Behavioral Real Options, Financial Literacy, and Investor's Inertia

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Abstract

I study a real-options model with a biased investor that faces uncertainty regarding the value of a project (VOP). I show that such a problem presents significant technical difficulties in using dynamic programming. By allowing the investor to compute the VOP from data, I transform the problem into one where dynamic programming is feasible. As she is biased, the investor's estimation process is subject to computation mistakes. I show that the biases lead to a wait-and-see approach: at the VOP at which a rational investor optimally exercises the option, the biased one is still unconvinced and waits for a more extreme valuation. Finally, I show that the wait-and-see approach explains the documented relationship between financial literacy and investors' inertia (investors with a poor understanding of financial concepts exhibit long inactivity spells).

Keywords: behavioral finance, real options, overconfidence, overprecision, financial literacy, investor's inertia

JEL Codes: G40, G50, G53, D14

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1 Introduction

The neoclassical theory of finance crucially relies on information-rationality assumptions. Examples are that investors perfectly observe relevant information, know how to interpret it, and make utility-maximizing decisions based on their unbiased perception of reality. In contrast, a large body of empirical evidence challenges these assumptions. Investors usually need to compute or interpret relevant information, a process that is subject to significant cognitive biases (Barberis and Thaler, 2003; Miller and Shapira, 2004; Szyszka, 2013). Specifically, the two most common biases affecting investors are the over-precision and inference biases. In the former, investors believe that their estimates are more precise than what they are (Freixas and Laux, 2011; Posen et al., 2018). In the latter, investors think that they can learn helpful information from irrelevant data (Dumas et al., 2009; Scheinkman and Xiong, 2003).[1]

In this paper, I study the effect of over-precision and inference biases on an investor's decisions. Specifically, I analyze a real-options model in which a biased investor has to estimate from data the evolution of the value of a potential investment project instead of perfectly observing it.[2] Every period, the investor updates her estimate of the VOP and decides whether to invest. I focus on the mechanisms through which the investor's biases affect her optimal investment rule, timing, and the expected value of the option.

My main finding is that the over-precision and inference biases induce a wait-and-see approach in the investor, which grows with her biases. At the VOP that convinces an unbiased investor to invest, the biased investor is still undecided and waits for a more favorable VOP. The wait-and-see approach emerges after the biased investor overestimates the volatility of the VOP. On the one hand, her over-precision bias makes her attribute part of her computation mistakes to the volatility of the VOP. On the other hand, her inference bias makes her believe that noisy information is informative of the VOP. As a result, she overestimates the probability of an extreme VOP occurring in the short term and finds it optimal to wait longer when an unbiased investor would exercise the option.

This finding also helps explain the relationship between financial literacy and investors' inertia. Prior work has documented a strong relationship between an investor's ability to understand basic financial concepts and the length of sub-optimal inactivity spells regarding her investments (Agnew et al., 2003; Bianchi, 2018; Bilias et al., 2010; Calcagno and Monticone, 2015; Calvet et al., 2009;

[1] More generally, over-precision and inference biases are particular forms of the overconfidence bias.

[2] A real option is the right—but not the obligation—to undertake certain business initiatives, such as deferring, abandoning, expanding, staging, or contracting a capital investment project. Examples are an investor considering whether to build a new factory, when to sell a machine, and whether to drill a new oil field.

Lusardi and Mitchell, 2014). This relationship is often associated with the status quo bias: investors with lower financial literacy resist change more. My model provides a micro-foundation for such a phenomenon: investors with less financial literacy are more prone to suffer from over-precision and inference biases, which exacerbates the wait-and-see approach. Exploiting this relationship, I show that an information provider can profit from strategically changing the information's complexity, as it modifies how long the investor waits before intervening in her investments.

I focus on real options throughout this article because they present a natural example of investments that require the investor to estimate the VOP. As such, the over-precision and inference biases are also naturally introduced. For instance, an investor deciding whether to open a new factory needs to estimate how doing so will affect her sales, costs, taxes, and brand value before deciding. These estimations depend on the potential mistakes of the investor and what information she believes is useful. However, my framework applies more widely. For instance, it pertains to financial options when the investor is allowed to misinterpret the market information.

My model is a generalization of the influential real-options model by McDonald and Siegel (1986) in that I allow the investor to have the over-precision and inference biases. The same generalization applies to other related work in real options, for instance, the one summarized in Dixit and Pindyck (1994) and Stokey (2009). An essential contrast of my work with this literature is that I study how the optimal investment rule and timing depend on the biases and, more generally, on the parameters of the VOP. In contrast, they focus on characterizing the investor's optimal behavior.

My work also connects with the literature of options with incomplete information (Danilova et al., 2010; Décamps et al., 2005; Ekström and Lu, 2011; Ekstrom and Vaicenavicius, 2016; Monoyios, 2009). Under incomplete information, a typical issue is that standard dynamic programming tools do not apply. To circumvent this issue, the literature usually assumes that the investor "filters" or learns about the VOP after progressively observing its realizations (cf. Bain and Crisan (2008)). However, the model solution under this approach often requires numerical methods that hinder the tractability of comparative statics exercises. In contrast, I show how when the investor estimates the VOP from data, the problem is re-casted into a standard optimal stopping problem with respect to her estimates.

The rest of the paper is organized as follows: Section 2 presents the theoretical model and Section 3 re-casts the imperfect information problem into a complete information one. Section 4 presents my theoretical results. Finally, Section 5 applies my model and results to the problem of financial literacy and investor's inertia.

2 Model

Time is continuous on an infinite horizon denoted by $t \geq 0$, and there is a biased investor considering whether to build a new factory. As such, the problem is one of American real options.

2.1 The Value of the Project

There is no objective source of information that tells the investor the value of having a new factory at a given point in time. Instead, she must estimate the value of the project (VOP) S_t from available data. Initially, the investor knows that two factors drive the VOP. On the one hand, industry-specific indicators like the manufacturing gross domestic product (GDP), employment, costs, and related. I call the summary of these indicators *the fundamental driver* of S_t, denoted f_t. On the other hand, the current overall state of the economy, denoted by $\sigma_s W_t$, where $\sigma_s \geq 0$ and W_t^s is a standard Brownian Motion. Then, the VOP evolves per the geometric Brownian Motion diffusion

$$\frac{dS_t}{S_t} = df_t + \sigma_s dW_t^s. \tag{1}$$

I assume that the investor knows the diffusion in equation 1, σ_s, and can perfectly observe the outcome of W_t^s every period. However, she must estimate f_t from data. Ex-ante, the investor only knows that the evolution of the fundamental driver evolves per the diffusion

$$df_t = \mu dt + \sigma_f dW_t^f, \tag{2}$$

where μdt is a trend, $\sigma_f > 0$ is the volatility, and W_t^f is a standard Brownian motion independent of everything else. However, she does not observe these factors and must estimate them from data. In my running example, μ can be the monthly increase in profit from having a new factory, W_t^s can be the state of the manufacturing sector at time t, and σ_f the sensitivity of the profits to shocks in the manufacturing sector. The investor does not directly observe these factors but needs to compute them with data from the Bureau of Economic Analysis and retrospective data from a factory she built before.

2.2 Decision Problem

The investor's problem is when, if at any time, to build the factory to maximize her expected gains from it. This problem is the American Call (AC) option problem. I analyze the case of the American Put in Appendix B as it is analogous.[3] Formally, letting $\mathbb{E}_s$ be a given expectation conditional on $S_0 = s$, $\mathcal{T}$ be the set of stopping times of the sigma-algebra generated by S_t, $\rho > 0$ be the investor's discount rate and $I(\Theta)$ the function that takes the value of one when the event Θ is true and zero otherwise,

Definition 1. Let ρ be a discount factor such that $\rho > \mu$. Subject to S_t evolving per equation 1, the optimal stopping problem for an investor with an American Call (AC) type of option is

$$V_{AC}(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s(e^{-\rho\tau}(S_\tau - E)^+ I(\tau < \infty)). \tag{3}$$

The (standard) assumption that $\mu < \rho$ ensures that the investor is impatient enough and has incentives to exercise the option in finite time. Solving the investor's decision problem presents an important technical difficulty. As a result of the uncertainty around f_t, the VOP's dynamics are not standard, and the problem cannot be solved using dynamic programming or asset pricing techniques. Specifically, $\sigma_s dW_t^s + \sigma_f dW_t^f$ is not a standard Brownian motion measurable with respect to the sigma-algebra generated by S_t, due to the fact that the investor cannot tell apart μdt from $\sigma_f dW_t^f$. In the following section, I show that this difficulty resolves when the investor solves the AC problem based on her estimates of the VOP, transforming the problem into a perfect information one.

3 Preliminaries: Data, VOP Estimation, and Investor's Biases

There are two sources of data available to the investor. On the one hand, a data set D_p that contains the realized data of a factory the investor built before. These data contains the information to estimate the parameters (μ, σ_f). On the other hand, there is a continuous-time data source D_w which contains period-to-period information about the state of the manufacturing sector W_t^f, for instance, the Bureau of Economic Analysis. After estimating (μ, σ_f) and inferring W_t^f, the investor uses the model in equation 2 to fit an estimate of f_t, and finally, to estimate S_t from equation 1. To introduce the biases, I assume that the investor has an overconfidence level $\phi \in [0, 1]$ that affects how she processes and what she believes she can learn from the data.

A key assumption of my model is that once the investor estimates (μ, σ_f) from D_p, she does not revise her estimates. This assumption does not affect—qualitatively—my results and allows for a clean comparative statics exercise. Allowing the investor to revise her estimates requires

[3] For short, in an American Call, the investor pays a fixed cost to enter a project. In an American Put, the investor receives a fixed payment to exit a project.

stochastic filtering techniques that rely on numeric methods. [4] My approach, instead, allows me to convert the problem into a perfect information one.

