$\alpha-$robust equilibrium in anonymous games

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Abstract

In this paper, we consider the notion of $\alpha-$ robust equilibrium for finite strategic players in anonymous games, where utility of a player depends on other players' actions only through the resulting distribution of actions played. This equilibrium is defined as the set of strategies of the players such that no user wants to deviate as long as $N-\alpha-1$ number are playing the equilibrium strategies. We provide sufficient conditions for the existence of this equilibrium. In addition, we prove a part of Berge's Maximal Theorem for correspondences.

I. Introduction

Von Neumann and Morgenstern defined the notion of saddle point equilibrium [1] for zero sum games and Nash defined the notion of Nash equilibrium [2] for general games. Nash equilibrium is defined as a set of strategies of the users such that if all other users play the equilibrium strategy, then the user also plays an equilibrium policy and has no incentive to deviate. Thus Nash equilibrium strategies of the players are defined as a fixed point of the best responses of the players and its existence in finite games is proved using Kakutani's fixed-point theorem [3]. Since then game theory has been developed as a powerful tool to model strategic interaction both for static and dynamic settings.

In this paper, we consider the notion of $\alpha-$ *robust Nash equilibrium* as set of strategies of the players such that no user wants to deviate as long as $N-\alpha-1$ number are playing the equilibrium strategies. This notion of equilibrium was introduced in [4] (We refer the reader to [4] for numerous examples). Thus it is robust to up to α number of players playing arbitrary strategies. This is motivated by users who are not necessarily rational or to accommodate modeling errors. We note that for $\alpha = 0$, this boils down to the standard Nash equilibrium. In this paper, we define this notion of $\alpha-$robust Nash equilibrium and provide sufficient conditions for its existence for *anonymous games* [5] with finite action spaces and symmetric pay-offs. These are a class of games in which each player's pay-off does not depend on the identity of other players, but only on the number of each player choosing a particular action. As pointed out in [5] such games arise in many economic models and are a natural setting to study this equilibrium concept as one can simply look at a fixed set of α players as deviating from the equilibrium, while in a more general game the impact of α players deviating may vary depending on which set of players deviates..

The paper is structured is as follows. In Section II, we define the model. In Section III, we define the equilibrium concepts both for finite and infinite horizon players. In Section IV, we provide sufficient conditions for the existence of the equilibrium. We conclude in Section VI.

II. Model

Consider a game $G(N, \mathcal{A}, u)$ with N players, each with a finite action space $\mathcal{A}$. Let each player i have a utility u that depends on other players' actions as

$$u(a^i, f(a^{-i})) \tag{1}$$

where $\forall i, a^i \in \mathcal{A}$, a^{-i} is the set of actions of all players other than i, and $f(a^{-i})$ is the frequency distribution of a^{-i}, i.e., this specifies the number of times each action in $\mathcal{A}$ appears in a^{-i}.

III. Equilibrium concept

Let $\bar{\sigma}^i \in \Delta(\mathcal{A})$ be a mixed strategy of player i, $\underline{\sigma} \in \Delta(\mathcal{A})^{N-\alpha-1}$ be a mixed strategy profile of any $N-\alpha-1$ defecting players, where $\Delta(\mathcal{A})$ is the space of probability distributions on the set $\mathcal{A}$. We define the notion of $\alpha-$ robust Nash equilibrium for finite players as a strategy profile $\underline{\sigma}^{*(\alpha)} = (\sigma^{k,*(\alpha)} \in \Delta(\mathcal{A}))_{k:|k|=N-\alpha}$ such that $\sigma^{i,*(\alpha)}$ maximizes the expected utility of player i, if $N-\alpha-1$ players play according to $\sigma^{-i,*(\alpha)} = (\sigma^{k,*(\alpha)})_{k\neq i,|k|=N-\alpha-1}$ and α players play arbitrary strategies, i.e.

$$\sigma^{i,*(\alpha)} \in \bigcap_{\underline{\sigma}} \arg\max_{\bar{\sigma}^i} \mathbb{E}^{\bar{\sigma}^i,\underline{\sigma}^{-i,*(\alpha)},\underline{\sigma},\alpha}[u(A^i, f(A^{-i}))] \tag{2}$$

where the above expectation is taken wrt $\bar{\sigma}^i(a^i)\prod_{j\neq i,|j|=\alpha}\sigma^j(a^j)\prod_{k\neq i,|k|=N-\alpha-1}\sigma^{k,*(\alpha)}(a^k)$, and $\underline{\sigma} = (\sigma^j)_{j\neq i,|j|=\alpha}, \underline{\sigma}^{-i,*(\alpha)} = (\sigma^{k,*(\alpha)})_{k\neq i,|k|=N-\alpha-1}$.

