Many-to-one assignment markets: extreme core allocations*

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Preliminary. Please do not circulate.

Abstract

This paper studies many-to-one assignment markets, or matching markets with wages. Although it is well-known that the core of this model is non-empty, the structure of the core has not been fully investigated. To the known dissimilarities with the one-to-one assignment game, we add that the bargaining set does not coincide with the core, the kernel may not be included in the core, and the τ-value may also lie outside the core. Besides, not all extreme core allocations can be obtained by a procedure of lexicographic maximization, as it is the case in the one-to-one assignment game. Our main results are on the extreme core allocations. First, we characterize the set of extreme core allocations in terms of a directed graph defined on the set of workers and also provide a necessary condition for each side-optimal allocation. Finally, we prove that each extreme core allocation is the result of sequentially maximizing or minimizing the core payoffs according to a given order on the set of workers.

Keywords: Many-to-one matching markets · extreme core allocations · side-optimal allocations · kernel · core

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1 Introduction

This paper studies many-to-one assignment markets that may represent job markets with wages (endogenous side transfers). In this problem, there are two sides of the market, a set of firms and a set of workers. Firms want to hire many workers, but each worker can work for only one firm. The main data of the market is the value that each firm-worker pair can attain when matched. A matching is an assignment of a group of workers to each firm and a coalitional game is introduced where the worth of a coalition is the value that can be obtained by matching firms and workers in the coalition without violating the capacity of each firm. A natural solution concept in this setting is the core, which is the set of allocations of the total value of the market that cannot be improved upon by any coalition. It is established in the literature that the core of a many-to-one assignment market is non-empty. Moreover, there exist optimal side allocations for each side of the market, the firm-optimal core allocation and the worker-optimal core allocation, at the former each firm obtains the maximum core payoff and at the latter each worker receives the maximum core payoff.

The many-to-one assignment market is an extension of the well-known (one-to-one) *assignment game* introduced by Shapley and Shubik (1971) to study two-sided markets where there are indivisible goods which are traded between sellers and buyers in exchange for money.[1] In their model, each buyer wants at most one unit of good, and each seller has exactly one indivisible good. They solve the linear assignment problem that maximizes the total weight generated by seller-buyer pairs to define the corresponding coalitional game (assignment game). [2] The question is how to share the total profit attained by the buyers and sellers. To this end, they show that the core of an assignment game is always non-empty. Furthermore, it coincides with the set of dual solutions to the linear assignment problem and has a lattice structure which guarantees the existence of two optimal allocations: one that is optimal for all buyers (buyer-optimal) and the other optimal for all sellers (seller-optimal).

There are in the literature several generalizations of the one-to-one assignment game to the many-to-one and to the many-to-many cases (see for instance Sotomayor, 1992; Bikhchandani and Ostroy, 2002; Jaume, Massó and Neme, 2012). Since in our model each worker can establish only one partnership, each firm-worker pair can make at most one agreement. Hence, our model lies in the intersection of the multiple-partners game (Sotomayor, 1992) and the transportation game (Sánchez-Soriano, Lopez and Garcia-Jurado, 2001; Sotomayor, 2002). Therefore, our negative results also hold for the multiple-partners game and also for the transportation game.

Sotomayor (2002) points out the similarities and differences between the one-to-one assignment market and the many-to-one case. Both have a non-empty core and an optimal core allocation for each side of the market but, on the other side, the opposition of interests between both sides in the core and some non-manipulability properties of the optimal stable rules do not carry over. We consider other solution concepts different from the core to show that more dissimilarities appear.

[1] Koopmans and Beckmann (1957) is one of the earliest works on assignment problems within an economic context.

[2] Núñez and Rafels (2015) is a recent survey on assignment markets and games.

We prove that, differently from the one-to-one assignment game, the kernel may not be a subset of the core for many-to-one assignment games. Therefore, the core and the classical bargaining set do not coincide (Example 5 and Corollary 6). Moreover, making use of the Example 5, we show that, unlike the assignment game, the single-valued solution known as the τ-value need not be a core allocation.

Beyond the non-emptiness of the core for many-to-one assignment markets and the existence of side-optimal allocations, little is known about the geometric structure of the core of these market games. To this end, we aim to study the extreme core allocations. First, we observe that, unlike the one-to-one case, in an extreme core allocation it may be the case that no agent achieves his/her marginal contribution and moreover extreme core allocations are not obtained by a lexicographic maximization procedure as it is the case for one-to-one markets (Núñez and Solymosi, 2017).

Based on the projection of the core to the space of workers' payoffs, and given a core allocation, we define a digraph where the set of workers is the set of nodes. Then, we show that a core allocation is an extreme point if and only if each component of the digraph defined on that core allocation contains either a worker with zero payoff or a worker with a payoff that equals the total surplus it creates with a firm under an optimal matching (Theorem 11). Following this result, we provide a necessary condition for each side-optimal allocation in terms of the digraph defined on the workers (Proposition 12 and Proposition 13, respectively). Example 14 shows that these necessary conditions for an extreme core allocation to be a side-optimal allocation are not sufficient. For each order on the set of workers, we define a payoff vector where each worker sequentially maximizes or minimizes its core payoff, preserving what has been allocated to its predecessors. Making use of the digraph introduced before, this set of max-min vectors is proved to contain all the extreme core points of the many-to-one assignment market, and this gives a procedure for the computation of these extreme points and as a consequence allows for a representation of the complete core.

The paper is organized as follows. In Section 2, some preliminaries on transferable utility games are provided. Section 3 introduces many-to-one assignment markets and games. In Section 4, we introduce our results on the core, the kernel, and the bargaining set. We provide our main results on extreme core allocations in Section 5. In Section 6 we define the set of max-min vectors and show each extreme core point is of this type. Finally, Section 7 concludes.

2 Notations and definitions

A *transferable utility (TU) cooperative game* (N, v) is a pair where N is a non-empty, finite set of *players (or agents)* and $v : 2^N \to \mathbb{R}$ is a *coalitional function* satisfying $v(\emptyset) = 0$. The number $v(S)$ is regarded as the worth of the coalition $S \subseteq N$. We identify the game with its coalitional function since the player set N is fixed throughout the paper. The game (N, v) is called *superadditive* if $S \cap T = \emptyset$ implies $v(S \cup T) \geq v(S) + v(T)$ for every two coalitions $S, T \subseteq N$. Coalition $R \subseteq N$ is called *inessential* in game v if it has a nontrivial partition $R = S \cup T$ with $S, T \neq \emptyset$ and $S \cap T = \emptyset$ such that $v(R) \leq v(S) + v(T)$. Notice that in a superadditive game the weak majorization can only happen as equality. Those non-empty coalitions which are not inessential are called *essential*. Note that the single-player coalitions are essential in any game, and any inessential coalitional value can be weakly majorized by the value of a partition composed only of essential coalitions.

Given a game (N, v), a *payoff allocation* $x \in \mathbb{R}^N$ represents the payoffs to the players. The total payoff to coalition $S \subseteq N$ is denoted by $x(S) = \sum_{i \in S} x_i$, in particular $x(\emptyset) = 0$, for throughout the paper we keep the convention that summing over the empty-set gives zero. In a game v, we say the payoff allocation x is *efficient*, if $x(N) = v(N)$. The set of *imputations*, denoted by $I(v)$, consists of all payoff vectors that are *individually rational*, that is, $x_i \geq v(\{i\})$ for all $i \in N$. The core $C(v)$ is the set of imputations that are *coalitionally rational*, that is, $x(S) \geq v(S)$ for all $S \subseteq N$. Observe that all the coalitional rationality conditions for inessential coalitions are implied by the inequalities related to essential coalitions, hence can be ignored: the core and the essential-core are always the same.

Given a game (N, v), the game (N, v^*) defined by $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$ is called the *dual game*. Notice that $v^*(\emptyset) = 0$ and $v^*(N) = v(N)$ for any game (N, v). It is easily seen that the core of any coalitional game coincides with the *anticore* of its dual game, that is,

$$\mathbf{C}(v) = \mathbf{C}^*(v^*) := \{x \in \mathbb{R}^N : x(N) = v^*(N),\ x(S) \leq v^*(S)\ \forall S \subseteq N\}. \tag{1}$$

It follows that if $i \in N$ is a *null player* in game v (i.e. $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$, in particular, $v(\{i\}) = 0$), its payoff is $x_i = 0$ in any core allocation $x \in C(v)$. Indeed, then $v(N) = v(N \setminus \{i\}) + v(\{i\}) \leq x(N \setminus \{i\}) + x_i = x(N) = v(N)$, implying both $x(N \setminus \{i\}) = v(N \setminus \{i\})$ and $x_i = v(i) = 0$.

An order on the set of players N is a bijection $\sigma : \{1, 2, \ldots, n\} \to N$, where for all $i \in \{1, 2, \ldots, n\}$, $\sigma_i = \sigma(i)$ is the player that occupies position i. For a given order σ, $P_i^\sigma = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$ denotes the set of predecessors of agent $i \in N$. For each order σ on the player set N of game (N, v), a *marginal payoff vector* $m^{\sigma,v}$ is defined by $m_i^{\sigma,v} = v(P_{\sigma_i}^\sigma \cup \{\sigma_i\}) - v(P_{\sigma_i}^\sigma)$ for all $i \in N$. Whenever a marginal payoff vector is in the core, then it is an extreme core allocation.

Hamers et al. (2002) showed that each extreme core allocation of an assignment game is a marginal payoff vector. Nevertheless, the opposite implication does not hold.

