# Differential games of public investment: Markovian best responses in the general case *

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**Abstract**

We define a differential game of public investment with a discontinuous Markovian strategy space. The best response correspondence for the game is well-behaved: a best response exists and maps a profile of opponents' strategies back to the strategy space. Our chosen strategy space thus makes the differential game well-formed as a normal form game, resolving a long-standing open problem in the literature. We provide a user-friendly necessary and sufficient condition for constructing the best response. Our methods do not require recourse to specific functional forms. Our theory has general applications, including to problems of noncooperative control of stock pollutants, harvesting of natural resources, and joint investment problems.

## 1 Introduction

The dynamic public goods game is an important economic problem which shows up in different settings, including joint investment projects between firms, allocation of effort among members of a team, harvesting renewable resources under common access, and non-cooperative mitigation of climate change. As with infinite-horizon dynamic games in general, these games typically admit the possibility of multiple equilibria—even under Markovian strategies, which condition each player's investment flow on the current state of the accumulating capital stock (so that equilibria in which strategies directly depend on past actions are not admitted). Even when the existence of multiple Markov-perfect Nash equilibria (MPE) is recognised, the extant literature has not been able to identify the entire set of equilibria except in special cases. This makes it difficult to draw clear positive predictions or normative recommendations.

In this paper, we initiate a research programme which makes progress on this issue. We are able to do this by resorting to a continuous-time framework; in other words, we study a differential game.[1] Continuous time makes equilibria more tractable to analysis, in particular because it allows us concentrate the analysis on local properties of value functions, without having to know the global properties (as

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[1]The textbook Dockner et al. (2000) and the handbook Başar and Zaccour (2018) contain extensive overviews of differential games. For reviews of application in environmental and resource economics, see (Long, 2011, 2012).

would be necessary in discrete time). These benefits mirror recent developments in continuous-time macroeconomics (Brunnermeier and Sannikov, 2016; Achdou et al., 2014, 2022).

We restrict the players to use Markovian strategies (or policy rules), so that the control vector $q(t)$ is given as a function $\phi$ of the scalar state variable $x(t)$, or $q(t) = \phi(x(t))$. The appropriate choice of strategy space—the set from which $\phi$ can be chosen—has long been an open question in differential games (Başar and Olsder, 1982; Fudenberg and Tirole, 1991; Dockner et al., 2000). To understand why, note that computing the payoffs in a differential game requires the determination of a trajectory which is a solution to the state dynamics $\dot{x}(t) = f(x(t), q(t))$. Under Markovian strategies, if the dynamic $f(x, q)$ and the strategies $\phi(x)$ are Lipschitz-continuous, then a unique classical solution exists. However, it is well-known that the optimal response to Lipschitz-continuous dynamics may involve an indifference point (often called a Skiba point), so that the best response cannot be described as a Lipschitz-continuous function of the state (Skiba, 1978; Mäler et al., 2003). Thus, the best-response correspondence does not, in general, map the space of Lipschitz-continuous functions back to itself. The resulting game (interpreted as a normal-form game in policy rules) is not well-formed.

We allow the players to use discontinuous Markovian strategies, so that player $i$'s policy rule $\phi_i(x)$ is selected from a space $\mathscr{S}$ of functions with a finite number of discontinuities or kinks. Our first result (Theorem 4.1) shows that the resulting game is well-formed, in that the best response of player $i$ to any profile of other players' strategies $\phi_{-i} \in \mathscr{S}^{N-1}$ is unique, and can be described as a Markovian policy rule $\phi_i(x) \in \mathscr{S}$. Thus the best-response correspondence maps $\mathscr{S}^N$ back into itself. It is then a meaningful exercise to look for fixed points of this correspondence, i.e. Nash equilibria in Markovian strategies or MPE. Interpreted as a normal-form game in policy rules, our results imply that the payoff matrix is complete (all strategy profiles induce a vector of payoffs); that each player has a best response in the set of available strategies $\mathscr{S}$; and that each player's available policy rules do not depend on the policy rules simultaneously chosen by the other players.[2]

Our second result (Theorem 4.3) is practical: a user-friendly necessary and sufficient condition for a best response. Even though we do not here talk much about equilibrium, the result goes quite a long way towards this goal, as identifying and constructing equilibria requires simply finding a set of mutual best responses. While the present paper is necessarily quite technical, this condition should be easy to use in applied settings. We provide the result under relatively general conditions, not requiring the use of particular functional forms.

The present paper establishes fundamentals. In a companion paper (Jaakkola and Wagener, 2020), we demonstrate that our framework is both useful and important. In particular, we show how our methods can be used to construct the entire set of symmetric equilibria to the canonical problem of noncooperative mitigation of climate change (Dockner and Long, 1993) under a general functional specification. Thus, the necessary conditions identified in the present paper are straightforward to use in applications. The importance is shown by the main result of this companion paper: the type of equilibrium most commonly discussed in the literature is Pareto-dominated by all other symmetric equilibria.[3] This result seems to us to be quite strong and surprising. It raises questions about the importance of this equilibrium—a focal point in the literature—from both positive and normative perspectives. In the companion paper, we also construct the Pareto-dominant equilibria and obtain meaningful intuition for them. There is a large literature of other applications; we believe it is worthwhile to also take a second look at these, using our methods. Moreover, we hope our methods can also shed light on asymmetric equilibria, something the existing literature has almost entirely ignored.

[2]Some authors use an admissibility criterion on strategy profiles, which implies that the set of allowed strategies may depend on strategies chosen by other players. See e.g. Dockner et al. (2000).

[3]In a linear-quadratic framework, this would be the linear strategy obtained by the guess-and-verify method.

