

Correlation-Robust Optimal Auctions

Wanchang Zhang[†]

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Abstract

I study the design of auctions in which the auctioneer is assumed to have information only about the marginal distribution of a generic bidder's valuation, but does not know the correlation structure of the joint distribution of bidders' valuations. I assume that a generic bidder's valuation is bounded and \bar{v} is the maximum valuation of a generic bidder. The performance of a mechanism is evaluated in the worst case over the uncertainty of joint distributions that are consistent with the marginal distribution. For the two-bidder case, *the second-price auction with the uniformly distributed random reserve* maximizes the worst-case expected revenue across *all* dominant-strategy mechanisms under certain regularity conditions. For the N -bidder ($N \geq 3$) case, *the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve* maximizes the worst-case expected revenue across *standard* (a bidder whose bid is not the highest will never be allocated) dominant-strategy mechanisms under certain regularity conditions. When the probability mass condition (part of the regularity conditions) does not hold, *the second-price auction with the s^* -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve* maximizes the worst-case expected revenue across standard dominant-strategy mechanisms, where $s^* \in (0, \bar{v})$.

Keywords: Robust mechanism design, dominant-strategy mechanism, second-price auction, random reserve, duality.

JEL Codes: C72, D82, D83.

[†]Department of Economics, University of California, San Diego. Email: waz024@ucsd.edu

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1 Introduction

The mechanism design literature assumes that bidders' valuation profile follows a *commonly known* joint distribution. For example, Myerson (1981) assumes that the auctioneer knows the marginal distribution of each bidder's valuation, and also knows that bidders' valuations are independently distributed. While the independent private value model is widely acknowledged as a useful benchmark, little is known about how the optimal mechanism would perform once the model is misspecified. In addition, it is not clear how the auctioneer should determine which model is the correct one to use.

In this paper, I study the single-object auction problem in the correlated private value environment in which bidders' valuation profile is drawn from a general joint distribution. I assume that the auctioneer knows the marginal distribution of a generic bidder's valuation, but does not have any knowledge about the correlation structure of different bidders' valuations¹. A joint distribution of bidders' valuation profile is said to be *conceivable* if it is consistent with the known marginal distribution of a generic bidder's valuation. The auctioneer seeks a dominant-strategy mechanism. A mechanism is evaluated according to the auctioneer's expected revenue in the dominant-strategy equilibrium derived in the worst case, referred to as the *revenue guarantee*, over all conceivable joint distributions. The objective of the auctioneer is to design a mechanism that maximizes the revenue guarantee across some general class of dominant-strategy mechanisms. I call such a mechanism a *maxmin mechanism*.

This framework is in the same spirit as the robust mechanism literature in that it assumes away detailed knowledge of the auctioneer (Wilson (1987)). It is motivated by the observation that the joint distribution is a much higher-dimensional object than the marginal distribution of a generic bidder. Therefore it is more difficult to estimate the joint distribution. Practically, it fits into the situations where the bidder pool changes constantly and then there is no data for estimating the correlation structure. Another situation where the auctioneer may only know the marginal distribution for each bidder is the one where the identities of the participating bidders cannot be observed.

The first main result (Theorem 1) is that, under certain regularity conditions on the marginal distribution, *the second-price auction with the uniformly distributed random reserve* is a maxmin mechanism across all dominant-strategy mechanisms for the two-bidder case. Under this mechanism, a random reserve is drawn from a uniform distribution on $[0, \bar{v}]$ where

¹The framework is originally proposed by Carroll (2017) for the multi-dimensional screening problem. His solution is simple and conveys a clear and intuitive message: if you do not know how to bundle, then do not. It is natural to adapt his framework to an environment with multiple bidders whose private valuations may be correlated.

\bar{v} is the maximum valuation.

The randomness in this mechanism hedges against uncertainty over correlation structures. Indeed, the specific random device in this mechanism exhibits a *full-insurance property*: the expected revenue is the same across all joint distributions consistent with the marginal distribution. To see this, consider a valuation profile (v_1, v_2) in which $v_1 > v_2$. Under this mechanism, if the random reserve r is lower than v_2 , occurring with a probability of $\frac{v_2}{\bar{v}}$, then bidder 1 pays v_2 ; if the random reserve r is between v_2 and v_1 , then bidder 1 pays r . Therefore the revenue from the valuation profile (v_1, v_2) is $v_2 \cdot \frac{v_2}{\bar{v}} + \int_{v_2}^{v_1} r \cdot \frac{1}{\bar{v}} dr = \frac{v_1^2 + v_2^2}{2\bar{v}}$, which is separable in v_1 and v_2 . This implies the full-insurance property. In addition, the expected revenue is the second moment of the marginal distribution over the maximum valuation.

I show that this mechanism is a maxmin mechanism across all dominant-strategy mechanisms under certain regularity conditions by constructing a conceivable joint distribution such that (i) the joint distribution minimizes the expected revenue under the mechanism across all conceivable joint distributions and (ii) the mechanism maximizes the expected revenue under the joint distribution across all dominant-strategy mechanisms. I call such a joint distribution a *worst-case correlation structure*. It is straightforward that (i) and (ii) imply that the mechanism is a maxmin mechanism.

To construct such a joint distribution, I first reformulate the problem of maximizing the expected revenue across all dominant-strategy mechanisms as the problem of maximizing the expected “virtual value” of the bidder who is allocated the object subject to that the allocation rule is monotone (a monotonicity constraint associated with dominant-strategy incentive compatibility), where the “virtual value” is that the bidder’s valuation less information rents that are pinned down by dominant-strategy incentive compatibility and the binding ex-post participation constraints of zero-valuation bidders. This is a straightforward adaption of the well-known revenue equivalence result of Myerson (1981). Importantly, this simplifies the problem in that one can now point-wise maximize the objective, ignoring the monotonicity constraint². Then such a joint distribution is obtained by letting the virtual value of the high bidder (the bidder with a higher valuation than that of her opponent) be zero except when the high bidder’s valuation is \bar{v} . The intuition behind this property is that the auctioneer is *indifferent* between allocating and not allocating the object to the high bidder as long as her valuation is below \bar{v} under the second-price auction with the uniformly distributed random reserve.

To illustrate, consider a special case in which the marginal distribution is an *equal-revenue*

²Of course, one need to check that the monotonicity constraint holds in the end.

distribution³, defined by the property of a unit-elastic demand: in the monopoly pricing problem, the monopoly’s revenue from charging any price in the support of this distribution is the same. Notably, there is a probability mass on the maximum valuation in an equal-revenue distribution. Under the joint distribution where the high bidder’s virtual value is 0 except when her valuation is \bar{v} , the two bidders’ valuations turn out to be *independent*. Notably, the low bidder’s virtual value also *equals* 0. The proposed regularity conditions generalize the special case: they guarantee that under the constructed joint distribution, the high bidder’s virtual value is 0 and the low bidder’s virtual value is *weakly negative*. Then if the proposed regularity conditions hold, the second-price auction with the uniformly distributed random reserve maximizes the expected revenue across all dominant-strategy mechanisms under the constructed joint distribution. Indeed, if the proposed regularity conditions hold, any dominant-strategy mechanism, in which 1) the ex-post participation constraints are binding for zero-valuation bidders and 2) the object is allocated with probability one to the high bidder with the valuation of \bar{v} and the object is never allocated to the low bidder, maximizes the expected revenue across all dominant-strategy mechanisms under the constructed joint distribution.

Notably, the proposed regularity conditions contain a probability mass condition on the maximum valuation. That is, the result requires that the marginal distribution have an atom on the maximum valuation and that the size of the atom be bounded from below. Indeed, these conditions capture many heavy-tailed distributions⁴, which are observed in many real-world auctions. For example, according to [Arnosti et al. \(2016\)](#), it has been observed that a huge fraction of the total value comes from a small number of very valuable impressions in online advertising. In addition, according to [Ibragimov and Walden \(2010\)](#), very diverse private valuations have been observed in markets for cultural and sport events as well as in those for antiques and collectibles and online auctions and marketplaces such as eBay and StubHub. Many papers have studied mechanism design problems when the distribution of values exhibits a heavy tail (e.g., [Arnosti et al. \(2016\)](#) and [Ibragimov and Walden \(2010\)](#)).

I extend the analysis to the case of general number of bidders ($N \geq 3$). For tractability, I restrict attention to a subclass of dominant-strategy mechanisms in which a bidder whose bid is not the highest is never allocated. A mechanism in this subclass is referred to as a *standard*⁵ dominant-strategy mechanism. The second main result (Theorem 2) is that, under certain regularity conditions on the marginal distribution, *the second-price auction with the*

³This distribution is identified as a buyer-optimal signal distribution in a monopoly selling problem by [Roesler and Szentes \(2017\)](#).

⁴I present a detailed discussion of these conditions in Section 4.1.

⁵The terminology of "standard" comes from [Bergemann et al. \(2019\)](#) who define standard mechanisms in a similar manner. [He and Li \(2022\)](#) also adopts this terminology.

\bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms. Under this mechanism, the cumulative distribution function of the random reserve r is $G(r) = (\frac{r}{\bar{v}})^{\frac{1}{N-1}}$ for $r \in [0, \bar{v}]$. Following a methodology similar to the two-bidder case, I show this result by constructing a worst-case correlation structure.

This mechanism embodies, albeit not the full-insurance property, a good hedging property: it yields the same expected revenue across a range of correlation structures. Precisely, as I will show, the expected revenue is the same for any conceivable joint distribution whose support lies in the set of valuation profiles in which either all bidders have the same valuations or there is a unique highest bidder and the other bidders have the same valuations. Indeed, the constructed worst-case correlation structure has such a support. Intuitively, given the restriction to standard mechanisms, only the highest bidders are possible to generate positive revenue to the auctioneer. Thus, the other bidders' valuations except for the highest one are "wasted". To reduce the expected revenue as much as possible while maintaining the consistency with the marginal distribution, a worst-case correlation structure maximizes the waste by increasing the other bidders' valuations as much as possible until all the other bidders' valuations are the same. Then similar to the two-bidder case, the worst-case correlation structure is obtained by requiring the highest bidder's virtual value be zero except when the highest bidder's valuation is \bar{v} . Here, the proposed regularity conditions on the marginal distribution guarantee that the construction is feasible. As I focus on standard dominant-strategy mechanisms, the bidders whose valuations are not the highest do not contribute to the expected revenue and therefore their virtual values do not matter. Then it is straightforward that the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve maximizes the expected revenue across standard dominant-strategy mechanisms under the constructed joint distribution.

Moreover, I show that the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is asymptotically optimal across all dominant-strategy mechanisms as the number of the bidders goes to infinity, regardless of the marginal distribution (Remark 5). To establish this, I show that the revenue guarantee of this mechanism converges to the expectation of a generic bidder's valuation. Indeed, the expectation of a generic bidder's valuation is an upper bound of the revenue guarantee for any dominant-strategy mechanism, as it is possible that the correlation structure is the maximally positively correlated distribution (that is, all bidders have the same valuations for any valuation profile in the support), under which the expectation of a generic bidder's valuation is the most surplus that the auctioneer can extract.

The first two main results both require the probability mass on the maximum valuation to be bounded from below. The third main result (Theorem 3) characterizes *the second-price*

auction with the s^* ⁶-scaled $\text{Beta}(\frac{1}{N-1}, 1)$ distributed random reserve as a maxmin mechanism across standard dominant-strategy mechanisms if the probability mass condition does not hold. Under this mechanism, the cumulative distribution function of the random reserve r is $G_{s^*}(r) = (\frac{r}{s^*})^{\frac{1}{N-1}}$ with support $[0, s^*]$ where $s^* \in (0, \bar{v})$. Notably, the highest bidder will be fully allocated the object provided that her valuation is higher than s^* .

Remarkably, a second-price auction (albeit with some random reserve), which is simple and widely adopted in practice, arises as a robustly optimal mechanism across some general class of mechanisms. Importantly, this is true for a wide range of marginal distributions. Therefore, the main results provide a positive explanation why the second-price auction is prevalent in the real world: it maximizes the worst-case expected revenue for a wide range of marginal distributions. Furthermore, the explanation is particularly convincing for the two-bidder case: the robust optimality is established across *all* dominant-strategy mechanisms.

In addition to the main results, I propose a family of second-price auctions with t -scaled $\text{Beta}(\frac{1}{N-1}, 1)$ distributed random reserves where $t \in (0, \bar{v})$. I identify a non-trivial lower bound of the revenue guarantee for each auction in this family. For any given marginal distribution, I am able to find an auction from this family that has a strictly higher revenue guarantee than that of any posted-price mechanism and any second-price auction with a non-negative deterministic reserve (Theorem 4 and Theorem 5).

The remainder of the introduction discusses related literature. Section 2 presents the model. Section 3 illustrates the methodology and conducts preliminary analysis. Section 4 characterizes the main results. Section 5 proposes a family of auctions and studies their performance in terms of the revenue guarantee. Section 6 is a conclusion. Omitted proofs are in Appendix A, B and C. Appendix D contains the result for the essential necessity of the regularity conditions.

