Multilateral War of Attrition with Majority Rule (preliminary and incomplete)

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Abstract

We analyze a multilateral war of attrition game with majority rule. A chair and two competing players decide how to split one unit of surplus over continuous time. Each player has an exogenously given demand that are incompatible. In each period, the players simultaneously choose whether to concede or continue. The chair can concede to either of the two competing players but the competing players can concede only to the chair. An agreement is reached when at least one player concedes. We characterize the equilibria of this game and establish the necessary and sufficient conditions under which equilibria with delay exists.

1 Introduction

In this paper, we study a concession game between three players in continuous time in which only two players are needed to reach an agreement on how to split the surplus. In their influential paper on reputational bargaining, Abreu and Gül (2000) study a bilateral concession game and show that players can benefit from building a reputation for being stubborn, i.e., irrationally following a strategy of demanding a high share of surplus and not conceding to any demand below a specified low level. In multilateral bargaining with majority rule, such a reputation has its costs since expensive players would be excluded from the winning coalition. When one of the players can be excluded from the agreement, players can benefit from building a reputation for being compliant, i.e., irrationally following a strategy of demanding a low share of surplus. Players prefer to split the surplus with compliant types by conceding to them, so compliant types may become more likely to be included in the agreement. As a preliminary

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for analyzing a reputational multilateral bargaining game with majority rule, we first analyze a multilateral war of attrition game with majority rule with complete information.

Even with complete information, a multilateral concession game has some interesting features not present in a bilateral concession game (Hendricks, Weiss and Wilson (1988)). We analyze a continuous time game in which three players are deciding how to split one unit of surplus. Each player has an exogenously given demand and the sum of the demands exceed one, i.e. meeting all their demands given the available surplus is not possible. In each period, the players simultaneously choose whether to concede by accepting the demand of another player or continue. We assume that one of the players (the chair) must be included in the agreement, but only one of the remaining two players (competing players) is required to reach an agreement. As a result, the competing players can only concede to the chair, while the chair can concede to either of the two competing players. The game ends as soon as at least one player concedes.

We first characterize the equilibria in which agreement is reached immediately. We show that there is a continuum of such equilibria. When the demands of the competing players are not equal to each other, or when their demands are sufficiently small, in every immediate-agreement equilibrium, both of the competing players concede at the beginning of the game with certainty and the chair concedes later. When the demands of the competing players are equal to each other and are sufficiently large, the type of equilibria just described exists, but there is also another equilibrium in which the chair concedes at the beginning of the game, and the competing players concede later.

We then turn to characterizing equilibria with delay. We start by showing that when the demands of the competing players differ from each other, delay equilibria cannot exist. As such, to characterize the equilibria with delay, we assume the demands of the competing players are identical. We show that if the game does not end immediately, while the total probability of concession by the chair has a constant hazard rate, the hazard rates of concession to each of the competing players can vary over the course of the game. Similarly, the hazard rates of concession by the competing players to the chair can also vary over the course of the game. We show that there exist equilibria in which the competing players (one at a time) have rates of concession arbitrarily close to zero. In such equilibria, the competing players alternate in "holding out", i.e., not conceding to the chair over non-trivial time intervals.

For delay equilibria in which all players gradually conceding throughout the game, we find that the magnitude of the interest incompatibility, measured by the total demands of two players in the agreement minus the total surplus to be divided, decreases the concession rate for the chair and the aggregate concession rate for competing players. This captures the sense that the agreement is harder to be achieved with more intense interest conflict. We also show that these concession rates strictly increase in the chair's impatience; in particular, the concession rate for the chair also strictly increases in both competing players' impatience. Furthermore, the chair obtains the same payoff in the two-player war of attrition and in the three-player war of attrition; whereas competing players obtain strictly less payoff in the three-player war of attrition due to the bargaining competition.

TO BE COMPLETED.

1.1 Literature

Papers to discuss among others:

Abreu and Gül (2000), Hendricks, Weiss and Wilson (1988), Özyurt (2015), Ellingsen and Miettinen (2008), Miettinen (2022), Miettinen and Vanberg (2020), Ma (2022)

2 The Model

Consider the following model of bargaining. Players in the set $N = \{0, 1, 2\}$ are deciding how to split one unit of surplus. Player 0 must receive a share of surplus, and at most two players can receive positive shares. We call player 0 the *chair*, and call player 1 and 2 *competing players*. Each player $i \in N$ has an exogenously given demand $\alpha_i \in (0, 1)$. These demands satisfy $\alpha_0 + \alpha_i > 1$ for $i = 1, 2$, that is, the demand of player 0 is incompatible with the demand of any other player.

The game is played in continuous time with player i discounting future at rate $r_i > 0$. In each period $t \geq 0$, players simultaneously choose whether to concede or continue. Players 1 and 2 ("he") can only concede to player 0 ("she"), while player 0 choose to concede to either of the other two players. The game ends as soon as at least one player concedes. There are a total of 12 action profiles, with all but one having at least one player conceding and the game ending. These action profiles can be written as: $(0 \to 1)$, $(0 \leftarrow 1)$, $(0 \leftrightarrow 1)$, $(0 \to 2)$, $(0 \leftarrow 2)$, $(0 \leftrightarrow 2)$, $(1 \to 0 \to 2)$, $(1 \to 0 \leftarrow 2)$, $(1 \leftarrow 0 \leftarrow 2)$, $(1 \leftrightarrow 0 \leftarrow 2)$, $(1 \to 0 \leftrightarrow 2)$, and (none concedes).

Fix $t \geq 0$ and consider a pair of players who can divide the surplus among themselves, i.e., fix $i = 1, 2$ and consider the pair $(0, i)$. Suppose that no concession has been made before time t and that $j \neq i$ does not concede at t. If player 0 concedes to i and player i does not concede, then player i receives his claim α_i and player 0 receives the remainder of surplus $1 - \alpha_i$. If player 0 does not concede to any player and player i concedes, then player 0 gets her claim α_0 and player i gets $1 - \alpha_0$. If player 0 concedes to j and player i does not concede, then player i gets 0 and player 0 gets $1 - \alpha_j$. These are *base* outcomes. If more than one player concedes at t, then the outcome is chosen uniformly from the set of base outcomes determined by concessions.

A pure strategy of player i specifies for each time t whether to concede (and to which player if $i = 0$) or continue given that no player has conceded prior to time t. Since the game ends after the first concession, all pure strategies of player i with the same earliest concession time are payoff-equivalent. Therefore, we can index each pure strategy by the earliest concession time, i.e., let t_i be a pure strategy such that t_i is the earliest time at which player $i = 1, 2$ concedes, and write $t_i = \infty$ if player i never concedes. Similarly, let (t_0, i) be a pure strategy of player 0 such that t_0 is the earliest time at which she concedes to player $i = 1, 2$. Again, $(t_0, i) = (\infty, i)$ if player 0 never concedes to i.

For any pure strategy profile $\boldsymbol{t} = ((t_0, \kappa), t_1, t_2) \in (\bar{\mathbb{R}}_+ \times \{1, 2\}) \times \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$, let $u_i(\boldsymbol{t})$ denote the payoff of player $i \in N$. Then,

$$u_0(\boldsymbol{t}) = \begin{cases} \alpha_0 e^{-r_0 \min\{t_1, t_2\}} & \text{if } t_0 > \min\{t_1, t_2\}, \\ (1 - \alpha_\kappa) e^{-r_0 t_0} & \text{if } t_0 < \min\{t_1, t_2\}, \\ (\frac{1}{2}\alpha_0 + \frac{1}{2}(1 - \alpha_\kappa)) e^{-r_0 t_0} & \text{if } t_0 = \min\{t_1, t_2\} < \max\{t_1, t_2\}, \\ (\frac{2}{3}\alpha_0 + \frac{1}{3}(1 - \alpha_\kappa)) e^{-r_0 t_0} & \text{if } t_0 = t_1 = t_2, \end{cases}$$

and for $i = 1, 2$ and $j \neq i$

$$u_i(\boldsymbol{t}) = \begin{cases} \alpha_i e^{-r_i t_0} & \text{if } t_0 < \min\{t_1, t_2\} \text{ and } i = \kappa, \\ (1 - \alpha_0) e^{-r_i t_0} & \text{if } t_i < \min\{t_0, t_j\}, \\ (\frac{1}{2}\alpha_i + \frac{1}{2}(1 - \alpha_0)) e^{-r_i t_0} & \text{if } t_0 = t_i < t_j \text{ and } i = \kappa, \\ \frac{1}{2}\alpha_i e^{-r_i t_0} & \text{if } t_0 = t_j < t_i \text{ and } i = \kappa, \\ \frac{1}{2}(1 - \alpha_0) e^{-r_i t_0} & \text{if } t_0 = t_i < t_j \text{ and } i \neq \kappa, \text{ or } t_1 = t_2 < t_0, \\ (\frac{1}{3}\alpha_i + \frac{1}{3}(1 - \alpha_0)) e^{-r_i t_0} & \text{if } t_0 = t_1 = t_2 \text{ and } i = \kappa, \\ \frac{1}{3}(1 - \alpha_0) e^{-r_i} t_0 & \text{if } t_0 = t_1 = t_2 \text{ and } i \neq \kappa, \\ 0 & \text{if } \min\{t_0, t_\kappa\} < t_i \text{ and } i \neq \kappa. \end{cases}$$

Denote a mixed strategy of player $i = 1, 2$ by $G_i : \bar{\mathbb{R}}_+ \to [0, 1]$, where $G_i(t)$ is the probability that player i concedes by time t. The probability that player i never concedes is given by $1 - \lim_{t\to\infty} G_i(t)$. Notice that G_i must be weakly increasing and right-continuous by definition. Notice that we don't require $\lim_{t\to\infty} G_i(t) = 1$.

Similarly, denote a mixed strategy of player 0 by $G_0 = (G_{0,1}, G_{0,2})$ where $G_{0,i} : \bar{\mathbb{R}}_+ \to [0, 1]$ and $G_{0,i}(t)$ denotes the probability that player 0 concedes to player i by time t. We assume that $G_{0,i}$ is weakly increasing and right-continuous for each $i = 1, 2$, and that $G_{0,1}(t) + G_{0,2}(t) \leq 1$ for all t. Define $\tilde{G}_0(t) = G_{0,1}(t) + G_{0,2}(t)$. Then, $\tilde{G}_0(t)$ denotes the probability that player 0 concedes by time t, and the probability that player 0 never concedes is given by $1 - \lim_{t\to\infty} \tilde{G}_0(t)$.

Let $\mathcal{G}_i$ denote the set of mixed strategies for player $i = 0, 1, 2$. For any $(G_1, G_2) \in \mathcal{G}_1 \times \mathcal{G}_2$ and $t \geq 0$, let $U_0(t, \kappa, G_1, G_2)$ denote the expected utility of player 0 from conceding to player κ at time t with certainty when players 1 and 2 use mixed strategies G_1 and G_2 respectively.

Then,

$$U_0(t,\kappa,G_1,G_2) = \iint u_0((t,\kappa),t_1,t_2)\mathrm{d}G_1(t_1)\mathrm{d}G_2(t_2).$$

Likewise, for $i=1,2$ and $j \neq i$, for any $(G_0,G_j) \in \mathcal{G}_0 \times \mathcal{G}_j$ and $t \geq 0$, let $U_i(t,G_0,G_j)$ denote the expected utility of player i from conceding at time t with certainty when the players 0 and j use mixed strategies G_0 and G_j respectively. Then,

$$U_i(t,G_0,G_j) = \iint u_i((t_0,\kappa),t,t_j)\mathrm{d}G_0(t_0,\kappa)\mathrm{d}G_j(t_j).$$

In what follows, for any $f : \mathbb{R}_+ \to \mathbb{R}$, any $G : \bar{\mathbb{R}}_+ \to [0,1]$ non-decreasing and right-continuous, and any $s < t$, let

$$\int_s^t f(v)\mathrm{d}G(v) = \lim_{\tau \uparrow t} \int_s^\tau f(v)\mathrm{d}G(v).$$

In other words, all integrals omit the mass point (if any) at the upper bound of integration.

3 Preliminaries

In this section, we define Nash equilibrium for this game and describe some of its properties.

Definition 1. *A mixed strategy profile $(G_0,G_1,G_2) \in \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$ is a Nash equilibrium if*

(i) $\int U_0(t,\kappa,G_1,G_2)\mathrm{d}G_0(t,\kappa) \geq \int U_0(t,\kappa,G_1,G_2)\mathrm{d}\hat{G}_0(t,\kappa)$ *for all* $\hat{G}_0 \in \mathcal{G}_0$*, and*

(ii) $\int U_i(t,G_0,G_j)\mathrm{d}G_i(t) \geq \int U_i(t,G_0,G_j)\mathrm{d}\hat{G}_i(t)$ *for* $i,j \in \{1,2\}$ *with* $i \neq j$*, and for all* $\hat{G}_i \in \mathcal{G}_i$.

A degenerate Nash equilibrium is a pure strategy Nash equilibrium.

For the rest of this section follows, fix a mixed strategy profile (G_0,G_1,G_2) and let $q_i(t)$ denote the probability that player $i=1,2$ concedes exactly at time t, $q_{0,i}(t)$ denote the probability that player 0 concedes to player i exactly at time t, and $\tilde{q}_0(t)$ denote the probability that player 0 concedes exactly at time t, i.e., $\tilde{q}_0(t) = q_{0,1}(t) + q_{0,2}(t)$. Formally, $q_i(t) = G_i(t) - \lim_{\tau\uparrow t} G_i(\tau)$, and $q_{0,i}(t) = G_{0,i}(t) - \lim_{\tau\uparrow t} G_{0,i}(\tau)$. The probability that player i never concedes is denoted by $q_i(\infty)$, and the probability that player 0 never concedes is $q_0(\infty)$. Clearly, if G_i (respectively $G_{0,i}$) is continuous, then $q_i(t)=0$ (respectively $q_{0,i}(t)=0$) for all $t \geq 0$.

For player $i \in \{1,2\}$, define $T_i^+ = \{t \in [0,\infty] : q_i(t) > 0\}$; and for player 0, define $T_{0,i}^+ = \{t \in [0,\infty] : q_{0,i}(t) > 0\}$. In what follows, we refer to a point $t \in T_i^+$ (or $t \in T_{0,i}^+$) as an atom point in concession time for player i (or for player 0 conceding to player i respectively).

Slightly abusing notation, let $G_i(\mathcal{T})$ denote the probability that player i assigns to a measurable set $\mathcal{T}$ according to a mixed strategy $G_i \in \mathcal{G}_i$. Likewise, let $G_{0,i}(\mathcal{T})$ denote the probability that player 0 assigns to a measurable set $\mathcal{T}$ according to a measure $G_{0,i}$.

Lemma 1. *Let (G_0, G_1, G_2) be a mixed strategy Nash equilibrium and fix player $i \in \{1,2\}$. For all measurable $\mathcal{T} \subseteq [0,\infty]$, we have:*

(i) if there exists $s \notin \mathcal{T}$ such that $U_i(s, G_0, G_j) > U_i(\tau, G_0, G_j)$ for all $\tau \in \mathcal{T}$, then $G_i(\mathcal{T}) = 0$;

(ii) if there exists $s \notin \mathcal{T}$ or $j \neq i$ such that $U_0(s, j, G_1, G_2) > U_0(\tau, i, G_1, G_2)$ for all $\tau \in \mathcal{T}$, then $G_{0,i}(\mathcal{T}) = 0$.

Lemma 1 implies that, in equilibrium, if a player concedes with a strictly positive probability at time t, then no other concession time can yield her a strictly higher utility. In other words, every atom point in concession time must be a best response to the strategies of other players, analogous to a well-known mixed strategy Nash equilibrium characterization for games in which players have a continuum of actions.[1] In fact, an equilibrium mixed strategy must place zero weight on any measurable set of pure strategies (here, concession times) that are not best responses to the strategies of other players.

Corollary 1. *Let (G_0, G_1, G_2) be a mixed strategy Nash equilibrium and fix player $i \in \{1,2\}$. For any time $s, \tau_1, \tau_2 \in \bar{\mathbb{R}}$ such that $\tau_1 < \tau_2$, we have:*

(i) if G_i is strictly increasing over (τ_1, τ_2), then $U_i(t, G_0, G_j) \geq U_i(s, G_0, G_j)$ for $t \in (\tau_1, \tau_2)$ almost everywhere;

(ii) if $G_{0,i}$ is strictly increasing over (τ_1, τ_2), then $U_0(t, i, G_0, G_j) \geq U_0(s, \kappa, G_0, G_{\bar{\kappa}})$ for $t \in (\tau_1, \tau_2)$ almost everywhere, and $j, \kappa, \bar{\kappa} \in \{1,2\}$, $j \neq i$, $\kappa \neq \bar{\kappa}$.

Lemma 2 below implies that if an equilibrium exists, then some player must necessarily concede with positive probability in that equilibrium. This is because all players receive zero payoffs when no player ever concedes, and thus any player can profitably deviate by conceding at some time.

Lemma 2. *There does not exist an equilibrium in which all players wait infinitely with certainty.*

This result is consistent with part (a) of Theorem 1 in Hendricks, Weiss and Wilson (1988) which provides necessary and sufficient conditions for an infinite delay in their framework. In our model, being the first player to concede is strictly better than an infinite delay (i.e., $L_\alpha(0) > S_\alpha(1)$ for $\alpha = a, b$ in their notation), so there is no equilibrium in which no one concedes.

We show that both competing players concede at time $t = 0$ in every pure-strategy Nash equilibrium. To see why this is true, note first that there is always a pure-strategy Nash equilibrium where player 1 and player 2 concede at the start and player 0 concedes later. This is an equilibrium since player 0 attains highest utility achievable, i.e., α_0, and thus has no strictly profitable deviation; and neither competing player prefers to concede later because

[1] See, for example, Proposition 142.2 in Osborne (2004).

$\frac{1-\alpha_0}{2} > 0$. Furthermore, this characterization covers all pure-strategy Nash equilibria, as we now show. Fix a pure-strategy equilibrium $((t_0, \kappa), t_1, t_2) \in (\bar{\mathbb{R}}_+ \times \{1, 2\}) \times \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$. First, from Lemma 2 we know that in every Nash equilibrium there exists a player who eventually concedes, that is, $\min\{t_0, t_1, t_2\} < \infty$. We denote the earliest concession time by $\underline{t}$. Next, player 0 cannot concede at $\underline{t}$. If $t_0 = \underline{t}$ and $t_j > \underline{t}$ for player $j \neq \kappa$, then j can deviate to $\underline{t}$ and get a payoff $\frac{1-\alpha_0}{2}$ instead of 0. And if $t_0 = \underline{t} = t_i$ for any $i \in \{1, 2\}$, then player 0 can deviate to $t > \underline{t}$ and get a payoff α_0, which is greater than any payoff player 0 can receive by conceding at $t_0 = \underline{t}$. Further, the game must end immediately, that is, $\underline{t} = 0$. If $\underline{t} > 0$, then each player $i \in \{1, 2\}$ has a profitable deviation to $t < \underline{t}$ because it saves the waiting cost and guarantees the highest possible share $1 - \alpha_0$ that player i can achieve by conceding before player 0. Finally, player 1 and player 2 must concede simultaneously, that is, $t_1 = t_2$. If $t_i < t_j$, then player j has a profitable deviation to $t = 0$. It follows that in every pure-strategy Nash equilibrium, player 1 and player 2 concede simultaneously at time $t = 0$, and player 0 concedes at some time $t > 0$. Pure-strategy Nash equilibria only differ in player 0's action, namely the concession time t_0 and the target player κ. These results give us Proposition 1.