We note that each $\alpha-$robust equilibrium is also a Nash equilibrium.

IV. Existence

It is known there always exists a Nash equilibrium for a finite game [2]. Thus there always exists an $\alpha = 0$ equilibrium. In this section, we ask the question, what are the values of α for which there exists an $\alpha-$ robust Nash equilibrium. In the following, we discuss the existence of such an equilibrium. Note that (2) can be written as

$$\sigma^{*(\alpha)} \in \bigcap_{\underline{\sigma}} \arg\max_{\bar{\sigma}^i} \mathbb{E}^{\bar{\sigma}^i,\sigma^{*(\alpha)},\underline{\sigma},\alpha}[u(A^i, f(A^{-i}))] \tag{3}$$

$$\in \bigcap_{\underline{\sigma}} \arg\max_{\bar{\sigma}^i} \sum_{a} \bar{\sigma}^i(a^i) \prod_{j\neq i,|j|=\alpha} \sigma^j(a^j) \prod_{k\neq i,|k|=N-\alpha-1} \sigma^{k,*(\alpha)}(a^k) u(a^i, f(a^{-i})). \tag{4}$$

Note that in the above definition, since the utility of the player i does not depend on its index i, the choice of defecting $\alpha-$ players doesn't matter.

Let

$$\begin{aligned} BR^{i,\alpha}(\underline{\sigma}, \underline{\hat{\sigma}}) &= \arg\max_{\bar{\sigma}^i} \mathbb{E}^{\bar{\sigma}^i,\underline{\sigma},\underline{\hat{\sigma}},\alpha}[u(A^i, f(A^{-i}))] \\ &= \arg\max_{\bar{\sigma}^i} \sum_{a^i} \bar{\sigma}^i(a^i) \sum_{a^{-i}} \prod_{j\neq i,|j|=\alpha} \sigma^j(a^j) \prod_{k\neq i,|k|=N-\alpha-1} \hat{\sigma}^k(a^k) u(a^i, f(a^{-i})) \\ &= \arg\max_{\bar{\sigma}^i} \sum_{a^i} \bar{\sigma}^i(a^i) u^{i,\alpha}(a^i, \underline{\sigma}, \underline{\hat{\sigma}}), \end{aligned} \tag{5}$$

where with some abuse of notation $u^{i,\alpha}(a^i, \underline{\sigma}, \underline{\hat{\sigma}}) = \prod_{j\neq i,|j|=\alpha} \sigma^j(a^j) \prod_{k\neq i,|k|=N-\alpha-1} \hat{\sigma}^k(a^k) u(a^i, f(a^{-i}))$. Let

$$T^{i,\alpha}(\underline{\hat{\sigma}}) := \bigcap_{\underline{\sigma}} BR^{i,\alpha}(\underline{\sigma}, \underline{\hat{\sigma}}) \tag{6}$$

Hence, $T^{i,\alpha}$ is the set of actions for player i, that are a best response to any mixed strategy profile of the other players in which the the non-defecting players follow the profile $\hat{\sigma}$ and the defecting players can select any mixed strategy profile. Using this definition the α-robust equilibrium can be equivalently written as:

$$\underline{\sigma}^{*,(\alpha)} \in T^{\alpha}(\underline{\sigma}^{*,(\alpha)}). \tag{7}$$

Theorem 1: If $T^{\alpha}(\underline{\hat{\sigma}})$ is non-empty for all $\underline{\hat{\sigma}}$, then there exists an $\alpha-$ robust equilibrium.

Proof: We first note that $\mathbb{E}^{\bar{\sigma}^i,\underline{\hat{\sigma}},\underline{\sigma},\alpha}[u(A^i, \max(A^{-i}))]$ is linear in $\bar{\sigma}^i, \underline{\sigma}, \underline{\hat{\sigma}}$. Since the maximization is done over the probability simplex which is compact, this implies from Berge's maximal theorem [3] that $BR^{i,\alpha}(\underline{\sigma}, \underline{\hat{\sigma}})$ is continuous in $(\sigma, \underline{\hat{\sigma}})$. Moreover, $BR^{i,\alpha}(\underline{\sigma}, \underline{\hat{\sigma}})$ is convex valued because of linearity of the objective function.