Núñez and Solymosi (2017) studied other lexicographic allocation procedures for coalitional games looking for a characterization and a computation procedure of their extreme core points. Given a game (N, v) and over the set $\mathbf{Ra}^*(N, v) = \{x \in \mathbb{R}^N : x(S) \leq v(N) - v(N \setminus S)$ for all $S \subseteq N\}$ of dual coalitionally rational payoff vectors, the following lexicographic maximization procedure is proposed: for any order σ of the players, the σ-*lemaral* vector $\overline{r}^{\sigma,v} \in \mathbb{R}^N$ is defined by, for all $i \in \{1, 2, \ldots, n\}$,

$$\overline{r}_{\sigma_i}^{\sigma,v} = \max\left\{x_{\sigma_i} \ :\ x \in \mathbf{Ra}^*(N, v),\ x_{\sigma_l} = \overline{r}_{\sigma_l}^{\sigma,v}\ \forall l \in \{1, \ldots, i-1\}\right\}, \tag{2}$$

which trivially leads to

$$\overline{r}_{\sigma_i}^{\sigma,v} = \min\left\{v^*(Q \cup \{\sigma_i\}) - \overline{r}^{\sigma,v}(Q) \ :\ Q \subseteq P_{\sigma_i}^\sigma\right\}. \tag{3}$$

A game with a non-empty empty core (N, v) is called *ONTO-lemaral* if all its extreme core allocations are lemaral vectors, and *INTO-lemaral* if all lemaral vectors are extreme core allocations of (N, v). It follows from Núñez and Solymosi (2017) that assignment games are ONTO-lemaral and also INTO-lemaral. Hence, for these games, the set of extreme core allocations coincides with the set of lemarals.

3 The many-to-one assignment game

We consider a market where there are two types of agents: a finite set of firms $F = \{f_1, f_2, \ldots, f_m\}$ and a finite set of workers $W = \{w_1, w_2, \ldots, w_n\}$ where the number of firms m can be different from the number of workers n. Let $N = F \cup W$ be the set of all agents. We sometimes denote a generic firm and a generic worker by i and j, respectively. A pair of firm $i \in F$ and worker $j \in W$ can generate a non-negative income a_{ij} if firm $i \in F$ hires worker $j \in W$. The valuation matrix denoted by $A = (a_{ij})_{(i,j) \in F \times W}$ represents the pairwise income for each possible pair of firm and worker when the firm hires the worker. Each firm $i \in F$ would like to hire up to $r_i \geq 0$ number of workers and each worker $j \in W$ can work for at most one firm. Then, a many-to-one assignment market is a quadruple $\gamma = (F, W, A, r)$.

A *matching* μ for the market $\gamma = (F, W, A, r)$ is a set of $F \times W$ pairs such that each firm $i \in F$ appears in at most r_i pairs and each worker $j \in W$ in at most one pair. We denote by $\mathcal{M}(F, W, r)$ the set of matchings for the market γ. A matching $\mu \in \mathcal{M}(F, W, r)$ is an *optimal matching* for γ if $\sum\limits_{(i,j)\in\mu} a_{ij} \geq \sum\limits_{(i,j)\in\mu'} a_{ij}$ holds for any other matching $\mu' \in \mathcal{M}(F, W, r)$. We denote by $\mathcal{M}_A(F, W, r)$ the set of optimal matchings for the market γ. Given a matching $\mu \in \mathcal{M}(F, W, r)$, the set of workers matched to firm $i \in F$ under μ is $\mu(i) = \{j \in W \mid \text{ there exists } (i, j) \in \mu\}$. It will be convenient to denote the set of workers unmatched under μ by $\mu(f_0)$, that is $\mu(f_0) = W \setminus \bigcup_{i \in F} \mu(i)$. Observe that $i \neq j \in F$ implies $\mu(i) \cap \mu(j) = \emptyset$, hence $\mu(f_0) \cup \bigcup_{i \in F} \mu(i) = W$ is a partition of the set of workers.

Given a many-to-one assignment market $\gamma = (F, W, A, r)$, we define the income maximization linear programming problem by

$$\begin{aligned} \mathcal{V}(F, W) = \max & \sum_{i\in F} \sum_{j \in W} a_{ij} h_{ij} \\ \text{s. t.} \quad & \sum_{j \in W} h_{ij} \leq r_i, \;\; i \in F \\ & \sum_{i \in F} h_{ij} \leq 1, \;\; j \in W \\ & h_{ij} \geq 0, \qquad (i, j) \in F \times W. \end{aligned} \tag{4}$$

It is well known that any variable in any basic feasible solution of this LP problem with integer right hand sides is integral, hence, by the worker capacity inequlaties, 0 or 1. Consequently, the relation $(i, j) \in \mu \leftrightarrow h_{ij} = 1$ defines a bijection between the set of basic feasible solutions to this LP problem and the set of matchings $\mu \in \mathcal{M}(F, W, r)$. Henceforth, the optimum value of (4) gives the maximimum of the sum of values of the matched pairs while respecting the capacities of firms.

Given market $\gamma = (F, W, A, r)$, we also apply the above notation and terminology for any submarket $\gamma_{(S,T)} = (S, T, A_{(S,T)}, r_S)$ with $S \subseteq F$, $T \subseteq W$, and accordingly restricted payoff matrix $A_{(S,T)}$ and capacity vector r_S.

Now, let us associate coalitional games with transferable utility (TU-game) with this type of two-sided matching markets. Given a many-to-one assignment market $\gamma = (F, W, A, r)$, its associated *many-to-one assignment game* is the pair (N, v_γ) where $N = F \cup W$ is the set of players and the coalitional function $v_\gamma(S \cup T) = \max\limits_{\mu \in \mathcal{M}(S,T,r_S)} \sum\limits_{(i,j)\in\mu} a_{ij}$ for all $S \subseteq F$ and $T \subseteq W$.[3] For brevity, we denote coalition $S \cup T$ with $S \subseteq F$ and $T \subseteq W$ by (S, T), in particular, one-sided coalitions by $(\emptyset, T)$ and $(S, \emptyset)$. As the union of matchings for disjoint coalitions is a matching for the union of the coalitions, i.e. $\mu \in \mathcal{M}(S, T, r_S)$ and $\mu' \in \mathcal{M}(S', T', r_{S'})$ with $S \cap S' = \emptyset$ and $T \cap T' = \emptyset$ implies $\mu \cup \mu' \in \mathcal{M}(S \cup S', T \cup T', r_{S\cup S'})$, it easily follows that many-to-one assignment games are superadditive. On the other hand, if $\nu \in \mathcal{M}(S, T, r_S)$ is an optimal matching for coalition (S, T), that is $v_\gamma(S, T) = \sum\limits_{(i,j)\in\nu} a_{ij} = \sum\limits_{i\in S} \sum\limits_{j\in\nu(i)} a_{ij}$, then it follows from $v_\gamma(i, \nu(i)) = \sum\limits_{j\in\nu(i)} a_{ij}$ for all $i \in S$ that $v_\gamma(S, T) = \sum\limits_{i\in S} v_\gamma(i, \nu(i))$. Since $(S, T) = (\emptyset, \nu(f_0)) \cup \bigcup\limits_{i\in S} (i, \nu(i))$ where $\nu(f_0)$ denotes the unmatched workers in T under ν, and $v_\gamma(\nu(f_0)) = 0 = \sum\limits_{j\in\nu(f_0)} v_\gamma(j)$, we get the following observations.

Proposition 1. *In many-to-one assignment games, the following types of coalitions are inessential:*

- *any coalition containing at least two firms,*
- *any single-firm coalition containing more workers than the capacity of the firm,*
- *any one-sided coalition containing at least two players.*

Consequently, in an $(m+n)$-player many-to-one assignment game, among the $2^{m+n} - 1$ non-empty coalitions, at most $\sum_{i=1}^{m} \sum_{t=1}^{r_i} \binom{n}{t} \leq 2^n - 2$ coalitions can be essential. However, this exponential upper bound is sharp (if all a_{ij} pairwise income values are positive, $m = 2$, and $n = r_1 + r_2$).

As in any coalitional game, the main concern is how to share the grand coalition (total income) among all agents. To do so, we focus on the solution concept that coincides with the stable allocations, the *core*. Different than two-sided assignment games, instability may arise by a group of workers and a firm, since they can be better off by recontracting instead of their prescribed agreements.

4 Core, kernel and bargaining set

In the setting of two-sided assignment games where each agent has a unit capacity, Shapley and Shubik (1971) show that the core is always non-empty. Furthermore, it coincides with the set of dual solutions to the assignment problem and it has a

[3] When no confusion arises, for a given market γ, we denote its corresponding coalitional function by v instead of v_γ.

complete lattice structure. Hence, there exists a unique firm-optimal (worker-optimal) core allocation such that the profit (salary) of every firm (worker) is at least as good as under any other core allocation. Moreover, there is an opposition of interest between the two sides of the market when comparing two core allocations: all agents in one side agree on which of the two they prefer. As a consequence, the best allocation for one side is the worst for the other side of the market. Demange (1982) and Leonard (1983) prove that in the firm-optimal core allocation each firm attains its marginal payoff and in the worker-optimal core allocation each worker attains her marginal payoff.

It is well-known (Sánchez-Soriano, Lopez and Garcia-Jurado, 2001; Sotomayor, 2002) that some of the existing results for one-to-one assignment games cannot be extended to the many-to-one assignment games and hence to the transportation games: the core need not coincide with the set of dual solutions and the opposition of interest between the two sides does not hold in general. As for the lattice structure, it is only preserved in the many-to-one case based on the partial order on the set of workers' payoffs, as we will point out later on. This guarantees in that case the existence of an optimal core allocation for each side of the market.

Another relevant result for the one-to-one assignment game is the coincidence between the core and the bargaining set (Solymosi, 1999). The bargaining set is a set-solution concept for coalitional games based on a notion of objections and counterobjections (Davis and Maschler, 1967). The coincidence between the core and the bargaining set, when it holds, is a robustness property of the core since it guarantees that whenever a core allocation presents some objection, then this objection can be counterobjected.

To investigate if this coincidence also holds for many-to-one markets, we need to analyze closely the structure of the core of these games.