Proposition 1. *There is a continuum of pure-strategy Nash equilibria. In every pure-strategy Nash equilibrium, player 1 and player 2 concede simultaneously at time $t = 0$ and player 0 concedes at a later time. The equilibrium payoffs are unique with player 0 receiving α_0, and players 1 and 2 receiving $\frac{1}{2}(1 - \alpha_0)$.*

Proposition 1 shows that there always exists an equilibrium that ends at time zero with probability 1. Furthermore, this equilibrium is in pure strategies and the players to concede at time 1 are the competing players, i.e., either player 1 or player 2. This is however not the only possible equilibria in which the game ends at time zero with probability 1. We call an equilibrium *immediate-agreement equilibrium* if the game ends at time zero with probability 1, otherwise it is a *delay equilibrium*. Formally,

Definition 2. *An equilibrium $(G_0, G_1, G_2) \in \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$ is an* **immediate-agreement equilibrium** *if and only if $(1 - G_{0,1}(0) - G_{0,2}(0))(1 - G_1(0))(1 - G_2(0)) = 0$. An equilibrium $(G_0, G_1, G_2) \in \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$ is a* **delay equilibrium** *if and only if $(1 - G_{0,1}(0) - G_{0,2}(0))(1 - G_1(0))(1 - G_2(0)) \neq 0$.*

Recall that $q_i(t)$ is the probability that player $i = 1, 2$ concedes exactly at time t, and $\tilde{q}_0(t)$ is the probability that player 0 concedes exactly at time t. It follows that an equilibrium (G_0, G_1, G_2) is a delay equilibrium if and only if, $q_1(0) < 1, q_2(0) < 1$, and $\tilde{q}_0(0) < 1$.

We fully characterize the immediate-agreement equilibria. The following proposition says that any immediate-agreement equilibrium is generically a pure-strategy equilibrium described in Proposition 1. Only when the demands of the two competing players are equal and are sufficiently high (i.e., $\alpha_1 = \alpha_2 \geq 2(1 - \alpha_0)$), it is ever possible for player 0 to concede at time $t = 0$ with certainty.

Proposition 2. *There is a continuum of immediate-agreement Nash equilibria. Moreover:*

(i) If $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2 < 2(1-\alpha_0)$, then in every immediate-agreement equilibrium, player 1 and player 2 concede at time $t = 0$ with certainty and player 0 concedes later.

(ii) If $\alpha_1 = \alpha_2 \geq 2(1-\alpha_0)$, then in any immediate-agreement equilibrium, either player 1 and player 2 concede at time $t = 0$ with certainty and player 0 concedes later, or player 0 concedes at time $t = 0$ with certainty and player 1 and player 2 concede later.

Note that there always exists an immediate-agreement equilibrium in which both competing players concede at the start of the game. By contrast, there need not exist an equilibrium in which player 0 concedes at the start of the game. Part (ii) of Proposition 2 gives the necessary and sufficient conditions for such an equilibrium to exist. Intuitively, if the demands of the competing players are not equal, then there cannot exists an equilibrium in which player 0 concedes because if player 0 ever concedes, she would concede to the competing player with the lower demand. Knowing this, the competing player with the higher demand would concede before player 0. If the demands of the competing players are equal and sufficiently low, then again there is an incentive for the competing players to concede, but this time it is because the gain from waiting is not worth the risk that player 0 might concede to the other player. Finally if the demands of the conceding players are equal and sufficiently high, then there is an equilibrium in which player 0 concedes to one of the competing players.

In the proof of Proposition 2, we show that if $\alpha_1 = \alpha_2 \geq 2(1-\alpha_0)$, then there exists an immediate-agreement equilibrium in which player 0 concedes to each other player with probability $\frac{1}{2}$ at the start of the game, and player 1 and player 2 concede later. When $\alpha_1 = \alpha_2 > 2(1-\alpha_0)$, there also exists other equilibria in which player 0 concedes to players i with probability $q_{0,i}(0) \neq \frac{1}{2}$ but yet $q_{0,1}(0) + q_{0,2}(0) = 1$ so that the equilibrium is still immediate-agreement.

Suppose $\alpha_1 = \alpha_2 = \alpha$. When the competing players do not concede at the beginning of the game, waiting must be sufficiently profitable for each of them compared to conceding. Thus, if both of the competing players are waiting and player 0 concedes with certainty, then the aggregate payoff of the competing players must be α. If instead one of them unilaterally deviates, their aggregate payoff would be $\frac{1}{2}\alpha + \frac{1}{2}(1-\alpha_0)$. For there to be an immediate-agreement equilibrium in which player 0 concedes at time $t = 0$ and the competing players wait, it must be the case that $\alpha \geq 2(1-\alpha_0)$.

Note that unlike in Proposition 1 and part (i) of Proposition 2, the multiplicity of equilibria in part (ii) of Proposition 2 is accompanied with multiple equilibrium outcomes and equilibrium payoffs.

4 Delay Equilibria

In this section, we characterize delay equilibria, i.e., a mixed strategy equilibrium (G_0, G_1, G_2) with $q_1(0) < 1, q_2(0) < 1$, and $\tilde{q}_0(0) < 1$. We start by showing that if $\alpha_1 \neq \alpha_2$, then it is not possible to have a delay equilibrium. In particular, if $\alpha_i > \alpha_j$, then player 0 either does not concede or concedes to player j leaving player i with nothing. In either case, player i has an incentive to concede immediately at the start of the game, which is not possible in any delay equilibrium. This intuition leads us to the following result.

Lemma 3. *If $\alpha_1 \neq \alpha_2$, then there does not exists a delay equilibrium.*

Consequently, when $\alpha_1 \neq \alpha_2$, the characterization of equilibria is complete by Proposition 2. The rest of this section is devoted to characterizing delay equilibria when $\alpha_1 = \alpha_2 = \alpha$. In particular, we study the following five scenarios:

(i) Start: What can happen at the start of the game? (Lemma 4).

(ii) Jumps: Do two players concede with strictly positive probability at the same time? Is it ever possible that some player concedes with strictly positive probability before the game ends? (Lemma 5 and Lemma 6).

(iii) Suspense: How will others react if some player has no intention to concede for a period of time? Is it ever possible that some player does not concede for a period of time before the game ends? (Lemma 7).

(iv) End: How will others react if some player concedes with strictly positive probability at the end of the game? Is it ever possible that some player ends the game in finite time? (Lemma 8).

(v) Gradual concession: How will others react to a player who concedes gradually over time? (Lemma 9).

After answering these questions, we characterize the mixed strategies for all players in any delay equilibrium in Proposition 3 using Lemma **??** to Lemma **??**.

For comparison, the answers to the questions above in Hendricks et al. (1988) are: (1) at most one player concedes with strictly positive probability at the start of the game; (2) no positive atom points for both players (and if someone concedes with strictly positive probability after the game starts, the other would end the game by that time); (3) no suspense occurs (and if someone suspends for a while before the game ends, the other would suspended for that period as well); (4) game does not terminate in finite time.

4.1 Properties of Best Responses

We start by answering the questions posed above to establish the properties of the best responses of players to the strategies of other players.

4.1.1 Start

Consider the case in which some player concedes with positive probability at the start of the game but does not concede with certainty. The following result states that the chair cannot concede at the start of the game when one of the competing players concedes.

Lemma 4. *If $q_i(0) > 0$, then $q_{0,1}(0) = q_{0,2}(0) = 0$.*

The intuition behind Lemma 4 is that if player 0 knows some player $i \in \{1, 2\}$ concedes at time 0 with positive probability, then she would rather wait for a little longer to concede than concede at time 0. That is because for player 0, the cost of waiting for an infinitesimal amount of time is negligible whereas its benefit discretely jumps when $q_i(0) > 0$. Therefore, if either of the competing players concede at time 0 with strictly positive probability, player 0 does not concede at time 0. The same logic applies when $t > 0$. See Lemma 5.

So what if $\tilde{q}_0(0) > 0$ in equilibrium? A direct implication of Lemma 4 is that: if $\tilde{q}_0(0) > 0$, then $q_1(0) = q_2(0) = 0$. To see this, suppose for the purpose of contradiction that $q_i(0) > 0$ for some $i \in \{1, 2\}$. Then by Lemma 4, we have $\tilde{q}_0(0) = q_{0,1}(0) + q_{0,2}(0) = 0$, a contradiction. Therefore, a competing player and the chair cannot concede simultaneously at the start of the game. Formally, we have $\tilde{q}_0(0)q_i(0) = 0$ for any $i \in \{1, 2\}$.

For the time being, we cannot say much about how the competing player $j \neq i$ would react in equilibrium if competing player i concedes with positive probability (i.e., $q_i(0) > 0$). Player j's best response depends on player 0's equilibrium strategy. If player 0 concedes to player j shortly after the game starts, player j will not concede at the start (i.e., $q_j(0) = 0$); however, if player 0 only concedes to player j after a long time, player j may be indifferent between conceding at the start or later (i.e., $q_j(0) \geq 0$). Later in Lemma **??**, we will establish that in all delay equilibria, $q_1(0) = q_2(0) = 0$, and $\tilde{q}_0(0) \leq \frac{\alpha+\alpha_0-1}{1-\alpha_0}$.

4.1.2 Jumps

So far we only considered the start of the game, which is special because it marks the earliest possible concession time. Now we move on to characterizing what happens after the game starts in any delay equilibrium. The following result is analogous to Lemma 4 but describes what happens when player $i \in \{1, 2\}$ concedes with a strictly positive probability at time $t > 0$ rather than $t = 0$. However, Lemma 5 takes into account the possibility that the game might have ended before time t, something that cannot happen at the start of the game.

Lemma 5. *For any $t > 0$, if $q_i(t) > 0$ and $\lim_{\tau \uparrow t} G_j(\tau) < 1$, then $q_{0,1}(t) = q_{0,2}(t) = 0$.*

The message from Lemma 5 is that whenever player $i \in \{1, 2\}$ concedes with strictly positive probability, player 0 won't concede at the same time as long as the game has not essentially ended by then. The intuition behind this result is similar to Lemma 4 since for player 0, the cost of waiting for an infinitesimal time is negligible while the benefit from it is strictly positive. The difference between these two lemmas just consists in the time under study: Lemma 5 describes delay equilibrium at $t > 0$ while Lemma 4 is for the start of the game.

We show that if player i concedes with positive probability at time $t > 0$, then the competing player j does not have an incentive to concede at the same time.

Lemma 6. *For any $t > 0$, if $q_i(t) > 0$ and $\lim_{\tau \uparrow t} G_{0,1}(\tau) + G_{0,2}(\tau) < 1$, then $q_j(t) = 0$.*

The message from Lemma 6 is that whenever player $i \in \{1, 2\}$ concedes with strictly positive probability, player $j \neq i$ won't concede at the same time as long as the game hasn't ended by then. This result looks quite similar to Lemma 5 but the intuition behind them is quite different. In Lemma 5, player 0 won't concede at the same time with player i because she has an incentive to wait a little *longer*, while in Lemma 6, player j won't concede at the same time with player i because he has the incentive to concede a little *earlier*.

4.1.3 Suspense

We study how the other players will react if some player does not move for an interval.

Lemma 7. *If the strategies of competing players are constant over some interval before the game ends, then the strategy of player 0 is constant in that interval, and vice versa. Formally,*

(i) If $G_1(t) = G_1(s) < 1$, $G_2(t) = G_2(s) < 1$ for some $t < s$, then $G_{0,\kappa}(t) = G_{0,\kappa}(s)$ for $\kappa = 1, 2$.

(ii) If $G_{0,i}(t) = G_{0,i}(s)$, $\lim_{w \uparrow s} G_{0,1}(w) + G_{0,2}(w) < 1$ and $\lim_{w \uparrow s} G_j(w) < 1$ for some $t < s$ and $j \in \{1, 2\}$, then $G_i(t) = G_i(s)$ for $i \in \{1, 2\}$ with $i \neq j$.

The message from Lemma 7 is that whenever players 1 and 2 do not move for some interval, player 0 does not move during the same interval as long as the game hasn't ended by then. Likewise, whenever player 0 does not move during some interval, then the other two players do not move during that same interval. The force driving Lemma 7 is that any player would rather concede earlier than concede within the suspense period in which (s)he is not conceded to. This is because by conceding earlier, the player saves waiting cost and does not harm his/her chance of being conceded to (since the opponent(s) concession choices are constant over this suspense period).

4.1.4 End

The next result shows that if a player concedes with strictly positive probability at time $t > 0$, then the game must end by time t.

Lemma 8. *The game ends before the first atom point in concession time among the three players. Formally, for any $t > 0$ and $i, j \in \{1, 2\}$ such that $j \neq i$ we have:*

(i) If $q_{0,i}(t) > 0$*, then* $\lim_{\tau\uparrow t} G_1(\tau) = 1$ *or* $\lim_{\tau\uparrow t} G_2(\tau) = 1$*.*

(ii) If $q_i(t) > 0$*, then* $\lim_{\tau\uparrow t} G_{0,1}(\tau) + G_{0,2}(\tau) = 1$ *or* $\lim_{\tau\uparrow t} G_j(\tau) = 1$*.*

Loosely speaking, the proof is as follows. Suppose player i concedes at time t with strictly positive probability to player j. Then j has an incentive to concede later. As such, if, to the contrary of the statement of the lemma, the game hasn't ended by t, then there is a short suspense period right before t during which player j does not move. Therefore, by Lemma 7 player i should not have moved for that period as well, which is a contradiction to i jumping at t.

4.1.5 Gradual Concession

Consider an interval $[t, s]$ during which at least one player concedes gradually, i.e., without jumps or suspense. In Lemma 9, we characterize the best responses to strictly increasing concession strategies which includes gradual concession as a special case. Note however that as long as the game has not ended, strictly increasing concession strategies are equivalent to gradual concession by Lemma 8.

Let $H(\tau)$ be the probability that at least one of the competing players concedes by time $\tau \in \mathbb{R}_+$:

$$\begin{aligned} H(\tau) &= G_1(\tau)(1 - G_2(\tau)) + G_2(\tau)(1 - G_1(\tau)) + G_1(\tau)G_2(\tau) \\ &= G_1(\tau) + G_2(\tau) - G_1(\tau)G_2(\tau). \end{aligned} \tag{1}$$

We show that when the chair concedes gradually, H must have a constant hazard rate on $[t, s]$ given by

$$\mu = \frac{(1-\alpha)r_0}{\alpha_0 + \alpha - 1} > 0. \tag{2}$$

Similarly, the chair's total probability of concession $\tilde{G}_0$ must have a constant hazard rate on $[t, s]$ when both competing players concede gradually. We show that this hazard rate is given by

$$\rho = \frac{\mu + r_1 + r_2}{\dfrac{\alpha}{1-\alpha_0} - 2}. \tag{3}$$

This result puts a restriction on the demands of players by requiring that $\alpha + 2\alpha_0 > 2$, which guarantees that the hazard rate ρ is positive.

For any $\tau \in [t, s]$, let $\lambda_i(\tau)$ denote the hazard rate of the concession strategy G_i of player $i \in \{1, 2\}$. Formally,

$$\lambda_i(\tau) = \frac{g_i(\tau)}{1 - G_i(\tau)}. \tag{4}$$

Similarly define $\lambda_{0,i}(\tau)$ as

$$\lambda_{0,i}(\tau) = \frac{g_{0,i}(\tau)}{1 - \tilde{G}_0(\tau)}. \tag{5}$$

Note that the sum of λ_1 and λ_2 is the hazard rate of the competing players' aggregate concession strategy H, and the sum of $\lambda_{0,1}$ and $\lambda_{0,2}$ is the hazard rate of the chair's aggregate concession strategy $\tilde{G}_0$. In Lemma 9, we show that the sum of λ_1 and λ_2 is constant and equal to μ when $\tilde{G}_0$ is strictly increasing. We also show that the sum of $\lambda_{0,1}$ and $\lambda_{0,2}$ is constant and equal to ρ when G_1 and G_2 are strictly increasing.

Lemma 9. *Consider an interval $[t, s]$ with $0 < t < s$ and suppose that $\lim_{\tau\uparrow s} G_1(\tau) < 1$, $\lim_{\tau\uparrow s} G_2(\tau) < 1$, and $\lim_{\tau\uparrow s} \tilde{G}_0(\tau) < 1$.*

(i) If $\tilde{G}_0$ is strictly increasing on $[t, s]$, then

$$\mu = \lambda_1(\tau) + \lambda_2(\tau) \tag{6}$$

for all $\tau \in [t, s)$ where μ is given by (2).

(ii) If G_i is strictly increasing on $[t, s]$ and G_j is constant on $[t, s]$, then

$$\lambda_i(\tau) = \mu, \tag{7}$$

$$\lambda_{0,i}(\tau) = \frac{(1-\alpha_0)(r_i + \lambda_{0,j}(\tau))}{\alpha_0 + \alpha - 1} = \frac{1-\alpha_0}{\alpha}(r_i + \lambda_0(\tau)) \tag{8}$$

for all $\tau \in [t, s)$ where μ is given by (2).

(iii) If G_1 and G_2 are both strictly increasing on $[t, s]$, then

$$\lambda_i(\tau) = -r_j - \rho + \frac{\alpha}{1-\alpha_0}\lambda_{0,j}(\tau), \quad i = 1, 2,\ j \neq i \tag{9}$$

for all $\tau \in [t, s)$ where ρ is given by (3).

By Lemma 7, $\tilde{G}_0$ must be strictly increasing when G_1 and G_2 are strictly increasing. Together with part (i) of Lemma 9, when G_1 and G_2 are strictly increasing, we must have $\lambda_1(\tau)+\lambda_2(\tau) = \mu$. By adding the two equations in part (iii) of Lemma 9 and using the observation that $\lambda_1(\tau)+\lambda_2(\tau) = \mu$, we obtain $\lambda_{0,1}(\tau)+\lambda_{0,2}(\tau) = \rho$ for all τ, that is, $\tilde{G}_0$ has a constant hazard rate. This in turn implies that for there to be an equilibrium in which G_1 and G_2 are strictly increasing, it must be the case that $\alpha + 2\alpha_0 > 2$, that is, the demands of players are sufficiently high. We already require that the demands are incompatible in the sense that $\alpha > 1 - \alpha_0$. Thus, for there to be an equilibrium in which players 1 and 2 gradually concede, a stronger restriction $\alpha > 2(1 - \alpha_0)$ must hold.[2]

To summarize, we have the following corollary.

Corollary 2. *Consider an interval* $[t, s]$ *with* $0 < t < s$ *and suppose that* $\lim_{\tau \uparrow s} G_1(\tau) < 1$, $\lim_{\tau \uparrow s} G_2(\tau) < 1$, *and* $\lim_{\tau \uparrow s} \tilde{G}_0(\tau) < 1$. *Both* G_1 *and* G_2 *are strictly increasing over the time interval* $[t, s]$ *only when* $\alpha > 2(1 - \alpha_0)$. *In this case,* $\tilde{G}_0$ *has a constant hazard rate* ρ *over* $[t, s)$*, i.e., for any time* $\tau \in [t, s)$*,*

$$1 - \tilde{G}_0(\tau) = (1 - \tilde{G}_0(t))e^{-\rho(\tau - t)}$$

where ρ *is given by* (3)*.*

4.2 Characterization of Mixed Strategy Equilibria with Delay

In this section, we characterize mixed strategy equilibria with delay. Throughout this section, fix an equilibrium mixed strategy profile $(G_{0,1}, G_{0,2}, G_1, G_2)$ with delay. As before, let $\tilde{G}_0(t) = G_{0,1}(t) + G_{0,2}(t)$ and $H(t) = G_1(t) + G_2(t) - G_1(t) - G_2(t)$ for all $t \in \mathbb{R}_+$.