By similar arguments as above, and using a version of Berge's maximal theorem for correspondences (Theorem 2 proved in Appendix A), $T^{\alpha}(\underline{\hat{\sigma}})$ is upper hemi-continuous in $\underline{\sigma}$ which implies it has the closed graph property. Moreover, $\forall i, \underline{\hat{\sigma}}$, $T^{i,\alpha}(\underline{\hat{\sigma}})$ are convex valued since $BR^{i,\alpha}$

are convex valued and intersection of convex sets is convex. The domain of T^α is the probability simplex which is non-empty, compact and convex. Thus, if $\forall \underline{\hat{\sigma}}$, $T^\alpha(\underline{\hat{\sigma}})$ is non-empty, this implies using Kakutani's fixed-point theorem [3] that there exists a fixed-point. ■

We note that in above Theorem, we could restrict $\underline{\hat{\sigma}}$ to be in any non empty compact set (not necessarily convex) $\mathcal{S}$ and as long as $T^\alpha(\underline{\hat{\sigma}})$ is non-empty for all $\underline{\hat{\sigma}}$ in that set and $T^\alpha : \mathcal{S} \to \mathcal{S}$, there exists an $\alpha-$ robust equilibrium from Glicksberg Fixed-point Theorem [6]. In the following lemma, we provide some sufficient conditions for the set $T^{i,\alpha}(\underline{\hat{\sigma}})$ to be non-empty. If we consider the class of linear programs (corresponding to the best response of a player) whose objective is parametrized by strategies of the defectors (and equivalently parametrized by their pure strategies) then the optimum solution set doesn't change by changing objective functions within this class. This is guaranteed by sensitivity analysis of the considered linear programs.

Lemma 1: For some given $\underline{a}$, let $c := [u(a^i, \underline{a}, \underline{\hat{\sigma}})]_{a^i}, \Delta c := max_{\underline{\sigma}}(c - [(u(a^i, \underline{\sigma}, \underline{\hat{\sigma}})]_{a^i})) = max_{\underline{a}}(c - ([u(a^i, \underline{a}, \underline{\hat{\sigma}})]_{a^i}))$ and $y = \max_{a^i} u(a^i, \underline{a}, \underline{\hat{\sigma}})$. Then $T^{i,\alpha}(\underline{\hat{\sigma}})$ is non-empty if $\Delta c \leq -(c - [11...1]^T y)$.

Proof: Note that $T^{i,\alpha}(\underline{\hat{\sigma}})$ is intersection of solutions of a class of linear programs defined in (5), where changing $\underline{\sigma}$ changes the direction of the plane in a convex and continuous fashion. The result is then implied from the sensitivity analysis of the linear program [7]. ■

Remark 1 (Sufficient Condition): $T^{i,\alpha}(\underline{\hat{\sigma}})$ is non-empty if $\frac{[u(a^i,\underline{\sigma},\underline{\hat{\sigma}})]_{\underline{\sigma}}}{norm([u(a^i,\underline{\sigma},\underline{\hat{\sigma}})]_{\underline{\sigma}})}$ doesn't change with $\underline{\sigma}$. This is because under this condition the orientation of the objective function doesn't change in the linear program in (5) and therefore $BR^i(\underline{\sigma}, \underline{\hat{\sigma}})$ is independent of $\underline{\sigma}$. This is true for $\alpha = 1$ case which is the Nash equilibrium.

The necessary and sufficient conditions for $T^{i,\alpha}(\underline{\hat{\sigma}})$ to be non-empty would be, where instead of asking if the set of optimum solutions does not change by changing objective functions, as in the above lemma, if the intersection of set of optimum solutions with respect to each objective in this class is non-empty. We leave this as an open problem.

V. Example

Consider an N player game of matching actions, where each player chooses an action from the set $\{1, 2, 3\}$, such that a player's utility is 1 if it plays action that is played by maximum number of people. Thus

$$u(a^i, f(a^{-i})) = \begin{cases} 1 \text{ if } a^i = \arg\max_{a \in \{1,2,3\}} f(a^{-i}) \\ 0 \text{ othwerwise} \end{cases} \tag{8}$$

Clearly all actions $a \in \{1,2,3\}$ are pure strategy Nash equilibria and $P(a=i) = 1/3$ for $i = 1,2,3$ is a mixed-strategy Nash equilibrium. We note that actions $a = 1,2,3$ are $(N-1)/2-$robust whereas $P(a=i) = 1/3$ for $i = 1,2,3$ is only 1-robust.

