Persuasion with Hard and Soft Information*

Sulagna Dasgupta

April 5, 2022

Abstract

A privately informed sender with state-independent preferences communicates with an uninformed receiver about a two-dimensional state. The sender can verifiably disclose the state's first dimension with some probability, and can communicate about both dimensions via cheap talk. When the two dimensions are positively dependent, unravelling occurs - i.e. the sender fully reveals evidence whenever he has it - if and only if the sender has evidence with probability one. When unravelling does not occur, the model features multiple equilibria. Varying across equilibria, I show that equilibria that feature more disclosure are worse for the sender, with the disclosure minimizing equilibrium being sender-best. Comparative statics results indicate a substitution effect between communication via cheap talk and disclosure. I fully characterize the sender-optimal equilibrium for a few applications, and provide an extension to multiple unverifiable dimensions and non-monotonic sender utility under certain equilibrium selection rules.

1 Introduction

In many situations, individuals or institutions might want to convince a third party about something they have to offer, where only some of the aspects of what they have to offer may be disclosed with evidence (*hard* information), but they can convey unverifiable information about any of the aspects (*soft* information). For

*I am indebted to Ben Brooks, Doron Ravid and Emir Kamenica for their guidance and support. I gratefully acknowledge the comments and suggestions from Phil Reny, Daniel Rappaport, Alex Frankel and Debasis Mishra. I also thank the seminar participants at I also thank the seminar participants at the European Winter Meeting of the Econometric Society (Dec 2021), Winter School 2021 by the Econometric Society and the Delhi School of Economics (Dec 2021), Annual Conference at the Indian Statistical Institute, Delhi (Dec 2021), Annual Meeting of the Illinois Economics Association (Oct 2021), Theory group seminar at the University of Chicago (Nov 2021) and brown bag seminar at the Colorado State University (Nov 2021). All errors are my own.

example, a student applying to a college can furnish her standardized test scores to the selection committee, which is verifiable, unlike her leadership skills, interpersonal skills etc., though she can write about both in her personal essays. A manufacturer of packaged food can disclose the calorific contents of its product to its potential customers with verifiable evidence in the form of lab tests etc., but it cannot provide such evidence about how good it tastes. However the manufacturer can try to convey information - though without evidence - about both aspects using advertising or other promotional communication. In this paper we analyze the impact of interaction of these two types of communication in such settings.

We consider a model where a privately informed sender wants to persuade an uninformed receiver about a two-dimensional state. Only one of the dimensions of the state is verifiably disclosable, with some probability. But the sender can communicate about both using cheap talk. We assume that the sender has state-independent preferences - i.e. his utility depends only on the receiver's action, which is taken to be her posterior mean state. Sender's motives are transparent - his utility is increasing in the both the components of the receiver's action, i.e. this posterior mean.

It is easy to see that due to the presence of the additional unverifiable dimension, this model will, in general, feature a large set of equilibria stemming from the freedom of choosing off path beliefs. Following Hart et al. (2017), we use the special significance of revealing to truth to select equilibria, called *truth-leaning equilibria* by Hart et al. (2017). Specifically, these obtain from limits of perturbed games with infinitesimal increases in the sender's utility when revealing evidence if he has it, and in his probability of doing so. Analogous to Hart et al. (2017), truth-leaning in our model amounts to the following two natural conditions: (i) when the rewards for revealing the evidence and concealing and engaging in cheap talk are the same, the sender prefers to reveal the whole truth; and (ii) there is a small positive probability that the evidence gets revealed. Again, as shown by Hart et al. (2017), truth-leaning turns out to be consistent with the various refinement conditions offered in the literature.

It is perhaps intuitive to conjecture, that when evidence of the verifiable dimension of the state is always available, the famous "unravelling" result of Milgrom (1981) and Grossman (1981) might ensue - i.e. the sender must always disclose his evidence - because the sender's utility is increasing in both components of the receiver's posterior mean. The existence of the unverifiable dimension renders this conjecture false, as we show by way of an example (Example 1). As may be intuitive, in this case unravelling is prevented by a negative correlation between the two dimensions - the sender does not necessarily want to reveal that he has a high first dimension, because this becomes a signal of a low second dimension. Therefore it turns out whenever the two dimensions are *non-negatively dependent*[1] - including the case when they are independent - unravelling does ensue, if and only if the sender obtains evidence with probability one (Proposition 2). This provides sufficient conditions under which the intuition of Dye (1985) carries over to the setting with additional unverifiable dimensions of the state.

Driven by the well-known multiplicity of equilibria of the pure cheap talk game, our model typically features a continuum of equilibria, with varying sender payoffs. We show that across these equilibria, the less the sender discloses, the higher is his payoff (Proposition 3), making the equilibrium with the *smallest disclosure set* (of the first dimension) sender-optimal. We call this the *Minimum Disclosure Principle*.

Subsequently, by way of applications, we use the Minimum Disclosure Principle to characterize the sender-best equilibria for several classes of sender preferences. We show that whenever the sender's utility is quasiconcave (specifically, including linear) in the posterior means, the cutoff at the sender-optimal equilibrium is decreasing in the probability of obtaining evidence (Corollary 4). Intuitively, When the sender is very likely to have evidence, it is difficult to convince the receiver that he does not. Therefore the reliability of his communication when he claims not to have evidence, goes down in the eyes of the receiver. To compensate, he must reveal more verifiable information - i.e. lower the threshold. Our second example is a generalized version of the two-goods seller example of Chakraborty and Harbaugh (2010), later also analyzed by Lipnowski and Ravid (2020). Specifically, we consider a seller who wants to maximize the probability of selling one of her two products to a buyer, the quality of only one of which he can verifiably disclose. In this setting, we first characterize the sender-optimal equilibrium for the pure cheap talk game for a general full support prior - which turns out to consist of a single linear partition.[2] We then use this to find the sender-optimal equilibrium for the example case when the two dimensions of the state are independently and uniformly distributed.

By way of an extension we explore the robustness of our results to the case of multiple unverifiable dimensions and/or non-monotonic sender utility. The key additional complexity in this case is that there can be multiple equilibria in the continuation game upon revealing the verifiable dimension, which generate different payoffs for the sender. Hence the Minimum Disclosure Principle does not hold in general - if some types of senders are convinced that an adversarial equilibrium is going to be selected in the continuation game, this would - on the one hand - deter them from disclosing, depriving them of their best equilibrium payoff in the continuation game upon disclosure but expanding the non-disclosing set of types

[1]To be precise, whenever the joint distribution of the two dimensions of the state is such that the conditional expectation of the second dimension is non-decreasing in the first dimension.

[2]To be precise, this is the least informative sender-optimal equilibrium. There are infinitely many equilibria, payoff equivalent to this one for the sender, which involve revealing more precise information to the receiver within each partitional element. See section 5.2 for details.

and potentially increasing the non-disclosure payoff. The result of the interaction of these two forces is ambiguous in general. However, if the sender can select the equilibrium in the continuation game, this interaction no longer exists and we get back the Minimum Disclosure Principle even in this context (Proposition 9).

I conclude by discussing avenues for future research.

1.1 Related literature

This paper brings notions arising from three distinct strands of literature together - Disclosure of verifiable information, cheap talk and imperfect evidence. The verifiable diclosure literature, arguably pioneered by the seminal works of Grossman (1981) and Milgrom (1981) study disclosure of a single-dimensional unknown state by an informed sender, when he can prove any true claim. The key insight of these papers is the famous “unravelling” result, which shows that under complete verifiability and monotone preferences, full information must be revealed in equilibrium (this notion was subsequently generalized by Hagenbach et al. (2014)).

Subsequently there has been a large literature studying failure of the unraveling result under various conditions. Verrecchia (1983) shows that when disclosure incurs a cost, the receiver cannot conclusively infer nondisclosure to a low state. Milgrom and Roberts (1986) show that in the presence of naively credulous decisionmakers, the interested party would like opaque information to elicit favorable decisions from them. Dye (1985), Dziuda (2011), Shin (1994), and Wolinsky (2003) trace the failure of unravelling respectively to uncertainty in obtaining evidence, preferences, and honesty, all within the world where the state is single-dimensional. Our paper adds to this literature by providing yet another avenue when the unravelling result can fail even with monotonic preferences and perfect evidence - if there is an additional, unverifiable dimension (Example 1).

Particularly of interest to us is the approach of Dye (1985) (later also Jung and Kwon (1988)), in which evidence is obtained with some probability, because this is the channel through which interesting communication arises in our model, bypassing unravelling. The difference between these models and ours is twofold - first, our sender knows the state fully, even when he cannot obtain evidence for it, unlike in Dye (1985) where, when the sender fails to obtain evidence, he is also in the dark about the state. Second, of course, is the existence of the additional unverifiable dimension, which allows non-trivial persuasion via cheap talk to happen, even though the sender’s preferences are transparent (monotonic in both components of the state).

Lastly, our paper contributes to the large literature on strategic communication in the absence of verifiability (cheap talk) and more specifically to transmission of multidimensional information (see Sobel (2013), for an extensive review of the literature on strategic communication). In this framework, Chakraborty and Harbaugh

(2007) show that in a multidimensional model of cheap talk communication, some information, in particular, relative statements about the dimensions of interest, may be transmitted.Chakraborty and Harbaugh (2010) further show the sender can always communicate some information credibly and influence the receiver's actions by trading off dimensions. Lipnowski and Ravid (2020) consider a more general, abstract formulation and characterize optimal equilibrium outcomes for the sender. Battaglini (2002) studies cheap talk with multiple senders and focuses on the possibility of full revelation. This paper builds on the insights of both Chakraborty and Harbaugh (2010) and Lipnowski and Ravid (2020).

2 Model

2.1 Notations

For any set X, $\mathcal{B}(X)$ denotes the Borel σ-algebra on it and $\mathcal{P}(X)$ denotes its power set, i.e. the set of all its subsets. $\mathbb{U}(\mu; U)$ denotes the set of cheap talk payoffs when the sender's utility function is $U(\cdot)$ and the receiver's prior is $\mu \in \Delta[0,1]^2$. Throughout the paper, for $\theta, \widetilde{\theta} \in [0,1]^2$, $\theta \leq \widetilde{\theta}$ is used to mean $\theta_i \leq \widetilde{\theta}_i$ for $i \in \{1,2\}$.

2.2 Setting

There is a sender (S, he), who wishes to persuade a receiver (R, she) about an two-dimensional state θ. Let $\theta \equiv (\theta_1, \theta_2)$, $\theta_i \in \Theta_i = [0,1]$. Only one of the components of the state, θ_1, is verifiable. However, S can use cheap talk to communicate with R about both θ_1 and θ_2, using messages from a sufficiently rich space M. S's utility depends only on R's posterior mean of θ, and is given by $U : [0,1]^2 \to \mathbb{R}_+$, which is continuous and weakly increasing in both components.[3] Let $F(\cdot)$ be the absolutely continuous, full support joint prior distribution with a continuous density. Let $F_i(\cdot)$ denote the i-th marginal of F. Let $\theta_{i0} = \mathbb{E}_F(\theta_i)$.

The timing of the game is as follows: First, $\theta \equiv (\theta_1, \theta_2)$ is realized and observed by S. S takes a test and obtains perfectly revealing verifiable evidence on θ_1 with exogenously fixed probability $q \leq 1$. Whether S has evidence is independent of θ. If S has evidence, he then chooses the probability with with to reveal his evidence, depending on his type $d : \Theta \to [0,1]$. Let $T = \{0,1\}$ capture the set of "evidence outcomes", with 1 denoting evidence is provided and 0 denoting it is not. Simultaneously S also chooses a distribution over cheap messages to send, again

[3]An *example* of a way in which U can be microfounded is as follows: Suppose there exists some publicly known function $v : [0,1]^2 \to \mathbb{R}$, strictly increasing in both arguments, such that R accepts if v of her posterior mean (conditional on all information known to R) $s \equiv (s_1, s_2)$ is weakly above a privately observed threshold $\omega \sim H$, a strictly increasing CDF, so S's utility is $U(s_1, s_2) = H(v(s_1, s_2))$.

depending on his type and whether evidence was revealed in the previous stage: $\sigma : \Theta \times T \to \Delta M$. Finally receiver takes actions based on her observed evidence and/or cheap messages, $\rho : T \times M \to \Theta$, and payoffs are realized.

We sometimes use $v : \Delta\Theta \to \mathbb{R}$ to denote the sender's payoff as a function of the receiver's belief. Clearly, $v(\mu) = U \circ E\mu$ where $E : \Delta\Theta \to \Theta$ is the expectation operator, which is continuous. $U(\cdot)$ is continuous by assumption. Therefore $v(\cdot)$ is a continuous function in our model. $\overline{v} : \Delta\Theta \to \mathbb{R}$ denotes sender's maximized cheap talk equilibrium payoff as a function of the prior, when his payoff function is $v(\cdot)$.[4]

2.3 Equilibrium

We focus on perfect Bayesian equilibria. Hence an equilibrium consists of four objects: A disclosure strategy of S, $d : \Theta \to [0, 1]$, a communication strategy of S, $\sigma : \Theta \times T \to \Delta M$, R's belief updating rule, $\mu : \Theta_1 \cup \emptyset \times M \to \Delta\Theta$ and her strategy $\rho : \Theta_1 \cup \emptyset \times M \to \Theta$, where we have assumed $\rho(t, m) = \mathbb{E}\mu(t, m)$, $t \in \Theta_1 \cup \emptyset$ where $\emptyset$ denotes no verifiable disclosure was made. For the quadruple (d, σ, μ, ρ) to constitute a PBE the following must hold.

1. μ is obtained from F, given σ, using Bayes' rule on the equilibrium path.
2. $\sigma(\theta)$ is supported on $\arg \max_{m \in M \times \Theta_1 \cup \emptyset} U(\mathbb{E}\mu(m))$ for all $\theta \in \Theta$.
3. ρ is consistent with R's utility maximization.
4. d is consistent with S's expected utility maximization, given μ.

Since we have assumed $\rho(t, m) = \mathbb{E}\mu(t, m)$ we drop ρ from the equilibrium quadruple.

Let $\mathbb{E}(\theta_2|\theta_1, D)$ denote the posterior expectation of θ_2 given θ_1 conditional on disclosure. Therefore, $\mathbb{E}(\theta_2|\theta_1, D) = \frac{\int_B \theta_2 d(\theta) dF(\theta_2|\theta_1)}{\int_{\Theta_2} d(\theta) dF(\theta_2|\theta_1)}$.

First we note that no persuasive communication is possible post verifiable disclosure of θ_1. This follows from Proposition 1 and Theorem 1 of Lipnowski and Ravid (2020), but a simple proof is provided in the appendix for our particular case, for completeness.

Since we are concerned with sender's payoffs in this paper, using the above observation, going forward we assume, without loss of generality (from the point of view of sender's payoffs) that no cheap talk communication occurs in equilibrium post disclosure of θ_1. Therefore in an equilibrium with disclosure strategy d, if θ_1 is disclosed with positive probability on the equilibrium path, the receiver's belief about θ_2 upon disclosure is given by:

[4]As shown by Lipnowski and Ravid (2020), this maximum exists under when $v(\cdot)$ is continuous.

$$\mu^D(B|\theta_1, d) = \frac{\int\limits_B d(\theta)dF(\theta_2|\theta_1)}{\int\limits_{\Theta_2} d(\theta)dF(\theta_2|\theta_1)} \; \forall \; B \in \mathcal{B}(\Theta_2). \tag{1}$$

If non-disclosure happens with positive probability on the path of some equilibrium (which must happen, for example, if $q < 1$), the receiver's belief conditional on the event of non-disclosure (denoted by ND), before observing any cheap talk communication, is given by:

$$\mu^{ND}(A|ND) = \frac{(1-q)F(A) + q\int\limits_A (1-d(\theta))dF(\theta)}{1-q+q\int\limits_\Theta (1-d(\theta))dF(\theta)} \; \forall \; A \in \mathcal{B}(\Theta). \tag{2}$$

Of course, R's posterior belief in any equilibrium, μ, consists of μ^D and μ^{ND} taken together. For ease of exposition, going forward, we denote an equilibrium by the quadruple $(d, \sigma, \mu^D, \mu^{ND})$ instead of the triple (d, σ, μ), as defined earlier.