1.1 Related Literature

The closest related paper is He and Li (2022), who study the design of auctions within the correlation-robust framework. They show, among others, that a second-price auction with a random reserve is a maxmin mechanism across standard dominant-strategy mechanisms under certain conditions on the marginal distribution. Methodologically, both papers use duality theory to proceed the analysis. The main differences can be summarized as follows. My setting is more general than theirs in that I allow the marginal distribution to have a probability mass on the maximum valuation⁷. More importantly, I obtain a new and strong result for the two-bidder case: I establish that, under certain regularity conditions

⁶ s^* is characterized by the known marginal distribution, details of which are given in Section 4.3.

⁷They assume continuous distributions and do not allow for a mass point on the maximum valuation.

(different from theirs), my proposed mechanism is a maxmin mechanism across *all* dominant-strategy mechanisms. In addition, I establish the main results under weaker conditions on the marginal distribution⁸.

This paper is closely related to Che (2020), Koçyiğit et al. (2020) and Zhang (2021). The first two papers both consider a model of auction design in which the auctioneer only knows the expectation of each bidder’s valuation. Specifically, Che (2020) shows that a second-price auction with an optimal random reserve is a maxmin mechanism within a class of competitive mechanisms⁹; Koçyiğit et al. (2020) characterize a maxmin mechanism across highest-bidder lotteries¹⁰ for the case where the known expectations are the same across bidders. Similarly, my paper also considers some general class of mechanisms. The main difference is that I assume that the auctioneer knows exactly the marginal distribution. That is, I assume that the auctioneer knows more and therefore the revenue guarantee in my setting is an upper bound of theirs. Zhang (2021) considers a model of bilateral trade in which the profit-maximizing intermediary only knows the expectations of each trader’s valuation. He characterizes maxmin trading mechanisms across all dominant-strategy mechanisms. The maxmin trading mechanism features fixed-commission fee, uniformly random spread and midpoint transaction price in the symmetric case. In contrast, this paper considers a model of auction design and assumes that the auctioneer knows exactly the marginal distribution. Like that paper, I consider all dominant-strategy mechanisms for the two-bidder case. In addition, this paper employs a similar methodology to proceed the analysis. More specifically, both papers use properties of “virtual value” to construct worst-case distributions.

This paper is also closely related to Bei et al. (2019), who study the design of auctions within the correlation-robust framework with a focus on *simple* mechanisms. They show, among others, that the revenue guarantee of the sequential posted-price mechanism is at least $\frac{1}{2\ln 4+2}$ times the revenue guarantee of the optimal dominant-strategy mechanism.

This paper is related to Bose et al. (2006) who study the design of auctions assuming that the bidders’ valuations are independently distributed but there may be ambiguity about the marginal distribution of a generic bidder’s valuation. In contrast, my paper assumes that the marginal distribution of a generic bidder’s valuation is known but the correlation structure of bidders’ valuations is unknown. Because of the different framework, the methodology of this paper differs significantly from that one. They show, among others, that an auction

⁸For valuations below the maximum valuation, both papers assume that the marginal distribution admits a density f . For the main results, I only require that $x^2f(x)$ be non-decreasing in x instead of that $xf(x)$ be non-decreasing in x , required in their paper.

⁹The class of dominant-strategy mechanisms is not a subset of the class of competitive mechanisms, and vice versa.

¹⁰This is the same as standard dominant-strategy mechanisms.

that “fully insures” the auctioneer is a maxmin mechanism when the auctioneer does not know the marginal distribution but the bidders know it. Similarly, in my framework, the first main result (Theorem 1) characterizes a maxmin mechanism exhibiting a full-insurance property. However, the notion of full-insurance is different from theirs. While full-insurance requires that the same *expected revenue* be obtained across all conceivable distributions in this paper, it requires that the same *ex-post revenue* be obtained across all valuation profiles in that one.

Broadly, this paper joins the robust mechanism design literature (Bergemann and Morris (2005)). There are other papers searching optimal solutions in the worst case over the space of parameters (e.g., Carroll and Meng (2016), Garrett (2014), Bergemann and Schlag (2011), Carroll (2017), Giannakopoulos et al. (2020), Chen et al. (2019)). Bergemann et al. (2016), Du (2018) and Brooks and Du (2020) consider a model of auction design with common values. They assume that bidders’ valuations for the object are drawn from a commonly known prior, but they may have arbitrary information (high-order beliefs) about the prior distribution unknown to the seller. An auction’s performance is measured by the worst expected revenue across a class of incomplete information correlated equilibria termed Bayes correlated equilibria (BCE) in Bergemann and Morris (2013). In this paper, I completely ignore the beliefs of bidders by focusing on the dominant-strategy mechanisms. This assumption is more appropriate for situations in which one not say much about bidders’ beliefs, as dominant-strategy mechanisms are robust to misspecification of bidders’ beliefs.

2 Preliminaries

2.1 Notation

I introduce the following technical notations. First, all spaces considered are polish spaces; I endow them with their Borel σ -algebra. Second, product spaces are endowed with product σ -algebra. Third, I use $\Delta(X)$ to denote the set of all probability measures over X .

2.2 Environment

I consider an environment where a single indivisible object is sold to $N \geq 2$ risk-neutral bidders. I denote by $I = \{1, 2, \dots, N\}$ the set of bidders. Each bidder i has private information about her valuation for the object, which is modeled as a random variable v_i with cumulative distribution function F_i ¹¹. Throughout the paper, I focus on symmetric environment, i.e.,

¹¹As will be discussed later, I allow distributions to have a probability mass on the maximum valuation. Furthermore, all results (with slight modifications) hold in discrete environments.

$F_i = F_j = F$ ¹² for any $i, j \in I$. I denote the support of F_i by V_i . I assume that $V_i = [0, \bar{v}]$ for some $\bar{v} > 0$. The joint support of all F_i is $V = \times_{i=1}^N V_i = [0, \bar{v}]^N$ with a typical valuation profile v . I denote bidder i 's opponents' valuation profiles by v_{-i} , i.e., $v_{-i} \in V_{-i} = \times_{j \neq i} V_j$.

The valuation profile v is drawn from a joint distribution \mathcal{P} , which may have an arbitrary correlation structure. The auctioneer only knows the marginal distribution F of each bidder's valuation but does not know how these bidders' valuations are correlated. To the auctioneer, any joint distribution is *conceivable* as long as the joint distribution is consistent with the marginal distribution. I denote by

$$\Pi(F) = \{\pi^{13} \in \Delta V : \forall i \in I, \forall A_i \subset V_i, \pi(A_i \times V_{-i}) = F(A_i)\}$$

the collection of conceivable joint distributions.

2.3 Marginal Distribution

I assume that the marginal distribution F admits a probability density function $f(x)$ for any $x \in [0, \bar{v})$. I allow F to have a probability mass on \bar{v} , the size of which is denoted by $Pr(\bar{v})$.¹⁴ Importantly, as will be seen, the first two main results (Theorem 1 and 2) require that the marginal distribution have an atom on \bar{v} and that the size of the atom be bounded from below. Indeed, distributions with a probability mass on the maximum valuation are familiar in many information design and robust mechanism design environments. [Roesler and Szentes \(2017\)](#) analyze a model where the buyer has a given value distribution and designs a signal structure to learn about her valuation. After observing the signal structure, the seller makes a take-it-or-leave-it offer. They show that an equal-revenue distribution is the buyer-optimal signal distribution. In a closely related work, [Condorelli and Szentes \(2020\)](#) analyze a model where the buyer can choose the probability distribution of her valuation for the good. After observing the buyer's choice of the distribution, the seller makes a take-it-or-leave-it offer. They show that an equal-revenue distribution is the buyer-optimal value distribution. Besides, [Bergemann and Schlag \(2008\)](#) consider a minimax regret design problem of selling a single object to a single buyer and find that an equal-revenue distribution is a worst-case distribution; [Zhang \(2022\)](#) considers a robust public good mechanism design problem and finds that the worst-case marginal distribution has a probability mass on the

¹²With slight abuse of notation, I also use F to denote the probability measure consistent with the distribution F .

¹³With slight abuse of notation, I also use π to denote the probability density of the probability measure π if the probability density exists.

¹⁴[He and Li \(2022\)](#) restrict attention to marginal distributions that admits a positive density function everywhere, whereas this paper allows a probability mass on the maximum valuation. That is, the assumption in this paper is more general.

maximum valuation.

Moreover, distributions with a probability mass on the maximum valuation admits an "approximating" interpretation as follows. Consider a marginal distribution \hat{F} with a non-negative density function \hat{f} everywhere on $[0, \infty)$. In the real world, it is reasonable to assume that bidders' valuations are bounded as the total wealth, which is physically impossible to be infinite, is an upper bound of bidders' valuations. Then the marginal distribution F with bounded support $[0, \bar{v}]$ is generated by truncating \hat{F} on \bar{v} as follows: $f(x) = \hat{f}(x)$ for $x < \bar{v}$ and $Pr(\bar{v}) = 1 - \hat{F}(\bar{v})$. When \bar{v} is large, F is a natural approximation of \hat{F} .

2.4 (Standard) Dominant-strategy Mechanisms

I focus on dominant-strategy mechanisms. The revelation principle holds and it is without loss of generality to restrict attention to direct mechanisms. A direct mechanism (q, t) is defined as an allocation rule $q : V \rightarrow [0, 1]^N$ and a payment function $t : V \rightarrow \mathbb{R}^N$. With slight abuse of notation, each bidder submits a sealed bid $v_i \in V_i$ to the auctioneer. Upon receiving the bids profile $v = (v_1, v_2, \dots, v_N)$, the allocation probabilities are $q(v) = (q_1(v), q_2(v), \dots, q_N(v))$ and the payments are $t(v) = (t_1(v), t_2(v), \dots, t_N(v))$ where $q(v) \geq 0$ and $\sum_i q_i(v) \leq 1$ for all $v \in V$. A direct mechanism is a dominant-strategy mechanism if for all $i \in I$, all $v \in V$, and all $v'_i \in V_i$,

$$v_i q_i(v) - t_i(v) \geq v_i q_i(v'_i, v_{-i}) - t_i(v'_i, v_{-i}),$$

$$v_i q_i(v) - t_i(v) \geq 0.$$

The set of all such mechanisms is denoted by \mathcal{D} . I say a direct mechanism (q, t) is *standard* if for any $v \in V$ and $i \in I$ such that $v_i < \max_{j \in I} v_j$, the allocation to the bidder i is $q_i(v) = 0$. That is, only the highest bidders are possible to be allocated in a standard mechanism. The set of all standard dominant-strategy mechanisms is denoted by \mathcal{E} .

2.5 Objective Function

I am interested in the auctioneer's expected revenue in the dominant-strategy equilibrium in which each bidder truthfully reports her valuation of the object. Then the expected revenue of a dominant-strategy mechanism (q, t) when the joint distribution is π is $U((q, t), \pi) = \int_{v \in V} \sum_{i=1}^N t_i(v) d\pi(v)$. The auctioneer evaluates a mechanism (q, t) by its worst-case expected revenue, referred to as the *revenue guarantee* of the mechanism (q, t) , over all conceivable joint distributions. Formally, the mechanism (q, t) 's revenue guarantee is $REG((q, t)) = \inf_{\pi \in \Pi(F)} U((q, t), \pi)$. The auctioneer's goal is to find a *maxmin mechanism*

from either \mathcal{D} or \mathcal{E} with the maximal revenue guarantee¹⁵. Formally, the auctioneer solves

$$\sup_{(q,t) \in \mathcal{D}(\text{or } \mathcal{E})} REG((q, t)). \quad (\text{MRG})$$

3 Methodology and Preliminary Analysis

The maxmin optimization problem (MRG) can be interpreted as a two-player sequential zero-sum game. The two players are the auctioneer and adversarial nature. The auctioneer first chooses a mechanism $(q, t) \in \mathcal{D}$ (or $(q, t) \in \mathcal{E}$). After observing the auctioneer's choice of the mechanism, adversarial nature chooses a conceivable joint distribution $\pi \in \Pi(F)$. The auctioneer's payoff is $U((q, t), \pi)$, and adversarial nature's payoff is $-U((q, t), \pi)$. One can also consider the simultaneous-move version of this zero-sum game, whose Nash equilibrium is indeed a *saddle point* of the payoff functional U , i.e., for any $(q, t) \in \mathcal{D}$ (or $(q, t) \in \mathcal{E}$) and any $\pi \in \Pi(F)$,

$$U((q^*, t^*), \pi) \geq U((q^*, t^*), \pi^*) \geq U((q, t), \pi^*).$$

The first inequality says the joint distribution π^* minimizes the expected revenue under the mechanism (q^*, t^*) , and the second inequality implies that, under the joint distribution π^* , the other dominant-strategy mechanisms cannot attain a strictly higher expected revenue. Hence, the auctioneer's equilibrium strategy in the simultaneous-move version of this zero-sum game, (q^*, t^*) , is a maxmin mechanism. π^* is referred to as a *worst-case correlation structure*. I will construct a saddle point for each of the main results.

