Multi-point solution concepts of incomplete cooperative games

Martin Černý

April 16, 2022

Abstract

The model of incomplete cooperative games incorporates uncertainty into the classical model of (complete) cooperative games by considering a partial characteristic function. This leaves values of some of the coalitions unknown. The main focus of this paper is to initiate the study of multi-point solution concepts of incomplete cooperative games.

We generalise the standard solution concepts into the incomplete setting in the following manner. For an incomplete game, we determine the set of all complete games which coincide with the incomplete game on the known values of coalitions and satisfy further properties (e.g. being a member of a subset of cooperative games). Such games are called extensions. Now, we compute a standard solution concept for every such extension and take its union. Similarly, an intersection is considered.

A systematic analysis is performed for a variety of standard solution concepts as the core, the Weber set or the (pre-)kernel. Different sets of extensions (namely 1-convex and positive) are considered. Surprisingly, many of such generalisations yield the imputation set when we restrict to minimal incomplete games.

1 Cooperative games

Cooperation of players is an important concept in the game theory. We briefly present necessary basics of the cooperative game theory. For more information on cooperative games, see [Gra16, PS07, Pet08].

Definition 1. *A* cooperative game *is an ordered pair* (N, v)*, where* $N = \{1, 2, \ldots, n\}$ *and* $v\colon 2^N \to \mathbb{R}$ *is the characteristic function of the cooperative game. Further,* $v(\emptyset) = 0$*.*

Subsets of N are called *coalitions* and N itself is called the *grand coalition*. We write v instead of (N, v) when there is no confusion over what the player set is and we shall associate the characteristic functions $v\colon 2^N \to \mathbb{R}$ with vectors $v \in \mathbb{R}^{2^{|N|}}$.

1-convex cooperative games The *utopia vector* $b^v \in \mathbb{R}^n$ defined by $b^v_i \coloneqq v(N) - v(N \setminus i)$ represents the marginal value of player i for coalition N.

Definition 2. *A cooperative game* (N, v) *is called* 1-convex game*, if for all coalitions* $\emptyset \neq S \subseteq N$*, it holds*

$$v(S) \leq v(N) - \sum_{i \in N \setminus S} b^v_i \tag{1}$$

and also

$$\sum_{i \in N} b^v_i \geq v(N). \tag{2}$$

The set of 1-convex n-person games is denoted by C^n_1*.*

Condition (1) represents that even after every player outside the coalition S gets paid his utopia value, there is still more left of the value of the grand coalition for players from S to split among themselves than if they decided to separate from N. This condition challenges the players to remain in the grand coalition and try to find a compromise in the payoff distribution. Also, from (2), it follows the utopia demands are always at least as large as the value of the grand coalition N. This was motivated by the idea that the study of possible distributions is not interesting if every player can obtain utopia value.

Positive cooperative games The set of all n-person cooperative games forms a vector space. Shapley [Sha53] introduced an interesting basis of the vector space, so called *unanimity games*. Those are games (N, u_T) defined for $\emptyset \neq T \subseteq N$ as $u_T(S) \coloneqq 1$ if $T \subseteq S$ and $u_T(S) \coloneqq 0$ otherwise. Any game (N, v) can be thus expressed as a linear combination $v = \sum_{\emptyset \neq T \subseteq N} d_v(T) u_T$, where the coefficients $d_v(T)$ are called *Harsanyi dividends*. Now, positive cooperative games correspond to non-negative linear combinations of unanimity games. Interestingly, the set of positive cooperative games is a subset of *convex cooperative games*.

Definition 3. *A cooperative game (N, v) is* positive*, if it holds for all coalitions $\emptyset \neq T \subseteq N$ that*

$$d_v(T) \geq 0.$$

We denote the set of all positive cooperative n-person games by P^n.

1.1 Solution concepts and payoff vectors

The main goal of the cooperative game theory is to introduce methods of payments for players based on the values of the characteristic function. To be able to work with individual payoffs more easily, we employs *payoff vectors* $x \in \mathbb{R}^n$ where x_i represents the individual payoff of player i.