We start by examining the possibility that no one moves in some interval by the end of the game. Let $\hat{t} \in \bar{\mathbb{R}}_+$ be the duration of the game, i.e.

$$\hat{t} = \inf\{t \geq 0 \ : \max\{\tilde{G}_0(t),\ H(t)\} = 1\}$$

with the convention that $\hat{t} = \infty$ if the set $\{t \geq 0 \ : \max\{\tilde{G}_0(t),\ H(t)\} = 1\}$ is empty. Since we focus on the delay equilibrium in this section, we must have $\hat{t} > 0$.

The following lemma shows that the aggregate strategies $\tilde{G}_0$ and H prescribe a gradual concession from the beginning of the game, $t = 0$, until its end, $t = \hat{t}$.

Lemma 10. *Aggregate strategies* $\tilde{G}_0$ *and* H *are strictly increasing on* $[0, \hat{t})$*.*

There two immediate implications of this lemma. First, player 0 must concede gradually throughout the game. Second, at any point during the game, at least one of the competing players must be conceding. Therefore, combined with Lemma 8, there are only two possible cases in delay equilibria:

1. Both competing players gradually concede throughout the game, i.e., G_1 and G_2 are both strictly increasing over $[0, \hat{t}]$.

[2]This condition is slightly stronger than the necessary condition for there to be an immediate agreement equilibrium in which player 0 concedes at time zero (recall part (ii) of Proposition 2) in that the inequality is strict.

2. One and only one competing player does not concede over some interval(s), i.e., $\exists a, b \in [0, \hat{t}]$ such that $G_i(a) = G_i(b)$ and G_j is strictly increasing over $[a, b]$.

In both cases, by Lemma 8, there is no atom point in any player's strategy before the end of the game, that is, $G_1, G_2, G_{0,1}$ and $G_{0,2}$ are all continuous over $[0, \hat{t})$. We next characterize delay equilibria in these two cases separately.

4.2.1 Both Competing Players Gradually Concede

Since G_1 and G_2 are strictly increasing, Lemma 7 implies that $G_{0,1}$ and $G_{0,2}$ are both strictly increasing over $[0, \hat{t})$. The next result shows that the probability that the game does not end by time t is strictly positive for each $t \in \mathbb{R}_+$, that is, $\hat{t} = \infty$.

Lemma 11. *In any delay equilibrium when both competing players gradually concede throughout the game, we have $\hat{t} = \infty$.*

We next reconsider what can happen at the start of the game. By Lemma 4, we have already shown that player 0 and any competing player cannot simultaneously concede at $t = 0$ with positive probability. The next result further establishes that no competing player concedes at the start of the game with positive probability. It also establishes that if player 0 ever concedes at the start of the game, she needs to concede to both competing players, and moreover, she the probability of conceding to player 1 must be "close" to the probability of conceding to player 2.

Lemma 12. *In any delay equilibrium when both competing players gradually concede throughout the game, we have $q_1(0) = q_2(0) = 0$ and*

$$\min\{q_{0,1}(0), q_{0,2}(0)\} \geq \frac{1 - \alpha_0}{\alpha + \alpha_0 - 1} \max\{q_{0,1}(0), q_{0,2}(0)\}.$$

Finally, we are ready to fully characterize the type of delay equilibrium in which both competing players gradually concede by the end of the game. Recall that by Lemma 3 and Corollary 2 we must have $\alpha_1 = \alpha_2 = \alpha > 2(1 - \alpha_0)$ for such an equilibrium to exists. The proposition characterizes other necessary conditions in addition sufficient conditions.

Proposition 3. *Fix a mixed strategy profile $(G_{0,1}, G_{0,2}, G_1, G_2)$ with $\tilde{q}_0(0) < 1$, $q_1(0) < 1$ and $q_2(0) < 1$. Let $\lambda_i(t) = \frac{G_i'(t)}{1 - G_i(t)}$ be the hazard rate at time $t \in \mathbb{R}_+$ for player $i \in \{1, 2\}$, and let $\lambda_{0,i}(t) = \frac{G_{0,i}'(t)}{1 - G_{0,1}(t) - G_{0,2}(t)}$ for any $t \in \mathbb{R}_+$. The mixed strategy profile $(G_{0,1}, G_{0,2}, G_1, G_2)$ is a Nash equilibrium profile in which both competing players concede throughout the game if and only if the following conditions hold:*

(i) $G_{0,1}, G_{0,2}, G_1$ *and* G_2 *are all continuous over* $(0, \infty)$*;*

(ii) $G_1(0) = G_2(0) = 0$*, and* $\min\{G_{0,1}(0), G_{0,2}(0)\} \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1} \max\{G_{0,1}(0), G_{0,2}(0)\}$*;*

(iii) $\lambda_1(t) + \lambda_2(t) = \mu$ *and* $\lambda_1(t), \lambda_2(t) > 0$ *for* $t \geq 0$ *almost everywhere, where* μ *is given by* (2)*;*

(iv) $\lambda_{0,j}(t) = \frac{1-\alpha_0}{\alpha}(\lambda_i(t) + r_j + \rho)$ *for* $t > 0$ *almost everywhere and* $i, j \in \{1, 2\}, j \neq i$*, where* ρ *is given by* (3)*.*

Example 1 (Constant Concession Rates)**.** *We can consider a special case when player 1's concession rate is constant over time as* $\lambda_1 \in (0, \lambda)$*. Then by Proposition 3,* $(G_{0,1}, G_{0,2}, G_1, G_2)$ *constitute a delay equilibrium in which for any* $t > 0$*,*

$$
\begin{aligned}
q_i(0) = 0 \ \textit{and}\ \frac{1-\alpha_0}{\alpha}\tilde{q}_0(0) \leq q_{0,i}(0) &\leq \frac{\alpha+\alpha_0-1}{\alpha}, \\
\lambda_2 &= \lambda - \lambda_1, \\
G_1(t) &= 1 - e^{-\lambda_1 t}, \\
G_2(t) &= 1 - e^{-\lambda_2 t} \quad \textit{for any } t > 0, \\
G_{0,1}(t) &= q_{0,1}(0) + \psi \int_0^t (\lambda_2 + \rho + r_1) e^{-\rho\tau} \mathrm{d}\tau = q_{0,1}(0) + \frac{\psi}{\rho}(\lambda_2 + \rho + r_1)(1 - e^{-\rho t}), \\
G_{0,2}(t) &= q_{0,2}(0) + \frac{\psi}{\rho}(\lambda_1 + \rho + r_2)(1 - e^{-\rho t}).
\end{aligned}
$$

From Proposition 3, it can be seen that there are multiple equilibria in which both competing players concede gradually throughout the game. There are two sources of indeterminacy. First, at the start of the game, player 0's strategy is not fully pinned down; the only requirement is that it must satisfy $\min\{G_{0,1}(0), G_{0,2}(0)\} \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1} \max\{G_{0,1}(0), G_{0,2}(0)\}$. Second, after the start of the game, there is one degree of freedom in nailing down $\lambda_{0,1}(t)$ $\lambda_{0,2}(t), \lambda_1(t)$, and $\lambda_2(t)$. Once one of these four are fixed, the remaining can be determined by the three equations in parts (iii) and (iv) of Proposition 3. Of course, once the hazard rates are determined, the mixed strategy profile $(G_{0,1}, G_{0,2}, G_1, G_2)$ is also nailed down.

Table 1 compares players' expected payoffs across different equilibria. Different rows correspond to different equilibrium types. The columns correspond to players,

As can be seen from the table, the agenda setter strictly prefers the equilibria in which one of the competing players concede immediately, but is indifferent between the equilibria in which she herself concedes immediately to one of the competing player and the gradual concession equilibria.

By contrast, the competing players prefer the gradual concession equilibria to the immediate concession equilibria. The fact that competing player i prefers the gradual

	Player 0	Player i
Competing Player Immediate Concession	α_0	$(1-\alpha_0)/2$
Agenda Setter Immediate Concession	$1-\alpha$	$\alpha q_{0,i}(0)$
Gradual Concession	$1-\alpha$	$\alpha q_{0,i}(0)+(1-\alpha_0)(1-\tilde{q}_0(0))$

Table 1: Equilibrium Expected Payoffs

concession equilibria to the equilibria in which the agenda setter concedes immediately follows from the fact that $\tilde{q}_0(0) > 1$. The fact that competing player i prefers the gradual concession equilibria to the equilibria in which the one of the competing players concedes immediately follows from part (ii) of Proposition 3.

Whether competing player i prefers the equilibria in which competing players concede immediately to the equilibria in which the agenda setter concedes immediately depends on the demands of the agenda setter and the competing players as well as the agenda setter's randomization behavior at the beginning of the game. If it is less likely that the agenda setter concedes to player i or if the agenda setter's demand is relatively low or if the competing players' demand is relatively low (i.e., $q_{0,i}(0) < \frac{1-\alpha_0}{2\alpha}$)), then competing player i prefers the equilibria in which one of the competing players immediately concede to the equilibria in which the agenda setter immediately concedes. Intuitively when the demands are relatively low, then the gain from being conceded to as opposed to conceding is not very high. In addition, when $q_{0,i}$ is relatively low, the chances of being left out from the coalition altogether is high in the equilibria in which the agenda setter concedes immediately.

It follows that equilibria with delay are the most preferable equilibria for the competing players whereas it is the least preferable equilibria for the agenda setter. Furthermore, note that the equilibrium payoff for player $i \in 1,2$ in the two-player war of attrition is given by

$$(\alpha_1+\alpha_2-1)q_j(0)+(1-\alpha_j) \text{ for } j \neq i.$$

Compared to the two-player case, player 0 obtains the same equilibrium payoff whereas competing players obtain strictly less payoff in the three-player game.

One may wonder how the concession rates for three players are compared with those for two players. By Proposition 3, the concession rate for any competing player $i \in \{1,2\}$ strictly decreases due to the introduction of another competing player in the bargaining process. Specifically, the aggregate concession rate for competing players is $\lambda = \frac{(1-\alpha)r_0}{\alpha_0+\alpha-1}$, which is the same as the concession rate for player i in the two-player war of attrition and larger than his concession rate in the three player game (since $\lambda_j > 0$ and $\lambda_1+\lambda_2=\lambda$).

However, concession rate for player 0 strictly increases ($\rho > \frac{(1-\alpha_0)r_i}{\alpha_0+\alpha-1}$). Overall speaking, the concession rate is higher in the three-player war of attribution than in the two-player version.

We end the discussion about the gradual concession equilibrium by the comparative statics of concession rates with regard to the magnitude of interest conflicts and player's impatience. Recall that the aggregate concession rate of competing players is $\lambda = \frac{(1-\alpha)r_0}{\alpha_0+\alpha-1}$ and the concession rate of player 0 is $\rho = \frac{\frac{1-\alpha_0}{\alpha_0+\alpha-1}r_0+r_1+r_2}{\frac{\alpha}{1-\alpha_0}-2}$. First, for a given demand of competing players α, a larger demand of player 0, α_0, increases the interest conflict and also decreases the concession rate for both competing player and player 0. Similarly, for a given demand of player 0 α_0, a larger demand of competing player α increases the interest conflict and also decreases the concession rates. Second, λ and ρ strictly increase in player 0's impatience (r_0); additionally, the concession rate for player 0, ρ, also strictly increases in both competing players' impatience (r_1 and r_2).

4.2.2 One Competing Player Does Not Concede In Some Interval

We next characterize the equilibria in which one and only one competing player does not concede over some interval(s), i.e., $\exists a, b \in [0, \hat{t}]$ such that $G_i(a) = G_i(b)$ and G_j is strictly increasing over $[a, b]$. To start with, we describe what happens in such equilibria at the start of the game.

Lemma 13. *In any delay equilibrium in which competing player i concedes but competing player j does not concede over $(0, s)$ for some $s \in (0, \hat{t}]$, we have $q_j(0) = 0$, and $q_{0,i}(0) \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1} q_{0,j}(0)$.*

Lemma 13 asserts that if a competing player does not concede for an interval immediately after the game starts, then at the start of the game, this player does not concede, and the chair concedes to the other competing player with a sufficiently high probability. Compared to Lemma 12, now the other competing player can concede at the start with strictly positive probability.

In contrast to the case in which both competing players gradually concede throughout the game, there exists a delay equilibrium with suspense in which one competing player never concedes.

Proposition 4. *If $\alpha > 2(1-\alpha_0)$, there exists a mixed-strategy equilibrium in which, with probability one, one completing player does not concede. That is, $q_1(0) < 1$, $q_2(0) < 1$, $\tilde{q}_0(0) < 1$, and $\exists j \in \{1, 2\}$ such that $q_j(\infty) = 1$.*

Proof. Consider a mixed-strategy profile $(\tilde{G}_0, G_1, G_2)$ in which for any $t \in \bar{\mathbb{R}}_+$, $G_i(t) = 1 - e^{-\mu t}$, $G_j(t) = \mathbb{1}\{t = \infty\}$ for any $t \in \bar{\mathbb{R}}_+$, $\tilde{G}_0(0) = 0$ and $\lambda_0(t) \geq \frac{(1-\alpha_0)(r_1+r_2+\mu)}{\alpha-2(1-\alpha_0)} = \rho$ (that is, the hazard rate for player 0 is sufficiently large). Then for player 0, by Equations A1,A2,A3, we have

$$
\begin{aligned}
U_0(0, \kappa, G_1, G_2) =& 1 - \alpha, \\
U_0(\infty, \kappa, G_1, G_2) =& \sum_{i=1}^{2} \int_0^\infty \alpha_0 e^{-r_0 v}(1 - G_j(v)) dG_i(v) = \int_0^\infty \alpha_0 e^{-r_0 v} dG_i(v). \\
=& \alpha_0 \mu \int_0^\infty e^{-(r_0+\mu)v} dv = \alpha_0 \mu \left. \frac{e^{-(r_0+\mu)v}}{r_0 + \mu} \right|_\infty^0 = \frac{\alpha_0 \mu}{r_0 + \mu} = 1 - \alpha. \\
U_0(t, \kappa, G_1, G_2) =& (1 - \alpha) e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) + \sum_{i=1}^{2} \int_0^t \alpha_0 e^{-r_0 v}(1 - G_j(v)) \mathrm{d}G_i(v) \\
=& (1 - \alpha) e^{-r_0 t} e^{-\mu t} + \alpha_0 \mu \left. \frac{e^{-(r_0+\mu)v}}{r_0 + \mu} \right|_t^0 \\
=& (1 - \alpha) e^{-(r_0+\mu)t} + (1 - \alpha)(1 - e^{-(r_0+\mu)t}) = 1 - \alpha
\end{aligned}
$$

for any $t > 0$. Therefore, player 0 is indifferent about the concession time — no matter when she concedes, her expected payoff is always $1 - \alpha$.

Next we consider player i. By Equations A4,A5, A6, and

$$
G_{0,i}(t) = q_{0,i}(0) + \frac{(1 - \alpha_0)(1 - \tilde{q}_0(0))}{\alpha} \int_0^t (r_i + \lambda_0(\tau)) e^{-\Lambda_0(\tau)} \mathrm{d}\tau,
$$

we have

$$
\begin{aligned}
U_i(0, G_0, G_j) =& (1 - \alpha_0)(1 - \tilde{q}_0(0)) + \frac{1 - \alpha_0}{2} q_{0,j}(0) + \Big(\frac{1 - \alpha_0 + \alpha}{2}\Big) q_{0,i}(0) = 1 - \alpha_0, \\
U_i(\infty, G_0, G_j) =& \alpha q_{0,i}(0) + \int_0^\infty \alpha e^{-r_i v}(1 - G_j(v)) \mathrm{d}G_{0,i}(v) = \int_0^\infty \alpha e^{-r_i v} \mathrm{d}G_{0,i}(v) \\
=& \int_0^\infty \alpha e^{-r_i v} \frac{(1 - \alpha_0)(1 - \tilde{q}_0(0))}{\alpha} (r_i + \lambda_0(v)) e^{-\tilde{\Lambda}_0(v)} \mathrm{d}v \\
=& (1 - \alpha_0) \int_0^\infty (r_i + \lambda_0(v)) e^{-\tilde{\Lambda}_0(v) - r_i v} \mathrm{d}v = 1 - \alpha_0,
\end{aligned}
$$

$$
\begin{aligned}
U_i(t,G_0,G_j) =&\alpha q_{0,i}(0) + \int_0^t \alpha e^{-r_i v}(1-G_j(v))dG_{0,i}(v) + (1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t))(1-G_j(t)) \\
=&\int_0^t \alpha e^{-r_i v} dG_{0,i}(v) + (1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t)) \\
=&(1-\alpha_0)\int_0^t (r_i+\lambda_0(v))e^{-\tilde{\Lambda}_0(v)-r_i v}\mathrm{d}v + (1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t)) \\
=&(1-\alpha_0)(1-e^{-\tilde{\Lambda}_0(t)-r_i t}) + (1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t)) \\
=&(1-\alpha_0)[1-(1-\tilde{G}_0(t))e^{-r_i t}] + (1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t)) = 1-\alpha_0
\end{aligned}
$$

for any $t>0$. Therefore, player i is indifferent about the concession time — no matter when he concedes, his expected payoff is always $1-\alpha_0$.

Finally we consider player j. By Equations A4,A5, A6, and $G_{0,j}(t) = \tilde{G}_0(t) - G_{0,i}(t) = \tilde{G}_0(t) - q_{0,i}(0) - \frac{(1-\alpha_0)(1-\tilde{q}_0(0))}{\alpha}\int_0^t (r_i+\lambda_0(\tau))e^{-\Lambda_0(\tau)}\mathrm{d}\tau$, we have

$$
\begin{aligned}
U_j(0,G_0,G_i) =&(1-\alpha_0)(1-\tilde{q}_0(0)) + \frac{1-\alpha_0}{2}q_{0,i}(0) + \Big(\frac{1-\alpha_0+\alpha}{2}\Big)q_{0,j}(0) = 1-\alpha_0 \\
U_j(\infty,G_0,G_i) =&\alpha q_{0,j}(0) + \int_0^\infty \alpha e^{-r_j v}(1-G_i(v))\mathrm{d}G_{0,j}(v) = \int_0^\infty \alpha e^{-(r_j+\mu)v}\mathrm{d}G_{0,j}(v) \\
U_j(t,G_0,G_i) =&\alpha q_{0,j}(0) + \int_0^t \alpha e^{-r_j v}(1-G_i(v))dG_{0,j}(v) + (1-\alpha_0)e^{-r_j t}(1-\tilde{G}_0(t))(1-G_i(t)) \\
=&\int_0^t \alpha e^{-(r_j+\mu)v}dG_{0,j}(v) + (1-\alpha_0)e^{-(r_j+\mu)t}(1-\tilde{G}_0(t)) \\
=&\int_0^t \alpha e^{-(r_j+\mu)v}[\lambda_0(v)e^{-\Lambda_0(v)} - \frac{(1-\alpha_0)}{\alpha}(r_i+\lambda_0(v))e^{-\Lambda_0(v)}]\mathrm{d}v \\
&+(1-\alpha_0)e^{-(r_j+\mu)t-\Lambda_0(t)} \\
=&\int_0^t [\alpha\lambda_0(v) - (1-\alpha_0)(r_i+\lambda_0(v))]e^{-(r_j+\mu)v-\Lambda_0(v)}\mathrm{d}v + (1-\alpha_0)e^{-(r_j+\mu)t-\Lambda_0(t)}
\end{aligned}
$$

for any $t>0$. Note that $U_j(t,G_0,G_j)$ is differentiable over $\mathbb{R}_{++}$ and is continuous at $t=0$ and $t=\infty$. We take first derivative of $U_j(t,G_0,G_j)$ with respect to t and get

$$
\begin{aligned}
\frac{\partial U_j(t,G_0,G_i)}{\partial t} =&[\alpha\lambda_0(t) - (1-\alpha_0)(r_i+\lambda_0(t))]e^{-(r_j+\mu)t-\Lambda_0(t)} \\
&-(1-\alpha_0)(r_j+\mu+\lambda_0(t))e^{-(r_j+\mu)t-\Lambda_0(t)} \\
=&[\alpha\lambda_0(t) - (1-\alpha_0)(r_i+2\lambda_0(t)+r_j+\mu)]e^{-(r_j+\mu)t-\Lambda_0(t)} \\
=&\{[\alpha-2(1-\alpha_0)]\lambda_0(t) - (1-\alpha_0)(r_1+r_2+\mu)\}e^{-(r_j+\mu)t-\Lambda_0(t)}
\end{aligned}
$$

Note if $\alpha \le 2(1-\alpha_0)$, then $\frac{\partial U_j(t,G_0,G_i)}{\partial t} < 0$, which implies the expected utility for player j is strictly decreasing over time, a contradiction to $q_j(\infty)=1$. As long as $\alpha > 2(1-\alpha_0)$, we can make $\lambda_0(t) \geq \frac{(1-\alpha_0)(r_1+r_2+\mu)}{\alpha-2(1-\alpha_0)} = \rho$. As such, $\frac{\partial U_j(t,G_0,G_i)}{\partial t} \geq 0$, which implies the expected utility for player j is weakly increasing over time. Therefore, $q_j(\infty) = 1$ is a best response for player j. ■

Remark 1. *In an equilibrium with delay in which both competing players gradually concede, player 0's concession rate is constant and equal to ρ throughout the game. In contrast, in an equilibrium with delay in which one competing player does not concede for some interval, player 0's concession rate can be higher than ρ.*

Additionally, we can specify another type of possible delay equilibria with suspense in which two competing players take turns to concede to the chair. The following proposition states the existence of such equilibria.