Proposition 1 (Revenue Equivalence). *When searching for a maxmin mechanism, it is without loss to restrict attention to mechanisms satisfying the following properties: 1) $q_i(\cdot, v_{-i})$ is non-decreasing in v_i for all v_{-i} and 2) $t_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(x, v_{-i}) dx$.*

Proof. The proof is in Appendix A. □

Proposition 1 simplifies the analysis by establishing two properties of a maxmin mechanism. The first property says the allocation rule is *monotone*, and the second property says that the payment rule can be characterized by the allocation rule and that the ex-post participation constraints are binding for zero-valuation bidders. This is standard in the mechanism design literature (e.g., Myerson (1981)).

Moreover, Proposition 1 allows me to obtain a virtual representation of the expected revenue, which is essential for my analysis. Precisely, consider the problem that fixing

¹⁵As will be seen later, the first main result (Theorem 1) characterizes a maxmin mechanism from \mathcal{D} , and either of the other main results (Theorem 2 and Theorem 3) characterizes a maxmin mechanism from \mathcal{E} .

an arbitrary joint distribution π , the auctioneer designs an optimal mechanism (q, t) . For exposition, I assume that π admits a positive density function¹⁶. The density function of v_i conditional on v_{-i} is denoted by $\pi_i(v_i|v_{-i})$, and the cumulative distribution function of v_i conditional on v_{-i} is denoted by $\Pi_i(v_i|v_{-i})$. Then an direct implication of Proposition 1 is that the expected revenue of (q, t) under the joint distribution π is

$$E\left[\sum_{i=1}^N t_i(v)\right] = E\left[\sum_{i=1}^N q_i(v)\phi_i(v)\right],$$

where $\phi_i(v) = v_i - \frac{1 - \Pi_i(v_i|v_{-i})}{\pi_i(v_i|v_{-i})}$ is the *virtual value* of bidder i when the valuation profile is v . Thus the problem of designing an optimal mechanism given a joint distribution is equivalent to maximizing the *expected total virtual surplus*, which refers to the expected sum of the allocation times the virtual value, subject to that the allocation rule is monotone.

4 Main Results

In Section 4.1, I characterize a maxmin mechanism across all dominant-strategy mechanisms under certain regularity conditions for the two-bidder case. In Section 4.2, I characterize a maxmin mechanism across standard dominant-strategy mechanisms under certain regularity conditions for the N -bidder ($N \geq 3$) case. In Section 4.3, I characterize a maxmin mechanism across standard dominant-strategy mechanisms condition for the N -bidder ($N \geq 2$) case when the probability mass condition (part of the regularity conditions) fails.

4.1 Two Bidders

I first define a mechanism and a joint distribution. Then I define regularity conditions under which the mechanism and the joint distribution form a saddle point (Theorem 1). Then I illustrate Theorem 1. Finally, I give a discussion of the regularity conditions.

The second-price auction with the uniformly distributed random reserve is defined as follows. The auctioneer first draws a random reserve r from the uniform distribution with support $[0, \bar{v}]$. Then the two bidders bid simultaneously. The high bidder (the bidder with a higher bid than that of her opponent) wins the object if her bid is also higher than r , and she pays the maximum of r and her opponent's bid; the low bidder loses the auction and pays nothing. In case of ties, each bidder wins the object with a half probability if the bid

¹⁶The virtual representation can be similarly derived for joint distributions in which there is a probability mass on $\underbrace{(1, \dots, 1)}_N$.

is higher than r , and the winner pays the bid.

Equivalently, it can be defined by (q^*, t^*) as follows. If $v_1 > v_2$, then $q_1^*(v_1, v_2) = \frac{v_1}{\bar{v}}$, $q_2^*(v_1, v_2) = 0$ and $t_1^*(v_1, v_2) = \frac{v_1^2 + v_2^2}{2\bar{v}}$, $t_2^*(v_1, v_2) = 0$; if $v_1 < v_2$, then $q_1^*(v_1, v_2) = 0$, $q_2^*(v_1, v_2) = \frac{v_2}{\bar{v}}$ and $t_1^*(v_1, v_2) = 0$, $t_2^*(v_1, v_2) = \frac{v_1^2 + v_2^2}{2\bar{v}}$; if $v_1 = v_2 = x$, then $q_1^*(v_1, v_2) = q_2^*(v_1, v_2) = \frac{x}{2\bar{v}}$ and $t_1^*(v_1, v_2) = t_2^*(v_1, v_2) = \frac{x^2}{2\bar{v}}$.

The joint distribution π^* is defined as follows¹⁷.

$$\pi^*(v_1, v_2) = \pi^*(v_2, v_1) = \begin{cases} f(0) & \text{if } v_1 = v_2 = 0; \\ 0 & \text{if } v_1 > v_2 = 0; \\ \frac{1}{v_1^2} (v_2 f(v_2) - \frac{\int_0^{v_2} x^2 f(x) dx}{v_2^2}) & \text{if } \bar{v} > v_1 \geq v_2 > 0; \\ \frac{1}{\bar{v}} (v_2 f(v_2) - \frac{\int_0^{v_2} x^2 f(x) dx}{v_2^2}) & \text{if } \bar{v} = v_1 > v_2 > 0. \end{cases}$$

$$Pr^*(\bar{v}, \bar{v}) = Pr(\bar{v}) - \frac{\int_{x \in (0, \bar{v})} x^2 f(x) dx}{\bar{v}^2}.$$

The two-bidder robust regularity conditions are defined as follows: $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$ and $Pr(\bar{v}) \geq \frac{\int_{x \in (0, \bar{v})} x^2 f(x) dx}{\bar{v}^2}$. I refer to the second part of the conditions as the probability mass condition.

Remark 1. Note that the probability mass condition will vanish as $\bar{v} \rightarrow \infty$ if $\int_{x \in (0, \bar{v})} x^2 f(x) dx$ is of order \bar{v}^γ with $\gamma < 2$.

Theorem 1. For the two-bidder case, the second-price auction with the uniformly distributed random reserve is a maxmin mechanism across all dominant-strategy mechanisms if the two-bidder robust regularity conditions hold. The revenue guarantee is $\frac{E[X^2]}{\bar{v}}$.¹⁸ The joint distribution π^* is a worst-case correlation structure.

Now I illustrate Theorem 1. I start with the illustration of the mechanism.

Definition 1. I say a dominant-strategy mechanism (q, t) exhibits the *full-insurance property* if the expected revenue of (q, t) is the same across all conceivable joint distributions.

¹⁷Here $\pi^*(v_1, v_2)$ denotes the density of the valuation profile (v_1, v_2) whenever the density exists and $Pr^*(v_1, v_2)$ denotes the probability mass of the valuation profile (v_1, v_2) whenever there is some probability mass on (v_1, v_2) . The marginal distributions that the result covers have a probability mass on the maximum valuation \bar{v} . In the joint distribution π^* , there is (non-negative) probability mass on the point (\bar{v}, \bar{v}) .

¹⁸The distribution of X is F .

Proposition 2. *For the two-bidder case, the second-price auction with the uniformly distributed random reserve exhibits the full-insurance property.*

Proof. Note that under the second-price auction with the uniformly distributed random reserve, the total revenue from a valuation profile (v_1, v_2) is $t^*(v_1, v_2) = t_1^*(v_1, v_2) + t_2^*(v_1, v_2) = \frac{v_1^2 + v_2^2}{2\bar{v}}$. Then fix any conceivable joint distribution π , the expected revenue is

$$\begin{aligned} \int_{[0, \bar{v}]^2} t^*(v_1, v_2) d\pi(v_1, v_2) &= \int_{[0, \bar{v}]^2} \frac{v_1^2 + v_2^2}{2\bar{v}} d\pi(v_1, v_2) \\ &= \int_{[0, \bar{v}]^2} \frac{v_1^2}{2\bar{v}} d\pi(v_1, v_2) + \int_{[0, \bar{v}]^2} \frac{v_2^2}{2\bar{v}} d\pi(v_1, v_2) \\ &= \int_{[0, \bar{v}]} \frac{v_1^2}{2\bar{v}} dF(v_1) + \int_{[0, \bar{v}]} \frac{v_2^2}{2\bar{v}} dF(v_2) \\ &= \frac{E[X^2]}{\bar{v}}. \end{aligned}$$

□

The joint distribution π^* is obtained by a condition requiring that the high bidder's virtual value be 0 except when her valuation is \bar{v} . Formally,

$$\phi_i^*(v_i, v_j) = 0 \quad \text{if } v_j \leq v_i < \bar{v}. \quad (1)$$

The property (1) is motivated by a property of the second-price auction with the uniformly distributed random reserve: the auctioneer is indifferent between allocating and not allocating the object to the high bidder as long as her valuation is not \bar{v} .

Furthermore, I impose a condition on the constructed joint distribution π^* that the virtual value of the low bidder is weakly smaller than that of the high bidder. Formally,

$$\phi_j^*(v_i, v_j) \leq \phi_i^*(v_i, v_j) \quad \text{if } v_j \leq v_i. \quad (2)$$

The property (2) is motivated by another property of the second-price auction with the uniformly distributed random reserve: the low bidder is never allocated the object.

Indeed, if the property (1) and the property (2) hold for a joint distribution, it is straightforward that the second-price auction with the uniformly distributed random reserve maximizes the expected revenue across all dominant-strategy mechanisms given this joint distribution. The two-bidder robust regularity conditions, as I will show, guarantee that the property (1) and the property (2) hold for the constructed joint distribution π^* . In summary, I obtain a virtual value matrix for π^* as follows if the two-bidder robust regularity conditions

hold.

$$\begin{pmatrix} (0,0)_{0,0} & (0,0)_{0,>0} & \cdots & \cdots & (0,0)_{0,<\bar{v}} & (0,0)_{0,\bar{v}} \\ (0,0)_{>0,0} & (0,0)_{v_1=v_2>0} & (-,0)_{v_1<v_2} & \cdots & (-,0)_{v_1<v_2<1} & (\leq,+)<\bar{v},\bar{v} \\ \vdots & (0,-)_{v_1>v_2} & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ (0,0)_{<\bar{v},0} & (0,-)_{<\bar{v},>0} & \cdots & \cdots & (0,0)_{v_1=v_2<\bar{v}} & (\leq,+)<\bar{v},\bar{v} \\ (0,0)_{\bar{v},0} & (+,\leq)_{\bar{v},>0} & \cdots & \cdots & (+,\leq)_{\bar{v},<\bar{v}} & (+,+)_{\bar{v},\bar{v}} \end{pmatrix}$$

Here “0” in the bracket means zero virtual value, “−” means a non-positive virtual value, “+” means a non-negative virtual value, “≤” means the virtual value of the bidder is weakly smaller than that of her opponent. The subscript denotes the corresponding valuation profile.

Remark 2. The probability mass condition arises because the property (1) requires that the conditional distribution of the high bidder’s valuation be an equal-revenue distribution, which has an atom on the maximum valuation \bar{v} .

Proposition 3. *If the two-bidder robust regularity conditions hold, the second-price auction with the uniformly distributed random reserve maximizes the expected revenue across all dominant-strategy mechanism under the joint distribution π^* .*

Proof. The proof is in Appendix B which presents details about the construction of π^* . \square

Theorem 1 follows immediately from Proposition 2 and Proposition 3.

There is an important special case where Theorem 1 applies: the equal-revenue distribution. Recall that the equal-revenue distribution is familiar in the information design literature and robust mechanism design literature (e.g., Roesler and Szentes (2017), Bergemann and Schlag (2008), Du (2018), etc).

Corollary 1. *For the two-bidder case, if the marginal distribution is an equal-revenue distribution with $\alpha \in (0, \bar{v})$:*

$$F(x) = \begin{cases} 1 - \frac{\alpha}{x} & \text{if } \alpha \leq x < \bar{v}; \\ 1 & \text{if } x = \bar{v}, \end{cases}$$

then the second-price auction with the uniformly distributed random reserve is a maxmin mechanism across all dominant-strategy mechanism. The revenue guarantee is $2\alpha - \frac{\alpha^2}{\bar{v}}$. The independent equal-revenue distribution¹⁹ is a worst-case correlation structure.

¹⁹That is, the marginal distribution of each bidder’s valuation is the known equal-revenue distribution; bidders’ valuations are independently distributed.

Proof. It is straightforward to show that an equal-revenue distribution satisfies the two-bidder robust regularity conditions. Then Theorem 1 implies Corollary 1. \square

There are many other distributions satisfying the two-bidder robust regularity conditions. I now provide some examples.

Example 1. Any (truncated) Pareto distribution with $\alpha \in (0, \bar{v}), \beta \in (0, 1)$:

$$F(x) = \begin{cases} 1 - \frac{\alpha^\beta}{x^\beta} & \text{if } \alpha \leq x < \bar{v}; \\ 1 & \text{if } x = \bar{v}. \end{cases}$$

To see this, note that $x^2 f(x) = \alpha^\beta \beta x^{1-\beta}$ is non-decreasing when $\beta \in (0, 1)$. For the probability mass condition, note that $Pr(\bar{v}) = (\frac{\alpha}{\bar{v}})^\beta \geq (\frac{\alpha}{\bar{v}})^\beta \frac{\beta}{2-\beta} [1 - (\frac{\alpha}{\bar{v}})^{2-\beta}] = \frac{\int_{(0, \bar{v})} x^2 f(x) dx}{\bar{v}^2}$ when $\beta \in (0, 1)$.

