A CONTINUUM MODEL OF STABLE MATCHING WITH FINITE CAPACITIES

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Abstract. We introduce a unified framework for stable matching, which nests the traditional definition of stable matching in finite markets and the continuum definition of stable matching from Azevedo and Leshno (2016) as special cases. Within this framework, we introduce a novel continuum model, which makes individual-level probabilistic predictions.

Our model always has a unique stable outcome, which can be found using an analog of the Deferred Acceptance algorithm. The crucial difference between our model and that of Azevedo and Leshno (2016) is that they assume that the amount of student interest at each school is deterministic, whereas we assume that it follows a Poisson distribution. As a result, our model accurately predicts the simulated distribution of cutoffs, even for markets with only ten schools and twenty students.

We use our model to generate new insights about the number and quality of matches. When schools are homogeneous, we provide upper and lower bounds on students' average rank, which match results from Ashlagi et al. (2017) but apply to more general settings. Our model also provides clean analytical expressions for the number of matches in a platform pricing setting considered by Marx and Schummer (2019).

1. Introduction

Ever since Gale and Shapley (1962) defined stability in two-sided matching markets, the topic has generated a great deal of interest from academics and practitioners alike: their paper has over 7,000 citations, and variants of their deferred acceptance algorithm are used to assign medical residencies in the United States and public school seats in cities across the globe. These developments prompted the award of the 2012 Nobel Prize to Alvin Roth and Lloyd Shapley "for the theory of stable allocations and the practice of market design."

Although many aspects of this theory are now well-understood, it remains difficult to predict how changes to market primitives will affect the set of stable outcomes. A recent report by the Brookings institution highlighted this as a key challenge facing school choice initiatives:

> *Even if DA [Deferred Acceptance] algorithms are relatively simple, predicting how student assignment policies will affect enrollment and outcomes is difficult... This creates challenges for policymakers to assess a priori how policy decisions will affect students and schools – and creates potential for unintended negative consequences. (Kasman and Valant, 2019)*

Along these lines, Kojima (2012) shows that affirmative action policies that boost the priority of students in a targeted group may result in worse outcomes for all members of this group. Even when the direction of an effect is known, seemingly small changes can have surprisingly large consequences: Ashlagi et al. (2017) show that in balanced markets with random preferences, adding a single student dramatically increases students' average rank. Conversely, seemingly large changes may have minimal impact. In Boston, a 1999 compromise granted students living within a school's walk zone priority for 50% of its seats. Over a decade later, Dur et al. (2018) demonstrated that this policy led to virtually identical outcomes as a policy that granted no walk zone priority at all!

In an effort to better understand two-sided matching markets, researchers have turned to a variety of "large market" and "continuum" models. An ideal approach would be *flexible* enough to incorporate complex preferences and priorities, able to *accurately* predict match outcomes, and would offer new *insights*.

One line of work studies outcomes in large finite markets. To maintain tractability, these papers typically impose strong assumptions. For example,Pittel (1989), Knuth (1996), Ashlagi et al. (2017), Ashlagi et al. (2019), Ashlagi and Nikzad (2019) and Kanoria et al. (2021) all assume that schools are symmetric (student preferences are iid and uniformly distributed), and all but Knuth (1996) assume that school priorities are either identical or drawn independently and uniformly at random. Recent work by Arnosti (2021) permits schools to differ along a one-dimensional measure of "popularity," but still imposes strong assumptions on preferences and priorities. These papers offer *insights* into how match outcomes depend on the choice of proposing side, the market imbalance, and the length of student lists. However, the analysis underlying these insights is not *flexible* enough to accommodate more realistic assumptions on preferences and priorities.

As a result, these models are of limited use when trying to tackle practical problems. For example, parents may want to know how likely their child is to be admitted to a particular school. Administrators may want to predict how a proposed policy change will affect the number of students who fail to match to any school on their list. For these problems, the best tool is often simulation. Abdulkadiroglu et al. (2009) use simulations to compare different tiebreaking procedures in New York City and Boston, and de Haan et al. (2018) do the same for Amsterdam. Ashlagi and Nikzad (2019) and Kanoria et al. (2021) use simulation data from New York to answer different questions. While simulation is a very *flexible* and *accurate* tool, it typically does not offer much *insight*. It shows what is true, but not why it is true. If a pattern is observed through simulation, it can be difficult to predict whether the same pattern will continue to hold in other settings.

1.1. **The Continuum Model of Azevedo and Leshno (2016).** Perhaps the work that comes closest to hitting the trifecta of flexibility, accuracy, and insightfulness is that of Azevedo and Leshno (2016). Students in their model are described by a "type" θ, which determines both their preferences and their priority score at each school. Student types are distributed according to an (almost) arbitrary measure η, giving the model *flexibility* to capture complex preferences and priorities. This measure determines the set of market-clearing "cutoff scores": school-specific scores such that if schools admit students with priority above their cutoff score, and students attend their favorite school where they are admitted, expected "demand" (enrollment) is equal to capacity at each school with a positive cutoff. This approach enables *tractable* analysis into the effect of market primitives on the final cutoffs.

The *accuracy* of their model at predicting outcomes in finite random markets depends on the number of seats at each school. We elaborate on this point, as it motivates our work. Hypothetically, if each school were to use its predicted cutoff score to determine admissions, and n student types were drawn iid from the measure η, then a school h with expected demand equal to its capacity C_h would have *realized* demand following a binomial distribution with parameters n and C_h/n. In

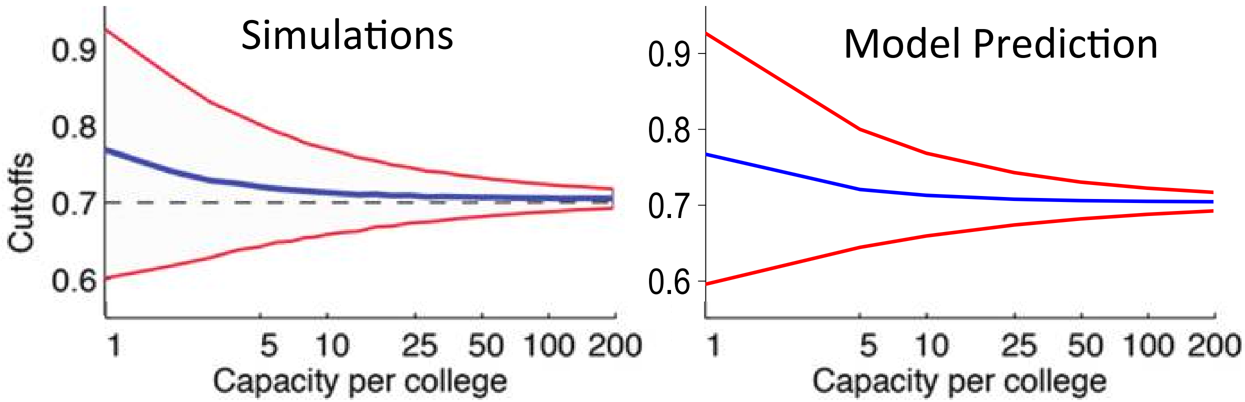

FIGURE 1. The distribution of "cutoff scores" required for admission in a class of examples considered by Azevedo and Leshno (2016). In this market there are ten schools, and the x-axis denotes the number of seats per school, which ranges from 1 to 200. In all scenarios, there are twice as many students as seats. Students submit complete lists drawn uniformly at random, and schools' priority scores are imperfectly correlated: they consist of the average of the student's quality (drawn uniformly on $[0,1]$) and iid student-school terms (also drawn uniformly on $[0,1]$). This correlation renders direct analysis of the finite random market intractable. The left panel displays simulation results reported by Azevedo and Leshno (2016): the blue line shows the empirical average cutoff score, with the red lines representing the 5^{th} and 95^{th} percentile of the empirical distribution. Their continuum model predicts a deterministic cutoff score, shown by the black dotted line. Note that this prediction does not depend on the number of seats at each school, and does not capture the uncertainty in cutoffs, which is significant unless school capacities are large. Our proposed alternative predicts the *distribution* of cutoff scores. The right panel shows the average, 5^{th} percentile and 95^{th} percentile of the predicted distribution.

reality, capacity constraints must hold for each realization (rather than only in expectation), so the fluctuations in realized demand translate to fluctuations in realized cutoff scores. As a result, a student whose priority is a bit above the predicted cutoff score is not truly "safe", and one with priority below the predicted cutoff is not hopeless. Rather, students' probability of admission varies continuously with their priority.

If C_h is large, then a Binomial with parameters n and C_h/n will be concentrated around its expectation, and the fluctuation in cutoff scores will be minor. That is, students with priority significantly above the predicted cutoff are almost certain to be admitted, while those with priority significantly below the predicted cutoff have almost no chance. This is formalized in Theorem 2 and Proposition 3 from Azevedo and Leshno (2016).

However, the capacity of each school is small, then random variation in student demand results in significant variation in schools' realized cutoff scores, as shown in Figure 1. Their model fails to capture this variability, causing it to produce inaccurate predictions. For a simple (if stylized) illustration of this point, consider a market where n schools each have capacity C and nC students each list a single school selected uniformly at random. Their model predicts a cutoff score of zero for each school, implying that all students will match to their first choice. If C is very large, this is nearly correct, but if $C = 1$, the expected fraction of students who match is $1-(1-1/n)^n \approx 1-1/e$.

1.2. **Our Contributions.** In Section 4, we introduce a model that addresses this shortcoming, and predicts a distribution (rather than a point estimate) for each school's cutoff score. As noted above, if $D_h(P)$ denotes *expected* demand at school h when all schools use cutoff scores given by P, then *realized* demand at h will follow a binomial distribution with parameters n and $D_h(P)/n$. If the number of students n is modestly large, this is well-approximated by a Poisson distribution with mean $D_h(P)$. Motivated by these observations, our model assumes that for any potential cutoff $p \in [0,1]$, the realized demand at school h follows a Poisson distribution.

We use this assumption to calculate a cutoff score distribution for each school. Of course, the calculation for school h does not assume that schools other than h use deterministic cutoff scores. Instead, we look for a set of "self-consistent" cutoff score distributions (in a sense made precise in Section 2).

Our model, like that of Azevedo and Leshno (2016), is *flexible* enough to permit a nearly arbitrary joint distribution of student preferences and priorities. We demonstrate its *tractability* by reproducing key insights from the recent work of Ashlagi et al. (2017) in Section 4.1 (and extending these insights to settings where students submit incomplete lists and schools have multiple seats), and by providing closed-form expressions for the number of matches in a setting considered by Marx and Schummer (2019) in Section 4.2. Finally, our new model is *accurate*. As school capacities grow, its predictions converge to those of Azevedo and Leshno (2016). When capacities are not large, we demonstrate numerically that our model accurately predicts a range of outcomes: Figure 1 compares its predictions for the distribution of school cutoffs to simulations from Azevedo and Leshno (2016), Figure 2 compares its predictions for students' average rank to simulations from Ashlagi et al. (2017), and Figure 5 compares its predictions for the number of matches formed to exact results and simulations from Marx and Schummer (2019). In all three cases, the match is excellent.

In addition to providing a new model which is flexible, tractable, and accurate, we provide a new framework for studying stable matching. This framework describes matchings through three different perspectives: *admissions functions* which describe the distribution of school cutoff scores, *interest functions* which describe the expected number of students at each priority level who wish to attend each school, and *matchings* which describe the probability that students are assigned to each school on their list.

Our framework identifies a large class of stable matching models, parameterized by a *measure of student types* η and a *vacancy function* $\mathcal{V}$. We show that suitable choices of these parameters recover the traditional model of stable matching in finite markets (Proposition 1) as well as the continuum model of Azevedo and Leshno (2016) (Proposition 2). Both of these definitions use a deterministic vacancy function $\mathcal{V}^{det}$ with range $\{0,1\}$. By contrast, our model from Section 4 uses a vacancy function $\mathcal{V}^{pois}$ based on the Poisson distribution, which allows it to make probabilistic predictions.

In addition to offering a unified perspective on several definitions of stable matching, our framework identifies other definitions associated with other choices of the vacancy function $\mathcal{V}$. We show that for *any* weakly decreasing vacancy function, classic results for stable matchings continue to hold: stable matchings always exist and form a lattice (Theorem 1), and the extreme points of this

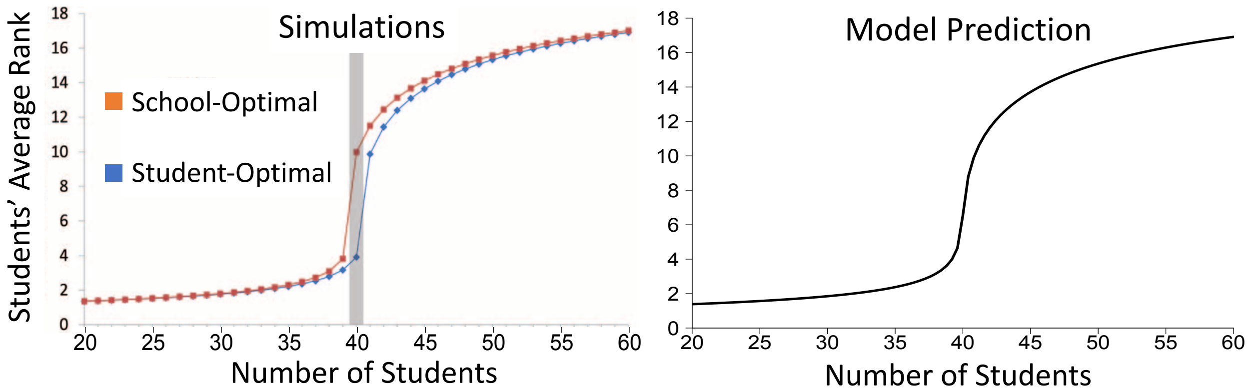

FIGURE 2. Students' average rank for their assigned school. In this market, there are 40 schools, each with a single seat. The number of students is given along the x axis, and both student preferences and school priorities are drawn iid and uniformly at random. The left panel shows simulation results from Ashlagi et al. (2017) demonstrating that (i) the difference between the school-optimal and student-optimal stable match is typically small, and (ii) in a balanced market (highlighted in gray), adding or removing one student has a dramatic effect. Our proposed model of stable matching generates a unique prediction (right), which closely matches the simulations and captures the dramatic effect of additional students in nearly-balanced markets.

lattice can be found using a generalization of the Deferred Acceptance algorithm (defined in Section 3.4). If η has no mass points, then the "rural hospital theorem" holds (Theorem 2). Finally, if the vacancy function $\mathcal{V}$ is *strictly* decreasing, then there is a unique stable matching (Theorem 3). This uniqueness result holds more generally than that of Azevedo and Leshno (2016), and helps to explain the small core observed empirically by Roth and Peranson (1999), and theoretically by Immorlica and Mahdian (2005), Kojima and Pathak (2009), and Ashlagi et al. (2017).

