From prejudice to racial profiling and back

A naïve intuitive statistician's curse

Manuel Foerster* Dominik Karos†

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Abstract

A designer conducts random searches to detect criminals, and may condition the search probability on individuals' appearance. She updates her belief about the distribution of criminals across appearances using her search results, but incorrectly takes her sample distribution for the population distribution. In equilibrium she employs optimal search probabilities given her belief, and her belief is consistent with her findings. We show that she will be discriminating an appearance if and only if she overestimates the probability of this appearance's being criminal. Moreover, in a linear model, tightening her budget will worsen the situation of those most discriminated against.

Keywords: Biased inference, police search, naïve intuitive statistics, racial profiling, discrimination.

1 Introduction

In the years 2014 to 2017, New York City (NYC) police arrested or summoned 19,328 individuals after stopping them, and possibly frisking them, in the street. Black people

*Center for Mathematical Economics, Bielefeld University, Germany, email: manuel.foerster@uni-bielefeld.de.

†Center for Mathematical Economics, Bielefeld University, Germany, email: dominik.karos@uni-bielefeld.de.

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Group	Population (NYC)	Arrests/Summons	Stops
White	1,875,108	1,938	10,228
Latino	2,346,883	6,540	26,181
Black	1,875,108	9,840	49,362
Others	1,245,527	1,010	6,612
Total	8,185,314	19,328	92,383

Table 1: Stops and arrests/summons by NYC police force, 2014–2017.

accounted for 9,840 or 52% of these arrests, while at the same time they represent only 23% of the population. Does this imply that black people are disproportionally criminal? Of course not! The reason lies in the last column of Table 1.[1] More than 53% of all stops targeted black people, that is, the sample distribution crucially deviates from the population distribution.[2]

The (misleading) calculation above illustrates the problem that this paper addresses: a *naïve (intuitive) statistician* (Juslin et al., 2007) will conclude that black people account for almost 52% of all (serious) infractions—ignoring the discrepancy between sample and population distribution. We do not expect our reader to make this mistake, but we do point out that this mistake is more common than one might hope: a growing body of evidence suggests that people lack the "metacognitive ability" to analyze and quantify sampling biases and to correct their judgments accordingly (Fiedler, 2000, 2012; Fiedler et al., 2019). Alarmingly, if this mistake is common among those who are responsible for conducting these stops, we will enter a vicious cycle: black people are stopped disproportionally often, they are perceived as disproportionally criminal, and, hence, will be stopped more often.

This vicious cycle is the topic of this paper. We start by considering a designer who faces a population in which each individual is endowed with a privately known and binary *conduct* (being criminal or not) and a publicly known *appearance* (race, age, sex, etc.).

[1]These numbers stem from the report "Stop-and-Frisk in the de Blasio Era" by the New York Civil Liberties Union, retrieved from https://nyclu.org/sites/default/files/field_documents/20190314_nyclu_stopfrisk_singles.pdf and https://nyclu.org/sites/default/files/field_documents/20190314_nyclu_stopfrisk_appendices_onlineonly.pdf, accessed December 15, 2020.

[2]This pattern is not unique to the NYC police, see Abrahams (2020), Horrace and Rohlin (2016), Pierson et al. (2020) and Sanga (2009) for evidence of discrimination/racial profiling in police stops. Lang and Kahn-Lang Spitzer (2020) provide a recent survey on racial discrimination.

The designer's objective is to identify criminals by conducting random searches. A *search rule* specifies the probabilities with which individuals of each appearance will be searched. We assume that the designer is endowed with a budget to conduct searches, and that she solves a constrained maximization problem: her objective function is increasing in the expected number of successful searches, and decreasing as the discrepancies in search probabilities across appearances become large, because discrimination might lead to legal or social sanctions.

Next, we allow the designer to update her belief about the conditional probability that an individual of a given appearance is criminal based on the outcome of her search. However, we assume that she makes the mistake above, i.e., she takes the sample distribution for the population distribution. A belief is *admissible* given a search rule if it coincides with the updated belief that is calculated in this fashion, i.e., if posterior and prior are identical. We provide necessary and sufficient conditions for the existence of an admissible belief, we show that this belief must be unique, and we provide a connection between the error in an admissible belief and the extent to which the underlying search rule is discriminatory.

Our main result combines the two steps: an *equilibrium* consists of a belief and a search rule such that the search rule solves the designer's maximization problem given her belief, and her belief is admissible given the search rule. We provide sufficient conditions for the existence of an equilibrium and show that the designer will target an appearance disproportionally if and only if she overestimates the probability of this appearance's being criminal (compared to the true distribution).

Our final result shows how model parameters such as sanctions for discriminatory search rules or budget cuts affect the equilibrium in a linear model. In particular, we show that simple budget cuts, that is, a reduction of the average search probability, will have the same effect as a decrease in sanctions for discrimination. Namely, while the overall percentage of the population that will be the victim of discrimination will decrease, the extent of discrimination facing the most-discriminated-against individuals will increase.

We demonstrate in this paper how racial profiling as documented above may be both the cause *and* the consequence of common racial prejudices, that is, the overestimation of black people's being criminal (Hurwitz and Peffley, 1997; Chiricos et al., 2001; Welch, 2007). Furthermore, our investigation of budget cuts reveals that "defunding the police" will not lead to less discrimination against marginalized groups. On the contrary, the situation of those who face the strongest discrimination will actually become worse.

The rest of the paper is structured as follows: after providing some related literature in Section 2 and some preliminaries in Section 3, we state and solve the designer's maximization problem in Section 4 and investigate her updated belief in Section 5. In Section 6 we introduce the equilibrium, and in Section 7 we consider the linear case. Section 8 concludes the paper with a brief discussion.

2 Related literature

This paper is related to the literature on optimal police search. Press (2009), Meng (2012) and Hoogstrate and Klaassen (2011, 2015) investigate the optimal search strategy for rare malfeasors, for instance at airport security checkpoints. These authors assume that there is exactly one criminal individual in a finite population and that beliefs are exogenous. They show that, similar to our solution of the designer's maximization problem, optimal search strategies which use only limited resources will be discriminatory. Knowles et al. (2001) and Persico and Todd (2005) investigate a game between the police and individuals, who decide whether or not to abide the law. They model prejudice as an exogenous difference in search costs and show that success rates are equal across appearances in equilibrium if search costs are equal, which suggests an empirical test of racial bias.[3] To the best of our knowledge, we are the first to study a model of police search in which prejudice may arise endogenously in the form of biased beliefs as a consequence of discrimination.

Discrimination will result in biased beliefs if the designer is a naïve statistician, that is, if she ignores discrepancies between sample and population distribution. Thus, our paper connects to the literature on sampling and statistical inference. In Osborne and Rubinstein (1998), players sample each of their available actions once and then choose the action with the best performance, ignoring that the samples are random; Spiegler (2006a,b) studies firm competition over consumers who choose according to this proce-

[3] The test uses success rates to distinguish between statistical discrimination due to differences in criminality and racist preferences. It is based on the idea that there is heterogeneity besides race, sex, age, etc. only visible to police officers, such that success rates can be expected to decrease as search rates increase. See also Anwar and Fang (2006), Dominitz and Knowles (2006) and Persico and Todd (2006) for generalizations of these models.

dure. In the sampling equilibrium of Osborne and Rubinstein (2003), players obtain a sample of other players' actions and best respond to the sample averages. In a recent contribution, Salant and Cherry (2020) incorporate a statistical inference procedure into sampling equilibrium. In contrast to these contributions, the sample distribution in our model is endogenous and may be biased due to discrimination.