The present paper is necessarily quite technical. In particular, as our ultimate goal is to help understand equilibria, in which the strategies are of course endogenous, we cannot rule out complicated cases *a priori*. The technical analysis is hidden in a long appendix; in the paper, we provide sketches of the proofs. We use two primary tools in our analysis. The first is the theory of viscosity solutions (Bardi and Capuzzo-Dolcetta, 2008). We apply viscosity theory to optimal control under discontinuous dynamics, relying heavily on Barles et al. (2013, 2014).[4] These methods allow us to construct the value function to a player's problem. To show that the best response can be described by a Markovian strategy, we use the theory of nonlinear dynamical systems, in particular the Centre Manifold Theorem.

In the wider scheme of things, the present paper thus makes two contributions. First, we put the theory of Markov-perfect equilibria in differential games on a sound theoretical footing, at least for the case in which the state variable is a scalar. This issue has been an open problem for decades (Başar and Olsder, 1982; see also Fudenberg and Tirole 1991; Dockner et al. 2000). Our results demonstrate the usefulness of continuous-time methods in deriving novel and general results in the analysis of dynamic strategic interactions. Multiplicity of Markovian equilibria is similarly present in (discrete-time) dynamic games, and their analysis in general is typically quite difficult. Our results ultimately flow from the fact that, in continuous time, the value function can be analysed and constructed using local information only.

Second, our paper consolidates and clarifies the long literature on multiple MPE in differential games, starting with Tsutsui and Mino (1990) and Dockner and Long (1993), and subsequently continued by e.g. Dockner and Sorger (1996), Sorger (1998), Rubio and Casino (2002), Rowat (2007), and Dockner and Wagener (2014).

The rest of the paper proceeds as follows. In Section 2, we set up the basic model. Section 3 develops the conceptual framework of the paper. The main results, existence of the best response and its characterisation, are given in Section 4. Proofs are relegated to the Appendix.

## 2 Model

In this section we present the fundamental features of the model.

Time is continuous and runs to infinity: $t \in [0, \infty)$. There are $N$ identical players, indexed by $i \in I \equiv \{1, \dots, N\}$. There is a scalar *state variable* $x \in \mathcal{X} \subset \mathbb{R}$. The state space $\mathcal{X}$ is a closed interval.

Player $i$ has access to an *action variable* $q_i \in \mathcal{Q}$ through a *action schedule* $a_i : [0, \infty) \to \mathcal{Q} \subset \mathbb{R}$. We assume the *control set* $\mathcal{Q} = [q_\ell, q_u]$ is convex and compact and of positive length, that is, $q_\ell < q_u$.[5] We collect actions and action schedules into vectors $q = (q_1, \dots, q_N)$ and $a(t) = (a_1(t), \dots, a_N(t))$.

The state evolution depends on the current state, but not on calendar time: given a vector of action schedules $a$, the differential equation governing the state evolution is

$$\dot{y}(t) = f(y(t), a(t)). \tag{1}$$

A function $y : [0, \infty) \to \mathcal{X}$ satisfying $y(0) = x$ and (1) almost everywhere is a *trajectory*, and $(y_{x,a}, a)$ a *trajectory–action pair*. When emphasising action and initial state, we sometimes write $y = y_{x,a}$. Note that we distinguish between state and action variables $x$ and $q$, and state trajectories $y$ and action schedules $a$.

[4] These papers consider exogenously given discontinuities in dynamics. In a game-theoretic equilibrium, the discontinuities in the dynamics will be endogenous.

[5] We could allow multivariate controls as in Dockner and Wagener (2014); our results would follow, given additional assumptions along the way. Our key insights are best conveyed without such complications.

A function is real analytic at a point if it can be represented, in a nonempty neighbourhood of the point, as a convergent power series with real coefficients. We shall say that the function $f(x, q)$ is real analytic in $x$ and $q$, piecewise with respect to $x$, or, more briefly, piecewise real analytic, if it is real analytic at all points, excepting a finite number of states.

**Assumption** ($\mathrm{A}_1$): *The function $f(x, q)$ is continuous, real analytic in $q$ for all fixed $x$, real analytic in $x$ and $q$, piecewise with respect to $x$, and satisfies $f_{q_i} > 0$ for all $i$ everywhere.*

The state variable is a public good (or public bad), in that the players' action variables reflect their contributions to investing in or disinvesting from it. The primitive of the payoffs is the flow felicity function:

**Definition 2.1.** *The* **felicity** *of player $i$ when playing $q_i$ at the state $x$ is $u_i(x, q_i)$.*

**Assumption** ($\mathrm{A}_2$): *The felicity $u_i$ is real analytic in $x$ and $q$ and satisfies $(u_i)_x < 0$ everywhere. Moreover, for every $x$ the set $\{\eta \in \mathbb{R}^2 : \eta_1 = f(x, q),\ \eta_2 \leq u_i(x, q_i),$ for some $q \in \mathcal{Q}\}$ is convex.*

The assumption of piecewise real analyticity covers the vast majority of parametrised models in the literature, which are usually specified using polynomial, rational, algebraic or elementary transcentdental functions. We do not allow $f$ or the $u_i$ to have singularities in the domain of definition: but such singularities are as a rule only included for analytic convenience considerations, which do not apply in our context.

When an action takes the state out of the state space at a boundary point $x$, the system is stopped and the players receive a boundary value payoff $\beta(x)$. Let $\Theta$ denote the infimum of the set $\{t : y(t) \notin \mathcal{X}\}$ if that quantity is finite, and $\infty$ otherwise.

All players discount future felicity and boundary payoffs at a common rate $\rho \in (0, \infty)$. In the absence of discontinuities, the overall payoff is be given by the sum of future discounted felicity, or

$$\int_0^{\Theta} \exp(-\rho t) u_i(y_{x,a}(t), a_i(t))\, \mathrm{d}t + \exp(-\rho\Theta)\beta(y_{x,a}(\Theta)). \tag{2}$$

For the payoffs to be consistent with the fundamentals of the model when strategies can be discontinuous, we will require a richer description of the payoffs in what follows. We thus postpone the full payoff specification until Section 3.2.