Proposition 5. *If $\alpha > 2(1-\alpha_0)$, there exists a mixed-strategy equilibrium in which two completing players take turns to concede. That is, $q_1(0) < 1$, $q_2(0) < 1$, $\tilde{q}_0(0) < 1$, for any time interval $[t_1, t_2]$ where $t_2 > t_1 \geq 0$, there exists a completing player $c \in \{1, 2\}$ such that $G_c(t_1) = G_c(t_2)$ and $G_{3-c}(\cdot)$ is strictly increasing over $[t_1, t_2]$.*

Proof. Fix $S > 0$. Consider a mixed-strategy profile $(\tilde{G}_0, G_1, G_2)$ in which

$$G_i(t) = \begin{cases} 1 - e^{-\mu t} & \text{if } t \in [0, S), \\ 1 - e^{-\mu S} & \text{if } t \in [S, \infty), \\ 1 & \text{if } t = \infty. \end{cases} \qquad G_j(t) = \begin{cases} 0 & \text{if } t \in [0, S), \\ 1 - e^{-\mu(t-S)} & \text{if } t \in [S, \infty]. \end{cases}$$

and $\tilde{G}_0(t) = 1 - e^{-\rho t}$ for $t \in \bar{\mathbb{R}}_+$. Under this strategy profile, before time S only player 1 gradually concedes, and after time S only player 2 gradually concedes. Now we consider player 0. By Equations A1,A2,A3, we have

$$\begin{aligned}
U_0(0, \kappa, G_1, G_2) =& 1 - \alpha, \\
U_0(\infty, \kappa, G_1, G_2) =& \sum_{i=1}^{2} \int_0^\infty \alpha_0 e^{-r_0 v}(1 - G_j(v)) dG_i(v) \\
=& \int_0^S \alpha_0 e^{-r_0 v} dG_i(v) + \int_S^\infty \alpha_0 e^{-r_0 v} e^{-\mu S} dG_j(v) \\
=& \alpha_0 \mu \int_0^S e^{-(r_0+\mu)v} dv + \alpha_0 \mu \int_S^\infty e^{-(r_0+\mu)v} dv \\
=& \alpha_0 \mu \int_0^\infty e^{-(r_0+\mu)v} dv = \alpha_0 \mu \left. \frac{e^{-(r_0+\mu)v}}{r_0+\mu} \right|_\infty^0 = \frac{\alpha_0 \mu}{r_0 + \mu} = 1 - \alpha,
\end{aligned}$$

$$\begin{aligned}U_0(t,\kappa,G_1,G_2) =&(1-\alpha)e^{-r_0t}(1-G_1(t))(1-G_2(t))+\sum_{i=1}^{2}\int_0^t \alpha_0 e^{-r_0v}(1-G_j(v))\mathrm{d}G_i(v)\\ =&(1-\alpha)e^{-r_0t}(1-H(t))+\alpha_0\mu\left.\frac{e^{-(r_0+\mu)v}}{r_0+\mu}\right|_t^0\\ =&(1-\alpha)e^{-(r_0+\mu)t}+(1-\alpha)(1-e^{-(r_0+\mu)t})=1-\alpha.\end{aligned}$$

Therefore, player 0 is indifferent about the concession time — no matter when she concedes, her expected payoff is always $1-\alpha$.

Next consider player i. When $t\in[0,S)$,

$$G_{0,i}(t)=q_{0,i}(0)+\frac{(1-\alpha_0)(1-\tilde{q}_0(0))}{\alpha}\int_0^t (r_i+\lambda_0(\tau))e^{-\Lambda_0(\tau)}\mathrm{d}\tau.$$

When $t\in[S,\infty]$,

$$\begin{aligned}G_{0,i}(t)=q_{0,i}(0)+&\frac{(1-\alpha_0)(1-\tilde{q}_0(0))}{\alpha}\int_0^S (r_i+\lambda_0(\tau))e^{-\Lambda_0(\tau)}\mathrm{d}\tau\\ +&\frac{1-\tilde{q}_0(0)}{\alpha}\int_S^t[(\alpha+\alpha_0-1)\lambda_0(\tau)-(1-\alpha_0)r_j]e^{-\Lambda_0(\tau)}\mathrm{d}\tau.\end{aligned}$$

Thus by Equations A4,A5, A6, we have

$$\begin{aligned}U_i(0,G_0,G_j) =&(1-\alpha_0)(1-\tilde{q}_0(0))+\frac{1-\alpha_0}{2}q_{0,j}(0)+\Big(\frac{1-\alpha_0+\alpha}{2}\Big)q_{0,i}(0)=1-\alpha_0.\\ U_i(\infty,G_0,G_j) =&\alpha q_{0,i}(0)+\int_0^\infty \alpha e^{-r_iv}(1-G_j(v))\mathrm{d}G_{0,i}(v)=\int_0^\infty \alpha e^{-r_iv}(1-G_j(v))\mathrm{d}G_{0,i}(v).\end{aligned}$$

For any $t\in(0,S)$,

$$\begin{aligned}U_i(t,G_0,G_j) =&\alpha q_{0,i}(0)+\int_0^t \alpha e^{-r_iv}(1-G_j(v))dG_{0,i}(v)+(1-\alpha_0)e^{-r_it}(1-\tilde{G}_0(t))(1-G_j(t))\\ =&\int_0^t \alpha e^{-r_iv}dG_{0,i}(v)+(1-\alpha_0)e^{-r_it}(1-\tilde{G}_0(t))\\ =&(1-\alpha_0)\int_0^t (r_i+\lambda_0(v))e^{-\Lambda_0(v)-r_iv}\mathrm{d}v+(1-\alpha_0)e^{-r_it}(1-\tilde{G}_0(t))\\ =&(1-\alpha_0)(1-e^{-\Lambda_0(t)-r_it})+(1-\alpha_0)e^{-r_it}(1-\tilde{G}_0(t))\\ =&(1-\alpha_0)[1-(1-\tilde{G}_0(t))e^{-r_it}]+(1-\alpha_0)e^{-r_it}(1-\tilde{G}_0(t))\\ =&1-\alpha_0=U_i(0,G_0,G_j).\end{aligned}$$

For any $t \in [S, \infty)$,

$$\begin{aligned}
U_i(t, G_0, G_j) =& \alpha q_{0,i}(0) + \int_0^t \alpha e^{-r_i v}(1 - G_j(v))dG_{0,i}(v) + (1-\alpha_0)e^{-r_i t}(1 - \tilde{G}_0(t))(1 - G_j(t)) \\
=& \int_0^S \alpha e^{-r_i v} dG_{0,i}(v) + \int_S^t \alpha e^{-r_i v}(1 - G_j(v))dG_{0,i}(v) + (1-\alpha_0)e^{-r_i t}(1 - \tilde{G}_0(t))e^{-\mu(t-S)} \\
=& (1-\alpha_0)[1 - (1 - \tilde{G}_0(S))e^{-r_i S}] + (1-\alpha_0)(1 - \tilde{q}_0(0))e^{-(r_i+\mu)t+\mu S - \Lambda_0(t)} \\
&+ (1 - \tilde{q}_0(0)) \int_S^t e^{-r_i v - \mu(v-S)}[(\alpha + \alpha_0 - 1)\lambda_0(\tau) - (1-\alpha_0)r_j]e^{-\Lambda_0(v)}dv.
\end{aligned}$$

Note that $U_i(t, G_0, G_j)$ is differentiable over (S, ∞) and is continuous at $t = S$ and $t = \infty$. We take first derivative of $U_i(t, G_0, G_j)$ with respect to t and get

$$\begin{aligned}
\frac{\partial U_i(t, G_0, G_j)}{\partial t} =& (1 - \tilde{q}_0(0))e^{-r_i t - \mu(t-S)}[(\alpha + \alpha_0 - 1)\lambda_0(t) - (1-\alpha_0)r_j]e^{-\Lambda_0(t)} \\
&- (1-\alpha_0)(1 - \tilde{q}_0(0))e^{-(r_i+\mu)t+\mu S-\Lambda_0(t)}(r_i + \mu + \lambda_0(t)) \\
=& e^{-r_i t - \mu(t-S) - \Lambda_0(t)}[(\alpha + \alpha_0 - 1)\lambda_0(t) - (1-\alpha_0)r_j - (1-\alpha_0)(r_i + \mu + \lambda_0(t))] \\
=& (1-\alpha_0)e^{-r_i t - \mu(t-S) - \Lambda_0(t)}[(\frac{\alpha}{1-\alpha_0} - 2)\lambda_0(t) - r_1 - r_2 - \mu].
\end{aligned}$$

Note if $\alpha \leq 2(1-\alpha_0)$, then $\frac{\partial U_i(t,G_0,G_j)}{\partial t} < 0$, which implies the expected utility for player i is strictly decreasing over time, a contradiction to $q_i(\infty) > 0$. For $\alpha > 2(1-\alpha_0)$, we can make $\lambda_0(t) = \frac{(1-\alpha_0)(r_1+r_2+\mu)}{\alpha - 2(1-\alpha_0)} = \rho$. As such, $\frac{\partial U_j(t,G_0,G_j)}{\partial t} = 0$, which implies the expected utility for player j is constant over time. Therefore, player i is indifferent about the concession time — no matter when she concedes, her expected payoff always remains $1 - \alpha_0$.

Lastly consider player j. When $t \in [0, S)$,

$$G_{0,j}(t) = \tilde{G}_0(t) - G_{0,i}(t) = \tilde{G}_0(t) - q_{0,i}(0) - \frac{(1-\alpha_0)(1-\tilde{q}_0(0))}{\alpha} \int_0^t (r_i + \lambda_0(\tau))e^{-\Lambda_0(\tau)}\mathrm{d}\tau.$$

When $t \in [S, \infty]$,

$$\begin{aligned}
G_{0,j}(t) =& G_{0,j}(S) + \int_S^t \lambda_{0,j}(\tau)(1 - \tilde{G}_0(\tau))d\tau \\
=& \tilde{G}_0(S) - q_{0,i}(0) - \frac{(1-\alpha_0)(1-\tilde{q}_0(0))}{\alpha} \int_0^S (r_i + \lambda_0(\tau))e^{-\Lambda_0(\tau)}\mathrm{d}\tau \\
&+ \frac{(1-\alpha_0)(1-\tilde{q}_0(0))}{\alpha} \int_S^t (r_j \\
&+ \lambda_0(\tau))e^{-\Lambda_0(\tau)}d\tau.
\end{aligned}$$

Thus by Equations A4,A5, A6, we have

$$
\begin{aligned}
U_j(0, G_0, G_i) =& (1-\alpha_0)(1-\tilde{q}_0(0)) + \frac{1-\alpha_0}{2} q_{0,i}(0) + \Big(\frac{1-\alpha_0+\alpha}{2}\Big) q_{0,j}(0) = 1-\alpha_0. \\
U_j(\infty, G_0, G_i) =& \alpha q_{0,j}(0) + \int_0^\infty \alpha e^{-r_j v}(1-G_i(v))\mathrm{d}G_{0,j}(v) = \int_0^\infty \alpha e^{-r_j v}(1-G_i(v))\mathrm{d}G_{0,j}(v).
\end{aligned}
$$

For any $t \in (0, S)$,

$$
\begin{aligned}
U_j(t, G_0, G_i) =& \alpha q_{0,j}(0) + \int_0^t \alpha e^{-r_j v}(1-G_i(v)) dG_{0,j}(v) + (1-\alpha_0)e^{-r_j t}(1-\tilde{G}_0(t))(1-G_i(t)) \\
=& \int_0^t \alpha e^{-(r_j+\mu)v} dG_{0,j}(v) + (1-\alpha_0)e^{-(r_j+\mu)t}(1-\tilde{G}_0(t)) \\
=& \int_0^t \alpha e^{-(r_j+\mu)v}[\lambda_0(v)e^{-\Lambda_0(v)} - \frac{(1-\alpha_0)}{\alpha}(r_i+\lambda_0(v))e^{-\Lambda_0(v)}]\mathrm{d}v \\
& +(1-\alpha_0)e^{-(r_j+\mu)t-\Lambda_0(t)} \\
=& \int_0^t [\alpha\lambda_0(v) - (1-\alpha_0)(r_i+\lambda_0(v))]e^{-(r_j+\mu)v-\Lambda_0(v)}\mathrm{d}v + (1-\alpha_0)e^{-(r_j+\mu)t-\Lambda_0(t)}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial U_j(t, G_0, G_i)}{\partial t} =& [\alpha\lambda_0(t) - (1-\alpha_0)(r_i+\lambda_0(t))]e^{-(r_j+\mu)t-\Lambda_0(t)} \\
& -(1-\alpha_0)(r_j+\mu+\lambda_0(t))e^{-(r_j+\mu)t-\Lambda_0(t)} \\
=& [\alpha\lambda_0(t) - (1-\alpha_0)(r_i+2\lambda_0(t)+r_j+\mu)]e^{-(r_j+\mu)t-\Lambda_0(t)} \\
=& \{[\alpha-2(1-\alpha_0)]\lambda_0(t) - (1-\alpha_0)(r_1+r_2+\mu)\}e^{-(r_j+\mu)t-\Lambda_0(t)}.
\end{aligned}
$$

Since $\lambda_0(t) = \frac{(1-\alpha_0)(r_1+r_2+\mu)}{\alpha-2(1-\alpha_0)} = \rho$, we can obtain $\frac{\partial U_j(t,G_0,G_j)}{\partial t} = 0$, which implies the expected utility for player j is constant over time $(0, S)$. Note that $U_j(t, G_0, G_i)$ is continuous at $t = 0$. $U_j(t, G_0, G_i) = U_j(0, G_0, G_i) = 1-\alpha_0$.

For any $t \in [S, \infty)$,

$$\begin{aligned}
U_j(t, G_0, G_i) =& \alpha q_{0,j}(0) + \int_0^t \alpha e^{-r_j v}(1 - G_i(v))dG_{0,j}(v) + (1-\alpha_0)e^{-r_j t}(1 - \tilde{G}_0(t))(1 - G_i(t)) \\
=& \alpha q_{0,j}(0) + \int_0^S \alpha e^{-r_j v}(1 - G_i(v))dG_{0,j}(v) + \int_S^t \alpha e^{-r_j v}(1 - G_i(v))dG_{0,j}(v) \\
&+ (1-\alpha_0)e^{-(r_j+\rho)t - \mu S} \\
=& 1 - \alpha_0 - (1-\alpha_0)e^{-(r_j+\rho+\mu)S} + (1-\alpha_0)e^{-\mu S}\int_S^t e^{-(r_j+\rho)v}(r_j+\rho)dv \\
&+ (1-\alpha_0)e^{-(r_j+\rho)t - \mu S} \\
=& (1-\alpha_0)[1 - e^{-(r_j+\rho+\mu)S} + e^{-\mu S}(e^{-(r_j+\rho)S} - e^{-(r_j+\rho)t}) + e^{-(r_j+\rho)t-\mu S}] \\
=& 1 - \alpha_0 = U_j(0, G_0, G_i).
\end{aligned}$$

Therefore, player i is indifferent about the concession time — no matter when she concedes, her expected payoff always remains $1 - \alpha_0$. ■

Note the switching equilibrium characterized here is a limiting case of the gradual concession equilibrium when λ_i goes to zero, $i \in \{1, 2\}$.

Now we can describe equilibria with delay by answering the questions at the start of Section 4 and compare these answers to those in Hendricks et al. (1988) for the two-player war of attrition.

1. Start:
   - (i) If both competing players gradually concede after the game begins, there is an asymmetry in concession choices between player 0 and competing players at the start of the game. Only player 0 can concede with strictly positive probability at the start of the game — when this happens, player 0 must concede to both competing players and the relative concession probability must fall in a moderate range $\frac{\min\{q_{0,1}(0), q_{0,2}(0)\}}{\max\{q_{0,1}(0), q_{0,2}(0)\}} \in [\frac{1-\alpha_0}{\alpha+\alpha_0-1}, 1]$.
   - (ii) If one competing player does not concede after the game begins, then this competing player cannot concede with strictly positive probability at the start of the game.
   - (iii) In contrast, for the two-player war of attrition, either (and at most one) player can concede with strictly positive probability at the start of the game, and the concession probability can be chosen freely in the range $[0, 1)$.
2. Atom points: no atom points for all players ($q_{0,1}(t) = q_{0,2}(t) = q_1(t) = q_2(t) = 0$ for any $t > 0$). This result is the same as the two-player case.

3. Suspense: suspense can occur for one competing player in equilibrium for three-player war of attrition. In contrast, both players must gradually concede in any delay equilibrium in the two-player case.
4. End: when both competing players gradually concede throughout the game, the game does not terminate in finite time for any delay equilibrium as in the two-player case.

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A Appendix

The following technical lemma is used extensively in our analysis.

Lemma A1. *Fix a mixed strategy profile* $G = (G_0, G_1, G_2) \in \mathcal{G}_0 \times \mathcal{G}_1 \times \mathcal{G}_2$. *Then, for each* $(a, b) \subseteq \mathbb{R}_+$ *with* $a < b$, *there exists* $t \in (a, b)$ *such that* G *is continuous at* t.

Proof of Lemma A1. Follows immediately from the monotonicity of each component of G and the fact that monotone functions are continuous almost everywhere.[3] ■

[3] See, e.g., Royden and Fitzpatrick (2010, p. 108).