3 Notation and preliminaries

Consider a population $N = [0,1]$. Individual $i \in N$'s *conduct* is an element $\alpha \in \{G, B\}$; her *appearance* is an element $\omega \in \Omega$, where Ω is finite. A *type* is a pair $(\alpha, \omega) \in \Theta \equiv \{G, B\} \times \Omega$ of a conduct and an appearance. Our interpretation of type (α, ω) is that ω consists of i's visible characteristics, such as race, sex, age, etc., while α captures whether i is "good" or "bad", i.e., abides the law or is criminal. Types are distributed according to some strictly positive distribution $p \in \Delta^\circ(\Theta)$. To keep notation simple, let $p(\omega) = p_\Omega(\omega) = p(G, \omega) + p(B, \omega)$ denote the marginal distribution over appearances.

We assume that the type of any $i \in N$ is exogenously given: individuals do not decide whether or not to abide the law. Thus, the objective of the *designer* in this paper is the detection of criminals rather than deterrence. For this purpose, she can randomly search individuals. The probability that an individual is searched can be contingent on the individual's appearance. Thus, a *search rule* is a map $\pi : \Omega \to \Delta(\{\text{search}, \text{don't search}\})$. To keep notation simple, we write $\pi^\omega = \pi(\text{search}|\omega)$, $\pi = (\pi^\omega)_{\omega\in\Omega} \in [0,1]^\Omega$, and we define $\bar{\pi} = \sum_{\omega\in\Omega} p(\omega)\,\pi^\omega$ as the average search probability given search rule π. We say that a search rule *discriminates against* appearance ω if $\pi^\omega > \bar{\pi}$.

Throughout the paper we work under the assumption that the true distribution p is unknown to the designer. Nevertheless, the designer has some belief $q \in \Delta(\Theta)$ about the distribution of types. While individuals' conducts are unknown, their appearances are not. Thus, we assume that q is correct at least across appearances.

Consistency. For all $\omega \in \Omega$ it holds that $q(\omega) = p(\omega)$.

For a given belief q the expected success rate and the expected variance of a search rule

π are given by

$$E = E(\pi) = \sum_{\omega \in \Omega} q(B|\omega)\, q(\omega)\, \pi^{\omega} \tag{1}$$

$$\text{and} \quad V = V(\pi) = \sum_{\omega \in \Omega} q(\omega)\, (\pi^{\omega} - \bar{\pi})^2, \tag{2}$$

respectively.

4 From disparities to discriminatory search rules

Suppose that the designer with given belief q has two objectives when choosing a search rule: first, to detect as many individuals with conduct B as possible, and second, to be not "too discriminatory", i.e., to use a search rule according to which the probabilities of being searched do not vary too much across appearances. Formally, let $U : [0,1]^2 \to \mathbb{R}$ be a quasi-concave and continuously differentiable utility function that maps the expected success rate E and the expected variance V of π to $U(E,V)$. We assume that $U_E > 0$ and $U_V < 0$, i.e., U is increasing in the expected success and decreasing in the expected variance of the search rule. At this point we can already provide some intuition about optimal behavior given such a rule: on the one hand, as $U_E > 0$, the designer will look for criminals where she deems it most promising, i.e., where $q(B|\omega)$ is large.[4] On the other hand, as $U_V < 0$, any discrepancies among the search probabilities for different appearances will negatively affect the designer's utility. (Maybe because such behavior comes with a flavor of discrimination and might lead to costly social or even legal sanctions.) Quasi-concavity of U means that, the more severely discrimination becomes, the more strongly the success rate must increase in order to make up for it in terms of utility.

With the notation above, the designer's maximization problem is

$$\max_{\pi} U(E(\pi), V(\pi)) \tag{3}$$

$$\text{s.t.} \quad 0 \le \pi^{\omega} \le 1 \text{ for all } \omega \in \Omega \tag{4}$$

$$\sum_{\omega \in \Omega} q(\omega)\, \pi^{\omega} \le k, \tag{5}$$

[4] Arrow (1973) coined the term *statistical discrimination* for such behavior.

where $k < 1$. The budget constraint (5) ensures that it is not optimal for the designer to search every individual with probability 1; otherwise the problem would be mathematically trivial and the solution useless for any practical purposes. Note that the budget constraint depends only on the correct parts of the designer's belief. In particular, it is equivalent to $\bar{\pi} \leq k$.

As this problem maximizes a continuous function on a compact set, it has a solution.[5] Our first result delivers an important property of this solution: a designer who *perceives* disparities in conduct across appearances (be her perception correct or not) will use a discriminatory search rule, and she will search those appearances more often that she believes to be more prone to being criminal. The proofs of all results can be found in the appendix.

Proposition 1. *For any given belief q, the maximization problem in (3)–(5) has a unique solution π, and this solution satisfies $\pi^{\omega} \geq \pi^{\omega'}$ whenever $q(B|\omega) \geq q(B|\omega')$. Moreover, there is $c \in \mathbb{R}$ such that $\pi^{\omega} \geq \bar{\pi}$ if and only if $q(B|\omega) \geq q(B) + c$. If π is an interior solution, then $c = 0$.*

A few remarks seem in order. First, the optimal search rule satisfies some monotonicity property with respect to the conditional beliefs $q(B|\omega)$. The more some appearance ω is perceived to have conduct B, the more likely an individual of this appearance will be searched.[6] Second, the constant c depends on the type of corner solutions: if, for instance, the budget k is small, then no appearance will be searched with probability 1, but some might be searched with probability 0. In this case c is positive, that is, an appearance ω will be searched disproportionally often if and only if the conditional belief $q(B|\omega)$ exceeds $q(B)$ by at least c; otherwise, ω will be searched less often than the average.

It is worth mentioning that interior solutions are particularly attractive for two reasons: first, here an appearance will be discriminated against, i.e., searched more often than the average, if and only if it is perceived to be positively correlated with conduct B. Second, for interior solutions the *(normalized) discrimination gap* of any appearance ω, that is,

[5] It is clear from the discussion above that for any solution of the maximization problem the budget constraint will be binding. When we later refer to an *interior solution* of the problem, we mean a solution for which the multipliers that correspond to the constraints in (4) are 0.

[6] The converse of this statement is not true as it might be the case, if k is very small, that $\pi^{\omega} = \pi^{\omega'} = 0$ even if $q(B|\omega) > q(B|\omega')$.

the relative difference between the average search probability and π^ω, is given by

$$\delta^\omega \equiv \frac{\pi^\omega - \bar{\pi}}{\bar{\pi}} = \frac{U_E(E,V)}{2kU_V(E,V)}\left(q(B) - q(B|\omega)\right) \tag{6}$$

and, hence, proportional to the difference between $q(B)$ and $q(B|\omega)$. The following corollary establishes a necessary and sufficient condition for the solution to be interior.

Corollary 2. *For any given belief q, the solution π of the maximization problem (3)–(5) is an interior solution if and only if*

$$-\frac{U_E(E,V)}{2U_V(E,V)}\left(q(B) - q(B|\omega)\right) \leq k \leq 1 - \frac{U_E(E,V)}{2U_V(E,V)}\left(q(B) - q(B|\omega)\right), \tag{7}$$

where E and V are defined as in (1) and (2), respectively. In particular, in this case, for all $\omega \in \Omega$ it holds that $\pi^\omega \geq \bar{\pi}$ if and only if $q(B|\omega) \geq q(B)$.

Of course, the conditions in Corollary 2 are difficult to verify ex ante for arbitrary utility functions U. However, if U is linear in both arguments, then (7) is independent of the solution π.