2. Model

There is a finite set of high schools $\mathcal{H}$. School $h \in \mathcal{H}$ has capacity $C_h \in \mathbb{N}$. We let $\mathcal{H}_0 = \mathcal{H} \cup \{\emptyset\}$ denote the set of schools along with the outside option of going unassigned, and define $C_\emptyset = \infty$. Let $\mathcal{R}$ be the set of complete orders over $\mathcal{H}_0$. We impose no restriction on the number of acceptable schools for each student (i.e. the number of schools preferred to the outside option $\emptyset$), and the order of schools ranked below $\emptyset$ will be irrelevant.

Students are characterized by their type $\theta = (\succ^\theta, \mathbf{p}^\theta)$, where $\succ^\theta \in \mathcal{R}$ indicates the student's preferences and $\mathbf{p}^\theta \in [0,1]^{\mathcal{H}}$ indicates the student's priority score at school h (higher is better). We let $\Theta = \mathcal{R} \times [0,1]^{\mathcal{H}}$ denote the space of student types. Students are distributed according to a positive finite measure η over Θ.

A fractional matching is a function M mapping each $\theta \in \Theta$ to a probability distribution on $\mathcal{H}_0$. For each $h \in \mathcal{H}_0$ and $\theta \in \Theta$, the quantity $M_h(\theta)$ can be interpreted as the probability that a student of type θ is assigned to h. Hereafter, we use "matching" to mean a fractional matching, and denote the space of matchings by $\mathfrak{M}$.

We now define what it means for a matching to be *stable*. Our definition uses two auxiliary concepts, which are based on the perspective of individual agents. What matters to each student is the set of schools that admit them. What matters to a school is the set of students who are "interested," meaning that they would attend if admitted. In our model, these are described by

- An *admissions function* $A : [0,1] \to [0,1]^{\mathcal{H}_0}$.
- An *interest function* $I : [0,1] \to \mathbb{R}_+^{\mathcal{H}_0}$.

Given $h \in \mathcal{H}$ and $p \in [0,1]$, $A_h(p)$ can be interpreted as the probability that a student with priority p at h will be admitted, while $I_h(p)$ can be interpreted as the measure of interest in school h from students whose priority at h exceeds p. Given these interpretations, it is natural that A_h should be increasing and I_h should be decreasing. We let $\mathfrak{A}$ denote the set of componentwise weakly increasing functions from $[0,1]$ to $[0,1]^{\mathcal{H}_0}$, and let $\mathfrak{I}$ denote the set of componentwise weakly decreasing functions from $[0,1]$ to $\mathbb{R}_+^{\mathcal{H}_0}$.

We will define consistency conditions that link a matching M to school interest I and student admissions decisions A. Formally, we define maps $\mathcal{I} : \mathfrak{M} \to I$, $\mathcal{A} : \mathfrak{I} \to \mathfrak{A}$ and $\mathcal{M} : \mathfrak{A} \to \mathfrak{M}$, and define a stable matching as a fixed point of the composition of these maps.

Although our approach may seem unfamiliar, our definition subsumes existing ones. The function $\mathcal{I}$ depends on the measure of student types η, and the function $\mathcal{A}$ depends on a *vacancy function* $\mathcal{V}$. Section 3.1 shows that for a particular choice of η and $\mathcal{V}$, our definition of a stable matching coincides with the absence of blocking pairs in a finite market. Section 3.2 shows that when η is changed to a continuous measure, any matching that is stable according to our definition is associated with a set of market-clearing cutoffs, and vice versa.

2.1. **Matching to Interest.** Given any matching $M \in \mathfrak{M}$, define $\mathcal{I}(M) \in \mathfrak{I}$ to be the interest function I^M such that for each $h \in \mathcal{H}_0$ and $p \in [0,1]$,

$$I_h^M(p) = \int \mathbf{1}(p_h^\theta > p)(1 - \sum_{h' \succ^\theta h} M_{h'}^\theta)\, d\eta(\theta). \tag{1}$$

Note that the sum in (1) gives the probability under matching M that student type θ matches to a school preferred to h, so the interpretation of (1) is that students are interested in h if they are not matched to any preferred school. The indicator ensures that the only students contributing to $I_h^M(p)$ are those with priority above p at h, allowing us to interpret $I_h^M(p)$ as the expected number of students with priority above p who are interested in h.

2.2. **Interest to Admissions.** The interest function I describes expected interest at each school $h \in \mathcal{H}$ and priority level $p \in [0,1]$. From this, we wish to determine an admissions function $A : [0,1] \to [0,1]^{\mathcal{H}}$, where $A_h(p)$ is interpreted as the probability that a student with priority p at h will be admitted to h (equivalently, the probability that school h has a final cutoff below p). We define A using a *vacancy function* $\mathcal{V} : \mathbb{R}_+ \times \mathbb{N} \to [0,1]$. Formally, we let $\mathcal{A}(I)$ be the admissions function $A^I \in \mathfrak{A}$ that satisfies, for each $h \in \mathcal{H}$ and $p \in [0,1]$,

$$A_h^I(p) = \mathcal{V}(I_h(p), C_h). \tag{2}$$

We define $A_\emptyset^I(p) = 1$ for all $p \in [0,1]$ (students are always admitted to the outside option).

The choice of vacancy function is an important feature of the model, and one of our key innovations. The quantity $\mathcal{V}(\lambda, C)$ is interpreted as the probability that when *expected* interest is equal to λ, *realized* interest will be below C. Thus, if schools consider students in descending order of priority, $\mathcal{V}(I_h(p), C_h)$ gives the probability that school h will still have at least one vacancy when it considers a student with priority p.

One natural choice of vacancy function is

$$\mathcal{V}^{det}(\lambda, C) = \mathbf{1}(\lambda < C) \qquad \forall \lambda \in \mathbb{R}_+, C \in \mathbb{N}, \tag{3}$$

In other words, realized interest is *det*erministically equal to expected interest, and there is still a vacancy if and only if expected interest is below the school's capacity. This choice produces a deterministic prediction for each student type θ and each school h. We show in Sections 3.1 and 3.2 that this choice of vacancy function allows us to recover the definition of stability in a finite market, as well as the definition used by Azevedo and Leshno (2016).

In Section 4, we use the following alternative choice of vacancy function, which assumes that when expected interest is equal to λ, realized interest follows a Poisson distribution with mean λ.

$$\mathcal{V}^{pois}(\lambda, C) = \sum_{k=0}^{C-1} \frac{e^{-\lambda}\lambda^k}{k!}. \tag{4}$$

Note that this choice produces admissions probabilities in $(0, 1]$, reflecting the uncertainty facing participants in finite random matching markets.

2.3. **Admissions to Matching.** Recall that an admissions function A describes the probability that a student of any given priority $p \in [0, 1]$ will be admitted to each school. From this, we construct an associated fractional matching $\mathcal{M}(A) = M^A$ given by

$$M^A_h(\theta) = A_h(p^\theta_h) \prod_{h' \succ^\theta h} (1 - A_{h'}(p^\theta_{h'})). \tag{5}$$

This says that a student matches to h if and only if she is admitted to h and not to any preferred school. Note that this formula implicitly assumes independence of admissions outcomes across schools. A straightforward inductive argument implies that for any $A \in \mathfrak{A}$, $\theta \in \Theta$ and $h \in \mathcal{H}_0$,

$$1 - \sum_{h' \succ^\theta h} M^A_{h'}(\theta) = \prod_{h' \succ^\theta h} (1 - A_{h'}(p^\theta_{h'})), \tag{6}$$

with both sides interpreted to be 1 if h is the first choice of θ.

2.4. **Definition of Stability.** The two key parameters to our model are the measure of student types η (which determines the function $\mathcal{I} : \mathfrak{M} \to \mathfrak{I}$), and vacancy function $\mathcal{V}$ (which determines the function $\mathcal{A} : \mathfrak{I} \to \mathfrak{A}$). For any η and $\mathcal{V}$, we employ the following definitions of stability.

Definition 1.

*A matching $M \in \mathfrak{M}$ is $(\eta, \mathcal{V})$-**stable** if $M = \mathcal{M}(\mathcal{A}(\mathcal{I}(M)))$.*

*An admissions function $A \in \mathfrak{A}$ is $(\eta, \mathcal{V})$-**stable** if $A = \mathcal{A}(\mathcal{I}(\mathcal{M}(A)))$.*

*An interest function $I \in \mathfrak{I}$ is $(\eta, \mathcal{V})$-**stable** if $I = \mathcal{I}(\mathcal{M}(\mathcal{A}(I)))$.*

*An outcome $(M, I, A) \in \mathfrak{M} \times \mathfrak{I} \times \mathfrak{A}$ is $(\eta, \mathcal{V})$-**stable** if $M = \mathcal{M}(A), I = \mathcal{I}(M)$, and $A = \mathcal{A}(I)$.*

Definition 1 implies that there is a one-to-one correspondence between stable matchings, stable admissions functions, stable interest functions, and stable outcomes. We include each of the definitions above because it is sometimes most convenient to work with stable matchings, and at other times simpler to work with stable interest functions or stable outcomes.

3. Results

Our definition of stability in Definition 1 may seem strange to those familiar with more traditional definitions based on the absence of blocking pairs, or cutoffs that clear the market. It more closely resembles fixed-point characterizations of stable matchings by Adachi (2000), Fleiner (2003), and Echenique (2004). Our first results bridge this divide by showing that when using the deterministic vacancy function $\mathcal{V}^{det}$ from (3), our definition encompasses more traditional definitions as special cases. Section 3.1 shows that in finite markets, our definition corresponds to the absence of blocking pairs. Section 3.2 shows that in continuum markets, our definition is equivalent to one based on market-clearing cutoffs.

While these results both assume that the vacancy function $\mathcal{V}$ is as given in (3), we proceed to establish that for *any* η and $\mathcal{V}$, several classic results hold: the set of stable matchings is a non-empty lattice, the extreme points of this lattice can be found using the deferred acceptance algorithm, and the rural hospital theorem applies. Finally, we prove that if η has strict priorities and $\mathcal{V}$ is strictly decreasing, there is a unique stable matching.

3.1. Finite Markets: Stability = No Blocking Pairs.

Traditionally, stable matching problems involve a finite set of students $\mathcal{S} \subset \Theta$. We adopt the standard assumption that $\mathcal{S}$ has "strict priorities": no two students in $\mathcal{S}$ have identical priority at any school. Given $h \in \mathcal{H}$ and $p \in [0,1]$, define

$$\Theta_h(p) = \{\theta : h \succ^\theta \emptyset, p_h^\theta = p\} \tag{7}$$

to be the set of student types that consider school h acceptable and have priority p at school h.

Definition 2 (Strict Priorities)**.**

A finite subset $\mathcal{S} \subset \Theta$ has **strict priorities** *if for each $h \in \mathcal{H}$ and $p \in [0,1]$, $|\mathcal{S} \cap \Theta_h(p)| \le 1$.*

A positive measure η on Θ has **strict priorities** *if for each $h \in \mathcal{H}$ and $p \in [0,1]$, $\eta(\Theta_h(p)) = 0$.*

The second part of this definition is motivated by the study of *random* finite matching markets, where $\mathcal{S}$ is generated by drawing student types iid from some measure η over Θ. In this case, the condition above ensures that $\mathcal{S}$ has strict priorities with probability one.[1]

We now give a version of the traditional definition of stability based on the absence of blocking pairs. We refer to this concept as "no blocking pairs" to distinguish it from the definition of stability in Definition 1.

Definition 3 (No Blocking Pairs)**.** *Given any finite set $\mathcal{S} \subset \Theta$, an $\mathcal{S}$-matching is a function $\mu : \mathcal{S} \to \mathcal{H}_0$. An $\mathcal{S}$-matching μ is* **feasible** *if for each $h \in \mathcal{H}_0$,*

$$|\{\theta \in \mathcal{S} : \mu(\theta) = h\}| \le C_h. \tag{8}$$

[1] The assumption that there are no ties is essential to many of our results. This is not an artifact of our definitions or proof techniques, but rather reflects fundamental challenges to defining stable matchings with indifferences.