Example 2. A combination of an uniform distribution on $[0, \bar{v})$ and a probability mass on \bar{v} with $Pr(\bar{v}) \geq \frac{1}{4}$.

To see this, note that the first part of the conditions holds trivially because it is uniformly distributed on $[0, \bar{v})$. For the probability mass condition, note that $Pr(\bar{v}) \geq \frac{\int_{(0, \bar{v})} x^2 \cdot \frac{1-Pr(\bar{v})}{\bar{v}} dx}{\bar{v}^2} = \frac{\int_{(0, \bar{v})} x^2 f(x) dx}{\bar{v}^2}$ when $Pr(\bar{v}) \geq \frac{1}{4}$.

As a final topic of this section, I discuss the two-bidder robust regularity conditions. Using the approximation interpretation in Section 2.3, F is obtained via a truncation of \hat{F} on \bar{v} . Now consider the following regularity conditions for \hat{F} : $x^2 \hat{f}(x)$ is non-decreasing on $[0, \infty)$ and $\frac{\int_0^s x^2 \hat{f}(x) dx}{s^2} \rightarrow 0$ as $s \rightarrow \infty$. These conditions imply that the two-bidder robust regularity conditions hold for any $\bar{v} > 0$.²⁰ It is straightforward to verify that these conditions hold for many heavy-tailed distributions including a family of power law distributions²¹, Cauchy distributions²², log-Cauchy distributions²³, Lévy distributions²⁴, etc. Thus, when \bar{v} is large,

²⁰Indeed, that $x^2 \hat{f}(x)$ is non-decreasing implies that the function $K(s) := \hat{F}(s) + \frac{\int_0^s x^2 \hat{f}(x) dx}{s^2}$ is non-decreasing. To see this, note that $K'(s) = 2[\hat{f}(s) - \frac{\int_0^s x^2 \hat{f}(x) dx}{s^3}] \geq 0$, shown in (B.9). Then $\hat{F}(\bar{v}) + \frac{\int_0^{\bar{v}} x^2 \hat{f}(x) dx}{\bar{v}^2} \leq 1$ for any $\bar{v} > 0$ if $\frac{\int_0^s x^2 \hat{f}(x) dx}{s^2} \rightarrow 0$ as $s \rightarrow \infty$.

²¹The power law distribution is given by $F(x) = 1 - \frac{\alpha^\beta}{x^\beta}$ with $\alpha > 0$ for $x \in [\alpha, \infty)$. The parameter β determines the weight of the tail. These conditions hold for any power law distribution with $\beta \in (0, 1]$.

²²The density function of the Cauchy distribution is given by $f(x) = \frac{2b}{\pi(b^2+x^2)}$ with $b > 0$ for $x \in [0, \infty)$. It is straightforward to show that these conditions hold for Cauchy distributions with any $b > 0$.

²³The density function of the log-Cauchy distribution is given by $f(x) = \frac{1}{\pi x} [\frac{\sigma}{(\ln x - \mu)^2 + \sigma^2}]$ with $\sigma > 0$ for $x \in (0, \infty)$. It is straightforward to show that these conditions hold for log-Cauchy distributions with any $\sigma \geq 1$ and any real number μ .

²⁴The density function of the Lévy distribution is given by $f(x) = \sqrt{\frac{c}{2\pi}} \frac{e^{-\frac{c}{2x}}}{x^{\frac{3}{2}}}$ with $c > 0$ for $x \in (0, \infty)$. It is straightforward to show that these conditions hold for Lévy distributions with any $c > 0$.

the two-bidder robust regularity conditions hold for a distribution that is an approximation of some heavy-tailed distribution.

4.2 N Bidders

The structure of this section is similar to Section 4.2: I first define a mechanism and a joint distribution. Then I define regularity conditions under which the mechanism and the joint distribution form a saddle point (Theorem 2). Then I illustrate Theorem 2.

The second-price auction with \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is defined as follows. The auctioneer first draws a random reserve r from the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distribution. That is, the cumulative distribution function of the random reserve r is $G(r) = (\frac{r}{\bar{v}})^{\frac{1}{N-1}}$ with support $[0, \bar{v}]$. Then the N bidders bid simultaneously. The highest bidder wins the object if her bid is also higher than r , and she pays the maximum of r and the second highest bid; a bidder whose bid is not the highest loses the auction and pays nothing. In case of ties, each bidder wins the object with an equal probability if the bid is higher than r , and the winner pays the bid.

Equivalently, it can be defined by (q^*, t^*) as follows. I denote the highest valuation in a valuation profile v by $v(1)$, and the second highest valuation (if any) by $v(2)$. If $\#\{k : v_k = v(1)\} = 1$, then $q_i^{**}(v) = (\frac{v(1)}{\bar{v}})^{\frac{1}{N-1}}$, $q_j^{**}(v) = 0$ and $t_i^{**}(v) = \frac{v(1)^{\frac{N}{N-1}} + (N-1)v(2)^{\frac{N}{N-1}}}{N\bar{v}^{\frac{1}{N-1}}}$, $t_j^{**}(v) = 0$ for $i \in \{k : v_k = v(1)\}$ and $j \notin \{k : v_k = v(1)\}$; if $\#\{k : v_k = v(1)\} = K \geq 2$, then $q_i^{**}(v) = \frac{1}{K}(\frac{v(1)}{\bar{v}})^{\frac{1}{N-1}}$, $q_j^{**}(v) = 0$ and $t_i^{**}(v) = \frac{v(1)^{\frac{N}{N-1}}}{K\bar{v}^{\frac{1}{N-1}}}$, $t_j^{**}(v) = 0$ for $i \in \{k : v_k = v(1)\}$ and $j \notin \{k : v_k = v(1)\}$.

The joint distribution π^{**} is symmetric and is defined as follows²⁵. The support of π^{**} is $V^+ := \{v \in V | v_i = v(1) \text{ for any } i \text{ or } \exists i \text{ s.t. } v_i = v(1) > v_j = v(2) \text{ for any } j \neq i\}$. That is, $v \in V^+$ if either all bidders have the same valuations or there is a unique highest bidder and all of the remaining bidders have the same valuations. If $v \notin V^+$, then $\pi^{**}(v) = 0$. If

²⁵Here $\pi^{**}(v)$ denote the density of the valuation profile v whenever the density exists and $Pr^{**}(v)$ denote the probability mass of the valuation profile v whenever there is some probability mass on v . The marginal distributions that this result covers have a probability mass on the maximum valuation \bar{v} . In the joint distribution π^{**} , there is a (non-negative) probability mass on the point $(\underbrace{\bar{v}, \dots, \bar{v}}_N)$.

$v \in V^+$, then

$$\pi^{**}(v_i, v_{-i}) = \begin{cases} f(0) & \text{if } v = (0, \dots, 0); \\ 0 & \text{if } 0 = v_j < v_i, \forall j \neq i; \\ \frac{1}{(N-1)v(1)^2}(v(2)f(v(2))) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1} \int_0^{v(2)} x^{\frac{N}{N-1}} f(x)dx & \text{if } 0 < v(2) = v_j \leq v_i = v(1) < \bar{v}, \forall j \neq i; \\ \frac{1}{(N-1)\bar{v}}(v(2)f(v(2))) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1} \int_0^{v(2)} x^{\frac{N}{N-1}} f(x)dx & \text{if } 0 < v(2) = v_j < v_i = \bar{v}, \forall j \neq i. \end{cases}$$

$$Pr^{**}(\underbrace{\bar{v}, \dots, \bar{v}}_N) = Pr(\bar{v}) - \frac{\int_{(0, \bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}.$$

The N -bidder robust regularity conditions (I) are defined as follows: $f(x) \geq \frac{x^{-\frac{2N-1}{N-1}}}{N-1} \int_0^x s^{\frac{N}{N-1}} f(s)ds$ for $x \in (0, \bar{v})$ and $Pr(\bar{v}) \geq \frac{\int_{(0, \bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$ ²⁶. The N -bidder robust regularity conditions (II) are defined as follows: $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$ and $Pr(\bar{v}) \geq \frac{\int_{(0, \bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$.

Remark 3. Here the probability mass condition will vanish as the number of the bidders goes to infinity. To see this, note that $\frac{\int_{(0, \bar{v})} x^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}} \leq \frac{\int_{(0, \bar{v})} \bar{v}^{\frac{N}{N-1}} f(x)dx}{(N-1)\bar{v}^{\frac{N}{N-1}}} \leq \frac{1}{N-1} \rightarrow 0$ as $N \rightarrow \infty$. Therefore, the probability mass condition is non-restrictive when the number of bidders is large.

Theorem 2. For the N -bidder case ($N \geq 3$), the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms if the N -bidder robust regularity conditions (I) hold. The revenue guarantee is $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$. The joint distribution π^{**} is a worst-case correlation structure. In addition, the N -bidder robust regularity conditions (II) imply the N -bidder robust regularity conditions (I).

Remark 4. It is straightforward that the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve converges to the second-price auction without a reserve as the number of bidders goes to infinity. The asymptotic behaviour of the random reserve is consistent with the empirical finding that reserve prices are substantially lower than

²⁶With slight abuse of notation, here the condition for $Pr(\bar{v})$ is also referred to as the probability mass condition.

the optimal ones under the estimated distribution of values (e.g., Paarsch (1997), McAfee et al. (2002), Haile and Tamer (2003) and Bajari and Hortacısu (2003)).

Definition 2. Given a marginal distribution F , suppose the revenue guarantee of a maxmin mechanism across all dominant-strategy mechanisms is $Opt_N(F)$ for each N -bidder case. Consider a dominant-strategy mechanism M_N for each N -bidder case. Suppose the revenue guarantee of M_N is $Reg_N(F)$. I say M_N is *asymptotically optimal* across all dominant-strategy mechanisms given the marginal distribution F if $Opt_N(F) - Reg_N(F) \rightarrow 0$ as $N \rightarrow \infty$.

Remark 5. The second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is asymptotically optimal across all dominant-strategy mechanisms, regardless of the marginal distribution. Furthermore, the rate of convergence is $O(\frac{1}{N})$ ²⁷. To see these, recall that $E[X]$ is an upper bound of the revenue guarantee for any dominant-strategy mechanism. This is because it is always possible that adversarial nature chooses the maximally positively correlated distribution. But by the Dominated Convergence Theorem, I have that

$$\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}} \rightarrow E[X]$$

as $N \rightarrow \infty$. Furthermore, let $j(x) := x - \frac{x^{\frac{N}{N-1}}}{\bar{v}^{\frac{1}{N-1}}}$. Because $j'(x) = 1 - \frac{Nx^{\frac{1}{N-1}}}{(N-1)\bar{v}^{\frac{1}{N-1}}}$ and $j''(x) = -\frac{Nx^{\frac{2-N}{N-1}}}{(N-1)^2\bar{v}^{\frac{1}{N-1}}} \leq 0$, $j(x)$ is maximized at $x = (\frac{N-1}{N})^{N-1}\bar{v}$ and the maximized value is $(\frac{N-1}{N})^{N-1} \cdot \frac{\bar{v}}{N}$ by simple calculation. Then I have that $E[X] - \frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}} \leq (\frac{N-1}{N})^{N-1} \cdot \frac{\bar{v}}{N} \leq \frac{\bar{v}}{N}$. Therefore the rate of convergence is $O(\frac{1}{N})$.

Now I illustrate Theorem 2. I start with the illustration of the mechanism.

This mechanism exhibits a robust property in the following sense: for any conceivable joint distribution whose support is V^+ , the expected revenue under this mechanism is the same. To see this, note that the total revenue from a valuation $v \in V^+$ is $t^{**}(v) = \sum_{i=1}^N t_i^{**}(v) = \sum_{i=1}^N \frac{v_i^{\frac{N}{N-1}}}{N\bar{v}^{\frac{1}{N-1}}}$. Therefore, the expected revenue is $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$ for any conceivable joint distribution whose support is V^+ . I will use the linear programming duality theorem to show that such a joint distribution indeed minimizes the expected revenue across all conceivable joint distributions.

Proposition 4. For the N -bidder case ($N \geq 3$), under the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve, any conceivable joint distribution whose

²⁷In addition, this rate of convergence is the fastest across all standard dominant-strategy mechanisms, as is shown in He and Li (2020).

support is V^+ minimizes the expected revenue across all conceivable joint distributions. The minimized expected revenue is $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$.

Proof. The proof is in Appendix B which presents the details about the construction of the mechanism. \square

Proposition 4 implies that the constructed joint distribution π^{**} minimizes the expected revenue across all conceivable joint distributions under the mechanism, as the support of the constructed joint distribution π^{**} is V^+ .

The joint distribution π^{**} is then obtained by a condition requiring that the highest bidder's virtual value be 0 except when her valuation is \bar{v} . Formally,

$$\phi_i^{**}(v_i, v_{-i}) = 0 \quad \text{if } v \in V^+, \max_{j \neq i} v_j \leq v_i < \bar{v}. \quad (1')$$

If the property (1') holds for a joint distribution, then it is straightforward that the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve maximizes the expected revenue across all standard dominant-strategy mechanisms given this joint distribution. Note that I do not impose any condition on lower bidders' virtual values. This is because I restrict attention to standard dominant-strategy mechanisms, and then a bidder whose bid is not the highest is not allocated the object and pays nothing. The N -bidder robust regularity conditions (I) guarantee that the property (1') holds for the constructed joint distribution π^{**} . The N -bidder robust regularity conditions (II) are simpler conditions, and, as I will show, they imply the N -bidder robust regularity conditions (I).