Example 3. Let $U(E,V) = aE - bV$ with $a \leq b$, and let $k \in \left[\frac{a}{2b}, 1 - \frac{a}{2b}\right]$. Then the conditions in Corollary 2 are satisfied for all beliefs q. In this case

$$\delta^\omega = \frac{a}{2bk}\left(q(B|\omega) - q(B)\right)$$

for all $\omega \in \Omega$. We refrain from any comparative statics at this point, as these will be dealt with in Section 7, where the designer's belief q will no longer be assumed exogenous. $\square$

5 From discriminatory search rules to biased beliefs

In the previous section we have found the optimal search rule for a given belief q. In this section we shall investigate how a given search rule may affect the designer's belief. The mathematically competent reader of our paper will probably argue that, as long as each appearance is searched with positive probability and the number of searches is sufficiently large, the designer will eventually learn the true distribution p, whatever her initial belief q might have been. The point we want to make here is that the designer might, unfortunately, not be that competent. In particular, she might behave as a naïve statistician and argue as follows:

> *I have conducted M searches, and a certain number R of these searches have led to the detection of an individual with conduct B. So, the overall criminality rate is $\tau(B) = \frac{R}{M}$. Moreover, among all those with conduct B, there were S individuals who had appearance ω. Thus, the conditional probability that an individual with conduct B has appearance ω is $\tau(\omega|B) = \frac{S}{R}$.*

Before we formalize these calculations, we want to point out the major flaws in the argument: the designer makes the implicit assumption that her sample is representative. The calculations are perfectly fine for calculating the (conditional) probabilities in the sample, they just might not be the same as in the overall population if the sample is biased. The second flaw is that the designer only considers individuals with conduct B. We will discuss the consequences of this feature below, after we have formalized the designer's argument.

Suppose the designer conducts searches according to some search rule π. Then the sample distribution is defined by

$$\tau^{\pi}(\alpha) = \frac{\sum_{\omega' \in \Omega} \pi^{\omega'} p(\omega') p(\alpha|\omega')}{\sum_{\omega' \in \Omega} \pi^{\omega'} p(\omega')} = \frac{\sum_{\omega' \in \Omega} \pi^{\omega'} p(\alpha, \omega')}{\bar{\pi}} \tag{8}$$

$$\tau^{\pi}(\omega|\alpha) = \frac{\pi^{\omega} p(\omega) p(\alpha|\omega)}{\sum_{\omega' \in \Omega} \pi^{\omega'} p(\omega') p(\alpha|\omega')} = \frac{\pi^{\omega} p(\alpha, \omega)}{\sum_{\omega' \in \Omega} \pi^{\omega'} p(\alpha, \omega')}, \tag{9}$$

for $\alpha = G, B$. Even if the search rule π is discriminatory, i.e., if $\pi^{\omega} > \bar{\pi}$ for some $\omega \in \Omega$, a designer who calculated τ^{π} according to these formulas for both G and B would soon realize that τ^{π} is not consistent, i.e., that her statistic is flawed. Indeed, in this case

$$\tau^{\pi}(\omega) = \tau^{\pi}(\omega|B)\,\tau^{\pi}(B) + \tau^{\pi}(\omega|G)\,\tau^{\pi}(G) = \frac{\pi^{\omega} p(B, \omega) + \pi^{\omega} p(G, \omega)}{\bar{\pi}} = \frac{\pi^{\omega}}{\bar{\pi}} p(\omega) \neq p(\omega).$$

So, a designer who discriminates against some appearances but who keeps book on all positive and negative findings while conducting searches will be able to figure out that something is wrong with the distribution τ^{π}.

But, as it happens, in practice there seldomly is a book for negative searches: only positive searches, that is, searches in which conduct B was detected, will be documented and counted. Our designer as well only makes her calculations for conduct B. This leaves plenty of room for errors that the designer will not be able to detect herself: she might have some belief q, conduct searches according to some rule π, calculate the (biased) distribution τ^{π} based on individuals with conduct B, and see herself confirmed in her belief if τ^{π} and q coincide. To make this statement more formal, we define:

Definition 4 (B-admissibility)**.** A belief $q \in \Delta(\Theta)$ is *B-admissible* (at π) if $q(B) = \tau^{\pi}(B)$ and $q(\omega|B) = \tau^{\pi}(\omega|B)$ for all $\omega \in \Omega$.

Our next proposition characterizes the search rules under which the designer can have a consistent B-admissible belief, such that the results of the search will confirm her belief.

Proposition 5. *For any search rule π, there is $q \in \Delta(\Theta)$ which is consistent and B-admissible at π if and only if*

$$\pi^{\omega} p(B|\omega) \leq \bar{\pi} \tag{10}$$

for all $\omega \in \Omega$. In this case, q is unique. Moreover, $q(B|\omega) \geq p(B|\omega)$ if and only if $\pi^{\omega} \geq \bar{\pi}$.

The first finding of this proposition deserves a brief discussion, as it does not seem entirely plausible at first sight. The necessity of (10) stems from the fact that if an appearance ω with high conditional probability $p(B|\omega)$ were searched too often, the conditional probability $q(B|\omega)$ would not be well-defined any more.

The last part of Proposition 5, on the other hand, is in line with our expectation: if an appearance ω is searched more often than the average, then the conditional probability that this appearance has conduct B is higher in the sample distribution than in the population distribution. This implies, in particular, that a consistent belief q that is B-admissible at some discriminatory search rule π is biased: there are at least two appearances for whom the conditional probabilities $q(B|\omega)$ are over- and underestimated, respectively.

6 Equilibrium beliefs and search rules

In the previous two sections we have made two observations: a designer who perceives disparities in conduct across appearances will use a discriminatory search rule; and a naïve statistician who uses a discriminatory search rule will develop a biased belief. In this section we will bring both parts of the problem together: we ask whether there are a belief and a search rule such that the former is B-admissible given the latter, and the latter is optimal given the former.

Definition 6. A pair (q, π) of a consistent belief q and a search rule π is an *equilibrium* if q is B-admissible at π, and π solves the maximization problem (3)–(5) for q.

And indeed: we answer our question with the affirmative. In particular, the designer will be discriminating against an appearance in equilibrium if and only if she overestimates the probability of this appearance's being criminal (compared to the true distribution).

Theorem 7. *Suppose that p and k are such that $p\,(B|\omega) \leq k$ for all $\omega \in \Omega$. Then there is an equilibrium (q, π). If, additionally, $k \in \left[-\frac{U_E(E,V)}{2U_V(E,V)}, 1 + \frac{U_E(E,V)}{2U_V(E,V)}\right]$ holds at equilibrium, then the following are equivalent:*

(i) $q\,(B|\omega) \geq q\,(B)$,

(ii) $q\,(B|\omega) \geq p\,(B|\omega)$,

(iii) $\pi^\omega \geq \bar{\pi}$.

The conditions in Theorem 7 are sufficient but not necessary. The additional conditions on U and k in the second part ensure that at equilibrium the optimal search rule is an interior solution of the maximization problem (3)–(5). In this case an appearance will be discriminated against by π if and only if it is perceived to be criminal more often than the average. Observe, however, that Theorem 7 does not guarantee that appearances which are more prone to have conduct B according to the true distribution p are searched more often than others in equilibrium. That is, without further assumptions it might be the case that $p\,(B|\omega) > p\,(B|\omega')$ and, at the same time, $q\,(B|\omega) < q\,(B|\omega')$ and $\pi^\omega < \pi^{\omega'}$. (Recall that the latter two inequalities are equivalent in equilibrium by Proposition 1.)

We close this section with a brief discussion on whether the true distribution p might arise in equilibrium. The answer is: it might, but generically it does not.

Corollary 8. *There is an equilibrium (q, π) with $q = p$ if and only if $p\,(B|\omega) = p(B)$ for all $\omega \in \Omega$.*

So, indeed, if appearance and conduct are correlated, then the correct distribution p cannot be part of an equilibrium. This means that, generically speaking, whenever our naïve statistician feels her belief confirmed, she must be wrong.