Define the aggregate equilibrium probability of disclosure given θ_1, $D(\theta_1) = \int\limits_{\theta_2 \in \Theta_2} d(\theta)dF(\theta_1, \theta_2)$. We call $D : \Theta_1 \to [0, 1]$ the **aggregate disclosure strategy** associated with d. Also, let $\widetilde{U}(\theta_1) = U(\theta_1, \mathbb{E}_F(\theta_2|\theta_1))$.

Let $\mathbb{E}(\theta_2|\theta_1, ND)$ denote the posterior expectation of θ_2 given θ_1 conditional on having evidence but not disclosing. Therefore, $\mathbb{E}(\theta_2|\theta_1, ND) = \frac{\int\limits_B \theta_2(1-d(\theta))dF(\theta_2|\theta_1)}{\int\limits_{\Theta_2} (1-d(\theta))dF(\theta_2|\theta_1)}$.

Because all θ_1-types must earn the same payoff upon disclosure and non-disclosure, if $D(\theta_1) \in (0, 1)$, the disclosure and non-disclosure payoffs given θ_1 must be equal, i.e. $U(\theta_1, \mathbb{E}(\theta_2|\theta_1, D)) = U_{ND}$, where U_{ND} is the non-disclosure payoff at this equilibrium. Formally,

Observation 1. $D(\theta_1) \in (0, 1) \implies U(\theta_1, \mathbb{E}(\theta_2|\theta_1, D)) = U_{ND}$.

2.4 Truth-leaning equilibria

Evidence games may have many uninteresting equilibria, owing to the often large set of possible off-equilibrium-path beliefs of the receiver.[5] We are interested in those in which the sender has a bias towards telling the truth, and as a result, enjoys the benefit of the doubt. To formalize this idea, we borrow from the notion of truth-leaning equilibria introduced by Hart et al. (2017). First, consider the class of equilibria satisfying the following conditions: whenever the sender has evidence of

[5]As an example, suppose $U(\theta) = \theta_1\theta_2$, and the joint density of (θ_1, θ_2) is f. Consider the off-path belief of the receiver - if $d(\theta) = 0$ in an equilibrium, $\mu^D(\cdot; \theta_1) = \delta_{(\theta_1, 0)}$, i.e. if a type θ_1 is not supposed to reveal evidence in an equilibrium, upon observing an off-path revelation of such evidence, the receiver assumes $\theta_2 = 0$. This is an example of a "punishment belief". Note that any pure cheap talk equilibrium with no revelation of evidence is an equilibrium of this game for any $q \in (0, 1]$, because any θ_1-type, including very high types, get a reward of zero if they deviate and reveal their evidence.

the verifiable dimension, first, if he is indifferent between revealing it and pretending not to have evidence, he prefers to reveal it (this happens, for instance, when the sender has a natural bias towards telling the truth - he always prefers a higher reward, but if the reward is the same whether he tells the truth about having evidence or not, he prefers to tell the truth). Secondly, there is an infinitesimal probability that it gets revealed (which happens, for example, when the sender is not strategic and instead reveals his information whenever he has it). We show that in the limit when the slight preference for truth and the infitinitesimal probability of evidence being revealed go to zero, these equilibria feature truth-leaning.

To formalize this we use a standard limit-of-small-perturbations approach, again following the framework of Hart et al. (2017). Given $\epsilon = (\epsilon_1, \epsilon_2) > 0$, let Γ^ε denote the following perturbation of the game Γ. The sender's payoff increases by ϵ_1 when he has evidence of θ_1 and reveals it. Secondly, his disclosure strategy $d(\cdot)$ is required to satisfy $d \geq \varepsilon_2$, i.e. $d(\theta) \geq \varepsilon_2$ for every $\theta \in \Theta$. A Nash equilibrium $(d, \sigma, \mu^D, \mu^{ND})$ of the original game Γ is truth-leaning if it is a limit point of Nash equilibria of Γ^ε as ϵ converges to $(0,0)$, i.e., if there are sequences $\varepsilon^n \to_{n\to\infty} (0,0)$, and $\left(d_n, \sigma_n, \mu_n^D, \mu_n^{ND}\right) \to_{n\to\infty} \left(d, \sigma, \mu^D, \mu^{ND}\right)$ such that $\left(d_n, \sigma_n, \mu_n^D, \mu_n^{ND}\right)$ is a Nash equilibrium of Γ^{ε^n} for every n.

In terms of the original game, truth-leaning turns out to be essentially equivalent to imposing the following two condition on a Nash equilibrium $\left(d_n, \sigma_n, \mu_n^D, \mu_n^{ND}\right)$ of Γ :

(S0) For every $\theta_1 \in \Theta_1$, if $\max_{m\in M} v\left(\mu^{ND}(m)\right) = U\left(\theta_1, \mathbb{E}\mu^D\left(\cdot;\theta_1\right)\right)$, then $d(\theta_1,\cdot) = 1$, i.e. $D(\theta_1) = 1$.

(R0) For every $\theta_1 \in \Theta_1$, if $D(\theta_1) = 0$ then, $\mu^D(\cdot;\theta_1) = F(\cdot)|_{\theta_1}$.

Proposition 1. *(i) Truth-leaning equilibria exist. (ii) If an equilibrium is truth-leaning it satisfies (S0) and (R0).*

The proof is relegated to the Appendix. Truth-leaning may thus be viewed as an equilibrium selection criterion (a "refinement"). As shown in Appendix C.5 of Hart et al. (2017), truth-leaning satisfies the requirements of most relevant equilibrium refinements that have been proposed in the literature, such as the intuitive criterion, the D1 condition, universal divinity, and the never-weak-best-reply criterion (Kohlberg and Mertens (1986),Banks and Sobel (1987),Cho and Kreps (1987)).

Going forward we analyze the truth-leaning equilibria of the game, which we refer to as its *equilibria*.

The following observation is immediate from (S0) and Observation 1:

Observation 2. *In any equilibrium, the sender's disclosure strategy $d(\cdot)$ cannot depend on θ_2.*

By Observation 2, $d(\theta_1, \theta_2) = D(\theta_1)$ for all $\theta_1 \in \Theta_1$. As such, we refer to $D(\cdot)$ as the sender's *disclosure strategy* going forward.

Clearly, in an equilibrium, if $D(\theta_1) > 0$, all θ_1-types disclosure payoff is $\widetilde{U}(\theta_1)$. Therefore by Observation 1, $D(\theta_1) \in (0,1) \implies \widetilde{U}(\theta_1) = U_{ND}$. Moreover, suppose in an equilibrium for some θ_1, $\widetilde{U}(\theta_1) > U_{ND}$. Therefore $D(\theta_1) \notin (0,1)$ by the previous line. If $D(\theta_1) = 0$, all types θ_1 can profitably deviate to disclosing. Therefore $D(\theta_1) = 1$. Similarly, $\widetilde{U}(\theta_1) < U_{ND} \implies D(\theta_1) = 0$. We put these facts together below.

Observation 1´. *In any truth-leaning equilibrium,*

- $D(\theta_1) \in (0,1) \implies \widetilde{U}(\theta_1) = U_{ND}$.
- $D(\theta_1) = 1$ *for* $\widetilde{U}(\theta_1) > U_{ND}$ *and* $D(\theta_1) = 0$ *for* $\widetilde{U}(\theta_1) < U_{ND}$.

3 Unravelling

3.1 Unravelling need not happen: An example

When there is an unverifiable dimension in addition to the verifiable one, unravelling need not happen even with perfect tests. The following example illustrates the above.

Example 1. *Suppose θ_1 is uniform on $[0,1]$ and θ_2 is uniform on $[1-\theta_1-\frac{1}{2}\theta_1(1-\theta_1), 1-\theta_1+\frac{1}{2}\theta_1(1-\theta_1)]$ for each θ_1. The mean and range of θ_2 conditional on θ_1 is shown in the figure below. Therefore $\mathbb{E}(\theta_2|\theta_1) = 1-\theta_1$ and the prior mean is $\theta_0 = (\frac{1}{2}, \frac{1}{2})$. Suppose $q = 1$, i.e. evidence is perfect.*

S's utility is given by $U(\theta_1, \theta_2) = \sqrt{\theta_1\theta_2}$. Therefore S's disclosure payoff at θ_1 is $\widetilde{U}(\theta_1) = U(\theta_1, \mathbb{E}(\theta_2|\theta_1)) = \sqrt{\theta_1(1-\theta_1)}$, which is maximized at $\theta_1 = \frac{1}{2}$. Due to the concavity of U, S's highest cheap talk payoff is equal to his payoff in the babbling equilibrium, $U(\theta_0) = \sqrt{\frac{1}{2} \times \frac{1}{2}} = \frac{1}{2} \geq \widetilde{U}(\theta_1)$ for all $\theta_1 \in [0,1]$. Therefore no-disclosure is an equilibrium of this game, where unravelling does not occur.

The above example illustrates that when the two components of the unknown state are known to be negatively correlated, it is possible that the existence of a verifiable disclosure technology on one of the components would make no difference and the cheap talk equilibria which would have ensued without such a technology could still ensue.[6] Intuitively, this happens because the forces leading to unravelling in case of a single-dimensional type - in the event of non-disclosure, R rationally estimates S's type to be worse than the mean of the non-disclosure set, which prompts

[6]As an aside, turns out, in this particular case, even the existence of a perfect disclosure technology for both the dimensions would not lead to unravelling (full disclosure), as S's unravelling payoff is strictly less than his payoff in the babbling equilibrium: $\mathbb{E}U(\theta_1,\theta_2) = \mathbb{E}(\sqrt{\theta_1\theta_2}) < \sqrt{\mathbb{E}(\theta_1\theta_2)} = \frac{1}{\sqrt{6}} < \frac{1}{2}$, where the second inequality follows from Jensen's inequality, due to the strict concavity of U. In that sense, this example provides yet another potential avenue for failure of unravelling, even with strictly monotonic preferences - a multidimensional state.

S to iteratively keep shrinking the non-disclosure set down to the empty set - are not necessarily present if there are additional components of the state which are negatively correlated with the disclosable component. In the latter case, if silence on the part of S sends a bad signal about θ_1, that also translates into a good signal about θ_2. Therefore depending on how the receiver values θ_1 and θ_2, she need not view silence entirely unfavorably, even when her (and therefore S's) preferences are strictly increasing in both θ_1 and θ_2. Examples of such situations include selling of packaged food, the value of which to the customer has both components - health and taste - only one of which (health) is verifiably disclosable (through calorie labels etc.).

3.2 When does unravelling happen?

The main driving force behind the failure of unravelling in the previous example was the negative correlation between the verifiable and unverifiable components. In this section we show that when the two components are non-negatively dependent, the unravelling result is restored.

Assumption 1. *We assume $\mathbb{E}(\theta_2|\theta_1)$ is a non-decreasing function of θ_1.*

Note that Assumption 1 subsumes the case when θ_1 and θ_2 are independent. The condition in Assumption 1 is known in the statistics literature as *positive dependence* (de Castro, 2009).[7]

We say the test is **perfect** if $q = 1$ and imperfect otherwise. We say **unravelling** occurs if S discloses evidence whenever he has it. Turns out that with perfect tests, unravelling must occur in the natural cases of θ_1 and θ_2 being independent or positively dependent, as observed in the proposition below.

Proposition 2. *Suppose assumption 1 holds and the sender's utility is strictly increasing in both components. Then, unravelling occurs if and only if the test is perfect. Moreover, whenever the test is imperfect, some types disclose.*

Proposition 2 can be thought of as an analog of Theorem 1 of Dye (1985), which gives conditions for unravelling. Note that Proposition 2 holds for *any* equilibrium, not just the sender preferred one.

The proof can be found in the Appendix.

4 Sender-preferred equilibria

For any equilibrium with disclosure strategy $D(\cdot)$, we call the set $\mathbb{D} = \{\theta_1 \in \Theta_1 : D(\theta_1) = 1\}$ its **disclosure set**. This is the set of θ_1-types which disclosure their

[7] To be precise, the statistical notion of positive dependence is slightly stronger than the condition in Assumption 1.

evidence in this equilibrium. We call an equilibrium an *unravelling equilibrium* if in it, the sender discloses evidence almost whenever he has it, i.e. if $F_1(\mathbb{D}) = 1$. Analogously, we call an equilibrium a *partial disclosure equilibrium/no disclosure equilibrium* if in it, $F_1(\mathbb{D}) \in (0,1)$ / $F_1(\mathbb{D}) = 0$.

Our main result for this section is the following.

Proposition 3. *If E_1 and E_2 are two partial disclosure equilibria with disclosure sets D_1 and D_2 and sender's ex-ante payoffs u_1 and u_2, then:*

- $u_1 > u_2 \implies D^1 \subsetneq D^2$.
- $D^1 \subsetneq D^2 \implies u_1 \geq u_2$.
- *Neither $D^1 \subsetneq D^2$ nor $D^2 \subsetneq D^1$* $\implies u_1 = u_2$.

Moreover, if there are unravelling (resp. no disclosure) equilibria, the sender weakly prefers all partial-disclosure (resp. no disclosure) equilibria to all unravelling (resp. partial disclosure) equilibria.

Note that if $\widetilde{U}(\cdot)$ does not have any flat regions, the above characterization can be strengthened as follows.

Corollary 1. *Suppose $\widetilde{U}(\cdot)$ is nowhere-constant. Then, the disclosure sets of the partial-diclosure equilibria are totally ordered under the inclusion order, with sender's ex-ante utility monotonically decreasing in this order.*

Proposition 3 implies that the sender-best equilibrium must feature the smallest disclosure set.

Corollary 2 (Minimum disclosure)**.** *The following characterizes the sender-optimal equilibria in terms of disclosure sets:*

- *If there exists a no disclosure equilibrium, the sender-optimal equilibrium features no disclosure.*
- *If the above does not hold but there exists a partial disclosure equilibrium, the sender-optimal equilibria are characterized by the smallest disclosure set out of all equilibria.*
- *Otherwise the sender-optimal equilibrium is an unravelling equilibrium with the sender-optimal cheap talk equilibrium played if and when evidence is not available.*

Let us call the set of such equilibria E^*. Obviously, they are all payoff-equivalent to the sender.

Let $\Theta_1(U_{ND}) = \{\theta_1 : \widetilde{U}(\theta_1) \leq U_{ND}\}$, and $\overline{U}(F_0)$ is the highest cheap talk payoff of a sender with utility function U, when the prior is F_0. As shown by Lipnowski and Ravid (2020), $\overline{U}(F_0)$ is well-defined. Let us define $\underline{U}_D = \inf_{\theta_1} \widetilde{U}(\theta_1), \overline{U}_D = \sup_{\theta_1} \widetilde{U}(\theta_1)$.

Using the logic of Proposition 3, the following also becomes obvious, owing to the cheap talk equilibrium payoff set for any prior being an interval, as shown by Lipnowski and Ravid (2020). This is illustrated in Figure 1 below for the special case when all the equilibria are partial-disclosure.