3.1 Parameter Estimation and Over-Precision Bias

The data about the older factory D_p has a complexity degree $n_p \geq 0$, which makes it hard to understand and process. For instance, D_p can represent a raw data set that requires cleaning and manipulation; the difficulty of the process is represented by n_p. The estimation process involves a mistake $\overline{\psi_p}(n_p) \in [0,1]$, which is increasing in n_p and unknown to the investor. By being overconfident, the investor suffers from an over-precision bias, namely, she believes that her mistakes are of size $\psi_p(n_p, \phi) \leq \overline{\psi_p}(n_p)$, for every n_p and ϕ and strict inequality whenever $\phi > 0$. $\psi_p \in [0,1]$ is strictly increasing in n_p and the more overconfident the investor, the smaller she believes her mistakes are, namely, ψ_p is strictly decreasing in ϕ. For consistency, I assume that an investor that does not suffer from overconfidence is fully aware of her mistakes, namely $\psi_p(n_p, 0) = \overline{\psi_p}(n_p)$.

For simplicity, I assume that $D_p = \{\gamma_i\}_{i=1}^N$ for N large. Each γ_i represents a data observation

$$\gamma_i \sim_{iid} \mu + \sigma_f \epsilon + \overline{\psi}_p(n_p)\nu, \tag{4}$$

with ϵ and ν standard Normal variables independent of everything else. Using the law of the large numbers, the investor estimates from D_p

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} \gamma_i \approx \mu \tag{5}$$

and

$$\hat{\Sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}(\gamma_i - \hat{\mu})^2 \approx \sigma_f^2 + (\overline{\psi}_p(n_p))^2. \tag{6}$$

The investor, believing that her estimation mistake is $\psi_p(n_p, \phi)$, corrects her standard deviation estimate to be

$$\hat{\sigma_f} = (\hat{\Sigma}^2 - (\psi_p(n_p, \phi))^2)^{1/2} = (\sigma_f^2 + \Psi_p(n_p, \phi))^{1/2}, \tag{7}$$

where $\Psi_p(n_p, \phi) = (\overline{\psi}_p(n_p))^2 - (\psi_p(n_p, \phi))^2$ is the investor's over-precision bias. Note that as $\Psi_p(n_p, \phi) \geq 0$, the investor over-estimates the variance of the system, namely, $\hat{\sigma_f} \geq \sigma_f$. Intuitively, due to the complexity of the data, the investor makes estimation mistakes, but she believes they are smaller than what they are, so she under-corrects. As a result, she wrongly attributes the remaining part of such mistakes to the variance of the system. Moving forward, I assume that the complexity

[4] See for instance Bain and Crisan (2008).

of the data affects the investor more when she is overconfident, namely that Ψ_p is strictly increasing in n_p for every $\phi > 0$.

3.2 Inference of W_t^f and Inference Bias

Every period, the investor observes a noisy signal of the state of the manufacturing sector

$$D_t = W_t^f + \epsilon_t, \tag{8}$$

for ϵ_t a standard Brownian motion independent of everything else. At a given period T, the investor observes $\{D_t\}_{t\leq T}$ and infers $\hat{W}_t^f = \mathbb{E}(W_t^f \mid \{D_t\}_{t\leq T})$. However, the investor's overconfidence leads to an inference bias: the investor believes she can learn about the manufacturing sector by looking at the services sector. In other words, there is an independent white noise variable ϵ_t which the investor believes to be correlated with W_t^f. To model this feature I follow (Dumas et al., 2009; Scheinkman and Xiong, 2003) and assume that the investor believes that ϵ_t has a correlation Ψ_w with W_t^f as in

$$\epsilon_t = \Psi_w(n_w, \phi)W_t^f + (1 - \Psi_w(n_w, \phi))\nu_t, \tag{9}$$

for ν_t a standard Brownian motion independent of everything else, and n_w the information complexity. $\Psi_w(n_w, \phi) \in [0, 1]$ is the investor's inference bias, with Ψ_w is strictly increasing in n_w and ϕ. As a result, the investor wrongfully believes that her data follows the structure

$$D_t = (1 + \Psi_w(n_w, \phi))W_t^f + (1 - \Psi_w(n_w, \phi))\nu_t. \tag{10}$$

Per Equation 10, the investor's inability to fully discard irrelevant information leads her to overestimate the informativeness of D_t about W_t^f. Based in equation 10, as shown in appendix A, at time t the investor estimates $\hat{W}_t^f$ and its dynamic to be

Lemma 1.

$$\hat{W}_t^f \equiv \mathbb{E}(W_t^f \mid \{D_k\}_{0\leq k\leq t}) = \underbrace{\frac{1 + \Psi_w(n_w, \phi)}{(1 + \Psi_w(n_w, \phi))^2 + (1 - \Psi_w(n_w, \phi))^2}}_{\equiv \alpha(\Psi_w)} D_t, \tag{11}$$

and

$$d\hat{W}_t^f = \frac{1 + \Psi_w(n_w, \phi)}{((1 + \Psi_w(n_w, \phi))^2 + (1 - \Psi_w(n_w, \phi))^2)^{1/2}} dB_t, \tag{12}$$

for B_t a standard Brownian motion independent of W_t^s. Moreover, the outcome of B_t is observable to the investor.

3.3 Estimation of the VOP and the Decision Problem Revisited

By substituting $(\hat{\mu}, \hat{\sigma}_f, d\hat{W}_t^f)$ from equations 5, 7, and 12 instead of (μ, σ_f, dW_t^f) in equations 1 and 2, the investor computes an estimate $\hat{S}_t$ of S_t. The next lemma shows that the investor's problem is a full-information one with respect to her estimates.

Lemma 2. There exists a standard Brownian motion W_t such that

$$\frac{d\hat{S}_t}{\hat{S}_t} = \mu dt + \sigma dW_t, \tag{13}$$

for

$$\sigma \equiv \sigma(n_p, n_w, \phi) = \sigma(\Psi_p, \Psi_w) = \left(\frac{(\sigma_f^2 + \Psi_p(n_p, \phi))(1 + \Psi_w(n_w, \phi))^2}{(1 + \Psi_w(n_w, \phi))^2 + (1 - \Psi_w(n_w, \phi))^2} + \sigma_s^2 \right)^{1/2}. \tag{14}$$

Moreover, σ is known to the investor and W_t is measurable with respect to $\hat{S}_t$.

Lemma 2 implies that the American option problems in Definition 4 are standard when the investor considers her estimates of the VOP. In this case, dynamic programming techniques apply to solve the problem. For that reason, moving forward, I focus on such problems subject to the dynamic in Equation 13. Moreover, I write S_t for the estimated VOP. The simplicity of my "estimation approach" contrasts with the often-difficult manipulations required to apply dynamic programming techniques in non-standard Markov scenarios (Danilova et al., 2010; Décamps et al., 2005; Ekström and Lu, 2011; Ekstrom and Vaicenavicius, 2016; Monoyios, 2009).

3.4 Benchmark Case: A Rational Investor

It is helpful to define a rational paradigm to analyze how the investor's biases affect her decisions. A rational investor is an unbiased one: in other words, she is aware of the full extent of her computation mistakes, and she understands what part of the signal is not informative when inferring the state of the world. Formally,

Definition 2. A rational investor is one for whom $\Psi_p = \Psi_w = 0$.

Directly computing from equations 11 and 14, the rational investor estimates the standard deviation of the system and the state of the world to be, respectively,

$$\sigma_R = (\frac{1}{2}\sigma_f^2 + \sigma_s^2)^{1/2} \tag{15}$$

and

$$\hat{W}_{R,t}^f = \frac{1}{2} D_t \tag{16}$$

4 Results

4.1 The Effect of the Biases on the Investor's Estimates

My following result analyzes how the complexity of the data and the investor's biases affect her parameters and state-of-the-world estimates. Formally,

Proposition 1. The following are true:

(i) A biased investor overestimates the variance of the system, namely $\sigma(\Psi_p, \Psi_w) > \sigma_R$ whenever $\Psi_p > 0$ or $\Psi_w > 0$.

(ii) A biased investor overweights the informativeness of D_t, namely, $\alpha(\Psi_w) > \frac{1}{2}$ for all $\Psi_w > 0$.

(iii) σ is strictly increasing in the size of the biases Ψ_p and Ψ_w, and thus, it is strictly increasing in ϕ, n_p and n_w.

(iv) There exists $\Psi_w^* \in (0, 1)$ such that $\alpha(\Psi_w)$ is strictly increasing for $\Psi_w < \Psi_w^*$ and strictly decreasing for $\Psi_w > \Psi_w^*$.

A biased investor mistakenly attributes part of her mistakes to the variance of data D_t. Likewise, she believes she can learn something from ϵ_t, attributing more variance to W_t^f than she should. As a result, she overestimates the variance of the system σ. Moreover, the miscalculations become more significant the greater the biases, leading to a more significant variance overestimation. This argument is behind parts (i), (ii), and (iii) of the proposition.