An $\mathcal{S}$-matching μ **has no blocking pairs** *if it is feasible, and for each $\theta' \in S$ and each $h \in \mathcal{H}_0$ such that $h \succ^{\theta'} \mu(\theta')$,*

$$|\{\theta \in \mathcal{S} : \mu(\theta) = h, p_h^\theta > p_h^{\theta'}\}| = C_h. \tag{9}$$

Definition 3 states that a feasible $\mathcal{S}$-matching has no blocking pairs if for each student $\theta' \in \mathcal{S}$, each school that θ' prefers to its assignment is filled with higher-priority students. Note that this implies individual rationality: because $C_\emptyset = \infty$, the outside option is never filled to capacity. Therefore, if μ has no blocking pairs, then it does not assign any student to a school that she considers inferior to the outside option.

Our first result is to show that our definition of stability corresponds with the traditional definition based on the absence of blocking pairs. To state this result formally, we note that each finite set $\mathcal{S} \subset \Theta$ is naturally associated with an associated counting measure $\eta^\mathcal{S}$ over Θ, defined by

$$\eta^\mathcal{S}(\tilde{\Theta}) = |\tilde{\Theta} \cap \mathcal{S}| \qquad \forall \tilde{\Theta} \subseteq \Theta. \tag{10}$$

Similarly, there is a natural correspondence between $\mathcal{S}$-matchings (which define an assignment only for student types in $\mathcal{S}$) and deterministic matchings (which define an assignment for all types in Θ). Any deterministic matching M naturally defines an $\mathcal{S}$-matching μ^M: for each $\theta \in \mathcal{S}$, let

$$\mu^M(\theta) = h \Leftrightarrow M_h(\theta) = 1. \tag{11}$$

Similarly, each $\mathcal{S}$-matching μ naturally induces a deterministic matching M^μ as follows. Define the admissions outcome A^μ by

$$A_h^\mu(p) = \mathbf{1}(|\{\theta \in \mathcal{S} : p_h^\theta > p, \mu(\theta) = h\}| < C_h), \tag{12}$$

and define $M^\mu = \mathcal{M}(A^\mu)$. In other words, (12) says that student $\theta \in \Theta$ is admitted to h if there are fewer than C_h higher-priority students from $\mathcal{S}$ matched to h under μ, and $M^\mu(\theta)$ is the matching that results when each student type θ is assigned to its most-preferred school among those where it is admitted.

The following result says that if priorities are strict, then the functions $M \to \mu^M$ and $\mu \to M^\mu$ define a bijection between the set of $(\eta^\mathcal{S}, \mathcal{V}^{det})$-stable matchings, and the set of $\mathcal{S}$-matchings with no blocking pairs. The proof of this result is deferred to Appendix A.1.

Proposition 1 (No Blocking Pairs Corresponds to a Stable Matching)**.** *Let $\mathcal{S}$ be a finite subset of Θ with strict priorities. If M is a $(\eta^\mathcal{S}, \mathcal{V}^{det})$-stable matching, then μ^M has no blocking pairs. If μ is an $\mathcal{S}$-matching with no blocking pairs, then M^μ is $(\eta^\mathcal{S}, \mathcal{V}^{det})$-stable, and $\mu^{M^\mu} = \mu$.*

3.2. **Continuum Markets: Stability = Market-Clearing Cutoffs.** Azevedo and Leshno (2016) provide a continuum model in which a market is described by a positive measure η over Θ and a stable matching is described by a vector of priority cutoffs $P \in [0,1]^\mathcal{H}$. Students are admitted to school h if and only if their priority at h exceeds its cutoff P_h. They define demand for school h

at cutoffs P by[2]

$$D_h(P) = \int \mathbf{1}(p_h^\theta > P_h) \prod_{h' \succ^\theta h} \mathbf{1}(p_{h'}^\theta \le P_{h'}) d\eta(\theta). \tag{13}$$

That is, demand for h at cutoffs P is equal to the measure of students who are admitted to h and are not admitted to any school that they prefer to h. Each cutoff vector is naturally associated with a deterministic matching in which students attend the school that they demand. The definition of stability used by Azevedo and Leshno (2016) is that the cutoff vector should clear the market.

Definition 4. *A cutoff $P \in [0,1]^{\mathcal{H}}$ is η-**market-clearing** if $D_h(P) \le C_h$ for all $h \in \mathcal{H}$, with equality if $P_h > 0$.*

In this section, we show that in continuum markets, a cutoff vector P is η-market-clearing if and only if a corresponding interest function is $(\eta, \mathcal{V}^{det})$-stable. To formalize this claim, we first define a natural associate between cutoff vectors and interest functions. Each cutoff vector P is naturally associated with an interest function I^P defined for each $h \in \mathcal{H}$ and $p \in [0,1]$ by

$$I_h^P(p) = \int \mathbf{1}(p_h^\theta > p) \prod_{h' \succ^\theta h} \mathbf{1}(p_{h'}^\theta \le P_{h'}) d\eta(\theta). \tag{14}$$

That is, students contribute to this quantity if they have priority above p at h and do not "clear the cutoff" at any school that they prefer. Note that when $p = P_h$, we have $I_h^P(P_h) = D_h(P)$.

Conversely, from any interest function $I \in \mathfrak{I}$, we can define the associated cutoffs $\mathcal{P}(I) = \{\mathcal{P}_h(I)\}_{h \in \mathcal{H}} \in [0,1]^{\mathcal{H}}$ by

$$\mathcal{P}_h(I) = \inf\{p \ge 0 : I_h(p) < C_h\}. \tag{15}$$

Equation (14) defines a mapping from cutoffs to interest functions, while (15) defines mapping from interest functions to cutoffs. It turns out that these mappings take stable interest functions to market-clearing cutoffs, and vice versa.[3]

Proposition 2 (Market-Clearing Cutoffs Correspond to Stable Interest Functions)**.** *Let η have strict priorities. If P is η-market-clearing, then I^P is $(\eta, \mathcal{V}^{det})$-stable. If I is $(\eta, \mathcal{V}^{det})$-stable, then $\mathcal{P}(I)$ is η-market-clearing, and $I = I^{\mathcal{P}(I)}$.*

We prove this result in Appendix A.2.

3.3. **Existence and Lattice Structure.** Having established that when $\mathcal{V} = \mathcal{V}^{det}$, our definition of stability nests existing definitions, we now prove results for general type measures η and vacancy

[2] An astute and informed reader might notice that our choice of A^P assumes that student types θ with $p_h^\theta = P_h$ are not admitted to h, whereas Azevedo and Leshno (2016) assume that they are admitted. Because η is a continuum measure with strict priorities in both cases, this distinction is of consequence only to sets of η-measure zero.

[3] We briefly comment on a subtlety that explains why Proposition 2 is stated in terms of the interest function I^P rather than the admissions function A^P or the matching M^P. In general, multiple market-clearing cutoffs may correspond to the same stable matching (up to a set of measure zero). For example, suppose that there is a single school h with capacity C, and that the total measure of students is $\eta(\Theta) = 2C$, with priorities uniformly distributed on $(0, 1/3) \cup (2/3, 1)$. Then any cutoff $P \in [1/3, 2/3]$ clears the market. Our definition of stability eliminates this redundancy: the unique $(\eta, \mathcal{V}^{det})$-stable matching corresponds to a cutoff of $2/3$ and leaves students in $[0, 2/3]$ unassigned. Thus, if $P \in [1/3, 2/3)$, P clears the market but M^P is not $(\eta, \mathcal{V}^{det})$-stable. By contrast, for any P, $I^P(p) = \eta(\{\Theta : p_h^\theta > p\})$ is a stable interest function.

functions $\mathcal{V}$. The first of these results shows that stable matchings always exist and form a lattice. To state this result, we define the following partial orders:

- $M \succeq^{\mathfrak{M}} \tilde{M}$ if for each $h \in \mathcal{H}_0$ and $\theta \in \Theta$,
$$\sum_{h' \succeq^{\theta} h} M_{h'}(\theta) \geq \sum_{h' \succeq^{\theta} h} \tilde{M}_{h'}(\theta).$$
That is, $M \succeq^{\mathfrak{M}} \tilde{M}$ if each student prefers M to $\tilde{M}$ in the sense of first-order stochastic dominance.
- $A \succeq^{\mathfrak{A}} \tilde{A}$ if for each $h \in \mathcal{H}_0$ and $p \in [0,1]$, $A_h(p) \geq \tilde{A}_h(p)$.
That is, $A \succeq^{\mathfrak{A}} \tilde{A}$ if admissions probabilities are uniformly higher under A.
- $I \succeq^{\mathfrak{I}} \tilde{I}$ if for each $h \in \mathcal{H}_0$ and $p \in [0,1]$, $I_h(p) \geq \tilde{I}_h(p)$.
That is, $I \succeq^{\mathfrak{I}} \tilde{I}$ if each school receives more interest at every cutoff under I.
- $(M, A, I) \succeq (\tilde{M}, \tilde{A}, \tilde{I})$ if $M \succeq^{\mathfrak{M}} \tilde{M}$, $A \succeq^{\mathfrak{A}} \tilde{A}$, and $\tilde{I} \succeq^{\mathfrak{I}} I$.

Theorem 1 (Existence and Lattice Structure). *If the vacancy function $\mathcal{V}$ is weakly decreasing in its first argument, then for any $(\mathcal{H}, \mathbf{C}, \eta)$, the set of $(\eta, \mathcal{V})$-stable outcomes is non-empty, and forms complete lattice with partial order $\succeq$.*

Proof of Theorem 1. Define the function $\xi : \mathfrak{M} \to \mathfrak{M}$ by

$$\xi(M) = \mathcal{M}(\mathcal{A}(\mathcal{I}(M))). \tag{16}$$

Note that

- By (1), $\tilde{M} \succeq^{\mathfrak{M}} M$ implies $\mathcal{I}(M) \succeq^{\mathfrak{I}} \mathcal{I}(\tilde{M})$.
- By (2) and monotonicity of $\mathcal{V}$, $I \succeq^{\mathfrak{I}} \tilde{I}$ implies $\mathcal{A}(\tilde{I}) \succeq^{\mathfrak{A}} \mathcal{A}(I)$.
- By (5), $\tilde{A} \succeq^{\mathfrak{A}} A$ implies $\mathcal{M}(\tilde{A}) \succeq^{\mathfrak{M}} \mathcal{M}(A)$.

From this, we draw two conclusions. First, if (M, I, A) and $(\tilde{M}, \tilde{I}, \tilde{A})$ are stable outcomes, then $(M, I, A) \succeq (\tilde{M}, \tilde{I}, \tilde{A})$ if and only if $M \succeq^{\mathfrak{M}} \tilde{M}$. Second, the function ξ is an order preserving function, so Tarski's fixed point theorem implies that the set of fixed points of ξ (that is, the set of stable matchings) forms a complete lattice with respect to $\succeq^{\mathfrak{M}}$ (and in particular is non-empty). $\square$

3.4. **Deferred Acceptance Algorithm.** Theorem 1 establishes the existence of stable outcomes, but does not address how to find them. However, the proof suggests a natural procedure: start from a matching M and repeatedly apply the function ξ defined by $\xi(M) = \mathcal{M}(\mathcal{A}(\mathcal{I}(M)))$. If one starts from the matching $\overline{M}$ which assigns each student to her most preferred school, then this procedure corresponds to the student-proposing deferred acceptance algorithm, and converges to the student-optimal stable matching. To see that it converges, note that $\xi(\overline{M}) \preceq^{\mathfrak{M}} \overline{M}$, from which the fact that ξ is order-preserving implies that the sequence $\{\xi^k(\overline{M})\}_{k=0}^{\infty}$ is decreasing. Therefore, it converges by completeness of $\mathfrak{M}$. Conversely, repeatedly applying ξ from the student-pessimal matching $\underline{M}$ (defined by $\underline{M}_{\emptyset}(\theta) = 1$ for all θ) produces an increasing sequence of matchings that converges to the school-optimal stable matching.

Although convergence is guaranteed, in general it does not occur in finitely many steps. In examples that we have tried, convergence happens quickly enough that this algorithm can be applied fruitfully. The main practical challenge is computing $\mathcal{I}(M)$, which requires taking an integral over

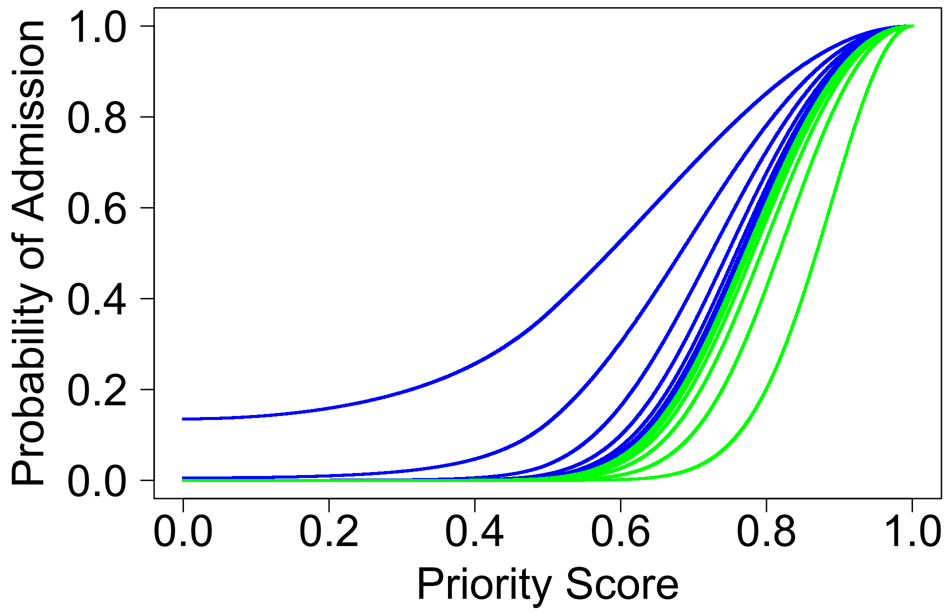

FIGURE 3. Finding a stable admissions function for the example from Azevedo and Leshno (2016), using vacancy function $\mathcal{V}^{pois}$ defined in (4). Because schools are symmetric, $A_h = A_{h'}$ for $h, h' \in \mathcal{H}$. We iterate the map $A \to \mathcal{A}(\mathcal{I}(\mathcal{M}(A)))$. Starting from $A(p) = 1$ results in a decreasing sequence of admissions functions (in blue) converging to the student-optimal stable admissions function. Starting from $A(p) = 0$ results in an increasing sequence of admissions functions (in green) converging to the school-optimal stable admissions function. Because $\mathcal{V}^{pois}$ is strictly decreasing, Theorem 3 guarantees that these admissions functions coincide. Notice that the admissions function is the CDF of the cutoff distribution whose 5^{th} and 95^{th} percentiles are shown in Figure 1.

student types. Although this may be challenging for arbitrary measures η, it is tractable for many cases of interest.