Corollary 3. *The sender's equilibrium payoff set is an interval.*

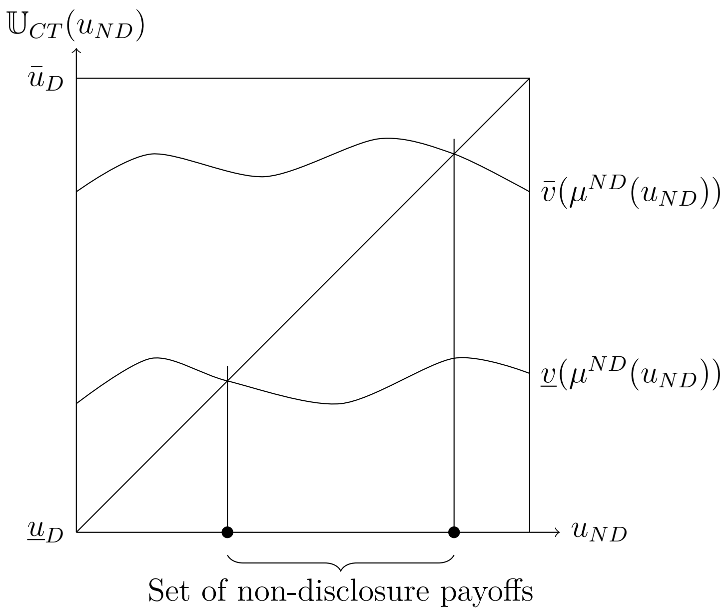

Figure 1: Sender's equilibrium payoff set is an interval

In the above figure, $\mu^{ND}(u_{ND})$ denotes receiver's belief upon observing non-disclosure, before observing any cheap talk communication, when the non-disclosure payoff is u_{ND}. Clearly,

$$\mu^{ND}(u_{ND}) = \frac{(1-q)F + qF_1(\{\theta_1 : \widetilde{U}(\theta_1) \leq u_{ND}\})F|_{\{\theta_1 : \widetilde{U}(\theta_1) \leq u_{ND}\}}}{1 - q + qF_1(\{\theta_1 : \widetilde{U}(\theta_1) \leq u_{ND}\})} \tag{3}$$

$\mathbb{U}_{CT}(u_{ND})$ denotes the set of cheap talk equilibrium payoffs when the prior is $\mu(u_{ND})$, which, we know from Lipnowski and Ravid (2020) is the interval $[\overline{v}(\mu(u_{ND})), \underline{v}(\mu(u_{ND}))]$.

The next proposition operationalizes Proposition 3 for practical use in finding sender optimal equilibria.

Proposition 4. *One of the following must hold at the sender-preferred equilibria:*

1. *Unravelling: Either* $\overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})}) < U_{ND}$ *for all* $U_{ND} \in [\underline{U}_D, \overline{U}_D]$*, in which case unravelling is the unique*[8] *- and therefore sender-preferred - equilibrium.*

2. *No disclosure: Or,* $\overline{U}(F) \geq \overline{U}_D$*, in which case the sender-preferred pure cheap talk equilibrium is the sender-preferred equilibrium.*

3. *Or, the sender-preferred equilibria are characterized by the non-disclosure payoff* U^*_{ND} *given by :*

$$U^*_{ND} = \max\{U_{ND} \in [\underline{U}_D, \overline{U}_D] : \overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})}) = U_{ND}\}$$

The proofs of both the propositions from this section are given in the appendix.

4.1 Comparative statics: Substitution effect between cheap talk and disclosure

In this section we use comparative statics to analyze the trade-off the sender faces in communicating via verifiable disclosure vs cheap talk. Using $\succsim_{MPS}$ to denote domination in the mean-preserving spread order, we have the following.

Proposition 5. *Suppose* F *and* G *are two priors with the same marginal on* θ_1 *such that* $G(\theta_2|\theta_1) \succsim_{MPS} F(\theta_2|\theta_1)$ *for all* θ_1*. Then the sender optimal equilibrium under* G *features a higher sender payoff and lower disclosure compared to that under* F*.*

The intuition behind Proposition 5 is as follows. A more informative distribution gives the sender more choices for cheap talk communication strategies, leading to a weakly higher payoff from cheap talk than he can get for any non-disclosure set. This shifts the trade-off between communication via cheap talk and disclosure in favor of cheap talk for the sender, leading to weakly lesser disclosure. Proposition 5 thus illustrates the substitution effect between information conveyed through cheap talk and disclosure.

5 Applications

5.1 Quasiconcave preferences

Proposition 6. *Suppose Assumption 1 holds. Without cheap talk, the disclosure threshold* $\underline{\theta}_1$ *in the sender-optimal equilibrium is non-increasing in the evidence probability* q*.*

[8]Unique in terms of payoffs and beliefs. Of course, infinitely many equilibria exist in the continuation game post-disclosure, across all of which receiver belief about θ_2 is the same - the prior belief on θ_2 conditional on disclosure of θ_1.

The intuition behind Proposition 6 is as follows. When q is very high, i.e. the sender is very likely to have evidence, it is difficult to convince the receiver that he doesn't. Therefore the reliability of his communication when he claims not to have evidence, goes down in the eyes of the receiver. To compensate, he must reveal more verifiable information - i.e. lower the threshold. The literature shows that this basic intuition holds even under other forms of communication - e.g. under full commitment - as shown by Shishkin (2019), who studies a similar model with faulty tests, but where the sender has full commitment power, and shows that the threshold below which test results are hidden goes down as the probability of obtaining result increases. The details of the proof are given in the appendix.

Clearly, for a quasiconcave U, the sender optimal cheap talk equilibrium conditional on a given cutoff $\underline{\theta}_1$ is the babbling equilibrium, for the same reason as it is in the pure cheap talk case (Chakraborty and Harbaugh, 2010). Therefore Proposition 6, combined with Proposition 4, leads to the following corollary.

Corollary 4. *Suppose Assumption 1 holds and U is quasiconcave. Then, the sender-optimal equilibrium is the highest-cutoff babbling equilibrium. Moreover, this cutoff is decreasing in the probability of obtaining evidence, q.*

Note further, that if $U(\cdot)$ is quasiconcave, in any sender-optimal cheap talk equilibrium under non-disclosure, no information is revealed. In addition, as noted in Corollary 4 above, the sender-optimal equilibrium is the lowest disclosure babbling equilibrium. Therefore in every other equilibrium, the sender either makes more verifiable disclosure, or reveals more information upon non-disclosure, or both - i.e. reveals more information to the receiver. This insight gives us the following corollary, where we call an equilibrium E *Blackwell-minimal* if the signal received by the receiver (through verifiable disclosure and/or cheap talk) in any other equilibrium is weakly Blackwell more informative than in E.

Corollary 5. *If $U(\cdot)$ is strictly quasiconcave, the sender-optimal equilibrium is Blackwell-minimal. If it is weakly quasiconcave, there exists a Blackwell-minimal equilibrium which is sender-best.*

5.2 The salesman: max utility function

In this section we adapt the famous salesman example analyzed by Chakraborty and Harbaugh (2010) and later expanded by Lipnowski and Ravid (2020) to our setting and use the logic of Proposition 4 to characterize the sender optimal equilibrium for this example. A buyer (R) can take an outside option or buy one of two goods from a salesman (S). The seller can provide verifiable proof of quality or value for only one of the products (say, a software) but not the other (say, a book of fiction). He can, of course, share unverifiable information with the customer - "talking up" or "talking down" - regarding both the products. Moreover, he may not have verifiable evidence of the quality of the first product with some probability.[9]

The seller knows the vector $\theta = (\theta_1, \theta_2)$, where θ_i denotes the buyer's net value from product i. Product values are jointly distributed according to some full support prior of continuous density over $[0,1]^2$. The seller wants to maximize the probability of a sale, but does not care which product is sold. Hence, the seller receives a value of 1 if the buyer chooses to purchase product $i \in \{1,2\}$, and 0 if the buyer chooses the outside option, which we denote by 0. Only the buyer knows her value from the outside option, ϵ, which is distributed independently from θ, uniformly on $[0,1]$. Chakraborty and Harbaugh (2010) study this example and show the seller can always benefit from communication. Lipnowski and Ravid (2020) show that the particular equilibrium identified by Chakraborty and Harbaugh (2010) in the case where θ_1 and θ_2 are i.i.d. is also sender-optimal. In this section, we use our tools to further generalize the analysis of both.

We first observe the following.

Proposition 7. *When the sender's utility is given by $U(\theta_1, \theta_2) = \max\{\theta_1, \theta_2\}$ and the prior F has full support, the sender's optimal cheap talk equilibrium payoff is given by the V-component of the highest-V solution to the following system of simultaneous equations, subject to $m > 0$:*

$$\mathbb{E}_F(\theta_2 | (\theta_2 - V) \geq m(\theta_1 - V)) = V,$$
$$\mathbb{E}_F(\theta_1 | (\theta_2 - V) \leq m(\theta_1 - V)) = V.$$

The intuition for the above result is as follows. In this setting, the receiver may buy one of the goods if its posterior mean quality is greater than her privately observed threshold. Therefore, even though - unlike in Bayesian persuasion settings - in cheap talk settings it is usually not without loss to focus on recommendation signals, in this case it is (from sender's perspective). We focus on "recommendation" messaging strategies by the sender and show that for any full support prior, it takes a simple form in the sender-optimal equilibrium. Specifically, it is characterized by a line of strictly positive slope through the point (V, V) where V is the sender's maximized payoff, such that good 2 (1) is recommended above (below) it. Intuitively, it consists of a degradation of information from the sender's optimal full-commitment strategy, which is, obviously, the same information structure, with $m = 1$ - i.e. the sender truthfully reveals which good is better. If sender had commitment power, he would have done exactly that, i.e. told the truth. But he cannot, due to the absence of commitment power, and hence needs to distort information by potentially rotating this line away from $m = 1$ about the point (V, V). Thus, the sender is forced to lie

[9] Test results of the software may be inconclusive or unavailable at the time of sale.

if the state falls in the region in between the full commitment line with unit slope and the rotated line.

We now apply the above characterization to find the sender-optimal equilibrium for the classic case when θ_1 and θ_2 are independently and uniformly distributed in $[0,1]$. As would be clear from Figure 2 below, in this case the sender's optimized equilibrium non-disclosure payoff is equal to the disclosure threshold $\underline{\theta}_1$ - the θ_1 below which evidence is hidden in the equilibrium. In the figure, $\mu^{ND}(\underline{\theta}_1)$ denotes the receiver's posterior when she knows states with the first component below $\underline{\theta}_1$ are hidden. Clearly,

$$\mu^{ND}(\underline{\theta}_1) = \frac{(1-q)F + qF_1(\underline{\theta}_1)F|_{\{\theta_1 \leq \underline{\theta}_1\}}}{1-q+qF_1(\underline{\theta}_1)}$$

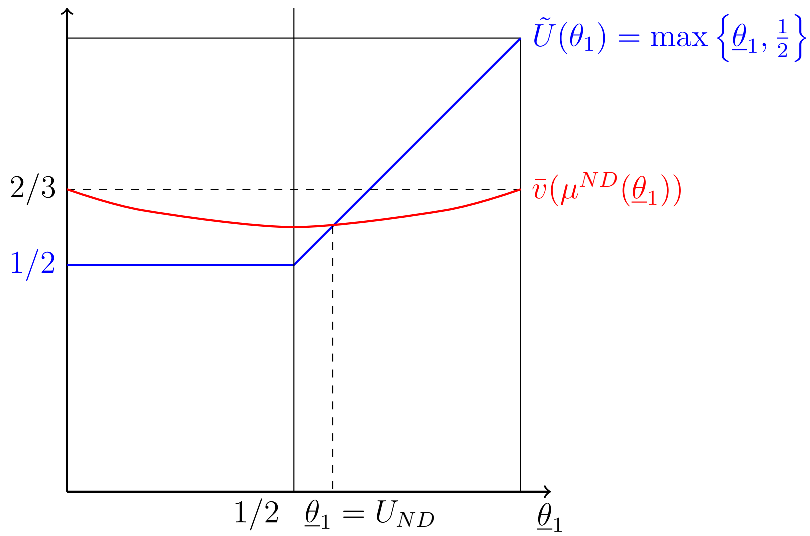

Figure 2: Determination of the sender-optimal disclosure cutoff for $U(\theta) = \max\{\theta_1, \theta_2\}$

Some algebra shows that when the disclosure threshold is $\underline{\theta}_1$, the equilibrium information structure used is the one specified in Proposition 7 and the sender's maximized cheap talk payoff given the disclosure threshold $\underline{\theta}_1$ is V, we have the following, using ND to denote the event of non-disclosure:

$$
\begin{aligned}
Pr(ND) &= 1 - q + q\underline{\theta}_1 \\
\mathbb{E}(\theta_1|ND) &= \frac{1 - q + q\underline{\theta}_1^2}{2Pr(ND)} \\
Pr(2) &= \frac{1}{Pr(ND)}\left(a + m\left(b(\underline{\theta}_1 - a) - \frac{1}{2}\left(\underline{\theta}_1^2 - a^2\right) + m(1-q)\left(b(b - \underline{\theta}_1) - \frac{1}{2}\left(b^2 - \underline{\theta}_1^2\right)\right)\right)\right) \\
\mathbb{E}(\theta_1|2) &= \frac{1}{Pr(2)Pr(ND)}\left(\frac{a^2}{2} + m\left(\frac{b}{2}\left(\underline{\theta}_1^2 - a^2\right) - \frac{1}{3}\left(\underline{\theta}_1^3 - a^3\right)\right)\right. \\
&\qquad \left. + m(1-q)\left(\frac{b}{2}\left(b^2 - \underline{\theta}_1^2\right) - \frac{1}{3}\left(b^3 - \underline{\theta}_1^3\right)\right)\right) \\
\mathbb{E}(\theta_2|2) &= \frac{1}{2}\left(1 + V(1-m) + m\mathbb{E}(\theta_1|2)\right) \\
\mathbb{E}(\theta_1|1) &= \frac{\mathbb{E}(\theta_1|ND) - Pr(2)\mathbb{E}(\theta_1|2)}{1 - Pr(2)}, \\
&\text{where } a = V\left(1 - \frac{1}{m}\right), b = V\left(1 - \frac{1}{m}\right) + \frac{1}{m}.
\end{aligned} \tag{4}
$$

Using equations (4) we can form a system of three simultaneous equations with three unknowns, m, V and $\underline{\theta}_1$, which we can then solve to obtain the sender optimal disclosure cutoff $\underline{\theta}_1$ and non-disclosure payoff V. This is formalized in the proposition below.

Proposition 8. *When the sender's utility is given by $U(\theta_1, \theta_2) = \max\{\theta_1, \theta_2\}$ and the prior is uniform on $[0,1]$ with θ_1 and θ_2 independent, the sender-optimal disclosure cutoff $\underline{\theta}_1$, which is also equal to the sender's maximized non-disclosure payoff V, is given by the solution to the following system of equations:*

$$
\begin{aligned}
\mathbb{E}(\theta_2|2) &= V \\
\mathbb{E}(\theta_1|1) &= V \\
V &= \underline{\theta}_1 \\
&\textit{subject to } (4).
\end{aligned} \tag{5}
$$

Simple algebra shows that the ex-ante sender payoff as a function of the disclosure cutoff and evidence probability is given by:

$$
\mathbb{V} = \underline{\theta}_1(q) + \frac{q}{2}(1 - \underline{\theta}_1(q))^2 \tag{6}
$$

where $\underline{\theta}_1(q)$ is the sender-optimal cutoff as a function of q, obtained from equation (5).

6 Discussion and Extensions

6.1 Significance of truth-leaning

In this section we present an example to show that the minimum-disclosure principle does not hold without restricting the class of equilibria to truth-leaning equilibria.