I let $\mathbb{P}_s$ be the probability distribution of S_t when $S_0 = s$ and simplify the notation to only write $\mathbb{P}$. I also let Φ be the standard Normal cumulative distribution function (CDF). Finally, I denote $S_t(\sigma_R)$ the VOP estimated by a rational investor and $S_t(\sigma(\Psi_p, \Psi_w)$ the one estimated by a biased investor. My first main result, Proposition 2, shows how the biases turn the investor conservative with her estimates, and such behavior gets exacerbated as time progresses. Formally,

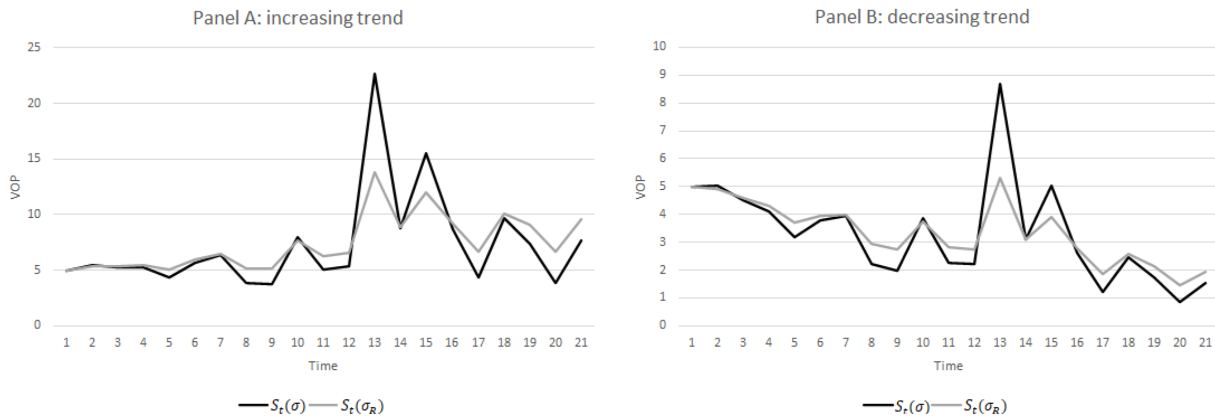
Proposition 2. The following are true when either $\Psi_p > 0$ or $\Psi_w > 0$:

(i) Compared to a rational investor, a biased investor is likely to be more conservative in her estimates of the VOP and, as time progresses, she is almost certain to have a smaller estimate. Formally, $P_t(\Psi_p, \Psi_w) \equiv \mathbb{P}(S_t(\sigma_R) \geq S_t(\sigma(\Psi_p, \Psi_w))) = \Phi(\frac{(\sigma(\Psi_p,\Psi_w)+\sigma_R)t}{2}) \geq \frac{1}{2}$ for every t with strict inequality whenver $(\Psi_w, \Psi_p) \neq (0, 0)$, and $P_t(\Psi_p, \Psi_w) \to 1$ as $t \to \infty$.

(ii) The investor becomes more conservative the bigger her biases. Formally, P_t is strictly increasing in Ψ_p and Ψ_w for every t.

(ii) As time progresses, the difference in estimates between a rational and a biased investor diverges independently of the size of the biases. Formally, whenever $(\Psi_w, \Psi_p) \neq (0, 0)$,

Figure 1: VOP undervaluation.



Note: Panel A shows the VOP systematic undervaluation for the case of an increasing trend where $\mu = 0.4$, $\sigma = 0.4$, $\sigma_R = 0.2$ and $s = 5$. Panel B shows the VOP systematic undervaluation for the case of a decreasing trend where $\mu = -0.4$, $\sigma = 0.4$, $\sigma_R = 0.2$ and $s = 5$. Implicit is the fact that $\sigma = \sigma(\Psi_p, \Psi_w)$.

$$\frac{S_t(\sigma_R)}{S_t(\sigma(\Psi_p,\Psi_w))} \to \infty \ \mathbb{P}\text{-almost-surely as } t \to \infty.$$

The facts that the biased investor is likely to underestimate the VOP and that her estimates diverge from the rational investor's stem from the *volatility drag* (Messmore, 1995) that is typical in Geometric Brownian Motion models like mine. Specifically, adverse shocks to the VOP disproportionately affect its estimated long-run compound (geometric) growth rate, and it takes the VOP longer to recover from decrements than in the case of a simple (arithmetic) growth rate. Moreover, the extent of this recovery delay is increasingly proportional to the volatility, as extreme negative realizations are more likely. As a result of these facts, in the long run, the difference between a rational and a biased investors' VOP becomes increasingly large in time.

4.2 Optimal Investment Rule and Time

Using standard dynamic programming techniques (Dixit and Pindyck, 1994; Peskir and Shiryaev, 2006), in Theorem 1 I characterize the optimal investment rule, the optimal investment time, the investor's value function, and the probability of exercising the option. Theorems 1 offers the classical solution to the American Options problem Dixit and Pindyck (1994); Peskir and Shiryaev (2006). It establishes an optimal stopping region because the investor does not exercise the option unless the VOP reaches a high critical value. As for the exercise time, if the starting VOP is over such a critical value, the investor exercises the option immediately. Meanwhile, suppose the starting VOP is below the critical value. In that case, the investor only exercises the option on finite time if the trend of the fundamental value is steep enough to increase the system's uncertainty. Formally,

Theorem 1. For an investor facing an AC problem it is optimal to exercise the option per the rule

$$D^*_{AC}(s) = \begin{cases} \text{Not invest}, & \text{if } s < s^*_{AC}(\Psi_p, \Psi_w), \\ \text{Invest}, & \text{if } s \geq s^*_{AC}(\Psi_p, \Psi_w), \end{cases}$$

for

$s^*_{AC}(\Psi_p, \Psi_w) = \dfrac{r}{r-1}E$ and $r = \left(\dfrac{1}{2} - \dfrac{\mu}{(\sigma(\Psi_p, \Psi_w))^2}\right) + \left[\left(\dfrac{1}{2} - \dfrac{\mu}{(\sigma(\Psi_p, \Psi_w))^2}\right)^2 + \dfrac{2\rho}{(\sigma(\Psi_p, \Psi_w))^2}\right]^{1/2} > 1.$

As a result, $\tau^*_{AC}(\Psi_p, \Psi_w) = \inf\{t \geq 0 : S_t \geq s^*_{AC}(\Psi_p, \Psi_w)\}$ is an optimal stopping time, with

$$\mathbb{P}_s(\tau^*_{AC}(\Psi_p, \Psi_w) < \infty) = \begin{cases} 1, & \text{if } s \geq s^*_{AC}(\Psi_p, \Psi_w) \text{ or } \mu \geq \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}, \\ \left(\frac{s}{s^*_{AC}(\Psi_p, \Psi_w)}\right)^{1 - \frac{2\mu}{(\sigma(\Psi_p, \Psi_w))^2}}, & \text{if } s < s^*_{AC}(\Psi_p, \Psi_w) \text{ and } \mu < \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}, \end{cases}$$

and

$$\mathbb{E}_s(\tau^*_{AC}(\Psi_p, \Psi_w)) = \begin{cases} \infty, & \text{if } s < s^*_{AC}(\Psi_p, \Psi_w) \text{ and } \mu \leq \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}, \\ \frac{\log\left(\frac{s^*_{AC}(\Psi_p, \Psi_w)}{s}\right)}{\mu - \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}}, & \text{if } s < s^*_{AC}(\Psi_p, \Psi_w) \text{ and } \mu > \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}, \\ 0, & \text{if } s \geq s^*_{AC}(\Psi_p, \Psi_w). \end{cases}$$

Finally,

$$V_{AC}(s) = \begin{cases} r^{-r}\left(\frac{r-1}{E}\right)^{r-1} s^r, & \text{if } s < s^*_{AC}(\Psi_p, \Psi_w), \\ s - E, & \text{if } \hat{s} \geq s^*_{AC}(\Psi_p, \Psi_w). \end{cases}$$

4.3 The Biases' Effect on the Investor's Decisions

My second main result follows. It establishes how the investor's biases affect her decision making. Formally,

Theorem 2. The following are ture:

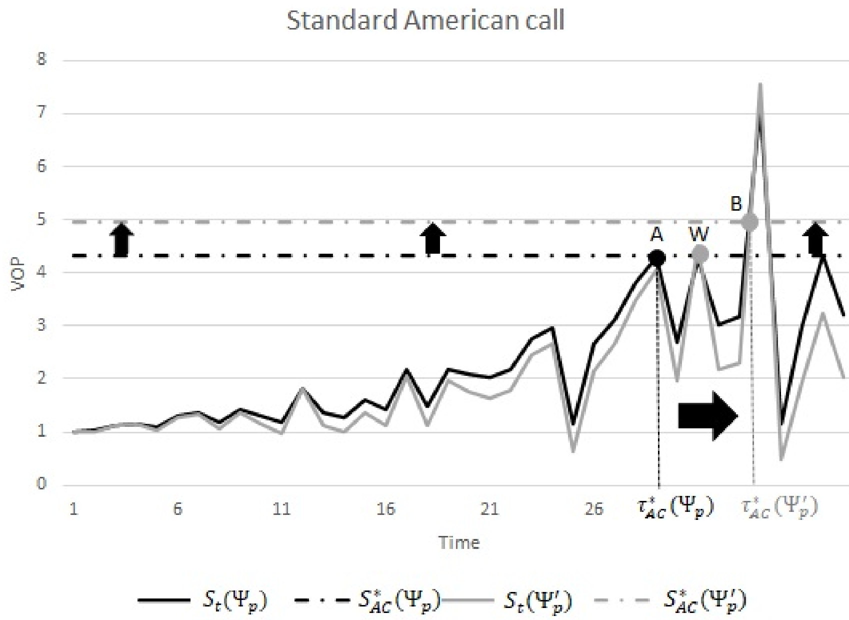
(i) The optimal stopping value $s^*_{AC}(\Psi_p, \Psi_w)$ is strictly increasing in Ψ_p and Ψ_w.

(ii) $s^*_{AC}(\Psi_p, \Psi_w) > s^*_R$ whenever $(\Psi_p, \Psi_w) \neq (0, 0)$, for s^*_R the rational investor's critical value.

(iii) When the investor exercises in strictly positive and finite time, $\mathbb{E}_s(\tau^*_{AC}(\Psi_p, \Psi_w))$ is strictly increasing in Ψ_p and Ψ_w, whereas it is constant otherwise.