Because of the correspondence between stable matchings, stable interest functions, and stable admissions functions, it is also possible to apply an analogous iterative process using the admissions function as the primitive of interest. In that case, one could start from $A_h(p) = 1$ for all h and p (resulting in convergence to the student-optimal stable matching) or $A_h(p) = 0$ for all h and p (resulting in convergence to the student-pessimal stable matching). Figure 3 shows the sequence of admissions functions that result when this iterative process is applied to an example from Azevedo and Leshno (2016).

3.5. **Rural Hospital Theorem.** We now establish a "rural hospital theorem," which states that for any two stable matchings, each student's probability of assignment and each school's measure of matched students is identical. This is a generalization of the corresponding result for finite markets, proved by McVitie and Wilson (1970) and Roth (1986).

Theorem 2 (Rural Hospital Theorem)**.** *If η has strict priorities and $\mathcal{V}$ is weakly decreasing in its first argument, then the set of matched agents is identical across stable outcomes: if (M, I, A) and*

$(\tilde{M}, \tilde{I}, \tilde{A})$ *are* $(\eta, \mathcal{V})$*-stable outcomes, then for each* $h \in \mathcal{H}_0$*,*

$$\int M_h(\theta) d\eta(\theta) = \int \tilde{M}_h(\theta) d\eta(\theta), \tag{17}$$

and for each $\theta \in \Theta$ *outside of a set of* η*-measure zero,*

$$\sum_{h \succ^\theta \emptyset} M_h(\theta) = \sum_{h \succ^\theta \emptyset} \tilde{M}_h(\theta). \tag{18}$$

In contrast to the existence result in Theorem 1, Theorem 2 requires an assumption of strict priorities. This assumption is essential for the result to hold.[4] The failure of the rural hospital theorem when there are ties in priority is not specific to our definition of stability: in finite markets with indifferences, it is known that strongly stable matchings may not exist (Irving, 1994), and weakly stable ones may not satisfy the rural hospital theorem (Manlove, 1999).

3.6. **Uniqueness.** Finally, we establish conditions under which there is a unique stable outcome.

Theorem 3 (Uniqueness)**.** *If* η *has strict priorities and* $\mathcal{V}$ *is strictly decreasing in its first argument, then there is a unique* $(\eta, \mathcal{V})$*-stable outcome.*

The intuition underlying this result is as follows. By Theorem 1, there are student-optimal and student-pessimal stable admissions functions A and $\tilde{A}$, with $A \succeq \tilde{A}$. It follows that all students will be weakly more likely to match under A. If $\mathcal{V}$ is strictly decreasing, then $A \succ \tilde{A}$ implies that some students will be strictly more likely to match under $\tilde{A}$. This contradicts the rural hospital theorem, implying that we must have $A = \tilde{A}$. The complete proof is provided in Appendix A.3.

If $\mathcal{V}$ is only weakly decreasing, it is possible that there are multiple stable matchings that lead to different outcomes for a positive η-measure of students: see Azevedo and Leshno (2016) for an example with $\mathcal{V} = \mathcal{V}^{det}$. However, their Theorem 1 shows that even in this case, there is typically a unique stable matching: this holds if η has full support, or for a generic set of school capacities.

4. A New Vacancy Function

In this section, we study model predictions when using the vacancy function $\mathcal{V}^{pois}$ given in (4), which assumes that realized interest follows a Poisson distribution.

To motivate this assumption, we revisit our discussion of the introduction. Suppose that schools post deterministic cutoff scores given by the vector $P \in [0,1]^{\mathcal{H}}$ and we generate a set $\mathcal{S}$ of n students by sampling iid from a probability measure $\tilde{\eta}$ on Θ. Then for each $h \in \mathcal{H}$ and each $p \in [0,1]$, the set of students who are in $\mathcal{S}$, interested in h, and have priority above p at h will follow a Binomial

[4] To see that the conclusion of Theorem 2 may fail to hold if η does not have strict priorities, consider an example with two schools, A and B, each with a single seat. The measure η corresponds to a finite market with three students, x, y, z. Students x and y prefer A to B, while student z prefers B to A. Student x has priority $1/4$ at school A and $3/4$ at school B. Student y has priority $1/4$ at school A and $2/4$ at school B. Student z has priority $3/4$ at school A and $1/4$ at school B.
We claim that there are two $(\eta, \mathcal{V}^{det})$-stable matchings, and that y is assigned in one and unassigned in the other. In the school-optimal stable matching, x goes to B, z goes to A, and y is unassigned. In the student-optimal stable matching, x and y go to A, and z goes to B. Note that the student-optimal stable matching is infeasible (two students are assigned to A). This illustrates that our definition of stability (which was intended for markets with strict priorities) does not enforce capacity constraints in markets with ties.

distribution with parameters n and $I_h^P(p)/n$ (where we define I^P as in (14), with $\eta = n\tilde{\eta}$). If n is modestly large, this will be well-approximated by a Poisson with parameter $I_h^P(p)$.

In the limit as the number of students and the capacity of each school grow, the predictions of this new model converge to those of Azevedo and Leshno (2016).[5] Conceptually, this is because Poisson random variables with large means are highly concentrated. The advantage of our model lies in its ability to generate probabilistic predictions for markets where capacities are modest and cutoffs are uncertain. We demonstrate its accuracy by using it to generate predictions about two measures of interest: the measure of matched students

$$\sum_{h \in \mathcal{H}} \int M_h(\theta) d\eta(\theta), \tag{19}$$

and the average rank of matched students, which we now define. Denote θ's rank of $h \in \mathcal{H}$ by

$$R_h(\theta) = |\{h' \in \mathcal{H}_0 : h' \succeq^\theta h\}|$$

and define[6]

$$AverageRank(M) = \frac{\int \sum_{h \in \mathcal{H}} M_h(\theta) R_h(\theta) d\eta(\theta)}{\int \sum_{h \in \mathcal{H}} M_h(\theta) d\eta(\theta)}. \tag{20}$$

We compare our predictions to existing analytical and simulation results for finite markets. To predict outcomes from random finite markets with n students whose types are drawn drawn iid from a probability distribution $\tilde{\eta}$ on Θ, we define the measure η by $\eta(\tilde{\Theta}) = n\tilde{\eta}(\tilde{\Theta})$ for all $\tilde{\Theta} \subseteq \Theta$, and study $(\eta, \mathcal{V}^{pois})$-stable matchings.

Section 4.1 provides results for students' average rank, using the results of Ashlagi et al. (2017) as the primary comparison. Section 4.2 provides results for the number of matches, and compares against findings from Marx and Schummer (2019). In both cases, our model accurately predicts simulation results for finite markets of moderate size. In addition, our model provides new analytical expressions and insights, described below.

4.1. **Average Rank.** We now present our results on average rank, using as a comparison the simulations and numerical bounds from Ashlagi et al. (2017). We first introduce some new definitions.

Definition 5.

[5]To be more precise, given a market $\mathcal{E} = (\mathcal{H}, \mathbf{C}, \eta)$, we can define a sequence of markets $\mathcal{E}^m = (\mathcal{H}, \mathbf{C}^m, \eta^m)$, with $C_h^m = mC_h$ for all $h \in \mathcal{H}$ and $\eta^m(\tilde{\Theta}) = m \cdot \eta(\tilde{\Theta})$ for all measurable $\tilde{\Theta} \subseteq \Theta$. That is, we simply scale up the number of students and number of seats per school by a factor of m. This is the same limiting regime as considered by Azevedo and Leshno (2016). If $\mathcal{E}$ has a unique $(\eta, \mathcal{V}^{det})$-stable matching M^*, then the $(\eta, \mathcal{V}^{pois})$-stable matchings of $\mathcal{E}^m$ converge to M^* as m grows. This is because Poisson random variables with large means are highly concentrated. More formally, for any $C \in \mathbb{N}$ and any $\lambda \in \mathbb{R}_+ \backslash \{C\}$, $\mathcal{V}^{pois}(m\lambda, mC) \to \mathcal{V}^{det}(\lambda, C)$ as $m \to \infty$.

[6]Note that (20) counts the average rank among matched students. We make this choice to align with the convention adopted by Ashlagi et al. (2017). One could instead work with the average rank across *all* students, counting a student who lists ℓ schools and goes unassigned as receiving their $(\ell+1)^{st}$ choice. In general, these two definitions are incomparable, meaning that each can be larger than the other. However, Propositions 3 holds for either definition. To see this, note that

$$\frac{\int \sum_{h \in \mathcal{H}_0} M_h(\theta) R_h(\theta) d\eta(\theta)}{\eta(\Theta)} = \frac{\sum_{h \in \mathcal{H}} I_h(0) + I_\emptyset(0)}{\sum_{h \in \mathcal{H}} \int M_h(\theta) d\eta(\theta) + I_\emptyset(0)} \leq \frac{\sum_{h \in \mathcal{H}} I_h(0)}{\sum_{h \in \mathcal{H}} \int M_h(\theta) d\eta(\theta)}.$$

$\lvert\mathcal{M}\rvert = \lvert\mathcal{W}\rvert + 10$	100	200	500	1000	2000	5000
MOSM	29.5	53.6	115.8	203.8	364.5	793.1
WOSM	30.1	54.7	118.0	207.5	370.8	804.7
AKL	25.3	45.7	98.2	175.2	314.6	690.5
Continuum Model	29.6	53.9	115.8	205.5	366.1	793.4

FIGURE 4. A reproduction of the first column of Table 1 from Ashlagi et al. (2017), along with our model predictions. Each column above corresponds to a different market size, holding fixed the absolute imbalance. The first two rows show the (simulated) average rank under the student optimal ("man optimal" in their paper) and school optimal ("woman optimal") stable matches, which differ by at most 2%. The third row shows predictions made by Ashlagi et al. (2017), which underestimate the average rank by approximately 15%. The final row shows our predictions, which always lie between the simulation results for the extremal stable matchings.

The measure η is a **symmetric IID measure** *if (i) the restriction of $\succ^\theta$ to $\mathcal{H}$ is uniformly distributed, and (ii) for each $\succ \in \mathcal{R}$, the conditional distribution of $\mathbf{p}^\theta$ given $\succ^\theta = \succ$ is uniform on $[0,1]^{\mathcal{H}}$.*

The measure η is a **symmetric RSD measure** *if (i) the restriction of $\succ^\theta$ to $\mathcal{H}$ is uniformly distributed, and (ii) for each $\succ^\theta \in \mathcal{R}$, the conditional distribution of $\mathbf{p}^\theta$ given $\succ^\theta = \succ$ is uniform on $\{\theta : p^\theta_h = p^\theta_{h'} \text{ for all } h, h' \in \mathcal{H}\}$.*

To predict the outcome of their simulations, we consider a market with $C_h = 1$ for all h, and η the symmetric iid measure in which all schools are preferred to the outside option, and analyze $(\eta, \mathcal{V}^{pois})$-stable matchings. There are several reasons that our model's predictions might not match the simulation results. First, our model generates a unique prediction, whereas finite markets may have multiple stable matchings. Although Ashlagi et al. (2017) establish that differences between stable matchings are relatively minor in imbalanced markets, our "prediction error" must at least be comparable to the variation across stable matchings. Second, the assumption of independent outcomes across schools introduces error: when the number of students is below 40, every student in the finite market must match, whereas our model predicts that each student has a positive (albeit small) probability of going unassigned.

Despite these concerns, the model prediction is excellent. Figure 2 displays their reported simulation results alongside our predictions. The curves do not merely appear qualitatively similar, they also match quantitatively. To emphasize this point, we reproduce their Table 1 in Figure 4. In their simulations, the average rank of students differs by at most 2% between student-optimal and school-optimal stable matches. Our model predictions always lie in between these two numbers.

Although the numerical predictions of our model are excellent, the resulting formulas are complex, and do not immediately offer insight about how students' average rank depends on market primitives. We address this by using our model to derive analytical bounds on students' average rank. Figure 2 makes it clear that the behavior of the market is very different depending on whether the number of students is less or greater than the number of seats. Accordingly, our analysis will consider these cases separately. We define ρ to be the ratio of students to schools:

$$\rho = \eta(\Theta)/|\mathcal{H}|, \tag{21}$$

and let C denote the (common) capacity at each school. We analyze the case with more seats ($\rho < C$) in Section 4.1.1, and the case with more students ($\rho > C$) in Section 4.1.2.