The definition of the payoff vector is quite general. This is why, for a cooperative game (N, v), one usually considers *preimputations* $\mathcal{I}^*(v)$ and *imputations* $\mathcal{I}(v)$,

- $\mathcal{I}^*(v) \coloneqq \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N)\}$,
- $\mathcal{I}(v) \coloneqq \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N)$ and $x_i \geq v(i)$ for $i \in N\}$.

Preimputations distribute the worth of the grand coalition N between the players. In addition, imputations also satisfy that every player receives higher profit than is his individual value. Subsets of (pre-)imputations may be viewed as payment methods and are called *solution concepts*.

1.1.1 The core and the Weber set

The core and the Weber set are closely intertwined solution concepts. The former one is a subset of the later and they are equal if and only if the cooperative game is *convex*.

Definition 4. *The* core *$\mathcal{C}(v)$ of a cooperative game (N, v) is the set*

$$\mathcal{C}(v) \coloneqq \{x \in \mathcal{I}(v) \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for } S \subseteq N\}.$$

Conditions $\sum_{i \in S} x_i \geq v(S)$ for $S \subseteq N$ assert *stability* of the concept. For any coalition S, it is disadvantageous to leave the grand coalition because the players in S are already receiving in total at least as much as they would if they decied to leave the grand coalition and distribute $v(S)$ among themselves.

The Weber set is defined as a convex hull of all marginal vectors. For a permutation $\sigma \in \Sigma_n$, we define the set of predecessors of i with respect to σ as $S_\sigma(i) \coloneqq \{j \in N \mid \sigma(j) < \sigma(i)\}$. A *marginal vector* $m^v_\sigma \in \mathbb{R}^n$ is then defined as $(m^v_\sigma)_i \coloneqq v(S_\sigma(i) \cup \{i\}) - v(S_\sigma(i))$.

Definition 5. *The* Weber set *$\mathcal{W}(v)$ of a cooperative game (N, v) is the set*

$$\mathcal{W}(v) \coloneqq conv\{m^v_\sigma \mid \sigma \in \Sigma_n\}.$$

1.1.2 The prekernel and the kernel

Both solution concepts were originally defined as easily computable subsets of the famous *bargaining set* and are closely connected. The definition of the kernel is slightly more restrictive than that of the prekernel. The main difference is in individual rationality of the payoff vectors. In both definitions, we employ the *excess* $e(S, x, v) \coloneqq v(S) - x(S)$ and the *maximal surplus of i over j at x*, $s_{ij}(x, v) \coloneqq \max_{S: i \in S, j \notin S} e(S, x, v)$.

Definition 6. *The* prekernel *$\mathcal{K}^*(v)$ of a cooperative game (N, v) is the set*

$$\mathcal{K}^*(v) \coloneqq \{x \in \mathcal{I}^*(v) \mid s_{ij}(x, v) \leq s_{ji}(x, v) \forall i \neq j\}$$

Definition 7. *The* kernel *$\mathcal{K}(v)$ of a cooperative game (N, v) is the set*

$$\mathcal{K}(v) \coloneqq \{x \in \mathcal{I}(v) \mid \forall i \neq j : (s_{ij}(x, v) - s_{ji}(x, v))(x_j - v_j) \leq 0 \text{ or } (s_{ij}(x, v) - s_{ij}(x, v))(x_i - v_i) \leq 0\}.$$

2 Incomplete cooperative games

Our model deals with following problems. First, consider there is a cooperative game, however, we manage to acquire only a subset of coalition values and some of the values remain unknown. Alternatively, we have to pay high prices for every such value, which we want to avoid if possible. Our goal is to distribute payoff between players according to known values. Also, we want to be able to say as much as possible about the underlying complete game. Second, we might consider incomplete games as a tool to analyse how certain properties of cooperative games depend on values of coalitions.

