call the condition role supermodularity because the roles that an H-type agent is willing to take determines the equilibrium matching pattern. Role supermodularity is a stronger necessary condition for positive assortative matching than the standard strict supermodularity condition,

$$f(H, H) + f(L, L) > f(H, L) + f(L, H),$$

because of the role choice—H-type agents are never willing to become workers to match with an L-type manager, as this is always less efficient than swapping roles. This is in contrast to the standard assignment

[4] See Appendix B for details on the knife-edge cases.

model, where $f(L,H)$ matters because the matching market is pre-determined and H-type agents may end up a worker because she has no role choice.[5] Hence, H-type agents are only willing to become workers if (i) $f(H,H) - f(H,L)$ is large and/or (ii) $f(H,L) - f(L,L)$ is small (L-type workers demand a high wage).

This theoretical prediction aligns with Adhvaryu et al. (2020)'s empirical study of a garment facturing firm in India. They find that negative assortative matching occurs even though the underlying production function displays complementarities between managers and workers, which is consistent with the hypothesis that a stronger condition than strict supermodularity (which can be interpreted informally as "inputs behaving more like complements than substitutes") such as role supermodularity (which informally requires that inputs behave *strongly* as complements) is needed to induce positive assortative matching in labor markets. In general, the model predicts that in labor markets in which agents must preemptively decide to enter as the lead role (e.g., entering the management track at a company may require extra training or external credentials), more complementarity between agents is needed to sustain positive clustering.

4 Wage Differentials and Productivity

Theorem 1 implies that two factors determine the extent of wage inequality: (1) differences in productivity and costs may cause matching patterns to differ across sectors, which drives wage differentials between agents of different types in the same role, and (2) the ability to pass on the cost of entering as a manager will cause wage differentials between agents of the same type in different roles. In both equilibria, the wage differential between agents of the same type in different roles is

$$v^*_{m\theta} - v^*_{w\theta} = \frac{n-1}{n+1}c(\theta).$$

Note that managers are able to pass a portion of costs onto workers whenever clusters occur.

In a clustering equilibrium, the wage differentials between agents of different types and the same role are

$$\begin{aligned} v^*_{mH} - v^*_{mL} &= \frac{1}{n+1}[f(H,nH) - f(L,nL)] - \frac{n}{n+1}[c(H) - c(L)] \text{ and} \\ v^*_{wH} - v^*_{wL} &= \frac{1}{n+1}[(f(H,nH) - f(L,nL)) - (c(H) - c(L))]. \end{aligned}$$

Changes to $f(\theta, n\theta)$ and/or $c(\theta)$ affect only wages of θ-type agents, but affect both sides of the market. Given that, the main productivity-related reason that clustering may switch to specialization is growth in $f(H,nL)$, which shrinks marginal productivity of H-type clusters.

In a specialization equilibrium, the wage differential between H-type and L-type managers is