7 The linear case

We have mentioned before that without further restrictions we might find an equilibrium in which an appearance ω is believed to be more criminal than another appearance ω', while in fact it is the other way around. A class of utility functions which rules out such a possibility are the linear functions that we considered in Example 3. In particular, in this case perceived disparities are larger than actual disparities.

Theorem 9. *Let $\Omega = \{\omega^1, \ldots, \omega^m\}$ and let p be such that $p(B|\omega^1) \geq \ldots \geq p(B|\omega^m)$ with at least one strict inequality. Let further $U(E, V) = aE - bV$ with $a < 2b$, and let k be such that $p(B|\omega^1) \leq k \leq 1 - (1 - p(\omega^1))\frac{a}{2b}$. Then there is a unique equilibrium (q, π), and this equilibrium has the following properties:*

(i) $q(B|\omega^1) \geq \ldots \geq q(B|\omega^m)$ and $\pi^{\omega^1} \geq \ldots \geq \pi^{\omega^m}$ with at least one strict inequality in each chain;

(ii) there is $m^* \geq 1$ such that $\pi^{\omega^\ell} > \bar{\pi}$ for all $\ell \leq m^*$ and $\pi^{\omega^\ell} \leq \bar{\pi}$ for all $\ell > m^*$;

(iii) if $\frac{a}{2bk}$ increases (within the boundaries above), then $q(B)$ increases, m^* (weakly) decreases, and δ^{w^1} increases.

Part (iii) of Theorem 9 is particularly intriguing. A decrease in budget k, ceteris paribus, causes an increase in $\frac{a}{2bk}$ and thereby an increase in the perceived average crime rate. At the same time, that part of the population whose crime rate is underestimated—that is, appearances ω with $\pi^\omega < \bar{\pi}$—increases as well. This means that those who remain facing discrimination will be perceived even more criminal. In particular, the discrimination gap (6) of those individuals who face the strongest discrimination will further increase, and they will hence suffer most from budget cuts. The intuition behind this mechanism is easily explained: as budgets are cut, i.e., as fewer people will be searched, incentives become stronger to search where it seems most promising. But this will shift searches towards those who have been discriminated against most before. Observe that a decrease in k has exactly the same consequences as a decrease in b, the parameter that determines the sanction for discrimination. Thus, increasing sanctions will have exactly the opposite effect: more appearances will be discriminated against, but less harshly, and the overestimation of the crime rate will be reduced.

The above discussion suggests that budget cuts or lower sanctions may overall increase discrimination. To investigate this question, we introduce a measure of discrimination that is comparable across different average search probabilities (and hence budgets). The *normalized expected variance*

$$\widetilde{V}(\pi) \equiv \sum_{\omega\in\Omega} q(\omega)\left(\frac{\pi^{\omega}-\bar{\pi}}{\bar{\pi}}\right)^{2}$$

of a search rule π measures the dispersion of search probabilities relative to the average search probability. We show that indeed budget cuts or lower sanctions increase the normalized expected variance.

Corollary 10. *Under the conditions of Theorem 9, the unique equilibrium (q,π) is such that $\widetilde{V}(\pi)$ is increasing in $\frac{a}{2bk}$ (within the stated boundaries).*

8 Discussion

In this paper we studied a designer who distributes resources to the searches of certain types. She is doing so by maximizing, subject to a budget constraint, a utility function that depends both on the success rate of her searches and the discrepancies in search probabilities across appearances. When evaluating the conducted searches, she makes the mistake of taking the distribution of her sample for the population distribution—which is wrong as long as she does not search all appearances with equal probability. Generically speaking, any belief that is self-confirming in the sense that it is based on the result of searches which in turn are based on the belief must be incorrect. In particular, the designer overestimates the probability with which an appearance is criminal if and only if she is discriminating this appearance. Moreover, we show that in a linear model, as the budget constraint becomes tighter, both the overestimation of the crime rate and the situation of those who face the strongest discrimination will worsen. The latter finding should be strongly considered when campaigning for the defunding of police forces due to alleged racial discrimination. At the same time, sanctioning discriminatory search rules will have the opposite effect: in this case more appearances will be discriminated, but they will be discriminated less strongly, and the overestimation of the crime rate will be reduced.

References

Abrahams, S. (2020). Officer differences in traffic stops of minority drivers. *Labour Economics 67*, 101912.

Anwar, S. and H. Fang (2006). An alternative test of racial prejudice in motor vehicle searches: Theory and evidence. *American Economic Review 96*(1), 127–151.

Arrow, K. (1973). The theory of discrimination. In O. Ashenfelter and A. Rees (Eds.), *Discrimination in Labor Markets*, pp. 3–33. Princeton University Press.

Chiricos, T., R. McEntire, and M. Gertz (2001). Perceived racial and ethnic composition of neighborhood and perceived risk of crime. *Social Problems 48*(3), 322–340.

Dominitz, J. and J. Knowles (2006). Crime minimisation and racial bias: what can we learn from police search data? *The Economic Journal 116*(515), F368–F384.

Fiedler, K. (2000). Beware of samples! A cognitive-ecological sampling approach to judgment biases. *Psychological Review 107*(4), 659–676.

Fiedler, K. (2012). *Psychology of Learning and Motivation*, Volume 57, Chapter 1, pp. 1–55. Elsevier.

Fiedler, K., M. Hütter, M. Schott, and F. Kutzner (2019). Metacognitive myopia and the overutilization of misleading advice. *Journal of Behavioral Decision Making 32*(3), 317–333.

Hoogstrate, A. J. and C. A. Klaassen (2011). Minimizing the average number of inspections for detecting rare items in finite populations. In *2011 European Intelligence and Security Informatics Conference*, pp. 203–208. IEEE.

Hoogstrate, A. J. and C. A. Klaassen (2015). Information weighted sampling for detecting rare items in finite populations with a focus on security. *European Journal of Operational Research 241*(3), 880–887.

Horrace, W. C. and S. M. Rohlin (2016). How dark is dark? bright lights, big city, racial profiling. *Review of Economics and Statistics 98*(2), 226–232.

Hurwitz, J. and M. Peffley (1997). Public perceptions of race and crime: The role of racial stereotypes. *American Journal of Political Science 41*(2), 375–401.

Juslin, P., A. Winman, and P. Hansson (2007). The naïve intuitive statistician: A naïve sampling model of intuitive confidence intervals. *Psychological Review 114*(3), 678.

Knowles, J., N. Persico, and P. Todd (2001). Racial bias in motor vehicle searches: Theory and evidence. *Journal of Political Economy 109*(1), 203–229.

Lang, K. and A. Kahn-Lang Spitzer (2020). Race discrimination: An economic perspective. *Journal of Economic Perspectives 34*(2), 68–89.

Meng, X.-L. (2012). Enhanced security checks at airports: minimizing time to detection or probability of escape? *Stat 1*(1), 42–52.

Osborne, M. J. and A. Rubinstein (1998). Games with procedurally rational players. *American Economic Review 88*(4), 834–847.

Osborne, M. J. and A. Rubinstein (2003). Sampling equilibrium, with an application to strategic voting. *Games and Economic Behavior 45*(2), 434–441.

Persico, N. and P. Todd (2006). Generalising the hit rates test for racial bias in law enforcement, with an application to vehicle searches in Wichita. *The Economic Journal 116*(515), F351–F367.

Persico, N. and P. E. Todd (2005). Passenger profiling, imperfect screening, and airport security. *American Economic Review: Papers and Proceedings 95*(2), 127–131.

Pierson, E., C. Simoiu, J. Overgoor, S. Corbett-Davies, D. Jenson, A. Shoemaker, V. Ramachandran, P. Barghouty, C. Phillips, R. Shroff, and S. Goel (2020). A large-scale analysis of racial disparities in police stops across the United States. *Nature Human Behaviour 4*, 1–10.