Theorem 4 establishes that the more biased the investor, the higher the critical VOP she waits for before exercising the option. Then, a biased investor is more strict with the critical value she requires before she is convinced of exercising the option. In other words, at the VOP the rational

Figure 2: Comparative statics when Ψ_p increases, all else constant.



Note: I consider an increase in the bias $\Psi'_p > \Psi_p = 0$, so that the black lines represent the baseline situation and the gray ones represent the new situation. I also consider $\mu = 3$, $\sigma(\Psi_p) = 1$, $\sigma(\Psi'_p) = 1.5$, $s = 1$, $\rho = 5$ and $E = 1.5$. The rational investor exercises the option at point A, whereas the biased investor exercises it at point B. At point W, the biased investor reaches a VOP at which a rational investor would exercise the option, point at which she is still unconvinced of exercising and waits for a more extreme value, namely, point B. This is the wait-and-see approach.

investor would exercise the option (point W in figure 2), the biased investor is still unconvinced and decides to wait for an additional period until a more favorable VOP arrives (point B in figure 2).

Such a wait-and-see approach is optimal for a biased investor because by overestimating the variance of the system, she overestimates the probability of getting a more favorable draw of the VOP in the short term, making the wait seem more attractive relative to the time discount. As a result, the investor's expected time-to-exercise increases with the biases: not only does she requires a higher VOP before exercising, but she systematically underestimates the VOP, per Proposition 1. In Appendix A, I present additional results that relate the biases to the investor's value function and probability of exercising the option, which is in line with Theorem 4.

5 Application: Financial Literacy and Investor's Inertia

The average economic agent typically faces investment decisions regularly, such as retirement plans, savings, stocks and bonds, mortgages, loans, small business, health insurance, college education, and so on (Amram et al., 1998; Szyszka, 2013; Thaler and Sunstein, 2009). However, there is strong evidence that shows that the average investor lacks *financial literacy*, understood as the inability to understand basic financial concepts such as compound interest (Agnew et al., 2003; Bianchi, 2018; Bilias et al., 2010; Calcagno and Monticone, 2015; Calvet et al., 2009; Lusardi and Mitchell, 2014).

The literature above also finds a strong relationship between financial literacy and investors' inertia: investors with a more limited understanding of financial concepts exhibit more extended inactivity periods in their investments. This relationship between financial literacy and investors' inertia is often attributed to the status quo bias, but how the former turns into the latter is an open question. Using my model, I provide an answer: a less financially literate investor is more prone to make computation mistakes and more likely to believe irrelevant information is relevant. Finally, I show how a bank can strategically exploit the connection between financial literacy and investors' inertia to profit.

5.1 Setup

There is one bank that administers a low-risk instrument that an investor holds. Alternatively, the investor can divest her low-risk investment to invest in cryptocurrency, a high-risk asset. The bank periodically informs the investor about the current state of the low-risk investment. In contrast, the investor has access to information about the high-risk instrument via the internet and other sources. I focus on how the bank tailors the complexity of the information it reports to the investor.

The returns of the low-risk and high-risk investments respectively evolve per the geometric Brownian motions

$$\frac{dR_t}{R_t} = \overline{\mu}dt + \overline{\sigma}dW_t^R \tag{17}$$

and

$$\frac{dL_t}{L_t} = \underline{\mu}dt + \underline{\sigma}dW_t^L, \tag{18}$$

where $\overline{\mu}$, $\underline{\mu}$, $\overline{\sigma}$, and $\underline{\sigma} > 0$ are constants and W_t^R and W_t^L are independent standard Brownian motions. By definition, the high-risk investment involves more variance, namely, $\overline{\sigma} > \underline{\sigma} > 0$.

Moreover, I assume that the high-risk investment has a higher return on average, this is, $\overline{\mu} > \underline{\mu}$. These dynamics lead to an evolution of the relative benefit of the high-risk investment over the low-risk one of

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{19}$$

for W_t a standard Brownian motion, $\sigma = (\underline{\sigma}^2 + \overline{\sigma}^2)^{1/2}$, and $\mu = \overline{\mu} - \underline{\mu}$.[5]

5.1.1 The Investor

As in the main setup, the investor does not observe μ, σ and dW_t but has related data with complexity $n_p = n_w = n \geq 0$ which is provided by the bank. The investor estimates S_t and decides when, if at any point, to divest the low-risk investment to invest in cryptocurrency. Switching investments involves a cost $E > 0$ for the investor. Then, the investor's problem is as in Definition 4, subject to the estimated version of equation 19. From Theorem 1, it follows that the investor's optimal switching time is a function of the information complexity via her biases, namely $\tau^*(n)$.

The investor has a degree of financial literacy $\gamma \in [0, 1]$. I assume that the more financially literate she is, the smaller her biases—she can better understand her computation mistakes to account for them and understand which information is irrelevant, to avoid relying on it to infer about the state of the world. As in the main model, I also assume that the investor's biases grow with data complexity n. In other words, $\Psi_p = \Psi_p(n, \phi; \gamma)$ and $\Psi_w = \Psi_w(n, \phi; \gamma)$ are both strictly increasing in n and decreasing in γ.[6] For simplicity, I keep overconfidence fixed at the level ϕ, as I am interested in the interaction of data complexity n with the degree of financial literacy γ. Alternatively, my result holds under the alternative assumption that ϕ is a non-increasing function of γ.

The following lemma shows that investors with less financial literacy have a longer time-to-action because financial literacy reduces the size of the investor's biases:

Lemma 5.1. *For a given information complexity n, $\mathbb{E}_s(\tau^*(n))$ is strictly decreasing in γ.*

[5] Let $S_t = \frac{R_t}{L_t}$. Then, $\frac{dS_t}{S_t} = d\log(S_t) = d\log(R_t) - d\log(L_t) = \frac{dR_t}{R_t} - \frac{dL_t}{L_t}$, and the equation follows from the properties of independent standard Brownian motions.

[6] For instance, if $\psi_p(n_p, \phi) = ((1-\phi)n)^{1/\gamma}$, then $\Psi_p(n, \phi; \gamma) = n^{2/\gamma}(1 - (1-\phi)^{2/\gamma})$ meets my requirements. Likewise, $\Psi_w = (\phi n)^{1/\gamma}$ does.

5.1.2 The Bank

The bank gains a revenue $R(\tau;\tau_B)$ that is uniquely maximized at $\tau = \tau_B$, the bank's target switching time.[7] I further assume that R is a strictly concave function. At the same time, the bank is endowed with a base information-complexity $n_0 > 0$ and achieving the information complexity $n \geq 0$ incurs in a cost $C(n;n_0)$.[8] I that C is strictly convex and uniquely minimized at n_0. Moreover, I assume that C and R are twice-continuously differentiable and that there is an information complexity level that is prohibitively expensive, namely, there is $\overline{n}$ such that $C(\overline{n}) > R(\tau^*(n))$.[9] The bank's profit when it chooses the information complexity n is just $\pi(n) = R(\tau^*(n));\tau_B) - C(n;n_0)$.

5.2 The Game

I analyze the next game. In the first stage, the bank decides $n \in [0,1]$ to maximize its profits $\pi(n)$. At the beginning of the second stage, the investor observes n and estimates μ and σ as in previous sections. After that, at period t, the investor infers W_t, computes S_t, and solves the AC problem in definition 4. When the investor's optimal time arrives—if it does—the game is over. A strategy for the bank is a complexity level n. A strategy for the investor is, for any given n, an optimal stopping time τ^*. The equilibrium concept for the game is the following:

Definition 3. A strategy profile (n^*, τ^*) is an equilibrium if

(i) Given n, $\tau^*(n)$ solves the AC problem in definition 4.

(ii) Given $\tau^*(n)$, $n^* \in \arg\max_n \pi(n)$.

(iii) $\tau^*(n^*) = \inf\{\tau : \tau \text{ is optimal in the investor's AC problem}\}$.

Part (iii) of the equilibrium concept is a refinement that requires τ^* to be the smallest stopping time solving the investor's AC problem in definition 4. This refinement is needed because existing mathematical tools to solve optimal stopping problems rely on guess-and-verify arguments, and

[7] For instance, the bank might want to keep the investor as long as possible, so τ_B is infinitely large. In another case, the bank might want to retire the low-risk investment from the market in the short future, and τ_B is small.

[8] For instance, if the bank uses financial software to track the market, the output report has a quality n_0. Making the information simpler for the investor ($n < n_0$) requires analysts and is costly, so as including additional technical complex information ($n > n_0$).

[9] One example with these characteristics when the bank has a quadratic loss function L around τ_B, a cost of information complexity C, and makes a revenue R_0 per period the investor holds the low-risk investment, namely $R = R_0\tau^*(n) - (\tau_B - \mathbb{E}_s(\tau^*))^2$ and $C = (n_0 - n)^2$.

general results are about existence, not uniqueness. Part (iii) allows me to focus on the smallest among every possible optimal stopping time. To rule out the trivial case where the investor never switches independently of n, I assume that $S_0 = s < E$ and that the volatility of the estimated VOP has a suitable upper bound, namely $\frac{\sup_{\Psi_p,\Psi_w}(\sigma(\Psi_p,\Psi_w))^2}{2} < \mu$.

5.3 Equilibrium

The equilibrium optimal stopping time for the investor is just the solution to her stopping problem, computed in Theorem 1, $\tau^*(n)$. Therefore, the equilibrium solution can be computed by equating the bank's marginal revenue from the information complexity, to its marginal cost, namely, $R'(\mathbb{E}(\tau^*(n))\frac{d\mathbb{E}(\tau^*(n))}{dn} = C'(n)$. The following proposition summarizes the solution:

Proposition 3. There exists at least one equilibrium (n^*, τ^*) of the game, with the following properties:

(i) When $\mathbb{E}_s(\tau^*(n_0)) = \tau_B$, the equilibrium is unique and $n^* = n_0$.