The basic set-up is thus one of dynamic public investment. We will formulate our leading example in terms of a global pollution stock (a public bad) which is built up as a result of economic activity and decays naturally. Adjusting the signs of the partial derivatives of the felicity function, or modifying the dynamics, will allow the model to be interpreted e.g. as one of joint investment into a common project (with depreciating capital), or as a model of renewable resource exploitation.

# 3 Markov-perfect Nash equilibrium

In this section, we first describe the strategy space and discuss some issues which arise. We then set up an individual player's optimisation problem and derive a best response. Finally, we present the equilibrium concept.

## 3.1 Markovian strategies

The players use Markovian strategies, conditioning their actions on the current state variable only. Let $\mathscr{S}$ be the set of functions $\phi : \mathcal{X} \to \mathcal{Q}$ such that there is a covering of $\mathcal{X}$ by a finite number of closed intervals that have non-empty and mutually disjunct interiors; restricted to one of these covering intervals, the function $\phi$ is the restriction of an infinitely often continuously differentiable function defined on an open interval containing the covering interval; the function $\phi$ is moreover real analytic in the interior of the covering interval. Such a covering is called *adapted* to $\phi$. Informally, a function $\phi$ is constructed of sections of analytic functions, but with the possibility of discontinuities where two adjacent sections are pieced together; however, even at points of discontinuity, one-sided derivatives exist.

**Definition 3.1.** *A* **Markovian strategy** *of player $i$ is a function $\phi_i \in \mathscr{S}$. A* **strategy profile** *is an $N$-tuple of Markovian strategies $\phi \equiv (\phi_1, \ldots, \phi_n) \in \mathscr{S}^N$.*

In the remainder of this subsection, we fix an $N$-tuple $\phi$ of Markovian strategies.

When using a Markovian strategy a player sets its action as $a_i(t) = \phi_i(y(t))$. Given a strategy profile $\phi$, the system therefore evolves according to[6]

$$\dot{y}(t) = F(y(t)) \equiv f(y(t), \phi(y(t))). \tag{3}$$

In optimal management problems with nonconcave dynamics, it can occur that the optimal policy is described by a discontinuous Markovian policy rule. Hence, when there are several players present, it can happen that the best response of a player is a discontinuous Markovian strategy. A description of possibly play therefore has to take such strategies into account.

When Markovian strategies are not required to be continuous, $F$ may have discontinuities, so that the evolution equation (3) may not have classical solutions, or may have a multiplicity of solutions. We therefore have to be precise about our notion of solution, and how payoffs must be adapted to that notion.

For every function $\psi(x)$ defined on the state space, we define for $x \in \mathcal{X}$ the lateral limits

$$\psi_-(x) \equiv \lim_{z \uparrow x} \psi(z), \qquad \psi_+(x) \equiv \lim_{z \downarrow x} \psi(z), \tag{4}$$

whenever they exist. We for instance have

$$F_-(x) = f(x, \phi_-(x)), \quad F_+(x) = f(x, \phi_+(x)). \tag{5}$$

These limits exist for all $x \in \mathcal{X}$, excepting the boundary points $\min \mathcal{X}$, $\max \mathcal{X}$, given the definition of the strategy space $\mathscr{S}$.

We now extend our original notion of 'trajectory' by specifying solutions to the dynamics given by equation (3).

**Definition 3.2.** *Given a strategy profile $\phi$ and an initial state $x$, a* **Filippov solution** *to equation* (3)*, or* **trajectory** *of the dynamics* (3)*, is an absolutely continuous function $y(t)$ which satisfies $y(0) = x$ and, for almost all $t$, the differential inclusion*

$$\dot{y}(t) \in \overline{\text{co}}\left(F_-(y(t)), F_+(y(t))\right), \tag{6}$$

*where $\overline{\text{co}}$ denotes the convex hull.*

Note that, for Filippov solutions (Filippov, 1988), pointwise deviations in the strategy do not affect the law of motion; in particular, the law of motion at any state $x$ is independent of $F(x)$, instead depending only on the left and right limits of $F(z)$ as $z \to x$.

[6] For brevity, we omit notation to indicate the dependence of $F$ on $\phi$.

We use $x_c$ to denote a general point in the state space at which there is no discontinuity, i.e. $F_-(x_c) = F_+(x_c)$; and $x_d$ to denote a point at which there is, so that $F_-(x_d) \neq F_+(x_d)$. Consider a trajectory $y$ such that $y(t_c) = x_c$ for some $t_c$. Then $\dot{y}(t_c)$ is uniquely pinned down by equation (6), and there exists a unique classical solution trajectory for $t$ close to $t_c$. Similarly a unique classical trajectory $y$ exists for $t$ near $t_d$, such that $y(t_d) = x_d$, if $F_-(x_d)$ and $F_+(x_d)$ have the same sign.[7]

Trajectories can be stabilised at $x_d$ if $F_-(x_d)$ and $F_+(x_d)$ have different signs. If a trajectory stays at such a point for a time interval $I$ with positive length, then for all $t \in I$ there exists *weights* $\mu_-(t), \mu_+(t) \in [0, 1]$ such that $\mu_-(t) + \mu_+(t) = 1$ and

$$\dot{y}(t) = \mu_-(t)F_-(y(t)) + \mu_+(t)F_+(y(t)) = 0. \tag{7}$$

If $\mu_-(t) > 0$ and $\mu_+(t) > 0$, this is known as a chattering solution.

Relying on Filippov solutions introduces three complications. First, the payoffs must be adapted to the Filippov dynamics, that is, the possibility of chattering. The weights $\mu_\pm(t)$ are central in this. Second, it is not clear that the best response to a set of Markov strategies is necessarily Markovian. Third, chattering solutions may not be unique.