Definition 8. *An incomplete game is a tuple $(N, \mathcal{K}, v)$ where $N = \{1, \ldots, n\}$, $\mathcal{K} \subseteq 2^N$ is the set of coalitions with known values and $v\colon 2^N \to \mathbb{R}$ is the characteristic function of the incomplete game. We further assume that $\emptyset \in \mathcal{K}$ and $v(\emptyset) = 0$.*

From now on, we restrict to $\mathcal{K} = \{\emptyset, N\} \cup \{\{i\} \mid i \in N\}$. Such games are called *minimal incomplete games*. A fundamental tool for the study of incomplete games are the C-extensions.

Definition 9. (C-extension) *Let C be a class of n-person cooperative games. A cooperative game $(N, w) \in C$ is a C-*extension *of an incomplete game $(N, \mathcal{K}, v)$ if $w(S) = v(S)$ for every $S \in \mathcal{K}$.*

Any C-extension may be viewed as a possible way to *reconstruct* missing values of a complete game for which $(N, \mathcal{K}, v)$ represents its partial information. The set of all C-extensions of an incomplete game $(N, \mathcal{K}, v)$ is denoted by $C(v)$.

One of the first questions when analysing an incomplete cooperative game is to describe sets of different C-extensions. Since in many cases those sets form convex polyhedrons they can be described using their extreme points and extreme rays.

1-convex extensions The set of 1-convex extensions is determined by n extreme points (N, v^i) and $2^n - 2n - 2$ extreme rays (N, e_T) (both defined in [Bv21]).

Theorem 1. *[Bv21] For a minimal incomplete game $(N, \mathcal{K}, v)$ satisfying $v(N) \geq \sum_{i \in N} v(i)$, the set of C_1^n-extensions can be described as*

$$C_1^n(v) = \left\{ \sum_{i \in N} \alpha_i v^i + \sum_{T \in E} \beta_T e_T \mid \sum_{i \in N} \alpha_i = 1 \text{ and } \alpha_i, \beta_T \geq 0 \right\}.$$

Positive extensions When non-empty, the set of positive extensions is always bounded. Thus, it is given by a set of $2^n - n - 1$ extreme points (N, v^T) (defined in [MI16]).

Theorem 2. *[MI16] For a minimal incomplete game $(N, \mathcal{K}, v)$ satisfying $v(N) \geq \sum_{i \in N} v(i)$ and $N_{>1} \coloneqq \{T \subseteq N \mid |T| > 1\}$, the set of P^n-extensions can be described as*

$$P^n(v) = \left\{ \sum_{T \in N_{>1}} \alpha_T v^T \mid \sum_{T \in N_1} \alpha_T = 1, \alpha_T \geq 0 \right\}. \tag{3}$$

3 Our results

We are concerned with the following question:

Based only on the minimal information, i.e. values $v(\{1\}), \ldots, v(\{n\})$ and $v(N)$, how can one distribute the value of the grand coalition $v(N)$ among the players?

It is reasonable to restrict ourselves to the set of imputations. This is because the values concerned in the definition of the imputation set are exactly those known for minimal incomplete games. The set of imputations is nonempty if $v(N) \geq \sum_{i \in N} v(\{i\})$ and in that case it can be expressed as

$$\mathcal{I}(v) \coloneqq \{I^\alpha \mid \alpha \in \mathbb{R}_+^n \text{ and } \sum_{i \in N} \alpha_i = 1\}, \tag{4}$$

where $I_i^\alpha \coloneqq v(\{i\}) + \alpha_i \Delta$ and $\Delta = v(N) - \sum_{i \in N} v(\{i\})$. The imputation set can be interpreted as the set of all payments where every player receives his singleton value $v(i)$ plus a share from the additional worth of the cooperation of all players Δ. This includes payments as $v(\{i\}) + \frac{\Delta}{n}$ or $v(\{i\}) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}$ but also payments with any non-trivial distribution of Δ. Can we do any better and further restrict this set in a reasonable way? Can we take advantage of the ideas behind standard multi-point solution concepts? To answer these questions, we take the following approach. As an example, imagine that a minimal incomplete game $(N, \mathcal{K}, v)$ represents partial information of a positive game (N, w^*). To approximate its core $\mathcal{C}(w)$, we can consider the union of cores over all P^n-extension of $(N, \mathcal{K}, v)$,

$$\cup\, \mathcal{C}(P^n)(v) \coloneqq \bigcup_{w \in P^n(v)} \mathcal{C}(w). \tag{5}$$

Such set includes $\mathcal{C}(w^*)$ as its subset, thus, in a way, it may be interpreted as an approximation from above. Similarly, if we consider an intersection of all cores, we are presented with payoff vectors that are guaranteed to be present in $\mathcal{C}(w^*)$. Possibly, this yields a subset of $\mathcal{C}(w^*)$, thus serves as an approximation from below.