Example 2. *Suppose $U(\theta_1, \theta_2) = \theta_1^2 + \theta_2^2$, θ_1 and θ_2 are distributed independently and uniformly on $[0, 1]$. Hence, $\widetilde{U}(\theta_1) = \theta_1^2 + \frac{1}{4}$. Suppose the evidence probability $q = 0.5$.*

First, consider an equilibrium with disclosure strategy of the following form, if it exists: There exist two thresholds α_1 and β_1, $0 < \alpha_1 < \beta_1 < 1$ such that the sender does not disclose θ_1 for $\theta_1 \in [0, \alpha_1)$, discloses only if θ_2 is below a certain threshold for $\theta_1 \in [\alpha_1, \beta_1]$, and discloses for all θ_2 for $\theta_1 \in (\beta_1, 1]$. Suppose the non-disclosure payoff in this equilibrium is u_1. We further assume for $\theta_1 = \alpha_1$, the sender is indifferent between fully disclosing and not disclosing, so $\widetilde{U}(\alpha_1) = u_1$, i.e.

$$\alpha_1 = \sqrt{1 - \frac{1}{4}} \tag{7}$$

As argued earlier, the partially disclosing θ_1-types, i.e $\theta_1 \in [\alpha_1, \beta_1]$ must be indifferent between disclosing and not disclosing. Therefore the disclosure threshold for θ_2 must vary with θ_1 for $\theta_1 \in [\alpha_1, \beta_1]$ such that this is satisfied, i.e. this threshold, say $\overline{\theta}_2(\theta_1)$ must satisfy, $U(\theta_1, \mathbb{E}(\theta_2|\theta_1, \theta_2 \leq \overline{\theta}_2(\theta_1))) = u_1$ for all $\theta_1 \in [\alpha_1, \beta_1]$. This boils down to $\overline{\theta}_2(\theta_1) = 2\sqrt{u_1 - \theta_1^2}$.

Therefore the marginal distributions of θ_1 and θ_2 conditional on having but not disclosing evidence (scaled by the total probability of this event, $F_0 = \alpha_1 + \int_{\alpha_1}^{\beta_1} (2\sqrt{u - \theta_1^2} - 1) d\theta_1$) is given by:

$$f_1(\theta_1) = \begin{cases} 1, \theta_1 \leq \alpha_1 \\ 1 - 2\sqrt{u_1 - \theta_1^2}, \theta_1 \in [\alpha_1, \beta_1] \\ 0, Otherwise. \end{cases} \qquad f_2(\theta_2) = \begin{cases} \alpha_1, \theta_2 \leq 2\sqrt{u - \beta_1^2} \\ \alpha_1 + \beta_1 - \sqrt{u - \left(\frac{\theta_2}{2}\right)^2}, Otherwise. \end{cases}$$

Total probability of non-disclosure is given by:

$$Pr(ND) = 1 - q + q\left(\alpha_1 + \int_{\alpha_1}^{\beta_1} (2\sqrt{u_1 - \theta_1^2} - 1) d\theta_1\right) \tag{8}$$

Using the above, the posterior means of θ_1 and θ_2 conditional on non-disclosure are given by:

$$\mathbb{E}(\theta_1|ND) = \frac{1}{Pr(ND)}\left(\frac{1}{2}(1-q) + q\left(\int_0^{\beta_1}\theta_1 d\theta_1 - 2\int_{\alpha_1}^{\beta_1}\theta_1\sqrt{u_1-\theta_1^2}\right)\right) \tag{9}$$

$$\mathbb{E}(\theta_2|ND) = \frac{1}{Pr(ND)}\left(\frac{1}{2}(1-q) + q\left(\alpha_1\int_0^1\theta_2 d\theta_2 + \int_{2\sqrt{u_1-\beta_1^2}}^1\theta_2\left(\beta_1 - \sqrt{u-\frac{1}{4}\theta_2^2}\right)\right)\right) \tag{10}$$

If such an equilibrium exists such that the sender babbles upon non-disclosure, we must have,

$$\mathbb{E}(\theta_1|ND)^2 + \mathbb{E}(\theta_2|ND)^2 = u_1 \tag{11}$$

Solving the system of equations (7)-(11), we find ($u = 0.43, \beta_1 = 0.5776$) is a solution. That is, such an equilibrium exists, in which the sender babbles upon non-disclosure, and his non-disclosure payoff is 0.43*. Let us call this equilibrium E_1. His ex-ante payoff in equilibrium E_1 is therefore, $V_1 = \int_0^{\beta_1} 0.43 d\theta_1 + \int_{\beta_1}^1 \left(\theta_1^2 + \frac{1}{4}\right) = 0.6231$.*

Now consider another equilibrium of the same game with a disclosure strategy of the following form, if any: There exist two thresholds α_2 and β_1, $0 < \alpha_2 < \beta_2 < 1$ such that the sender does not disclose θ_1 for $\theta_1 \in [0, \alpha_2)$, discloses only if θ_2 is ***above*** *a certain threshold for $\theta_1 \in [\alpha_2, \beta_2]$, and discloses for all θ_2 for $\theta_1 \in (\beta_2, 1]$. Suppose the non-disclosure payoff in this equilibrium is u_2. We further assume for $\theta_1 = \beta_2$, the sender is indifferent between fully disclosing and not disclosing, so $\widetilde{U}(\beta_2) = u_2$, i.e.*

$$\beta_2 = \sqrt{u_2 - \frac{1}{4}} \tag{12}$$

Similarly as above, the disclosure threshold for θ_2 must vary with θ_1 for $\theta_1 \in [\alpha_2, \beta_2]$ in a way such that those θ_1-types are indifferent between disclosing and non-disclosing, i.e. this threshold, say $\underline{\theta}_2(\theta_1)$ must satisfy, $U(\theta_1, \mathbb{E}(\theta_2|\theta_1, \theta_2 \leq \underline{\theta}_2(\theta_1))) = u_2$ for all $\theta_1 \in [\alpha_2, \beta_2]$. This boils down to:

$$\underline{\theta}_2(\theta_1) = 2\sqrt{u_2 - \theta_1^2} - 1 \tag{13}$$

Consider the two-message signal under which the sender reveals if $\theta_2 \geq \theta_1$ or not, upon non-disclosure. Of course, this need not be a cheap talk equilibrium. But as shown by Lipnowski and Ravid (2020), since this is a feasible signal, the minimum of the payoffs following the two messages under this signal constitutes a cheap talk equilibrium payoff. Suppose such an equilibrium exists of the continuation game

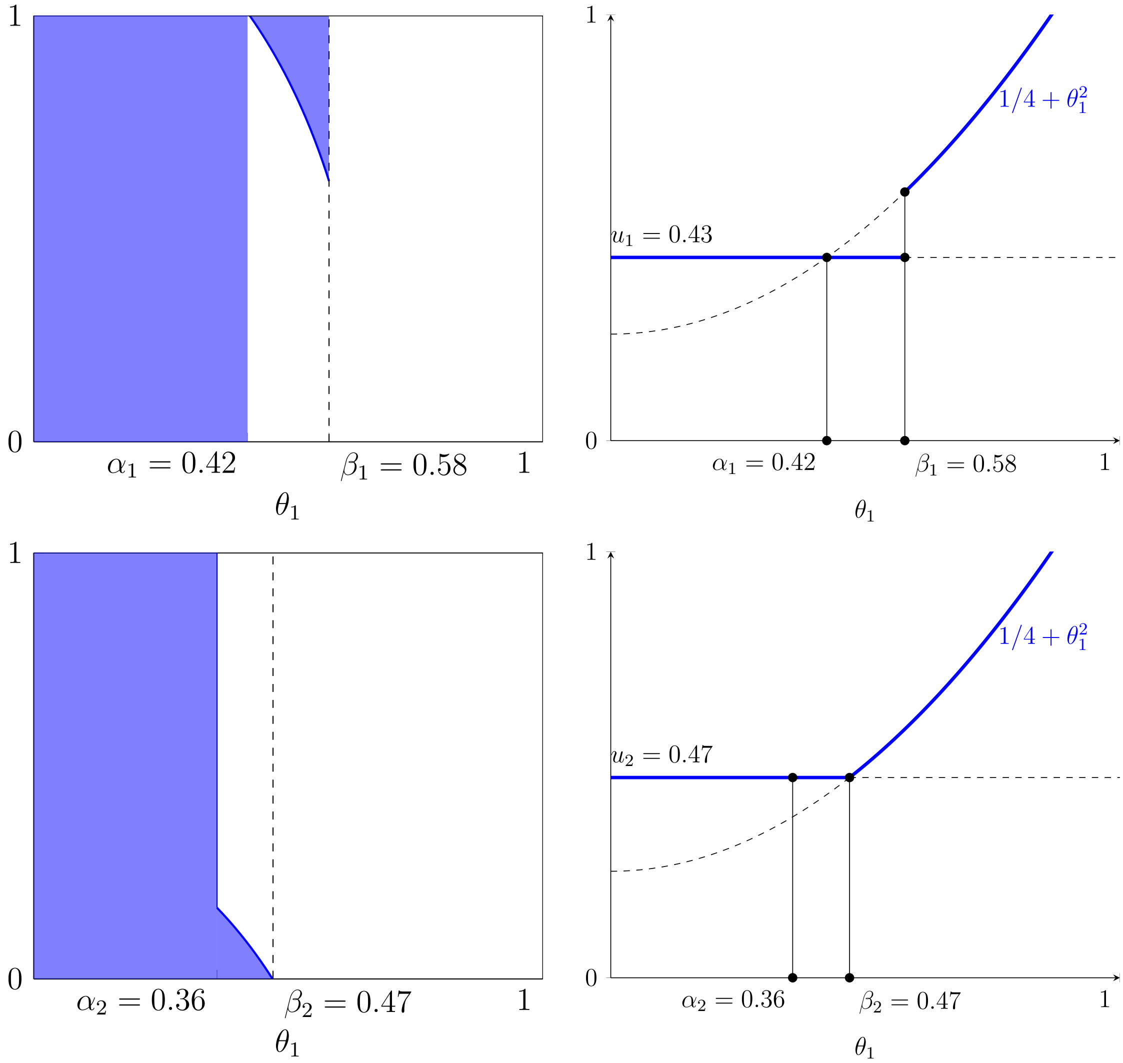

Figure 3: The minimum-disclosure principle does not hold without truth-leaning

upon non-disclosure, where this payoff is attained by the sender. In the next steps we compute this payoff and find a solution for (u_2, β_2) *which shows that such an equilibrium indeed exists.*[10]

Given the model parameters, it turns out the sender's payoff upon revealing $\theta_1 \geq \theta_2$ *is the minimum of the two payoffs. Therefore that is the one we focus on below.*

The point (γ, γ) *at which* $\underline{\theta}_2(\theta_1)$ *intersects the* $45°$ *line is given by,*

$$\gamma = 2\sqrt{u_2 - \gamma^2} - 1 \tag{14}$$

The above equation has at most one solution in $[0, 1]$ *for feasible* u_2*, i.e.* $u_2 \in \left[\min_{\theta \in [0,1]^2} U(\theta), \max_{\theta \in [0,1]^2} U(\theta)\right] \equiv [0, 2]$*. Let us denote this solution by* $\gamma(u_1)$*.*

The marginal distributions of θ_1 *and* θ_2 *conditional on non-disclosure and* $\theta_1 \geq \theta_2$ *are given by:*

[10] We could have explicitly constructed a cheap talk equilibrium of the continuation game upon non-disclosure, but have used this simplification to avoid additional algebraic complexity.

$$f_1(\theta_1) = \begin{cases} \theta_1, \theta_1 \leq \max\{\alpha_2, \gamma(u_2)\} \\ q\theta_2(\theta_1) + (1-q)\theta_1, \theta_1 \in [\max\{\alpha_2, \gamma(u_2)\}, \beta_2] \\ (1-q)\theta_1, Otherwise. \end{cases} \tag{15}$$

$$f_2(\theta_2) = \begin{cases} q\sqrt{u_2 - \left(\frac{\theta_2+1}{2}\right)^2} - \theta_2 + 1 - q, \theta_2 \leq \min\{\underline{\theta}_2(\alpha_2), \gamma(u_2)\} \\ q\alpha - \theta_2 + 1 - q, \theta_2 \in [\underline{\theta}_2(\alpha_2), \alpha] \\ (1-\theta_2)(1-q), \theta_2 \geq \max\{\alpha_2, \gamma(u_2)\}. \end{cases} \tag{16}$$

Note that in equation (16), the interval $[\underline{\theta}_2(\alpha_2), \alpha_2]$ may be empty for some values of α.

Using the above we conclude that the following must hold if such an equilibrium exists,

$$\mathbb{E}(\theta_1|ND, \theta_1 \geq \theta_2) = \frac{\int_0^1 \theta_1 f_1(\theta_1) d\theta_1}{\int_0^1 f_1(\theta_1) d\theta_1} \tag{17}$$

$$\mathbb{E}(\theta_2|ND, \theta_1 \geq \theta_2) = \frac{\int_0^1 \theta_2 f_2(\theta_2) d\theta_2}{\int_0^1 f_2(\theta_2) d\theta_2} \tag{18}$$

$$\mathbb{E}(\theta_1|ND, \theta_1 \geq \theta_2)^2 + \mathbb{E}(\theta_2|ND, \theta_1 \geq \theta_2)^2 = u_2. \tag{19}$$

Numerically we find that system of equations (12)-(19) has a solution ($u_2 = 0.47, \alpha_2 = 0.3578$). That is such an equilibrium exists and has a non-disclosure payoff of 0.47. Let us call this equilibrium E_2. The sender's ex-ante payoff in equilibrium E_2 is therefore, $V_2 = \int_0^{\beta_2} 0.47 d\theta_1 + \int_{\beta_2}^1 \left(\theta_1^2 + \frac{1}{4}\right) = 0.6521 > 0.6231 = V_1$.

We calculate $\alpha_1 = \sqrt{u_1 - \frac{1}{4}} = 0.4243 > \alpha_2 = 0.3578$ and $\beta_2 = \sqrt{u_2 - \frac{1}{4}} = 0.4690 < \beta_1 = 0.5776$. Therefore, using $\mathbb{D}_i$ and $\mathbb{ND}_i$ to denote the disclosure and non-disclosure sets of equilibrium i respectively, $i \in \{1, 2\}$, we have, $\mathbb{D}_1 = [\beta_1, 1] \subsetneq [\beta_2, 1] = \mathbb{D}_2$ and $\mathbb{ND}_1 = [0, \alpha_1] \supsetneq [0, \alpha_2] = \mathbb{ND}_2$, but the sender is better of in E_2 than in $E1$, i.e. the minimum disclosure principle is violated in this case.

6.2 Multiple unverifiable dimensions and/or non-monotonic sender utility

Consider the generalized model of the one presented in the main text in which, in addition to the verifiable dimension θ_1, there are $n-1$ unverifiable dimensions, with the state being denoted by $\theta \equiv (\theta_1, \cdots, \theta_n) \in [0,1]^n$, which is publicly known to follow a full support distribution $F \in \Delta[0,1]^n$ with a continuous density. Sender's utility is still a continuous function $U(\cdot)$ of the posterior mean of θ. However, in this section we relax the assumption that $U(\cdot)$ must be increasing in all its components and allow for non-monotonic variation of sender's utility with different components of θ.

A key aspect of the two-dimensional monotonic model which simplified our analysis was the fact that upon disclosure of θ_1, the sender's interim payoff was fixed to be $U(\theta_1, \mathbb{E}(\theta_2|\theta_1))$, because as is well known, and as we have shown in Lemma 2, because of the monotonicity of $U(\theta_1, \cdot)$, persuasive communication is not possible on the second dimension, once the first has been verifiably disclosed.

If we have non-monotonic preferences or if the number of unverifiable dimensions is more than one, this is no longer the case. A continuum of cheap talk equilibria may arise in the cheap talk game upon disclosing θ_1. Therefore $\widetilde{U}(\cdot)$ - sender's disclosure payoff as a function of the verifiable dimension - is no longer a function but a correspondence. Hence the Minimum Disclosure Principle no longer holds in general. The intuition for it is illustrated in the Figure below where $\overline{v}(F|_{\theta_1}; \theta_1)$ and $\underline{v}(F|_{\theta_1}; \theta_1)$ are respectively the highest and lowest cheap talk equilibrium payoffs of the sender in the continuation game upon revealing θ. Note that, sender optimality is no longer characterized by the highest non-disclosure payoff. As shown in the figure, depending on the prior distribution, between an equilibrium with a smaller non-disclosure set and a lower non-disclosure payoff and one with a larger non-disclosure set and a correspondingly larger non-disclosure payoff, either can be preferred by the sender. If an equilibrium features a larger non-disclosure set *and* a larger non-disclosure payoff than another, the former is, of course, sender-preferred due to the same reason as in Proposition 3.