4.1.1. *More Seats than Students.* We define

$$Enrollment(\lambda, C) = \int_0^\lambda \mathcal{V}(x, C)dx. \tag{22}$$

To state our bounds, for $C \in \mathbb{N}$ and $\rho < C$, define $\Lambda(\rho, C)$ to be the smallest solution λ to

$$Enrollment(\lambda, C) = \rho.$$

Note that the definition of $\mathcal{V}^{pois}$ in (4) implies that for any $C \in \mathbb{N}$,

$$\int_0^\infty \mathcal{V}^{pois}(\lambda, C) = C, \tag{23}$$

so $\Lambda(\rho, C)$ is defined for $\rho < C$.

Proposition 3. *Fix $C \in \mathbb{N}$, and let $C_h = C$ for all $h \in \mathcal{H}$. Let η^{IID} be a symmetric iid measure, and let M^{IID} be the unique $(\eta^{IID}, \mathcal{V}^{pois})$-stable matching guaranteed by Theorem 3. If there are more seats than students ($\rho < C$), then*

$$AverageRank(M^{IID}) \leq \Lambda(\rho, C)/\rho.$$

To clarify the relationship with results from Ashlagi et al. (2017), we parameterize ρ and apply Proposition 3 to the special case where $C = 1$.

Corollary 1. *In a symmetric iid market where schools have a single seat and $\rho = \frac{n}{n+k} < 1$,*

$$AverageRank(M^{IID}) \leq \frac{n+k}{n} \log\left(\frac{n+k}{k}\right).$$

This upper bound exactly matches that from Theorem 2 of Ashlagi et al. (2017).[7] Because we work with different models, neither result directly implies the other. However, the bound in Proposition 3 provides insight beyond the cases considered by Ashlagi et al. (2017). First, it does not assume that students submit complete (or long) lists: the distribution of list lengths can be arbitrary.[8] Instead, fixing school capacity C, the bound depends only on the ratio of students to schools ρ. Second, our bound improves as school capacity grows: fixing the ratio of students to seats $\rho/C = 0.97$, the bound on average rank is approximately 3.6 when $C = 1$, 2.0 when $C = 3$, 1.4 when $C = 10$, and tends to 1 as $C \to \infty$.

[7]Also matches heuristic bound derived in conclusion of Wilson (1972) when $\rho < 1$.

[8]While it is well known that increasing a student's list makes outcomes worse for all other students, this does not imply that extending a list increases the average rank, because average rank is calculated only for matched students. If some students submit short lists, extending their list may cause these students to match in place of (or in addition to) others with longer lists, thereby decreasing the average rank.

4.1.2. *More Students than Seats.* We now turn to the case where students outnumber seats. We consider the two special cases studied by Ashlagi et al. (2017): priorities are either generated iid $U[0,1]$ (IID), or are identical across schools, and independent of the length of student lists (RSD). The following result implies that RSD results in a much lower average rank than IID.

Proposition 4. *Suppose that $C_h = C_{h'}$ for all $h, h' \in \mathcal{H}$ and that there are more students than seats ($\rho > C$). Let η^{IID} be a symmetric IID measure in which all students list ℓ schools, and let M^{IID} be the unique $(\eta^{IID}, \mathcal{V}^{pois})$-stable matching. Then*

$$AverageRank(M^{IID}) \geq \ell \left(1 - \frac{\rho}{C} - \frac{1}{\log\left(1 - C/\rho\right)}\right).$$

Let η^{RSD} be a symmetric RSD measure in which all students list ℓ schools, and let M^{RSD} be the unique $(\eta^{RSD}, \mathcal{V}^{pois})$-stable matching. Then

$$AverageRank(M^{RSD}) \leq 1 + \log(\ell).$$

To facilitate a comparison with bounds from Ashlagi et al. (2017), we parameterize ρ and consider the special case where $C = 1$.

Corollary 2. *In a symmetric iid market where $C = 1$, $\rho = \frac{n+k}{n}$ and all students list n schools,*

$$AverageRank(M^{IID}) \geq \frac{n}{\log\left(\frac{n+k}{k}\right)} - k.$$

This lower bound is very similar to the bound of $\frac{n}{1+\frac{n+k}{n}\log\left(\frac{n+k}{k}\right)}$ from Theorem 2 of Ashlagi et al. (2017) (in fact, our bound is tighter for $k \leq n$).[9] More importantly, Proposition 4 clarifies the effect of list length, school capacity, and market size.

In the model of Ashlagi et al. (2017), n represents both the number of schools and the length of student lists, and it is assumed that each school has a single seat. It is not a priori clear how their bound changes if students list only a subset of the market, or if schools have multiple seats. Proposition 4 establishes that students' average rank under RSD is logarithmic in the *list length* (rather than the market size). Meanwhile, under IID priorities, students' average rank is linear in the list length, with proportionality constant that depends on the ratio of students to seats ρ/C. This implies that with IID priorities, large capacities don't result in meaningfully better outcomes: given a fixed ratio of students to seats ρ/C, the lower bound does not depend on whether schools are small or large.[10]

4.2. **Number of Matches.** Another metric of interest is the number of matches. Few theoretical papers study this quantity, despite its salience in many applications. In fact, many papers assume that students submit complete lists, or at leasts lists that are long enough that the short side of the market matches fully.

One recent exception is Marx and Schummer (2019). They consider the problem facing a matching platform that helps to match men and women, and charges prices p_m and p_w to each matched

[9] Letting $\beta = k/n$, algebra reveals that the bound in Corollary 2 is tighter than the bound from Theorem 2 of Ashlagi et al. (2017) so long as $(1+\beta)\beta\log^2(1+\beta) \leq 1$, which holds for $\beta \leq 1$.

[10] This is in contrast to RSD. In this case, fixing the ratio of students to seats ρ/C, the average rank is decreasing in C, and converges to one as C grows (although this fact is not reflected in the bound in Proposition 4).

man and woman, respectively. A man and woman can only be matched if they are both willing to pay the fee, which occurs with probability $1 - q(p_m, p_w)$ independently across man-woman pairs. The platform thus faces a tradeoff: if its prices are too high, there will be few mutually acceptable pairs, and few matches will form. The goal of the platform is to choose prices to maximize its revenue.

Marx and Schummer (2019) assume that both sides have uniformly random preferences over mutually acceptable partners, and that the final matching is stable. Let $V^{IID}(W, M, q)$ be the expected number of matches with W women, M men, and probability of mutual compatibility $1 - q$. The main technical challenge they confront is analyzing $V^{IID}(W, M, q)$. They argue that an analytical result is intractible, and instead analyze an algorithm in which men declare all acceptable partners, and then women are placed in a random order and sequentially allowed to choose their favorite unmatched mutually acceptable man. In essence, they study the outcome of a woman-selecting random serial dictatorship. Letting $V^{RSD}(W, M, q)$ be the expected number of matches that form in this case, Marx and Schummer (2019) note that

$$V^{RSD}(W, M, q) = \sum_{j=1}^{\min(M,W)} \frac{\prod_{i=0}^{j-1}(1 - q^{M-i}) \prod_{i=0}^{j-1}(1 - q^{W-i})}{1 - q^j}. \tag{24}$$

They use simulations to argue that $V^{RSD}(W, M, q)$ is a reasonable approximation of $V^{IID}(W, M, q)$. Figure 5 includes two plots from their paper: the first displays V^{RSD}, and the second displays simulation results showing the relative difference between V^{RSD} and V^{IID}.

To generate corresponding predictions from our model, we let W be the mass of students, M be the number of schools, and assume that schools are symmetric and have capacity $C = 1$. In their simulations, student list length is Binomial with parameters M and $1 - q$. When q is large, this is close to a Poisson with mean $M(1 - q)$, so we generate predictions assuming that student list lengths follow the latter distribution. We let $\hat{V}^{RSD}(W, M, q)$ and $\hat{V}^{IID}(W, M, q)$ be the mass of matches in the unique $(\eta^{RSD}, \mathcal{V}^{pois})$ and $(\eta^{IID}, \mathcal{V}^{pois})$ stable matchings of this market, and plot our predictions in Figure 5, alongside the original graphs from Marx and Schummer (2019). Despite the error in approximating the binomial distribution with a Poisson, the model predictions are very close to the exact expression and the simulation results.

Figure 5 suggests that IID preferences result in more matches than RSD. This can be proven using our model. Arnosti (2021) derives expressions which correspond to the predictions of our model with vacancy function $\mathcal{V}^{pois}$, when student preferences are uniformly random and school priorities are either IID or RSD. Theorem 3 and Proposition 1 from Arnosti (2021) imply that if the list length distribution has an increasing hazard rate, then IID priorities result in more matches than RSD. Both the binomial and Poisson distributions have an increasing hazard rate.

In addition to being numerically accurate, the predictions $\hat{V}^{RSD}(W, M, q)$ and $\hat{V}^{IID}(W, M, q)$ are analytically tractable. Arnosti (2021) shows that when $W \leq M$,[11]

$$\hat{V}^{RSD}(W, M, q) = W - \frac{\log(1 + e^{-(M-W)(1-q)} - e^{-M(1-q)})}{1 - q}. \tag{25}$$

[11]The choice $W \leq M$ is without loss of generality, and cleans up the expression in (25) by allowing the use of W in place of $\min(W, M)$ and M in place of $\max(W, M)$.

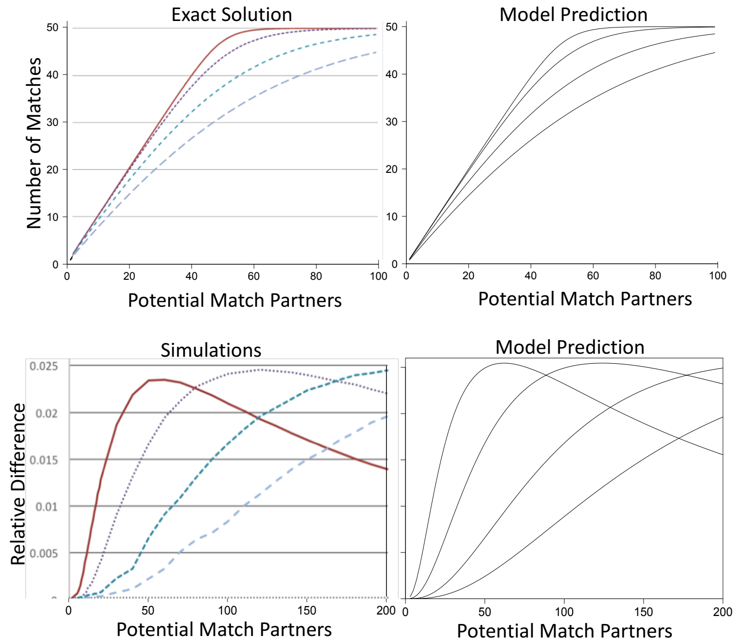

FIGURE 5. Studying the number of matches that form, with 50 participants on one side and varying the number of participants on the other side. Upper left: exact expressions from Marx and Schummer (2019) for V^{RSD} (each line represents a different probability of incompatibility q). Upper right: the corresponding predictions $\hat{V}^{RSD}$ from our proposed continuum model. Lower left: Marx and Schummer (2019) have no analytical expression for V^{IID}, but their simulations study the relative difference $(V^{IID} - V^{RSD})/V^{RSD}$. In our proposed continuum model, it is possible to prove that $\hat{V}^{IID} \geq \hat{V}^{RSD}$: the lower right plot shows the predicted relative difference. By comparison, the continuum model of Azevedo and Leshno (2016) predicts that $V^{IID} = V^{RSD}$.

This expression is much simpler than that in (24), and more amenable to optimization. Furthermore, whereas calculating $V^{IID}(W, M, q)$ is intractable, Arnosti (2021) shows that $\hat{V}^{IID}(W, M, q)$ is the unique solution to

$$(1-q)\hat{V}^{IID}(W,M,q) = \log\left(1 - \frac{\hat{V}^{IID}(W,M,q)}{W}\right)\log\left(1 - \frac{\hat{V}^{IID}(W,M,q)}{M}\right). \tag{26}$$

Note that for any desired match rate $V \leq \min(W, M)$, (26) gives a concise closed-form expression for the corresponding probability of incompatibility q.

5. Conclusion

Stable matching algorithms are used to assign students to schools in cities across the globe. In theory, the design of school priorities offers a flexible tool for encoding policy objectives. In practice, the benefits of designing priorities are limited by the fact that the relationship between priorities and the final outcome is complex and poorly understood.

This paper provides a model that can begin to address these questions. Our model has three desirable features: it is *flexible* enough to accommodate complex preferences and priorities, its numerical predictions are extremely *accurate*, and it tractable enough to offer new *insights*. We use a novel framework for stable matching to show that the only difference between our model and that of Azevedo and Leshno (2016) is that they assume that interest at each school is deterministic, whereas we assume that it follows a Poisson distribution. This difference allows our model to make probabilistic predictions that reflect the uncertainty in finite random markets.

Much work remains, including the establishment of rigorous accuracy guarantees in settings with small school capacities. However, the formal guarantees of our model match those of Azevedo and Leshno (2016), and its numerical accuracy is far superior when school capacities are modest. We believe that this model offers a fundamentally new perspective on stable matching, which will enable the study of settings – such as those with small and asymmetrical schools – which cannot be readily studied using prior techniques.

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Appendix A. Proofs from Section 3

A.1. **Proof of Proposition 1.** The following Lemma states that for any $(\eta^{\mathcal{S}}, \mathcal{V}^{det})$-stable outcome, the enrollment at each school will be the minimum of the school's capacity and the number of interested students.