Proposition 5. *For the N -bidder case ($N \geq 3$), if the N -bidder robust regularity conditions (I) hold, then the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve maximizes the expected revenue across all standard dominant-strategy mechanisms under the joint distribution π^{**} . In addition, the N -bidder robust regularity conditions (II) imply the N -bidder robust regularity conditions (I).*

Proof. The proof is in Appendix B which presents details about the construction of the joint distribution π^{**} . \square

Theorem 2 follows immediately from Proposition 4 and Proposition 5.

Now I present the result for the special case: the equal-revenue distribution. In contrast to the two-bidder case, bidders' valuations are not independently distributed in the worst-case correlation structure for the N -bidders case ($N \geq 3$).

Corollary 2. For the N -bidders case ($N \geq 3$), if the marginal distribution is an equal-revenue distribution ($\alpha \in (0, \bar{v})$):

$$F(v) = \begin{cases} 1 - \frac{\alpha}{v} & \text{if } \alpha \leq v < \bar{v}; \\ 1 & \text{if } v = \bar{v}, \end{cases}$$

then the second-price auction with the \bar{v} -scaled $\text{Beta}(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms. The revenue guarantee is $N\alpha - \frac{(N-1)\alpha^{\frac{N}{N-1}}}{\bar{v}^{\frac{1}{N-1}}}$. The worst-case correlation structure is symmetric and is defined as follows.

$$\pi(v_i, v_{-i}) = \begin{cases} \frac{\left(\frac{v(2)}{\alpha}\right)^{-\frac{N}{N-1}}}{(N-1)v(1)^2} & \text{if } \alpha \leq v(2) = v_j \leq v_i = v(1) < \bar{v}, \forall j \neq i; \\ \frac{\left(\frac{v(2)}{\alpha}\right)^{-\frac{N}{N-1}}}{(N-1)\bar{v}} & \text{if } \alpha \leq v_j = v(2) < v_i = \bar{v}, \forall j \neq i; \\ 0 & \text{if otherwise.} \end{cases}$$

$$Pr(\underbrace{\bar{v}, \dots, \bar{v}}_N) = \left(\frac{\alpha}{\bar{v}}\right)^{\frac{N}{N-1}}.$$

Proof. It is straightforward to show that an equal-revenue distribution satisfies the N -bidder robust regularity conditions (I) (and (II)). Then Theorem 2 implies Corollary 2. \square

4.3 When Probability Mass Condition Fails

When the probability mass condition fails, I characterize a maxmin mechanism across standard dominant-strategy mechanisms.

Theorem 3. For the N -bidder case ($N \geq 2$), if $x^2 f(x)$ is non-decreasing for $x \in (0, \infty)$ and $Pr(\bar{v}) < \frac{\int_{(0, \bar{v})} x^{\frac{N}{N-1}} f(x) dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$, then the second-price auction with the s^* -scaled $\text{Beta}(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms where $s^* \in (0, \bar{v})$ is a solution to

$$\frac{\int_{(0, s)} x^{\frac{N}{N-1}} dF(x)}{s^{\frac{N}{N-1}}} = (N-1)(1 - F(s)). \quad (\text{SD})$$

In addition, the revenue guarantee is $Ns^*(1 - F(s^*))$.

Proof. The proof is in Appendix B. □

Remark 6. This result generalizes and strengthens Theorem 3 in He and Li (2022). First, the conditions in this result allow for a probability mass on the maximum valuation. Second, the condition for $x < \bar{v}$ is weaker than that $xf(x)$ is non-decreasing²⁸, required in their result.

Under this mechanism, the cumulative distribution function of the random reserve r is $G_{s^*}(r) = (\frac{r}{s^*})^{\frac{1}{N-1}}$ with support $[0, s^*]$. Therefore, the highest bidder will be allocated the object with a probability less than one if her valuation is less than s^* , and will be allocated the object with probability one if her valuation is weakly higher than s^* . I follow the saddle point approach to show Theorem 3. Notably, the constructed worst-case correlation structure exhibits the property that the highest bidder’s virtual value is 0 when her valuation is less than s^* , and is weakly positive when her valuation is weakly higher than s^* . I relegate the details to Appendix B.

5 Robust Dominance

I have shown that the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism under certain regularity conditions. Then what if the regularity conditions do not hold? How does this mechanism perform? As a first topic of this section, I compare the performance of this mechanism with that of the posted-price mechanism, which is a dominant-strategy mechanism commonly used in practice. For exposition, I assume that the marginal distribution admits a positive density function everywhere on $[0, \bar{v}]$ in this section. I denote the set of all such distributions by $\Delta^c[0, \bar{v}]$.

Definition 3. I say a mechanism M dominates a family of mechanisms \mathcal{M} for a set of marginal distributions if for any marginal distribution in this set, the revenue guarantee of M is strictly greater than that of any mechanism in the family \mathcal{M} .

Definition 4. I say a family of mechanisms \mathcal{M}_1 universally dominates another family of mechanisms \mathcal{M}_2 if for any $F \in \Delta^c[0, \bar{v}]$, there exists a mechanism in \mathcal{M}_1 with a revenue guarantee strictly greater than that of any mechanism in \mathcal{M}_2 .

Proposition 6. For any $N \geq 2$, the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve dominates the family of posted-price mechanisms for the set $\mathcal{H} = \{F \in \Delta^c[0, \bar{v}] \mid \text{the revenue function } x \cdot (1 - F(x)) \text{ is strictly concave}\}$.

²⁸It is straightforward that $xf(x)$ is non-decreasing implies that $x^2f(x)$ is non-decreasing, but not vice versa.

Proof. The proof is in Appendix C. □

Motivated by the main idea embedded in the construction, I propose a family of second-price auctions with t -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserves where $t \in (0, \bar{v})$, denoted by $\mathcal{M}_{SP-\beta}$. As a second topic of this section, I study the performance of the proposed family of auctions. Formally, the cumulative distribution function of the random reserve r in this family is $G_t(r) = (\frac{r}{t})^{\frac{1}{N-1}}$ with support $[0, t]$ for some $t \in (0, \bar{v})$. Under such a random reserve, if the valuation of the highest bidder is above the threshold t , the object will be fully allocated to her. For each t , I am able to identify a non-trivial lower bound of the revenue guarantee by constructing a set of feasible dual variables.

Lemma 1. *A lower bound of the revenue guarantee of the second-price auction with t -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve where $t \in (0, \bar{v})$ is*

$$\int_{(0,t)} \frac{x^{\frac{N}{N-1}}}{t^{\frac{1}{N-1}}} dF(x) + t(1 - F(t)).$$

Proof. The proof is in Appendix C. □

Lemma 1 suggests a potential criterion under which the auctioneer selects an auction from this family. Although the revenue guarantee of an auction in this family may depend on the details of the marginal distribution and thus may be hard to be identified, it has a non-trivial lower bound. Then I can compare this lower bound with the revenue guarantees of some dominant-strategy mechanisms commonly used in practice.

Theorem 4. *For any $N \geq 2$, $\mathcal{M}_{SP-\beta}$ universally dominates the family of posted-price mechanisms.*

Proof. The proof is in Appendix C. □

Theorem 5. *For any $N \geq 2$, $\mathcal{M}_{SP-\beta}$ universally dominates the family of second-price auctions with non-negative deterministic reserves.*

Proof. The proof is in Appendix C. □

Strikingly, I do not require any distributional assumption for these two theorems to hold. Moreover, Theorem 4 and 5 imply that for a given marginal distribution, the auctioneer can find a second-price auction with a t -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve whose revenue guarantee is strictly higher than that of any posted-price mechanism and any second-price auction with a non-negative deterministic reserve. In addition, Theorem 4 can be interpreted as that the competition effect dominates the adversarial correlation effect. To

see this, note that Theorem 4 implies that there exists an auction for the two-bidder case that generates strictly higher revenue guarantee than the monopoly revenue from one bidder, regardless of the marginal distribution. Thus, even if nature picks the worst-case correlation structure, it is always *strictly* more desirable for the auctioneer to have just one more bidder.

6 Concluding Remarks

In this paper, I consider the correlation-robust framework and show, among others, that the second-price auction with the uniformly distributed random reserve maximizes the revenue guarantee across all dominant-strategy mechanisms for the two-bidder case, and that the second-price auction with the $Beta(\frac{1}{N-1}, 1)$ distributed random reserve maximizes the revenue guarantee across standard dominant-strategy mechanisms for the N -bidder ($N \geq 3$) case. These auctions have familiar formats and admit simple descriptions which do not require the information of the marginal distribution except for the maximum valuation of a generic bidder. Thus, these auctions are more practical compared with, for example, Myerson's auction, which often requires the full information of the marginal distribution to calculate the optimal reserve. To my knowledge, this paper is the first to characterize a maxmin mechanism across *all* dominant-strategy mechanism in the correlation-robust framework. It remains an open question what the maxmin mechanisms across all dominant-strategy mechanisms are for general number of bidders. The constructive method may shed light on other robust design problems and general robust optimization problems.

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A Proof for Section 3: Proposition 1

1) Dominant-strategy incentive compatibility (DSIC) for a type v_i requires that for any $v'_i \neq v_i$:

$$v_i q_i(v_i, v_{-i}) - t_i(v_i, v_{-i}) \geq v_i q_i(v'_i, v_{-i}) - t_i(v'_i, v_{-i}).$$

DSIC also requires that:

$$v'_i q_i(v'_i, v_{-i}) - t_i(v'_i, v_{-i}) \geq v'_i q_i(v_i, v_{-i}) - t_i(v_i, v_{-i}).$$

Adding the two inequalities, I obtain that

$$(v_i - v'_i)(q_i(v_i, v_{-i}) - q_i(v'_i, v_{-i})) \geq 0.$$

It follows that $q_i(v_i, v_{-i}) \geq q_i(v'_i, v_{-i})$ whenever $v_i > v'_i$.

2) Fix any v_{-i} , and define

$$U_i(v_i) = v_i q_i(v_i, v_{-i}) - t_i(v_i, v_{-i}).$$

By the two inequalities in 1), I obtain that

$$(v'_i - v_i)q_i(v_i, v_{-i}) \leq U_i(v'_i) - U_i(v_i) \leq (v'_i - v_i)q_i(v'_i, v_{-i}).$$

Dividing throughout by $v'_i - v_i$, I obtain that

$$q_i(v_i, v_{-i}) \leq \frac{U_i(v'_i) - U_i(v_i)}{(v'_i - v_i)} \leq q_i(v'_i, v_{-i}).$$

As $v \uparrow v'$, I have that

$$\frac{dU_i(v_i)}{dv_i} = q_i(v_i, v_{-i}).$$

Then I obtain that

$$t_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(s, v_{-i}) ds - U_i(0).$$

Note that $U_i(0) \geq 0$ by the ex-post individually rational constraint. If $U_i(0) > 0$, then I can reduce it to 0 so that I can increase the revenue from any valuation profile in which the others' valuation profile is v_{-i} . And the revenue guarantee will be weakly greater. Thus, when searching for a maxmin mechanism, it is without loss to let $U_i(0) = 0$. Then I obtain that $t_i(v_i, v_{-i}) = v_i q_i(v_i, v_{-i}) - \int_0^{v_i} q_i(s, v_{-i}) ds$.

B Proofs for Section 4

B.1 Proof of Proposition 3

First, I illustrate the details about the construction of π^* . Note that by allocating all marginal density $f(0)$ to valuation profile $(0, 0)$, I have that $\phi_i^*(v_i, 0) = \phi_j^*(0, v_j) = 0$ for any v_i and v_j . Thus, the property (1) trivially holds for any one of these valuation profiles. Now let $A_{kj} := \{v | k \leq v_1 \leq j, v_2 = k\}$, and define $c(0) := f(0)$ and $c(k) := \int_{A_{k\bar{v}}} d\pi^*$ for $k > 0$. Consider the valuation profile (v_1, v_2) where $0 < v_2 \leq v_1 < \bar{v}$. In order for the virtual value to satisfy the property (1), I have that

$$\phi_1^*(v_1, v_2) = v_1 - \frac{c(v_2) - \int_{v_2}^{v_1} \pi^*(x, v_2) dx}{\pi^*(v_1, v_2)} = 0, \quad \forall 0 < v_2 \leq v_1 < \bar{v}.$$

These equations are essentially a system of ordinary differential equations, whose solution is well known²⁹:

$$\pi^*(v_1, v_2) = \frac{v_2 c(v_2)}{v_1^2}, \quad \forall 0 < v_2 \leq v_1 < \bar{v}, \quad (\text{B.1})$$

$$\pi^*(\bar{v}, v_2) = \frac{v_2 c(v_2)}{\bar{v}}, \quad \forall 0 < v_2 < \bar{v}. \quad (\text{B.2})$$

By symmetry, I also obtain $\pi^*(v_2, v_1) = \pi^*(v_1, v_2)$ for $0 < v_2 \leq v_1 < \bar{v}$ and $\pi^*(v_2, \bar{v}) = \pi^*(\bar{v}, v_2)$ for $0 < v_2 < \bar{v}$. Finally,

$$Pr^*(\bar{v}, \bar{v}) = Pr(\bar{v}) - \frac{\int_{j \in (0, \bar{v})} j c(j) dj}{\bar{v}}. \quad (\text{B.3})$$

Now I solve for $c(k)$. Note since the marginal distribution is the same across the two bidders, given the above derivation, $c(k)$ must satisfy the following condition:

$$c(k) = f(k) - \frac{\int_0^k j c(j) dj}{k^2}, \quad \forall 0 < k < \bar{v}. \quad (\text{B.4})$$

To see this, note $f(k) = \int_{\{0 \leq v_1 \leq \bar{v}, v_2 = k\}} d\pi^* = \int_{A_{k\bar{v}} \cup \{0 \leq v_1 < k, v_2 = k\}} d\pi^* = \int_{A_{k\bar{v}}} d\pi^* + \int_{\{0 \leq v_1 < k, v_2 = k\}} d\pi^* = \int_{A_{k\bar{v}}} d\pi^* + \int_{\{v_1 = k, 0 \leq v_2 < k\}} d\pi^*$ where the last equality follows from symmetry. Multiplying both sides of (B.4) by k , I obtain that

$$kc(k) = kf(k) - \frac{\int_0^k j c(j) dj}{k}, \quad \forall 0 < k < \bar{v}.$$

²⁹The solution is reminiscent of the equal-revenue distribution.