(ii) When $\mathbb{E}_s(\tau^*(n_0)) > \tau_B$, the equilibrium is unique and $n^* < n_0$.

(iii) When $\mathbb{E}_s(\tau^*(n_0)) < \tau_B$, the equilibrium may not be unique but $n^* > n_0$.

Proposition 3 shows that in equilibrium, the bank exploits the relationship between financial literacy and the investor's biases to tailor the complexity of the information profitably. Specifically, in equilibrium, when the bank's goal is to make the investor wait for less, it invests in decreasing the complexity of the information. Meanwhile, when the goal is to make the investor wait longer, it invests in increasing the complexity of the information, and the optimal complexity level might not be unique. The potential multiplicity in the latter case follows from the fact that $\mathbb{E}(\tau(n))$ is convex in n, allowing for the possibility that the bank's cost grows at the same rate as the revenue when the information complexity increases. One way of ensuring the uniqueness of the equilibrium, in that case, is by assuming that R is concave enough so that $R(\mathbb{E}(\tau^*(n))$ is concave in n.

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A Mathematical proofs

Proof of Lemma 1. As W_t^f and ν_t are independent Standard Brownian motions. As a result, $t^{-1/2}W_t^f$ and $t^{-1/2}\nu_t$ are independent standard normal random variables. Therefore,

$$\begin{pmatrix} W_t^f \\ \nu_t \end{pmatrix} = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix} \cdot \begin{pmatrix} t^{-1/2}W_t^f \\ t^{-1/2}\nu_t \end{pmatrix} \Rightarrow \begin{pmatrix} W_t^f \\ \nu_t \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \right).$$

Thus, from equation 10 we have that

$$\begin{pmatrix} W_t^f \\ D_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1+\Psi_w & 1-\Psi_w \end{pmatrix} \cdot \begin{pmatrix} W_t^f \\ \nu_t \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & (1+\Psi_w) \\ (1+\Psi_w) & (1+\Psi_w)^2 + (1-\Psi_w)^2 \end{pmatrix} t \right).$$

As the Brownian motion has independent stationary increments, $\mathbb{E}(W_t^f \mid \{D_k\}_{0\le k\le t}) = \mathbb{E}(W_t^f \mid D_t)$. Then, the properties of the multivariate Normal imply that

$$\hat{W}_t^f \equiv \mathbb{E}(W_t^f \mid D_t) = \frac{1+\Psi_w(n_w,\phi)}{(1+\Psi_w(n_w,\phi))^2 + (1-\Psi_w(n_w,\phi))^2} D_t.$$

Moreover, from the multivariate normal distribution we have that

$$(1+\Psi_w)W_t^f + (1-\Psi_w)\nu_t \sim \mathcal{N}(0, [(1+\Psi_w)^2 + (1-\Psi_w)^2]t)$$

Therefore,

$$B_t \equiv \frac{(1+\Psi_w)W_t^f + (1-\Psi_w)\nu_t}{[(1+\Psi_w)^2 + (1-\Psi_w)^2]^{1/2}} \sim \mathcal{N}(0,t).$$

Note that the numerator of the B_t expression is simply D_t, whose outcome is observable to the investor. Therefore, the outcome of B_t is as well. As affine transformations don't affect the almost-surely continuity, start at zero or independent and stationary increments of standard Brownian motion, B_t is one. Rewriting

$$((1+\Psi_w)^2 + (1-\Psi_w)^2)^{1/2} B_t = (1+\Psi_w)W_t^f + (1-\Psi_w)\nu_t = D_t,$$

by plugging it in equation 11, we have that

$$\hat{W}_t^f = \frac{1+\Psi_w(n_w,\phi)}{((1+\Psi_w)^2 + (1-\Psi_w)^2)^{1/2}} B_t.$$

Finally, an application of Ito Lemma yields

$$d\hat{W}_t^f = \frac{1+\Psi_w(n_w,\phi)}{((1+\Psi_w)^2 + (1-\Psi_w)^2)^{1/2}} dB_t.$$

■

Proof of Lemma 2. Substituting equations 5, 7 and 12 in equation 2, and this last one into equation 1 we get

$$\frac{d\hat{S}_t}{\hat{S}_t} = \mu dt + \frac{(\sigma_f^2 + \Psi_p)^{1/2}(1+\Psi_w)}{((1+\Psi_w)^2 + (1-\Psi_w)^2)^{1/2}} dB_t + \sigma_s dW_t^s.$$

As B_t and W_t^s are independent Brownian motions,

$$\frac{(\sigma_f^2+\Psi_p)^{1/2}(1+\Psi_w)}{((1+\Psi_w)^2+(1-\Psi_w)^2)^{1/2}}B_t+\sigma_s W_t^s \sim \mathcal{N}(0,(\frac{(\sigma_f^2+\Psi_p)(1+\Psi_w)^2}{(1+\Psi_w)^2+(1-\Psi_w)^2}+\sigma_s^2)t).$$

Thus,

$$W_t \equiv \left(\frac{(\sigma_f^2+\Psi_p)(1+\Psi_w)^2}{(1+\Psi_w)^2+(1-\Psi_w)^2}+\sigma_s^2\right)^{-1/2}\left(\frac{(\sigma_f^2+\Psi_p)^{1/2}(1+\Psi_w)}{((1+\Psi_w)^2+(1-\Psi_w)^2)^{1/2}}B_t+\sigma_s W_t^s\right)$$
$$\sim \mathcal{N}(0,t),$$

and therefore, as affine transformations do not affect the almost-surely continuity, start at zero or stationary independent increments of the standard Brownian motion, W_t is one. As $(\sigma_f^2+\Psi_p)=\hat{\sigma}_f$, Ψ_w, σ_s, the outcome of B_t, and the outcome of W_t^s are observable to the investor, so is the outcome of W_t.

Applying Ito's Lemma and rearranging,

$$\left(\frac{(\sigma_f^2+\Psi_p)(1+\Psi_w)^2}{(1+\Psi_w)^2+(1-\Psi_w)^2}+\sigma_s^2\right)^{1/2}dW_t=\frac{(\sigma_f^2+\Psi_p)^{1/2}(1+\Psi_w)}{((1+\Psi_w)^2+(1-\Psi_w)^2)^{1/2}}dB_t+\sigma_s dW_t^s$$

Substituting this expression back in $\frac{d\hat{S}_t}{\hat{S}_t}$, we have that

$$\frac{d\hat{S}_t}{\hat{S}_t}=\mu dt+\underbrace{\left(\frac{(\sigma_f^2+\Psi_p)(1+\Psi_w)^2}{(1+\Psi_w)^2+(1-\Psi_w)^2}+\sigma_s^2\right)^{1/2}}_{\equiv\sigma}dW_t.$$

Finally, as the outcome of W_t, σ, and μ are observable to the investor, $\frac{d\hat{S}_t}{\hat{S}_t}$ is a standard Ito Diffusion and W_t is measurable with respect to the sigma algebra generated by $\hat{S}_t$.

■

Proof of Proposition 1. *Part (i).* Note that $\sigma(\Psi_p,\Psi_w)>\sigma^R$ if and only if

$$\frac{(\sigma_f^2+\Psi_p(n_p,\phi))(1+\Psi_w(n_w,\phi))^2}{(1+\Psi_w(n_w,\phi))^2+(1-\Psi_w(n_w,\phi))^2}>\frac{\sigma_f^2}{2}. \tag{20}$$

We show that equation 20 always holds if $\Psi_p>0$ or $\Psi_w>0$. To show it, we rewrite

$$\frac{(1+\Psi_w(n_w,\phi))^2}{(1+\Psi_w(n_w,\phi))^2+(1-\Psi_w(n_w,\phi))^2}=\frac{1}{1+(\frac{1-\Psi_w(n_w,\phi)}{1+\Psi_w(n_w,\phi)})^2}\geq\frac{1}{2}, \tag{21}$$

where the inequality follows from the fact that $\frac{1-\Psi_w(n_w,\phi)}{1+\Psi_w(n_w,\phi)}$ is maximized in $[0,1]$ at $\Psi_w=0$. Then, equality in equation 21 only holds if $\Psi_w=0$. If $\Psi_w>0$, equation 21 holds strictly so, along with $\Psi_p\geq 0$, it implies that equation 20 holds true. If $\Psi_w=0$, equation 21 holds with equality and then the fact that $\Psi_p>0$ makes equation 20 hold true.

Part (ii). $\alpha(\Psi_w)>\frac{1}{2}$ if and only if

$$\frac{1+\Psi_w}{(1+\Psi_w)^2+(1-\Psi_w)^2}>\frac{1}{2}\iff 2\Psi_w(\Psi_w-1)<0,$$

which holds true as $\Psi_w \in (0,1)$.

Part (iii). by writing

$$\sigma(\Psi_p, \Psi_w) = \frac{\sigma_f^2 + \Psi_p(n_p, \phi)}{1 + (\frac{1-\Psi_w(n_w,\phi))}{1+\Psi_w(n_w,\phi)})^2},$$

it is clear that σ is strictly increasing in Ψ_p and Ψ_w. Particularly, as Ψ_p and Ψ_w are strictly increasing in ϕ and n_p and n_w respectively, σ is as well.

Part (iv). It follows from direct computation through calculus that $\alpha(\Psi_w)$ is uniquely maximized at $\Psi_w^* = \sqrt{2} - 1$, while α is strictly concave in Ψ_w.