Lemma 1. *If $\mathcal{S}$ is a finite subset of Θ with strict priorities, and (M, I, A) is $(\eta^{\mathcal{S}}, \mathcal{V}^{det})$-stable, then for any $h \in \mathcal{H}$ and $p \in [0,1]$,*

$$\sum_{\theta \in \mathcal{S}} \mathbf{1}(p_h^\theta > p) M_h(\theta) = \min(I_h(p), C_h).$$

Proof of Lemma 1. Define $\bar{p} = \inf\{p \in [0,1] : I_h(p) < C_h\}$. Note that (1) and (10) imply that I_h is right-continuous, and therefore $I_h(\bar{p}) < C_h$. It follows from (2) and (3) that for $p \in [0,1]$,

$$A_h(p) = \mathcal{V}^{det}(I_h(p), C_h) = \mathbf{1}(I_h(p) < C_h) = \mathbf{1}(p \geq \bar{p}). \tag{27}$$

Combining (6) and (27) we see that if $I_h(p) < C_h$ then $p \geq \bar{p}$ and

$$\sum_{\theta \in \mathcal{S}} \mathbf{1}(p_h^\theta > p) M_h(\theta) = \sum_{\theta \in \mathcal{S}} \mathbf{1}(p_h^\theta > p)(1 - \sum_{h' \succ^\theta h} M_{h'}(\theta)) = I_h(p), \tag{28}$$

where the final inequality uses (1) and (10).

Meanwhile, if $I_h(p) \geq C_h$, then $\bar{p} > p$, and (6) and (27) imply that

$$\begin{aligned}\sum_{\theta \in \mathcal{S}} \mathbf{1}(p_h^\theta > p) M_h(\theta) &= \sum_{\theta \in \mathcal{S}} \mathbf{1}(p_h^\theta > \bar{p})(1 - \sum_{h' \succ^\theta h} M_{h'}(\theta)) + \sum_{\theta \in \mathcal{S}} \mathbf{1}(p_h^\theta = \bar{p})(1 - \sum_{h' \succ^\theta h} M_{h'}(\theta)), \\ &= I_h(\bar{p}) + \sum_{\theta \in \mathcal{S}} \mathbf{1}(p_h^\theta = \bar{p})(1 - \sum_{h' \succ^\theta h} M_{h'}(\theta)), \qquad (29)\end{aligned}$$

where the second line also follows from (27).

Note that because $\mathcal{V}^{det}(\lambda, C) \in \{0,1\}$, (2) and (5) imply that $M_h(\theta) \in \{0,1\}$ and therefore (1) implies that $I_h(p) \in \mathbb{N}$. Furthermore, the fact that $\mathcal{S}$ has strict priorities implies that at discontinuities of I_h, it decreases by exactly one. We know from the definition of $\bar{p}$ that $I_h(\bar{p}) < C_h$ but $I_h(p) \geq C_h$ for all $p < \bar{p}$, so $I_h(\bar{p}) = C_h - 1$ and $\sum_{\theta \in \mathcal{S}} \mathbf{1}(p_h^\theta = \bar{p})(1 - \sum_{h' \succ^\theta h} M_{h'}(\theta)) = 1$. This implies that the expression in (29) is equal to C_h, completing the proof. □

Proof of Proposition 1. We first suppose that (M, I, A) is $(\eta^{\mathcal{S}}, \mathcal{V}^{det})$-stable, and show that μ^M has no blocking pairs. Note that the definition of $\mathcal{V}^{det}$ in (3) implies that for all $\lambda \in \mathbb{R}_+, C \in \mathbb{N}$ we have $\mathcal{V}^{det}(\lambda, C) \in \{0,1\}$, so by (2) and (5), M is deterministic and μ^M is well-defined. We now show that μ is feasible. Note that

$$\begin{aligned}\sum_{\theta \in \mathcal{S}} M_h(\theta) &= \sum_{\theta \in \mathcal{S}} M_h(\theta) \mathbf{1}(p_h^\theta > 0) + \sum_{\theta \in \mathcal{S}} M_h(\theta) \mathbf{1}(p_h^\theta = 0) \\ &\leq \sum_{\theta \in \mathcal{S}} M_h^\theta \mathbf{1}(p_h^\theta > 0) + \eta^{\mathcal{S}}(\Theta_h(0)) A_h(I_h(0)) \\ &= \min(I_h(0), C_h) + \eta^{\mathcal{S}}(\Theta_h(0)) \mathbf{1}(I_h(0) < C_h) \\ &\leq \min(I_h(0), C_h) + \mathbf{1}(I_h(0) < C_h) \\ &\leq C_h.\end{aligned}$$

The second line follows from $M_h(\theta) \leq A_h(p_h^\theta)$ (see (5)) and $\sum_{\theta\in\mathcal{S}} \mathbf{1}(p_h^\theta = 0) = \eta^{\mathcal{S}}(\Theta_h(0))$ (see (10)); the third from Lemma 1, along with (2) and (3), and the fourth because $\mathcal{S}$ has strict priorities.

Finally, we show that μ^M has no blocking pairs. By definition, if $h \succ^{\theta'} \mu^M(\theta')$ then $M_h(\theta') = 0$, and $A_h(p_h^{\theta'}) = 0$ by (5). From this, (2) and (3) imply that $I_h(p_h^{\theta'})) \geq C_h$, so by Lemma 1,

$$|\{\theta \in \mathcal{S} : \mu^M(\theta) = h, p_h^\theta > p_h^{\theta'}\}| = \sum_{\theta\in\mathcal{S}} M_h(\theta)\mathbf{1}(p_h^\theta > p_h^{\theta'}) = \min(I_h(p_h^{\theta'}), C_h) = C_h.$$

That is, (9) holds, so μ^M has no blocking pairs.

Next, we assume that μ is an $\mathcal{S}$-matching with no blocking pairs, and show that

i) M^μ "agrees" with μ: for $\theta \in \mathcal{S}, h \in \mathcal{H}_0$, we have

$$M_h^\mu(\theta) = \mathbf{1}(\mu(\theta) = h). \tag{30}$$

ii) M^μ is $(\eta^{\mathcal{S}}, \mathcal{V}^{det})$-stable.

We start by showing (30). Fix $\theta' \in \mathcal{S}$. Then for any $h \succ^{\theta'} \mu(\theta')$, the fact that μ has no blocking pairs implies that (9) holds, from which the definition of A^μ in (12) implies that $A_h^\mu(p_h^{\theta'}) = 0$, so $M_h^\mu(\theta') = 0$. Meanwhile, for $h' = \mu(\theta')$, feasibility of μ implies

$$|\{\theta \in \mathcal{S} : \mu(\theta) = h', p_{h'}^\theta > p_{h'}^{\theta'}\}| < |\{\theta \in \mathcal{S} : \mu(\theta) = h'\}| \leq C_{h'}.$$

Therefore, the definition of A^μ in (12) implies that $A_{h'}^\mu(p_{h'}^{\theta'}) = 1$, from which (5) implies that $M_{h'}(\theta') = 1$ (and that $M_h(\theta) = 0$ for all h such that $\mu(\theta') \succ^{\theta'} h$). Thus, (30) holds, implying that $\mu = \mu^{M^\mu}$.

We now show that M^μ is $(\eta^{\mathcal{S}}, \mathcal{V}^{det})$-stable. Define $I^\mu = \mathcal{I}(M^\mu)$. Then we have

$$I_h^\mu(p) = \sum_{\theta\in\mathcal{S}} \mathbf{1}(p_h^\theta > p)(1 - \sum_{h'\succ^\theta h} M_h^\mu(\theta)) \tag{31}$$

$$\geq \sum_{\theta\in\mathcal{S}} \mathbf{1}(p_h^\theta > p)\mathbf{1}(\mu(\theta) = h) = |\{\theta \in \mathcal{S} : p_h^\theta > p, \mu(\theta) = h\}|, \tag{32}$$

where the first equality follows from (1) and the definition of $\eta^{\mathcal{S}}$ in (10), and the inequality from (30). We claim that for $h \in \mathcal{H}_0$, $p \in [0, 1]$,

$$A_h^\mu(p) = \mathbf{1}(I_h^\mu(p) < C_h) = \mathcal{V}^{det}(I_h^\mu(p), C_h), \tag{33}$$

implying that $A^\mu = \mathcal{A}(I^\mu) = \mathcal{A}(\mathcal{I}(\mathcal{M}(A^\mu)))$, so A^μ is $(\eta^{\mathcal{S}}, \mathcal{V}^{det})$-stable, and therefore so is M^μ. To establish (33), we show that $A_h^\mu(p) = 0$ implies $I_h^\mu(p) \geq C_h$, and $A_h^\mu(p) = 1$ implies $I_h^\mu(p) < C_h$.

If $A_h^\mu(p) = 0$, then by definition of A^μ in (12),

$$|\{\theta \in \mathcal{S} : p_h^\theta > p, \mu(\theta) = h\}| \geq C_h.$$

By (32), this implies that $I_h^\mu(p) \geq C_h$. Conversely, if $A_h^\mu(p) = 1$, then by definition

$$|\{\theta \in \mathcal{S} : p_h^\theta > p, \mu(\theta) = h\}| < C_h.$$

This implies that for each $\theta \in \mathcal{S}$ that contributes to the sum in (31), $\mu(\theta) = h$ (otherwise, (9) would be violated). Therefore, the inequality in (32) is an equality, implying that $I_h^\mu(p) < C_h$. $\square$

A.2. **Proof of Proposition 2.** We start by establishing an analog to Lemma 1, which says that for any $(\eta, \mathcal{V})$-stable outcome, the measure of students matched to school h can be determined by the measure of interest in h.

Lemma 2. *If η is a continuum measure with strict priorities and $\mathcal{V}$ is weakly decreasing in its first argument, then for any $(\eta, \mathcal{V})$-stable outcome (M, I, A), any school $h \in \mathcal{H}$, and any $p \in [0,1]$,*

$$\int \mathbf{1}(p_h^\theta > p) M_h(\theta) d\eta(\theta) = \int_0^{I_h(p)} \mathcal{V}(\lambda, C_h) d\lambda.$$

If $\mathcal{V} = \mathcal{V}^{det}$, then the expression on the right is $\min(I_h(p), C_h)$, matching that in Lemma 1. However, Lemma 2 provides a more general expression that holds for any monotone vacancy function.

Proof of Lemma 2. Fix $n \in \mathbb{N}$, define $m = \lceil n I_h(p) \rceil$, and define $\{L_i\}_{i=0}^m$ by $L_i = i/n$ for $i < m$, and $L_m = I_h(p)$. Note that if η is a continuum measure with strict priorities, then (1) implies that I is continuous and decreasing. In particular, this implies that we can choose $1 = P_0 > P_1 > \cdots P_m = p$ such that $I_h(P_i) = L_i$ for each i. We claim the following chain of inequalities:

$$\begin{aligned}
\int \mathbf{1}(p_h^\theta > p) M_h(\theta) d\eta(\theta) &= \int \mathbf{1}(p_h^\theta > p) A_h(p_h^\theta) \prod_{h' \succ^\theta h} (1 - A_{h'}(p_{h'}^\theta)) d\eta(\theta) \\
&= \int \mathbf{1}(p_h^\theta > p) \mathcal{V}(I_h(p_h^\theta), C_h) \prod_{h' \succ^\theta h} (1 - A_{h'}(p_{h'}^\theta)) d\eta(\theta) \\
&= \int \sum_{i=1}^m \mathbf{1}(P_{i-1} \geq p_h^\theta > P_i) \mathcal{V}(I_h(p_h^\theta), C_h) \prod_{h' \succ^\theta h} (1 - A_{h'}(p_{h'}^\theta)) d\eta(\theta) \\
&\geq \sum_{i=1}^m \mathcal{V}(L_i, C_h) \int \mathbf{1}(P_{i-1} \geq p_h^\theta > P_i) \prod_{h' \succ^\theta h} (1 - A_{h'}(p_{h'}^\theta)) d\eta(\theta). \qquad (34)
\end{aligned}$$

The first equality holds from (5), the second from (2), and the third by definition of P_i. The final inequality comes from the fact that $\mathcal{V}$ is weakly decreasing in its first argument and I_h is weakly decreasing, and thus $p_h^\theta > P_i$ implies $\mathcal{V}(I_h(p_h^\theta), C_h) \geq \mathcal{V}(I_h(P_i), C_h) = \mathcal{V}(L_i, C_h)$. Note that

$$\begin{aligned}
&\int \mathbf{1}(P_{i-1} \geq p_h^\theta > P_i) \prod_{h' \succ^\theta h} (1 - A_{h'}(p_{h'}^\theta)) d\eta(\theta) \\
&\quad = \int \mathbf{1}(p_h^\theta > P_i) \prod_{h' \succ^\theta h} (1 - A_{h'}(p_{h'}^\theta)) d\eta(\theta) - \int \mathbf{1}(p_h^\theta > P_{i-1}) \prod_{h' \succ^\theta h} (1 - A_{h'}(p_{h'}^\theta)) d\eta(\theta) \\
&\quad = I_h(P_i) - I_h(P_{i-1}) \\
&\quad = L_i - L_{i-1}. \qquad (35)
\end{aligned}$$

The second equality follows from (1) and the third from the choice of P_i. Combining (34) and (35), and noting that $L_i - L_{i-1} = 1/n$ for $i < m$, we get

$$\int \mathbf{1}(p_h^\theta > p) M_h(\theta) d\eta(\theta) \geq \frac{1}{n} \sum_{i=1}^{m-1} \mathcal{V}(L_i, C_h).$$

This holds for any $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ yields

$$\int \mathbf{1}(p_h^\theta > p) M_h(\theta) d\eta(\theta) \geq \int_0^{I_h(p)} \mathcal{V}(\lambda, C_h) d\lambda. \tag{36}$$

The inequality in (34) can be reversed if we replace $\mathcal{V}(L_i, C_h)$ with $\mathcal{V}(L_{i-1}, C_h)$. From there, an analogous argument implies (36) with the inequality reversed, completing the proof.