Press, W. H. (2009). Strong profiling is not mathematically optimal for discovering rare malfeasors. *Proceedings of the National Academy of Sciences 106*(6), 1716–1719.

Salant, Y. and J. Cherry (2020). Statistical inference in games. *Econometrica 88*(4), 1725–1752.

Sanga, S. (2009). Reconsidering racial bias in motor vehicle searches: Theory and evidence. *Journal of Political Economy 117*(6), 1155–1159.

Spiegler, R. (2006a). Competition over agents with boundedly rational expectations. *Theoretical Economics 1*(2), 207–231.

Spiegler, R. (2006b). The market for quacks. *The Review of Economic Studies 73*(4), 1113–1131.

Welch, K. (2007). Black criminal stereotypes and racial profiling. *Journal of Contemporary Criminal Justice 23*(3), 276–288.

A Appendix

Proof of Proposition 1. For uniqueness it is sufficient to show that U is strictly quasi-concave in π. To this end note that E is linear and V is strictly convex in π. So, for any two search rules π_1, π_2 and $\alpha \in (0,1)$, it holds that

$$U\left(x\left(\alpha\pi_1 + (1-\alpha)\pi_2\right)\right) > U\left(\alpha x\left(\pi_1\right) + (1-\alpha)x\left(\pi_2\right)\right) \geq \min\{U\left(x\left(\pi_1\right)\right), U\left(x\left(\pi_2\right)\right)\},$$

where the first inequality follows from U's being strictly decreasing in V, and the second inequality follows from U's being quasi-concave in x. So, the solution is unique.

For any $\omega \in \Omega$, let $\underline{\lambda}^\omega, \bar{\lambda}^\omega$ denote the Lagrange multipliers that correspond to the constraints $\pi^\omega \geq 0$ and $\pi^\omega \leq 1$, respectively; and let μ denote the multiplier that corresponds to (5). Then, using that $q(\omega) = p(\omega)$ for all $\omega \in \Omega$, the first order conditions are given as

$$q\left(B|\omega\right)p(\omega)U_E\left(E,V\right) + 2p\left(\omega\right)\left(\pi^\omega - \bar{\pi}\right)U_V\left(E,V\right) + \underline{\lambda}^\omega - \bar{\lambda}^\omega - \mu p\left(\omega\right) = 0$$

for all $\omega \in \Omega$. Summing up over all $\omega \in \Omega$ delivers

$$q(B)U_E\left(E,V\right) + \sum_{\omega\in\Omega}\left(\underline{\lambda}^\omega - \bar{\lambda}^\omega\right) = \mu.$$

Thus,

$$q\left(B|\omega\right)p(\omega)U_E\left(E,V\right) + 2p\left(\omega\right)\left(\pi^\omega - \bar{\pi}\right)U_V\left(E,V\right) + \underline{\lambda}^\omega - \bar{\lambda}^\omega$$

$$= p(\omega)\left(q(B)U_E(E,V) + \sum_{\omega\in\Omega}\left(\underline{\lambda}^{\omega} - \bar{\lambda}^{\omega}\right)\right)$$

or, equivalently,

$$\pi^{\omega} - \bar{\pi} = \frac{U_E(E,V)\left(q(B) - q(B|\omega)\right) + \sum_{\omega'\in\Omega}\left(\underline{\lambda}^{\omega'} - \bar{\lambda}^{\omega'}\right)}{2U_V(E,V)} - \frac{\underline{\lambda}^{\omega} - \bar{\lambda}^{\omega}}{2p(\omega)U_V(E,V)}. \tag{11}$$

We first show that $\pi^{\omega} \geq \pi^{\omega'}$ whenever $q(B|\omega) \geq q(B|\omega')$. So, let $q(B|\omega) \geq q(B|\omega')$. Suppose first that $\underline{\lambda}^{\omega} = \bar{\lambda}^{\omega} = 0$ and recall that $U_V(E,V) < 0$. Assume for a moment that $\bar{\lambda}^{\omega'} > 0$, i.e., $\pi^{\omega'} = 1$. Then the right hand side of (11) is strictly smaller for ω' than for ω. Thus, $1 = \pi^{\omega'} < \pi^{\omega} \leq 1$, which is impossible. So, $\bar{\lambda}^{\omega'} = 0$ must hold. If $\underline{\lambda}^{\omega'} > 0$, then $\pi^{\omega'} = 0 \leq \pi^{\omega}$ as required. If $\underline{\lambda}^{\omega'} = 0$, then the right hand side of (11) is weakly smaller for ω' than for ω, so that $\pi^{\omega'} \leq \pi^{\omega}$. Suppose next that $\bar{\lambda}^{\omega} > 0$. Then $\pi^{\omega} = 1 \geq \pi^{\omega'}$. Finally, suppose that $\underline{\lambda}^{\omega} > 0$, i.e., $\pi^{\omega} = 0$. Assume that $\pi^{\omega'} > 0$, that is, $\underline{\lambda}^{\omega'} = 0$. Then the right hand side of (11) is strictly smaller for ω' than for ω, so that $\pi^{\omega'} < \pi^{\omega} = 0$, which is impossible. So, $\pi^{\omega} \geq \pi^{\omega'}$ whenever $q(B|\omega) \geq q(B|\omega')$.

We next show that $\pi^{\omega} \geq \bar{\pi}$ if and only if

$$q(B|\omega) \geq q(B) + \frac{1}{U_E(E,V)}\sum_{\omega'\in\Omega}\left(\underline{\lambda}^{\omega'} - \bar{\lambda}^{\omega'}\right). \tag{12}$$

Indeed, by Equation (11), and since $U_V < 0$, this is true if $\underline{\lambda}^{\omega} = \bar{\lambda}^{\omega} = 0$, i.e., if $\pi^{\omega} \in (0,1)$. Suppose that $\underline{\lambda}^{\omega} > 0$, i.e., $\pi^{\omega} = 0$ and $\bar{\lambda}^{\omega} = 0$. Then $\pi^{\omega} < \bar{\pi}$ is true. On the other hand, the left hand side of (11) is negative so that the reverse of (12) must hold as well. Similarly, if $\bar{\lambda}^{\omega} > 0$, i.e., $\pi^{\omega} = 1$ and $\underline{\lambda}^{\omega} = 0$, then $\pi^{\omega} > \bar{\pi}$. Thus, the left hand side of (11) is positive and the strict version of (12) must hold. Hence, the proposition is true with $c = \frac{1}{U_E(E,V)}\sum_{\omega'\in\Omega}\left(\underline{\lambda}^{\omega'} - \bar{\lambda}^{\omega'}\right)$. In particular, if π is an interior solution, i.e., if $\underline{\lambda}^{\omega} = \bar{\lambda}^{\omega} = 0$ for all $\omega \in \Omega$, then $c = 0$. ■

Proof of Corollary 2. It is sufficient to show that a solution to the maximization problem that consists only of (3) and (5) satisfies the constraints in (4) if and only if the inequalities in (7) hold. From (11) one finds that any solution to the problem without the constraints in (4) satisfies

$$\pi^{\omega} = k + \frac{U_E(E,V)}{2U_V(E,V)}\left(q(B) - q(B|\omega)\right)$$

for all $\omega \in \Omega$, where we use that the budget constraint (5) is binding. In particular, we find that $k \geq -\frac{U_E(E,V)}{2U_V(E,V)}\left(q(B) - q(B|\omega)\right)$ if and only if

$$\pi^{\omega} \geq \frac{U_E(E,V)}{2U_V(E,V)}\left(q(B) - q(B|\omega) - q(B) + q(B|\omega)\right) = 0.$$