Proposition 2. *Given a many-to-one assignment market $\gamma = (F, W, A, r)$, let $\mu \in \mathcal{M}_A(F, W, r)$ be an optimal matching. Then, $(x, y) \in \mathbb{R}^F_+ \times \mathbb{R}^W_+$ is in the core $C(v_\gamma)$ of the associated game if and only if for every firm $i \in F$,*

(i) $x_i + \sum_{j\in\mu(i)} y_j = \sum_{j\in\mu(i)} a_{ij}$,

(ii) $x_i = 0$ *if* $\mu(i) = \emptyset$,

(iii) $x_i + \sum_{j\in T} y_j \geq \sum_{j\in T} a_{ij} = v_\gamma(i, T)$ *for all* $T \subseteq W$ *with* $|T| \leq r_i$,

(iv) $y_j = 0$ *if* $j \in \mu(f_0)$*, i.e.* $j \in W$ *is unmatched under* μ.

Proof. First, we prove the "*if*" part. Let $(x, y) \in \mathbb{R}^F \times \mathbb{R}^W$ be an allocation that satisfies conditions *(i)*, *(ii)*, *(iii)*, and *(iv)*. In order to prove that it is a core allocation, we need to show that it satisfies both efficiency and coalitional rationality. Notice first that *(i)*, *(ii)*, and *(iv)* together imply efficiency, $\sum_{i\in F} x_i + \sum_{j\in W} y_j = \sum_{(i,j)\in\mu} a_{ij}$. It remains to prove $\sum_{i\in R\cap F} x_i + \sum_{j\in R\cap W} y_j \geq v_\gamma(R)$ for all $R \subset N$. Take $S \subseteq F$ and $T \subseteq W$, and $\mu_{(S,T)}$ be an optimal matching for the submarket $(S, T, A_{|S,T}, r_{|s})$. Then,

$$\begin{aligned}
\sum_{i\in S} x_i + \sum_{j\in T} y_j &= \sum_{\substack{i\in S\\ i \text{ matched by } \mu_{(S,T)}}} \left(x_i + \sum_{j\in\mu_{(S,T)}(i)} y_j \right) + \sum_{\substack{i\in S\\ i \text{ unmatched by } \mu_{(S,T)}}} x_i + \sum_{\substack{j\in T\\ j \text{ unmatched by } \mu_{(S,T)}}} y_j \\
&\geq \sum_{\substack{i\in S\\ i \text{ matched by } \mu_{(S,T)}}} v_\gamma\Big(\{i\} \cup \mu_{(S,T)}(i)\Big) + 0 + 0 \\
&= v_\gamma(S,T),
\end{aligned}$$

where the inequality follows from the assumption that all conditions hold. Together with efficiency, we get that (x, y) is a core allocation, $(x, y) \in C(v_\gamma)$.

Next, we prove the "*only if*" part. Let $(x, y) \in C(v_\gamma)$ be a core allocation and the matching μ be an optimal matching. Then, from efficiency, $v_\gamma(F, W) = \sum\limits_{(i,j)\in\mu} a_{ij} = \sum\limits_{i\in F} x_i + \sum\limits_{j\in W} y_j$. Then,

$$\begin{aligned}
\sum_{i\in F} x_i + \sum_{j\in W} y_j &= \sum_{\substack{i\in F\\ i \text{ matched by } \mu}} \left(x_i + \sum_{j\in\mu(i)} y_j \right) + \sum_{i \text{ unmatched by } \mu} x_i + \sum_{j \text{ unmatched by } \mu} y_j \\
&\geq \sum_{\substack{i\in F\\ i \text{ matched by } \mu}} \left(\sum_{j\in\mu(i)} a_{ij} \right) + 0 + 0 \\
&= \sum_{(i,j)\in\mu} a_{ij},
\end{aligned}$$

where the inequality follows from the assumption $(x, y) \in C(v_\gamma)$. Together with $v_\gamma(F, W) = \sum\limits_{(i,j)\in\mu} a_{ij} = \sum\limits_{i\in F} x_i + \sum\limits_{j\in W} y_j$, it implies that $x_i = 0$ if i is unmatched and hence *(ii)* holds, $y_j = 0$ if j is unmatched and hence *(iv)* holds, and $x_i + \sum\limits_{j\in\mu(i)} y_j = \sum\limits_{j\in\mu(i)} a_{ij}$ for all $i \in F$ matched by μ and hence *(i)* holds. Moreover, for all $i \in F$, $x_i + \sum\limits_{j\in T} y_j \geq v_\gamma(T, \{i\})$ for all $T \subseteq W$ with $|T| \leq r_i$ where the inequality follows from the definition of the core, and hence *(iii)* holds. □

The above description of the core of a many-to-one assignment game is based on Proposition 1 and the general equivalence of the core and the essential-core. As we remarked there, this description is still of exponential size. Next, we explore further reduction possibilities, and give a polynomial-size equivalent description of the core. It will contain only the payoffs of the workers, and will allow us to study the extreme core allocations.

For brevity of exposition, first we balance the model, if needed. In case the total capacity of the firms $\sum_{i\in F} r_i$ exceeds the number of workers n, we introduce $n - \sum_{i\in F} r_i > 0$ fictitious workers who can only generate zero income with any firm. Exclusively from this situation, in case of $n < \sum_{i\in F} r_i$, we introduce a fictitious firm, say f_0, requiring at most $r_0 = \sum_{i\in F} r_i - n > 0$ number of workers, but who can only generate zero income with any worker. Technically, if needed, we extend matrix A with $n - \sum_{i\in F} r_i > 0$ full 0 columns, or with one full 0 row. This clearly means that we extend the associated many-to-one assignment game v with one or more null players. Since the core payoff to any null player j is $x_j = 0$, the core of the original game is precisely the $x_j = 0$ section of the core of the extended game, we can assume without loss of generality that the market $\gamma = (F, W, A, r)$ is balanced ($n = \sum_{i\in F} r_i$). To keep the exposition simple, we do not introduce any new notation for the possible extended models.

Given a balanced market $\gamma = (F, W, A, r)$ ($n = \sum_{i\in F} r_i$), we will always assume, due to the non-negativity of matrix A without loss of generality, that in any optimal matching $\mu \in \mathcal{M}_A(F, W, r)$ any firm $i \in F$ is assigned precisely r_i workers ($|\mu(i)| = r_i$), and no worker is unmatched under μ.

The next description of the core in terms of the workers' payoffs follows easily from the Proposition 2 and is a simplification of the one already given in Sotomayor (2002) for the general, not necessarily balanced, market.

Proposition 3. *Given a balanced many-to-one assignment market $\gamma = (F, W, A, r)$, let $\mu \in \mathcal{M}_A(F, W, r)$ be an optimal matching. Then, $(x, y) \in \mathbb{R}^F_+ \times \mathbb{R}^W_+$ is in the core of the associated game $C(v_\gamma)$ if and only if*

(i) $y_j \leq a_{ij}$ *for any* $i \in F$ *and* $j \in \mu(i)$;

(ii) $y_j - y_k \leq a_{ij} - a_{ik}$ *for any* $i \in F$ *and* $j \in \mu(i)$, $k \notin \mu(i)$;

(iii) $x_i = \sum_{j\in\mu(i)} (a_{ij} - y_j)$ *for all* $i \in F$.

Notice that the number of constraints is $n = \sum_{i\in F} r_i$ in item (i), $\sum_{i\in F} r_i(n - r_i) \leq \frac{m\cdot n^2}{4}$ in item (ii), and m in item (iii), altogether at most $m \cdot n^2$.

Also, given any vector of wages $y \in \mathbb{R}^W$ that satisfies constraints (i) and (ii) above for some optimal matching μ, then the payoff to each firm is uniquely determined. Let us denote by $C(W)$ the set of wages that satisfy (i) and (ii), that is, the projection of the core to the space of workers' payoffs. It is proved in Sotomayor (2002) that $C(W)$ is endowed with a lattice structure under the partial order induced by $\mathbb{R}^W$. In fact, the reader will see that constraints (i) and (ii) in Proposition 3 define what is named a 45-degree polytope in Quint (1991).

The main consequence of this lattice structure of $C(W)$ is the existence of a firm-optimal core allocation and a worker-optimal core allocation for any many-to-one assignment game, which is not known to be true for the general many-to-many assignment games.

Here is an illustrative example of the above core description.

Example 4. Consider a many-to-one assignment market $\gamma = (F, W, A, r)$ where $F = \{f_1, f_2\}$, $W = \{w_1, w_2, w_3\}$ are the set of firms and the set of workers respectively, and the capacities of the firms are $r = (2, 1)$. The pairwise valuation matrix is the following:

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 8 & 6 & 3 \\ 7 & 6 & 4 \end{pmatrix} \end{array}.$$

Since the (unique) optimal matching assigns workers w_1 and w_2 to firm f_1, and w_3 to firm f_2, in any core allocation $x_1 = (8+6) - y_1 - y_2$ and $x_2 = 4 - y_3$ hold. Henceforth, in terms of the workers payoffs, the core is described by the following system (given in two equivalent forms):

$$
\begin{array}{rrrcl}
y_1 & y_2 & y_3 & \geq & 0 \\
\hline
y_1 & & & \leq & 8 \\
& y_2 & & \leq & 6 \\
& & y_3 & \leq & 4 \\
\hline
y_1 & & -y_3 & \leq & 5 = 8 - 3 \\
& y_2 & -y_3 & \leq & 3 = 6 - 3 \\
\hline
-y_1 & & y_3 & \leq & -3 = 4 - 7 \\
& -y_2 & y_3 & \leq & -2 = 4 - 6 \\
\hline
\end{array}
\qquad
\begin{array}{rcrrrcl}
\hline
0 & \leq & y_1 & & & \leq & 8 \\
0 & \leq & & y_2 & & \leq & 6 \\
0 & \leq & & & y_3 & \leq & 4 \\
\hline
3 & \leq & y_1 & & -y_3 & \leq & 5 \\
2 & \leq & & y_2 & -y_3 & \leq & 3 \\
\hline
\end{array}
$$

Notice the similarities to the one-to-one assignment case, but due to the capacity $r_1 = 2$ of firm f_1, there is no direct relation between the payoffs to its optimally matched workers, two-way direct pairwise comparisions are only between workers assigned to different firms.