■

Proof of Proposition 2. For a given σ, the unique strong solution of equation 13 when $S_0 = s$ (cf. Øksendal (2003)) is

$$S_t(\sigma) = se^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}. \tag{22}$$

Thus, for a given t and a draw of W_t,

$$\begin{aligned} P_t(\Psi_p, \Psi_w) \equiv \mathbb{P}_s(S_t(\sigma_R) \geq S_t(\sigma)) = \mathbb{P}_s\big(\frac{(\sigma_R^2 - \sigma^2)t}{2} \geq (\sigma - \sigma_R)W_t\big) = \mathbb{P}_s(\frac{(\sigma_R + \sigma)t}{2} > W_t) \\ = \Phi(\frac{(\sigma_R + \sigma)t}{2}) \geq \frac{1}{2}, \end{aligned}$$

where the second-to-last equality follows from proposition 1's $\sigma_R < \sigma$ when $\Psi_w > 0$ or $\Psi_p > 0$, and Φ is the standard Normal CDF. From the continuity of Φ it is clear that $P_t(\Psi_p, \Psi_w) \to 1$ as $t \to \infty$; this completes the proof of part (i). As Φ is strictly increasing in its argument, it is strictly increasing in σ. Proposition 1 shows that σ is strictly increasing in Ψ_p and Ψ_w so Φ is as well; this completes part (ii).

Finally, from equation 22 we write

$$\frac{S_t(\sigma_R)}{S_t(\sigma(\Psi_p, \Psi_w))} = \exp\big((\sigma - \sigma_R)[(\sigma + \sigma_R)t - W_t]\big),$$

so $\frac{S_t(\sigma_R)}{S_t(\sigma(\Psi_p,\Psi_w))} \to \infty$ if and only if $(\sigma + \sigma_R)t - W_t \to \infty$, provided that $\sigma_R < \sigma$ by proposition 1. The latter statement is true $\mathbb{P}_s$-almost-surely; to see it, we write

$$(\sigma + \sigma_R)t - W_t = \underbrace{(\sqrt{2t\log(\log(t))})}_{A}\Big(\underbrace{\frac{(\sigma + \sigma_R)t}{\sqrt{2t\log(\log(t))}}}_{B} - \underbrace{\frac{W_t}{\sqrt{2t\log(\log(t))}}}_{AC}\Big).$$

Then, as $t \to \infty$, $A \to \infty$ and $B \to \infty$, and $C \to c \in [-1,1]$ $\mathbb{P}_s$-almost-surely by the law of iterated logarithm for standard Brownian motion.■

Proof of Theorem 1. The proof for the formulas of s^*_{AC}, $V_{AC}(s)$, $\mathbb{P}_s(\tau^*_{AC}(\Psi_p, \Psi_w) < \infty)$ and $\tau^*_{AC}(\Psi_p, \Psi_w)$ is an application of the theorem in section VII.2a.4 of Shiryaev (1999) taking $\lambda = \rho - \mu$

and the dynamics of S_t given by equation 13. The optimality of $D^*_{AC}(s)$ follows from standard arguments about continuation and stopping regions; specifically, from section 25 in Peskir and Shiryaev (2006).

For completeness, however, we present the key step behind the computations. Per standard arguments (Dixit and Pindyck, 1994; Peskir and Shiryaev, 2006), we guess that the value function $V_{AC}(s)$ meets the Hamilton-Jacobi Bellman equation (HJBE), the smooth pasting (SP) and value-matching (VM) conditions, and the boundary condition (BC) defined as follows:

$$\rho V_{AC}(S) = \frac{1}{2}(\sigma(\Psi_p, \Psi_w))^2 S^2 V''_{AC}(S) + \mu S V'_{AC}(S), \tag{HJBE}$$

$$V'_{AC}(s^*_{AC}) = 1, \tag{SP}$$

$$V_{AC}(s^*_{AC}) = s^*_{AC} - E, \tag{VM}$$

$$V_{AC}(0) = 0. \tag{BC}$$

Noticing that HJBE is a second-order Cauchy-Euler differential equation, we guess that $V_{AC}(S) = AS^r$ for some constant A; plugging back this proposed solution in the HJBE we get the fundamental quadratic form

$$Q(\sigma, \mu, \rho, r) \equiv \frac{1}{2}(\sigma(\Psi_p, \Psi_w))^2 r(r-1) + \mu r - \rho = 0. \tag{23}$$

It is easy to see that equation 23 has two solutions: $r_1 < 0$ and $r > 1$. Therefore, the general solution for HJBE is of the form $V_{AC}(S) = AS^r + BS^{r_1}$, for some constant B; BC however implies that $B = 0$, as $r_1 < 0$. From equation 23, one can see that r as defined in the statement of the theorem is the root bigger than one. The determination of A follows from SP and VM. Finally, a standard verification argument is done to show that $V_{AC}(S)$ as just computed is optimal (Shiryaev, 1999).

To finish the proof, we proceed to compute $\mathbb{E}_s(\tau^*_{AC})$. Suppose that $s < s^*_{AC}$ and $\mu < \frac{(\sigma(\Psi_p,\Psi_w))^2}{2}$; by the law of total probability, we write

$\mathbb{E}_s(\tau^*_{AC}(\Psi_p, \Psi_w)) = \mathbb{E}_s(\tau^*_{AC} \mid \tau^*_{AC} < \infty)\mathbb{P}_s(\tau^*_{AC} < \infty) + \mathbb{E}_s(\tau^*_{AC} \mid \tau^*_{AC} = \infty)\mathbb{P}_s(\tau^*_{AC} = \infty) = \infty$,

where the last equality follows from the fact that $\mathbb{P}_s(\tau^*_{AC} = \infty) > 0$. Now suppose that $s \geq s^*_{AC}(\Psi_p, \Psi_w)$: per $D^*_{AC}(s)$, it is optimal to stop immediately and thus $\tau^*_{AC}(\Psi_p, \Psi_w) = 0$, so that $\mathbb{E}_s(\tau^*_{AC}(\Psi_p, \Psi_w)) = 0$. Finally, if $s < \hat{s}^*_{AC}(\Psi_p, \Psi_w)$ and $\mu \geq \frac{(\sigma(\Psi_p,\Psi_w))^2}{2}$, we rely on martingale techniques to compute the expected stopping time. We show the result in several steps:

1. Note that when $S_0 = s$, $S_t = se^{(\mu - \frac{1}{2}(\sigma(\Psi_p,\Psi_w))^2)t + \sigma(\Psi_p,\Psi_w)W_t}$ is the unique strong solution to equation 13 (cf. Øksendal (2003)).
2. Observe that as a consequence we can write

$$\tau^*_{AC} = \{t \geq 0 : S_t \geq s^*_{AC}(\Psi_p, \Psi_w)\} = \{t \geq 0 : se^{(\mu - \frac{1}{2}(\sigma(\Psi_p,\Psi_w))^2)t + \sigma(\Psi_p,\Psi_w)W_t} \geq s^*_{AC}(\Psi_p, \Psi_w)\} =$$
$$\{t \geq 0 : W_t = (\sigma(\Psi_p, \Psi_w))^{-1}\big[\log\big(\frac{s^*_{AC}(\Psi_p, \Psi_w)}{s}\big) - (\mu - \frac{(\sigma(\Psi_p, \Psi_w))^2}{2})t\big]\},$$

where the last equality follows from the almost-surely continuity of the standard Brownian motion W_t.

3. Recall the exponential martingale (cf. Durrett (2019)) $M_t = e^{\theta W_t - \frac{1}{2}\theta^2 t}$ for $\theta \in \mathbb{R}$ and W_t an adapted standard Brownian motion; the Optional Sampling Theorem (cf. Klenke (2013)) implies that $M_{t \wedge \tau^*_{AC}(\Psi_p,\Psi_w)}$ is a martingale as well, for $t \wedge \tau^*_{AC}(\Psi_p, \Psi_w) \equiv \min\{t, \tau^*_{AC}(\Psi_p, \Psi_w)\}$. Then, the martingale property of $M_{t \wedge \tau^*_{AC}(\Psi_p,\Psi_w)}$ implies that

$$\mathbb{E}_s(M_{t \wedge \tau^*_{AC}(\Psi_p,\Psi_w)}) = \mathbb{E}_s(M_0) = 1.$$

4. Notice that step 2 implies—by the definition of $\tau^*_{AC}(\Psi_p, \Psi_w)$—that

$$W_t \leq (\sigma(\Psi_p, \Psi_w))^{-1}\big[\log\big(\frac{s^*_{AC}(\Psi_p, \Psi_w)}{s}\big) - (\mu - \frac{(\sigma(\Psi_p, \Psi_w))^2}{2})t\big],$$

for $t \leq \tau^*_{AC}(\Psi_p, \Psi_w)$. Thus, if $\theta > 0$,

$$0 \leq M_{t \wedge \tau^*_{AC}(\Psi_p,\Psi_w)} = e^{\theta W_{t \wedge \tau^*_{AC}(\Psi_p,\Psi_w)} - \frac{1}{2}\theta^2 t \wedge \tau^*_{AC}(\Psi_p,\Psi_w)} \leq e^{\theta W_{t \wedge \tau^*_{AC}(\Psi_p,\Psi_w)}} \leq$$
$$e^{\theta[(\sigma(\Psi_p,\Psi_w))^{-1}(\log\left(\frac{s^*_{AC}(\Psi_p,\Psi_w)}{s}\right) - (\mu - \frac{(\sigma(\Psi_p,\Psi_w))^2}{2})\tau^*_{AC})]} \leq e^{\theta(\sigma(\Psi_p,\Psi_w))^{-1}\log\left(\frac{s^*_{AC}(\Psi_p,\Psi_w)}{s}\right)} < \infty,$$

where the second-to-last inequality follows from the fact that $\theta(\mu - \frac{(\sigma(\Psi_p,\Psi_w))^2}{2})\tau^*_{AC} \geq 0$.