□

Proof of Proposition 2. Given cutoffs $P \in [0,1]^{\mathcal{H}}$, we define

$$A_h^P(p) = \mathbf{1}(p > P_h). \tag{37}$$

$$M^P = \mathcal{M}(A^P). \tag{38}$$

We begin by noting two equalities that will repeatedly prove useful. Note that the definition of $\mathcal{A}$ through (2) and (3), and the definition of A^P and $\mathcal{P}(I)$ in (37) and (15) imply that for any interest function $I \in \mathfrak{I}$,

$$\mathcal{A}(I) = A^{\mathcal{P}(I)}. \tag{39}$$

Furthermore, the definition of I^P, A^P and M^P in (14), (37) and (38), along with (1) and (5), imply that for any cutoff vector P,

$$I^P = \mathcal{I}(\mathcal{M}(A^P)). \tag{40}$$

We first assume that I is $(\eta, \mathcal{V}^{det})$-stable, and show that $I = I^{\mathcal{P}(I)}$. By Definition 1, if I is $(\eta, \mathcal{V}^{det})$-stable, then so is

$$\mathcal{M}(\mathcal{A}(I)) = \mathcal{M}(A^{\mathcal{P}(I)}) = M^{\mathcal{P}(I)}, \tag{41}$$

where the equalities follow from (39) and (38), respectively. Furthermore, stability of I implies that

$$I = \mathcal{I}(\mathcal{M}(\mathcal{A}(I))) = \mathcal{I}(\mathcal{M}(A^{\mathcal{P}(I)})) = I^{\mathcal{P}(I)}, \tag{42}$$

where the second and third equalities follow from (41) and (40), respectively.

Next, we show that $\mathcal{P}(I)$ is market-clearing. We claim that

$$D_h(\mathcal{P}(I)) = \int M_h^{\mathcal{P}(I)}(\theta) d\eta(\theta) = \int_0^{I_h(0)} \mathcal{V}^{det}(\lambda, C_h) d\lambda = \min(I_h(0), C_h) \leq C_h. \tag{43}$$

The first equality follows from (13), the second from Lemma 2 and the fact that $\mathcal{I}(M^{\mathcal{P}}(I)) = I$ (by (41) and (42)), and the third from the definition of $\mathcal{V}^{det}$ in (3). Furthermore, if $\mathcal{P}_h(I) > 0$, it follows from definition of $\mathcal{P}$ in (15) that $I_h(P_h) \geq C_h$ (this also uses the fact that I_h is continuous, which follows from (1) and the fact that η is a continuum measure with strict priorities). Because (1) implies that I_h is weakly decreasing, it follows that $I_h(0) \geq C_h$, and therefore the inequality in (43) is tight. Therefore, $\mathcal{P}(I)$ is market-clearing.

Finally, we show that if $P \in [0,1]^{\mathcal{H}}$ is market-clearing, then I^P is $(\eta, \mathcal{V}^{det})$-stable. By (39) and (40),

$$\mathcal{I}(\mathcal{M}(\mathcal{A}(I^P))) = \mathcal{I}(\mathcal{M}(A^{\mathcal{P}(I^P)})) = I^{\mathcal{P}(I^P)}. \tag{44}$$

We wish to show that this is equal to I^P. For less cumbersome notation, we define the cutoff vector

$$\tilde{P} = \mathcal{P}(I^P). \tag{45}$$

The steps to prove that $I^{\tilde{P}} = I^P$ are as follows:

I. Show that $I_h^P(P_h) = D_h(P)$. Conclude that

$$\tilde{P}_h \geq P_h. \tag{46}$$

II. Define

$$\Delta_h = \{\theta : p_h^\theta \in (P_h, \tilde{P}_h], p_{h'}^\theta \leq P_{h'} \text{ for all } h' \succ^\theta h\} \tag{47}$$

to be the η-measure of students who have priority between P_h and $\tilde{P}_h$ at h, and are not admitted to any school preferred to h under either P or $\tilde{P}$. Establish that I_h^P is constant on $(P_h, \tilde{P}_h]$, and therefore that

$$0 = I_h^P(P_h) - I_h^P(\tilde{P}_h) = \eta(\Delta_h). \tag{48}$$

III. Conclude that for all $h \in \mathcal{H}$ and $p \in [0, 1]$,

$$I_h^{\tilde{P}}(p) = I_h^P(p). \tag{49}$$

We now establish step I. Note that

$$\begin{aligned} I_h^P(p) &= \int \mathbf{1}(p_h^\theta > p)(1 - \sum_{h' \succ^\theta h} M_{h'}^P(\theta))d\eta(\theta) \\ &= \int \mathbf{1}(p_h^\theta > p) \prod_{h' \succ^\theta h} (1 - A_{h'}^P(p_{h'}^\theta))d\eta(\theta), \end{aligned}$$

where the first line follows from (40) and (1), and the second from (6). It follows that

$$\begin{aligned} I_h^P(P_h) &= \int A_h^P(p_h^\theta) \prod_{h' \succ^\theta h} (1 - A_{h'}^P(p_{h'}^\theta))d\eta(\theta) \\ &= \int M_h^P d\eta(\theta) = D_h(P), \end{aligned} \tag{50}$$

where the second line follows from (5) and the definitions of A^P and $D_h(P)$ in (37) and (13). From (50) and the definition of $\mathcal{P}(\cdot)$ in (15), (46) follows.

Next, we move to step II. Because η is a continuous measure with strict priorities, (1) implies that I_h^P is continuous for each $h \in \mathcal{H}_0$. Therefore, the definition of $\mathcal{P}(\cdot)$ in (15) implies that either $\tilde{P}_h = 0$ (in which case (46) implies that $P_h = \tilde{P}_h$), or $I_h^P(\tilde{P}) = C_h$. But then (46) and the fact that I^P is decreasing imply that

$$I_h^P(P_h) \geq I_h^P(\tilde{P}_h) = C_h.$$

Because P is market-clearing, $D_h(P) \leq C_h$, implying that the inequality above must hold with equality. Therefore, I_h^P is constant on $(P_h, \tilde{P}_h]$. In particular, applying the definition of I^P in (14) reveals that (48) holds.

Finally, we move to step III. By (14), for any $h \in \mathcal{H}$ and $p \in [0, 1]$ we have

$$I_h^{\tilde{P}}(p) - I_h^P(p) = \eta(\Delta), \tag{51}$$

where

$$\Delta = \{\theta : p_h^\theta > p, \prod_{h' \succ^\theta h} \mathbf{1}(p_{h'}^\theta \leq \tilde{P}_{h'}) - \prod_{h' \succ^\theta h} \mathbf{1}(p_{h'}^\theta \leq P_{h'}) = 1\}$$

That is, the difference $I_h^{\tilde{P}}(p) - I_h^P(p)$ is the measure of students who have priority above p at h, and are admitted to a school preferred to h under cutoffs P, but are not admitted to any such school under $\tilde{P}$. For any $\theta \in \Delta$, there is some most-preferred school h' where θ is admitted under P but not under $\tilde{P}$. Then (47) implies that $\theta \in \Delta_{h'}$. In other words, $\Delta \subseteq \bigcup_{h \in \mathcal{H}} \Delta_h$. From this, (48) implies that $\eta(\Delta) = 0$, and (51) implies that (49) holds. This establishes that $I^P = I^{\tilde{P}} = I^{\mathcal{P}(I^P)}$, from which (44) implies that I^P is stable.

□

A.3. **Proof of Theorems 2 and 3.**

Proof of Theorem 2. By Theorem 1, there exist maximal and minimal stable outcomes, corresponding to the school-optimal and student-optimal stable outcomes, respectively. Denote these outcomes by $(M^H, I^H, A^H) \succeq (M^L, I^L, A^L)$, respectively. It is enough to prove the result for these outcomes. Note that

$$\int \sum_{h \succ^\theta \emptyset} M_h^L(\theta) d\eta(\theta) = \sum_{h \in \mathcal{H}} \int M_h^L(\theta) d\eta(\theta) = \sum_{h \in \mathcal{H}} \int_0^{I_h^L(0)} \mathcal{V}(\lambda, C_h) d\lambda$$

$$\geq \sum_{h \in \mathcal{H}} \int_0^{I_h^H(0)} \mathcal{V}(\lambda, C_h) = \sum_{h \in \mathcal{H}} \int M_h^H(\theta) d\eta(\theta) = \int \sum_{h \succ^\theta \emptyset} M_h^H(\theta) d\eta(\theta). \tag{52}$$

The first and last equalities hold because $A_\emptyset(p) = 1$ for all p, so by (5), $M_h(\theta) = 0$ if $\emptyset \succ^\theta h$. The second and second-to-last equalities hold by Lemma 2. The inequality follows from the fact that $I^L \succeq^{\mathfrak{I}} I^H$. But $M^H \succeq^{\mathfrak{M}} M^L$ implies

$$\sum_{h \succ^\theta \emptyset} M_h^L(\theta) \leq \sum_{h \succ^\theta \emptyset} M_h^H(\theta) \qquad \forall \theta \in \Theta. \tag{53}$$

Therefore, the inequality in (52) must hold with equality. In particular, this implies that (17) holds for each $h \in \mathcal{H}$. Furthermore, this implies that (18) holds for all θ except possibly a set of η-measure zero. □

Proof of Theorem 3. It suffices to show that there is a unique stable interest function: that is, if I^H and I^L are the largest and smallest stable interest functions according to $\succeq^{\mathfrak{I}}$, then $I^H = I^L$. We let A^H, M^H be the admissions function and matching associated with I^H, and define A^L, M^L analogously. We note that by (2) and the fact that $\mathcal{V}$ is decreasing in its first argument, $I^H \succeq^{\mathfrak{I}} I^L$ implies that

$$A^L \succeq^{\mathfrak{A}} A^H. \tag{54}$$

The proof proceeds by contradiction, showing that $I^H \succ^I I^L$ implies that Theorem 2 does not hold. That is, $I^H \succ^I I^L$ implies the existence of a set $\tilde{\Theta}$ with $\eta(\tilde{\Theta}) > 0$ such that

$$\sum_{h \in \mathcal{H}} M_h^H(\theta) < \sum_{h \in \mathcal{H}} M_h^L(\theta) \text{ for all } \theta \in \tilde{\Theta}. \tag{55}$$

We establish existence of such a $\tilde{\Theta}$ in three steps.

I. Note that Definition 2 and (1) imply that
   a) the stable interest functions I^H and I^L are component-wise continuous, and
   b) for all $h \in \mathcal{H}$, $I_h^H(1) = I_h^L(1) = 0$.

II. These jointly imply that if $I_h^H(p) > I_h^L(p)$ for some $p \in [0,1]$ and $h \in \mathcal{H}$, then there must exist an interval $(\underline{p}, \overline{p})$ such that:
   a) $I_h^H(p) > I_h^L(p)$ for $p \in (\underline{p}, \overline{p})$, and
   b) $I_h^H(\overline{p}) > I_h^H(\underline{p})$.

   That is, I_h^H is not constant and strictly larger than I_h^L on this interval.

III. Define

$$S = \{\theta : M_\emptyset^H(\theta) = 0\}\} \tag{56}$$

to be the set of student types who are sure to be matched. Define

$$\tilde{\Theta} = \{\theta : h \succ^\theta \emptyset, p_h^\theta \in (\underline{p}, \overline{p})\} \backslash S. \tag{57}$$

We will show that (55) holds, and that $\eta(\tilde{\Theta}) > 0$.

To see that (55) holds, note that if $\theta \in \tilde{\Theta}$,

$$A_h^H(p_h^\theta) = \mathcal{V}(I^H(p_h^\theta), C_h) < \mathcal{V}(I^L(p_h^\theta), C_h) = A_h^L(p_h^\theta), \tag{58}$$

where the equalities hold by (2) and the inequality follows from II.a), the fact that $p_h^\theta \in (\underline{p}, \overline{p})$, and the fact that $\mathcal{V}(\cdot, C_h)$ is strictly decreasing. Thus, when comparing A^H to A^L, each student in $\tilde{\Theta}$ is

i. weakly less likely to be admitted to each school under A^H by (54),
ii. strictly less likely to be admitted to h under A^H by (58), and
iii. not certain to be admitted to any school by definition of $\tilde{\Theta}$ in (57).

From this, (5) implies that each $\theta \in \tilde{\Theta}$ is strictly less likely to match under A^H. That is, (55) holds.

Finally, we show that $\eta(\tilde{\Theta}) > 0$. By definition of $\tilde{\Theta}$ in (57)

$$\begin{aligned} \eta(\tilde{\Theta} \cup S) &\geq \int \mathbf{1}(h \succ^\theta \emptyset, p_h^\theta \in (\underline{p}, \overline{p})) d\eta(\theta) \\ &\geq \int \mathbf{1}(h \succ^\theta \emptyset, p_h^\theta \in (\underline{p}, \overline{p}))(1 - \sum_{h' \succ^\theta h} M_{h'}^H(\theta)) d\eta(\theta) \\ &= I_h^H(\underline{p}) - I_h^H(\overline{p}) \\ &> 0, \end{aligned} \tag{59}$$

where the equality follows from (1) and stability of I^H, and the final line follows from II.b). We complete the proof by showing that $\eta(S) = 0$.