Clearly, this issue disappears if we fix an equilibrium selection rule in the continuation game upon disclosure of θ_1. It is also easy to see that any sender-optimal equilibrium must feature the sender-best equilibrium in the continuation game upon disclosure. Putting these two insights together we get back the Minimum Disclosure Principle even when there are multiple unverifiable dimensions or the sender's utility is non-monotonic. This is formalized below.

Assumption 2. *We assume the sender can choose the equilibrium in the continuation game upon disclosure.*

Proposition 9. *Suppose Assumption 2 holds. Then, the non-disclosure sets of the equilibria are totally ordered under the inclusion order, with sender's ex-ante utility monotonically increasing in this order.*

The proof is analogous to that of Proposition 7.

6.3 The single-dimensional case

In this section we consider the case where instead of two-dimensions, the unknown state θ has just one dimension - i.e. $\theta \in \Theta \equiv [0, 1]$ - but $U(\cdot)$ can be non-monotonic.

Using the analogous logic and the same notation as in Section 4 we have the equilibrium conditions as:

$$U_{ND} \in \mathbb{U}_{CT}(\mu^{ND}(U_{ND})),$$

$$\text{where } \Theta_{ND} = \{\theta_1 : U(\theta) \leq U_{ND}\} \text{ and}$$

$$\mu^{ND}(U_{ND}) = \frac{qF(\Theta_{ND})}{1 - q + qF(\Theta_{ND})} \times F|_{\theta \in \Theta_{ND}} + \frac{1 - q}{1 - q + qF(\Theta_{ND})} \times F$$

Analogously as before, for any equilibrium with disclosure strategy $D : \Theta \to [0, 1]$, we call the set $\mathbb{D} = \{\theta \in \Theta : D(\theta) = 1\}$ its **disclosure set**. The definition on unravelling, partial disclosure and no disclosure equilibria are also analogous to those used before.

By analogous reasoning as before, Proposition 3 and therefore Corollary 2 go through in this case as well. That is, the disclosure-minimizing equilibrium (in case there are multiple equilibria) is sender-best.

Observation 3. *Proposition 3 and Corollary 2 hold in the one-dimensional case as well.*

It is easy to see that if $U(\cdot)$ is quasiconvex, unravelling must occur if $q = 1$.

7 Conclusion

This paper studies a communication game where an informed sender with monotonic and state-independent preferences attempts to persuade an uninformed receiver about a two-dimensional state, only one of the dimensions of which he can verifiably disclose with some probability, while being able to use cheap talk to communicate about both. We show that the equilibria of this game are ordered in the inclusion order of the set of types which disclose, and that the sender's payoff is decreasing in this order - i.e. the smaller the disclosure set, the better off the sender - with the least-disclosure equilibrium being sender best. Under some equilibrium selection rules, this insight generalizes to settings with multiple unverifiable dimensions and/or non-monotonic sender utility. This principle allows us to fully characterize the sender-best equilibria for the adapted versions of a rich class of examples previously considered in the literature.

To the best of my knowledge, this paper is the first to study the interaction between cheap talk and verifiable disclosure in the presence of imperfections in the evidence technology. Several avenues of future research suggest themselves. While we show that if evidence is perfect, for an important class of dependence types between the two dimensions - namely, non-negative dependence including the independent case - interesting interaction between the two types of communication is lost due to unravelling, it would be interesting to study this interaction even with perfect evidence, when positive dependence is not satisfied. We also fix the evidence type to be perfectly revealing. What if the sender has access to more complex evidence structures, for example the option to reveal an interval containing this true type (of the verifiable dimension) or even more generally, any feasible experiment that reveals his true type? Finally, we have not modelled the receiver explicitly in this set up. It would also be interesting to study how receiver welfare varies across equilibria and with evidence probability.

Appendix

Throughout the appendices we denote $\min \widetilde{U}(\theta_1)$ by $\underline{U}_D$ and $\max \widetilde{U}(\theta_1)$ by $\overline{U}_D$. (which are well-defined, by continuity of $\widetilde{U}(\cdot)$.)

A.1 Preliminaries

Lemma 1. *If F has a continuous density, $\widetilde{U}(\theta_1) = U(\theta_1, \mathbb{E}_F(\theta_2|\theta_1))$ is continuous in θ_1.*

Proof. Let the joint density of θ_1 and θ_2 be denoted by $f_{1,2}(\cdot)$. We have,

$$\mathrm{E}(\theta_2 \mid \theta_1) = \int\limits_{\theta_2} \theta_2 \frac{f_{1,2}(\theta_1, \theta_2)}{f_1(\theta_1)} d\theta_2,$$

taking $f_{2|1}(\theta_2 \mid \theta_1) := \frac{f_{1,2}(\theta_1,\theta_2)}{f_1(\theta_1)} = 0$ when $f_1(\theta_1) = 0$, where $f_1(\cdot)$ is the marginal density of θ_1, $f_1(\theta_1) = \int\limits_{\Theta_2} f_{1,2}(\theta_1, \theta_2) d\theta_2$. Now, taking limits inside the integral in the above equation, continuity of $f_{1,2}(\theta_1, \theta_2)$ leads to continuity of $\mathrm{E}(\theta_2 \mid \theta_1)$ in θ_1. By the assumption of continuity of $U(\cdot, \cdot)$ in both the components, the claim follows. □

Lemma 2. *For any θ_1, all equilibria in which θ_1 is verifiably disclosed are payoff-equivalent for the sender.*

Proof. Consider an equilibrium in which θ_1 has been disclosed. Let $\sigma_2 : \Theta_2 \to \Delta M$ be the sender's cheap talk communication strategy in this equilibrium, and $\mu^D : M \to \Delta\Theta_2$ R's posterior belief about θ_2. σ_2 can, in general, depend on θ_1.

Note that in order for σ_2 to be a cheap talk equilibrium strategy in the continuation game after revealing θ_1, S must be indifferent among all the messages sent under σ_2 with positive probability. If $U(\theta_1, \cdot)$ is strictly increasing, this can happen only if $\mathbb{E}\mu^D(m_2) = \mathbb{E}\mu^D(m_2')$ for all $m_2, m_2' \in supp\ \sigma_2$. By feasibility, $\mathbb{E}_{m_2 \sim \sigma_2}\mathbb{E}\mu^D(m_2) =: \mathbb{E}(\theta_2|\theta_1, D)$. Therefore $\mathbb{E}\mu^D(m_2) = \mathbb{E}(\theta_2|\theta_1, D)$ for all $m_2 \in supp\ \sigma_2$. Therefore any cheap talk equilibrium strategy σ_2 in the continuation game after revealing θ_1 is payoff equivalent to the babbling equilibrium, to the sender.

By assumption $U(\cdot)$ is weakly increasing in both components. Therefore, if there exist $m_2, m_2' \in supp\ \sigma_2, m_2 \neq m_2'$ such that $\mathbb{E}\mu^D(m_2) \neq \mathbb{E}\mu^D(m_2')$, then σ_2 can be a cheap talk equilibrium policy only if $U(\theta_1, \mathbb{E}\mu^D(m_2)) = U(\theta_1, \mathbb{E}\mu^D(m_2'))$, i.e. there exists $[\underline{\theta}_2, \overline{\theta}_2] \subseteq \Theta_2$ such that $U(\theta_1, \cdot)$ is constant over $[\underline{\theta}_2, \overline{\theta}_2]$. Therefore every posterior mean induced with positive probability under σ_2 must lie on the (vertical) line segment $\{(\theta_1, \theta_2) : \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]\}$. They must average back to the mean of θ_2 conditional on verifiable disclosure of θ_1, given the equilibrium disclosure strategy (i.e. unconditional on cheap talk messages sent subsequently), $\mathbb{E}(\theta_2|\theta_1, D)$, which must lie on the same line segment. Therefore informative communication using cheap talk and babbling, conditional on disclosure of θ_1, are payoff-equivalent for the sender. □

A.2 Proofs for section 2.4

So far assuming the simple case where $\widetilde{U}(\cdot)$ does not have flat regions.

Lemma 3. *There exists a simple equilibrium, i.e. an equilibrium where the sender's disclosure strategy does not depend on θ_2.*

Proof. It follows from Observation 1´ that in any partial disclosure equilibrium, sender's equilibrium non-disclosure payoff set is given by $\{U_{ND} : U_{ND} \in \mathbb{U}_{CT}(\mu^{ND}(U_{ND}))\}$ where $\mu^{ND}(U_{ND})$ is the receiver's belief conditional on non-disclosure, when the non-disclosure payoff is U_{ND}, and $\mathbb{U}_{CT}(\cdot)$ is the set of cheap talk equilibrium payoffs as a function of the prior. That is, $\mu^{ND}(U_{ND}) = \frac{qF_1(\Theta_{1,ND})}{1-q+qF_1(\Theta_{1,ND})} \times F|_{\theta_1 \in \Theta_{1,ND}} + \frac{1-q}{1-q+qF_1(\Theta_{1,ND})} \times F$ where $\Theta_{1,ND} = \{\theta_1 : \widetilde{U}(\theta_1) \leq U_{ND}\}$. By continuity of $\widetilde{U}(\cdot)$, $\mu^{ND}(U_{ND})$ is a continuous function of U_{ND}. Expectation is a continuous operator, and $U(\cdot, \cdot)$ is continuous. So $U(\mathbb{E}(\mu^{ND}(U_{ND})))$ is a continuous function of U_{ND}. If $U(\mathbb{E}(\mu^{ND}(U_{ND}))) <$ (resp. $>$) U_{ND} for all $U_{ND} \in [\underline{U}_D, \overline{U}_D]$ there exists an unravelling (resp. no disclosure) equilibrium where the sender babbles conditional on non-disclosure. If, on the other hand, the continuous function $U(\mathbb{E}(\mu^{ND}(U_{ND}))) - U_{ND}$ crosses 0 in $[\underline{U}_D, \overline{U}_D]$, there exists a partial disclosure equilibrium where the sender babbles conditional on non-disclosure. Any of these equilibria is one where the sender's disclosure strategy does not depend on θ_2. □

Proof of Proposition 1. By Lemma 3 Γ has a simple equilibrium. We consider three cases, where this simple equilibrium is unravelling, partial disclosure and no disclosure respectively, and perturb it to construct a corresponding equilibrium of Γ^{ε} in each case.

Define $\widetilde{F}_1(\cdot)$ as $\widetilde{F}_1(u) = F_1\left(\{\theta_1 : \widetilde{U}(\theta_1) \leq u\}\right)$ for all $u \in \mathbb{R}$.

Suppose Γ has an unravelling equilibrium. Then, it is an equilibrium of Γ^{ϵ} as well.

Next suppose Γ has a partial disclosure simple babbling equilibrium. Therefore the following equation has a solution $U_{ND} \in (\underline{U}_D, \overline{U}_D)$.

$$U\left(\frac{(1-q)\theta_0 + q\widetilde{F}_1(U_{ND})\mathbb{E}(F|_{\widetilde{U}(\theta_1)\leq U_{ND}})}{1-q+q\widetilde{F}_1(U_{ND})}\right) = U_{ND}$$

Γ^{ε} has a partial disclosure babbling equilibrium with non-disclosure payoff U_{ND} if and only if the following equation has a solution $U_{ND} \in (\underline{U}_D, \overline{U}_D)$.

$$U\left(\frac{(1-q)\theta_0 + q(1-\epsilon_1)\widetilde{F}_1(U_{ND}-\epsilon_2)\mathbb{E}\left(F|_{\widetilde{U}(\theta_1)\leq U_{ND}-\epsilon_2}\right)}{1-q+q(1-\epsilon_1)\widetilde{F}_1(U_{ND}-\epsilon_2)}\right) = U_{ND} - \epsilon_2$$

The LHS is continuous in ϵ as a function of U_{ND}. Therefore if the previous equation has a solution $U_{ND} \in (\underline{U}_D, \overline{U}_D)$ so does this one, for small enough ϵ_1, ϵ_2.

If there exists a babbling no-disclosure equilibrium in the original game Γ, that means $U(\theta_0) > \overline{U}_D$. Therefore $U(\theta_0) > \overline{U}_D - \epsilon_2$, i.e. $D^{\epsilon}(\theta_1) = \epsilon_1$ for all θ_1 constitutes an equilibrium of Γ^{ϵ}.

To show that every simple equilibrium is truth-leaning, i.e. limit of an equilibrium of Γ^{ϵ}:

Fix a signal structure - a family of distributions $H(\cdot|\theta)_{\theta\in\Theta}$ with a finite number of signal realizations (this is a simplifying assumption for now) i.e. *supp* $H(\cdot|\theta) = \{m_1, \cdots, m_k\}$ for some k, for all θ. For any distribution $G \in \Delta\Theta$, define:

$$\widehat{U}(G) = \min\{U(\mathbb{E}(\theta|m_1; H), \cdots, U(\mathbb{E}(\theta|m_k; H))\}$$

$\widehat{U}\left(\frac{(1-q)F+q(1-\epsilon_1)\widetilde{F}_1(U_{ND}-\epsilon_2)F|_{\widetilde{U}(\theta_1)\leq U_{ND}-\epsilon_2}}{1-q+q(1-\epsilon_1)\widetilde{F}_1(U_{ND}-\epsilon_2)}\right)$ is continuous in (U_{ND}, ϵ).

If U_{ND} is a non-disclosure payoff in a partial disclosure simple equilibrium of Γ, the following must hold for *some* H.

$$\widehat{U}\left(\frac{(1-q)\theta_0 + q\widetilde{F}_1(U_{ND})\mathbb{E}(F|_{\widetilde{U}(\theta_1)\leq U_{ND}})}{1-q+q\widetilde{F}_1(U_{ND})}\right) = U_{ND}$$

.

Therefore by continuity of $\widehat{U}\left(\frac{(1-q)F+q(1-\epsilon_1)\widetilde{F}_1(U_{ND}-\epsilon_2)F|_{\widetilde{U}(\theta_1)\leq U_{ND}-\epsilon_2}}{1-q+q(1-\epsilon_1)\widetilde{F}_1(U_{ND}-\epsilon_2)}\right)$ in ϵ as a function of U_{ND} the equation below must have a solution for $U_{ND} \in (\underline{U}_D, \overline{U}_D)$ as well, for small enough ϵ_1, ϵ_2:

$$\widehat{U}\left(\frac{(1-q)F+q(1-\epsilon_1)\widetilde{F}_1(U_{ND}-\epsilon_2)F|_{\widetilde{U}(\theta_1)\leq U_{ND}-\epsilon_2}}{1-q+q(1-\epsilon_1)\widetilde{F}_1(U_{ND}-\epsilon_2)}\right)=U_{ND}-\epsilon_2$$

Lemma 4. *In any truth-leaning equilibrium, the payoffs of the hiding and no-evidence types must be equal, which must also equal the disclosure payoff of the threshold types.*

Proof. In any equilibrium, *each* of the no-evidence and the hiding types must be indifferent across all messages sent with positive probability by *either* type - otherwise there exist profitable deviations. Because the two types have the same preferences, all the messages sent in equilibrium must then lie on the same indifference curve of S, which corresponds to a utility of U_{ND}, say. Recall that the threshold type $\underline{\theta}_1$ must be indifferent across disclosing and not disclosing, so the disclosure payoff of type $\underline{\theta}_1$ must also equal U_{ND}. □

A.3 Proofs for section 3

Lemma 5. *If Assumption 1 holds, then in any equilibrium there exists* $0 \leq \underline{\theta}_1' \leq \underline{\theta}_1 \leq 1$ *such that* $D(\theta_1) = 1$ *for* $\theta_1 > \underline{\theta}_1$ *and* $D(\theta_1) = 0$ *for* $\theta_1 < \underline{\theta}_1'$.