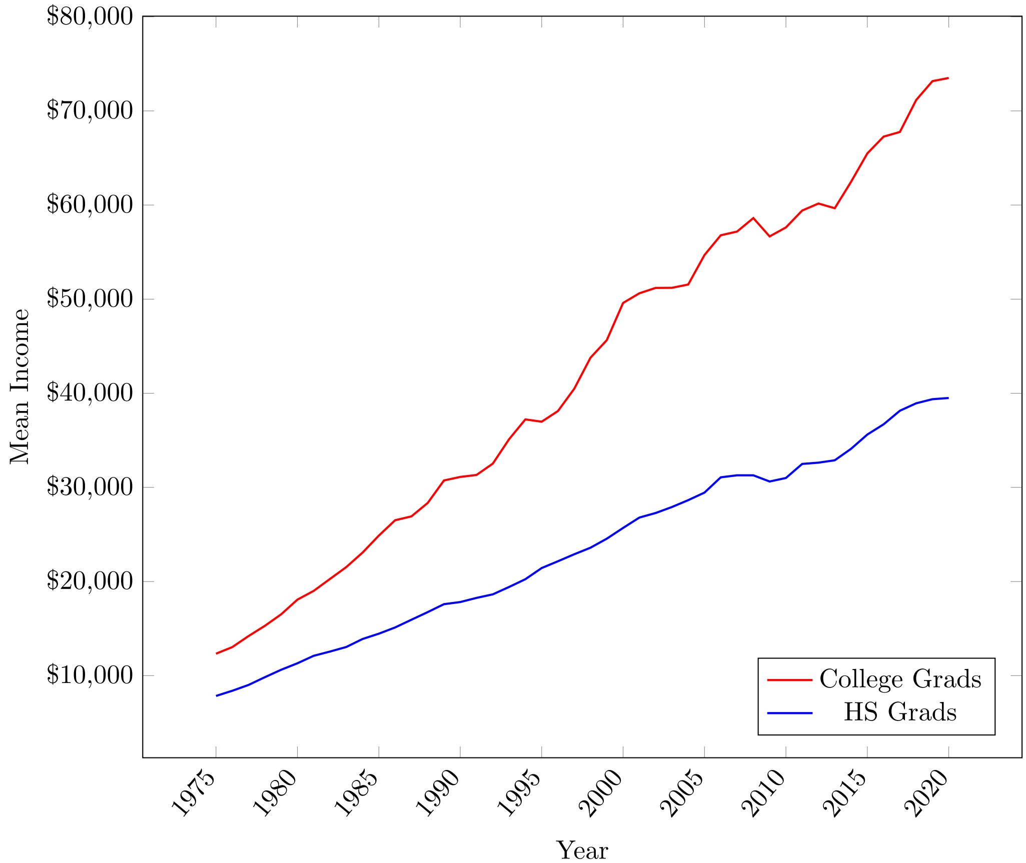
$$v^*_{mH} - v^*_{wH} = f(H,nL) - f(L,nL),$$

and $c(H)$ does not enter into the expression because H-type managers are unable to directly pass on $c(H)$ to workers, while managers of both types continue to pass a fraction of $c(L)$ onto L-type workers. Changes to $f(H,nL)$ and $c(H)$ affect only H-type managers' wages, while changes to $f(L,nL)$ or $c(L)$ affect all agents

[5]Consider, for example, a marriage market. A high-skilled woman may prefer to marry a high-skilled man, but she may be unable to if supply of such men is scarce and she must consider her utility from marrying a low-skilled man.

in the market: if L-type clusters become more productive, all L-type agents' outcomes improve while H-type managers are worse off, while increasing L-type cost works in the opposite direction. This suggests there are two productivity-related reasons that specialization may switch to clustering: (1) growth in $f(H, nH)$ or (2) growth in $f(L, nL)$.