Lemma A2. *For any $\varepsilon > 0$, any $t \geq 0$, and any $j \in \{1, 2\}$, there exists $\delta > 0$ such that:*

(a) $q_j(t+\delta) = 0$, $q_{0,j}(t+\delta) = 0$;

(b) $G_j(t+\delta) - G_j(t) < \varepsilon$, $G_{0,j}(t+\delta) - G_{0,j}(t) < \varepsilon$;

(c) $1 - e^{-r_0\delta} < \varepsilon$, $1 - e^{-r_j\delta} < \varepsilon$;

(d) if $t > 0$, *then* $q_j(t-\delta) = 0$ *and* $q_{0,j}(t-\delta) = 0$;

(e) if $t > 0$, *then* $G_j(t) - G_j(t-\delta) - q_j(t) < \varepsilon$ *and* $G_{0,j}(t) - G_{0,j}(t-\delta) - q_{0,j}(t) < \varepsilon$.

Proof of Lemma A2. Fix any $\varepsilon > 0$, any t, any $j \in \{1, 2\}$. To prove (a), (b), and (c), first notice that by the definition of right-continuity there exists $\delta_1 > 0$ such that $G_j(t+\delta) - G_j(t) < \varepsilon$ and $G_{0,j}(t+\delta) - G_{0,j}(t) < \varepsilon$ for all $\delta < \delta_1$. Next, by continuity there exists $\delta_2 > 0$ such that $1 - e^{-r_0\delta} < \varepsilon$ for all $\delta < \delta_2$. Finally, by Lemma A1 there exists $t_\varepsilon \in (t, t + \min\{\delta_1, \delta_2\})$ such that $q_j(t_\varepsilon) = 0$. Letting $\delta = t_\varepsilon - t$, we conclude that (a), (b), and (c) hold.

To prove (d) and (e), suppose $t > 0$. First, by the definitions of $q_j(\cdot)$ and $q_{0,j}(\cdot)$, there exists $\delta_1 > 0$ such that $G_j(t) - G_j(t-\delta) - q_j(t) < \varepsilon$ and $G_{0,j}(t) - G_{0,j}(t-\delta) - q_{0,j}(t) < \varepsilon$ for all $\delta < \delta_1$. Next, by continuity there exists $\delta_2 > 0$ such that $1 - e^{-r_0\delta} < \varepsilon$ for all $\delta < \delta_2$. Finally, by Lemma A1 there exists $t_\varepsilon \in (t - \min\{\delta_1, \delta_2\}, t)$ such that $q_j(t_\varepsilon) = 0$. Letting $\delta = t - t_\varepsilon$, we conclude that (d) and (e) hold. ■

A.1 Expected Payoff in a Delay Equilibrium

For any delay equilibrium, players' expected payoffs depend on their concession time: whether to concede at the start, concede later, or never concede.

Player 0

$$
\begin{aligned}
&U_0(0, \kappa, G_1, G_2) && \text{(A1)} \\
&= (1-\alpha_\kappa)(1-q_1(0))(1-q_2(0)) && \text{[neither player concedes at 0]} \\
&\quad + \Big(\frac{1-\alpha_\kappa}{3} + \frac{2\alpha_0}{3}\Big) q_1(0) q_2(0) && \text{[both concede at 0]} \\
&\quad + \Big(\frac{1-\alpha_\kappa}{2} + \frac{\alpha_0}{2}\Big) q_1(0)(1-q_2(0)) && \text{[player 1 concedes at 0]} \\
&\quad + \Big(\frac{1-\alpha_\kappa}{2} + \frac{\alpha_0}{2}\Big) q_2(0)(1-q_1(0)). && \text{[player 2 concedes at 0]}
\end{aligned}
$$

$$
\begin{aligned}
&U_0(\infty, \kappa, G_1, G_2) && \text{(A2)}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_0(\infty, \kappa, t_1, t_2) dG_1(t_1) dG_2(t_2)\\
&= \alpha_0\Big[1-(1-q_1(0))(1-q_2(0))\Big] && \text{[at least one player concedes at 0]}\\
&\quad + \sum_{v>0} \alpha_0 e^{-r_0 v} q_1(v) q_2(v) && \text{[both players concede at } v\text{]}\\
&\quad + \sum_{i=1}^{2}\int_0^{\infty}\int_0^{t_j} \alpha_0 e^{-r_0 t_i} dG_i(t_i) dG_j(t_j) && [i \text{ concedes before } j \text{ concedes]}\\
&\quad + \sum_{i=1}^{2}\int_0^{\infty} \alpha_0 e^{-r_0 t_i} q_j(\infty) dG_i(t_i) && [i \text{ concedes and } j \text{ never concedes]}\\
&= \alpha_0\Big[1-(1-q_1(0))(1-q_2(0))\Big]\\
&\quad + \sum_{v>0} \alpha_0 e^{-r_0 v} q_1(v) q_2(v)\\
&\quad + \sum_{i=1}^{2}\int_0^{\infty} \alpha_0 e^{-r_0 v}(1-q_j(\infty)-G_j(v)) dG_i(v)\\
&\quad + \sum_{i=1}^{2}\int_0^{\infty} \alpha_0 e^{-r_0 v} q_j(\infty) dG_i(v)\\
&= \alpha_0\Big[1-(1-q_1(0))(1-q_2(0))\Big] && \text{[at least one player concedes at 0]}\\
&\quad + \sum_{v>0} \alpha_0 e^{-r_0 v} q_1(v) q_2(v) && \text{[both players concede at } v\text{]}\\
&\quad + \sum_{i=1}^{2}\int_0^{\infty} \alpha_0 e^{-r_0 v}(1-G_j(v)) dG_i(v). && \text{[only } i \text{ concedes at } v\text{]}
\end{aligned}
$$

To see why the third equation holds, i.e.,

$$\int_0^{\infty}\int_0^{t_j} \alpha_0 e^{-r_0 t_i} dG_i(t_i) dG_j(t_j) = \int_0^{\infty} \alpha_0 e^{-r_0 v}(1-q_j(\infty)-G_j(v)) dG_i(v),$$

we provide two approaches to prove:

Approach 1: *Change the Order of Integration*

$$\begin{aligned}\int_0^\infty \int_0^{t_j} \alpha_0 e^{-r_0 t_i} dG_i(t_i) dG_j(t_j) &= \int_0^\infty \int_{t_i}^\infty \alpha_0 e^{-r_0 t_i} dG_j(t_j) dG_i(t_i) \\ &= \int_0^\infty \alpha_0 e^{-r_0 t_i}(1 - q_j(\infty) - G_j(t_i)) dG_i(t_i).\end{aligned}$$

Approach 2: *Integral by Parts*

$$\begin{aligned}\int_0^\infty \int_0^{t_j} \alpha_0 e^{-r_0 t_i} dG_i(t_i) dG_j(t_j) &= \int_\infty^0 \int_0^{t_j} \alpha_0 e^{-r_0 t_i} dG_i(t_i) d(1 - G_j(t_j)) \\ &= \int_0^{t_j} \alpha_0 e^{-r_0 t_i} dG_i(t_i)(1 - G_j(t_j))\Big|_\infty^0 - \int_\infty^0 (1 - G_j(t_j)) \alpha_0 e^{-r_0 t_j} dG_i(t_j) \\ &= -\int_0^\infty \alpha_0 e^{-r_0 t_i} dG_i(t_i) q_j(\infty) + \int_0^\infty (1 - G_j(v)) \alpha_0 e^{-r_0 v} dG_i(v) \\ &= \int_0^\infty \alpha_0 e^{-r_0 v}(1 - q_j(\infty) - G_j(v)) dG_i(v).\end{aligned}$$

For any $t > 0$, we have:

$$\begin{aligned}&U_0(t, \kappa, G_1, G_2) && \text{(A3)} \\ &= \alpha_0 \Big[1 - (1 - q_1(0))(1 - q_2(0))\Big] && \text{[at least one player concedes at 0]} \\ &\quad + (1 - \alpha_\kappa) e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) && \text{[neither concedes by } t\text{]} \\ &\quad + \sum_{i=1}^2 \int_0^t \alpha_0 e^{-r_0 v}(1 - G_j(v)) \mathrm{d}G_i(v) && \text{[only } i \text{ concedes at } v < t\text{]} \\ &\quad + \sum_{0<v<t} \alpha_0 e^{-r_0 v} q_1(v) q_2(v) && \text{[both concede at } v < t\text{]} \\ &\quad + \sum_{i=1}^2 \Big(\frac{1 - \alpha_\kappa}{2} + \frac{\alpha_0}{2}\Big) e^{-r_0 t}\Big[q_i(t)(1 - G_j(t))\Big] && \text{[only } i \text{ concedes at } t\text{]} \\ &\quad + \Big(\frac{1 - \alpha_\kappa}{3} + \frac{2\alpha_0}{3}\Big) e^{-r_0 t} q_1(t) q_2(t). && \text{[both concede at } t\text{]}\end{aligned}$$

Player $i = 1, 2$

$$
\begin{aligned}
&U_i(0, G_0, G_j) \qquad\qquad \text{(A4)}\\
&= (1-\alpha_0)\underbrace{(1-\tilde{q}_0(0))(1-q_j(0))}_{\text{neither 0 nor } j \text{ concedes at 0}} + \frac{1-\alpha_0}{2}\Big[\underbrace{q_{0,j}(0)(1-q_j(0))}_{i \to 0 \to j \text{ at } 0} + \underbrace{(1-\tilde{q}_0(0))q_j(0)}_{i \to 0 \leftarrow j \text{ at } 0}\Big]\\
&+ \Big(\frac{1-\alpha_0}{2} + \frac{\alpha_i}{2}\Big)\underbrace{q_{0,i}(0)(1-q_j(0))}_{i \leftrightarrow 0 \not\leftarrow j \text{ at } 0} + \Big(\frac{1-\alpha_0}{3} + \frac{\alpha_i}{3}\Big)\underbrace{q_{0,i}(0)q_j(0)}_{i \leftrightarrow 0 \leftarrow j \text{ at } 0} + \frac{1-\alpha_0}{3}\underbrace{q_{0,j}(0)q_j(0)}_{i \to 0 \leftrightarrow j \text{ at } 0}.
\end{aligned}
$$

$$
\begin{aligned}
U_i(\infty, G_0, G_j) = &\alpha_i \underbrace{q_{0,i}(0)(1-q_j(0))}_{j \not\to 0 \to i \text{ at } 0} + \frac{\alpha_i}{2}\underbrace{q_{0,i}(0)q_j(0)}_{j \to 0 \to i \text{ at } 0} + \int_0^\infty \alpha_i e^{-r_i v}\underbrace{(1-G_j(v))\mathrm{d}G_{0,i}(v)}_{j \not\to 0 \to i \text{ by } v} \qquad \text{(A5)}\\
&+ \sum_{v>0} \frac{\alpha_i}{2} e^{-r_i v}\underbrace{q_{0,i}(v)q_j(v)}_{j \to 0 \to i \text{ by } v}.
\end{aligned}
$$

$$
\begin{aligned}
&U_i(t, G_0, G_j) && \text{(A6)}\\
&= \alpha_i q_{0,i}(0)(1-q_j(0)) && [j \not\to 0 \to i \text{ at } 0]\\
&\quad + \frac{\alpha_i}{2} q_{0,i}(0)q_j(0) && [j \to 0 \to i \text{ at } 0]\\
&\quad + \int_0^t \alpha_i e^{-r_i v}(1-G_j(v))\mathrm{d}G_{0,i}(v) && [j \not\to 0 \to i \text{ by } v]\\
&\quad + \sum_{0<v<t} \frac{\alpha_i}{2} e^{-r_i v} q_{0,i}(v)q_j(v) && [j \to 0 \to i \text{ by } v]\\
&\quad + (1-\alpha_0)e^{-r_i t}(1-\tilde{G}_0(t))(1-G_j(t)) && [\text{neither 0 nor } j \text{ concedes by } t]\\
&\quad + \frac{1-\alpha_0}{2} e^{-r_i t} q_{0,j}(t)(1-G_j(t)) && [i \to 0 \to j \text{ at } t]\\
&\quad + \frac{1-\alpha_0}{2} e^{-r_i t}(1-\tilde{G}_0(t))q_j(t) && [i \to 0 \leftarrow j \text{ at } t]\\
&\quad + \Big(\frac{1-\alpha_0}{2} + \frac{\alpha_i}{2}\Big) e^{-r_i t} q_{0,i}(t)(1-G_j(t)) && [i \leftrightarrow 0 \not\leftarrow j \text{ at t}]\\
&\quad + \Big(\frac{1-\alpha_0}{3} + \frac{\alpha_i}{3}\Big) e^{-r_i t} q_{0,i}(t)q_j(t) && [i \leftrightarrow 0 \leftarrow j \text{ at t}]\\
&\quad + \frac{1-\alpha_0}{3} e^{-r_i t} q_{0,j}(t)q_j(t). && [i \to 0 \leftrightarrow j \text{ at t}]
\end{aligned}
$$

A.2 Omitted Proofs

Proof of Lemma 1. We only consider player $i \in \{1,2\}$, the proof for player 0 is analogous. Suppose that $G_i(\mathcal{T}) > 0$. Then, player i can profitably deviate by shifting weight from $\mathcal{T}$ to s. Let a mixed strategy $\hat{G}_i \in \mathcal{G}_i$ be such that: (1) $\hat{G}_i(B) = G_i(B) + G_i(\mathcal{T})$ for all measurable $B \subseteq [0,\infty]$ containing s, and (2) $\hat{G}_i(B) = G_i(B \setminus \mathcal{T})$ for all measurable $B \subseteq [0,\infty]$ not containing s. We have

$$\begin{aligned}
&\int_0^\infty U_i(t, G_0, G_j)\mathrm{d}\hat{G}_i(t) \\
&= \int_0^\infty U_i(t, G_0, G_j)\mathrm{d}G_i(t) - \int_{\mathcal{T}} U_i(\tau, G_0, G_j)dG_i(\tau) + U_i(s, G_0, G_j)G_i(\mathcal{T}) \\
&> \int_0^\infty U_i(t, G_0, G_j)\mathrm{d}G_i(t) - \int_{\mathcal{T}} U_i(s, G_0, G_j)dG_i(\tau) + U_i(s, G_0, G_j)G_i(\mathcal{T}) \\
&= \int_0^\infty U_i(t, G_0, G_j)\mathrm{d}G_i(t),
\end{aligned}$$

which implies that G_i is not an equilibrium strategy. ■

Proof of Proposition 2. First, we establish that in any immediate-agreement equilibrium, either $q_1(0) = q_2(0) = 1$ or $q_1(0), q_2(0) < 1$. To see this, notice that if $q_i(0) = 1$ and $q_j(0) < 1$, then an argument in the proof of Proposition 1 shows that player j has a profitable deviation to an immediate concession. Moreover, Proposition 1 implies that there exists a continuum of pure-strategy Nash equilibria, in all of which $q_1(0) = q_2(0) = 1$ and $\tilde{q}_0(0) = q_{0,1}(0) + q_{0,2}(0) = 0$. Therefore, there always exists a continuum of immediate-agreement Nash equilibria in which players 1 and 2 concede at time t = 0 with certainty and player 0 concedes later. To complete the characterization of immediate-agreement equilibria, we only need to check if there exist equilibria with $q_1(0), q_2(0) < 1$ and $\tilde{q}_0(0) = 1$.

(i) Suppose first $\alpha_1 \neq \alpha_2$ and assume without loss of generality that $\alpha_1 < \alpha_2$. We claim that there does not exists an equilibrium in which $q_1(0), q_2(0) < 1$. Suppose to the contradiction that there exists such an equilibrium, i.e. $q_1(0), q_2(0) < 1$. Since $1 - \alpha_1 > 1 - \alpha_2$, we have $q_{0,1}(0) = 1$. But then player 2 can profitably deviate to conceding at $t = 0$ with certainty since $\frac{1-\alpha_0}{2} > 0$, which is a contradiction.

Suppose next $\alpha_1 = \alpha_2 < 2(1 - \alpha_0)$, and again suppose to the contrary of the claim that there exists an equilibrium with $q_1(0), q_2(0) < 1$. Assume without loss of generality that $q_{0,1}(0) \leq q_{0,2}(0)$, and let $\gamma = q_{0,1}(0) \in [0, \frac{1}{2}]$, which implies $q_{0,2}(0) = 1 - \gamma$. Notice that for $q_1(0) < 1$ to hold in equilibrium, player 1 must weakly prefer conceding at time

$t > 0$ to conceding at $t = 0$, i.e., we must have

$$q_2(0)\Big(\frac{\gamma}{3}\alpha_1 + \frac{1}{3}(1-\alpha_0)\Big) + (1-q_2(0))\Big(\frac{\gamma}{2}\alpha_1 + \frac{1}{2}(1-\alpha_0)\Big) \leq q_2(0)\frac{\gamma}{2}\alpha_1 + (1-q_2(0))\gamma\alpha_1 \tag{A7}$$

where the left hand side is player 1's expected payoff if she concedes at time $t = 0$, and the right hand side is his expected payoff if he concedes at time $t > 0$. However, since $\gamma \in [0, \frac{1}{2}]$ and $\alpha_1 < 2(1-\alpha_0)$, we have $\frac{\gamma}{3}\alpha_1 + \frac{1}{3}(1-\alpha_0) > \frac{\gamma}{2}\alpha_1$ and $\frac{\gamma}{2}\alpha_1 + \frac{1}{2}(1-\alpha_0) > \gamma\alpha_1$, a contradiction to the inequality A7.

(ii) Let $\alpha_1 = \alpha_2 \equiv \alpha \geq 2(1-\alpha_0)$. We will construct an immediate-agreement equilibrium in which $\tilde{q}_0(0) = 1$. Since $0 < 1-\alpha < \alpha_0$, we can find $T \in \mathbb{R}_+$ such that $1 - \alpha \geq \alpha_0 e^{-r_0 T}$. Then, for any $t_1, t_2 \geq T$, the following comprises an immediate-agreement Nash equilibrium: $q_{0,1}(0) = q_{0,2}(0) = \frac{1}{2}$, $q_1(t_1) = 1$, $q_2(t_2) = 1$. Player 0 does not have an incentive to deviate since any deviation gives her utility not greater than $\alpha_0 e^{-r_0 T}$ which is in turn less than $1 - \alpha$. Consider a deviation by player $i = 1, 2$ to conceding at time $t = 0$. The payoff of player i along the path is $\frac{1}{2}\alpha$, and the payoff from the deviation is $\frac{1}{4}\alpha + \frac{1}{2}(1-\alpha_0) \leq \frac{1}{2}\alpha$, so player i will not deviate either. ■

Proof of Lemma 3: Assume $\alpha_i > \alpha_j$. Let $\hat{t} \equiv \sup\{t \geq 0 : G_1(t) < 1, G_2(t) < 1\}$ with the convention that $\hat{t} = 0$ if the set $\{t \geq 0 : G_1(t) < 1, G_2(t) < 1\}$ is empty. In particular, the equilibrium is immediate-agreement if $\hat{t} = 0$. Consider a delay equilibrium, i.e., $\hat{t} > 0$.

Step 1*: Player* 0 *does not concede to player* i *strictly before time* $\hat{t}$*, that is,* $\lim_{t\uparrow\hat{t}_0} G_{0,i}(t) = 0$*. Additionally,* $q_{0,i}(\hat{t}) > 0$ *implies that there exists* $k \in \{1, 2\}$ *such that* $\lim_{t\uparrow\hat{t}} G_k(t) = 1$.

Since player 0 strictly prefers to concede to player j rather than concede to player i prior to $\hat{t}$, player 0 would never concede to player i with positive probability. Formally, for any $t \in (0, \hat{t})$ we have

$$\begin{aligned} U_0(t, j, G_1, G_2) - U_0(t, i, G_1, G_2) &= \frac{\alpha_i - \alpha_j}{2} e^{-r_0 t}\Big[q_1(t)(1 - G_2(t)) + q_2(t)(1 - G_1(t))\Big] \\ &\quad + \frac{\alpha_i - \alpha_j}{3} e^{-r_0 t} q_1(t) q_2(t) + (\alpha_i - \alpha_j) e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) \\ &\geq (\alpha_i - \alpha_j) e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) > 0, \end{aligned}$$

where the last inequality holds since $G_1(t) < 1, G_2(t) < 1$ by the definition of $\hat{t}$.