Similarly, $k \leq 1 - \frac{U_E(E,V)}{2U_V(E,V)}\left(q(B) - q(B|\omega)\right)$ if and only if

$$\pi^{\omega} \leq 1 - \frac{U_E(E,V)}{2U_V(E,V)}\left(q(B) - q(B|\omega)\right) + \frac{U_E(E,V)}{2U_V(E,V)}\left(q(B) - q(B|\omega)\right) = 1.$$

Thus, π is an interior solution of the maximization problem (3)–(5) if and only if it is a solution that satisfies (7). ■

Proof of Proposition 5. Consistency and B-admissibility together uniquely define q by

$$\begin{aligned} q(B,\omega) &= \tau^{\pi}(\omega|B)\,\tau^{\pi}(B) = \frac{\pi^{\omega}}{\bar{\pi}}p(B,\omega) \\ q(G,\omega) &= p(\omega) - \tau^{\pi}(\omega|B)\,\tau^{\pi}(B) = p(\omega) - \frac{\pi^{\omega}}{\bar{\pi}}p(B,\omega) \end{aligned} \tag{13}$$

for all $\omega \in \Omega$. We have to show that q is a probability distribution if and only if (10) is satisfied. We have to show that q is a probability distribution if and only if (10) is satisfied. Independently of (10) we have that $q(B,\omega) \geq 0$ for all $\omega \in \Omega$ and

$$\sum_{\omega\in\Omega} q(B,\omega) + q(G,\omega) = \sum_{\omega\in\Omega} p(\omega) = 1.$$

Moreover, $q(G,\omega) \geq 0$ if and only if $p(\omega) \geq \frac{\pi^{\omega}}{\bar{\pi}}p(B,\omega)$, which is equivalent to (10). By consistency of q we have that $q(B|\omega) \geq p(B|\omega)$ if and only if $q(B,\omega) \geq p(B,\omega)$, which is equivalent to $\pi^{\omega} \geq \bar{\pi}$ by (13). ■

Proof of Theorem 7. Let Π be the set of search rules that satisfy $\bar{\pi} = k$. Let $\phi : \Delta(\Theta) \rightarrow \Pi$ be the map that maps q to the optimal search rule π with respect to the problem

(3)–(5). This is well defined as the budget constraint (5) ensures that $\pi \in \Pi$. Moreover, as the objective function in (3) is continuous in q, and the constraints are independent of q, ϕ is continuous in q.

Let $\psi : \Pi \to \Delta(\Theta)$ be the map that maps π to the unique B-admissible $q \in \Delta(\Theta)$. Observe that this is well defined by Proposition 5 as $\pi^\omega p(B|\omega) \leq p(B|\omega) \leq k = \bar{\pi}$ for all $\pi \in \Pi$. In particular, ψ is continuous in π.

The map $\phi \circ \psi : \Pi \to \Pi$ is, thus, a continuous map from a compact convex set into itself. Hence, it has a fixed point π and a corresponding belief $q = \psi(\pi)$, such that, by construction, q is B-admissible at π and π solves the maximization problem (3)–(5).

If, additionally, $k \in \left[-\frac{U_E(E,V)}{2U_V(E,V)}, 1 + \frac{U_E(E,V)}{2U_V(E,V)}\right]$ at equilibrium, then π is an interior solution by Corollary 2. Thus, by Proposition 1, (i) and (iii) are equivalent; and by Proposition 5, (ii) and (iii) are equivalent. ■

Proof of Corollary 8. If $p(B|\omega) = p(B)$ for all $\omega \in \Omega$ one finds that the search rule π with $\pi^\omega = k = \bar{\pi}$ for all $\omega \in \Omega$ solves the maximization problem (3)–(5). For this search rule Equations (8) and (9) deliver $\tau^\pi = p$.

On the other hand, suppose that there is ω with $p(B|\omega) \neq p(B)$ and assume that there is an equilibrium (p, π). Then there are ω, ω' such that $p(B|\omega) > p(B|\omega')$, which implies $\pi^\omega > \pi^{\omega'}$ by Proposition 1. Thus, as not all π^ω are equal, there is ω^* such that $\pi^{\omega^*} > \bar{\pi}$. By Proposition 5, $p(B|\omega^*) = \tau^\pi(B|\omega^*) > p(B|\omega^*)$, which is impossible. ■

Proof of Theorem 9. Since $k \geq p(B|\omega^1) \geq p(B|\omega)$ for all $\omega \in \Omega$, an equilibrium exist by Theorem 7. Because of Example 3, Equation (13), and the consistency of q, any equilibrium (q, π) such that π is an inner solution of the maximization problem in (3)–(5) must satisfy the equation system

$$\pi^\omega = k + \frac{a}{2b}\left(q(B|\omega) - q(B)\right) \tag{14}$$

$$q(B|\omega) = \frac{\pi^\omega}{k} p(B|\omega) \tag{15}$$

$$q(B) = \sum_{\omega \in \Omega} p(\omega)\, q(B|\omega)\,. \tag{16}$$

We show that this equation system has a unique solution (q, π) and that this solution is an equilibrium. This then implies that π is an interior solution for the maximization problem in (3)–(5).

Substituting (15) into (14) and solving for π^ω delivers

$$\pi^\omega = \frac{k - \frac{a}{2b}q(B)}{1 - \frac{a}{2b}\frac{p(B|\omega)}{k}} = k\frac{1 - \frac{a}{2bk}q(B)}{1 - \frac{a}{2bk}p\left(B|\omega\right)}$$

and

$$q\left(B|\omega\right) = p\left(B|\omega\right)\frac{1 - \frac{a}{2bk}q(B)}{1 - \frac{a}{2bk}p\left(B|\omega\right)}. \tag{17}$$

(Since $a < 2b$ and $k \geq p\left(B|\omega\right)$ for all $\omega \in \Omega$ all fractions are well defined.) Substituting (17) into (16) delivers

$$\begin{aligned} q\left(B\right) &= \sum_{\omega\in\Omega} p\left(\omega\right) p\left(B|\omega\right)\frac{1 - \frac{a}{2bk}q(B)}{1 - \frac{a}{2bk}p\left(B|\omega\right)} \\ &= \sum_{\omega\in\Omega}\frac{p\left(\omega\right) p\left(B|\omega\right)}{1 - \frac{a}{2bk}p\left(B|\omega\right)} - q(B)\frac{a}{2bk}\sum_{\omega\in\Omega}\frac{p\left(\omega\right) p\left(B|\omega\right)}{1 - \frac{a}{2bk}p\left(B|\omega\right)}, \end{aligned}$$

and solving for $q(B)$ we find

$$q(B) = \sum_{\omega\in\Omega} p\left(\omega\right)\frac{\frac{p(B|\omega)}{1-\frac{a}{2bk}p(B|\omega)}}{1 + \frac{a}{2bk}\sum_{\omega'\in\Omega}\frac{p(\omega')p(B|\omega')}{1-\frac{a}{2bk}p(B|\omega')}}.$$

Thus, with (17) we find

$$\begin{aligned} q\left(B|\omega\right) &= \frac{p\left(B|\omega\right)}{1 - \frac{a}{2bk}p\left(B|\omega\right)}\left(1 - \frac{a}{2bk}q(B)\right) \\ &= \frac{p\left(B|\omega\right)}{1 - \frac{a}{2bk}p\left(B|\omega\right)}\left(1 - \frac{\frac{a}{2bk}\sum_{\omega'\in\Omega}\frac{p(\omega')p(B|\omega')}{1-\frac{a}{2bk}p(B|\omega')}}{1 + \frac{a}{2bk}\sum_{\omega'\in\Omega}\frac{p(\omega')p(B|\omega')}{1-\frac{a}{2bk}p(B|\omega')}}\right) \\ &= \frac{p\left(B|\omega\right)}{1 - \frac{a}{2bk}p\left(B|\omega\right)}\frac{1}{1 + \frac{a}{2bk}\sum_{\omega'\in\Omega}\frac{p(\omega')p(B|\omega')}{1-\frac{a}{2bk}p(B|\omega')}} \\ &= \frac{\frac{p(B|\omega)}{1-\frac{a}{2bk}p(B|\omega)}}{1 + \frac{a}{2bk}\sum_{\omega'\in\Omega}\frac{p(\omega')p(B|\omega')}{1-\frac{a}{2bk}p(B|\omega')}}. \end{aligned} \tag{18}$$