Next figure illustrates the $C(W)$ of this example, where the 45-degree lattice structure can be seen.

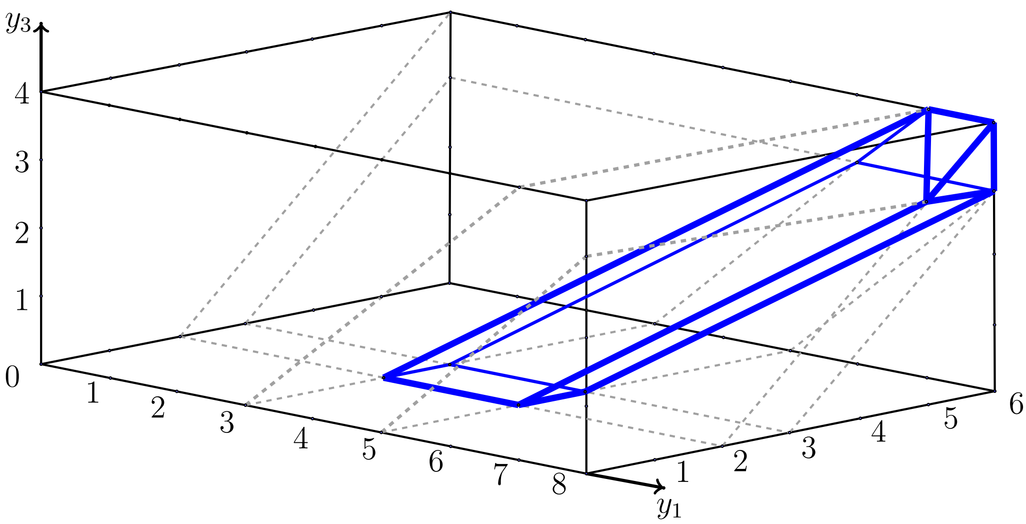


Figure 1: The core of the many-to-one market of Example 4

Next, we focus on the relationship between the core and other set-wise solution concepts. The *kernel*, introduced by Davis and Maschler (1965), is a non-empty subset of the *classical bargaining set* of Davis and Maschler (1967) for all games with a non-empty set of efficient payoffs.[4]

[4]The kernel is a set-wise solution concept. Whenever the core is non-empty, the intersection between

For one-to-one assignment games, Driessen (1998) showed that the kernel is a subset of the core and Solymosi (1999) proved the coincidence of the classical bargaining set and the core. We now investigate if these results are preserved in the many-to-one assignment game.

To this end, let us recall the definition of the kernel of a coalitional game. Since our game is zero-monotonic, the kernel is defined as follows:

$$\mathcal{K}(v) = \{z \in I(v) \mid s_{ij}(z) = s_{ji}(z), \text{ for all } i, j \in F \cup W\},$$

where $s_{ij}(z) = \max_{i \in S, j \notin S}\{v(S) - z(S)\}$, that is, the maximum excess at imputation z of coalitions containing i and not containing j. If an imputation is in the kernel, it is somehow balanced in the sense that the maximum excess that agent i can attain with no cooperation of j equals the maximum excess that j can attain without the cooperation of i.

Next example shows that the inclusion of the kernel in the core, and also the coincidence of the core with the bargaining set, do not carry over to the many-to-one assignment game and hence to the transportation game.

Example 5. Consider a many-to-one assignment market $\gamma = (F, W, A, r)$ where $F = \{f_1, f_2\}$ is the set of firms, $W = \{w_1, w_2, w_3\}$ is the set of workers, and the capacities of the firms are $r = (2, 2)$. The per-unit pairwise valuation matrix is the following:

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}.$$

The core contains a unique point, $C(v_\gamma) = \{(0, 0, 1, 1, 1)\}$. One can easily check that $(x, y) = (1, 1; 1/3, 1/3, 1/3)$ is not a core allocation, but it lies in the (pre)kernel since for any pair of agents the maximum surplus is the same. Let us show that it is indeed the case. The corresponding many-to-one assignment game $(F \cup W, v_\gamma)$ and excesses at the allocation $(x, y) = (1, 1; 1/3, 1/3, 1/3)$, $e(S, (x, y))$, are:

the kernel and the core is non-empty. Classical bargaining set is a set-valued solution for cooperative games. It is non-empty for all games with a non-empty set of imputations. Whenever the core is non-empty, the bargaining set contains the core.

$v_\gamma(S)$	x_1	x_2	y_1	y_2	y_3	$e(S,(x,y))$
0	1	.	.	.	.	-1
0	.	1	.	.	.	-1
0	.	.	1	.	.	$-1/3$
0	.	.	.	1	.	$-1/3$
0	.	.	.	.	1	$-1/3$
0	1	1	.	.	.	-2
1	1	.	1	.	.	$-1/3$
1	1	.	.	1	.	$-1/3$
1	1	.	.	.	1	$-1/3$
1	.	1	1	.	.	$-1/3$
1	.	1	.	1	.	$-1/3$
1	.	1	.	.	1	$-1/3$
0	.	.	1	1	.	$-2/3$
0	.	.	1	.	1	$-2/3$
0	.	.	.	1	1	$-2/3$

$v(S)$	x_1	x_2	x_3	x_4	x_5	$e(S,(x,y))$
1	1	1	1	.	.	$-4/3$
1	1	1	.	1	.	$-4/3$
1	1	1	.	.	1	$-4/3$
2	1	.	1	1	.	$1/3$
2	1	.	1	.	1	$1/3$
2	1	.	.	1	1	$1/3$
2	.	1	1	1	.	$1/3$
2	.	1	1	.	1	$1/3$
2	.	1	.	1	1	$1/3$
0	.	.	1	1	1	-1
2	1	1	1	1	.	$-2/3$
2	1	1	1	.	1	$-2/3$
2	1	1	.	1	1	$-2/3$
2	1	.	1	1	1	0
2	.	1	1	1	1	0
3	1	1	1	1	1	0

Next, we calculate the maximum excess at the point $(x,y) = (1,1;1/3,1/3,1/3)$ for all possible pair of agents. Since firms (workers) have the same payoff, $x_1 = x_2 = 1$ ($y_1 = y_2 = y_3 = 1/3$), it is sufficient to check the maximum excess for the following pairs:

- If $i = f_1$ and $j = f_2$, then $\max\limits_{\substack{f_1\in S\\ f_2\notin S}} e(S,(x,y)) = 1/3 = \max\limits_{\substack{f_2\in S\\ f_1\notin S}} e(S,(x,y))$.
- If $i = w_1$ and $j = w_2$, then $\max\limits_{\substack{w_1\in S\\ w_2\notin S}} e(S,(x,y)) = 1/3 = \max\limits_{\substack{w_2\in S\\ w_1\notin S}} e(S,(x,y))$.
- If $i = f_1$ and $j = w_1$, then $\max\limits_{\substack{f_1\in S\\ w_1\notin S}} e(S,(x,y)) = 1/3 = \max\limits_{\substack{w_1\in S\\ f_1\notin S}} e(S,(x,y))$.

Then, for any pair of agents the maximum excess over coalitions containing one of them but not the other is equal at the point $(x,y) = (1,1;1/3,1/3,1/3)$. Hence, the point $(x,y) = (1,1;1/3,1/3,1/3)$ lies in the (pre)kernel. Since it is not a core allocation, it implies that the kernel is not a subset of the core.

Since the kernel is a subset of the classical bargaining set, the point $(1,1;1/3,1/3,1/3)$ that is in the kernel is also in the bargaining set. This implies that the core and the classical bargaining set does not coincide.

Corollary 6. In the many-to-one assignment game,

(i) The kernel needs not be a subset of the core.

(ii) The classical bargaining set needs not coincide with the core.

Corollary 6 implies that the coincidence between the classical bargaining set and the core cannot be carried over from the one-to-one case to the many-to-one and to the many-to-many case. Remarkably, our result is the first showing that the coincidence between the core and the classical bargaining set is not satisfied for a class of combinatorial optimization games. Solymosi, Raghavan and Tijs (2003) showed the coincidence result for permutation games. Solymosi (2008) proved (among other variants) that the classical bargaining set coincides with the core for the classes including one-to-one assignment games, tree-restricted superadditive games, and simple network games. Bahel (2016) obtained the same result for veto games. Recently, Bahel (2021) obtained the coincidence result for the so-called quasi-hyperadditive games which contain one-to-one assignment games. Atay and Solymosi (2018) extended the coincidence result from one-to-one assignment games to a class of multi-sided matching games known as the supplier-firm-buyer games.

Although we have learned that the kernel may have imputations outside the core, we know that, whenever the core is non-empty, like in the case of the many-to-one assignment games, the kernel always contains some core elements. The reason is that, for games with non-empty core, it is well-known that the nucleolus is always in the intersection of the kernel and the core.

As in the one-to-one assignment game (Granot and Granot, 1992) some simplifications can be done to obtain those core allocations that also belong to the kernel. First, only essential coalitions are to be taken into account, and secondly, not all pairs of agents need to be considered.

We have defined a matching as a set of firm-worker pairs that do not violate the capacities of firms and workers, but we can also understand it as a partition of $F \cup W$ in essential coalitions. If $(i, j_1), (i, j_2), \ldots, (i, j_k)$ are pairs in a matching μ, then the coalition $T = \{i, j_1, j_2, \ldots, j_k\}$ is one element of the partition of $F \cup W$ induced by μ. This fact will simplify notations in the next result.

Proposition 7. *Let $\gamma = (F, W, A, r)$ be a many-to-one assignment game. Then,*

$$\mathcal{K}(v_\gamma) \cap C(v_\gamma) = \{z \in C(v_\gamma) \mid s_{ij}(z) = s_{ji}(z) \text{ for all } \{i, j\} \in T \subseteq \Phi(A)\},$$

where $\Phi(A)$ is the set of essential coalitions that belong to all optimal matching.