Because $\mathcal{V}(\cdot, C_h)$ is strictly decreasing, (2) implies that $A_h^H(p) = 1$ if and only if $I_h^H(p) = 0$. Define $p_h = \inf\{p : I_h^H(p_h^\theta) = 0\}$ to be the lowest priority at school h that guarantees admission, and note that

$$0 = I_h^H(p_h) = \int \mathbf{1}(p_h^\theta > p_h)(1 - \sum_{h' \succ^\theta h} M_{h'}^H(\theta)) d\eta(\theta), \tag{60}$$

where the second equality comes from stability of I^H. Define

$$S_h = \{\theta : p_h^\theta > p_h, \sum_{h' \succ^\theta h} M_{h'}^H(\theta) < 1\}$$

to be the set of agents who are certain to be admitted to h, and not certain to be admitted to any option that they prefer to h. Note that

$$\eta(S_h) = \int \mathbf{1}(p_h^\theta > p_h)\mathbf{1}\left(\sum_{h' \succ^\theta h} M_{h'}^H(\theta) < 1\right) d\eta(\theta) = 0,$$

where the second equality follows from (60). Because $S = \bigcup_{h \in \mathcal{H}} S_h$, it follows that $\eta(S) = 0$ and thus $\eta(\tilde{\Theta}) > 0$ by (59).

□

Appendix B. Proofs from Section 4

Define

$$AcceptanceRate(\lambda, C) = Enrollment(\lambda, C)/\lambda = \frac{1}{\lambda}\int_0^\lambda \mathcal{V}(x, C)dx, \tag{61}$$

where the second equality follows from the definition of *Enrollment* in (22). The following result implies that *AcceptanceRate* is decreasing in its first argument.

Lemma 3. *Given $f : \mathbb{R}_+ \to \mathbb{R}_+$, define $g : \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$g(y) = \frac{1}{y}\int_0^y f(x)dx.$$

If f is weakly increasing, then so is g. If f is weakly decreasing, then so is g.

Proof of Lemma 3. Note that

$$g'(y) = \frac{yf(y) - \int_0^y f(x)dx}{y^2}.$$

If f is weakly increasing, then $yf(y) \geq \int_0^y f(x)dx$; if f is weakly decreasing, the inequality reverses.

□

Proof of Proposition 3. Let $\ell(\theta) = |\{h : h \succ^\theta \emptyset\}|$ be the number of schools listed by type θ. We note that for any individually rational matching M and any θ,

$$\begin{aligned}
\sum_{h' \succ^\theta \emptyset} M_{h'}(\theta) R_{h'}(\theta) &\leq \ell(\theta)(1 - \sum_{h' \succ^\theta \emptyset} M_{h'}(\theta)) + \sum_{h' \succ^\theta \emptyset} M_{h'}(\theta) R_{h'}(\theta) \\
&= \ell(\theta) - \sum_{h' \succ^\theta \emptyset} M_{h'}(\theta)(\ell(\theta) - R_{h'}(\theta)) \\
&= \ell(\theta) - \sum_{h' \succ^\theta \emptyset} \sum_{h' \succ^\theta h \succ^\theta \emptyset} M_{h'}(\theta) \\
&= \sum_{h \succ^\theta \emptyset} (1 - \sum_{h' \succ^\theta h} M_{h'}(\theta)) \qquad (62) \\
&= \sum_{h \in \mathcal{H}\emptyset} (1 - \sum_{h' \succ^\theta h} M_{h'}(\theta)) \qquad (63)
\end{aligned}$$

where the third line follows from the fact that $\ell(\theta) - R_{h'}(\theta)$ is the number of acceptable schools that rank below h', the fourth follows by exchanging the order of summation, and the last uses the fact that M is individually rational.

From (63) and the definition of *AverageRank* in (20), it follows that if (M, I, A) is a stable outcome,

$$AverageRank(M) \leq \frac{\int \sum_{h\in\mathcal{H}} M_h(\theta)(1 - \sum_{h' \succ^{\theta} h} M_{h'}(\theta))d\eta(\theta)}{\int \sum_{h\in\mathcal{H}} M_h(\theta) d\eta(\theta)} = \frac{\sum_{h\in\mathcal{H}} I_h(0)}{\sum_{h\in\mathcal{H}} \int_0^{I_h(0)} \mathcal{V}(\lambda, C_h)d\lambda}. \tag{64}$$

Note that the final equality follows by the fact that M is stable, (1) and Lemma 2. In a symmetric iid market, $C_h = C_{h'}$ and $I_h = I_{h'}$ for all $h, h' \in \mathcal{H}$, so implies that for any $h \in \mathcal{H}$,

$$AverageRank(M) \leq \frac{I_h(0)}{\int_0^{I_h(0)} \mathcal{V}(\lambda, C_h)d\lambda} = \frac{1}{AcceptanceRate(I_h(0), C_h)}. \tag{65}$$

Lemma 3 implies that *AcceptanceRate* is decreasing in its first argument, so we can obtain an upper bound on this expression by obtaining an upper bound on $I_h(0)$. But it is clear that the denominator in (64) is at most $\eta(\Theta)$, from which symmetry implies that $Enrollment(I_h(0), C_h) \leq \rho = Enrollment(\Lambda(\rho, C_h), C_h)$. Because *Enrollment* is increasing in its first argument, this implies that $I_h(0) \leq \Lambda(\rho, C)$. Plugging this into (65) completes the proof. □

Lemma 4. *The function $AR : (0, 1] \to \mathbb{R}_+$ defined by*

$$AR(q) = \frac{1}{q} - \frac{\ell(1-q)^\ell}{1-(1-q)^\ell}, \tag{66}$$

is decreasing in q.

Proof of Proposition 4. Let (M, I, A) be the unique $(\eta^{IID}, \mathcal{V}^{pois})$-stable outcome. Note that by symmetry, $I_h(p) = I_{h'}(p)$ and $A_h(p) = A_{h'}(p)$ for all $h, h' \in \mathcal{H}$ and $p \in [0, 1]$, so in what follows, we write $I(p)$ and $A(p)$ in place of $I_h(p)$ and $A_h(p)$.

Define

$$q = \int_0^1 \mathcal{V}^{pois}(I(p), C)dp. \tag{67}$$

Note that

$$\begin{aligned}
\int \sum_{h\in\mathcal{H}} M_h(\theta)d\eta(\theta) &= \int (1 - \prod_{h \succ^{\theta} \emptyset} (1 - A(p_h^\theta))d\eta^{IID}(\theta) \\
&= \int (1 - \prod_{h \succ^{\theta} \emptyset} (1 - \mathcal{V}^{pois}(I(p_h^\theta), C))d\eta^{IID}(\theta) \\
&= \eta^{IID}(\Theta)(1 - (1-q)^\ell), \qquad (68)
\end{aligned}$$

where the first equality follows from (5), the second from (2), and the last from the fact that we assume all students list ℓ schools, and in an iid market, the priorities p_h are drawn iid $U[0, 1]$.

Furthermore, we have

$$\int \sum_{h\in\mathcal{H}} M_h(\theta)R_h(\theta)d\eta(\theta) = \int \sum_{h\in\mathcal{H}} R_h(\theta)A_h(p_h^\theta) \prod_{h'\succ^\theta h}(1-A_{h'}(p_{h'}^\theta))d\eta^{IID}(\theta)$$
$$= \eta^{IID}(\Theta)\sum_{k=1}^{\ell} kq(1-q)^{k-1}$$
$$= \eta^{IID}(\Theta)(1-(1-q)^\ell)AR(q). \tag{69}$$

Jointly, (68) and (69) imply that

$$AverageRank(M) = AR(q). \tag{70}$$

Note that (23) implies that

$$Enrollment(\lambda, C) \le C \text{ for all } \lambda \in \mathbb{R}_+, \tag{71}$$

and therefore

$$\eta^{IID}(\Theta)(1-(1-q)^\ell) = |\mathcal{H}|Enrollment(I(0), C) \le |\mathcal{H}|C. \tag{72}$$

Define α and α' as the solutions to

$$\eta^{IID}(\Theta)(1-(1-\alpha)^\ell) = |\mathcal{H}|C = \eta^{IID}(\Theta)(1-e^{-\alpha'\ell}). \tag{73}$$

Then it follows that

$$q \le \alpha \le \alpha', \tag{74}$$

with the last inequality holding because $e^{-\alpha'\ell} \ge (1-\alpha')^\ell$. Because AR is decreasing by Lemma 4, equations (70) and (74) imply that

$$AverageRank(M) = AR(q) \ge AR(\alpha) = 1/\alpha - \ell(\rho/C - 1) \ge 1/\alpha' + \ell(1-\rho/C), \tag{75}$$

where the second equality follows from the definition of α. Finally, noting that $\alpha' = \frac{-\log(1-C/\rho)}{\ell}$ completes the proof.

We now turn to the case where priorities are identical across schools. We define

$$q(u) = \mathcal{V}^{pois}(\Lambda(u, C), C). \tag{76}$$

We will prove that the following chain of inequalities hold:

$$AverageRank(M) = \frac{1}{Enrollment(I(0), C)} \int_0^{Enrollment(I(0),C)} AR(q(u))du \tag{77}$$
$$\le \frac{1}{C}\int_0^C AR(q(u))du \tag{78}$$
$$\le \int_0^1 AR(q)dq \tag{79}$$
$$\le 1 + \log(\ell). \tag{80}$$

To evaluate $AverageRank(M)$, we note that

$$\int \sum_{h\in\mathcal{H}} M_h(\theta)R_h(\theta)d\eta^{RSD}(\theta) = |\mathcal{H}|Enrollment(I(0),C). \tag{81}$$

$$\int \sum_{h\in\mathcal{H}} M_h(\theta)R_h(\theta)d\eta^{RSD}(\theta) = \eta^{RSD}(\Theta)\int_0^1 (1-(1-\mathcal{V}^{pois}(I(p),C))^\ell)AR(\mathcal{V}^{pois}(I(p),C))dp.$$

We apply u-substitution to the latter integral, with $u = Enrollment(I(p),C)$, so that

$$\frac{du}{dp} = \mathcal{V}^{pois}(I(p),C)I'(p) = -\frac{\eta^{RSD}(\Theta)}{|\mathcal{H}|}(1-(1-\mathcal{V}^{pois}(I(p),C))^\ell).$$

This yields

$$\eta^{RSD}(\Theta)\int_0^1 (1-(1-\mathcal{V}^{pois}(I(p),C))^\ell)AR(\mathcal{V}^{pois}(I(p),C))dp = |\mathcal{H}|\int_0^{Enrollment(I(0),C)} AR(q(u))du,$$

which when combined with (81) yields (77).

We now establish (78). The function q given in (76) is decreasing, as is AR by Lemma 4. Therefore, (71) and Lemma 3 imply (78).

We move on to establishing (79). Define $f:[0,1]\to\mathbb{R}_+$ implicitly by

$$\mathcal{V}^{pois}(f(q),C) = q. \tag{82}$$

We note that

$$-\frac{d}{d\lambda}\mathcal{V}^{pois}(\lambda,C) = \mathcal{V}^{pois}(\lambda,C) - \mathcal{V}^{pois}(\lambda,C-1) \le \mathcal{V}^{pois}(\lambda,C). \tag{83}$$

Therefore, by (76) and (82), we have

$$\frac{1}{C}\int_0^C AR(q(u))du = \frac{1}{C}\int_0^1 AR(q)\frac{\mathcal{V}^{pois}(f(q),C)}{\mathcal{V}^{pois}(f(q),C)-\mathcal{V}^{pois}(f(q),C-1)}dq. \tag{84}$$

(85)

The function[12]

$$h(\lambda) = \frac{1}{C}\frac{\mathcal{V}^{pois}(\lambda,C)}{\mathcal{V}^{pois}(\lambda,C)-\mathcal{V}^{pois}(\lambda,C-1)}$$

is decreasing in λ , from which it follows that $h(f(q))$ is increasing in q. Meanwhile, AR is decreasing in q by Lemma 4. It follows that

$$\int_0^1 AR(q)h(f(q))dq \le \int_0^1 AR(q)dq\int_0^1 h(f(q))dq = \int_0^1 AR(q)dq. \tag{86}$$

The final inequality follows because by (82) and (23) we have

$$\int_0^1 h(f(q))dq = \frac{1}{C}\int_0^\infty \mathcal{V}^{pois}(\lambda)d\lambda = 1.$$

Finally, we show (80). We note that by u-substitution with $1-u=(1-q)^\ell$,

$$\int_\epsilon^1 \frac{\ell(1-q)^\ell}{1-(1-q)^\ell}dq = \int_{1-(1-\epsilon)^\ell}^1 \frac{(1-u)^{1/\ell}}{u}du.$$

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Thus, we can write

$$\int_{\epsilon}^{1} AR(q)dq = \int_{\epsilon}^{1} \frac{1}{q} - \frac{\ell(1-1)^{\ell}}{1-(1-q)^{\ell}} dq = \int_{\epsilon}^{1-(1-\epsilon)^{\ell}} \frac{1}{q} dq + \int_{1-(1-\epsilon)^{\ell}}^{1} \frac{1-(1-u)^{1/\ell}}{u} du$$
$$\leq \log\left(\frac{1-(1-\epsilon)^{\ell}}{\epsilon}\right) + 1,$$

where the second line follows by evaluating the first integral and bounding the second using the fact that $(1-(1-u)^{1/\ell})/u \leq (1-(1-u))/u = 1$. Combining this with the fact that[13]

$$\int_{0}^{1} AR(q)dq = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} AR(q)dq.$$

implies (80).

□

[13]Despite appearances, AR is well-behaved at zero: for $q > 0$,

$$1 \leq \frac{1}{q} - \frac{\ell(1-q)^{\ell}}{1-(1-q)^{\ell}} \leq \frac{\ell+1}{2}.$$