Proof. Consider any equilibrium with a non-disclosure payoff U_{ND}. By Observation 1´, $D(\theta_1) = 1$ for $\widetilde{U}(\theta_1) > U_{ND}$ and $D(\theta_1) = 0$ for $\widetilde{U}(\theta_1) < U_{ND}$. By Assumption 1 $\widetilde{U}(\cdot)$ is non-decreasing. Suppose $\widetilde{U}(0) > U_{ND}$. Therefore $\widetilde{U}(\theta_1) > U_{ND}$ for all θ_1 and therefore by Observation 1´ $\underline{\theta}_1' = \underline{\theta}_1 = 0$. Alternatively if $\widetilde{U}(1) < U_{ND}$, $\widetilde{U}(\theta_1) < U_{ND}$ for all θ_1 and therefore by Observation 1´ $\underline{\theta}_1' = \underline{\theta}_1 = 1$. Otherwise, $\widetilde{U}(0) \leq U_{ND}$ and $\widetilde{U}(1) \geq U_{ND}$. So, by continuity of $\widetilde{U}(\cdot)$[11], the intermediate value theorem, and non-decreasingness of $\widetilde{U}(\cdot)$, there exists $0 \leq \underline{\theta}_1' \leq \underline{\theta}_1 \leq 1$ such that $\widetilde{U}(\theta_1) = U_{ND}$ for $\theta_1 \in [\underline{\theta}_1', \underline{\theta}_1]$, $\widetilde{U}(\theta_1) < U_{ND}$ for $\theta_1 < \underline{\theta}_1'$ and $\widetilde{U}(\theta_1) > U_{ND}$ for $\theta_1 > \underline{\theta}_1$. Hence the claim follows. □

Lemma 6. *Suppose assumption 1 holds. Then, S's payoff in any equilibrium of the pure cheap talk game is strictly greater than his payoff upon disclosure, when he has the lowest* θ_1*-type, i.e.* $\theta_1 = 0$.

Proof. First, consider the babbling equilibrium. Sender's payoff, $U(\theta_{0,1}, \theta_{0,2}) > U(0, \mathbb{E}(\theta_2|\theta_1 = 0))$ ($\because$ $\theta_{0,1} > 0$ by the full support assumption on μ, and $\mathbb{E}(\theta_2|\theta_1 = 0) \leq \mathbb{E}_{\theta_1}(\mathbb{E}(\theta_2|\theta_1)) = \theta_{0,2}$, by Assumption 1), which is S's payoff when he discloses evidence for $\theta_1 = 0$.

Next, consider an influential equilibrium. Let S's payoff in this equilibrium be $\underline{U}$ and let us denote the corresponding indifference curve by $\underline{I} \equiv \{(\theta_1, \theta_2) : U(\theta_1, \theta_2) = \underline{U}\}$. If $\underline{I}$ does not intersect the $\theta_1 = 0$ line then it is obviously a higher indifference

[11] See Lemma 1 in Appendix for proof of continuity of $\widetilde{U}(\cdot)$.

curve than the one through $(0, \mathbb{E}(\theta_2|\theta_1 = 0))$, and we are done. Therefore let us assume $\underline{I}$ intersects the $\theta_1 = 0$ line at $\underline{\theta}_2$. By strict increasingness of U in both θ_1 and θ_2, for all $(\theta_1, \theta_2) \in \underline{I} \setminus (0, \underline{\theta}_2)$, $\theta_2 < \underline{\theta}_2$ ($\because\ \ \theta_1 > 0$). Because this is an influential equilibrium, there exist at least two distinct points on $\underline{I}$ such that the average of their θ_2-components is $\theta_{0,2}$ (because posterior means must average back to the prior mean). Therefore at least one of them must have its θ_2-component strictly greater than $\theta_{0,2}$. Since $\theta_2 \leq \underline{\theta}_2$ for all $(\theta_1, \theta_2) \in \underline{I}$, $\underline{\theta}_2 > \theta_{0,2}$. Therefore $\underline{U} = U(0, \underline{\theta}_2) > U(0, \theta_{0,2}) \geq U(0, \mathbb{E}(\theta_2|\theta_1 = 0))$. □

Proof of Proposition 2. First, suppose the test is perfect, i.e. $q = 1$. Suppose, by way of contradiction, there exists $0 < \underline{\theta}_1' \leq \underline{\theta}_1 \leq 1$ such that $D(\theta_1) = 1$ for all $\theta_1 > \underline{\theta}_1$ and $D(\theta_1) = 0$ for all $\theta_1 < \underline{\theta}_1'$, i.e. at $\theta_1 = \underline{\theta}_1$ S is indifferent between disclosing and not disclosing. Below, we show that for any such $\underline{\theta}_1$, S strictly prefers disclosing when he has evidence of $\underline{\theta}_1$, to cheap talk, which is a contradiction. In doing so, we consider S's highest payoff cheap talk equilibrium, because if S strictly prefers disclosing at $\theta_1 = \underline{\theta}_1$ at his highest payoff cheap talk equilibrium, he does so at any other cheap talk equilibrium.

Because $q = 1$, whenever R does not receive any evidence, she knows $\theta_1 \leq \underline{\theta}_1$ with probability 1. Let $\underline{U}$ denote S's highest possible payoff from using cheap talk upon non-disclosure. Let us denote the corresponding indifference curve by $\underline{I} \equiv \{(\theta_1, \theta_2) : U(\theta_1, \theta_2) = \underline{U}\}$.

If $\underline{I}$ does not intersect the $\theta_1 = \underline{\theta}_1$ line in the $\theta_1 - \theta_2$ space (i.e. there exists some $\theta_1'' < \underline{\theta}_1$ such that $U(\theta_1'', 0) = \underline{U}$), then the point $(\underline{\theta}_1, \mathbb{E}(\theta_2|\theta_1 = \underline{\theta}_1))$ is obviously on a higher indifference curve than $\underline{I}$, so S cannot be indifferent between disclosing $\underline{\theta}_1$ and using cheap talk, upon non-disclosure. This is a contradiction, and we are done. So for the rest of the proof we assume $\underline{I}$ intersects the $\theta_1 = \underline{\theta}_1$ line at $\underline{\theta}_2 \geq 0$, i.e. $U(\underline{\theta}_1, \underline{\theta}_2) = \underline{U}$.

Suppose S's highest payoff cheap talk equilibrium is a babbling equilibrium. In that case the posterior mean upon non-disclosure, $\mathbb{E}(\theta|ND)$, lies *on* the indifference curve $\underline{I}$. Along $\underline{I}$, θ_2 is a strictly decreasing function of θ_1, due to the strict increasingness of U. By assumption, $F(\{D(\theta_1) = 0\}) > 0$, $\therefore\ \ F(\{\theta_1 \leq \underline{\theta}_1\}) > 0, \because$ $\{D(\theta_1) = 0\} \subseteq \{\theta_1 \leq \underline{\theta}_1\}$. Hence $\mathbb{E}(\theta_1|ND) < \underline{\theta}_1$. Therefore $\mathbb{E}(\theta_2|ND) > \underline{\theta}_2$. Using $F_1(\cdot|ND)$ to denote the marginal distribution of θ_1 conditional on non-disclosure we have,

$$\mathbb{E}(\theta_2|ND) = \frac{\int_0^{\underline{\theta}_1} \mathbb{E}(\theta_2|\theta_1)dF_1(\theta_1|ND)}{\int_0^{\underline{\theta}_1} dF_1(\theta_1|ND)} \text{ By Observation 1}'$$

$$\leq \sup_{\theta_1 \leq \underline{\theta}_1} \mathbb{E}(\theta_2|\theta_1)$$

$$= \mathbb{E}(\theta_2|\theta_1 = \underline{\theta}_1) \text{ ,By non-decreasingness of } \mathbb{E}(\theta_2|\theta_1).$$

Therefore, $U(\underline{\theta}_1, \mathbb{E}(\theta_2|\theta_1 = \underline{\theta}_1)) \geq U(\underline{\theta}_1, \mathbb{E}(\theta_2|ND) > U(\underline{\theta}_1, \underline{\theta}_2) = \underline{U}$.

Next, suppose S's highest payoff cheap talk equilibrium is an influential equilibrium. Here we use almost identical reasoning as in Lemma 6. Similarly as argued in Lemma 6, by Bayes plausibility, there exists $I \subseteq \underline{I}$ (I consists of the posterior means following the messages sent with positive probability in equilibrium), consisting of at least two elements, such that the "prior" mean conditional on non-disclosure is an interior point of the convex hull of I. For all $(\theta_1, \theta_2) \in I$, $\theta_2 \geq \underline{\theta}_2$ and there exists at most one point $(\theta_1, \theta_2) \in I$, such that $\theta_2 = \underline{\theta}_2$. Therefore the θ_2-component of $\mathbb{E}(\theta|\theta_1 \leq \underline{\theta}_1)$ is strictly greater than $\underline{\theta}_2$. $\therefore$ $\underline{U} = U(\underline{\theta}_1, \underline{\theta}_2) < U(\underline{\theta}_1, \mathbb{E}(\theta_2|ND)) \leq U(\underline{\theta}_1, \mathbb{E}(\theta_2|\theta_1 = \underline{\theta}_1))$, where the last inequality follows from the calculations in the previous part.

Alternatively, suppose $q < 1$. By way of contradiction, suppose there exists an equilibrium where unravelling occurs, i.e. S discloses evidence whenever he has it. Therefore whenever no evidence is provided, R knows S genuinely does not have evidence. Hence S without evidence can get one of his pure cheap talk payoffs, say U_0. By Lemma 6, $U_0 > U(0, \mathbb{E}(\theta_2|\theta_1 = 0))$, which is S's payoff when he has evidence for $\theta_1 = 0$, so the $\theta_1 = 0$ type S with evidence can profitably deviate to sending one of the messages sent in equilibrium when he does not have evidence. Therefore unravelling cannot be an equilibrium with $q < 1$.

To see the moreover part, assume, by way of contradiction, that there exists an equilibrium in which no θ_1-types fully disclose. Now consider the same arguments used previously to show that for any $\underline{\theta}_1 \in (0, 1]$, disclosing evidence of $\underline{\theta}_1$ is strictly preferred by S over using cheap talk when R knows $\theta_1 \leq \underline{\theta}_1$, and put $\underline{\theta}_1 = 1$. Therefore the disclosure payoff when $\theta_1 = 1$ is strictly higher than S's highest cheap talk payoff when (F_1-almost) no types fully disclose, say U_0. By continuity of $\widetilde{U}(\cdot)$, there exists $\epsilon > 0$ such that $\widetilde{U}(\theta_1) > U_0$ for all $\theta_1 \in (1 - \epsilon, 1]$. Therefore $\underline{\theta}_1 \leq 1 - \epsilon < 1$, i.e. there exists a positive F_1-measure of θ_1-types for which $D(\theta_1) = 1$ in any equilibrium.

A.4 Proofs for section 4

Proof of Proposition 3. Consider any two equilibria E_1 and E_2 such that non-disclosure happens on path for both of them. Let E_1 and E_2 have disclosure strategies D^1 and D^2, sender's ex-ant payoff u_1 and u_2, non-disclosure payoffs U^1_{ND} and U^1_{ND}, and disclosure sets $\mathbb{D}^1$ and $\mathbb{D}^2$ respectively.

Claim 1. $u_1 > u_2 \iff U^1_{ND} > U^2_{ND}$.

Proof. Fix any equilibrium with disclosure strategy D and non-disclosure payoff U_{ND}. By Observation 1´, for all θ_1 with $D(\theta_1) < 1$, the interim payoff is U_{ND}, and the interim payoff of all θ_1 with $D(\theta_1) = 1$ is $\widetilde{U}(\theta_1) \geq U_{ND}$. Therefore the sender's ex-ante payoff is $\mathbb{V}(U_{ND}) = q \int\limits_{\theta_1 \in \Theta_1} \max\{\widetilde{U}(\theta_1), U_{ND}\} + (1-q)U_{ND}$. Therefore $\mathbb{V}(U_{ND})$ is strictly increasing in U_{ND} for $q < 1$ and weakly increasing for $q = 1$, with $U^1_{ND} > U^2_{ND}$ *and* $u_1 = u_2$ possible only if $U^2_{ND} < U^1_{ND} \leq \underline{U}_D$. But if $U^2_{ND} < \underline{U}_D$ non-disclosure cannot happen on-path for E_2 by Observation 1´. The claim follows. □

Claim 2. *If E_2 is an unravelling (partial-disclosure) equilibrium and E_1 is a non-unravelling (no-disclosure) equilibrium, then $U^1_{ND} \geq U^2_{ND}$.*

Proof. We first show the first statement.

E_2 is an unravelling equilibrium, which means $D^2(\theta_1) = 1$ for all $\theta_1 \in \Theta_1$. Hence by Observation 1´, $\widetilde{U}(\theta_1) \geq U^2_{ND}$ for all $\theta_1 \in \Theta_1$, i.e. $U^2_{ND} \leq \underline{U}_D$. Similarly, if E_1 is non-unravelling, by Observation 1´ $\exists\, \theta_1$ such that $D^1(\theta_1) < 1$, which implies $\widetilde{U}(\theta_1) \leq U^1_{ND}$. Putting the last two statements together, we must have $U^1_{ND} \geq U^2_{ND}$.

It analogously follows that if E_2 is a partial-disclosure equilibrium and E_1 is a no-disclosure equilibrium, then $U^1_{ND} \geq U^2_{ND}$. □

Combining claims 1 and 2 the "moreover" part of the statement of Proposition 3 follows.

For the rest of the proof we assume both E_1 and E_2 are partial-disclosure equilibria.

Claim 3. *If both E_1 and E_2 are partial-disclosure equilibria, the following hold:*

- $u_1 > u_2 \implies D^1 \subsetneq D^2$.
- $D^1 \subsetneq D^2 \implies u_1 \geq u_2$.
- *Neither* $D^1 \subsetneq D^2$ *nor* $D^2 \subsetneq D^1 \implies u_1 = u_2$.

Proof. As argued in the proof of Claim 2, since E_i is non-unravelling, $\exists\, \theta_1$ such that $\widetilde{U}(\theta_1) \leq U^i_{ND}$, which implies $U^i_{ND} \geq \underline{U}_D$, $i \in \{1, 2\}$. By definition of partial-disclosure equilibria, $\exists\, \theta_1$ such that $D^i(\theta_1) = 1$ and by Observation 1´ that means $\exists\, \theta_1$ such that $\widetilde{U}(\theta_1) \geq U^i_{ND}$, therefore $U^i_{ND} \leq \overline{U}_D$.

First bullet point. By Claim 1, $u_1 > u_2 \implies U^1_{ND} > U^2_{ND}$. Combining with the previous paragraph, $\underline{U}_D \leq U^2_{ND} < U^1_{ND} \leq \underline{U}_D$. By continuity of $\widetilde{U}(\cdot)$, $\{\theta_1 : U^2_{ND} < \widetilde{U}(\theta_1) < U^1_{ND}\}$ is non-empty. $\therefore$ $\{\widetilde{U}(\theta_1) > U^2_{ND}\} \supsetneq \{\widetilde{U}(\theta_1) \geq U^1_{ND}\}$.

Combining the above with Observation 1´ we have,

$$\{D_2(\theta_1)=1\} \supseteq \{\widetilde{U}(\theta_1) > U^2_{ND}\} \supsetneq \{\widetilde{U}(\theta_1) \geq U^1_{ND}\} \supseteq \{D_1(\theta_1)=1\}.$$

This establishes the first bullet point.