Proof. Let $z \in C(v_\gamma)$, S an arbitrary coalition of $F \cup W$ and $\mu_S = \{T_1, T_2, \ldots, T_r\}$ an optimal matching for coalition S. Then,

$$e(S, z) = v_\gamma(S) - z(S) = \sum_{T \in \mu} v_\gamma(T) - z(T) \leq v_\gamma(T_k) - z(T_k), \text{ for all } T_k \in \mu_S,$$

where the inequality follows from the fact that excesses at a core allocation are always non-positive. Then, the maximum excess at z over coalitions containing agent i and not containing agent j is always attained at an essential coalition. This implies that only essential coalitions are to be considered to find those core elements that belong to the kernel.

Moreover, if we take two firms $i_1, i_2 \in F$, then

$$s_{i_1 i_2}(z) = e(S, z) = 0 = e(R, z) = s_{i_2 i_1}(z),$$

where $S = \{i_1\} \cup \mu(i_1)$ and $T = \{i_2\} \cup \mu(i_2)$, for any optimal matching μ.

Similarly, if we take two workers that are not assigned to the same firm in an optimal matching μ, that is, $(i_1, j_1) \in \mu$ and $(i_2, j_2) \in \mu$, then

$$s_{j_1 j_2}(z) = e(S, z) = 0 = e(R, z) = s_{j_2 j_1}(z),$$

where $S = \{i_1\} \cup \mu(i_1)$ and $T = \{i_2\} \cup \mu(i_2)$.

Finally, if we take a firm i_1 and a worker j_2 that are not matched in some optimal matching μ, that is, there is $\mu \in \mathcal{M}_A(F, W, r)$ and $i_2 \in F \setminus \{i_1\}$ such that $(i_2, j_2) \in \mu$, then also $s_{i_1 j_2}(z) = e(S, z) = 0 = e(R, z) = s_{j_2 i_1}(z)$, where $S = \{i_1\} \cup \mu(i_1)$ and $T = \{i_2\} \cup \mu(i_2)$.

To sum up, only firm-worker pairs that are not matched in all optimal matchings and pairs of workers that are not matched to the same firm in all optimal matchings are to be considered. $\square$

As a consequence, if a market (F, W, A, r) is such that no agent is matched to the same partner in all optimal matchings, then all core elements are in the kernel. This is precisely the case of Example 5.

To complete this section, we include some remarks on some single-valued solutions: the fair-division point, the nucleolus, and the τ-value. [5]

Regarding the computation of the nucleolus of the many-to-one assignment game, we simply point out that only the essential coalitions need to be considered in the lexicographic minimization procedure. This is a notable simplification but still of exponential size.

Secondly, because of the lattice structure of $C(W)$ and the existence of an optimal core allocation for each side, which is the worst core allocation for the opposite side, the fair division point can be defined, as in the one-to-one assignment game, as the midpoint of these two core allocations. The convexity of the core guarantees this midpoint is also a core allocation.

For the one-to-one assignment game, the fair division point coincides with the τ-value, which is the only efficient point in the segment between the utopia point, where each agent obtains his/her marginal contribution, and the minimum rights point (see Tijs (1981) for the formal definition).

Consider the many-to-one assignment market introduced in Example 5. One can easily calculate that the fair division point is $(4.5, 2; 5.5, 4, 2)$, the nucleolus is $(4, 2.25; 5.75, 4.25, 1.75)$ [6], and the τ-value is $(143/28, 52/28; 149/28, 108/28, 52/28)$. The τ-value does not coincide with the fair division point and moreover it does not lie in the core since, for instance, the core constraint for the coalition $\{f_2, w_3\}$ does not hold as $52/28 + 52/28 =$

[5] The *nucleolus* of a coalitional game (N, v) is the payoff vector $\eta(v) \in \mathbb{R}^N$ that lexicographically minimizes the vector of decreasingly ordered excesses of coalitions among all possible efficient payoff vectors (Schmeidler, 1969). The *fair-division point* of a one-to-one assignment market is the midpoint of the buyer-optimal and the seller-optimal core allocations (Thompson, 1981). The τ-value is a single-valued solution for coalitional games introduced in Tijs (1981). It is known that for one-to-one assignment games the τ-value and the fair-division point coincide (Núñez and Rafels, 2002).

[6] The nucleolus can be calculated efficiently by means of the algorithm introduced by Benedek, Fliege and Nguyen (2021).

$104/28 < 4 = v_\gamma(\{f_2, w_3\})$. Thus, we observe that for the many-to-one assignment game, and hence for the transportation game, the τ-value need not be a core allocation. Finally, we remark that the Shapley value is typically not a core allocation, not even in one-to-one assignment games. It is always the case, for example, in a glove game with different number of players on the two sides.

5 The extreme core allocations

In this section we study the set of extreme core allocations of the many-to-one assignment markets. A natural first approach is to consider the relationship between the extreme core allocations and lexicographic allocation procedures. This approach has been applied by Hamers et al. (2002) and Núñez and Solymosi (2017) to show that each extreme core allocation of a one-to-one assignment game is a marginal payoff vector and the coincidence between the extreme core allocations and the set of lemarals, respectively. Next, we provide a counter-example to show that, unlike the one-to-one assignment game, an agent may not achieve her marginal payoff at any core allocation. Furthermore, many-to-one assignment games (and hence transportation games) are neither INTO-lemaral nor ONTO-lemaral.

Example 8. Consider a many-to-one assignment market $\gamma = (F, W, A, r)$ where $F = \{f_1, f_2\}$, $W = \{w_1, w_2\}$ are the set of firms and the set of workers respectively, and the capacities of the firms are $r = (2, 1)$. The per-unit pairwise valuations are given in the following matrix:

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{cc} w_1 & w_2 \end{array} \\ \left(\begin{array}{cc} 4 & 3 \\ 3 & 2 \end{array} \right) \end{array}.$$

The corresponding many-to-one assignment game (N, v_γ) and its dual game are:

v	x_1	x_2	y_1	y_2	v^*
0	1	.	.	.	4
0	.	1	.	.	0
0	.	.	1	.	4
0	.	.	.	1	3
0	1	1	.	.	7
4	1	.	1	.	5
3	1	.	.	1	4
3	.	1	1	.	4
2	.	1	.	1	3
0	.	.	1	1	7
4	1	1	1	.	7
3	1	1	.	1	7
7	1	.	1	1	7
3	.	1	1	1	7
7	1	1	1	1	7

The marginal payoff of firm f_1 is $4 = v^*(f_1) = v(N) - v(N \setminus f_1)$ but it is not achievable in the core. The core-maximum for firm f_1 is $\max_C x_1 = 2 = v^*(\{f_2\}) + v^*(\{f_1, w_1\}) + v^*(\{f_1, w_2\}) - v^*(N)$. This shows that the marginal payoff of a player to the grand coalition may not be the core maximum payoff of the corresponding player for the many-to-one assignment game, and neither for the transportation game.

Now, take any order that starts with the firm f_1, $\sigma = (f_1, arbitrary)$. For that given order, the payoff of f_1 is 4 which cannot be attained at a core allocation. Hence, a lemaral obtained by an order $\sigma = (f_1, arbitrary)$ cannot be a core allocation. For instance, take the order $\sigma = (f_1, w_1, f_2, w_2)$. Then, the corresponding lemaral is $(4, 0; 1, 2)$ and it is not a core allocation. Hence, $(F \cup W, v_\gamma)$ is not INTO-lemaral.

Next, take the extreme core allocation $(2, 0; 3, 2)$. Notice that $\min_C y_2 = 2 = v(\{f_2, w_2\}) + v(\{f_1, w_1, w_2\}) - v(N)$ and both f_1 and f_2 obtain their core maximum allocations, and hence $(2, 0; 3, 2)$ is an extreme core allocation. We will try to construct a lemaral vector $(x, y) \in \mathbb{R}^N$ that coincide with the aforementioned extreme core allocation. First notice that f_2 is the only player that achieve her marginal payoff in the core. Hence, we only take into account orders that start with player f_2:

- Player 2 achieves her marginal payoff under an order $\sigma = (f_2, arbitrary)$: $x_2 = 0$,
- $\sigma = (f_2, f_1, \ldots)$: Then,
$$x_1 = \min\{v^*(f_1), v^*(\{f_1, f_2\}) - x_2\} = \min\{4, 7 - 0\} = 4 \neq 2 = x_1,$$
- $\sigma = (f_2, w_1, \ldots)$: Then,
$$y_1 = \min\{4, 4 - 0\} = 4 \neq 3 = y_1,$$
- $\sigma = (f_2, w_2, \ldots)$: Then,
$$y_2 = \min\{3, 3 - 0\} = 3 \neq 2 = y_2.$$

As a a consequence, there does not exist an order to construct a lemaral vector that coincides with the extreme core allocation $(2, 0; 3, 2)$. Hence, (N, v_γ) is not ONTO-lemaral.

Corollary 9. Many-to-one assignment games are neither INTO-lemaral nor ONTO-lemaral.

The above corollary shows that the relationship between the lemaral vectors and the extreme core allocations does not carry over from the one-to-one assignment game to the many-to-one assignment game, and hence to the transportation game. Hence, in order to characterize the set of extreme core allocations for many-to-one assignment markets, we will take a different approach. First, we introduce a directed tight graph following the description of the core introduced in Proposition 3.

Definition 10. Let $\gamma = (F, W, A, r)$ be a balanced many-to-one assignment market, $\mu \in \mathcal{M}(F, W, r)$ be an optimal matching, and $(F \cup W, v_\gamma)$ be its associated many-to-one assignment game. For each core allocation $(x, y) \in C(v_\gamma)$, we define the directed tight graph E^y with the set of nodes W and such that there exists a directed edge from the worker j to the worker k, $(j, k) \in E^y$, if $y_j - y_k = a_{ij} - a_{ik}$ where $j \in \mu(i)$ and $k \notin \mu(i)$.

This tight digraph is inspired by the one introduced in Balinski and Gale (1990) and also used in Hamers et al. (2002) to study extreme core points of the one-to-one assignment game. There, the nodes of the graph consist of the agents on both sides of the market, not just from one side as we do for the many-to-one case. And also, their graph is not directed since it is based on the constraints $x_i + y_j \geq a_{ij}$ where both variables have the same sign. They find that the extreme core points are those core points with a tight graph that has an agent with zero payoff in each component.