Generally, for a solution concept $\mathcal{S}$ and a set of C-extensions, we define the *weak C-solution* as

$$\cup\, \mathcal{S}(C)(v) \coloneqq \bigcup_{C\text{-extension } (N,w)} \mathcal{S}(w) \tag{6}$$

and the *strong C-solution* as

$$\cap\, \mathcal{S}(C)(v) \coloneqq \bigcap_{C\text{-extension } (N,w)} \mathcal{S}(w). \tag{7}$$

3.1 Weak and strong solution concepts

We focus on four solution concepts (the core, the Weber set, the prekernel and the kernel) and two sets of C-extensions (1-convex, positive) of minimal incomplete games. It seems the minimal information, despite its natural essence for many applications, does not yield an improvement over the set of imputations. The weak C_1^n-Weber set is even provably its superset. The only hope in yielding a proper subset of the imputation set is for the weak P^n-kernel, where we could not prove the equality with the imputation set, however, we have supporting arguments the equality might hold.

$\bigcup$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{K}$	$\mathcal{K}^*$
C_1^n	$\mathcal{I}(v)$	$\supsetneq$	$\mathcal{I}(v)$	$\mathcal{I}(v)$
P^n	$\mathcal{I}(v)$	$\mathcal{I}(v)$	$\subseteq$	$\subseteq$

While focused on the study of weak C-solution, we consider additional questions connected to them. For C_1^n-extensions, it makes sense to consider the union of solution concepts for a subset of C_1^n-extensions, where $\beta_T = 0$ for every $T \in E$. As these are both 1-convex and convex, they satisfy further properties. Further, we consider an explicit expression of the solution concept for every (or at least some) C-extension. This is done in situations where we fail to compute the weak C-solution exactly or in situations where this brings additional information to understanding the weak C-solution.

$\bigcap$	$\mathcal{C}$	$\mathcal{W}$	$\mathcal{K}$	$\mathcal{K}^*$
C_1^n	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
P^n	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Strong C-solutions seem to be too strong for minimal incomplete games in a sense that the intersection is empty in every setting we considered. One might believe the reason for this lies in extreme games, where the conditions of the solution concepts are too restrictive and yield a single payoff vector. However, we show that in many situations the solution concept contains one payoff vector for every C-extension as is the case in the following theorem.

Theorem 3. *For an incomplete game $(N, \mathcal{K}, v)$ satisfying $v(N) \geq \sum_{i \in N} v(\{i\})$, it holds for every C_1^n-extension (N, w) that $\mathcal{C}(w) = \{b^w\}$.*

Both weak and strong C-solutions support the argument that the imputation set is the best we can do when restricted to minimal incomplete games. Our method is also an interesting way to evaluate what can we say about multi-point solution concepts of cooperative games given only partial information.

References

[Bv21] J. Bok and M. Černý. 1-convex extensions of incomplete games and the average value. *arXiv preprint arXiv:2107.04679*, 2021.

[Gra16] M. Grabisch. *Set functions, games and capacities in decision making*. Springer, 2016.

[MI16] S. Masuya and M. Inuiguchi. A fundamental study for partially defined cooperative games. *Fuzzy Optimization Decision Making*, 15(1):281–306, 2016.

[Pet08] H. Peters. *Games Theory: A Multi-Leveled Approach*, volume 28. Springer Berlin Heidelberg, 2008.

[PS07] B. Peleg and P. Sudhölter. *Introduction to the Theory of Cooperative Games*, volume 34. Springer Science & Business Media, 2007.

[Sha53] L. S. Shapley. A value for n-person game. *Annals of Mathematical Studies*, 28:307–317, 1953.