Figure 1: Mean Income by Level of Education

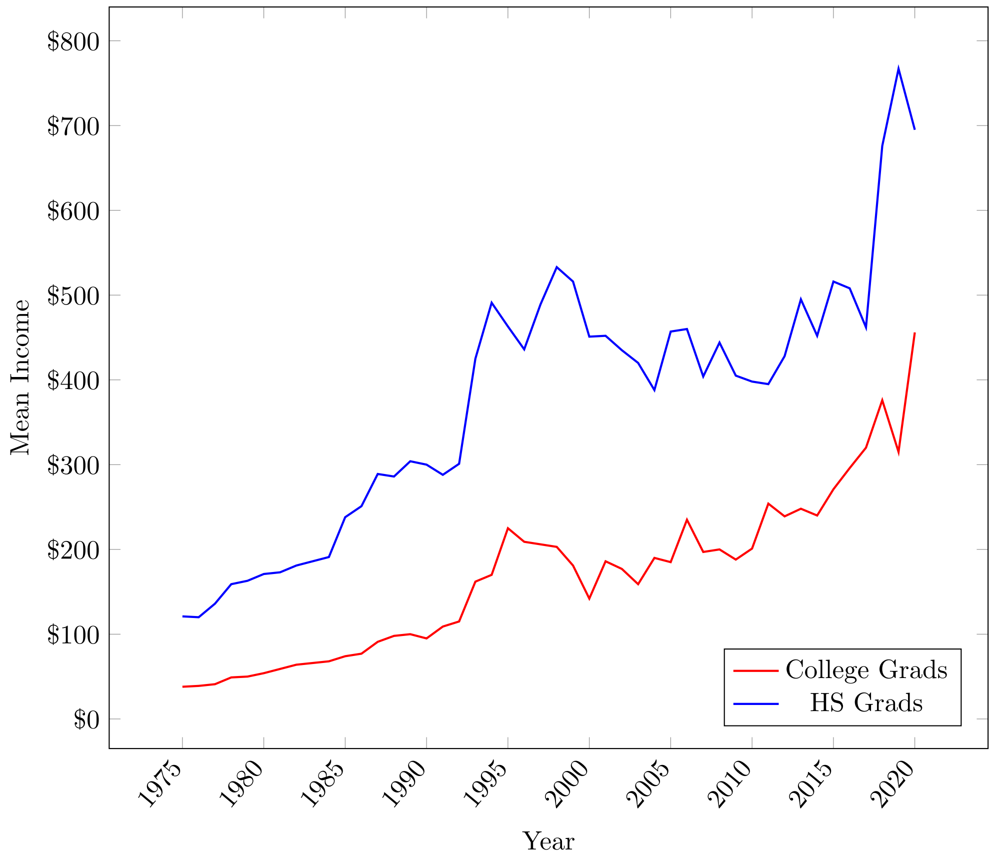


How do these comparative statics match up with observed wage patterns in the U.S.? Data from U.S. Census Bureau (2022) as shown in Figure 1 shows that the mean income of college graduates has increased faster than that of high school graduates over the last four decades.[6] I also show the time trend in standard errors of mean income in Figure 2. In my model, agents of the same type have variance in wages due to $c(\theta)$. Interestingly, the pattern of dispersion in wages across agents of the same education level is similar—suggesting that the cost of becoming a manager is increasing in general—but has consistently been higher for high school graduates. This implies that the cost of becoming a manager is larger for L-type agents, perhaps because they face a larger opportunity cost or higher risk in entering into the market as a manager.

That college educated adults are experiencing faster income growth and variance in both groups' incomes is increasing over time are together consistent with the hypothesis that the overall landscape of the U.S. labor market has moved towards clustering equilibria over time. Specialization is ruled out because variance in college graduates' incomes would be steady in this case. This may be driven by fields like technology-based

[6] Individuals in the dataset are 18+.

Figure 2: Standard Error of Mean Income by Level of Education



start-ups or finance, in which productivity has increased the fastest among high-skilled matches.[7] This is also consistent with Song et al. (2019), who find that between 1978 and 2013, increased within-firm positive sorting correlates with increases in between-firm wage disparity.[8]

The model has a notable policy implication: making it less costly for low-skilled agents to enter in the lead role can simultaneously increase productivity and decrease wage differentials. As previously noted, H-type clustering is always the most productive assignment disregarding costs; however, it may not occur in equilibrium because H-type agents prefer to manage L-type workers who demand lower wages. Suppose that firms and/or policymakers want to push the matching pattern towards clustering—for instance, if the most technically demanding projects are expected to generate positive externalities. If so, then they should subsidize L-type agents who wish to enter in the lead role: by decreasing $c(L)$, role supermodularity is easier to attain because v^*_{wL} is always decreasing in $c(L)$ and the H-type manager's trade-off between higher productivity and paying higher wages to H-type workers shifts towards the former. This simultaneously pushes the matching pattern towards the most "high powered" arrangement, clustering, and decreases wage differentials.

As an example, consider a field like web or mobile app development, which has multiple entry points depending on experience and training. Also, cost to receive credentials is increasing in type—aspiring developers

[7] I borrow the term "clustering" equilibrium from Silicon Valley tech clusters.
[8] Card, Heining, and Kline (2013) find similar patterns in Western Germany from 1985 to 2009.

without experience may attend short-term coding bootcamps to get their foot in the door of an entry-level job, but high profile jobs may require years of university education.[9] For example, senior mobile developers at Google require a bachelor's degree at minimum and consider an advanced degree substitutable with professional experience.[10] This model advises policymakers to subsidize short-term programs like coding bootcamps rather than providing scholarships for advanced degrees in computer science. By making entry-level coders better off, higher-level coders will prefer to group together.

5 Conclusion and Future Work

I present a two-sided matching model of a labor market in which agents can choose their role. The pre-matching strategic decision causes positive assortative matching to become more difficult to attain in equilibrium compared to the standard assignment model; this is primarily driven by H-type agents' incentives. The production technology must satisfy role supermodularity, a stronger condition than the standard strict supermodularity condition, for positive assortative matching to occur.