Given a Markovian strategy profile $\phi(x)$, suppose a trajectory is stabilised at $x_d$ with weights $\mu_-(t)$ and $\mu_+(t)$. The resulting payoff flow must be adapted to the dynamics and the weights given in (7). Define the left and right action schedules

$$a_k(t) = (a_{1,k}(t), \ldots, a_{N,k}(t)) \equiv \big(\phi_{1,k}(y(t)), \ldots, \phi_{N,k}(y(t))\big), \quad k \in \{-, +\}. \tag{8}$$

Note that the functions $a_-$ and $a_+$ are not lateral limits, as they are functions of $t$; rather, they are defined through equation (8). Then, the *discontinuity flow payoff* of player $i$ is defined in terms of the weights $\mu_-$ and $\mu_+$, player $i$'s left action $a_{i,-}$ and right action $a_{i,+}$ as

$$u_i^d(y(t), a_{i,-}(t), a_{i,+}(t), \mu_-(t), \mu_+(t)) = \mu_-(t)u_i(y(t), a_{i,-}(t)) + \mu_+(t)u_i(y(t), a_{i,+}(t)). \tag{9}$$

The specification captures the following notion: the system never sits still exactly at $x = x_d$; instead, it chatters between the two sides, with the action switching very rapidly between $q_{i,-} = \phi_{i,-}(x_d)$ and $q_{i,+} = \phi_{i,+}(x_d)$.[8] The overall payoff is then a weighted sum of the left and right limits of the payoffs, with the weights given by the force with which the dynamics push the system towards $x = x_d$. If the force pushing the system to the right is much stronger than the force pushing back to the left, then the system spends more time on the right side of $x = x_d$ and consequently the payoff reflects that. This outcome would result if, for example, time were partitioned into infinitesimal intervals during which the action variables were held constant at the value specified by the strategy for the state at the first moment of each interval (a so-called 'sample-and-hold' solution; see Clarke, 2004). Our notion of payoffs is used in the literature on optimal control over multidomains (Barles et al., 2013, 2014), whose results we use. However, in a game-theoretic context, the division of the state space into multiple domains is endogenous, instead of being given as a primitive of the problem.

At points $x_c$ where $\phi$ is continuous, the flow payoff for player $i$ is given simply by the felicity function $u_i(x, \phi_i(x))$. At points $x_d$ where strategies are discontinuous but $\mathrm{sign}(F_-(x_d)) = \mathrm{sign}(F_+(x_d)) \neq 0$, the dynamics move the system past the point of discontinuity in an instant of time of measure zero, so that

[7] To be precise, if $F_-(x_d)$ and $F_+(x_d)$ have the same sign and $F(x_d) = 0$, there is also a second stationary trajectory which stabilises at $y(t) = x_d$ for $t \geq t_c$. This trajectory is not a Filippov solution, however.

[8] With classical solutions, actual chattering would only occur if also $F(\phi(x_d), x_d) \neq 0$. In the case $F(\phi(x_d), x_d) = 0$, however, trajectories (and hence payoffs) would not be well-defined for small deviations in strategies at point $x = x_d$. Hence such deviations could not be evaluated. The usual admissibility requirement would simply rule out such deviations, thus limiting a player's action set in a way which depended on the other players' actions, purely due to a technical requirement.

this discontinuity has no impact on the time integral of discounted flow payoff. As explained above, in these cases equation (1) has a unique classically defined solution. In other words, for classically admissible strategies, our definition of Filippov solutions and accordingly adapted payoffs give exactly the outcome which the previous literature has used.[9] Thus, our specification of dynamic trajectories and payoffs nests the notion of utilities and equilibrium outcomes for strategy profiles commonly called 'admissible'.

To close off this subsection, we emphasise the key difference in the present paper compared to most of the literature on differential games: we explicitly deal with cases in which strategy profiles can lead to chattering solutions. Typically, such complications are assumed away by either requiring strategies to be Lipschitz continuous, or with an admissibility requirement on strategy profiles, so that strategy profiles which lead to chattering solutions are not admitted. The latter approach implies that the strategies player $i$ can choose depend on the strategies chosen by the other players. Our approach allows the space of admissible strategy profiles to simply be the product set of individual strategy spaces, as is standard in game theory, while allowing for non-Lipschitz strategies.[10]

### 3.2 Best response

We now consider the optimisation problem of player $i$, given the strategy profile

$$\phi_{-i} \equiv (\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_N)$$

of the remaining players.

The standard approach is to determine player $i$'s best response to $\phi_{-i}$ as a Markovian strategy $\phi_i^*$. However, it is not immediately clear whether the best response to a Markovian strategy is itself Markovian. In particular, suppose that $\phi$ is a strategy profile such that $\phi_i$ is a best response to $\phi_{-i}$ and that there is a discontinuity at $x_d$. With the payoff flow (9) and a $t_d$ such that $y(t_d) = x_d$, it is not obvious that a player $i$ will optimally set $a_{i,+}(t_d) = \lim_{x \downarrow x_d} \phi_i(x)$ and $a_{i,-}(t_d) = \lim_{x \uparrow x_d} \phi_i(x)$[11]. We will thus first tackle player $i$'s dynamic optimisation problem more generally, allowing the player to choose (possibly non-Markovian) action schedules $a_{i,-}$ and $a_{i,+}$ freely, that is, not subjected to a restriction like (8). The value $a_{i,-}(t)$ is then interpreted as the action of player $i$ at time $t$ if $y(t) \leq x_d$, and it is irrelevant if $y(t) > x_d$; similarly $a_{i,+}(t)$ is the action of player $i$ at time $t$ if $y(t) \geq x_d$, etc. If $y(t) = x_d$, then both $a_{i,-}(t)$ and $a_{i,+}(t)$ are relevant actions.