We now establish a relation between the extreme core allocations of the many-to-one assignment game and the components of the corresponding tight digraph.

Theorem 11. *Let $\gamma = (F, W, A, r)$ be a balanced many-to-one assignment market and let (x, y) be a core allocation, $(x, y) \in C(v_\gamma)$. Then, (x, y) is an extreme core allocation if and only if each component of the tight digraph E^y contains either a worker $j \in W$ with the zero payoff $y_j = 0$ or a worker $j \in W$ such that $y_j = a_{\mu^{-1}(j)(j)}$.*

Proof. First, we prove the "*only if*" part. Suppose on the contrary that $0 < y_j < a_{ij}$ for all $j \in W$ where $(i, j) \in \mu$. Now, define $\varepsilon' > 0$ small enough for all $j \in W$ such that $0 \leq y'_j = y_j + \varepsilon' \leq a_{ij}$ and $x'_i = \sum_{j\in\mu(i)} a_{ij} - \sum_{j\in\mu(i)} y'_j \geq 0$. Then, $(x', y') \in C(v_\gamma)$. Define also $\varepsilon'' > 0$ small enough for all $j \in W$ such that $0 \leq y''_j = y_j - \varepsilon'' \leq a_{ij}$ where $(i, j) \in \mu$ and $x''_i = \sum_{j\in\mu(i)} a_{ij} - \sum_{j\in\mu(i)} y''_j$. Then, $(x'', y'') \in C(v_\gamma)$. It is then straightforward to check that $(x, y) = \frac{1}{2}(x', y') + \frac{1}{2}(x'', y'')$, which contradicts the assumption that (x, y) is an extreme core allocation.

To see the converse implication, in each component of E^y take an arborescence of the digraph.[7] We can assume without loss of generality that E^y consists of only one component. Let $j_0 \in W$ be the worker in this component such that $v_{j_0} = 0$ or $v_{j_0} = a_{ij}$ for $(i, j) \in \mu$. Notice that this linear equality constraint together with the constraints $y_j - y_k = a_{ij} - a_{ik}$ for (j, k) in the arborescence of E^y form an independent system of as many linear equations as the cardinality of W. Since (x, y) is assumed to be in the core, the above remark implies it is an extreme core allocation. □

We have provided the first characterization for extreme core allocations of the many-to-one assignment game. Next, we study the two outstanding core allocations: the firm-optimal core allocation and the worker-optimal core allocation. Making use of the components of the tight digraph we have defined, we provide a necessary condition for each side-optimal core allocation.

Proposition 12. *Let $\gamma = (F, W, A, r)$ be a balanced many-to-one assignment market in which μ is an optimal matching, and let (x, y) be a core allocation, $(x, y) \in C(v_\gamma)$. If (x, y) is the firm-optimal core allocation, then each component of the tight digraph E^y contains a worker $j \in W$ with zero payoff $y_j = 0$.*

Proof. Take (x, y) to be the firm-optimal core allocation. Assume without loss of generality that E^y has only one component. If $y_j > 0$ for all $j \in W$, then we can define the payoff vector (x', y') by $y'_j = y_j - \varepsilon$ for all $j \in W$ with ε small enough such that $y'_j \geq 0$

[7] An arborescence in a digraph $D = (V, A)$ is a set B of arcs such that (V, B) is a rooted tree. See page 34 of Schrijver (2003) for formal definition.

for all $j \in W$. Notice that $y'_j < y_j \leq a_{ij}$ where $(i,j) \in \mu$, and $y'_j - y'_k = y_j - y_k \leq a_{ij} - a_{ik}$ for all $j, k \in W$ such that $(i,j) \in \mu$ and $(i,k) \notin \mu$. Hence $y' \in C(W)$ and if we define $x'_i = \sum_{j \in \mu(i)} (a_{ij} - y_j)$ we get that $(x', y') \in C(v_\gamma)$ and $y'_j < y_j$ for all $j \in W$ which contradicts that (x, y) is the firm-optimal core allocation. $\square$

Proposition 13. *Let $\gamma = (F, W, A, r)$ be a balanced many-to-one assignment market in which μ is an optimal matching, and let (x, y) be a core allocation, $(x, y) \in C(v_\gamma)$. If (x, y) is the worker-optimal core allocation, then each component of the tight digraph E^y contains a worker $j \in W$ whose payoff is $y_j = a_{\mu^{-1}(j)(j)}$.*

Proof. Take (x, y) to be the worker-optimal core allocation. Assume without loss of generality that E^y has only one component. If $y_j < a_{ij}$ for all $j \in W$, and $(i,j) \in \mu$, then we can define the payoff vector (x', y') by $y'_j = y_j + \varepsilon$ for all $j \in W$ with ε small enough such that $y'_j \leq a_{ij}$ for all $j \in W$ and $(i,j) \in \mu$. Notice that $y'_j > y_j \geq 0$ and $y'_j - y'_k = y_j - y_k \leq a_{ij} - a_{ik}$ for all $j, k \in W$ such that $(i,j) \in \mu$ and $(i,k) \notin \mu$. Hence $y' \in C(W)$ and if we define $x'_i = \sum_{j \in \mu(i)} (a_{ij} - y_j)$ we get that $(x', y') \in C(v_\gamma)$ and $y'_j > y_j$ for all $j \in W$ which contradicts that (x, y) is the worker-optimal core allocation. $\square$

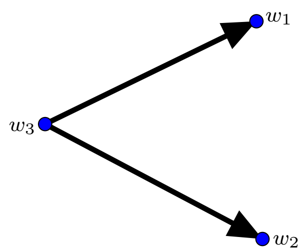
Next example shows the above necessary conditions for an extreme core allocation to be one of the side-optimal allocations are not sufficient.

Example 14. Let us recall the market situation of Example 4 with the pairwise valuation matrix

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 8 & 6 & 3 \\ 7 & 6 & 4 \end{pmatrix} \end{array},$$

the capacities of the firms $r = (2, 1)$, and under the unique optimal matching μ, workers w_1 and w_2 are matched with firm f_1, and w_3 with f_2. The firm-optimal core allocation is $(9, 4; 3, 2, 0)$ and the worker-optimal core allocation is $(0, 0; 8, 6, 4)$.

First, take the extreme core allocation $(x, y) = (1, 0; 7, 6, 4)$. Now, we construct the tight digraph on E^y. For the firm f_1, $y_3 - y_1 = -3 \neq -5 = a_{13} - a_{11}$ and $y_3 - y_2 = -2 \neq -3 = a_{13} - a_{12}$ and hence there is no directed edge. For the firm f_2, $y_1 - y_3 = 3 = a_{21} - a_{23}$ and $y_2 - y_3 = 2 = a_{22} - a_{23}$, and hence there are directed arcs from worker w_3 to w_1 and w_2:



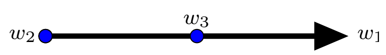


Figure 2: Digraphs E^y for extreme points $(1, 0; 7, 6, 4)$ and $(8, 4; 3, 3, 0)$.

Although $y_3 = 4 = a_{23}$ and $(f_2, w_3) \in \mu$, $(1, 0; 7, 6, 4)$ is not the worker-optimal allocation, and hence containing a worker $j \in W$ whose payoff is $y_j = a_{\mu^{-1}(j)(j)}$ in each component of the tight digraph E^y is not a sufficient condition for an extreme core allocation to be the worker-optimal allocation.

Next, take the extreme core allocation $(x, y) = (8, 4; 3, 3, 0)$. Now, we construct the digraph on E^y. For the firm f_1, $y_3 - y_1 = -3 \neq -5 = a_{13} - a_{11}$ and $y_3 - y_2 = -3 = a_{13} - a_{12}$, and hence there is a directed edge from worker w_2 to w_3. For the firm f_2, $y_1 - y_3 = 3 = a_{21} - a_{23}$ and $y_2 - y_3 = 3 \neq 2 = a_{22} - a_{23}$, and hence there is a directed edge from worker w_3 to w_1.

Although w_3 is a node of E^y obtained from the extreme point $(x, y) = (8, 4; 3, 3, 0)$ and its allotment at this extreme point is $y_3 = 0$, $(x, y) = (8, 4; 3, 3, 0)$ is not the firm-optimal allocation. Thus, containing a worker $j \in W$ with zero payoff $y_j = 0$ in each component of the tight digraph E^y is not a sufficient condition for an extreme core allocation to be the firm-optimal core allocation.

A natural question arises after Propositions 12 and 13 above: for which many-to-one markets all workers attain a zero payoff in the core and for which markets all workers j attain $a_{\mu^{-1}(j)j}$ as a core payoff? To this end we generalize the known condition for one-to-one assignment games due to Solymosi and Raghavan (2001).

Definition 15. A balanced many-to-one assignment market (F, W, A, r) has a dominant diagonal if and only if there exists an optimal matching μ such that

1. $\sum_{j \in \mu(i)} a_{ij} \geq \sum_{j \in T} a_{ij}$ for all $T \subseteq W$ with $|T| \leq r_i$, and
2. $a_{\mu^{-1}(j)j} \geq a_{ij}$ for all $j \in W$ and $i \in F$.

The above property characterizes those many-to-one markets where each agent attains a zero payoff in the core.

Proposition 16. *Let (F, W, A, r) be a balanced many-to-one assignment market. Then, it has a dominant diagonal if and only if every agent attains a zero payoff in the core.*

Proof. Take an optimal matching μ. If there is a core allocation where all workers $j \in W$ get $y_j = 0$, then, each firm $i \in F$ gets $x_i = \sum_{j \in \mu(i)} a_{ij}$. Now, such a payoff vector belongs to the core if and only if the core constraints for essential coalitions are satisfied, which means that for all $i \in F$, $\sum_{j \in \mu(i)} a_{ij} + 0 \geq \sum_{j \in T} a_{ij}$ for all $T \subseteq W$ with $|T| \leq r_i$.