VI. Conclusion

In this paper, we consider the notion of $\alpha-$ robust equilibrium for both finite strategic players for anonymous games where no user wants to deviate as long as $N-\alpha-1$ number are playing the equilibrium strategies. We provide sufficient conditions for existence of this equilibrium. In addition, we prove an extension of part of Berge's theorem for correspondences.

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Appendix A

Let X, Θ, and Y be topological spaces, $f : X \times \Theta \rightrightarrows Y$ be a continuous correspondence on the product $X \times \Theta$, and $C : \Theta \rightrightarrows X$ be a compact-valued correspondence such that $C(\theta) \neq \emptyset$ for all $\theta \in \Theta$. Define the marginal function (or value function) $f^* : \Theta \rightrightarrows Y$ by

$f^*(\theta) = \bigcap_{x \in C(\theta)} f(x,\theta)$ and the set of maximizers $C^* : \Theta \rightrightarrows X$ by

$C^*(\theta) = \{x \in C(\theta) : f(x,\theta) = f^*(\theta)\}$.[1] .

Theorem 2: If f is upper hemicontinuous and C is upper hemicontinuous, nonempty and compact-valued, then f^* is upper hemicontinuous.

Proof: Fix $\theta \in \Theta$, and consider a neighborhood W of $f^*(\theta)$ and W_x of $f(x,\theta)$ such that $W_x \subseteq W$. For each $x \in C(\theta)$, there exists a neighborhood $U_x \times V_x$ of (θ, x) such that whenever $(\theta', x') \in U_x \times V_x$, we have $f(x,\theta) \subseteq W_x$. The set of neighborhoods $\{V_x : x \in C(\theta)\}$ covers $C(\theta)$, which is compact, so $V_{x_1}, \ldots, V_{x_n}$ suffice. Furthermore, since C is upper hemicontinuous, there exists a neighborhood U' of θ such that whenever $\theta' \in U'$ it follows that $C(\theta') \subseteq \bigcup_{k=1}^{n} V_{x_k}$. Let $U = U' \cap U_{x_1} \cap \cdots \cap U_{x_n}$. Then for all $\theta' \in U$, we have $f(x',\theta') \in W_{x_k}$ for each $x' \in C(\theta')$, as $x' \in V_{x_k}$ for some k. It follows that

$f^*(\theta') = \bigcap_{x' \in C(\theta')} f(x',\theta') \subseteq \bigcap_{k=1,\ldots,n} W_{x_k} \subseteq W$, which proves upper hemicontinuity of f^*. ■

Theorem 3: If f is lower hemi-continuous and C is non empty and lower hemi-continuous, then f^* is lower hemi-continuous.

[1]The following proofs are adapted from the proofs of Berge's optimal theorem for functions.

Proof: Fix $\theta \in \Theta$. Let $\hat{W}$ be any open set such that $f^*(\theta) \bigcap \hat{W} \neq 0$. By definition of f^*, there exists $x \in C(\theta)$ such that for any neighborhood W of $f(x,\theta)$, $f^*(\theta) \subseteq W$. Thus $W \bigcap \hat{W} \neq \emptyset$. Now, since f is lower hemicontinuous, for every open set W' such that $f(x,\theta) \bigcap W' \neq \emptyset$, there exists a neighborhood $U_1 \times V$ of (θ, x) such that whenever $(\theta', x') \in U_1 \times V$ we have $f(x', \theta') \bigcap W' \neq \emptyset$. Observe that $C(\theta) \cap V \neq \emptyset$ (in particular, $x \in C(\theta) \cap V$). Therefore, since C is lower hemi-continuous, there exists a neighborhood U_2 such that whenever $\theta' \in U_2$, $C(\theta') \cap V \neq \emptyset$. Let $U = U_1 \cap U_2$. Consider $W' = \hat{W}$. Then whenever $\theta' \in U$, and $\forall x' \in C(\theta') \cap V$, $f(x', \theta') \bigcap \hat{W} \neq \emptyset$. Since $f^*(\theta') = \bigcap_{x \in C(\theta')} f(x, \theta') = \bigcap_{x \in C(\theta') \cap V} f(x, \theta')$, $f^*(\theta') \bigcap \hat{W} \neq \emptyset$, which proves the lower hemicontinuity of f^*.

■

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