We shall be working with finitely many points at which strategy profiles may be discontinuous, and our notation has to reflect this. Introduce therefore a covering $\mathscr{X}_{-i} = \{\mathcal{X}_j\}_{j=0}^J$ of $\mathcal{X}$ adapted to the strategy profile $\phi_{-i}$: each $\mathcal{X}_j$ is a closed interval in $\mathcal{X}$ with the intersection $\mathcal{X}_j \cap \mathcal{X}_k$ consisting of a single common boundary point if $j$ and $k$ differ by one, and being empty otherwise. Denote by $\mathcal{I}_j = \mathcal{X}_{j-1} \cap \mathcal{X}_j$ the *interface* between two adjacent regions, and by $\mathcal{I} = \bigcup_{j=1}^J \mathcal{I}_j$ the collection of all *interface points*.

To be *adapted* to the strategy profile, the covering must satisfy that the restriction of $\phi_{-i}$ to $\operatorname{int} \mathcal{X}_j$ is the restriction of an analytic function

$$\phi_{-i,j} = (\phi_{1,j}, \ldots, \phi_{i-1,j}, \phi_{i+1,j}, \ldots, \phi_{N,j})$$

[9] However, our specification does not allow for equilibria with isolated stationary points, with the dynamics pushing in the same direction at a strictly positive rate on both sides. Dutta and Sundaram (1993) show in a discrete-time model how such equilibria could help avoid the tragedy of the commons. However, they would not be robust to vanishingly small noise.

[10] Klein and Rady (2011) impose payoffs of $-\infty$ for strategy profiles which do not generate classically defined trajectories for the state. Thus, while the set of admissible strategy profiles coincides with the product of the strategy sets, the payoffs are very discontinuous in the space of strategy profiles, a feature which is a technical artefact rather than following from the economic fundamentals of the model.

[11] See e.g. the example in Barles et al. (2013), Section 5.4

defined on an open interval containing $\mathcal{X}_j$. We will consider the minimal covering, i.e. the covering with the least elements (i.e. ignoring trivial interface points at which $\phi_{-i}$ is in fact analytic).

The *local action* $q_{i,j}$ for player $i$ in region $\mathcal{X}_j$ is the action played when the state satisfies $x \in \operatorname{int} \mathcal{X}_j$. On $\mathcal{X}_j$, the *local dynamics* for player $i$ are given by

$$f_{i,j}(x, q_{i,j}) = f(x, \phi_{1,j}(x), \ldots, q_{i,j}, \ldots, \phi_{N,j}(x)). \tag{10}$$

Note that $f_{i,j}$ is conditional on $\phi_{-i}$. We do not explicitly indicate this, to avoid notational clutter.

**Definition 3.3.** *An* **admissible trajectory** *is an absolutely continuous function $y : [0, \infty) \to \mathcal{X}$ such that*

$$\dot{y}(t) \in f_{i,j}(y(t), \mathcal{Q})$$

*for almost all $t \geq 0$ such that $y(t) \in \operatorname{int} \mathcal{X}_j$, and*

$$\dot{y}(t) \in \overline{\mathrm{co}} \left( f_{i,j-1}(y(t), \mathcal{Q}) \cup f_{i,j}(y(t), \mathcal{Q}) \right),$$

*for almost all $t \geq 0$ such that $y(t) \in \mathcal{I}_j$.*

If the state is in the interface point belonging to $\mathcal{X}_{j-1}$ and $\mathcal{X}_j$, that is if $x \in \mathcal{I}_j$, then two local actions are active and the local discontinuity dynamics for $i$ are given by

$$\lambda_{i,j-1} f_{i,j-1}(x, q_{i,j-1}) + \lambda_{i,j} f_{i,j}(x, q_{i,j}), \tag{11}$$

where $\lambda_{i,k} \in [0, 1]$ are such that $\lambda_{i,j-1} + \lambda_{i,j} = 1$.

Although the definition of local actions is motivated by the fact that these are the actual actions if the trajectory $y(t)$ is in the relevant parts of the state space, they are not conditioned on the value of $y(t)$, rather being functions of time only.

**Definition 3.4.** *A* **control schedule** $c_i = (a_i, \mu_i)$ *of player $i$ consists of a vector*

$$a_i(t) = (a_{i,0}(t), \ldots, a_{i,J}),$$

*with $a_{i,j} \in L^\infty(0, \infty; \mathcal{Q})$* **local action schedules***, and a vector*

$$\mu_i(t) = (\mu_{i,0}(t), \ldots, \mu_{i,J}(t)),$$

*with $\mu_{i,j} \in L^\infty(0, \infty; [0, 1])$ and $\sum_{j=0}^{J} \mu_{i,j}(t) = 1$* **local weights***. The set of control schedules of player $i$ is denoted $\mathcal{C}^i$.*

*The control schedule $c_i = (a_i, \mu_i)$ is* **admissible** *if $\mu_{i,j}(t) = 1$ if $y(t) \in \operatorname{int} \mathcal{X}_j$, $\mu_{i,j}(t) = 0$ if $y(t) \notin \mathcal{X}_j$, and if there is an admissible trajectory $y$ such that*

$$\dot{y}(t) = \sum_{j=0}^{J} \mu_{i,j}(t) f_{i,j}(y(t), a_{i,j}(t)) \qquad \text{for almost all } t.$$

*The pair $(y, c_i)$ is called an admissible* **trajectory–control pair***.*

We note that the local action schedules $a_{i,j}$ are the natural generalisations of the left and right action schedules $a_{i,-}$ and $a_{i,+}$, introduced in Equation (8), to a situation with finitely many interfaces.