However, these results follow from stylized assumptions on the production technology. I specifically rule out the possibility that it may be optimal for H-type managers to lead a mix of H-type and L-type workers by letting the marginal productivity of H-type workers weakly increase in n via assumption MP1. This assumption becomes less plausible as n increases—generally, one would expect to eventually see diminishing marginal returns for n sufficiently large. I discuss how to loosen MP1 in Appendix C. Another strong assumption is that managers must match with *exactly* n workers. In Appendix D, I show that n can be endogenized when considering specific functional forms.

A natural extension of the paper is to add skill types. To conclude, I give some informal discussion about how I anticipate results would extrapolate to a model with more skill types. To simplify discussion, I assume one worker and only discuss matching patterns. Suppose there are three skill types, $H > M > L$, and f satisfies the following: (i) if $\theta_m > \theta'_m$, then $f(\theta_m, \bar{\theta}_w) > f(\theta'_m, \bar{\theta}_w)$, (ii) if $\theta_w > \theta'_w$, then $f(\bar{\theta}_m, \theta_w) > f(\bar{\theta}_m, \theta'_w)$, and (iii) if $\theta > \theta'$, then $f(\theta, \theta') > f(\theta', \theta)$.[11] I anticipate five possible equilibrium matching patterns:

1. Full clustering: All types match with their own type.
2. H-clustering: H-type agents match with their own type. M-type managers match with L-type workers.
3. M-clustering: H-type managers match with L-type workers. M-type agents match with their own type.
4. L-clustering: H-type managers match with M-type workers. L-type agents match with their own type.
5. Full specialization: H-type managers match with M-type workers. M-type managers match with L-type workers.

By adding the assumption $f(H, M) - f(M, M) > f(H, L) - f(M, L)$, (i.e., M-type clustering is inefficient; this rules out M-clustering), the production assumptions are analogous to the setting in Anderson (2020),

[9]See `https://brainstation.io/get-hired` for examples of coding bootcamps. Most are several weeks long.

[10]See `https://careers.google.com/jobs/results/107199685019476678-senior-software-engineer-android-applications/`.

[11]Unlike the case with two types, these assumptions are not enough for a complete ordering. For example, $f(L, H)$ cannot be compared to $f(M, M)$.

which generalizes Kremer and Maskin (1997). Moreover, the proposed equilibria matching patterns align with the matching pattern that Anderson (2020) derives: skill types in a connected interval form groups, and within those groups, specialization occurs.[12] This suggests that as more and more skill types are added, equilibria matching patterns in a two-sided framework may remain efficient.

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[12] Anderson (2020) terms this matching pattern "positive clustering". Unfortunately, our shared usage of the term "clustering" refer to opposite scenarios. In Anderson (2020)'s model, a cluster is a group of (potentially multiple) skill types. In my model, a cluster is a single skill type that prefers to match with own type if possible.

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A Main Theorem - Proof

I first establish two basic facts about any potential equilibrium, then prove the main result.

Lemma 1. *If σ induces an unbalanced market, then σ cannot be part of an equilibrium.*

Proof. If the market is unbalanced, then some agents will be unmatched and receive 0. All unmatched agents must share the same role r. Suppose some of the unmatched agents are θ-type. Since θ-type agents are perfect substitutes for each other and the market is competitive, $v_{r\theta} = 0$ in any stable market outcome. σ_θ is suboptimal, as each individual θ-type agent in role r can do strictly better by deviating to the other role and matching with another θ-type agent who was previously unmatched with certainty, as by Equations 1 and 2, her wage from deviating is $f(\theta, \theta) > 0$. $\square$

Lemma 2. *Let the matching market be balanced. If $\sigma_\theta \in (0, 1)$ is optimal, then the expected payoff vector that the agents satisfies $v_{m\theta} - c(\theta) = v_{w\theta}$.*

Proof. As the market is balanced, agents are matched with certainty. A θ-type agent is indifferent between pure strategies if and only if $v_{m\theta} - c(\theta) = v_{w\theta}$. $\square$

Theorem 1. *Outside of knife-edge cases, a unique equilibrium always exists.*[13]

1. *If Equation 3 holds, then there exists a unique* ***clustering equilibrium****. The equilibrium strategies are $\sigma^*_{CE} = (\frac{1}{n+1}, \frac{1}{n+1})$. In the unique stable market outcome, all agents match with their own type, and wages are*

$$v^*_{m\theta} = \frac{1}{n+1}(f(\theta, n\theta) + n \cdot c(\theta)),$$
$$v^*_{w\theta} = \frac{1}{n+1}(f(\theta, n\theta) - c(\theta)).$$