For $t = 0$, we have

$$
\begin{aligned}
&U_0(0, j, G_1, G_2) - U_0(0, i, G_1, G_2)\\
=&(\alpha_i - \alpha_j)(1 - q_1(0))(1 - q_2(0)) + \frac{\alpha_i - \alpha_j}{2}\Big[q_1(0)(1 - q_2(0)) + q_2(0)(1 - q_1(0))\Big] + \frac{\alpha_i - \alpha_j}{3}q_1(0)q_2(0)\\
\geq&(\alpha_i - \alpha_j)(1 - q_1(0))(1 - q_2(0)) > 0,
\end{aligned}
$$

where the last inequality holds since $q_1(0) < 1$ and $q_2(0) < 1$ for any non-degenerate equilibrium. Therefore, by Lemma 1, $\lim_{t\uparrow\hat{t}} G_{0,i}(t) = G_{0,i}\Big([0, \hat{t})\Big) = 0$.

From Lemma 1, we also know that $q_{0,i}(\hat{t}) > 0$ only if $U_0(\hat{t}, j, G_1, G_2) - U_0(\hat{t}, i, G_1, G_2) \leq 0$, that is

$$
\begin{aligned}
0 \geq\ & U_0(\hat{t}, j, G_1, G_2) - U_0(\hat{t}, i, G_1, G_2)\\
=\ & \frac{\alpha_i - \alpha_j}{2}e^{-r_0\hat{t}}\Big[q_1(\hat{t})(1 - G_2(\hat{t})) + q_2(\hat{t})(1 - G_1(\hat{t}))\Big]\\
&+ \frac{\alpha_i - \alpha_j}{3}e^{-r_0\hat{t}}q_1(\hat{t})q_2(\hat{t}) + (\alpha_i - \alpha_j)e^{-r_0\hat{t}}(1 - G_1(\hat{t}))(1 - G_2(\hat{t})).
\end{aligned}
$$

Since $\alpha_i > \alpha_j$, there exists $k \in \{1, 2\}$ such that $G_k(\hat{t}) = 1$ and $q_k(\hat{t}) = 0$, that is, $\lim_{t\uparrow\hat{t}} G_k(t) = 1$.

Step 2: *Player i concedes before time $\hat{t}$ with certainty, that is, $G_i(\hat{t}_0) = 1$.*

If $\hat{t} = \infty$, then $G_i(\hat{t}) = 1$ holds trivially. If $\hat{t} < \infty$, suppose by contradiction that $G_i(\hat{t}) < 1$. By the definition of $\hat{t}$, we have $G_j(\hat{t}) = 1$. In turn, *Step 1* implies that whenever $q_{0,i}(\hat{t}) > 0$ we must also have $q_j(\hat{t}) = 0$, that is, $q_{0,i}(\hat{t})q_j(\hat{t}) = 0$. Since $\lim_{t\uparrow\hat{t}} G_{0,i}(t) = 0$ and $\hat{t} > 0$, we have $q_{0,i}(v) = G_{0,i}(v) = 0$ for all $v \in [0, \hat{t})$. Therefore, we have $U_i(t, G_0, G_j) = 0$ for all $t \in (\hat{t}, \infty]$. And we also have

$$
U_i(0, G_0, G_j) = (1 - \alpha_0)(1 - q_j(0)) + \frac{1 - \alpha_0}{3}q_{0,j}(0)q_j(0) > 0
$$

since $q_j(0) < 1$ in any non-degenerate equilibrium. It follows that $U_i(0, G_0, G_j) - U_i(t, G_0, G_j) > 0$ for all $t \in (\hat{t}, \infty]$ and thus, by Lemma 1, $G_i\Big((\hat{t}, \infty]\Big) = 0$, i.e., $G_i\Big([0, \hat{t}]\Big) = G_i(\hat{t}) = 1$, which is a contradiction.

Step 3: *Player i concedes at time $\hat{t}$ with zero probability, that is $q_i(\hat{t}) = 0$.*

Suppose, by contradiction, that $q_i(\hat{t}) > 0$. There are two distinct cases: (i) $q_{0,i}(\hat{t}) > 0$; (ii) $q_{0,i}(\hat{t}) = 0$. If $q_{0,i}(\hat{t}) > 0$, we have $G_j(\hat{t}) = 1$ and $q_j(\hat{t}) = 0$, and thus $U_i(\hat{t}, G_0, G_j) = 0 < U_i(0, G_0, G_j)$, similar to *Step 2*. But then $q_i(\hat{t}) = 0$ by Lemma 1, a contradiction.

And if $q_{0,i}(\hat{t}) = 0$, then we have $G_{0,i}(\hat{t}) = 0$ by *Step 1*. Hence,

$$\begin{aligned} U_i(\hat{t}, G_0, G_j) = & (1-\alpha_0)e^{-r_i\hat{t}}(1 - G_{0,j}(\hat{t}))(1 - G_j(\hat{t})) \\ & + \frac{1-\alpha_0}{2}e^{-r_i\hat{t}}\Big[q_{0,j}(\hat{t})(1 - G_j(\hat{t})) + (1 - G_{0,j}(\hat{t}))q_j(\hat{t})\Big] \\ & + \frac{1-\alpha_0}{3}e^{-r_i\hat{t}}q_{0,j}(\hat{t})q_j(\hat{t}) \\ \leq & (1-\alpha_0)e^{-r_i\hat{t}}\Big(1 - \lim_{s\uparrow\hat{t}} G_{0,j}(s)\Big)\Big(1 - \lim_{s\uparrow\hat{t}} G_j(s)\Big). \end{aligned}$$

On the other hand, we can find $\delta > 0$ sufficiently small such that $q_j(\delta) = 0$, $q_{0,j}(\delta) = 0$, and thus

$$\begin{aligned} U_i(\delta, G_0, G_j) = & (1-\alpha_0)e^{-r_i\delta}(1 - G_{0,j}(\delta))(1 - G_j(\delta)) \\ > & (1-\alpha_0)e^{-r_i\hat{t}}\Big(1 - \lim_{s\uparrow\hat{t}} G_{0,j}(s)\Big)\Big(1 - \lim_{s\uparrow\hat{t}} G_j(s)\Big). \end{aligned}$$

Again, Lemma 1 implies $q_i(\hat{t}_0) = 0$, a contradiction.

Step 4*: Player i must concede at time 0 with certainty, that is, $G_i(0) = 1$, contradicting $\hat{t} > 0$.*

For any $t \in (0, \hat{t}_0)$, and for any $\zeta \in (0, 1)$ there exists a sufficiently small number $\delta \in (0, \zeta t)$ such that $q_j(\delta) = q_{0,j}(\delta) = 0$. Then, for any $\tau \in [\zeta t, t]$ and analogous to *Step 3*, we have

$$U_i(\tau, G_0, G_j) < U_i(\delta, G_0, G_j).$$

Therefore, by Lemma 1, $G_i\Big([\zeta t, t]\Big) = 0$ for any $t \in (0, \hat{t}_0)$, and any $\zeta \in (0, 1)$. Finally, it follows that

$$1 - G_i(0) = G_i\Big((0, \hat{t}_0)\Big) = G_i\Big(\bigcup_{k=3}^{\infty}[\frac{1}{k}\hat{t}_0, \frac{k-1}{k}\hat{t}_0]\Big) = \lim_{k\to\infty} G_i\Big([\frac{1}{k}\hat{t}_0, \frac{k-1}{k}\hat{t}_0]\Big) = 0,$$

which implies that player i concedes at the start of the game with certainty. This implies that $\hat{t} = 0$, which is a contradiction. Therefore, we must have $\alpha_1 = \alpha_2$ in every delay equilibrium. ■

Proof of Lemma 4*.* By Lemma 1, it is sufficient to show that player 0 strictly prefers conceding to player $\kappa \in \{1, 2\}$ at some time $\delta > 0$ rather than conceding to player κ at time 0.

Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta > 0$ such that for any $j \in \{1, 2\}$:

(i) $q_j(\delta) = 0$;

(ii) $G_j(\delta) - q_j(0) < \varepsilon$;

(iii) $1 - e^{-r_0\delta} < \varepsilon$.

Using the definitions in (A1) and (A3), and substituting $q_1(\delta) = q_2(\delta) = 0$ by (i), we can write:

$$
\begin{aligned}
U_0(\delta, \kappa, G_1, G_2) - U_0(0, \kappa, G_1, G_2) &= \alpha_0 \left[1 - (1 - q_1(0))(1 - q_2(0))\right] \\
&+ (1-\alpha)e^{-r_0\delta}(1 - G_1(\delta))(1 - G_2(\delta)) \\
&+ \sum_{i=1}^{2} \int_0^\delta \alpha_0 e^{-r_0 v}(1 - G_i(v))\mathrm{d}G_j(v) + \sum_{0<v<\delta} \alpha_0 e^{-r_0 v} q_1(v) q_2(v) \\
&- (1-\alpha)(1 - q_1(0))(1 - q_2(0)) - \left(\frac{1-\alpha}{3} + \frac{2\alpha_0}{3}\right) q_1(0) q_2(0) \\
&- \left(\frac{1-\alpha}{2} + \frac{\alpha_0}{2}\right) \left[q_1(0)(1 - q_2(0)) + q_2(0)(1 - q_1(0))\right].
\end{aligned}
$$

Note that the expressions in the third line are non-negative, and the expression in the second line can be expressed as $(1-\alpha)(1-q_1(0))(1-q_2(0)) + o(\varepsilon)$ by conditions (ii) and (iii). Combining this with the remaining terms, we obtain that $U_0(\delta, \kappa, G_1, G_2) - U_0(0, \kappa, G_1, G_2)$ is bounded below by

$$
\frac{\alpha_0 + \alpha - 1}{3} q_1(0) q_2(0) + \frac{\alpha_0 + \alpha - 1}{2} \left[q_1(0)(1 - q_2(0)) + q_2(0)(1 - q_1(0))\right] + o(\varepsilon)
$$

which is positive since $\alpha_0 + \alpha > 1$, $q_i(0) > 0$ and ε is arbitrarily small. ■

Proof of Lemma 5. The proof is similar to the proof of Lemma 4. By Lemma 1, it is sufficient to show that player 0 strictly prefers to concede to player $\kappa \in \{1, 2\}$ at time $t + \delta$ where $\delta > 0$ rather than concede to player κ at time t.

Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta > 0$ such that for any $i \in \{1, 2\}$:

(i) $q_i(t + \delta) = 0$,

(ii) $G_i(t + \delta) - G_i(t) < \varepsilon$,

(iii) $1 - e^{-r_0\delta} < \varepsilon$.

Using the definition in (A3) for $t + \delta$ and t, and substituting $q_1(t+\delta) = q_2(t+\delta) = 0$

by (i), we have:

$$
\begin{aligned}
&U_0(t+\delta,\kappa,G_1,G_2)-U_0(t,\kappa,G_1,G_2)\\
&=(1-\alpha)e^{-r_0(t+\delta)}(1-G_1(t+\delta))(1-G_2(t+\delta))-(1-\alpha)e^{-r_0t}(1-G_1(t))(1-G_2(t))\\
&\quad+\sum_{i=1}^{2}\int_t^{t+\delta}\alpha_0e^{-r_0v}(1-G_i(v))\mathrm{d}G_j(v)+\sum_{t\le v<t+\delta}\alpha_0e^{-r_0v}q_1(v)q_2(v)\\
&\quad+\sum_{i=1}^{2}\alpha_0e^{-r_0t}(1-G_i(t))q_j(v)+\alpha_0e^{-r_0v}q_1(v)q_2(v)\\
&\quad-\sum_{i=1}^{2}\left(\frac{1-\alpha}{2}+\frac{\alpha_0}{2}\right)e^{-r_0t}(1-G_i(t))q_j(t)-\left(\frac{1-\alpha}{3}+\frac{2\alpha_0}{3}\right)e^{-r_0t}q_1(t)q_2(t).
\end{aligned}
$$

The two expressions in the second line are non-negative. The first expression in the first line can be written as $(1-\alpha)e^{-r_0t}(1-G_1(t))(1-G_2(t))+o(\varepsilon)$ by (ii) and (iii). Combining these with the remaining terms, we obtain that $U_0(t+\delta,\kappa,G_1,G_2)-U_0(t,\kappa,G_1,G_2)$ is bounded below by

$$
\sum_{i=1}^{2}\frac{\alpha_0+\alpha-1}{2}e^{-r_0t}q_i(t)(1-G_j(t))+\frac{\alpha_0+\alpha-1}{3}e^{-r_0t}q_1(t)q_2(t)+o(\varepsilon).
$$

For sufficiently small ε, this expression is stricly positive since $\alpha_0+\alpha>1$, $q_i(t)>0$, and $\lim_{\tau\uparrow t}G_j(\tau)<1$, because the latter implies that either $q_j(t)>0$ or $G_j(t)<1$ or both.

■

Proof of Lemma 6. The proof is similar to the proofs of Lemma 4 and Lemma 5. By Lemma 1, it is sufficient to show that player j strictly prefers to concede at an earlier time $t-\delta>0$.

Fix $\varepsilon>0$. By Lemma A2, there exists $\delta>0$ such that:

(i) $1-e^{-r_0\delta}<\varepsilon$,

(ii) $q_i(t-\delta)=q_{0,i}(t-\delta)=0$,

(iii) $G_i(t)-G_i(t-\delta)-q_i(t)<\varepsilon$, and $G_{0,\kappa}(t)-G_{0,\kappa}(t-\delta)-q_{0,\kappa}(t)<\varepsilon$ for any $\kappa\in\{1,2\}$.

First, using the definition in (A6) for $t-\delta$ and t, and substituting $q_i(t-\delta)=$

$q_{0,i}(t-\delta) = 0$ from (ii), we have:

$$
\begin{aligned}
&U_j(t-\delta, G_0, G_i) - U_j(t, G_0, G_i) \\
&= -\int_{t-\delta}^{t} \alpha e^{-r_j v}(1 - G_i(v))\mathrm{d}G_{0,j}(v) - \sum_{t-\delta<v<t} \frac{\alpha_j}{2} e^{-r_j v} q_{0,j}(v) q_i(v) \\
&\quad + (1-\alpha_0)\Big[e^{-r_j(t-\delta)}(1-\tilde{G}_0(t-\delta))(1-G_i(t-\delta)) - e^{-r_j t}(1-\tilde{G}_0(t))(1-G_i(t))\Big] \\
&\quad - \frac{1-\alpha_0}{2} e^{-r_j t}\Big[q_{0,i}(t)(1-G_i(t)) + (1-\tilde{G}_0(t))q_i(t)\Big] \\
&\quad - \Big(\frac{1-\alpha_0}{2} + \frac{\alpha}{2}\Big) e^{-r_j t} q_{0,j}(t)(1-G_i(t)) - \Big(\frac{1-\alpha_0}{3} + \frac{\alpha}{3}\Big) e^{-r_j t} q_{0,j}(t) q_i(t) - \frac{1-\alpha_0}{3} e^{-r_j t} q_{0,i}(t) q_i(t).
\end{aligned}
$$

Next, $e^{-r_j v}(1 - G_i(v)) \leq 1$ for all v implies that

$$
\int_{t-\delta}^{t} \alpha e^{-r_j v}(1-G_i(v))\mathrm{d}G_{0,j}(v) \leq \int_{t-\delta}^{t} \alpha \mathrm{d}G_{0,j}(v) = \alpha(G_{0,j}(t) - q_{0,j}(t) - G_{0,j}(t-\delta)) < \alpha\varepsilon
$$

where the last inequality uses (iii). Therefore, for all $i \neq j \in \{1, 2\}$ we have

$$
\int_{t-\delta}^{t} \alpha e^{-r_j v}(1-G_i(v))\mathrm{d}G_{0,j}(v) = o(\varepsilon).
$$

Also, $e^{-r_j v} \leq 1$ for all v implies that

$$
\sum_{t-\delta<v<t} e^{-r_j v} q_{0,j}(v) q_i(v) \leq \sum_{t-\delta<v<t} q_{0,j}(v) q_i(v) \leq \Bigg(\sum_{t-\delta<v<t} \sqrt{q_{0,j}(v) q_i(v)}\Bigg)^2 \leq \sum_{t-\delta<v<t} q_{0,j}(v) \sum_{t-\delta<v<t} q_i(v) < \varepsilon^2
$$

where the third inequality follows from the Cauchy-Schwarz inequality and the fourth inequality uses (iii). Therefore,

$$
\sum_{t-\delta<v<t} \frac{\alpha_j}{2} e^{-r_j v} q_{0,j}(v) q_i(v) = \frac{\alpha_j}{2} \sum_{t-\delta<v<t} e^{-r_j v} q_{0,j}(v) q_i(v) = o(\varepsilon).
$$

Finally, (i) and (iii) imply that

$$
e^{-r_j(t-\delta)}(1-\tilde{G}_0(t-\delta))(1-G_i(t-\delta)) = e^{-r_j t}(1-\tilde{G}_0(t)+\tilde{q}_0(t))(1-G_i(t)+q_i(t)) + o(\varepsilon).
$$

Combining these results, we obtain:

$$
\begin{aligned}
&U_j(t-\delta, G_0, G_i) - U_j(t, G_0, G_i) \\
&= o(\varepsilon) + (1-\alpha_0)e^{-r_j t}\left[(1-\tilde{G}_0(t))q_i(t) + (1-G_i(t))\tilde{q}_0(t) + \tilde{q}_0(t)q_i(t)\right] \\
&\quad - \frac{1-\alpha_0}{2}e^{-r_j t}\left[q_{0,i}(t)(1-G_i(t)) + (1-\tilde{G}_0(t))q_i(t)\right] \\
&\quad - \Big(\frac{1-\alpha_0}{2} + \frac{\alpha}{2}\Big)e^{-r_j t}q_{0,j}(t)(1-G_i(t)) - \Big(\frac{1-\alpha_0}{3} + \frac{\alpha}{3}\Big)e^{-r_j t}q_{0,j}(t)q_i(t) - \frac{1-\alpha_0}{3}e^{-r_j t}q_{0,i}(t)q_i(t).
\end{aligned}
$$

Suppose, by contradiction, $q_j(t) > 0$. Then $\lim_{\tau\uparrow t} G_j(\tau) < 1$, and thus by Lemma 5, we have $q_{0,1}(t) = q_{0,2}(t) = 0$ and $\tilde{G}_0(t) = \lim_{\tau\uparrow t}\tilde{G}_0(\tau) < 1$. Therefore,

$$U_j(t-\delta, G_0, G_i) - U_j(t, G_0, G_i) = \frac{1-\alpha_0}{2}e^{-r_j t}(1-\tilde{G}_0(t))q_i(t) + o(\varepsilon)$$

For sufficiently small ε, this expression is stricly positive since $\alpha_0 < 1$, $\tilde{G}_0(t) < 1$ and $q_i(t) > 0$ (by assumption). This contradicts Lemma 1.

■

Proof of Lemma 7: Fix player $i \in \{1,2\}$ and player $j \in \{1,2\}$ such that $i \neq j$. We prove part (ii). The proof of part (i) is analogous. Fix player $i \in \{1,2\}$ and player $j \in \{1,2\}$ such that $i \neq j$. Lemma A1 implies that for any $\zeta \in (t,s)$ there is an earlier time $\delta \in (t,\zeta)$ such that $q_j(\delta) = 0$ and $\tilde{q}_0(\delta) = 0$.