Define $g(k) := \int_0^k j c(j) dj$ for $0 < k < \bar{v}$. Then I have that

$$g'(k) = k f(k) - \frac{g(k)}{k}, \quad \forall 0 < k < \bar{v}.$$

Note this is an ordinary differential equation, and I solve for $g(k)$:

$$g(k) = \frac{1}{k} \int_0^k j^2 f(j) dj, \quad \forall 0 < k < \bar{v}. \quad (\text{B.5})$$

From this I compute $c(k)$:

$$c(k) = f(k) - \frac{\int_0^k j^2 f(j) dj}{k^3}, \quad \forall 0 < k < \bar{v}. \quad (\text{B.6})$$

Plugging (B.6) to (B.1), (B.2) and (B.3), I obtain the joint distribution π^* .

To guarantee that it is possible to construct π^* , it has to be a feasible joint distribution in that the density (or probability mass) has to be non-negative for all valuation profiles, i.e., $\pi^*(v_1, v_2) \geq 0$ for $0 \leq v_1, v_2 < \bar{v}$ and $Pr^*(\bar{v}, \bar{v}) \geq 0$. Therefore, I have that

$$f(k) - \frac{\int_0^k j^2 f(j) dj}{k^3} \geq 0, \quad \forall 0 < k < \bar{v}, \quad (\text{B.7})$$

$$Pr(\bar{v}) \geq \frac{\int_{x \in (0,1)} x^2 f(x) dx}{\bar{v}^2}. \quad (\text{B.8})$$

Now I show that the first part of the two-bidder robust regularity conditions implies (B.7). To see this, note that if $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$, then for any $0 < k < \bar{v}$, I have that

$$f(k) - \frac{\int_0^k j^2 f(j) dj}{k^3} \geq f(k) - \frac{\int_0^k k^2 f(k) dj}{k^3} = 0, \quad (\text{B.9})$$

where the inequality follows from that $j^2 f(j) \leq k^2 f(k)$ if $j \leq k$.

Now given that the construction is feasible, I argue that the two-bidder robust regularity conditions guarantee that the property (2) holds. Given that the property (1) holds for π^* , it suffices to show

$$\phi_2(v_1, v_2) \leq 0$$

if $0 < v_2 \leq v_1 < \bar{v}$. I now calculate $\phi_2(v_1, v_2)$ for $0 < v_2 \leq v_1 < \bar{v}$:

$$\begin{aligned}\phi_2(v_1, v_2) &= v_2 - \frac{f(v_1) - \int_0^{v_2} \pi^*(v_1, t) dt}{\pi^*(v_1, v_2)} \\ &= v_2 - \frac{f(v_1) - \int_0^{v_2} c(t) \frac{t}{v_1^2} dt}{c(v_2) \frac{v_2}{v_1^2}} \\ &= v_2 - \frac{f(v_1) - \frac{1}{v_1^2 v_2} \int_0^{v_2} t^2 f(t) dt}{\left(f(v_2) - \frac{\int_0^{v_2} s^2 f(s) ds}{v_2^3}\right) \frac{v_2}{v_1^2}},\end{aligned}$$

where the second equality follows from (B.1) and the third equality follows from (B.5) and (B.6). Now it is straightforward that for any $0 < v_2 \leq v_1 < \bar{v}$:

$$\phi_2(v_1, v_2) \leq 0 \iff v_2^2 f(v_2) \leq v_1^2 f(v_1).$$

B.2 Proof of Proposition 4

First, I illustrate the details about the construction of the mechanism. I first write down the primal minimization problem for adversarial nature given a mechanism (q, t) and derive its dual maximization problem. Formally, let $\{\lambda_i(v_i)\}_{i \in \{1, 2, \dots, N\}, v_i \in [0, \bar{v}]}$ be dual variables.

$$(P) \quad \inf_{\pi \in \Pi(F)} \int_{v \in [0, \bar{v}]^N} \sum_{i=1}^N t_i(v) d\pi(v)$$

subject to

$$\int_{[0, \bar{v}]^{N-1}} d\pi(v_i, v_{-i}) = f(v_i), \quad \forall v_i \in [0, \bar{v}],$$

$$\int_{[0, \bar{v}]^{N-1}} d\pi(v_i = \bar{v}, v_{-i}) = Pr(v_i = \bar{v}).$$

$$(D) \quad \sup_{\{\lambda_i(v_i)\}} \sum_{i=1}^N \int_0^{\bar{v}} \lambda_i(v_i) dF(v_i)$$

subject to

$$\sum_{i=1}^N \lambda_i(v_i) \leq \sum_{i=1}^N t_i(v), \quad \forall v \in [0, \bar{v}]^N. \tag{B.10}$$

It is straightforward to show that weak duality holds ³⁰. The mechanism is constructed by a complementary slackness condition as follows.

$$\sum_{i=1}^N \lambda_i(v_i) = \sum_{i=1}^N t_i(v), \quad \forall v \in V^+. \quad (\text{B.11})$$

I assume that $\lambda_i = \lambda$ for all $i \in I$, and that the mechanism is a second-price auction with a random reserve whose cumulative distribution function is G , then (B.11) implies

$$N\lambda(v_i) = v_i G(v_i), \quad \forall v_i \in [0, \bar{v}], \quad (\text{B.12})$$

$$\lambda(v(1)) + (N-1)\lambda(v(2)) = v(1)G(v(1)) - \int_{v(2)}^{v(1)} G(s)ds, \quad \forall 0 \leq v(2) < v(1) \leq 1. \quad (\text{B.13})$$

Note by (B.12), I have that for $v_i \in [0, \bar{v}]$,

$$\lambda(v_i) = \frac{v_i G(v_i)}{N}. \quad (\text{B.14})$$

Plugging (B.14) to (B.13), I obtain that for $0 \leq v(2) < v(1) \leq \bar{v}$,

$$\frac{v(1)G(v(1)) + (N-1)v(2)G(v(2))}{N} = v(1)G(v(1)) - \int_{v(2)}^{v(1)} G(s)ds. \quad (\text{B.15})$$

Taking first order derivatives with respect to $v(1)$ and $v(2)$, I obtain the same ordinary differential equation that for $x \in [0, \bar{v}]$,

$$(N-1)xG'(x) = G(x). \quad (\text{B.16})$$

Given that G is a distribution, the solution to (B.16) is

$$G(x) = \left(\frac{x}{\bar{v}}\right)^{\frac{1}{N-1}}, \quad \forall x \in [0, \bar{v}].$$

This is the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distribution.

Now I show that under the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve, any conceivable joint distribution with the support V^+ minimizes the expected revenue across all conceivable joint distributions. As I have argued in Section 4.2, the expected revenue given such a joint distribution is $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$, then it suffices to show the value of (D) is also $\frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$. To this end, I construct the dual variables as follows. For

³⁰See, for example, He and Li (2022).

all $i \in \{1, 2, \dots, N\}$ and $x \in [0, 1]$,

$$\lambda_i(x) = \frac{x^{\frac{N}{N-1}}}{N\bar{v}^{\frac{1}{N-1}}}.$$

Note that $\sum_{i=1}^N \int_0^{\bar{v}} \lambda_i(v_i) dF(v_i) = \frac{E[X^{\frac{N}{N-1}}]}{\bar{v}^{\frac{1}{N-1}}}$ under the constructed dual variables. Then, it suffices to show that (B.10) holds under the constructed dual variables. To see this, I divide valuation profiles into two cases.

Case 1: $\#\{k : v_k = v(1)\} = 1$.

In this case, $t(v) = \frac{v(1)^{\frac{N}{N-1} + (N-1)v(2)^{\frac{N}{N-1}}}}{N\bar{v}^{\frac{1}{N-1}}}$. The L.H.S. of (B.10) is maximized when all bidders except the highest bidder have the same valuations, and (B.10) holds with equality when the L.H.S. of (B.10) is maximized. Therefore (B.10) holds.

Case 2: $\#\{k : v_k = v(1)\} \geq 2$.

In this case, $t(v) = \frac{v(1)^{\frac{N}{N-1}}}{\bar{v}^{\frac{1}{N-1}}}$. The L.H.S. of (B.10) is maximized when all bidders have the same valuations, and (B.10) holds with equality when the L.H.S. of (B.10) is maximized. Therefore (B.10) holds.

B.3 Proof of Proposition 5

First, I illustrate the details about the construction of π^{**} . Note that by allocating all marginal density $f(0)$ to the valuation profile $\underbrace{(0, \dots, 0)}_N$, I have that $\phi_i^{**}(v) = 0$ for any i and $v_i \geq 0, v_j = 0, \forall j \neq i$. Thus, the property (1') trivially holds for any one of these valuation profiles. Now let $B_{kj} := \{v | k \leq v_1 \leq j, v_i = k, \forall i \neq 1\}$, and define $d(0) := f(0)$ and $d(k) := \int_{B_{k\bar{v}}} d\pi^*$ for $k > 0$. Consider the valuation profile $(v_1, \underbrace{v_2, \dots, v_2}_{N-1})$ where $0 < v_2 \leq v_1 < \bar{v}$. In order for the virtual value of bidder 1 to satisfy the property (1'), I have that

$$\phi_1^{**}(v_1, \underbrace{v_2, \dots, v_2}_{N-1}) = v_1 - \frac{d(v_2) - \int_{v_2}^{v_1} \pi^{**}(s, \underbrace{v_2, \dots, v_2}_{N-1}) ds}{\pi^{**}(v_1, \underbrace{v_2, \dots, v_2}_{N-1})} = 0, \quad \forall 0 < v_2 \leq v_1 < \bar{v}.$$

These equations are essentially a system of ordinary differential equations, whose solution is well known:

$$\pi^{**}(v_1, \underbrace{v_2, \dots, v_2}_{N-1}) = \frac{v_2 d(v_2)}{v_1^2}, \quad \forall 0 < v_2 \leq v_1 < \bar{v}, \quad (\text{B.17})$$

$$\pi^{**}(1, \underbrace{v_2, \dots, v_2}_{N-1}) = \frac{v_2 d(v_2)}{\bar{v}}, \forall 0 < v_2 < \bar{v}. \quad (\text{B.18})$$

By symmetry, I also obtain that $\pi^{**}(v) = \pi^{**}(v_1, \underbrace{v_2, \dots, v_2}_{N-1})$ for $0 < v_j = v_2 \leq v_i = v_1 < \bar{v}, \forall j \neq i, \forall i$ and $\pi^{**}(v) = \pi^*(1, \underbrace{v_2, \dots, v_2}_{N-1})$ for $0 < v_j = v_2 < v_i = \bar{v}, \forall j \neq i, \forall i$. Finally,

$$Pr^{**}(\underbrace{\bar{v}, \dots, \bar{v}}_N) = Pr(\bar{v}) - \frac{\int_{j \in (0, \bar{v})} j d(j) dj}{\bar{v}}. \quad (\text{B.19})$$

Now I solve for $d(k)$. Note that $d(k)$ must satisfy the following condition:

$$f(k) = (N-1)d(k) + \frac{\int_0^k j d(j) dj}{k^2}, \quad \forall 0 < k < \bar{v}.$$

To see this, suppose that the bidder 1's valuation is k . Then either k is the highest valuation and other bidders all have a valuation of $j \in [0, k]$ (with a probability of $\frac{\int_0^k j d(j) dj}{k^2}$) or k is the second highest valuation and one of the other bidders has the highest valuation (with a probability of $(N-1)d(k)$). Multiplying both sides of (B.3) by k , I obtain that

$$k f(k) = (N-1)k d(k) + \frac{\int_0^k j d(j) dj}{k}, \quad \forall 0 < k < \bar{v}.$$

Define $h(k) := \int_0^k j d(j) dj$ for $0 < k < \bar{v}$. Then I have that

$$k f(k) = (N-1)h'(k) + \frac{h(k)}{k}, \quad \forall 0 < k < \bar{v}.$$

Note that this is an ordinary differential equation, and I solve for $h(k)$:

$$h(k) = \frac{\int_0^k j^{\frac{N}{N-1}} f(j) dj}{(N-1)k^{\frac{1}{N-1}}}, \quad \forall 0 < k < \bar{v}. \quad (\text{B.20})$$

From this I compute $d(k)$:

$$d(k) = \frac{1}{N-1} \left(f(k) - \frac{\int_0^k j^{\frac{N}{N-1}} f(j) dj}{(N-1)k^{1+\frac{N}{N-1}}} \right), \quad \forall 0 < k < \bar{v}. \quad (\text{B.21})$$

Plugging (B.21) to (B.17), (B.18) and (B.19), I obtain the joint distribution π^{**} .