**Proposition 3.5.** *For every initial point $x \in \mathcal{X}$ there is an admissible trajectory–control pair $(y, c)$ such that $y(0) = x$.*

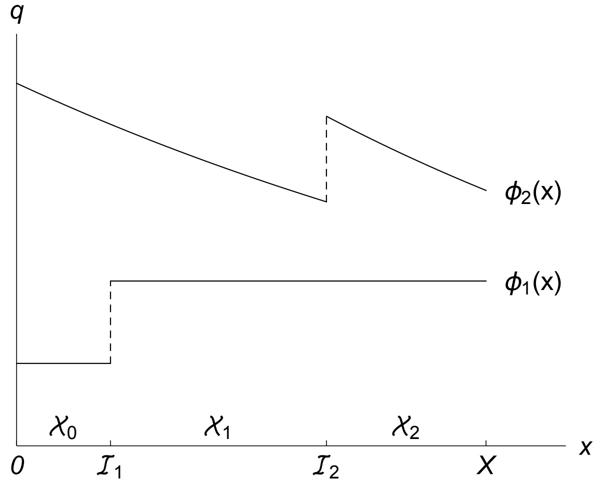


Figure 1: Covering adapted to strategies.

*Proof.* This is a direct consequence of Barles et al. (2013, Theorem 2.1). □

*Example.* Figure 1 shows a three-player game with $\mathcal{X} = (0, X)$, two strategies with discontinuties, and a covering $\{\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2\}$ adapted to these. Player 3's control schedule is then

$$c_3(t) = (a_3(t), \mu_3(t)) = (a_{3,0}(t), a_{3,1}(t), a_{3,2}(t), \mu_{3,0}(t), \mu_{3,1}(t), \mu_{3,2}(t)).$$

Of course, at a given time, at most two of the $a_{3,j}(t)$'s and the $\mu_{3,j}(t)$'s are active, depending on whether $y(t)$ is at an interface point $\mathcal{I}_j$ or in the interior of some interval $\mathcal{X}_j$.

At any point in time $t$, the local flow payoff and dynamics, for player $i$, can now be constructed based on the control schedule and the state variable:

**Definition 3.6.** *The* **flow payoff** *for player $i$ is a function $v_i : \mathcal{X} \times \mathcal{Q}^{J+1} \times [0,1]^J \to \mathbb{R}$ given by*

$$v_i(x, q_i, \lambda_i) = \begin{cases} u_i(x, q_{i,j}) & \text{if } x \in \operatorname{int} \mathcal{X}_j, \\ \lambda_{i,j-1} u_i(x, q_{i,j-1}) + \lambda_{i,j} u_i(x, q_{i,j}) & \text{if } x \in \mathcal{I}_j, \end{cases}$$

*where $u_i$ is the felicity function.*

*The* **dynamics** *for player $i$ are given by a function $g_i : \mathcal{X} \times \mathcal{Q}^{J+1} \times [0,1]^J \to \mathbb{R}$ defined by*

$$g_i(x, q_i, \lambda_i) = \begin{cases} f_{i,j}(x, q_{i,j}) & \text{if } x \in \operatorname{int} \mathcal{X}_j, \\ \lambda_{i,j-1} f_{i,j-1}(x, q_{i,j-1}) + \lambda_{i,j} f_{i,j}(x, q_{i,j}) & \text{if } x \in \mathcal{I}_j, \end{cases}$$

*where $f_{i,j}$ is given by* (10).

We now define trajectories:

**Definition 3.7.** *Given an initial state $x$, a strategy profile $\phi_{-i}$ and adapted covering $\mathscr{X}_{-i}$, and a control schedule $c_i = (a_i, \mu_i)$, a* **trajectory** *for player $i$ is a map $y : [0, \infty) \to \mathcal{X}$, such that $y(0) = x$ and*

$$\dot{y}(t) = g_i(y(t), c_i(t)), \tag{12}$$

*for almost all $t \geq 0$. If* (12) *holds, the pair $(y, c_i)$ is a* **trajectory–control** *pair.*

*If moreover for almost all $t \geq 0$ such that $y(t) \in \mathcal{I}_j$ for some $j$ we have*

$$f_{i,j-1}(y(t), a_{i,j-1}(t)) \geq 0 \quad \textit{and} \quad f_{i,j}(y(t), a_{i,j}(t)) \leq 0, \tag{13}$$

*then the trajectory is called* **regular**. *The set of all trajectories, respectively all regular trajectories, given $x$, $\phi_{-i}$ and $c_i$, is denoted $\mathcal{T}_{x,\phi_{-i},c_i}$, respectively $\mathcal{T}^{\mathrm{reg}}_{x,\phi_{-i},c_i}$.*

The following result is a direct consequence of Theorem 2.1 of Barles et al. (2013).

**Proposition 3.8.** *The sets $\mathcal{T}_{x,\phi_{-i},c_i}$ is nonempty.*

The inequalities (13) require that stabilisation cannot occur with controls which are 'pulling away' from an interface point. Such a situation is inherently unstable and would be immediately resolved by the slightest perturbation.

Total welfare is integrated discounted utility:

**Definition 3.9.** *Given a trajectory–control pair $(y, c_i)$ with initial state $x$, the* **total payoff** *from the pair for player $i$ is given by*

$$U_i(y, c_i) = \int_0^\Theta \exp(-\rho t) v_i(y(t), c_i(t))\,\mathrm{d}t + \exp(-\rho\Theta)\beta(y(\Theta)). \tag{14}$$

The value of the strategy profile $\phi_{-i}$ to player $i$ is now just a function mapping the initial state to the overall maximal payoff:

**Definition 3.10.** *The* **value** *at the initial state $x_0$ of the profile $\phi_{-i}$ to player $i$ is*

$$V_i(x) = \sup_{\mathcal{C}^i} \sup_{\mathcal{T}_{x,\phi_{-i},c_i}} U(y, c_i) \tag{15}$$

*where the first supremum is taken over the control schedules $c_i$, and the second over the set of trajectories $y$ for player $i$, given $x$, $\phi_{-i}$, and $c_i$.*

*The* **regular value** *$V_i^{\mathrm{reg}}$ is defined analogously, with the set $\mathcal{T}_{x,\phi_{-i},c_i}$ of trajectories replaced by the set $\mathcal{T}^{\mathrm{reg}}_{x,\phi_{-i},c_i}$ of regular trajectories.*

*A control schedule $c_i$ for which the supremum is realised is called a* **best response of player** *$i$. The set of all trajectories, respectively all regular trajectories, that are associated to a best response is denoted $\mathcal{T}^*_{i,x,\phi_{-i}}$, respectively $\mathcal{T}^{\mathrm{reg},*}_{i,x,\phi_{-i}}$.*

An important technical result will be to show that the condition $(u_i)_x < 0$ implies that $V_i$ and $V_i^{\mathrm{reg}}$ are identical.