Second bullet point. Similarly as above, by Observation 1´,

$$\{\widetilde{U}(\theta_1) \geq U^2_{ND}\} \supseteq \{D_2(\theta_1)=1\} \supseteq \{D_1(\theta_1)=1\} \supseteq \{\widetilde{U}(\theta_1) > U^1_{ND}\}.$$

$\{\widetilde{U}(\theta_1) \geq U^2_{ND}\} \supseteq \{\widetilde{U}(\theta_1) > U^1_{ND}\} \implies U^1_{ND} \geq U^2_{ND}$. Therefore by Claim 1 the second bullet point follows.

Third bullet point. Suppose, by way of contradiction, $D^1 \not\subseteq D^2, D^2 \not\subseteq D^1$ and $U^1_{ND} > U^2_{ND}$. Therefore, by Claim 1 $u_1 > u_2$. Hence by the first bullet point $D^1 \subsetneq D^2$ - a contradiction. Hence $D^1 \not\subseteq D^2, D^2 \not\subseteq D^1 \implies U^1_{ND} = U^2_{ND} \implies u_1 = u_2$ by Claim 1. □

The first statement of proposition 3 follows immediately from Claim 3.

Proof of Proposition 4. We know from Proposition 3 that the sender-preferred equilibrium is any equilibrium featuring the highest possible equilibrium non-disclosure payoff. We know the set of possible sender-payoffs in the cheap talk game with prior $F|_{\theta_1 \in \Theta_1(U_{ND})}$, say $V(F|_{\theta_1 \in \Theta_1(U_{ND})})$, is a compact interval (Lipnowski and Ravid, 2020). Below we show that the highest equilibrium non-disclosure payoff is obtained by selecting the highest value in $V(F|_{\theta_1 \in \Theta_1(U_{ND})})$ for every U_{ND}.

Case 1. $\overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})}) < U_{ND} \implies V(F|_{\theta_1 \in \Theta_1(U_{ND})}) < U_{ND}$ for all $U_{ND} \in [\underline{U}_D, \overline{U}_D]$. Therefore there cannot be any equilibrium with on-path voluntary non-disclosure. The claim follows.

Case 2. If $\overline{U}(F) \geq \overline{U}_D$, in the sender-preferred pure cheap talk equilibrium, *no* θ_1-type has an incentive to deviate and disclose. The claim follows.

Case 3. $\widetilde{U}(\theta_1)$ is a continuous function of θ_1 on a compact interval, therefore $\min \widetilde{U}(\theta_1) = \underline{U}_D, \max \widetilde{U}(\theta_1) = \overline{U}_D$. Clearly, for $U_{ND} = \underline{U}_D$ or $U_{ND} = \overline{U}_D$, $\overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})}) = \overline{U}(F)$.

$\overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})})$ is a continuous function of U_{ND}. To see why, note that $F|_{\theta_1 \in \Theta_1(U_{ND})}$ is continuous is U_{ND} by continuity of U. Moreover, $\overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})})$ is upper semicontinuous in $F|_{\theta_1 \in \Theta_1(U_{ND})}$, also by continuity of U (Lipnowski and Ravid, 2020).

Because $\overline{U}(F) < \overline{U}_D$ for Case 3, the set $\{U_{ND} : U_{ND} \in V(F|_{\theta_1 \in \Theta_1(U_{ND})})\}$ is non-empty, and gives the set of equilibrium non-disclosure payoffs of the sender. Below we show that $\max\{U_{ND} : U_{ND} \in V(F|_{\theta_1 \in \Theta_1(U_{ND})})\} = \max\{\max V(F|_{\theta_1 \in \Theta_1(U_{ND})}) = U_{ND}\}$.

Under Case 3, $\overline{U}(F|_{\theta_1 \in \Theta_1(\overline{U}_D)}) < \overline{U}_D$ and $\overline{U}(F|_{\theta_1 \in \Theta_1(\overline{U}_{ND})}) \geq \overline{U}_{ND}$ for some $U_{ND} \in [\underline{U}_D, \overline{U}_D)$. By continuity of $\overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})})$ in U_{ND}, the set $\{U_{ND} \in$

$[\underline{U}_D, \overline{U}_D) : \overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})}) = U_{ND}\}$ is non-empty and compact. Suppose there exists an equilibrium with a strictly higher non-disclosure payoff - which is equivalent to strictly preferred by sender, by Proposition 3 - than $\max\{U_{ND} \in [\underline{U}_D, \overline{U}_D) : \overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})}) = U_{ND}\}$. But for all $U_{ND} > \max\{U_{ND} \in [\underline{U}_D, \overline{U}_D) : \overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})}) = U_{ND}\}$, $\overline{U}(F|_{\theta_1 \in \Theta_1(U_{ND})}) < U_{ND}$ by case 3 assumption, and therefore $u < U_{ND}$ for all $u \in V(F|_{\theta_1 \in \Theta_1(U_{ND})}) < U_{ND}$. This is a contradiction.

Proof of Proposition 5. Fix $U_{ND} \in [\underline{U}_D, \overline{U}_D]$. Let $\Theta_1^{ND} := \Theta_1(U_{ND}) = \left\{\theta_1 : \widetilde{U}(\theta_1) \leq U_{ND}\right\}$. Let the common marginal distribution of F and G over Θ_1 be denoted by $F_1(\cdot)$. Consider the distributions $F' := F|_{\theta_1 \in \Theta_1^{ND}}$ and $G' := G|_{\theta_1 \in \Theta_1^{ND}}$. Note that, G' is a mean-preserving spread of F'. To see why, consider a convex function $u(\theta_1, \theta_2)$. So $u(\theta_{1c}, \theta_2)$ is a convex function of θ_2 for all $\theta_{1c} \in \Theta_1$.

$$\begin{aligned}
\therefore\ \mathbb{E}_{G'} u(\theta_1, \theta_2) &= \mathbb{E}_{\theta_1 \in \Theta_1^{ND}} \mathbb{E}_{\theta_2 \sim G(\cdot|\theta_1)} u(\theta_1, \theta_2) \\
&= \int\limits_{\theta_1 \in \Theta_1^{ND}} \left(\int\limits_{\theta_2 = 0}^{1} u(\theta_1, \theta_2) dG(\theta_2|\theta_1) \right) dF_1(\theta_1) \\
&\geq \int\limits_{\theta_1 \in \Theta_1^{ND}} \left(\int\limits_{\theta_2 = 0}^{1} u(\theta_1, \theta_2) dF(\theta_2|\theta_1) \right) dF_1(\theta_1) = \mathbb{E}_{F'} u(\theta_1, \theta_2).
\end{aligned}$$

where the inequality follows from the fact that $G(\cdot|\theta_1) \succsim_{MPS} F(\cdot|\theta_1)$ for all θ_1 and convexity of $u(\theta_1, \cdot)$.

The above shows that G' is above F' in the convex order, which is equivalent to the mean-preserving spread order. Therefore the set of Bayes plausible information policies under the prior F' is a subset of that under the prior G'. Hence the set of securable payoffs (Lipnowski and Ravid, 2020) under the prior F' is also a subset of that under the prior G'. Therefore $\overline{v}(G') \geq \overline{v}(F')$. The choice of $U_{ND} \in [\underline{U}_D, \overline{U}_D]$ was arbitrary, so the function $\overline{v}(\mu^{ND}(\cdot|G))$ majorizes the function $\overline{v}(\mu^{ND}(\cdot|F))$, where $\mu^{ND}(\cdot)$ is given by equation (6.3) in the main text. Therefore by Proposition 4 the highest equilibrium non-disclosure payoff is weakly higher for G than F. Hence by proposition 3 the disclosure set at the sender-best equilibrium under G is a subset of that under F.

A.5 Proofs for section 5.1

*Proof of Proposition 6.*The setting without cheap talk is equivalent to only the babbling equilibrium existing conditional on non-disclosure. The cutoff $\underline{\theta}_1$ for the babbling equilibrium is defined by the equation:

$$U\left(\frac{qF_1(\underline{\theta}_1)\mathbb{E}(\theta|\theta_1 \leq \underline{\theta}_1) + \theta_0(1-q)}{(1-q) + qF_1(\underline{\theta}_1)}\right) = U\left(\underline{\theta}_1, \mathbb{E}(\theta_2|\theta_1)\right)$$

Clearly, the LHS of the above equation equals $U(\theta_0)$ at both $\underline{\theta}_1 = 0$. By Lemma 6 and arguments in the proof of Proposition 2, $\underline{\theta}_1 = 1$ and is higher (resp. lower) than the RHS at $\underline{\theta}_1 = 0$ (resp. $\underline{\theta}_1 = 1$). Therefore the highest $\underline{\theta}_1$ at which the two are equal must be at a point where the LHS is either decreasing in $\underline{\theta}_1$, or increasing in $\underline{\theta}_1$ but flatter than RHS. The highest-cutoff babbling equilibrium is the sender-optimal babbling equilibrium, by Proposition 3. Below we show that in either of the aforementioned two cases, the highest point of intersection between LHS and RHS is falling in q.

Note that $\mathbb{E}(\theta|\theta_1 \leq \underline{\theta}_1) \leq \theta_0$ and monotonically increases to θ_0 as $\underline{\theta}_1$ increases from 0 to 1. For any fixed $\underline{\theta}_1$, as q increases, the weight on $\mathbb{E}(\theta|\theta_1 \leq \underline{\theta}_1)$ in the convex combination of θ_0 and $\mathbb{E}(\theta|\theta_1 \leq \underline{\theta}_1)$ which is the argument of U on the LHS, increases. Therefore the convex combination decreases and therefore the LHS decreases. The RHS is increasing in $\underline{\theta}_1$ and unaffected by q. Therefore the *highest* solution to the above equation must be falling in q.

A.6 Proofs for section 5.2

Proof of Proposition 7. Receiver may buy one of the goods if its posterior mean quality is greater than her privately observed threshold. Therefore it is without loss (from sender's perspective) to consider recommendation signals - at every state the sender recommends to the receiver which good to buy. The receiver either accepts the recommendation or does not buy. Therefore it is without loss to consider the message space $M = \{1, 2\}$ where message i stands for recommending good i.

Claim A.6.1. *If $\overline{v}(F)$ is implemented by a finite number of messages, then $\overline{v}(\cdot)$ is continuous at F.*

Proof. Let the set of messages used in some sender-optimal cheap talk equilibrium be $M = \{m_1, \cdots, m_k\}$. Let the conditional distribution over M at this sender-optimal equilibrium, when the prior is F, be $\sigma \equiv \{\sigma(m_1|\theta), \cdots, \sigma(m_k|\theta)\}_\theta$.

Consider a sequence of distributions $F_n \Longrightarrow F$. Therefore the minimum sender payoff under the joint distribution over signals and states induced by σ when the prior is F_n, i.e. $\min_i v(\mu(m_i; \sigma, F_n))$ - where $\mu(m_i; \sigma, F_n)$ is the receiver's belief induced by the messaging strategy σ when the prior is F_n - is close to that when the prior is F, by continuity of $v(\cdot)$ is receiver beliefs. Passing to a subsequence if necessary, we have, $\min_i v(\mu(m_i; \sigma, F_n)) \geq \min_i v(\mu(m_i; \sigma, F)) - \frac{1}{n} = \overline{v}(F) - \frac{1}{n}$ for all n, where the equality comes from the fact that $v(\mu(m_i; \sigma, F)) = v(\mu(m_j; \sigma, F)) = \overline{v}(F)$ for all $i, j \leq k$, by the fact that σ is an equilibrium strategy and is also sender-optimal, under the prior F.

By securability (Lipnowski and Ravid, 2020), $\overline{v}(F_n) \geq \min_i v(\mu(m_i; \sigma, F_n))$ because $\min_i v(\mu(m_i; \sigma, F_n))$ is a securable payoff under the prior F_n. Therefore,

$$\overline{v}(F_n) \geq \overline{v}(F) - \frac{1}{n} \; \forall \, n.$$
$$\implies \liminf_n \overline{v}(F_n) \geq \overline{v}(F).$$

But we know from Lipnowski and Ravid (2020) that $\overline{v}(\cdot)$ is upper semi-continuous, so $\limsup_n \overline{v}(F_n) \leq \overline{v}(F)$. Combining with the previous statement we have, $\lim_n \overline{v}(F_n)$ exists and is equal to $\overline{v}(F)$, proving continuity. □

Claim A.6.2. *If $F_n \implies F$, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\overline{v}(F_n) < 1 - \epsilon$ for all $n \geq N$.*

Proof. Clearly, $\overline{v}(F) < 1$.[12] Therefore there exists ϵ such that $\overline{v}(F) < 1 - \epsilon$. By continuity of $\overline{v}(\cdot)$ at F (Claim A.6.1), if $F_n \implies F$, there exists $N \in \mathbb{N}$ such that $\overline{v}(F_n) < 1 - \epsilon$ for all $n \geq N$. □

Let $p(\theta) \in [0, 1]$ denote the probability of recommending good 1 at state θ.

The sender's problem is therefore:

$$\begin{gathered}
\max \int_\Theta \theta_1 p(\theta) dF(\theta) + \int_\Theta \theta_2 (1 - p(\theta)) dF(\theta) \\
\text{s.t.} \int_\Theta \theta_1 p(\theta) dF(\theta) \geq \int_\Theta \theta_2 p(\theta) dF(\theta) \\
\int_\Theta \theta_2 (1 - p(\theta)) dF(\theta) \geq \int_\Theta \theta_1 (1 - p(\theta)) dF(\theta) \\
\left(\int_\Theta (1 - p(\theta)) dF(\theta) \right) \left(\int_\Theta \theta_1 p(\theta) dF(\theta) \right) = \left(\int_\Theta \theta_2 (1 - p(\theta)) dF(\theta) \right) \left(\int_\Theta p(\theta) dF(\theta) \right).
\end{gathered}$$

The first two inequality constraints ensure that when i is recommended, the posterior mean of i is at least as large as that of j and vice versa, $i, j \in \{1, 2\}, i \neq j$. The last equality constraint captures the standard cheap talk equilibrium constraint of sender indifference among signals sent with positive probability. It is written in multiplication form instead of equality-of-means form to accommodate the cases where one of the signals may be sent with probability zero at the optimal solution.

Consider the relaxed problem:

[12] Either no persuasion is possible, and $\overline{v}(F) = \max\{\mathbb{E}_F(\theta_1), \mathbb{E}_F(\theta_2)\} < 1$ by the fact that F is full support. Or, there is a persuasive sender-optimal recommendation equilibrium, in which case both signals 1 and 2 are sent with positive probability, in which case $\overline{v}(F) = \mathbb{E}(\theta_1 | 1) < 1$, because the support of signal 1 has positive F-measure and so $\mu(\theta_1 = 1 | 1) = 0$ and $\mu(\theta_1 < 1 | 1) = 1$, where $\mu(\cdot)$ is receiver's posterior belief after observing signal 1.

$$\max \int_\Theta \theta_1 p(\theta) dF(\theta) + \int_\Theta \theta_2 (1-p(\theta)) dF(\theta)$$
$$\text{s.t.} \left(\int_\Theta (1-p(\theta))dF(\theta)\right)\left(\int_\Theta \theta_1 p(\theta)dF(\theta)\right) = \left(\int_\Theta \theta_2(1-p(\theta))dF(\theta)\right)\left(\int_\Theta p(\theta)dF(\theta)\right). \tag{20}$$

Claim A.6.3. *Any solution to the relaxed problem is a solution to the original problem.*

Proof. Suppose v and v' are the maximized values of the original and relaxed problems. $\therefore$ $v' \geq v$.

First, consider a solution to the relaxed problem, if any, where both the signals are sent with positive probability.