2. *If not, then there exists a unique* ***specialization equilibrium****. The equilibrium strategies are $\sigma^*_{SE} = (1, \frac{1-(n+1)M_H}{(n+1)(1+M_H)})$. In the unique stable market outcome, all H-type agents become managers, L-type agents mix such that the market clears, and the wages are*

$$v^*_{mH} = f(H, nL) - \frac{n}{n+1}(f(L, nL) - c(L)),$$
$$v^*_{mL} = \frac{1}{n+1}(f(L, nL) + n \cdot c(L)),$$
$$v^*_{wL} = \frac{1}{n+1}(f(L, nL) - c(L)).$$

Equilibrium matching patterns are socially efficient.

Proof. **Part 1.** Suppose that f satisfies Equation 3. First, I show σ^*_{PCE} is an equilibrium. σ_L and σ_H are mutual best responses by Lemma 1. The proposed wage structure follows from pairwise efficiency and Lemma 2. This is a stable payoff vector in the assignment game, hence it is sustainable in rational expectations. Also, since all agents have the same utility on both sides of the market, there is no incentive

[13] See Appendix B for details on the knife-edge cases.

for any individual agent to deviate. So the proposed strategies and stable market outcome are together an equilibrium.

Next, I show uniqueness. I break up other possible strategies according to σ_H case by case and show they cannot be equilibria.

1. Suppose $\sigma_H = 0$ is part of an equilibrium. L-type agents match with groups of n H-type workers or groups of n L-type workers. Then by pairwise efficiency and Lemma 2, $v_{mL} = \frac{1}{n+1}(f(L, nL) - c(L))$ and $v_{wL} = \frac{1}{n+1}(f(L, nL) - n \cdot c(L))$. By pairwise efficiency, $v_{wH} = \frac{1}{n}[f(L, nH) - v_{mL}]$. But any H-type agent have an incentive to deviate to becoming a manager, because she will be able to partner with n H-type workers for an outcome $f(H, nH) - [f(L, nH) - v_{mL}] - c(H)$.

2. Suppose $\sigma_H \in (0, \frac{1}{n+1})$ is part of an equilibrium. H-type managers match with groups of n H-type workers, and L-type managers match with groups of n H-type workers or groups of n L-type workers. But then pairwise efficiency and Lemma 2 cannot both hold in general.

3. Suppose $\sigma_H \in (\frac{M_H}{n}, 1]$ is part of an equilibrium. Two sub-cases follow:
   (a) n and M_H are such that there are no L-type managers. H-type managers work with groups of n H-type workers or groups of n L-type workers. Then by pairwise efficiency and Lemma 2, $v_{mH} = \frac{1}{n+1}[f(H, nH) + c \cdot c(H)]$ and $v_{wL} = \frac{1}{n}(f(H, nL) - \frac{1}{n+1}[f(H, nH) + c(H)])$. But any L-type agent has an incentive to deviate to becoming a manager, because she will be able to partner with n L-type workers for an outcome $f(L, nL) - [f(H, nL) - n \cdot v_{wL}] - c(L)$.
   (b) n and M_H are such that there are some L-type managers. H-type managers match with groups of n H-type workers or groups of n L-type workers, and L-type managers work with groups of n L-type workers. But then pairwise efficiency cannot hold in general.

Part 2. Suppose f doesn't satisfy Equation 3. First, I show that σ^*_{PSE} is an equilibrium. σ_L and σ_H are mutual best responses by Lemma 1. The proposed wage structure follows from pairwise efficiency and Lemma 2. This is a stable payoff vector in the assignment game, hence it is sustainable in rational expectations. L-type agents receive the same wage on both sides of the market, so there is no incentive for any individual L-type agent to deviate. There is also no incentive for nay individual H-type manager to deviate, because her marginal productivity as a manager is greater than her marginal productivity as a worker, hence it would be an unprofitable deviation. So the proposed strategies and stable market outcome are an equilibrium.

Next, I show uniqueness. Again, I check case by case.