### 3.3 Markovian best responses and MPE

We next consider which trajectory–control pairs are compatible with Markov strategy profiles and the Filippov dynamics (6). Let therefore a trajectory–control pair $(y, c_i)$ be given, as well as a strategy profile $\phi$ and a covering $\mathscr{X}$ adapted to it.

The first compatibility condition is the standard feedback relation

$$a_{i,j}(t) = \phi_{i,j}(y(t)) \tag{16}$$

for all $i$ and $j$, and for almost all $t \geq 0$ such that $y(t) \in \mathcal{X}_j$.

The dynamics $f(x, \phi(x))$ may fail to be continuous at interface points. Writing $\mathcal{I}_j = \{\bar{x}_j\}$, let $T_j = \{t \geq 0 : y(t) = \bar{x}_j\}$ be the set of times for which the trajectory is at the $j$'th interface. For almost all $t \in T_j$ we have $\dot{y}(t) = 0$, which implies the second compatibility condition

$$\mu_{i,j-1}(t) = F_{i,j}(t)/(F_{i,j}(t) - F_{i,j-1}(t)), \qquad \mu_{i,j}(t) = F_{i,j-1}(t)/(F_{i,j-1}(t) - F_{i,j}(t)), \tag{17}$$

where $F_{i,k}(t) = f_{i,k}(y(t), \phi_{i,k}(y(t)))$.

**Definition 3.11.** *Given an initial state $x$ and a strategy profile $\phi_{-i}$ of the remaining players, a* **Markovian control schedule** *$c_i = (a_i, \mu_i)$ induced by a strategy $\phi_i$ is a control schedule such that for every trajectory–control pair $(y, c_i^\phi)$ with $y(0) = x$ equations* (16) *and* (17) *are satisfied almost everywhere.*

*The set of Markovian control schedules for player $i$ induced by $\phi_i$ is denoted $\mathcal{MC}_{x,\phi_{-i},\phi_i}$.*

In particular, a strategy profile $\phi$ and an initial state $x$ uniquely specify a trajectory, except if there are $i$ and $j$ such that for $x \in \mathcal{I}_j$ we have $f_{i,j-1}(x, \phi_{i,j-1}(x)) \leq 0$ and $f_{i,j}(x, \phi_{i,j}(x)) \geq 0$, with one of the inequalities strict. At such points, there are infinitely many trajectories that remain at $x$ for an initial time interval of positive length, before moving either to the right or to the left. Even restricting to regular trajectories does not fully eliminate the multiplicity: if both inequalities are strict, there is one regular trajectory that moves immediately to the right, and another that moves immediately to the left.

We now define player $i$'s payoffs and optimal trajectories when restricted to Markovian control schedules.

**Definition 3.12.** *Given an initial state $x$ and a strategy profile $\phi_{-i}$ of the remaining players, the* **utility** *of a strategy $\phi_i$ to player $i$ is*

$$V_i^\phi(x) = \sup_{\mathcal{MC}_{x,\phi_{-i},\phi_i}} \sup_{\mathcal{T}_{x,\phi_{-i},c_i}} U(y, c_i). \tag{18}$$

Player $i$'s best response is Markovian if, for any initial state $x \in \mathcal{X}$, the player cannot do better than choose a Markovian control schedule:

**Definition 3.13.** *A* **Markovian best response** *by player $i$ to a strategy profile $\phi_{-i}$ is a strategy $\phi_i$ such that, for all $x \in \mathcal{X}$, $V_i(x) = V_i^\phi(x)$.*

*Given an initial state $x$, a strategy profile $\phi_{-i}$, and a Markovian best response $\phi_i$, let $c_i$ be the control schedule induced by $\phi_i$. A* **Markovian best response trajectory** *for player $i$ is a trajectory $y$ with $y(0) = x$ such that the pair $(y, c_i)$ realises the supremum in* (18). *The set of Markovian best response trajectories for player $i$ is denoted by $\mathcal{MT}^*_{i,x,\phi_{-i},\phi_i}$.*

Finally, we define the game and our equilibrium concept.

**Definition 3.14.** *The tuple $\Gamma = (N, \mathcal{X}, \mathcal{Q}, g, v, \rho)$ defines a* **differential game**. *If $g_i = g_k$ and $v_i = v_k$ for all $1 \leq i, k \leq N$, the game is* **symmetric**.

**Definition 3.15.** *A* **stationary Markov-perfect Nash equilibrium**, *or MPE, of the differential game $\Gamma$ is a strategy profile $\phi \in \mathscr{S}^N$ such that, first, for any player index $i \in I$, the strategy $\phi_i$ is a Markovian best response to $\phi_{-i}$ and, second, the set of* **Markovian equilibrium trajectories** *$\mathcal{T}^*_x = \bigcap_{i=1}^N \mathcal{T}^*_{i,x,\phi_{-i}} \cap \mathcal{MT}^*_{i,x,\phi_{-i},\phi_i}$ is nonempty for every $x \in \mathcal{X}$.*

*An MPE is* **continuous** *if all $\phi_i$ are continuous; otherwise it is* **discontinuous**. *A* **symmetric** *MPE is an MPE of a symmetric differential game such that $\phi_i = \phi_j$ for all $i, j \in I$.*

The definition is standard and requires that, starting from any state $x$ in the state domain, each player's strategy is a best response to the other players' strategies.