To guarantee that π^* is a feasible joint distribution in that the density (or probability mass) has to be non-negative for all valuation profiles, it is straightforward that the N -bidder

robust regularity conditions (I) have to hold. Now I show that the N -bidder robust regularity conditions (II) imply the N -bidder robust regularity conditions (I). To see this, note that if $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$, then for any $0 < k < \bar{v}$, I have that

$$f(k) - \frac{\int_0^k j^{\frac{N}{N-1}} f(j) dj}{(N-1)k^{1+\frac{N}{N-1}}} \geq f(k) - \frac{\int_0^k j^{\frac{N}{N-1}-2} k^2 f(k) dj}{(N-1)k^{1+\frac{N}{N-1}}} = 0, \quad (\text{B.22})$$

where the inequality follows from that $j^2 f(j) \leq k^2 f(k)$ if $j \leq k$.

B.4 Proof of Theorem 3

Lemma 2. *If $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$ and $Pr(\bar{v}) < \frac{\int_{(0, \bar{v})} x^{\frac{N}{N-1}} f(x) dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$, then there exists $s^* \in (0, \bar{v})$ that is a solution to (SD).*

Proof. First, note that if $s \uparrow \bar{v}$, the R.H.S of (SD) converges to $(N-1)Pr(\bar{v})$, thus the L.H.S. of (SD) > the R.H.S of (SD).

Next, take a monotone sequence $\{s_n\}_{n \in \mathbb{N}}$ where $s_n \downarrow 0$ as $n \rightarrow \infty$, $s_1 \in (0, \bar{v})$ and $\frac{s_{n+1}}{s_n} \leq \frac{1}{2}$ for any n .³¹ I will prove that $\limsup_{n \rightarrow \infty} s_n f(s_n) = 0$ by contradiction. Suppose that $\limsup_{n \rightarrow \infty} s_n f(s_n) = c > 0$, then for any $\epsilon > 0$, there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} f(s_{n_k}) - c \geq \epsilon$ for any k . So $f(s_{n_k}) \geq \frac{c-\epsilon}{s_{n_k}}$ for any k . Let ϵ be $\frac{c}{2}$. That $x^2 f(x)$ is non-decreasing implies that for any $x \in (s_{n_{k+1}}, s_{n_k})$, $f(x) \geq \frac{s_{n_{k+1}}^2 f(s_{n_{k+1}})}{x^2} \geq \frac{s_{n_{k+1}}(c-\epsilon)}{x^2} = \frac{c s_{n_{k+1}}}{2x^2}$ for any k . Therefore $\int_{s_{n_{k+1}}}^{s_{n_k}} f(x) \geq \frac{c s_{n_{k+1}}}{2} \left(\frac{1}{s_{n_{k+1}}} - \frac{1}{s_{n_k}} \right) \geq \frac{c}{4}$. Thus, $\int_0^{\bar{v}} dF(x) \geq \sum_{k=1}^K \int_{s_{n_{k+1}}}^{s_{n_k}} f(x) \geq \frac{cK}{4} \rightarrow \infty$ as $K \rightarrow \infty$, a contradiction to the fact that F is a probability measure. Therefore $\limsup_{n \rightarrow \infty} s_n f(s_n) = 0$. This implies that $\lim_{n \rightarrow \infty} s_n f(s_n) = 0$. Now, by L'Hôpital's rule,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\int_{(0, s_n)} x^{\frac{N}{N-1}} dF(x)}{s_n^{\frac{N}{N-1}}} &= \lim_{n \rightarrow \infty} \frac{s_n^{\frac{N}{N-1}} f(s_n)}{\frac{N}{N-1} s_n^{\frac{1}{N-1}}} \\ &= \lim_{n \rightarrow \infty} \frac{(N-1) s_n f(s_n)}{N} \\ &= 0. \end{aligned}$$

Then, if $s_n \downarrow 0$, the L.H.S. of (SD) < the R.H.S of (SD). By the Intermediate Value Theorem, there exists $s^* \in (0, \bar{v})$ that is a solution to (SD). \square

Proposition 7. *If $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$ and $Pr(\bar{v}) < \frac{\int_{(0, \bar{v})} x^{\frac{N}{N-1}} f(x) dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$, then*

³¹For example, $s_n = \frac{\bar{v}}{2^n}$ for $n \in \mathbb{N}$.

the revenue guarantee of the second-price auction with the s^* -scaled $\text{Beta}(\frac{1}{N-1}, 1)$ distributed random reserve is at least $(N-1)s^*(1-F(s^*))$.

Proof. This follows immediately from Lemma 1 and Lemma 2. \square

Proposition 8. *If $x^2 f(x)$ is non-decreasing for $x \in (0, \bar{v})$ and $\text{Pr}(\bar{v}) < \frac{\int_{(0, \bar{v})} x^{\frac{N}{N-1}} f(x) dx}{(N-1)\bar{v}^{\frac{N}{N-1}}}$, then there exists a joint distribution $\pi^{***} \in \pi(F)$ under which the second-price auction with the s^* -scaled $\text{Beta}(\frac{1}{N-1}, 1)$ distributed random reserve maximizes the expected revenue across standard dominant-strategy mechanisms. In addition, the maximized expected revenue is $(N-1)s^*(1-F(s^*))$.*

Proof. The joint distribution π^{***} is symmetric and is defined as follows. The support of π^{***} is V^+ . If $v \notin V^+$, then $\pi^{***}(v) = 0$. If $v \in V^+$, then

$$\pi^{***}(v_i, v_{-i})^{32} = \begin{cases} f(0) & \text{if } v = (0, \dots, 0); \\ 0 & \text{if } 0 = v_j < v_i, \forall j \neq i; \\ \frac{1}{(N-1)v(1)^2} (v(2)f(v(2)) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1} \int_0^{v(2)} x^{\frac{N}{N-1}} f(x) dx) & \text{if } 0 < v(2) = v_j \leq v_i = v(1) \leq s^*, \forall j \neq i; \\ \frac{f(v(1))}{(N-1)s^*(1-F(s^*))} (v(2)f(v(2)) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1} \int_0^{v(2)} x^{\frac{N}{N-1}} f(x) dx) & \text{if } 0 < v(2) = v_j \leq s^* < v_i < \bar{v}, \forall j \neq i; \\ \frac{\text{Pr}(\bar{v})}{(N-1)s^*(1-F(s^*))} (v(2)f(v(2)) - \frac{v(2)^{-\frac{N}{N-1}}}{N-1} \int_0^{v(2)} x^{\frac{N}{N-1}} f(x) dx) & \text{if } 0 < v(2) = v_j \leq s^* < v_i = \bar{v}, \forall j \neq i. \end{cases}$$

It is straightforward to verify that $\pi^{***} \in \Pi(F)$. When $v_i = v(1) \leq s^*$, the density function coincides with π^{**} . Therefore by the proof of Proposition 5,

$$\phi_i^{***}(v) = 0 \quad \text{for } v \in V^+, v_i = v(1) \leq s^*.$$

Note that under π^{***} , when $v_i = v(1) > s^*$, v_i and v_{-i} are independent. Therefore $\phi_i^{***}(v) = v_i - \frac{1-F(v_i)}{f(v_i)}$ for $s^* < v_i = v(1) < \bar{v}$ and $\phi_i^{***}(v) = \bar{v}$ for $v_i = v(1) = \bar{v}$.

Now I show that $\phi_i^{***}(v) \geq 0$ for $s^* < v_i = v(1) < \bar{v}$. First I show that $1 - F(x) - xf(x)$

³²The density function π^{***} in the region $(0, \bar{v})^N$ is similar to the density function η_F^* in the region $(0, 1)^N$ in He and Li (2022).

is non-increasing if $x^2 f(x)$ is non-decreasing. To see this, note that for any $0 < x_1 \leq x_2 < \bar{v}$,

$$\begin{aligned}
1 - F(x_2) - x_2 f(x_2) - [1 - F(x_1) - x_1 f(x_1)] &= x_1 f(x_1) - x_2 f(x_2) - \int_{x_1}^{x_2} f(x) dx \\
&\leq x_1 f(x_1) - x_2 f(x_2) - \int_{x_1}^{x_2} \frac{x_1^2 f(x_1)}{x^2} dx \\
&= \frac{x_1^2 f(x_1)}{x_2} - x_2 f(x_2) \\
&\leq 0,
\end{aligned}$$

where the first inequality follows from that $x^2 f(x) \geq x_1^2 f(x_1)$ for $x_1 \leq x \leq x_2$ and the second inequality follows from that $x_2^2 f(x_2) \geq x_1^2 f(x_1)$.

Recall that

$$\frac{\int_{(0, s^*)} x^{\frac{N}{N-1}} dF(x)}{(s^*)^{\frac{N}{N-1}}} = (N-1)(1 - F(s^*)).$$

Subtracting $(N-1)s^* f(s^*)$ from both sides, I obtain that

$$\frac{\int_{(0, s^*)} x^{\frac{N}{N-1}} dF(x)}{(s^*)^{\frac{N}{N-1}}} - (N-1)s^* f(s^*) = (N-1)[1 - F(s^*) - s^* f(s^*)].$$

The L.H.S. of the above equation is weakly negative, shown in (B.22). Together with that $1 - F(x) - x f(x)$ is non-increasing, I have that $1 - F(x) - x f(x) \leq 0$ for any $x \geq s^*$. Hence,

$$\phi_i^{***}(v) = v_i - \frac{1 - F(v_i)}{f(v_i)} \geq 0 \quad \text{for } v \in V^+, s^* < v_i = v(1) < \bar{v}.$$

Then, any standard dominant-strategy mechanism, in which 1) the ex-post participation constraints are binding for zero-valuation bidders and 2) the highest bidder with a valuation higher than s^* is allocated with probability one, maximizes the expected revenue across standard dominant-strategy mechanisms under π^{***} . And the second-price auction with the s^* -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is such a mechanism. Finally, it is straightforward to show that the maximized expected revenue is $Ns^*(1 - F(s^*))$ by calculating the expected total virtual surplus. Indeed, under this mechanism and the joint distribution

π^{***} , the expected total virtual surplus is

$$\begin{aligned}
& N \left\{ \int_{(0, \bar{v})} \int_{(0, s^*)} \frac{f(v_1)}{(N-1)s^*(1-F(s^*))} \left[v_2 f(v_2) - \frac{v_2^{-\frac{N}{N-1}}}{N-1} \int_0^{v_2} x^{\frac{N}{N-1}} f(x) dx \right] \cdot \left(v_1 - \frac{1-F(v_1)}{f(v_1)} \right) dv_2 dv_1 + \right. \\
& \left. \int_{(0, s^*)} \frac{Pr(\bar{v})}{(N-1)s^*(1-F(s^*))} \left[v_2 f(v_2) - \frac{v_2^{-\frac{N}{N-1}}}{N-1} \int_0^{v_2} x^{\frac{N}{N-1}} f(x) dx \right] \cdot \bar{v} dv_2 \right\} \\
&= \frac{N}{(N-1)s^*(1-F(s^*))} \cdot \left[\int_{(0, \bar{v})} (v_1 f(v_1) - 1 + F(v_1)) dv_1 + Pr(\bar{v})\bar{v} \right] \cdot \frac{\int_{(0, s^*)} s^{\frac{N}{N-1}} f(s) ds}{(s^*)^{\frac{1}{N-1}}} \\
&= \frac{N}{(N-1)s^*(1-F(s^*))} \cdot \left[\int_{(0, \bar{v})} (v_1 f(v_1) - 1 + F(v_1)) dv_1 + Pr(\bar{v})\bar{v} \right] \cdot (N-1)s^*(1-F(s^*)) \\
&= N \cdot \left[\int_{(0, \bar{v})} (v_1 f(v_1) - 1 + F(v_1)) dv_1 + Pr(\bar{v})\bar{v} \right] \\
&= N s^*(1-F(s^*)),
\end{aligned}$$

where the first equality follows from (B.20), the second equality follows from (SD) and the last equality uses integration by parts. \square

Theorem 3 follows immediately from Proposition 7 and Proposition 8.