1. Suppose $\sigma_H = 0$ is part of an equilibrium. By the same reasoning as in Part 1, this is suboptimal.

2. Suppose $\sigma_H \in (0, 1)$, and n and M_H are such that all H-type managers work with $n - m$ H-type workers and n L-type workers. Because $M_H \in (0, \frac{1}{n+1})$, there must be L-type clusters in the market for it to be balanced. But every H-type agent's marginal productivity as a manager is greater than her marginal productivity as a worker, hence H-type workers have an incentive to deviate for a higher wage.

3. Suppose $\sigma_H \in (0, 1)$, and n and M_H are such that some H-type managers work with n H-type workers, and others work with n L-type workers. Because $M_H \in (0, \frac{1}{n+1})$, there must be L-type clusters in the market for it to be balanced. But then pairwise efficiency cannot both hold in general.

□

B Equilibria in the Knife-Edge Case

Suppose $f(H, nH) + n \cdot f(L, nL) + n[c(H) - c(L)] = (n+1)f(H, nL)$. Both the clustering and specialization equilibria can occur, and so can equilibria where the matching pattern is mixed between the clustering and specialization cases. To see this, set $n = 1$ and costs to 0. Agents expect to face the same wage regardless of role, $v_\theta = \frac{1}{2}f(\theta, \theta)$. Since f satisfies $f(H, H) + f(L, L) = 2f(H, L)$, $v_H = \frac{1}{2}f(H, H) = f(H, L) - \frac{1}{2}$. Then any $\sigma_H \in [\frac{1}{2}, 1]$ can be sustained in equilibrium. To give an example, let $\sigma_H = \frac{3}{4}$ and σ_L be such that the market is balanced. Then a measure $\frac{1}{4}M_H$ of H-type managers match with H-type workers, a measure $\frac{1}{2}M_H$ of H-type managers match with L-type workers, and any remaining L-type workers match with L-type managers. No agents have an incentive to deviate, and the payoff vector is sustainable in rational expectations, as the matching and payoff vector together are a stable market outcome in the matching market induced by the agents' strategies.

Because of the multiplicity in equilibria that can arise, I rule out the knife-edge case altogether as a matter of convenience.

C Worker Supermodularity

The assumption that mixed worker compositions are inefficient is strong, especially as n increases and/or if H-type agents become less scarce. I briefly discuss here how worker composition can be handled more flexibly. Let $n = 2$, $M_H \in [\frac{1}{3}, \frac{1}{2})$, let $c(\theta) = 0$ for all θ, and drop the marginal productivity assumption on H-type workers.[14] Then the efficient way to match H-type agents is determined by a *worker supermodularity* condition. If

$$f(H, 2H) + f(H, 2L) > 2f(H, H + L) \qquad \text{(Worker supermodularity)}$$

is satisfied, then H-type workers only have H-type agents as co-workers. Note that this exactly aligns with the marginal productivity assumption: when worker supermodularity is satisfied, the marginal benefit from adding a second H-type worker is weakly larger than the marginal benefit from adding the first H-type worker, hence H-type workers should be grouped together. If worker supermodularity is not satisfied, then it is more efficient to have one H-type worker and one L-type worker in each match. I claim these conditions also determine the equilibrium market outcome.

Proposition 1. *Let f satisfy role supermodularity. Outside of knife-edge cases, a unique equilibrium always exists. Furthermore, equilibrium wages only depend on type.*

1. *If worker supermodularity holds, then there exists a unique* ***impure clustering equilibrium****. There are two types of matches, H-type managers to two H-type workers and H-type managers to two L-type workers. The payoff vector is $v_H^* = \frac{1}{3}f(H, 2H)$ and $v_L = \frac{1}{2}(f(H, 2L) - \frac{1}{3}f(H, 2H))$.*

2. *If not, then there exists a unique* ***impure specialization equilibrium****. There are two types of matches,*

[14] Note that dropping costs doesn't affect results, it only simplifies presentation.

H-type managers to one H-type and one L-type worker, and H-type managers to two L-type workers. The payoff vector is $v_H^* = \frac{1}{3}(2f(H, H+L) - f(H, 2L))$ *and* $v_L^* = \frac{1}{3}(2f(H, 2L) - f(H, H+L))$.

Equilibrium matching patterns are socially efficient.