Suppose both of the inequality constraints are violated at an optimal solution to the relaxed problem. Therefore $v \leq v' = \mathbb{E}(\theta_1|1) = \mathbb{E}(\theta_2|2) < \min\{\mathbb{E}(\theta_2|1), \mathbb{E}(\theta_1|2)\}$. Therefore any solution to the relaxed problem is an information policy, the minimum payoff under which is $\min\{\max\{\mathbb{E}(\theta_1|1), \mathbb{E}(\theta_2|1)\}, \max\{\mathbb{E}(\theta_1|2), \mathbb{E}(\theta_2|2)\}\} = \min\{\mathbb{E}(\theta_2|1), \mathbb{E}(\theta_1|2)\}$. Therefore $\min\{\mathbb{E}(\theta_2|1), \mathbb{E}(\theta_1|2)\}$ is a securable payoff (Lipnowski and Ravid, 2020) which is strictly greater than the maximum payoff v. This is a contradiction.

Suppose one of them is violated at a solution to the relaxed program (20). First, assume that, at this solution, both signals 1 and 2 are sent with positive probability.

Suppose $\mathbb{E}(\theta_1|1) < \mathbb{E}(\theta_2|1)$ but $\mathbb{E}(\theta_2|2) \geq \mathbb{E}(\theta_1|2)$ at this solution. Therefore at this solution we have,

$$\mathbb{E}(\theta_1|2) \leq \mathbb{E}(\theta_2|2) = \mathbb{E}(\theta_1|1) < \mathbb{E}(\theta_2|1). \tag{21}$$

The unconditional prior mean of θ_1 (resp. θ_2) is an interior convex combination of $\mathbb{E}(\theta_1|2)$ and $\mathbb{E}(\theta_1|1)$ (resp. $\mathbb{E}(\theta_2|2)$ and $\mathbb{E}(\theta_2|1)$). Therefore given (21), this is possible only if $\mathbb{E}(\theta_2) > \mathbb{E}(\theta_1)$. Therefore in the babbling equilibrium, the sender can get a payoff of $\mathbb{E}(\theta_2)$ which is strictly greater than $\mathbb{E}(\theta_2|2)$ at the optimal solution to (20), by (21). But the maximized value of the relaxed program must be weakly greater than that of the original program, which must be weakly greater than the sender's babbling payoff - a contradiction.

Next, consider a solution to the relaxed problem, if any, where one of the signals are sent zero probability.

Whenever either of the signals is sent with zero probability, the only constraint of the relaxed problem is satisfied, as both the sides of the required equality equals zero. Therefore in this case the optimized value of the relaxed problem is $\max\{\mathbb{E}\theta_1, \mathbb{E}\theta_2\}$, i.e. the sender's babbling payoff. The sender's optimal cheap talk payoff is bounded below by his babbling payoff, therefore $v \geq v'$. But we know $v' \geq v$. Therefore $v = v'$. $\square$

Claim A.6.4. *There exists m such that in the sender-optimal equilibrium, good 2 (good 1) is recommended for θ's above (below) the line $(\theta_2 - V) = m(\theta_1 - V)$, where V is the sender's maximized value.*

Proof. Consider the following discretized problem. Let $\Theta = \left\{ \left(\frac{i}{n}, \frac{j}{n}\right) \right\}_{(i,j)=(1,1)}^{(n,n)}$. $Pr\left(\left(\frac{i}{n}, \frac{j}{n}\right)\right) = f_{i,j}$. For large enough n, $f_{ij} > 0$ for all i, j by full support assumption on the limiting distribution.

In the problem below $k \equiv (k(1), k(2))$, k goes from $(1,1)$ to (n,n). p_k is the probability of recommending good 1 in state $\theta_k = \left(\frac{k(1)}{n}, \frac{k(2)}{n}\right)$. For ease of exposition, we sometimes use the notation $\theta_{i,n'} \equiv \theta_{i,k}$ for any k such that $k(i) = n'$, $i \in \{1, 2\}$. Below is the descretized version of the relaxed problem (20).

$$\max \sum_{k=(1,1)}^{(n,n)} \theta_{1,k} p_k f_k + \sum_{k=(1,1)}^{(n,n)} \theta_{2,k}(1 - p_k) f_k$$

$$\text{s.t.} \left(\sum_{k=(1,1)}^{(n,n)} \theta_{1,k} p_k f_k\right)\left(\sum_{k=(1,1)}^{(n,n)} (1 - p_k) f_k\right) = \left(\sum_{k=(1,1)}^{(n,n)} \theta_{2,k}(1 - p_k) f_k\right)\left(\sum_{k=(1,1)}^{(n,n)} p_k f_k\right),$$

$$0 \leq p_k \leq 1 \;\forall\; k.$$

Using λ to denote the Lagrange multiplier for the e quality constraint in the above program, and $\mu_{0,k} f_k$ and $\mu_{1,k} f_k$ (we can do this as $f_k > 0$ for all k) respectively for the ≥ 0 and ≤ 1 constraints on p_k, we have the Lagrangian for the above problem:

$$\begin{aligned}
\mathscr{L} &= \sum_k \theta_{1,k} p_k f_k + \sum_k \theta_{2,k}(1 - p_k) f_k \\
&\quad - \lambda\left(\left(\sum_k \theta_{1,k} p_k f_k\right)\left(\sum_k (1 - p_k) f_k\right) - \left(\sum_k \theta_{2,k}(1 - p_k) f_k\right)\left(\sum_k p_k f_k\right)\right) \\
&\quad - \sum_k \mu_{1,k} f_k\,(p_k - 1) + \sum_k \mu_{0,k} f_k p_k, \\
&\mu_{1,k}, \mu_{0,k} \geq 0 \;\forall\; k.
\end{aligned}$$

Taking FOC w.r.t p_k:

$$\frac{\partial \mathscr{L}}{\partial p_k} = 0 \iff \underbrace{(\lambda Pr(1) + 1)}_{\beta \text{ (say)}}(\theta_{2k} - V) - \underbrace{(1 - \lambda Pr(2))}_{\alpha \text{ (say)}}(\theta_{1k} - V) = \mu_{0,k} - \mu_{1,k}. \quad (22)$$

where $Pr(i)$ denotes the probability of signal $i, i \in \{1, 2\}$, i.e. $Pr(1) = \sum_k p_k f_k$ and $Pr(2) = 1 - Pr(1)$.

From the above equation we can conclude the following: At the optimal solution p^*,

- If $p_k^* \in (0,1)$, both $0 \le p_k$ and $p_k \le 1$ constraints are slack. Therefore by the complementary slackness condition $\mu_{0,k} = \mu_{1,k} = 0$. Hence θ_k must lie on the line $\beta(\theta_{2k} - V) - \alpha(\theta_{1k} - V) = 0$.

- If the $0 \le p_k$ constraint is slack, i.e. $p_k^* \in (0,1]$, $\mu_{0,k} = 0$. Hence, by (22), θ_k must lie weakly below line $\beta(\theta_{2k} - V) - \alpha(\theta_{1k} - V) = 0$. But by the previous point, if $p_k^* \in (0,1)$, θ_k cannot lie strictly below that line. Therefore θ_k can lie strictly below the line only if $p_k^* = 1$.

- Similarly θ_k can lie strictly above the line only if $p_k^* = 0$.

Combining the above we have the desired claim for the discretized problem. $\square$

Going forward we refer to the class of information structures $p(\theta) = \mathbb{1}\{(\theta_2 - V) < m(\theta_1 - V)\}, p(\theta) \in (0,1)$ if $(\theta_2 - V) = m(\theta_1 - V)$ by the tuple (m, V).

Claim A.6.5. *There exists m such that in the sender-optimal equilibrium for F, good 2 (good 1) is recommended for θ's above (below) the line $(\theta_2 - V) = m(\theta_1 - V)$, where V is the sender's maximized value.*

Proof. We know $\lim \overline{v}(F_n) = \overline{v}(F)$ (by Claim A.6.1). Let (m_n, V_n) be a sender-optimal equilibrium information structure for prior F_n. By Claim A.6.4, $\overline{v}(F_n) = \mathbb{E}_{F_n}(\theta_2 | (\theta_2 - V_n) \ge m_n(\theta_1 - V_n))$. Define the functional $\tilde{v} : \Delta\Theta \times [0,\infty) \times [0,1] \to [0,1]$ as $\tilde{v}(G, m, V) = \mathbb{E}_G(\theta_2 | (\theta_2 - V) \ge m(\theta_1 - V))$. Clearly $\tilde{v}(\cdot)$ is continuous in (G, m, V). (This line requires proof.) Therefore $\lim \overline{v}(F_n) = \lim \tilde{v}(F_n, m_n, V_n)$. Since we know the left hand limit exists, so must the right hand limit. Suppose $\lim(F_n, m_n, V_n)$ does not exist. This is possible only when $\lim m_n$ does not exist, because $\lim F_n = F$ by assumption and $\lim V_n = V$ by Claim A.6.4. Let m be a limit point of the sequence $\{m\}_n$.

Note that m cannot be ∞ or $-\infty$, because $\mathbb{E}_F(\theta_1 | \theta_1 \le V)$ or $\mathbb{E}_F(\theta_1 | \theta_1 \ge V)$ cannot be equal to V by the full support assumption on F and the fact that $V \notin \{1, 0\}$.

Suppose m and m' are two limit points of $\{m_n\}_n$. Therefore there are two subsequences of (F_n, m_n, V_n) with limits (F, m, V) and (F, m', V). By continuity of $\tilde{v}(\cdot), \mathbb{E}_F(\theta_2 | (\theta_2 - V) \ge m(\theta_1 - V)) = \mathbb{E}_F(\theta_2 | (\theta_2 - V) \ge m'(\theta_1 - V)) = \lim \tilde{v}(F_n, m_n, V_n) = \overline{v}(F)$. By A.6.7 if $m \ne m'$, $\mathbb{E}_F(\theta_2 | (\theta_2 - V) \ge m(\theta_1 - V)) = \mathbb{E}_F(\theta_2 | (\theta_2 - V) \ge m'(\theta_1 - V)) = V$ must hold.

Therefore the equilibrium information structures (m, V) and (m', V) are optimal for F. $\square$

Claim A.6.6. *If in a sender-optimal equilibrium for F, good 2 (good 1) is recommended for θ's above (below) the line $(\theta_2 - V) = m(\theta_1 - V)$, where V is the sender's maximized value, then $m > 0$.*

Proof. Let us denote the line $(\theta_2 - V) = m(\theta_1 - V)$ by l.

Suppose first, by way of contradiction, that $m < 0$. Draw a horizontal line $l' : \theta_2 = \overline{\theta}_2$ such that it intersects l at a point in $\{\theta_1 \in (0,1)\}$, say $\overline{\theta}_1$. Let $A = \{(\theta_2 - V) \leq m(\theta_1 - V), \theta_2 \geq \overline{\theta}_2\}$ and $B = \{(\theta_2 - V) \geq m(\theta_1 - V), \theta_2 \leq \overline{\theta}_2\}$, i.e. A and B are the two triangles (or one triangle and one trapezium, depending on where l intersects the boundaries of the square $[0,1]^2$) to the left and right of $(\overline{\theta}_1, \overline{\theta}_2)$ respectively, between the lines l and l'.

By the full support assumption on F and the fact that $m < 0$, both A and B have positive mass. Select open balls $B_1 \in Int\ A$ and $B_2 \in Int\ B$ such that $F(B_1) = F(B_2) = \epsilon > 0$. We can do this for small enough ϵ, by absolute continuity of F. Clearly, for all $\theta \in B_1, \theta' \in B_2$, $\theta'_1 > \theta_1$ and $\theta'_2 < \theta_2$. Therefore $\mathbb{E}(\theta_1|B_2) > \mathbb{E}(\theta_1|B_1)$ and $\mathbb{E}(\theta_2|B_1) > \mathbb{E}(\theta_2|B_2)$.

Let us denote the area on which 1 (2) is recommended under the original signal - i.e. the area below (above) l - as **1** (**2**) respectively. Let us consider a new signal under which 1 (2) is recommended on $\mathbf{1} \cup B_2 \setminus B_1$ ($\mathbf{2} \cup B_1 \setminus B_2$). Let us denote the posterior mean of θ_1 (θ_2) after receiving signal 1 (2) under this new signal be denoted by $\mathbb{E}(\theta_1|1')(\mathbb{E}(\theta_2|2'))$.

The sender's value under the original signal which recommends 2 (1) above (below) l is $V = \mathbb{E}(\theta_1|1) = \mathbb{E}(\theta_2|2)$. Therefore,

$$\begin{aligned}
\mathbb{E}(\theta_1|1) &= \mathbb{E}(\theta_1|\mathbf{1} \setminus B_1)(Pr(1) - \epsilon) + \mathbb{E}(\theta_1|B_1)\epsilon \\
&< \mathbb{E}(\theta_1|\mathbf{1} \setminus B_1)(Pr(1) - \epsilon) + \mathbb{E}(\theta_1|B_2)\epsilon \\
&= \mathbb{E}(\theta_1|1').
\end{aligned}$$

Similarly we can see that $\mathbb{E}(\theta_2|2) < \mathbb{E}(\theta_2|2')$, i.e. both the posterior means of θ_1 and θ_2 strictly increase under the new signal. Therefore by securability (Lipnowski and Ravid, 2020), sender's optimized value must be weakly higher than $\min\{\mathbb{E}(\theta_1|1'), \mathbb{E}(\theta_2|2')\}$, and therefore strictly higher than that under the original signal. Therefore the original signal cannot be optimal.

Next suppose $m = 0$. $\therefore$ $l : \theta_2 = V$. So we must have at the optimum, $\mathbb{E}(\theta_2|\theta_2 \geq V) = V$. But the full support assumption, this is possible only if $V = 1$. In that case, $\mathbb{E}(\theta_1|1) = \mathbb{E}(\theta_1|\theta_2 \leq 1) = \mathbb{E}(\theta_1) < 1$ by the full support assumption, which is a contradiction. Therefore $m \neq 0$.

Combining the above two parts, the claim follows. $\square$

Claim A.6.7. *The sender-optimal recommendation equilibrium is unique, in which good 2 (1) is recommended above (below) the line $(\theta_2 - V) = m(\theta_1 - V)$, for some $m > 0$.*

Proof. Suppose, by way of contradiction, there are multiple sender-optimal recommendation equilibria. Therefore by the above claim there is at least one other line, with slope $m' \neq m, m' > 0$, through the point (V, V), such that they both give the sender a payoff of V. Suppose $m' > m$. Then for any splitting of points along the m'-line, the support of the signal 1 under the m' IS includes more points with $\theta_1 > V$ and excludes point with $\theta_1 < V$, compared to m. So the new $\mathbb{E}(\theta_1|1)$ cannot be equal to V, which is a contradiction. $\square$

Note that while the above is the unique sender-optimal equilibrium within the class of *recommendation* equilibria, there are infinitely many sender-optimal equilibria outside this class - obtained, .e.g., by splitting the set on which each good is recommended further into sets across which the sender is indifferent, as shown in Fig.

Claim A.6.8. *The sender-optimal recommendation equilibrium must be informative, i.e. both good 1 and 2 must recommended with positive probability in it.*

Proof. Suppose not. Since the unique sender-optimal recommendation equilibrium for F consists of recommending good 2 (1) above (below) the line $(\theta_2 - V) = m(\theta_1 - V)$ for some $m > 0$, can be uninformative only if the point (V, V) is on the boundary of the set $\Theta = [0, 1]^2$. That is possible only if $V \in \{0, 1\}$. $V \geq \max\{\mathbb{E}(\theta_1), \mathbb{E}(\theta_2)\} > 0$ because $\{\mathbb{E}(\theta_1), \mathbb{E}(\theta_2)\} \in (0, 1)$ by the fact that F has full support. $V < 1$ as discussed earlier. Therefore the claim follows. $\square$

whatever

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