5. Notice that as $\mathbb{P}_s(\tau^*_{AC}(\Psi_p, \Psi_w) < \infty) = 1$, then $t \wedge \tau^*_{AC}(\Psi_p, \Psi_w) \to \tau^*_{AC}(\Psi_p, \Psi_w)$ almost-surely when $t \to \infty$: as a result, $M_{t \wedge \tau^*_{AC}(\Psi_p,\Psi_w)} \to M_{\tau^*_{AC}(\Psi_p,\Psi_w)}$ almost-surely when $t \to \infty$.

Part 4 implies that we can apply the Dominated Convergence Theorem (cf. Durrett (2019)) to the expression found in part 3, which by part 5 yields

$$\mathbb{E}_s(M_{\tau^*_{AC}}(\Psi_p, \Psi_w)) = \mathbb{E}_s(e^{\theta(\sigma(\Psi_p,\Psi_w))^{-1}[\log\left(\frac{s^*_{AC}(\Psi_p,\Psi_w)}{s}\right) - (\mu - \frac{(\sigma(\Psi_p,\Psi_w))^2}{2})\tau^*_{AC}(\Psi_p,\Psi_w)] - \frac{1}{2}\theta^2\tau^*_{AC}(\Psi_p,\Psi_w)}) = 1.$$

By letting $a = (\sigma(\Psi_p, \Psi_w))^{-1}\log\big(\frac{s^*_{AC}(\Psi_p,\Psi_w)}{\hat{s}}\big)$ and $b = (\sigma(\Psi_p, \Psi_w))^{-1}(\mu - \frac{(\sigma(\Psi_p,\Psi_w))^2}{2}) > 0$, the last expression is transformed into

$$\mathbb{E}_s(e^{-(b\theta + \frac{1}{2}\theta^2)\tau^*_{AC}(\Psi_p,\Psi_w)}) = e^{-\theta a}.$$

By defining $\psi = -(b\theta + \frac{1}{2}\theta^2) < 0$ so that $\theta = b + (b^2 - 2\psi)^{1/2} > 0$, the last expression becomes

$$\mathbb{E}_s(e^{\psi\tau^*_{AC}(\Psi_p,\Psi_w)}) = e^{-(b + (b^2 - 2\psi)^{1/2})a}.$$

This last expression is the moment generating function of $\tau^*_{AC}(\Psi_p, \Psi_w)$ when $\psi \leq \frac{b^2}{2}$. By taking the derivative of this expression with respect to ψ and evaluating at $\psi = 0$ we find that

$$\mathbb{E}_s(\tau^*_{AC}(\Psi_p, \Psi_w)) = \frac{\partial \mathbb{E}_s(e^{\psi \tau^*_{AC}(\Psi_p,\Psi_w)})}{\partial \psi}\bigg|_{\psi=0} = \frac{a}{b} = \frac{\log\left(\frac{s^*_{AC}(\Psi_p,\Psi_w)}{s}\right)}{\mu - \frac{(\sigma(\Psi_p,\Psi_w))^2}{2}},$$

which completes the proof. ■

Theorem 2. The following are true:

(i) The optimal stopping value $s^*_{AC}(\Psi_p, \Psi_w)$ is strictly increasing in Ψ_p and Ψ_w.

(ii) $s^*_{AC}(\Psi_p, \Psi_w) > s^*_R$ whenever $(\Psi_p, \Psi_w) \neq (0, 0)$, for s^*_R the rational investor's critical value.

(iii) When the investor exercises in strictly positive and finite time, $\mathbb{E}_s(\tau^*_{AC}(\Psi_p, \Psi_w))$ is strictly increasing in Ψ_p and Ψ_w, whereas it is constant otherwise.

(iv) $V_{AC}(s)$ is increasing in Ψ_p and Ψ_w, with the relationship strictly monotonic when $s < s^*_{AC}$.

(v) $\mathbb{P}_s(\tau^*_{AC}(\Psi_p, \Psi_w) < \infty)$ is increasing in Ψ_p and Ψ_w, with the relationship strictly monotonic if $s < s^*_{AC}(\Psi_p, \Psi_w)$ and $\mu > \frac{(\sigma(\Psi_p,\Psi_w))^2}{2}$.

Proof of Theorem 2. *Part (i).* Observe two things: first, from proposition 1, $\sigma(\Psi_p, \Psi_w)$ is strictly increasing in Ψ_p and Ψ_w. Second, from theorem 1, Ψ_p and Ψ_w only affect our objects of interest through σ. These two observations imply that it is enough to show the result for changes in σ. The first thing we show is that $r > 1$ from theorem 1 is strictly decreasing in σ. To see it, note that $x = r$ solves the HJBE from the proof of theorem 1, and thus, it solves the fundamental quadratic equation 23, namely

$$Q(\sigma, \mu, \rho, x) = \frac{1}{2}(\sigma(\Psi_p, \Psi_w))^2 x(x-1) + \mu x - \rho = 0.$$

We can compute then, through the Implicit Function Theorem,

$$\frac{dr}{d\sigma} = -\frac{\sigma r(r-1)}{\frac{1}{2}\sigma^2(2r-1)+\mu} = -\frac{\frac{2}{\sigma}r(r-1)}{2r-1+\frac{2\mu}{\sigma^2}} = -\frac{\frac{2}{\sigma}r(r-1)}{2[\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{(\frac{1}{2}-\frac{\mu}{\sigma^2})^2+\frac{2\rho}{\sigma^2}}]-1+\frac{2\mu}{\sigma^2}}$$

$$= -\frac{r(r-1)}{\sigma\sqrt{(\frac{1}{2}-\frac{\mu}{\sigma^2})^2+\frac{2\rho}{\sigma^2}}} < 0,$$

where $\frac{dr}{d\sigma} < 0$ follows from $r > 1$. Therefore, it follows immediately by definition that s^*_{AC} is strictly increasing in σ.

Part (ii). It follows immediately from part (i) and the fact that $\sigma_R = \sigma(0, 0)$.

Part (iii). From its definition, it is straightforward to see that $\frac{d\mathbb{E}_s(\tau^*_{AC}(\Psi_p,\Psi_w))}{d\sigma} = 0$ when $s \geq s^*_{AC}$ or $\mu \leq \frac{\sigma^2}{2}$. Direct derivative computation when $s < s^*_{AC}$ and $\mu > \frac{\sigma^2}{2}$ implies the remaining case.

Parts (iv) and (v) follow directly from algebra and the definitions, relying on $\frac{dr}{d\sigma} < 0$. In particular, part (iv) is a standard result quantitative finance theory that states that the value of a standard American option increases with the volatility (cf. The Greeks of the Black and Scholes model in Lee and Lee (2010)). ■

Proof of Lemma 3. The result follows from the fact that $\mathbb{E}_s(\tau^*)$ is strictly increasing in σ, which is strictly increasing in Ψ_p and Ψ_w, which are strictly decreasing in γ.

■

Proof of Proposition 3. For any given n, τ^*_{AC} from theorem 1 solves the investor's problem. Theorem 2.4 in Peskir and Shiryaev (2006) guarantees that if τ is another optimal stopping time for the AC problem in definition 4, $\tau^*(n) \leq \tau$. This shows that, for any given n, the only possible optimal stopping time for the investor meeting requirement (iii) of the equilibrium definition is $\tau^*(n) = \tau^*_{AC}$. Let $\hat{\tau} \equiv \mathbb{E}_s(\tau^*_{AC})$.

From part (iii) of Theorem 4, under our assumptions, $\hat{\tau}$ is strictly increasing in n because Ψ_p and Ψ_w are as well. Therefore, for each n, there exists a unique optimal $\hat{\tau}$ induced by it and there exists a unique n_B such that $\hat{\tau}(n_B) = \tau_B$. Moreover,

$$\pi'(n) = R'(\hat{\tau}(n))\hat{\tau}'(n) - C'(n)$$

and

$$\pi'(n) = R''(\hat{\tau}(n))[\hat{\tau}'(n)]^2 + R'(\hat{\tau}(n))\hat{\tau}''(n) - C''(n).$$

Part (i). As $\hat{\tau}$ is strictly increasing, when $\hat{\tau}(n_0) = \tau_B$, $n_B = n_0$. At $n^* = n_B = n_0$, then $\pi'(n^*) = 0$ so $n^* = n_0$ is a critical point of π. Moreover, $\pi''(n^*) < 0$ because $R'(\tau_B) = 0$, $C''(n) > 0$ and $R''(\tau) < 0$ for all τ and n. Thus, $n^* = n_0$ is a local maximum of π. Moreover, $\pi(n)$ is strictly quasi-concave when $n_B = n_0$, so the maximizer is unique.

Part (ii). As $\hat{\tau}$ is strictly increasing, when $\hat{\tau}(n_0) > \tau_B$, $n_B < n_0$. For $n_B \leq n < n_0$, $C'(n) \leq 0$ with equality only when $n = n_0$. At the same time, $R'(\tau_B) = 0$ and $R'(\hat{\tau}(n_0)) < 0$. As C', R', and $\hat{\tau}'$ are continuous in n, the intermediate value theorem implies that there is $n^* \in (n_B, n_0)$ such that $\pi'(n^*) = 0$. Moreover, as $\hat{\tau}''(n) > 0$ for any n and $R'(\hat{\tau}(n)) < 0$ for $n > n_B$, n^* is the only point in (n_B, n_0) where $\pi(n) = 0$. For the same reason, $\pi''(n^*) < 0$, so it is a local maximum. In fact, as $R'(\hat{\tau}(n)) > 0$ and $C'(n) < 0$ for $n < n_B$ and vice versa for $n > n_0$, n^* is the only critical point of π.