Together with (15) we see that q and π are uniquely determined. It is left to show that they form an equilibrium, i.e., that π is a well-defined search rule and that q is a probability distribution. Observe from (18) that $q\left(B|\omega\right) \geq 0$ for all $\omega \in \Omega$ since $a < 2b$ and $k \geq p\left(B|\omega\right)$ for all $\omega \in \Omega$. Hence, by (15), $\pi^{\omega} \geq 0$ for all $\omega \in \Omega$. Moreover, by (18) it holds that

$$q\left(B|\omega^1\right) \geq \ldots \geq q\left(B|\omega^m\right), \tag{19}$$

and, thus, by (15),

$$\pi^{\omega^1} \geq \ldots \geq \pi^{\omega^m}. \tag{20}$$

In order to show that π is a search rule, it is therefore sufficient to show that $\pi^{\omega^1} \leq 1$. Equations (15) and (18) together with $p\left(B|\omega^1\right) \leq k$ yield

$$\begin{aligned}
\pi^{\omega^1} &= \frac{k}{p\left(B|\omega^1\right)} q\left(B|\omega^1\right) = \frac{\frac{k}{1-\frac{a}{2bk}p(B|\omega^1)}}{1+\frac{a}{2bk}\sum_{\omega'\in\Omega}\frac{p(\omega')p(B|\omega')}{1-\frac{a}{2bk}p(B|\omega')}} \\
&\leq \frac{\frac{k}{1-\frac{a}{2bk}p(B|\omega^1)}}{1+\frac{a}{2bk}\frac{p(\omega^1)p(B|\omega^1)}{1-\frac{a}{2bk}p(B|\omega^1)}} = \frac{k}{1-\left(1-p\left(\omega^1\right)\right)\frac{a}{2bk}p\left(B|\omega^1\right)} \\
&\leq \frac{k}{1-\left(1-p\left(\omega^1\right)\right)\frac{a}{2b}} \leq 1,
\end{aligned}$$

where the last inequality follows from the upper bound on k. So, π is indeed a search rule. Finally, $q\left(B|\omega\right) = \pi^{\omega}\frac{p(B|\omega)}{k} \leq \pi^{\omega} \leq 1$ by (15). Thus, (q, π) is indeed an equilibrium. Moreover, we have shown claim (i) in Equations (19) and (20).

The second assertion follows immediately from the first as $\pi^{\omega^1} > \bar{\pi} > \pi^{\omega^m}$.

We next show that $q(B)$ increases as $\frac{a}{2bk}$ increases. Let $f_{\omega}(x) = \frac{p(\omega)p(B|\omega)}{1-p(B|\omega)x}$ and $g\left(x\right) = \sum_{\omega\in\Omega}\frac{f_{\omega}(x)}{1+x\sum_{\omega'}f_{\omega'}(x)}$, and observe that $q(B) = g\left(\frac{a}{2bk}\right)$. We have that

$$f'_{\omega}(x) = \frac{p\left(\omega\right)p\left(B|\omega\right)^2}{\left(1-p\left(B|\omega\right)x\right)^2} = \frac{p\left(B|\omega\right)}{1-p\left(B|\omega\right)x}f_{\omega}(x)$$

and

$$\left(xf_{\omega}(x)\right)' = f_{\omega}(x) + xf'_{\omega}(x) = \frac{1}{1-p\left(B|\omega\right)x}f_{\omega}(x).$$

Thus,

$$
\begin{aligned}
g'(x) =& \left(\sum_{\omega\in\Omega} \frac{f_\omega(x)}{1+x\sum_{\omega'\in\Omega} f_{\omega'}(x)}\right)' \\
=& \left(1+x\sum_{\omega'\in\Omega} f_{\omega'}(x)\right)^{-2} \\
& \sum_{\omega\in\Omega}\left[\frac{p\left(B|\omega\right)}{1-p\left(B|\omega\right)x} f_\omega(x)\left(1+x\sum_{\omega'\in\Omega} f_{\omega'}(x)\right) - f_\omega(x)\sum_{\omega'\in\Omega}\frac{f_{\omega'}(x)}{1-p\left(B|\omega'\right)x}\right] \\
=& \left(1+x\sum_{\omega'\in\Omega} f_{\omega'}(x)\right)^{-2}\left[\sum_{\omega\in\Omega} f_\omega(x)\left(\frac{p\left(B|\omega\right)x}{1-p\left(B|\omega\right)x}\sum_{\omega'\in\Omega} f_{\omega'}(x) - \sum_{\omega'\in\Omega}\frac{f_{\omega'}(x)}{1-p\left(B|\omega'\right)x}\right)\right. \\
& \left. + \sum_{\omega\in\Omega}\frac{p\left(B|\omega\right)}{1-p\left(B|\omega\right)x} f_\omega(x)\right]. \\
=& \left(1+x\sum_{\omega'\in\Omega} f_{\omega'}(x)\right)^{-2}\left[\sum_{\omega'\in\Omega} f_{\omega'}(x)\sum_{\omega\in\Omega}\frac{p\left(B|\omega\right)x-1}{1-p\left(B|\omega\right)x} f_\omega(x) + \sum_{\omega\in\Omega}\frac{p\left(B|\omega\right)}{1-p\left(B|\omega\right)x} f_\omega(x)\right] \\
=& \left(1+x\sum_{\omega'\in\Omega} f_{\omega'}(x)\right)^{-2}\sum_{\omega\in\Omega} f_\omega(x)\left(-\sum_{\omega'\in\Omega} f_{\omega'}(x) + \frac{p\left(B|\omega\right)}{1-p\left(B|\omega\right)x}\right) \\
=& \left(1+x\sum_{\omega'\in\Omega} f_{\omega'}(x)\right)^{-2}\left(\sum_{\omega\in\Omega} p\left(\omega\right)\left(\frac{p\left(B|\omega\right)}{1-p\left(B|\omega\right)x}\right)^2 - \left(\sum_{\omega\in\Omega} p\left(\omega\right)\frac{p\left(B|\omega\right)}{1-p\left(B|\omega\right)x}\right)^2\right) \\
>& 0,
\end{aligned}
$$

where the last inequality comes from Jensen's inequality. So, $q(B)$ is indeed increasing in $\frac{a}{2bk}$.

We show that m^* decreases as $\frac{a}{2bk}$ increases. For this purpose, by the continuity of the solution in $\frac{a}{2bk}$, it is sufficient to show that if $\pi^\omega = \bar{\pi}$, i.e., if ω is exactly on the brink of being discriminated against, ω will not be discriminated after an increase in $\frac{a}{2bk}$. So, let $\pi^\omega = \bar{\pi}$. By Equations (15) and (18) this is equivalent to

$$
\frac{p\left(B|\omega\right)}{1-\frac{a}{2bk}p\left(B|\omega\right)} = \sum_{\omega'\in\Omega} p\left(\omega'\right)\frac{p\left(B|\omega'\right)}{1-\frac{a}{2bk}p\left(B|\omega'\right)}. \tag{21}
$$

Let $x = \frac{a}{2bk}$. Differentiating the left hand side with respect to x yields

$$\begin{aligned}\frac{d}{dx}\left[\frac{p\left(B|\omega\right)}{1-xp\left(B|\omega\right)}\right] = \frac{\left(p\left(B|\omega\right)\right)^2}{\left(1-xp\left(B|\omega\right)\right)^2} &= \left(\sum_{\omega'\in\Omega} p\left(\omega'\right)\frac{p\left(B|\omega'\right)}{1-xp\left(B|\omega'\right)}\right)^2 \\ &< \sum_{\omega'\in\Omega} p\left(\omega'\right)\left(\frac{p\left(B|\omega'\right)}{1-xp\left(B|\omega'\right)}\right)^2 \\ &= \frac{d}{dx}\left[\sum_{\omega'\in\Omega} p\left(\omega'\right)\frac{p\left(B|\omega'\right)}{1-xp\left(B|\omega'\right)}\right],\end{aligned}$$

where the second equality follows from (21). Thus, ω will not be discriminated after an increase in $\frac{a}{2bk}$.