Similarly, when all firms get zero in a core allocation, then each worker gets $y_j = a_{\mu^{-1}(j)j}$. This allocation belongs to the core if and only if for all $i \in F$, $0 + \sum_{j \in T} a_{\mu^{-1}(j)j} \geq \sum_{j \in T} a_{ij}$, for all $T \subseteq W$ with $|T| \leq r_i$. This is equivalent to $a_{\mu^{-1}(j)j} \geq a_{ij}$ for all $j \in W$ and all $i \in F$. $\square$

Notice that the market of Example 4 satisfies condition (2) of Definition 15, but it does not satisfy condition (1), since for instance $\sum_{j \in \mu(2)} a_{ij} = a_{23} + a_{20} < a_{21} + a_{22}$. This is the reason why we observe in Figure 4 there is no core element where all firms get zero payoff.

It is important to remark that, since the above proposition shows that the dominant diagonal property is equivalent to a property of the core, which does not depend on the optimal matching we select for its representation, the diagonal dominant property is also independent of the optimal matching.

Let us finally point out, without entering into formal details, that, as in the one-to-one assignment game, the dominant diagonal property is a necessary condition for the stability of the core. Indeed, if the point (x, y), where $x_i = 0$ for all $i \in F$ and $y_j = a_{\mu^{-1}(j)j}$ for all $j \in W$, is not in the core, then it cannot be dominated by any core allocation, which implies the core is not a stable set. The same happens if the point (x, y), with $x_i = \sum_{j\in\mu(i)} a_{ij}$ for all $i \in F$ and $y_j = 0$ for all $j \in W$, does not belong to the core.

6 The max-min vectors

The characterization of the extreme core allocations of the many-to-one assignment game given in Theorem 11 will allow to describe a procedure to obtain all these extreme points. At the same time, we will see that the extreme core allocations of these games also correspond to a sequence of lexicographic optimization over the core, where some workers maximize their payoff while some other workers minimize it.

We have already pointed out that although each extreme core allocation of the one-to-one assignment game is the result of a lexicographic maximization over the set of dual rational allocations (Núñez and Solymosi, 2017), this is not the case in many-to-one assignment games, as shown in Example 8. Another approach to the extreme core points of the one-to-one assignment game is to consider, for each order on the set of agents, the lexicographic minimization procedure on the set of rational allocations. That is, let the payoff to the first player in the order be zero and, for each following agent, compute the minimum payoff that satisfies all core inequalities with his/her predecessors while preserving the payoffs that they have already been allocated. It is proved in Izquierdo, Núñez and Rafels (2007) that every extreme core point is of this form, although not all the vectors obtained in this way are extreme points of the core.

This inspires the following definition of the max-min vectors with two differences with respect the one-to-one assignment game: only workers are considered in the procedure and each order on the set of workers must be completed with an indication of whether the worker in this position maximizes or minimizes his/her payoff.

Let $\theta : \mathbb{N} \longrightarrow W$ be an order on the set of workers, where $\theta(i)$ is the worker in the ith-position, and we can also write $\theta = (j_1, j_2, \ldots, j_n)$. We denote by Σ_W the set of all orders on W. Given a worker $j \in W$, $P_j^\theta = \{k \in W \mid \theta^{-1}(k) < \theta^{-1}(j)\}$ is the set of predecessors of j according the order θ.

Then, an extension of the order θ is

$$\begin{array}{rcl} \tilde{\theta} : \mathbb{N} & \longrightarrow & W \times \{\min, \max\} \\ i & \mapsto & \tilde{\theta}(i) = \begin{cases} (\theta(i), \min) = \underline{\theta}(i) \\ \text{or} \\ (\theta(i), \max) = \overline{\theta}(i), \end{cases} \end{array}$$

where $\underline{\theta}(i)$ means that worker is in ith position and will minimize his/her payoff under some constraints. Similarly, $\overline{\theta}(i)$ means that the ith player in the order will maximize his/her payoff under some constraints. We denote by $\tilde{\Sigma}_W$ the set of all extended orders on W.

Definition 17. Let (F, W, A, r) be a many-to-one assignment game, μ an optimal matching, $\theta = (j_1, j_2, \ldots, j_n)$ an order on W and $\tilde{\theta}$ an extension of θ. The related *max-min payoff* $y^{\tilde{\theta}}$ satisfies

$$y^{\tilde{\theta}}_{j_1} = \begin{cases} 0 & \text{if} \quad \tilde{\theta}(1) = \underline{\theta}(1) \\ a_{\mu^{-1}(j_1)j_1} & \text{if} \quad \tilde{\theta}(1) = \overline{\theta}(1), \end{cases}$$

and for all $1 < r \leq n$,

$$y^{\tilde{\theta}}_{j_r} = \begin{cases} \max_{j \in P^{\theta}_{j_r}, \mu^{-1}(j) \neq \mu^{-1}(j_r)} \{y_j - a_{\mu^{-1}(j)j} + a_{\mu^{-1}(j)j_r}, 0\} & \text{if} \quad \tilde{\theta}(r) = \underline{\theta}(r) \\ \min_{j \in P^{\theta}_{j_r}, \mu^{-1}(j) \neq \mu^{-1}(j_r)} \{y_j - a_{\mu^{-1}(j_r)j} + a_{\mu^{-1}(j_r)j_r}, a_{\mu^{-1}(j_r)j_r}\} & \text{if} \quad \tilde{\theta}(r) = \overline{\theta}(r). \end{cases}$$

To give an interpretation to these vectors, recall from Proposition 3 that the core constraints worker j_r must satisfy are $0 \leq y_{j_r} \leq a_{\mu^{-1}(j_r)j_r}$ and

$$a_{\mu^{-1}(j)j_r} - a_{\mu^{-1}(j)j} \leq y_{j_r} - y_j \leq a_{\mu^{-1}(j_r)j_r} - a_{\mu^{-1}(j_r)j}, \text{ for all } j \in W, \, \mu^{-1}(j) \neq \mu^{-1}(j_r).$$

Then, when we reach worker j_r following order θ, the max-min vector procedure determines a payoff for j_r that satisfies (in a tight way) either one lower core bound or one upper core bound, depending on whether the extended order $\tilde{\theta}$ determines j_r is a maximizer or a minimizer. It is not surprising that a max-min vector may not be in the core, since one half of the core constraints are not checked during the procedure that builds such vector. However, we show next that if a max-min vector is in the core, then it is an extreme core point. This same property is satisfied by the marginal worth vectors in arbitrary coalitional games and by the max-payoffs vectors in one-to-one assignment games, which are also collections of vectors that are defined for each possible order on a player set.

Proposition 18. *Let $\gamma = (F, W, A, r)$ be a many-to-one assignment game, θ an order on W and $\tilde{\theta}$ an extension of θ. If $y^{\tilde{\theta}} \in C(W)$, then $y^{\tilde{\theta}} \in Ext(C(W))$.*

Proof. Let $y^{\tilde{\theta}} \in C(W)$. By definition of the max-min vectors, at each step of the procedure one core constraint is tight at $y^{\tilde{\theta}}$. Moreover, these equations are linearly independent since each of them involves a new worker whose payoff does not take part in the previous equations. Since the core membership is guaranteed by the assumption, the fact that n linearly independent core constraints are tight at $y^{\tilde{\theta}}$ implies that this is an extreme point of the core. □

Now the question is whether all extreme core points of the many-to-one assignment game are of this type, that is, are max-min vectors related to some extended order on the set of workers. Let us consider again the market of Example 4.

Example 19. Consider again the many-to-one assignment market $\gamma = (F, W, A, r)$ with set of firms $F = \{f_1, f_2\}$ with capacities $r = (2, 1)$, set of workers $W = \{w_1, w_2, w_3\}$ with unitary capacity and pairwise valuation matrix

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 8 & 6 & 3 \\ 7 & 6 & 4 \end{pmatrix} \end{array}.$$

We can obtain the extreme core points from the picture of the core in Figure 4 and then check that all core vertices are max-min vectors. In the following table we indicate for each core vertex at least one extended order such that the related max-min vector coincides with that vertex.

x_1	x_2	y_1	y_2	y_3	extended order
0	0	8	6	4	$(\overline{1},\overline{2},\overline{3})$ in any permutation, $(\overline{1},\overline{3},\underline{2})$
0	1	8	6	3	$(\overline{1},\overline{2},\underline{3})$, $(\overline{2},\overline{1},\underline{3})$
1	1	8	5	3	$(\overline{1},\underline{3},\underline{2})$
1	0	7	6	4	$(\overline{2},\overline{3},\underline{1})(\overline{3},\overline{2},\underline{1}$
2	1	6	6	3	$(\overline{2},\underline{3},\underline{1})$
9	4	3	2	0	$(\underline{3},\underline{2},\underline{1})$, $(\underline{3},\underline{1},\underline{2})$
7	4	5	2	0	$(\underline{3},\underline{2},\overline{1})$
6	4	5	3	0	$(\underline{3},\overline{2},\overline{1})$
8	4	3	3	0	$(\underline{3},\overline{2},\underline{1})$

Take for instance the extended order $\tilde{\theta} = (\overline{2},\overline{3},\underline{1})$. The related max-min vector is

$$\begin{aligned} y_2 &= a_{12} = 6, \\ y_3 &= \min\{y_2 - a_{22} + a_{23}, a_{23}\} = \min\{4,4\} = 4, \\ y_1 &= \max\{y_3 - a_{23} + a_{21}, 0\} = \max\{7,0\} = 7, \end{aligned}$$

which leads to the extreme core point $(1,0;7,6,4)$.

This example also shows that not all max-min vectors belong to the core, and hence they may not lead to an extreme core allocation. Take for instance the extended order $\tilde{\theta} = (\underline{1},\overline{2},\underline{3})$. Then,

$$\begin{aligned} y_1 &= 0, \\ y_2 &= a_{12} = 6, \\ y_3 &= \max\{y_1 - a_{11} + a_{13}, y_2 - a_{12} + a_{13}, 0\} = \max\{-5,3,0\} = 3. \end{aligned}$$

The related max-min vector is $y^{\tilde{\theta}} = (0,6,3)$ and it does not lead to a core payoff since the constraint $y_3 - y_1 \leq a_{23} - a_{21}$ is not satisfied.