Part (iii). As $\hat{\tau}$ is strictly increasing, when $\hat{\tau}(n_0) < \tau_B$, $n_B > n_0$. Note that $\pi'(n) > 0$ for $n < n_0$, whereas $\pi'(n) < 0$ for $n > n_B$. As a result, no maximizer can exist outside $[n_0, n_B]$. Moreover, $\pi(n)$ is continuous in $[n_0, n_B]$, so by the extreme value theorem, a maximizer exists in such a range. Uniqueness, however, is not guaranteed as $R'(\hat{\tau}(n))\hat{\tau}'(n)$ is not necessarily monotonic in n.

$\blacksquare$

B The American Put Option

Definition 4. Let ρ be a discount factor such that $\rho > \mu$. Subject to S_t evolving per equation 1, the optimal stopping problem for an investor with an American Put (AP) type of option is

$$V_{AP}(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s(e^{-\rho\tau}(X - S_\tau)^+ I(\tau < \infty)). \tag{24}$$

The AP problem models situations where the investor is considering dropping a project whose value is S_t to recover a scrape value of X_t; a classic example of an AP problem is a firm considering selling a machine for its scrape value. The following Theorem is the analogous to Theorem 1.

Theorem 3. For an investor facing an AP problem it is optimal to exercise the option per the rule

$$D^*_{AP}(s) = \begin{cases} \text{Not invest,} & \text{if } s > s^*_{AP}(\Psi_p, \Psi_w), \\ \text{Invest,} & \text{if } s \le s^*_{AP}(\Psi_p, \Psi_w), \end{cases}$$

for

$$s^*_{AP}(\Psi_p, \Psi_w) = \frac{q}{q-1} X \text{ and } q = \left(\frac{1}{2} - \frac{\mu}{(\sigma(\Psi_p, \Psi_w))^2}\right) - \left[\left(\frac{1}{2} - \frac{\mu}{(\sigma(\Psi_p, \Psi_w))^2}\right)^2 + \frac{2\rho}{(\sigma(\Psi_p, \Psi_w))^2}\right]^{1/2} < 0.$$

As a result, $\tau^*_{AP}(\Psi_p, \Psi_w) = \inf\{t \ge 0 : S_t \le s^*_{AP}(\Psi_p, \Psi_w)\}$ is an optimal stopping time, with

$$\mathbb{P}_s(\tau^*_{AP}(\Psi_p, \Psi_w) < \infty) = \begin{cases} 1, & \text{if } s \le s^*_{AP}(\Psi_p, \Psi_w) \text{ or } \mu \le \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}, \\ \left(\frac{s^*_{AP}(\Psi_p, \Psi_w)}{s}\right)^{\frac{2\mu}{(\sigma(\Psi_p, \Psi_w))^2} - 1}, & \text{if } s > s^*_{AP}(\Psi_p, \Psi_w) \text{ and } \mu > \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}, \end{cases}$$

and

$$\mathbb{E}_s(\tau^*_{AP}(\Psi_p, \Psi_w)) = \begin{cases} \infty, & \text{if } s > s^*_{AP} \text{ and } \mu \ge \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}, \\ \frac{\log\left(\frac{s}{s^*_{AP}(\Psi_p, \Psi_w)}\right)}{\frac{(\sigma(\Psi_p, \Psi_w))^2}{2} - \mu}, & \text{if } s > s^*_{AP}(\Psi_p, \Psi_w) \text{ and } \mu < \frac{(\sigma(\Psi_p, \Psi_w))^2}{2}, \\ 0, & \text{if } s \le s^*_{AP}(\Psi_p, \Psi_w). \end{cases}$$

Finally,

$$V_{AP}(s) = \begin{cases} |q|^{|q|}\left(\frac{X}{|q|+1}\right)^{1+|q|} s^{|q|}, & \text{if } s > s^*_{AP}(\Psi_p, \Psi_w), \\ X - s, & \text{if } s \le s^*_{AP}(\Psi_p, \Psi_w). \end{cases}$$

Proof of Theorem 3. The proof is completely analogous to the proof of Theorem 1, just taking $\theta < 0$ in the proof of $\mathbb{E}_s(\tau^*_{AP}(\Psi_p, \Psi_w))$.■

The following Theorem is the analogous of Theorem 4 for the case of the AP option. However, the result depends on how far the initial condition $S_0 = s$ is from the optimal stopping boundary s^*_{AP}: this is because the biases induce undervaluation but at the same time, the investor waits for a more extreme lower valuation. As a result, it is not clear ex-ante if the optimal stopping time decreases or increases. If s is "close enough" to s^*_{AP}, the effect of s^*_{AP} dominates and the average time-to-exercise increases, whereas if s is "far enough" of s^*_{AP}, the latter effect dominates and the average time-to-exercise decreases.[10]

Theorem 4. The following are true:

(i) The optimal stopping value $s^*_{AP}(\Psi_p, \Psi_w)$ is strictly decreasing in Ψ_p and Ψ_w.

(ii) $s^*_{AP}(\Psi_p, \Psi_w) < s^*_R$ whenever $(\Psi_p, \Psi_w) \neq (0, 0)$, for s^*_R the rational investor's critical value.

(iii) When the investor exercises in strictly positive and finite time, $\mathbb{E}_s(\tau^*_{AP}(\Psi_p, \Psi_w))$ is is strictly increasing in Ψ_p and Ψ_w when $\frac{s}{s^*_{AP}(\Psi_p,\Psi_w)} < \overline{s}(\Psi_p, \Psi_w)$ and strictly decreasing when $\frac{s}{s^*_{AP}(\Psi_p,\Psi_w)} > \overline{s}(\Psi_p, \Psi_w)$, for $\overline{s}(\Psi_p, \Psi_w) = \exp\left(\frac{(\frac{\sigma(\Psi_p,\Psi_w)^2}{2} - \mu)}{\sigma^2\sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2\rho}{\sigma^2}}}\right)$. Otherwise, it is constant.

(iv) $V_{AP}(s)$ is increasing in Ψ_p and Ψ_w, with the relationship strictly monotonic when $s > s^*_{AP}$.

(v) $\mathbb{P}_s(\tau^*_{AP}(\Psi_p, \Psi_w) < \infty)$ is decreasing in Ψ_p and Ψ_w, with the relationship strictly monotonic if $s > s^*_{AP}(\Psi_p, \Psi_w)$ and $\mu < \frac{(\sigma(\Psi_p,\Psi_w))^2}{2}$.

Proof of Theorem 4. *Part (i).* Observe two things: first, from proposition 1, $\sigma(\Psi_p, \Psi_w)$ is strictly increasing in Ψ_p and Ψ_w. Second, from theorem 3, Ψ_p and Ψ_w only affect our objects of interest through σ. These two observations imply that it is enough to show the result for changes in σ. The first thing we show is that $q < 0$ from theorem 3 is strictly increasing in σ. To see it, note that $x = q$ solves the HJBE from the proof of theorem 1, and thus, it solves the fundamental quadratic in equation 23, namely

$$Q(\sigma, \mu, \rho, x) = \frac{1}{2}(\sigma(\Psi_p, \Psi_w))^2 x(x-1) + \mu x - \rho = 0.$$

We can compute then, through the Implicit Function Theorem,

$$\frac{dq}{d\sigma} = -\frac{\sigma q(q-1)}{\frac{1}{2}\sigma^2(2q-1) + \mu} = -\frac{\frac{2}{\sigma}q(q-1)}{2q - 1 + \frac{2\mu}{\sigma^2}} = -\frac{\frac{2}{\sigma}q(q-1)}{2[\frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2\rho}{\sigma^2}}] - 1 + \frac{2\mu}{\sigma^2}}$$
$$= \frac{q(q-1)}{\sigma\sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2\rho}{\sigma^2}}} > 0,$$

where $\frac{dq}{d\sigma} > 0$ follows from $q < 0$. Therefore, it follows immediately by definition that s^*_{AP} is strictly decreasing in σ; this shows part (i).

Part (ii). It follows immediately from part (i) and the fact that $\sigma_R = \sigma(0, 0)$.

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[10] It is worth noting that the critical value $\overline{s}$ that defines if s is far or close to s^*_{AP} is not absolute; it depends on the size of the biases, as well as on the rest of the parameters of the model.

Part (iii). Noticing that $\frac{ds^*_{AP}}{d\sigma} = -\frac{X}{(q-1)^2}\frac{dq}{d\sigma}$, when $s < s^*_{AC}$ and $\mu > \frac{\sigma^2}{2}$, we have

$$\frac{d\mathbb{E}_s(\tau^*_{AP}(\Psi_p, \Psi_w))}{d\sigma} = \frac{-(s^*_{AP})^{-1}\frac{ds^*_{AP}}{d\sigma}(\frac{\sigma^2}{2} - \mu) - \sigma \log(\frac{s}{s^*_{AP}})}{(\frac{\sigma^2}{2} - \mu)^2} =$$

$$\frac{\frac{(\frac{\sigma^2}{2} - \mu)}{\sigma\sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2\rho}{\sigma^2}}} - \sigma \log(\frac{s}{s^*_{AP}})}{(\frac{\sigma^2}{2} - \mu)^2} = \frac{\log(\bar{s}) - \log(\frac{s}{s^*_{AP}})}{\sigma(\frac{\sigma^2}{2} - \mu)^2},$$

which can be positive or negative depending on the numerator; this completes part (iii).

Parts (iv) and (v) follow directly from algebra and the definitions, relying on $\frac{dr}{d\sigma} < 0$ and $\frac{dq}{d\sigma} > 0$. In particular, part (iv) is a standard result quantitative finance theory that states that the value of a standard American option increases with the volatility (cf. The Greeks of the Black and Scholes model in Lee and Lee (2010)). ■