Proof. **Part 1**. Suppose f satisfies worker supermodularity. σ_H and σ_L are mutual best responses by Lemma 1. By Lemma 2, $v_H = \frac{1}{3}f(H, 2H)$. By pairwise efficiency, $v_L = \frac{1}{2}(f(H, 2L) - \frac{1}{3}f(H, 2H))$. This is a stable payoff vector in the assignment game, hence it is sustainable in rational expectations. H-type agents receive the same wage on both sides of the market, so there is no incentive for any individual H-type agent to deviate. I claim there is also no incentive for any individual L-type agent to deviate. To see this, suppose not. By Equation 2, her wage from deviating is $f(L, 2H) - \frac{2}{3}f(H, 2H)$. Then

$$f(L, 2H) - \frac{2}{3}f(H, 2H) > \frac{1}{2}f(H, 2L) - \frac{1}{6}f(H, 2H)$$
$$f(L, 2H) - f(H, 2H) > f(H, 2L) - f(L, 2H)$$

contradicting the assumption that the marginal productivity of an H-type is higher as a manager than as a worker. So the proposed strategies and stable market outcome are an equilibrium.

Uniqueness follows from the fact that f satisfies both role and worker supermodularity. Specializing is better than clustering, but pure clustering is not possible. Given that, the only room for H-types to deviate is to split a strategy such that each H-type manager matches with one H-type worker and one L-type worker, and that strategy profile induces the impure specialization equilibrium. Suppose for a contradiction that this is a profitable deviation. Then

$$\frac{1}{3}f(H, 2H) < \frac{1}{3}(2f(H, H+L) - f(H, 2L))$$
$$f(H, 2H) + f(H, 2L) < 2f(H, H+L)$$

contradicting that worker supermodularity holds.

Also, L-type agents do not have a profitable deviation to clustering for a wage of $\frac{1}{3}f(L, 2L)$; to see this, suppose not. Then

$$\frac{1}{2}(f(H, 2L) - \frac{1}{3}f(H, 2H)) < \frac{1}{3}f(L, 2L)$$
$$3f(H, 2L) < f(H, 2H) + 2f(L, 2L)$$

which contradicts that f satisfies role supermodularity.

Part 2. Suppose f doesn't satisfy worker supermodularity. σ_H and σ_L are mutual best responses by Lemma 1. By pairwise efficiency and Lemma 2,

$$f(H, H+L) = 2v_H + v_L,$$
$$f(H, 2L) = v_H + 2v_L.$$

Solving the system of equations, $v_H = \frac{1}{3}(2f(H, H+L) + f(H, 2L))$ and $v_L = \frac{1}{3}(2f(H, 2L) - f(H, H+L)$. This is a stable payoff vector in the assignment game, hence it is sustainable in rational expectations. All agents receive the same wage on both sides of the market, so there is no incentive for any individual agent to deviate.

Uniqueness follows from the fact that f satisfies both role and worker supermodularity; it is the inverse of the uniqueness argument in Part 1. Also, L-type agents do not have a profitable deviation to clustering for a wage of $\frac{1}{3}f(L, 2L)$; to see this, suppose not. Then

$$\begin{aligned} 2f(H, 2L) - f(H, H + L) &< f(L, 2L) \\ f(H, 2L) - f(L, 2L) &< f(H, H + L) - f(H, 2L) \end{aligned}$$

contradicting the assumption that the marginal productivity of an H-type is higher as a manager than as a worker. □

D Endogenizing n

In the main model, one manager must match with exactly n workers. Now let $n = \{1, 2\}$ be endogenous. The intuition behind Theorem 1 is that H-type agents compare whether they are better off under clustering or specialization and choose their strategy accordingly. However, if a manager can choose up to n workers to match with, then the manager now compares up to four different possible matching patterns: one worker clustering, one worker specialization, two worker clustering, and two worker specialization.

Fix $L = 1$, $M_H \in (0, \frac{1}{3})$, relax the assumption that mixed coworker groups are inefficient, and let the production technology to be constant returns to scale Cobb-Douglass: $f(\theta_m, \theta_W) = \theta_m^\alpha \theta_W^{1-\alpha}$ for $\alpha \in (\frac{1}{2}, 1)$. Recall that constant returns to scale Cobb-Douglas production functions have the property that if inputs are scaled up by a given factor, then productivity is scaled up by the same factor; although the comparison is not quite one-to-one as only additional workers can be added, a key question of interest is whether it is always preferable to add as many workers as possible in this setting. I am most interested in comparing one worker clustering with two worker specialization and the implications on wage inequality.