C Proofs for Section 5

C.1 Proof of Proposition 6

Note that under a posted-price mechanism, the maximally positively correlated distribution (the valuations of the bidders are always the same) is a worst-case correlation structure, and the revenue guarantee of any posted-price mechanism is thus at most $\max_{x \in [0, \bar{v}]} x(1-F(x))$, which is the monopoly profit when there is only one bidder. It is straightforward to show that $\frac{x^{\frac{N}{N-1}}}{\bar{v}^{\frac{1}{N-1}}} \geq \frac{x^2}{\bar{v}}$ for any $x \in [0, \bar{v}]$ and $N \geq 2$. Thus, it suffices to compare $\frac{E[X^2]}{\bar{v}}$ with $\max_{x \in [0, \bar{v}]} x \cdot (1-F(x))$ if the revenue function $R(x) = x \cdot (1-F(x))$ is strictly concave. Using integration by parts, I obtain that

$$E[X^2] = 2 \int_0^{\bar{v}} x(1-F(x)) dx.$$

Let x^* denote the unique solution to $\max_{x \in [0, \bar{v}]} R(x)$. Then using graph (see Figure 1) it is straightforward that

$$\int_0^{\bar{v}} R(x) dx > \frac{1}{2} \cdot \bar{v} \cdot R(x^*). \quad (\text{D.1})$$

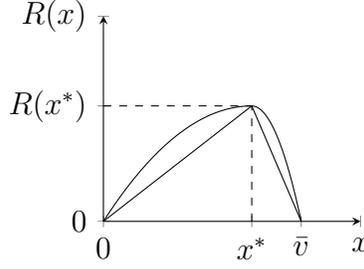


Figure 1: The curve is a strictly concave revenue function. The L.H.S. of (D.1) is the area under the curve. The R.H.S. of (D.1) is the area of the triangle.

(D.1) is equivalent to that

$$\frac{E[X^2]}{\bar{v}} = \frac{2 \int_0^{\bar{v}} x(1 - F(x))dx}{\bar{v}} > \max_{x \in [0, \bar{v}]} x(1 - F(x)).$$

C.2 Proof of Lemma 1

For each t , I construct the dual variables for the second-price auction with the random reserve whose cumulative distribution function is $G_t(r) = (\frac{r}{t})^{\frac{1}{N-1}}$ as follows:

$$\lambda_i(x) = \frac{x^{\frac{N}{N-1}}}{Nt^{\frac{1}{N-1}}} \quad \text{if } 0 \leq x \leq t, \forall i \in I,$$

$$\lambda_i(x) = \frac{t}{N} \quad \text{if } t < x \leq \bar{v}, \forall i \in I.$$

Given the constructed dual variables above, the value of (D) is

$$\int_0^t \frac{x^{\frac{N}{N-1}}}{t^{\frac{1}{N-1}}} dF(x) + t(1 - F(t)).$$

Then it suffices to show that the constructed dual variables are feasible, or (B.10) holds. I divide the valuation profiles into three cases.

Case 1: $v(1) \leq t$.

(B.10) holds by a similar argument with that in the proof of Proposition 3.

Case 2: $v(1) > t, \#\{k : v_k = v(1)\} = 1$.

When $v(2) > t$, then $t(v) = v(2)$. The L.H.S. of (B.10) is maximized when $v_i \geq t$ for all i , and the maximized value is $N \cdot \frac{t}{N} = t < t(v)$. When $v(2) \leq t$, then $t(v) = v(1) \cdot 1 - \int_t^{v(1)} dx - \int_{v(2)}^t (\frac{x}{t})^{\frac{1}{N-1}} dx = \frac{t}{N} + \frac{(N-1)v(2)^{\frac{N}{N-1}}}{Nt^{\frac{1}{N-1}}}$. The L.H.S. of (B.10) is maximized

when $v_i = v(2)$ for all $i \notin \{k : v_k = v(1)\}$, and the maximized value is $\frac{t}{N} + \frac{(N-1)v(2)^{\frac{N}{N-1}}}{Nt^{\frac{1}{N-1}}} = t(v)$. Therefore (B.10) holds.

Case 3: $v(1) > t$, $\#\{k : v_k = v(1)\} \geq 2$.

Now $t(v) = v(1)$. The L.H.S. of (B.10) is maximized when $v_i \geq t$ for all i , and the maximized value is $N \cdot \frac{t}{N} = t < t(v)$. Therefore (B.10) holds.

C.3 Proof of Theorem 4

Recall that the revenue guarantee of a posted-price mechanism is at most $\max_{x \in [0, \bar{v}]} x \cdot (1 - F(x))$ for a given $F \in \Delta^c[0, \bar{v}]$. Denote a solution to $\max_{x \in [0, \bar{v}]} x \cdot (1 - F(x))$ as x^* . For a given $F \in \Delta^c[0, \bar{v}]$, consider the second-price auction with the x^* -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve, and I have that

$$\int_0^{x^*} \frac{x^{\frac{N}{N-1}}}{(x^*)^{\frac{1}{N-1}}} dF(x) + x^*(1 - F(x^*)) > x^*(1 - F(x^*)),$$

where the inequality follows from that $x^* > 0$.

C.4 Proof of Theorem 5

By He and Li (2022), given a $F \in \Delta^c[0, \bar{v}]$, the revenue guarantee of the second-price auction with the optimal deterministic reserve for the N -bidder case is as follows:

$$\frac{N}{N-1} \int_{r^*}^{c(r^*)} x dF(x),$$

where r^* satisfies $F(Nr^*) = F(\frac{N-1+F(r^*)}{N})$, and $c(r^*) = F^{-1}(\frac{N-1+F(r^*)}{N})$.

Define $J(x) := \frac{N}{N-1}x - \frac{x^{\frac{N}{N-1}}}{c(r^*)^{\frac{1}{N-1}}}$. Because $J'(x) = \frac{N}{N-1}(1 - (\frac{x}{c(r^*)})^{\frac{1}{N-1}})$ and $J''(x) = -\frac{Nx^{\frac{1}{N-1}-1}}{(N-1)^2c(r^*)^{\frac{1}{N-1}}} \leq 0$, $J(x)$ is maximized at $x = c(r^*)$ and the maximized value is $\frac{1}{N-1}c(r^*)$ by simple calculation. For a given $F \in \Delta^c[0, \bar{v}]$, consider the second-price auction with the $c(r^*)$ -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve, and I have that

$$\begin{aligned} \int_0^{c(r^*)} \frac{x^{\frac{N}{N-1}}}{[c(r^*)]^{\frac{1}{N-1}}} dF(x) + c(r^*)[1 - F(c(r^*))] &> \int_{r^*}^{c(r^*)} \frac{x^{\frac{N}{N-1}}}{[c(r^*)]^{\frac{1}{N-1}}} dF(x) + c(r^*)[1 - F(c(r^*))] \\ &\geq \int_{r^*}^{c(r^*)} [\frac{N}{N-1}x - \frac{1}{N-1}c(r^*)] dF(x) + c(r^*)[1 - F(c(r^*))] \\ &= \frac{N}{N-1} \int_{r^*}^{c(r^*)} x dF(x), \end{aligned}$$

where the first inequality follows from that $r^* > 0$, the second inequality follows from that $J(x) \leq \frac{1}{N-1}c(r^*)$ and the equality follows from that

$$\begin{aligned} \int_{r^*}^{c(r^*)} \frac{1}{N-1}c(r^*)dF(x) &= \frac{1}{N-1}c(r^*)[F(c(r^*)) - F(r^*)] \\ &= \frac{1}{N-1} \left[\frac{N-1+F(r^*)}{N} - F(r^*) \right] F^{-1} \left(\frac{N-1+F(r^*)}{N} \right) \\ &= \left[\frac{1-F(r^*)}{N} \right] F^{-1} \left(\frac{N-1+F(r^*)}{N} \right) \\ &= c(r^*)[1 - F(c(r^*))]. \end{aligned}$$

D “Necessity” of Robust Regularity Conditions

Definition 5. I say the allocation rule q is *strictly monotone* if for any i , any v_{-i} and any pair of valuation v_i and v'_i in which $q_i(v_i, v_{-i}) > 0$ and $q_i(v'_i, v_{-i}) > 0$, I have that $q_i(v_i, v_{-i}) < q_i(v'_i, v_{-i})$ whenever $v_i < v'_i$.

Proposition 9. *For the two-bidder case, if the second-price auction with the uniformly distributed random reserve is a maxmin mechanism across dominant-strategy mechanisms, then the two-bidder robust regularity conditions hold almost surely.*

Proof. The intuition behind this is the observation that under the second-price auction with the uniformly distributed random reserve, the allocation rule is strictly monotone. In addition, the high bidder’s allocation is positive but less than 1 when her valuation is positive but less than \bar{v} . Thus in a Nash equilibrium, the high bidder’s virtual value has to be 0 for these valuations under the joint distribution, otherwise Myerson’s ironing argument implies that allocation rule in equilibrium should exhibit “flatness” across some range. Formally, I will establish Lemma 3 and Lemma 4 below.

Lemma 3. *For the two-bidder case, for the second-price auction with the uniformly distributed random reserve to be part of a Nash equilibrium across dominant-strategy mechanisms, the equilibrium joint distribution has to be π^* almost surely.*

Proof. let π be a best response of adversarial nature to the second-price auction with the uniformly distributed random reserve. Suppose (1) does not hold for a set of (v_1, v_2) where $\bar{v} > v_1 \geq v_2$ with some positive measure. If virtual values of bidder 1 for these valuation profiles are all positive, then consider a modified allocation exhibiting the property that the allocation to bidder 1 is one from the valuation profile in which the virtual value of bidder 1 becomes positive for the first time. Formally, let $\overline{v_1(v_2)} = \inf\{v_1 : \phi_1(v_1, v_2) > 0, v_1 \geq v_2\}$. Let $\tilde{q}_1(v_1, v_2) = 1$ for $v_1 > \overline{v_1(v_2)}$ and $\tilde{q}(v) := q^*(v)$ otherwise. Such modification is feasible

since bidder 2 gets zero allocation for any one of these valuation profiles in the second-price auction with the uniformly distributed random reserve. Thus I have a profitable and feasible deviation. If virtual values of bidder 1 for these valuation profiles are all negative, by a similar argument, I rule out the possibility that the second-price auction with the uniformly distributed random reserve is a best response of the auctioneer to π . Now If virtual values of bidder 1 for these valuation profiles are not all positive and not all negative, I have to discuss two cases. The first case is that the virtual value is still (weakly) monotone. Then by a similar argument, the second-price auction with the uniformly distributed random reserve can not a best response to π . The second case is that the virtual value is not monotone, then a best response has to exhibit flatness across a range of valuation profiles, which can be done by Myerson's ironing procedure. Recall that the allocation rule of the second-price auction with the uniformly distributed random reserve is strictly monotone. Thus, it cannot be a best response. To illustrate this, suppose $\phi_1(\cdot, v_2)$ is decreasing in v_1 for $v_1 \in (a(v_2), b(v_2))$ and $\phi_1(a(v_2), v_2) > \phi_1(\hat{v}_1(v_2), v_2) = 0 > \phi_1(b(v_2), v_2)$ for some $\hat{v}_1(v_2) \in (a(v_2), b(v_2))$. Then let $\tilde{q}_1(v_1, v_2) = q^*(\hat{v}_1(v_2), v_2)$ for $v_1 \in [a(v_2), b(v_2)]$ and $\tilde{q}(v) = q^*(v)$ otherwise. Since this is a feasible and profitable deviation, I conclude that the second-price auction with the uniformly distributed random reserve can not be a best response.

Together with the proof of Proposition 3, the equilibrium joint distribution is π^* almost surely. \square

Lemma 4. *For the two-bidder case, for the second-price auction with the uniformly distributed random reserve to be part of a Nash equilibrium across dominant-strategy mechanisms, π^* exhibits (2) almost surely.*

Proof. Suppose not. Then, there exists a set of (v_1, v_2) where $0 < v_2 < v_1 < \bar{v}$ but $\phi_2(v_1, v_2) > 0$ with some positive measure. Then by increasing the allocation to bidder 2 by a small amount ϵ when the valuation profile lies in this set, I have a feasible and profitable deviation. Thus, the second-price auction with the uniformly distributed random reserve is not a best response. \square

Proposition 9 follows immediately from Lemma 3, Lemma 4 and the proof of Proposition 3. \square

Proposition 10. *For the N -bidder ($N \geq 3$) case, If the second-price auction with the \bar{v} -scaled $Beta(\frac{1}{N-1}, 1)$ distributed random reserve is a maxmin mechanism across standard dominant-strategy mechanisms, then the N -bidder robust regularity conditions (I) hold almost surely.*

Proof. First, I establish Lemma 5 below.

Lemma 5. *For the N -bidder ($N \geq 3$) case, for the second-price auction with the $\text{Beta}(\frac{1}{N-1}, 1)$ distributed random reserve to be part of a Nash equilibrium across standard dominant-strategy mechanisms, the equilibrium joint distribution has to be π^{**} almost surely.*

Proof. As shown in the proof of Proposition 4, (B.10) holds with equality if and only if $v \in V^+$. This implies the equilibrium joint distribution has the support V^+ . Then this lemma follows from a similar argument to the proof of Lemma 3. \square

Proposition 10 follows immediately from Lemma 5 and the proof of Proposition 5. \square