To begin, we show that for any time $\tau \in (\zeta, s]$, player i strictly prefers to concede at time δ. Fix $\tau \in (\zeta, s]$. Since $G_{0,i}(t) = G_{0,i}(s)$, and $t < \delta < \tau \leq s$, we have $G_{0,i}(\delta) = G_{0,i}(\tau)$ and $q_{0,i}(v) = q_{0,j}(v) = 0$ for any $v \in (t,s]$. Then, (A6) implies

$$
\begin{aligned}
U_i(\delta, G_0, G_j) - U_i(\tau, G_0, G_j) &= (1-\alpha_0)e^{-r_i\delta}(1-\tilde{G}_0(\delta))(1-G_j(\delta)) \\
&\quad -(1-\alpha_0)e^{-r_i\tau}(1-\tilde{G}_0(\tau))(1-G_j(\tau)) - \frac{1-\alpha_0}{2}e^{-r_i\tau}(1-\tilde{G}_0(\tau))q_j(\tau) \\
&\quad -\frac{1-\alpha_0}{2}e^{-r_i\tau}q_{0,j}(\tau)(1-G_j(\tau)) - \frac{1-\alpha_0}{3}e^{-r_i\tau}q_{0,j}(\tau)q_j(\tau) \\
&\geq (1-\alpha_0)e^{-r_i\delta}(1-\tilde{G}_0(\delta))(1-G_j(\delta)) \\
&\quad -(1-\alpha_0)e^{-r_i\tau}(1-\lim_{w\uparrow\tau}\tilde{G}_0(w))(1-\lim_{w\uparrow\tau}G_j(w)) \\
&\geq (1-\alpha_0)(e^{-r_i\delta} - e^{-r_i\tau})(1-\lim_{w\uparrow\tau}\tilde{G}_0(w))(1-\lim_{w\uparrow\tau}G_j(w)) \\
&> 0,
\end{aligned}
$$

where $1-\lim_{w\uparrow\tau}\tilde{G}_0(w) = 1-\tilde{G}_0(\tau)+\tilde{q}_0(\tau) = 1-\tilde{G}_0(\tau)+q_{0,j}(\tau)$, and $1-\lim_{w\uparrow\tau}G_j(w) = 1-G_j(\tau)+q_j(\tau)$. The last strict inequality holds since $\lim_{w\uparrow\tau}\tilde{G}_0(w) \leq \lim_{w\uparrow s}\tilde{G}_0(w) < 1$,

$\lim_{w \uparrow \tau} G_j(w) \leq \lim_{w \uparrow s} G_j(w) < 1$, $\alpha_0 < 1$, and $e^{-r_i \delta} > e^{-r_i \tau}$.

Next, by part (i) of Lemma 1, for any $\zeta \in (t, s)$ we have $G_i(s) - G_i(\zeta) = G_i((\zeta, s]) = 0$. Finally we have

$$G_i(s) - G_i(t) = G_i((t, s]) = G_i\left(\bigcup_{k=1}^{\infty}\left(t + \frac{1}{k}\left(\frac{s-t}{2}\right), s\right]\right) = \lim_{k\to\infty} G_i\left(\left(t + \frac{1}{k}\left(\frac{s-t}{2}\right), s\right]\right) = 0,$$

which concludes the proof. ■

Proof of Lemma 8: We prove part (ii). The proof of part (i) is analogous.

The proof is by contradiction. Suppose $q_i(t) > 0$, $\lim_{\tau \uparrow t} G_{0,1}(\tau) + G_{0,2}(\tau) < 1$, and $\lim_{\tau \uparrow t} G_j(\tau) < 1$. Since $q_i(t) > 0$, Lemma 5 and Lemma 6 imply $q_j(t) = q_{0,1}(t) = q_{0,2}(t) = 0$ and therefore $G_{0,1}(t) + G_{0,1}(t) < 1$ and $G_j(t) < 1$.

We first establish that there exists $\delta > 0$ such that $G_{0,1}(t) = G_{0,1}(t-\delta)$ and $G_{0,2}(t) = G_{0,2}(t-\delta)$.

Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta \in (0, \varepsilon)$ such that:

(i) $G_i(t+\delta) - G_i(t-\delta) = G_i(t+\delta) - G_i(t) + q_i(t) + \lim_{\tau\uparrow t} G_i(\tau) - G_i(t-\delta) < \frac{\varepsilon}{2} + q_i(t) + \frac{\varepsilon}{2} = q_i(t) + \varepsilon$, and similarly, $G_j(t+\delta) - G_j(t-\delta) < q_j(t) + \varepsilon = \varepsilon$;

(ii) $q_i(\tau) \in [0, \frac{\varepsilon}{2})$ for $\tau \in [t-\delta, t+\delta] \backslash \{t\}$, and $q_j(\tau) = q_{0,1}(\tau) = q_{0,2}(\tau) \in [0, \frac{\varepsilon}{2})$ for $\tau \in [t-\delta, t+\delta]$;

(iii) $1 - e^{-r_0\delta} < \varepsilon$.

Then, for any $\tau \in (t-\delta, t)$ we have for any $\kappa \in \{1, 2\}$:

$$\begin{aligned}
& U_0(t+\delta, \kappa, G_1, G_2) - U_0(\tau, \kappa, G_1, G_2) \\
= \; & \int_{\tau}^{t+\delta} \alpha_0 e^{-r_0 v}(1 - G_2(v))\mathrm{d}G_1(v) + \int_{\tau}^{t+\delta} \alpha_0 e^{-r_0 v}(1 - G_1(v))\mathrm{d}G_2(v) \\
& +(1-\alpha)e^{-r_0(t+\delta)}(1 - G_1(t+\delta))(1 - G_2(t+\delta)) - (1-\alpha)e^{-r_0\tau}(1 - G_1(\tau))(1 - G_2(\tau)) + o(\varepsilon) \\
= \; & \alpha_0 e^{-r_0 t}(1 - G_j(t))q_i(t) \\
& +(1-\alpha)e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) - (1-\alpha)e^{-r_0 t}(1 - G_i(t) + q_i(t))(1 - G_j(t)) + o(\varepsilon) \\
= \; & (\alpha_0 + \alpha - 1)e^{-r_0 t}(1 - G_j(t))q_i(t) + o(\varepsilon) \\
> \; & 0,
\end{aligned}$$

where the last inequality holds when ε is sufficiently small, $q_i(t) > 0$, and $G_j(t) < 1$. Then, Lemma 1 implies $G_{0,\kappa}(t) = G_{0,\kappa}(t-\delta)$.

We are now ready to show a contradiction. Since $G_{0,k}(t-\delta) = G_{0,k}(t)$ for $k \in \{1, 2\}$, $G_{0,1}(t) + G_{0,2}(t) < 1$, and $G_j(t) < 1$. Therefore $G_i(t-\delta) = G_i(t)$ by Lemma 7, which contradicts $q_i(t) > 0$.

$\blacksquare$

Lemma A3. *Let G_i be strictly increasing on $[t, s]$ for some $i \in \{1, 2\}$ and $0 < t < s$. If $\lim_{\tau \uparrow s} G_j(\tau) < 1$ for $j \in \{1, 2\}$, $j \neq i$ and $\lim_{\tau \uparrow s} \tilde{G}_0(\tau) < 1$, then for almost all $\tau \in [t, s]$ we have:*

$$\frac{g_j(\tau)}{1 - G_j(\tau)} = -r_i - \frac{\tilde{g}_0(\tau)}{1 - \tilde{G}_0(\tau)} + \frac{\alpha}{1 - \alpha_0} \frac{g_{0,i}(\tau)}{1 - \tilde{G}_0(\tau)}.$$

Proof. Since $G_j(\tau) < 1$ and $\tilde{G}_0(\tau) < 1$ for all $\tau \in [t, s]$, Lemma 8 implies that $q_j(\tau) = \tilde{q}_0(\tau) = 0$ for all $\tau \in [t, s]$. Since G_i is strictly increasing on $[t, s]$, we have $G_i(\tau) < G_i(s) \leq 1$ for all $t \leq \tau < s$ and player i must attain her maximal utility almost everywhere on $[t, s)$. Therefore, player i must be indifferent between conceding at times τ and $\tau + \mathrm{d}\tau$ such that $t \leq \tau < \tau + \mathrm{d}\tau < s$, that is, we must have:

$$\begin{aligned}
0 &= U_i(\tau + \mathrm{d}\tau, G_0, G_j) - U_i(\tau, G_0, G_j) \\
&= \int_{\tau}^{\tau + \mathrm{d}\tau} \alpha_i e^{-r_i v}(1 - G_j(v)) \mathrm{d}G_{0,i}(v) \\
&\quad + (1 - \alpha_0) \Big[e^{-r_i(\tau + \mathrm{d}\tau)} (1 - G_{0,1}(\tau + \mathrm{d}\tau) - G_{0,2}(\tau + \mathrm{d}\tau))(1 - G_j(\tau + \mathrm{d}\tau)) \\
&\quad - e^{-r_i \tau}(1 - G_{0,1}(\tau) - G_{0,2}(\tau))(1 - G_j(\tau)) \Big]
\end{aligned}$$

Since G_j, $G_{0,1}$, and $G_{0,2}$ are monotone, they are differentiable almost everywhere on (t, s). Denote the corresponding derivatives g_j, $g_{0,1}$, and $g_{0,2}$. For sufficiently small $\mathrm{d}\tau > 0$, we have:

$$\begin{aligned}
0 &= \alpha e^{-r_i \tau}(1 - G_j(\tau)) g_{0,i}(\tau) \mathrm{d}\tau \\
&\quad - (1 - \alpha_0) e^{-r_i \tau} \Big[r_i (1 - \tilde{G}_0(\tau))(1 - G_j(\tau)) + \tilde{g}_0(\tau)(1 - G_j(\tau)) + g_j(\tau)(1 - \tilde{G}_0(\tau)) \Big] \mathrm{d}\tau.
\end{aligned}$$

It follows that

$$\alpha(1 - G_j(\tau)) g_{0,i}(\tau) = (1 - \alpha_0) \Big[r_i (1 - \tilde{G}_0(\tau))(1 - G_j(\tau)) + \tilde{g}_0(\tau)(1 - G_j(\tau)) + g_j(\tau)(1 - \tilde{G}_0(\tau)) \Big].$$

$$\iff \quad \frac{g_j(\tau)}{1 - G_j(\tau)} = -r_i - \frac{\tilde{g}_0(\tau)}{1 - \tilde{G}_0(\tau)} + \frac{\alpha}{1 - \alpha} \frac{g_{0,i}(\tau)}{1 - \tilde{G}_0(\tau)}.$$

for almost all $\tau \in [t, s)$. $\blacksquare$

Proof of Lemma 9: Since G_i and $G_{0,i}$, $i = 1, 2$ are cumulative distribution functions, they are increasing and therefore they are differentiable almost everywhere. Let g_i

denote the derivative of G_i when it exists, and let $g_{0,i}$ denote the derivative of $G_{0,i}$ when it exists. Also, let $\tilde{g}_0(\cdot) = g_{0,1}(\cdot) + g_{0,2}(\cdot)$.

(i) By hypothesis, no one ends the game by time s (i.e., $\lim_{\tau \uparrow s} G_1(\tau) < 1$, $\lim_{\tau \uparrow s} G_2(\tau) < 1$, and $\lim_{\tau \uparrow s} \tilde{G}_0(\tau) < 1$). Therefore, part (ii) of Lemma 8 implies that G_1 and G_2 are continuous on $(0, s)$. Since $\tilde{G}_0$ is strictly increasing over the interval $[t, s]$, the chair is indifferent between conceding at time τ and conceding time $\tau + \mathrm{d}\tau$ with $t \leq \tau < \tau + \mathrm{d}\tau < s$. She is also indifferent between conceding to player 1 and conceding to player 2 at any of those times since both of the competing players have the same demand. Therefore, for all $i, j \in \{1, 2\}$ we have

$$\begin{aligned}
0 &= U_0(\tau + \mathrm{d}\tau, i, G_1, G_2) - U_0(\tau, j, G_1, G_2) \\
&= \int_\tau^{\tau+\mathrm{d}\tau} \alpha_0 e^{-r_0 v}(1 - G_2(v))\mathrm{d}G_1(v) + \int_\tau^{\tau+\mathrm{d}\tau} \alpha_0 e^{-r_0 v}(1 - G_1(v))\mathrm{d}G_2(v) \\
&\quad + (1 - \alpha)\Big[e^{-r_0(\tau+\mathrm{d}\tau)}(1 - G_1(\tau + \mathrm{d}\tau))(1 - G_2(\tau + \mathrm{d}\tau)) - e^{-r_0\tau}(1 - G_1(\tau))(1 - G_2(\tau))\Big].
\end{aligned}$$

For sufficiently small $\mathrm{d}\tau > 0$, the right hand side is equal to

$$\begin{aligned}
&\alpha_0 e^{-r_0\tau}[(1 - G_2(\tau))g_1(\tau) + (1 - G_1(\tau))g_2(\tau)]\mathrm{d}\tau \\
&- (1 - \alpha)e^{-r_0\tau}\Big[r_0(1 - G_1(\tau))(1 - G_2(\tau)) + g_1(\tau)(1 - G_2(\tau)) + g_2(\tau)(1 - G_1(\tau))\Big]\mathrm{d}\tau.
\end{aligned}$$

It follows that for almost all $\tau \in [t, s)$ we have

$$(1-G_1(\tau))g_2(\tau)+(1-G_2(\tau))g_1(\tau) = \frac{(1 - \alpha)r_0}{\alpha_0 + \alpha - 1}(1-G_1(\tau))(1-G_2(\tau)) = \mu(1-G_1(\tau))(1-G_2(\tau)).$$

Dividing both sides by $(1 - G_1(\tau))(1 - G_2(\tau))$, we find

$$\lambda_1(\tau) + \lambda_2(\tau) = \mu.$$

(ii) Since G_i is strictly increasing on $[t, s]$, Lemma 7 implies that $\tilde{G}_0$ is also strictly increasing on $[t, s]$. Moreover, since G_j is constant on $[t, s]$, we have $\lambda_j(\tau) = 0$ for almost all $\tau \in [t, s]$. Therefore, part (i) implies that $\lambda_i(\tau) = \mu$ for all $\tau \in [t, s]$.

From Lemma A3, we have

$$\lambda_j(\tau) = -r_i - \frac{\tilde{g}_0(\tau)}{1 - \tilde{G}_0(\tau)} + \frac{\alpha}{1 - \alpha_0}\frac{g_{0,i}(\tau)}{1 - \tilde{G}_0(\tau)}.$$

Using the definitions of $\lambda_{0,i}$ and $\lambda_{0,j}$, since $\lambda_j(\tau) = 0$ on $[t, s]$, we can rewrite the above equality as

$$0 = -r_i + \lambda_{0,i}(\tau) + \lambda_{0,j}(\tau) + \frac{\alpha}{1-\alpha_0}\lambda_{0,i}(\tau).$$

Solving for $\lambda_{0,i}(\tau)$, we find

$$\lambda_{0,i}(\tau) = \frac{(1-\alpha_0)(t_i + \lambda_{0,j}(\tau))}{\alpha_0 + \alpha - 1}.$$

(iii) Since G_1 and G_2 are both strictly increasing over $[t, s]$, by Lemma A3 we have

$$\begin{aligned}
\frac{g_1(\tau)}{1-G_1(\tau)} + \frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} &= -r_2 + \frac{\alpha g_{0,2}(\tau)}{(1-\alpha_0)(1-\tilde{G}_0(\tau))}, \\
\frac{g_2(\tau)}{1-G_2(\tau)} + \frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} &= -r_1 + \frac{\alpha g_{0,1}(\tau)}{(1-\alpha_0)(1-\tilde{G}_0(\tau))}.
\end{aligned}$$

Adding these equations, we obtain

$$\frac{g_1(\tau)}{1-G_1(\tau)} + \frac{g_2(\tau)}{1-G_2(\tau)} + 2\frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} = -r_1 - r_2 + \frac{\alpha\tilde{g}_0(\tau)}{(1-\alpha_0)(1-\tilde{G}_0(\tau))}.$$

Integrating both sides yields

$$\ln(1-G_1(\tau)) + \ln(1-G_2(\tau)) = (r_1+r_2)\tau + C_2 + (\frac{\alpha}{1-\alpha_0} - 2)\ln(1-\tilde{G}_0(\tau)).$$

Taking the exponentials of both sides, we obtain

$$(1-G_1(\tau))(1-G_2(\tau)) = C_3 e^{(r_1+r_2)\tau}(1-\tilde{G}_0(\tau))^{\frac{\alpha}{1-\alpha_0}-2}.$$

Using the initial condition at t, we have

$$\left[\frac{1-\tilde{G}_0(\tau)}{1-\tilde{G}_0(t)}\right]^{\frac{\alpha}{1-\alpha_0}-2} = \frac{(1-G_1(\tau))(1-G_2(\tau))}{(1-G_1(t))(1-G_2(t))}e^{-(r_1+r_2)(\tau-t)}.$$

By part (i)

$$\frac{1-\tilde{G}_0(\tau)}{1-\tilde{G}_0(t)} = e^{-\frac{\lambda+r_1+r_2}{\frac{\alpha}{1-\alpha_0}-2}(\tau-t)} = e^{-\rho(\tau-t)}.$$

It follows that $\frac{\tilde{g}_0(\tau)}{1-\tilde{G}_0(\tau)} = \rho$. Using Lemma A3 again, we obtain the desired result. ■

Proof of Lemma 10: We only prove that $\tilde{G}_0$ must be strictly increasing on $[0, \hat{t})$. The proof for H is analogous.

The proof is by contradiction. Suppose $\tilde{G}_0(t) = \tilde{G}_0(s)$ for some $t, s \in [0, \hat{t})$ with $t < s$.

The definition of $\hat{t}$ implies $\tilde{G}_0(t) = \tilde{G}_0(s) < 1$. This in turn implies by Lemma 7 that $H(t) = H(s)$. Using the definition of $\hat{t}$ one more time, we must have $H(t) = H(s) < 1$. Define $s^* = \sup\{s' > 0 \ : \ \tilde{G}_0(s') = \tilde{G}_0(t) < 1, \ H(s') = H(t) < 1\}$. We next show that $s^* \leq \hat{t}$. If not, then take $\tilde{t} \in (\hat{t}, s^*)$. Since $t < \tilde{t} < s^*$, the definition of s^* implies $\tilde{G}_0(\tilde{t}) = \tilde{G}_0(t) < 1$ and $H(\tilde{t}) = H(t) < 1$. But since $\tilde{t} > \hat{t}$ we also have $\max\{\tilde{G}_0(\tilde{t}), \ H(\tilde{t})\} = 1$ by the definition of $\hat{t}$ which is a contradiction.

Note that $\tilde{G}_0$ and H cannot have an atom point at s^* by Lemma 8 and the fact that $\lim_{s' \uparrow s^*} \tilde{G}_0(s') = \tilde{G}_0(t) < 1$ and $\lim_{s' \uparrow s^*} H(s') = H(t) < 1$. Therefore, we have $\tilde{G}_0(s^*) = \tilde{G}_0(t) < 1$ and $H(s^*) = H(t) < 1$, and $s^* < \infty$. We will use these facts to show that there exists $\delta > 0$ such that $\tilde{G}_0$ and H are constant on $[t, s^* + \delta)$, which is a contradiction of the definition of s^*.