Finally, let again $x = \frac{a}{2bk}$ and observe from Equation (17) that

$$q\left(B|\omega\right) - q(B) = p\left(B|\omega\right)\frac{1-xq(B)}{1-xp\left(B|\omega\right)} - q(B)\frac{1-xp\left(B|\omega\right)}{1-xp\left(B|\omega\right)} = \frac{p\left(B|\omega\right)-q(B)}{1-xp\left(B|\omega\right)}.$$

Hence, as π is an interior solution of the designer's maximization problem, Equation (6) delivers

$$\delta^\omega = x\left(q\left(B\right) - q\left(B|\omega\right)\right) = x\frac{p\left(B|\omega\right)-q(B)}{1-xp\left(B|\omega\right)}.$$

Thus, we find

$$\begin{aligned}\frac{d\delta^\omega}{dx} &= \frac{\left(p\left(B|\omega\right)-q(B)+x\left(-\frac{dq(B)}{dx}\right)\right)\left(1-xp\left(B|\omega\right)\right) - x\left(p\left(B|\omega\right)-q(B)\right)\left(-p\left(B|\omega\right)\right)}{\left(1-xp\left(B|\omega\right)\right)^2} \\ &= \frac{p\left(B|\omega\right)-q(B)-x\frac{dq(B)}{dx}\left(1-xp\left(B|\omega\right)\right)}{\left(1-xp\left(B|\omega\right)\right)^2} \\ &= \frac{p\left(B|\omega\right)\left(1+x^2\frac{dq(B)}{dx}\right)-q(B)-x\frac{dq(B)}{dx}}{\left(1-xp\left(B|\omega\right)\right)^2}.\end{aligned}$$

for all $\omega \in \Omega$. Observe that $\frac{\partial\delta^\omega}{\partial x} \geq 0$ implies $\frac{\partial\delta^{\omega'}}{\partial x} > \frac{\partial\delta^\omega}{\partial x}$ for all $\omega' \in \Omega$ with $p\left(B|\omega'\right) > p\left(B|\omega\right)$. In particular, it is impossible that $\frac{\partial\delta^\omega}{\partial x} = 0$ for all $\omega \in \Omega$ since $p\left(B|\omega^1\right) > p\left(B|\omega^m\right)$. Furthermore, since $\sum_{l=1}^n p\left(\omega^l\right)\delta^{\omega^l} = 0$, at least one of the derivatives must be positive and one must be negative. But this implies that $\frac{\partial\delta^{\omega^1}}{\partial x} > 0$. ■

Proof of Corollary 10. Let $x = \frac{a}{2bk}$ and $h_\omega(x) = \frac{1}{1-xp(B|\omega)}$, then (18) simplifies to

$$q(B|\omega) = \frac{p(B|\omega)\, h_\omega(x)}{1 + x\sum_{\omega'\in\Omega} p(\omega')\, p(B|\omega')\, h_{\omega'}(x)} = \frac{p(B|\omega)\, h_\omega(x)}{\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)}.$$

Together with (15) we obtain

$$\pi^\omega = k\frac{q(B|\omega)}{p(B|\omega)} = \frac{kh_\omega(x)}{\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)}.$$

Consistency of q hence yields

$$\widetilde{V}(\pi) = \sum_{\omega\in\Omega} q(\omega)\left(\frac{\pi^\omega - \bar{\pi}}{\bar{\pi}}\right)^2 = \frac{1}{\bar{\pi}^2}\sum_{\omega\in\Omega} p(\omega)\,(\pi^\omega)^2 - 1 = \frac{\sum_{\omega\in\Omega} p(\omega)\, h_\omega(x)^2}{\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)^2} - 1.$$

Note that

$$h'_\omega(x) = \frac{p(B|\omega)}{(1 - xp(B|\omega))^2} = p(B|\omega)\, h_\omega(x)^2.$$

Hence,

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{\sum_{\omega\in\Omega} p(\omega)\, h_\omega(x)^2}{\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)^2} - 1\right)\\
=&2\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)^{-4}\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)\cdot\\
&\left[\left(\sum_{\omega\in\Omega} p(\omega)\, p(B|\omega)\, h_\omega(x)^3\right)\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)\right.\\
&\left.-\left(\sum_{\omega'\in\Omega} p(\omega')\, p(B|\omega')\, h_{\omega'}(x)^2\right)\left(\sum_{\omega\in\Omega} p(\omega)\, h_\omega(x)^2\right)\right]\\
=&2\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)^{-4}\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)\cdot\\
&\left[\sum_{\omega\in\Omega}\sum_{\omega'\in\Omega} p(\omega)\, p(\omega')\, p(B|\omega)\, h_\omega(x)^2 h_{\omega'}(x)\,(h_\omega(x) - h_{\omega'}(x))\right]\\
=&2\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)^{-4}\left(\sum_{\omega'\in\Omega} p(\omega')\, h_{\omega'}(x)\right)\cdot
\end{aligned}$$

$$
\begin{aligned}
&\Bigg[\sum_{\substack{\omega,\omega'\in\Omega:\\ p(B|\omega)>p(B|\omega')}} p\left(\omega\right)p\left(\omega'\right)\underbrace{p\left(B|\omega\right)h_{\omega}(x)^2 h_{\omega'}(x)}_{>p(B|\omega')h_{\omega}(x)h_{\omega'}(x)^2}\left(h_{\omega}(x)-h_{\omega'}(x)\right)\\
&-\sum_{\substack{\omega,\omega'\in\Omega:\\ p(B|\omega)<p(B|\omega')}} p\left(\omega\right)p\left(\omega'\right)p\left(B|\omega\right)h_{\omega}(x)^2 h_{\omega'}(x)\left(h_{\omega'}(x)-h_{\omega}(x)\right)\Bigg]\\
>&2\left(\sum_{\omega'\in\Omega}p\left(\omega'\right)h_{\omega'}(x)\right)^{-4}\left(\sum_{\omega'\in\Omega}p\left(\omega'\right)h_{\omega'}(x)\right)\cdot\\
&\Bigg[\sum_{\substack{\omega,\omega'\in\Omega:\\ p(B|\omega)>p(B|\omega')}} p\left(\omega\right)p\left(\omega'\right)p\left(B|\omega'\right)h_{\omega}(x)h_{\omega'}(x)^2\left(h_{\omega}(x)-h_{\omega'}(x)\right)\\
&-\sum_{\substack{\omega,\omega'\in\Omega:\\ p(B|\omega)<p(B|\omega')}} p\left(\omega\right)p\left(\omega'\right)p\left(B|\omega\right)h_{\omega}(x)^2 h_{\omega'}(x)\left(h_{\omega'}(x)-h_{\omega}(x)\right)\Bigg]\\
=&\,0,
\end{aligned}
$$

where the inequality follows from $h_{\omega}(x)>1$ and $p\left(B|\omega\right)\geq p\left(B|\omega'\right)\Leftrightarrow h_{\omega}(x)\geq h_{\omega'}(x)$ under the conditions of Theorem 9. ■