Comparing the clustering matching patterns, one worker clustering is always more efficient than two worker clustering for both types, since

$$\begin{aligned} Surplus(1cl) &= \frac{1}{2}(M_H H + (1 - M_H)) \\ Surplus(2cl) &= \frac{2^{1-\alpha}}{3}(M_H H + (1 - M_H)). \end{aligned}$$

Also, all agents prefer one worker clustering over two worker clustering, since $v_\theta^{1cl} = \frac{1}{2}\theta > \frac{2^{1-\alpha}}{3}\theta = v_L^{2cl}$ for all $\alpha \in (\frac{1}{2}, 1)$.

Next, I compare the specialization matching patterns. By the above, it is efficient and better off for excess

L-type agents to form one worker clusters. Comparing the surpluses under one and two worker specialization,

$$\begin{aligned}
Surplus(1sp) &= M_H H^\alpha + \frac{1}{2}(1 - 2M_H) \\
Surplus(2sp) &= 2^{1-\alpha} M_H H^\alpha + \frac{1}{2}(1 - 3M_H) \\
&= M_H H^\alpha + \frac{1}{2}(1 - 2M_H) + M_H(H^\alpha(2^{1-\alpha} - 1) - \frac{1}{2}),
\end{aligned}$$

so two-worker specialization is more efficient if and only if $H^\alpha(2^{1-\alpha} - 1) > \frac{1}{2}$. As for H-types, $v_H^{2sp} = 2^{1-\alpha}H^\alpha - 1$ and $v_H^{1sp} = H^\alpha - \frac{1}{2}$, so H-type agents prefer two worker specialization over one worker specialization if it is also more efficient. Since I am interested in comparing two worker specialization with one worker clustering, impose the condition and set $H^\alpha(2^{1-\alpha} - 1) > \frac{1}{2}$.

Finally, I compare one worker clustering and two worker specialization. Since

$$\begin{aligned}
Surplus(2sp) &= 2^{1-\alpha} M_H H^\alpha + \frac{1}{2}(1 - 3M_H) = 2^{1-\alpha} M_H H^\alpha + \frac{1}{2}(1 - M_H) - M_H \\
Surplus(1cl) &= \frac{1}{2}(M_H H + (1 - M_H)),
\end{aligned}$$

one worker clustering is more efficient if and only if $2^{1-\alpha} M_H H^\alpha - M_H < \frac{1}{2} M_H H \Rightarrow H + 2 > 2^{2-\alpha} H^\alpha$.

The wage rate for L-type agents is the same in both cases, $v_L^{1cl} = v_L^{2sp} = \frac{1}{2}$. As for H-types, $v_H^{1cl} = \frac{1}{2}H$ and $v_H^{2sp} = 2^{1-\alpha}H^\alpha - 1$, so the market outcome is one worker clustering if and only if $H + 2 > 2^{2-\alpha}H^\alpha$, hence the market outcome is efficient. Since $2^{2-\alpha}$ is bounded from above by $2^{1.5} < 3$, the condition holds whenever role supermodularity with $n = 2$ holds.

To summarize, there are two possible matching patterns in equilibrium. If $H + 2 > 2^{2-\alpha}H^\alpha$, then the market outcome is that θ-type managers match with one θ-type worker. If not, then the market outcome is that one H-type manager matches with two L-type workers, and excess L-type agents form one-worker clusters.

Example 1. *Let $f(\theta_m, \theta_W) = \theta_m^{\frac{3}{4}} \theta_W^{\frac{1}{4}}$. To guarantee that mixed worker compositions remain inefficient, fix $H \in (0, 30]$.*

Let $H = 1.5$. If the manager must hire two workers, then two worker specialization is the efficient market outcome. However, if the manager can hire either one or two workers, then two worker clustering is the efficient market outcome. Furthermore, the same is true for any $H \in (23, 30]$.

As α changes, the intervals of H that guarantee one worker clustering occurs instead of two worker specialization remains similar: if H is sufficiently large and if $H \in (1, 2)$. That is, if H-types are not that much more productive or significantly more productive than L-types, it is better to have H-type works together and take advantage of the fact that the H-type manager is most productive with an H-type worker. However, if H is in a "middle ground", then specialization becomes more productive because worker skill is additive.