It suffices to show that $\tilde{G}_0$ is constant on $[t, s^* + \delta)$, because Lemma 7 implies that if $\tilde{G}_0$ is constant, then H is constant as well. Since $\tilde{G}_0$ is constant over $[t, s^*]$, it is turn sufficent to show that $\tilde{G}_0[s^*, s^* + \delta) = 0$. By Lemma 1, we only need to show that player 0 strictly prefers to concede at time $t \notin [s^*, s^* + \delta)$ rather than conceding at any time $\tau \in [s^*, s^* + \delta)$. Fix $\varepsilon > 0$. By Lemma A2, there exists $\delta > 0$ such that $H(s^* + \delta) - H(s^*) = H(s^* + \delta) - H(t) < \varepsilon$. For any $\tau \in [s^*, s^* + \delta)$ and κ, we can write

$$\begin{aligned} U_0(\tau, \kappa, G_1, G_2) - U_0(t, \kappa, G_1, G_2) &= (1-\alpha)e^{-r_0\tau}(1 - H(\tau))) - (1-\alpha)e^{-r_0 t}(1 - H(t)) \\ &\quad + \int_t^\tau \alpha_0 e^{-r_0 v} \mathrm{d}H(v) + o(\varepsilon) \\ &= (1-\alpha)(e^{-r_0\tau} - e^{-r_0 t})(1 - H(t)) + o(\varepsilon) < 0, \end{aligned}$$

as desired. ■

Proof of Lemma 11: To establish a contradiction, suppose that $\hat{t} < \infty$. First, this implies that $\tilde{q}_0(\hat{t}) > 0$, or $q_i(\hat{t}) > 0$ for some player $i \in \{1, 2\}$. To see why, suppose $\tilde{q}_0(\hat{t}) = q_1(\hat{t}) = q_2(\hat{t}) = 0$. Then, by the definition of $\hat{t}$, either $\lim_{t \uparrow \hat{t}} \tilde{G}_0(t) = \tilde{G}_0(\hat{t}) = 1$ or there must exist a player $i \in \{1, 2\}$ such that $\lim_{t \uparrow \hat{t}} G_i(t) = G_i(\hat{t}) = 1$. There are two cases, each leading to a contradiction.

Suppose first $\lim_{t \uparrow \hat{t}} \tilde{G}_0(t) = 1$. By the definition of $\hat{t}$, $G_1(t) < 1$, $G_2(t) < 1$, $\tilde{G}_0(t) < 1$ for all $t < \hat{t}$. Then, since G_1 and G_2 are strictly increasing, by Corollary 2, $\tilde{G}_0(t) = 1 - (1 - \tilde{G}_0(0))e^{-\rho t}$. Taking the limit as t converges to $\hat{t}$, we establish contradiction as $\lim_{t \uparrow \hat{t}} \tilde{G}_0(t) = 1 - (1 - \tilde{G}_0(0))e^{-\rho \hat{t}} < 1$.

Suppose now $\lim_{t \uparrow \hat{t}} G_i(t) = 1$ for some $i = 1, 2$. This implies $\lim_{t \uparrow \hat{t}} H(t) = 1$. By the definition of $\hat{t}$, $G_1(t) < 1$, $G_2(t) < 1$, $\tilde{G}_0(t) < 1$ for all $t < \hat{t}$. Then, since $\tilde{G}_0$ is strictly increasing over $[0, \hat{t})$, part (i) of Lemma 9 implies $H(t) = 1 - (1 - H(0))e^{-\mu t} < 1$ for any $t \in [0, \hat{t})$. Taking the limit as t converges to $\hat{t}$ and recalling that $H(0) < 1$, we have $\lim_{t \uparrow \hat{t}} H(t) = 1 - (1 - H(0))e^{-\mu \hat{t}} < 1$, which is a contradiction.

Thus, we have shown that either $\tilde{q}_0(\hat{t}) > 0$ or $q_i(\hat{t}) > 0$ for some $i \in \{1, 2\}$. Then Lemma 8 implies that $\lim_{t \uparrow \hat{t}} G_j(t) = 1$ for some $j \in \{1, 2\}$ or $\lim_{t \uparrow \hat{t}} \tilde{G}_0(t) = 1$. But we have already shown that this leads to a contradiction. It follows that $\hat{t} = \infty$. ■

Proof of Lemma 12: Fix competing player $i = 1, 2$, and let $j \neq i$. Since G_i is strictly increasing, there exists $t > 0$ such that player i weakly prefers conceding at time t to conceding at time 0 by Lemma 1, that is $U_i(t, G_0, G_j) - U_i(0, G_0, G_j) \geq 0$. Note that by Lemma 4, we have $q_j(0)\tilde{q}_0(0) = 0$. In addition, Lemma 8, Lemma 10, and Lemma 11 imply that $\tilde{q}_0(\tau) = q_j(\tau) = 0$ for any $\tau > 0$. Finally, $1 - \tilde{G}_0(t) = (1 - \tilde{q}_0(0))e^{-\rho t}$ by Lemma 9. Together with A4 and A6 these imply that $U_i(t, G_0, G_j) - U_i(0, G_0, G_j)$ is equal to

$$\begin{aligned} &\alpha q_{0,i}(0) \int_0^t \alpha e^{-r_i v}(1 - G_j(v))dG_{0,i}(v) + (1 - \alpha_0)e^{-r_i t}(1 - \tilde{q}_0(0))e^{-\rho t}(1 - G_j(t)) \\ &\quad -(1 - \alpha_0)(1 - \tilde{q}_0(0))(1 - q_j(0)) - \frac{1 - \alpha_0}{2}[q_{0,j}(0) + q_j(0)] - \frac{1 - \alpha_0 + \alpha}{2} q_{0,i}(0). \quad \text{(A8)} \end{aligned}$$

By equation (9), the definition of $\lambda_{0,i}$, and Corollary 2), we have

$$G_{0,i}(t) = q_{0,i}(0) + \frac{(1 - \alpha_0)(1 - \tilde{q}_0(0))}{\alpha} \int_0^t (\lambda_j(\tau) + \rho + r_i)e^{-\rho \tau} \mathrm{d}\tau.$$

By the definition of λ_j, we have

$$1 - G_j(t) = (1 - q_j(0))e^{-\Lambda_j(t)}$$

where $\Lambda_j(t) = \int_0^t \lambda_j(\tau)\mathrm{d}\tau$. Using these facts, (A8) becomes

$$\begin{aligned} &\alpha q_{0,i}(0) + (1 - \alpha_0)(1 - \tilde{q}_0(0))(1 - q_j(0))(1 - e^{-(r_i + \rho)t - \Lambda_j(t)}) + (1 - \alpha_0)(1 - \tilde{q}_0(0))(1 - q_j(0))e^{-(r_i + \rho)t - \Lambda_j(t)} \\ &\quad -(1 - \alpha_0)(1 - \tilde{q}_0(0))(1 - q_j(0)) - \frac{1 - \alpha_0}{2}[q_{0,j}(0) + q_j(0)] - \frac{1 - \alpha_0 + \alpha}{2} q_{0,i}(0) \end{aligned}$$

It follows (after some algebra) that the condition $U_i(t, G_0, G_j) - U_i(0, G_0, G_j) \geq 0$ is equivalent to the condition

$$\frac{\alpha}{2} q_{0,i}(0) - \frac{1 - \alpha_0}{2}[\tilde{q}_0(0) + q_j(0)] \geq 0.$$

For this to be possible, we must have $q_j(0) = 0$. Otherwise, $q_j(0) > 0$ implies $\tilde{q}_0(0) = 0$ by Lemma 4, and then $U_i(t, G_0, G_j) - U_i(0, G_0, G_j) = -\frac{1 - \alpha_0}{2} q_j(0) < 0$, which is a contradiction. Therefore, $q_j(0) = 0$ and $\frac{\alpha}{2}q_{0,i}(0) \geq \frac{1-\alpha_0}{2}\tilde{q}_0(0)$. Since i was arbitrary, we have $q_1(0) = q_2(0) = 0$, and $q_{0,i}(0) \in [\frac{1-\alpha_0}{\alpha+\alpha_0-1}q_{0,j}(0), \frac{\alpha+\alpha_0-1}{1-\alpha_0}q_{0,j}(0)]$ as desired. ■

Proof of Proposition 3. Necessity: (i) By Lemma 8, there are no atom points in $G_{0,1}, G_{0,2}, G_1$ and G_2 over $(0, \hat{t})$ after the start of the game and before the end of the game. By Lemma 11, when both competing players gradually concede throughout the game we must have $\hat{t} = \infty$. Therefore, $G_{0,1}, G_{0,2}, G_1$ and G_2 must be continuous over $(0, \infty)$ in an equilibrium in which both competing players concede throughout the game. (ii) Immediately follows from Lemma 12. (iii) The fact that $\lambda_1(t) + \lambda_2(t) = \mu$ immediately follows from part (i) of Lemma 9. Since both competing players concede throughout the game, we must have $\lambda_1(t), \lambda_2(t) > 0$ for $t \geq 0$ almost everywhere. (iv) Immediately follows from part (iii) of Lemma 9.

Sufficiency: By condition (i), G_1 and G_2 are continuous over $(0, \infty)$, and thus λ_1 and λ_2 are well defined over $(0, \infty)$. Additionally, by (iii), $\lambda_i(t) = \frac{G_i'(t)}{1-G_i(t)} > 0$ for $i = 1, 2$. Therefore, $G_1' > 0$ and $G_2' > 0$, which implies that G_1 and G_2 are strictly increasing on $\bar{\mathbb{R}}_+$.

We now establish that no player has strictly profitable deviation from the proposed strategy profile $(G_{0,1}, G_{0,2}, G_1, G_2)$. First consider player 0. By (A1) and condition (ii), we have

$$U_0(0, \kappa, G_1, G_2) = 1 - \alpha.$$

By (A2) and conditions (i) and (ii), we have

$$U_0(\infty, \kappa, G_1, G_2) = \sum_{i=1}^{2} \int_0^\infty \alpha_0 e^{-r_0 v}(1-G_j(v))dG_i(v) = \alpha_0 \int_0^\infty -e^{-r_0 v} d(1-G_1(v))(1-G_2(v))$$

where the second inequality follows from the product rule for derivatives. Condition (iii) gives us $(1 - G_1(v))(1 - G_2(v)) = e^{-\mu v}$. Therefore,

$$U_0(\infty, \kappa, G_1, G_2) = \alpha_0 \int_0^\infty e^{-r_0 v} \mu e^{-\mu v} dv = \alpha_0 \mu \left. \frac{e^{-(r_0+\mu)v}}{r_0 + \mu} \right|_\infty^0 = \frac{\alpha_0 \mu}{r_0 + \mu} = 1 - \alpha$$

where the last equality follows from the definition of μ given in (2).

By (A3) and conditions (i) and (ii), for any > 0 we have

$$U_0(t, \kappa, G_1, G_2) = (1 - \alpha)e^{-r_0 t}(1 - G_1(t))(1 - G_2(t)) + \sum_{i=1}^{2} \int_0^t \alpha_0 e^{-r_0 v}(1 - G_j(v))\mathrm{d}G_i(v)$$

Similar to above, using the product rule for derivatives, the right hand side is equal to

$$(1-\alpha)e^{-r_0 t}(1-G_1(t))(1-G_2(t)) + \alpha_0 \int_0^t -e^{-r_0 v} d(1-G_1(v))(1-G_2(v)).$$

Using the fact that $(1-G_1(t))(1-G_2(t)) = e^{-\mu t}$, this is equal to

$$(1-\alpha)e^{-(r_0+\mu)t} + \alpha_0 \int_0^t \mu e^{-(r_0+\mu)v} dv = (1-\alpha)e^{-(r_0+\mu)t} + (1-\alpha)(1-e^{-(r_0+\mu)t}).$$

Thus, we conclude that

$$U_0(t, \kappa, G_1, G_2) = 1-\alpha.$$

for all $t \geq 0$. Therefore, given that players 1 and 2 gradually concede using the proposed mixed strategy profile (G_1, G_2), player 0 is indifferent about the concession time: no matter when she concedes, her expected payoff is always $1-\alpha$.

Next we consider the competing players $i = 1, 2$. Similar to derivations above, by (A4) and condition (ii), we have

$$U_i(0, G_0, G_j) = (1-\alpha_0)(1-\tilde{q}_0(0)) + \frac{1-\alpha_0}{2} q_{0,j}(0) + \Big(\frac{1-\alpha_0+\alpha}{2}\Big) q_{0,i}(0).$$

By (A5) and conditions (i) and (ii), we have

$$\begin{aligned} U_i(\infty, G_0, G_j) &= \alpha q_{0,i}(0) + \int_0^\infty \alpha e^{-r_i v}(1-G_j(v)) \mathrm{d}G_{0,i}(v) \\ &= \alpha q_{0,i}(0) + \int_0^\infty \alpha e^{-r_i v} e^{-\Lambda_j(v)} \mathrm{d}G_{0,i}(v) \qquad \text{(A9)} \end{aligned}$$

where $\Lambda_j(v) \equiv \int_0^v \lambda_j(\tau)\mathrm{d}\tau$ and the second equation follows from the definition of λ_j which implies $1-G_j(v) = (1-q_j(0))e^{-\Lambda_j(v)} = e^{-\Lambda_j(v)}$ by condition (ii) for any $v \in (0,\infty)$. By the definition of $\lambda_{0,i}$, we have $G'_{0,i}(v) = \lambda_{0,i}(v)[1-\tilde{G}_0(v)]$. By Corollary 2, the right hand side is equal to $\lambda_{0,i}(v)(1-\tilde{q}_0(0))e^{-\rho v}$. Thus using condition (iv), we obtain $G'_{0,i}(v) = \frac{1-\alpha_0}{\alpha}(\lambda_j(v)+r_i+\rho)(1-\tilde{q}_0(0))e^{-\rho v}$. Plugging in (A9), we obtain

$$\begin{aligned} U_i(\infty, G_0, G_j) &= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0)) \int_0^\infty e^{-r_i v} e^{-\Lambda_j(v)}(\lambda_j(v)+r_i+\rho)e^{-\rho v}\mathrm{d}v \\ &= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0))\; e^{-(r_i+\rho)v-\Lambda_j(v)}\big|_\infty^0 \\ &= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0)) \\ &\geq U_i(0, G_0, G_j) \end{aligned}$$

where the inequality follows from condition (ii).

Finally, by (A6) and conditions (i) and (ii), for any $t > 0$, we have

$$
\begin{aligned}
U_i(t, G_0, G_j) &= \alpha q_{0,i}(0) + \int_0^t \alpha e^{-r_i v}(1 - G_j(v))dG_{0,i}(v) + (1-\alpha_0)e^{-r_i t}(1 - \tilde{G}_0(t))(1 - G_j(t)) \\
&= \alpha q_{0,i}(0) + \int_0^t \alpha e^{-r_i v} e^{-\Lambda_j(v)} G'_{0,i}(v)dv + (1-\alpha_0)e^{-r_i t}(1 - \tilde{q}_0(t))e^{-\rho t}e^{-\Lambda_j(t)} \\
&= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0))(1 - e^{-(r_i+\rho)t - \Lambda_j(t)}) + (1-\alpha_0)(1-\tilde{q}_0(0))e^{-(r_i+\rho)t-\Lambda_j(t)} \\
&= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0)) \\
&= U_i(\infty, G_0, G_j) \\
&\geq U_i(0, G_0, G_j)
\end{aligned}
$$

where the second equation follows from $1 - G_j(v) = e^{-\Lambda_j(v)}$, and the third equation follows from $G'_{0,i}(v) = \frac{1-\alpha_0}{\alpha}(\lambda_j(v) + r_i + \rho)(1 - \tilde{q}_0(0))e^{-\rho v}$. Therefore, player i weakly prefers to concede after the start of the game, and is indifferent about the concession time after the start of the game. Thus, any (mixed) strategy that assigns zero probability mass at $t = 0$, as in the proposed strategy profile, is a best response. ■

Proof of Lemma 13. Fix $s \in (0, \hat{t}]$ and $t \in (0, s)$. By the definition of $\hat{t}$, we have $G_1(t) < 1, G_2(t) < 1, \tilde{G}_0(t) < 1$. Then by Lemma 10, $H = 1-(1-G_1)(1-G_2)$ is strictly increasing over $[0, t]$. Since G_j is constant over $(0, t]$, G_i must be strictly increasing over $[0, t]$. Therefore, by Corollary 1, for $t \in (0, s)$ almost everywhere, player i weakly prefers conceding at time t to conceding at time 0, i.e. $U_i(t, G_0, G_j) - U_i(0, G_0, G_j) \geq 0$. By (A4) and (A6), this condition can be written as

$$
\begin{aligned}
U_i(t, G_0, G_j) - U_i(0, G_0, G_j) &= \alpha q_{0,i}(0) + \int_0^t \alpha e^{-r_i v}(1 - G_j(v))dG_{0,i}(v) \\
&+ (1-\alpha_0)e^{-r_i t}(1 - \tilde{G}_0(t))(1 - G_j(t)) \\
&- (1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0)) - \frac{1-\alpha_0}{2}[q_{0,j}(0) + q_j(0)] - \frac{1-\alpha_0+\alpha}{2}q_0 \\
&= \alpha q_{0,i}(0) + \int_0^t \alpha e^{-r_i v}(1 - q_j(0))\frac{(1-\alpha_0)(1-\tilde{q}_0(0))}{\alpha}(r_i + \lambda_0(v))e^{-\Lambda_0(v)}\mathrm{d}v \\
&+ (1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0))e^{-r_i t - \Lambda_0(t)} \\
&- (1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0)) - \frac{1-\alpha_0}{2}[q_{0,j}(0) + q_j(0)] - \frac{1-\alpha_0+\alpha}{2}q_0 \\
&= \alpha q_{0,i}(0) + (1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0))(1 - e^{-r_i t-\Lambda_0(t)} + e^{-r_i t - \Lambda_0(t)}) \\
&- (1-\alpha_0)(1-\tilde{q}_0(0))(1-q_j(0)) - \frac{1-\alpha_0}{2}[q_{0,j}(0) + q_j(0)] - \frac{1-\alpha_0+\alpha}{2}q_0 \\
&= \frac{\alpha}{2}q_{0,i}(0) - \frac{1-\alpha_0}{2}[\tilde{q}_0(0) + q_j(0)] \geq 0 \qquad (A
\end{aligned}
$$

where $\Lambda_0(v) = \int_0^v \lambda_{0,1}(\tau) + \lambda_{0,2}(\tau)d\tau$. Here, the first equation follows from $q_1(0)\tilde{q}_0(0) = q_2(0)\tilde{q}_0(0) = 0$ (by Lemma 4) and $\tilde{q}_0(\tau) = q_1(\tau) = q_2(\tau) = 0$ for any $\tau \in (0, \hat{t})$ (by Lemma 8); and the second equation follows from $G_j(t) = q_j(0)$ and $G'_{0,i}(v) = \lambda_{0,i}(v)[1-\tilde{G}_0(v)] = \frac{1-\alpha_0}{\alpha}(r_i + \lambda_0(v))(1 - \tilde{q}_0(0))e^{-\Lambda_0(v)}$ (by Equation (8)).

We now show that $q_j(0) = 0$. Suppose, by contradiction, that $q_j(0) > 0$. By Lemma 4, we have $\tilde{q}_0(0) = 0$, and then $U_i(t, G_0, G_j) - U_i(0, G_0, G_j) = -\frac{1-\alpha_0}{2}q_j(0) < 0$, which gives a contradiction to $q_j(0) > 0$ by Lemma 1. Therefore, we have $q_j(0) = 0$. Then (A10) gives us $\frac{\alpha}{2}q_{0,i}(0) - \frac{1-\alpha_0}{2}\tilde{q}_0(0) \geq 0$, i.e., $q_{0,i}(0) \geq \frac{1-\alpha_0}{\alpha+\alpha_0-1}q_{0,j}(0)$. ■