Next theorem shows that although a max-min vector may not be an extreme core allocation, the converse inclusion always holds. As the example above illustrates, every extreme core point is a max-min vector related to one extended order, or maybe to several of them.

Theorem 20. *Let $\gamma = (F, W, A, r)$ be a many-to-one assignment game. Then,*

$$Ext(C(W)) \subseteq \{y^{\tilde{\theta}}\}_{\tilde{\theta}\in\tilde{\Sigma}_W}.$$

Proof. Let $y \in Ext(C(W))$ and consider the tight graph E^y. From Theorem 11, there exists $j_1 \in W$ such that either $y_{j_1} = 0$ or $y_{j_1} = a_{\mu^{-1}(j_1)j_1}$. In the first case define

$\tilde{\theta}(1) = \underline{\theta}(1) = j_1$ and in the second case $\tilde{\theta}(1) = \overline{\theta}(1) = j_1$. Notice that in both cases $y^{\tilde{\theta}}_{j_1} = y_{j_1}$.

For $1 < r \leq n-1$, assume by induction hypothesis that there exists $\tilde{\theta} \in \tilde{\Sigma}_W$ with $\tilde{\theta}(k) = j_k$ for all $1 \leq k \leq r$ such that $y^{\tilde{\theta}}_{\theta(k)} = y^{\tilde{\theta}}_{j_k} = y_{j_k}$, and let us see this also holds for $r+1$.

Case 1: There exists some $j \in W \setminus \{j_1, j_2, \ldots, j_r\}$ and some $j_k \in \{j_1, j_2, \ldots, j_r\}$ such that $(j, j_k) \in E^y$.

In this case, $y_j - y_{j_k} = a_{\mu^{-1}(j)j} - a_{\mu^{-1}(j)j_k}$, which implies $y_j = y_{j_k} + a_{\mu^{-1}(j)j} - a_{\mu^{-1}(j)j_k}$. Then, set $\tilde{\theta}(r+1) = \overline{\theta}(r+1) = j$, that is, $j_{r+1} = j$, and notice that, since y is in the core, the inequalities $y_j \leq a_{\mu^{-1}(j)j}$ and $y_j \leq y_{j_l} + a_{\mu^{-1}(j)j} - a_{\mu^{-1}(j)j_l}$, for $j_l \in \{j_1, \ldots, j_r\}$ with $\mu^{-1}(j_l) \neq \mu^{-1}(j)$, hold. This shows that $y_{j_{r+1}} = y^{\tilde{\theta}}_{j_{r+1}}$.

Case 2: There exists some $j \in W \setminus \{j_1, j_2, \ldots, j_r\}$ and some $j_k \in \{j_1, j_2, \ldots, j_r\}$ such that $(j_k, j) \in E^y$.

In this case, $y_{j_k} - y_j = a_{\mu^{-1}(j_k)j_k} - a_{\mu^{-1}(j_k)j}$, which implies $y_j = y_{j_k} + a_{\mu^{-1}(j_k)j} - a_{\mu^{-1}(j_k)j_k}$. Then, set $\tilde{\theta}(r+1) = \underline{\theta}(r+1) = j$, that is, $j_{r+1} = j$, and notice that, since y is in the core, the inequalities $y_j \geq 0$ and $y_j \geq y_{j_l} + a_{\mu^{-1}(j_l)j_l} - a_{\mu^{-1}(j_l)j}$ hold for all $j_l \in \{j_1, \ldots, j_r\}$ with $\mu^{-1}(j_l) \neq \mu^{-1}(j)$. This guarantees that $y_{j_{r+1}} = y^{\tilde{\theta}}_{j_{r+1}}$.

Case 3: There is no $j \in W \setminus \{j_1, j_2, \ldots, j_r\}$ connected in E^y to some $j_k \in \{j_1, j_2, \ldots, j_r\}$.

This means that all agents in $W \setminus \{j_1, j_2, \ldots, j_r\}$ belong to a different component of E^y than any agent in $\{j_1, j_2, \ldots, j_r\}$. From Theorem 11, there exists some $j \in W \setminus \{j_1, j_2, \ldots, j_r\}$ such that either $y_j = 0$ or $y_j = a_{\mu^{-1}(j)j}$. In the first case set $\tilde{\theta}(r+1) = \underline{\theta}(r+1) = j$ and in the second $\tilde{\theta}(r+1) = \overline{\theta}(r+1) = j$. It remains to see that in both cases $y^{\tilde{\theta}}_j = y_j$.

If $y_j = 0$, and taking into account the core inequalities $y_j \geq y_{j_l} - a_{\mu^{-1}(j_l)j_l} + a_{\mu^{-1}(j_l)j}$ hold for all $j_l \in \{j_1, \ldots, j_r\}$ with $\mu^{-1}(j_l) \neq \mu^{-1}(j)$, we have $0 = y_j \geq y_{j_l} - a_{\mu^{-1}(j_l)j_l} + a_{\mu^{-1}(j_l)j}$ for all $j_l \in \{j_1, \ldots, j_r\}$ with $\mu^{-1}(j_l) \neq \mu^{-1}(j)$, which means $y_j = y^{\tilde{\theta}}_j$.

Similarly, if $y_j = a_{\mu^{-1}(j)j}$, the core inequalities imply $y_j \leq y_{j_l} + a_{\mu^{-1}(j)j} - a_{\mu^{-1}(j)j_l}$ for all $j_l \in \{j_1, \ldots, j_r\}$ with $\mu^{-1}(j_l) \neq \mu^{-1}(j)$. Then, $a_{\mu^{-1}(j)j} = y_j \leq y_{j_l} + a_{\mu^{-1}(j)j} - a_{\mu^{-1}(j)j_l}$ guarantees $y_j = y^{\tilde{\theta}}_j$. □

A consequence of the above theorem is that each extreme core point of a many-to-one assignment game is the result of a lexicographic optimization procedure carried out by the workers over the core. This somehow resembles the one-to-one assignment game, where each extreme core point can be obtained from a lexicographic maximization over the core (Núñez and Rafels, 2003), or also from a lexicographic minimization (Núñez and Solymosi, 2017), but in both cases, all agents, firms and workers, take part in the optimization procedure.

Based on the notion of max-min vectors and the result in Theorem 20, we can propose a sort of algorithm to find all extreme core points of a many-to-one assignment game (F, W, A, r).

Algorithm:

Step 0: Take an extended order $\tilde{\theta}$.
Step 1: Compute $y^{\tilde{\theta}}_{\theta(1)}$.

STEP $k > 1$: Compute $y^{\tilde{\theta}}_{\theta(k)}$.

If $y^{\tilde{\theta}}_{\theta(k)}$ satisfies all core constraints with its predecessors,

if $k < n$, GO TO STEP k+1

if $k = n$, OUTPUT $y^{\tilde{\theta}}$,

otherwise, choose a different extended order $\tilde{\theta}$ and GO TO STEP 1.

Since this procedure is based on the max-min vectors, we obtain the extreme core allocations only making use of the valuation matrix, with no need to obtain the characteristic function of the game.

The above algorithm is not very efficient when computing all extreme core points, but may be useful to obtain the firm-optimal core allocation. Recall that in the one-to-one assignment game both optimal core allocations are easy to obtain since it is well-known that the maximum core payoff of any agent is its marginal payoff. In the many-to-one case, only workers (because they have unitary capacity) always attain their marginal payoffs in the core. But this is no longer the case for firms with capacity greater than one.

Given a market (F, W, A, r), if we take any order θ on W and consider the extended order $\tilde{\theta} = (\underline{\theta}(1), \underline{\theta}(2), \ldots, \underline{\theta}(n))$, the related max-min vector $y^{\tilde{\theta}}$ satisfies $y^{\tilde{\theta}}_{\theta(k)} \leq y_{\theta(k)}$ for all $(x, y) \in C(v_\gamma)$. This is because each worker's payoff at $y^{\tilde{\theta}}$ tightly satisfies some lower core bound, given the payoff of his/her predecessors. As a consequence, whenever $y^{\tilde{\theta}}$ belongs to the core, it is the worse core allocation for workers, and hence the firm-optimal core allocation.

7 Concluding remarks

The core of many-to-one assingment markets has been studied to some extent, but little is known about its structure. In this paper, we have expanded the known results on (dis)similarities between the one-to-one case and the many-to-one case (and hence the many-to-many case). First, we have studied the relationship between the core and other solution concepts. We have shown the τ-value need not be a core element, the kernel may not be included in the core, and remarkably, the coincidence between the core and the bargaining set does not hold. Second, we have observed that a firm may not achieve its marginal payoff in an extreme core allocation. When exploring the extreme core allocations, we have shown that a lexicographic maximization procedure does not always lead to such extreme core point as in one-to-one assignment markets. Instead, we have provided a characterization of extreme core allocations based on a tight digraph defined on the projection of the core to the space of workers' payoffs. Finally, we have shown that each extreme core point can be obtained by a procedure in which all workers sequentially maximize or minimize their payoff over the core. As a by-product we obtain a way to compute the firm-optimal core allocation.

It can be found in the literature other results on extreme allocations for related models. Baïou and Balinski (2000) provided a characterization of the convex hull of the stable assignments of ordinal many-to-one matching problems. Baïou and Balinski (2002) introduced an algorithm to calculate efficiently two side-optimal allocations of a given ordinal transportation problem. Izquierdo, Núñez and Rafels (2007) provided an efficient algorithm which returns among others all extreme core allocations for the one-to-one assignment game. Trudeau and Vidal-Puga (2017) proposed a method to generate all extreme core allocations for minimal cost spanning tree problems.

A possible direction for future research is to study the computational aspects of max-min payoff algorithm and how to extend these results to the many-to-many case.

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