It is conceivable that the set of equilibrium trajectories $\mathcal{T}^*_x$ may contain multiple elements. This gives rise to a problem of trajectory selection, akin to equilibrium selection: different players could choose different trajectories that are consistent with the same strategy profile $\phi$. We sidestep this question by assuming that the players are able to coordinate on a joint best response trajectory.

## 4 Results

This section states the main results of our article.

**Theorem 4.1.** *Let a differential game* $\Gamma$ *be given for which Assumptions* $(\mathrm{A}_1)$ *and* $(\mathrm{A}_2)$ *hold. Then the Markovian best response mapping* $\mathscr{B}_i : \mathscr{S}^{N-1} \to \mathscr{S}$ *is well-defined: for every strategy profile* $\phi_{-i}$*, there is exactly one Markovian best response* $\phi_i$ *by player* $i$*, and this best response is an element of* $\mathscr{S}$*.*

The theorem is proved in Appendices A–D. Appendix A shows that the value function $V_i$ of player $i$ is non-increasing. Appendix B introduces the notion of viscosity solution to the Hamilton–Jacobi–Bellman equation of player $i$. In Appendix C, we show first that the value function satisfies a number of additional properties: it is left continuous and continuous everywhere except for a finite number of points which we characterise. Then we show that the value function is the unique viscosity solution to that equation in the class of functions with the above properties. From this, we derive in Appendix D that $V_i$ is differentiable almost everywhere, and, using centre manifold theory, we strengthen this to piecewise analytic. From this the result easily follows.

The theorem shows that our specification of a differential game, and the Markovian strategy space $\mathscr{S}$, is well-formed in the sense that each player will have a best response in $\mathscr{S}$ to any profile of the other players' strategies $\phi_{-i} \in \mathscr{S}^{N-1}$. Note that this property does not hold when the Markovian strategies are required to be Lipschitz-continuous: it is straightforward to construct Lipschitz-continuous strategy profiles $\phi_{-i}$ such that the best response $\phi_i$ has a Skiba point and is not continuous (Wagener, 2003). Our specification thus connects the literature on differential games to the standard notion of normal-form games and (pure-strategy) equilibria, with the 'actions' being policy rules $\phi_i$. The payoffs for each player are then defined for all initial states $x$—this is just the value function— and, for any $i$, a strategy is preferred if it yields a higher payoff for every initial state.

Another implication of the theorem is that, while we have to set up in Section 3.2 the rather convoluted technical apparatus for dealing with potentially non-Markovian best responses, ultimately the best responses turn out to be Markovian, so that for applications it suffices to rely on the far simpler Markovian best responses and the associated Filippov dynamics (Definition 3.1).

As the best response is piecewise analytic, it can be characterised by classical conditions in the regions of analyticity, and by 'pasting' conditions at the interfaces. These conditions are formulated in terms of the utility $V_i^\phi$ of strategy $\phi_i$ to player $i$. Recall that a function is piecewise real analytic on a set $\mathcal{X}$, if there are open intervals $\hat{\mathcal{X}}_j$ such that $\mathcal{X} \setminus \bigcup_j \hat{\mathcal{X}}_j$ is a finite set and the function is real analytic on each $\hat{\mathcal{X}}_j$. Introduce the notation

$$f_{i,j}^\phi(x) = f_{i,j}(x, \phi_{i,j}(x)).$$

Then:

**Proposition 4.2.** *If* $\phi \in \mathscr{S}^N$*, then for each* $j$ *the function* $V_i^\phi$ *is bounded and real analytic at all but finitely many points of* $\mathcal{X}_j$*, and at these points* $f_{i,j}^\phi(x) = 0$*.*

The proof of this proposition is given in Appendix E.

The second main result gives necessary and sufficient conditions for $\phi_i$ to be the best response $\mathscr{B}(\phi_{-i})$.

**Theorem 4.3.** *Let the same conditions hold as for Theorem 4.1, and let* $\{\mathcal{X}_j\}_{j=0}^J$ *with* $\mathcal{X}_j = [x_j, x_{j+1}]$ *a covering of* $\mathcal{X}$ *adapted to* $\phi = (\phi_i, \phi_{-i})$*.*

*We have that* $\phi_i = \mathscr{B}(\phi_{-i})$ *if and only if the following hold.*

*a. Maximum principle: If $x \in \operatorname{int} \mathcal{X}_j$ and $V_i^\phi$ is differentiable at $x$, then $\phi_i(x)$ maximises*

$$q_i \mapsto u_i(x, q_i) + (V_i^\phi)'(x) f(x, q_i, \phi_{-i}(x)) \quad \text{on } \mathcal{Q}.$$

*b. $V_i^\phi$ is non-increasing.*

*c. Boundary values: If $x = \bar{x}_0$, then either $V_i^\phi(x) = \beta_i(x)$ or $f_{i,0}^\phi(x) \geq 0$.*

*Likewise, if $x = \bar{x}_{J+1}$, then either $V_i^\phi(x) = \beta_i(x)$ or $f_{i,J+1}^\phi(x) \leq 0$.*

*d. Value discontinuities: If $V_i^\phi$ is not continuous at $x$, then $x = \bar{x}_j$ for some $j \in \{1, \ldots, n\}$, $f_{i,j-1}^\phi(x) \leq 0$ and $f_{i,j}(x, q_\ell) \geq 0$.*

*e. Value at Filippov steady states: If $x = \bar{x}_j$ and $V_i^\phi$ is continuous at $x$, then*

$$\rho V_i^\phi(x) \geq \max\{v_i(x, q_i, \lambda_i) \,:\, g_i(x, q_i, \lambda_i) = 0\}. \tag{19}$$

*f. Strong push–push steady state: If $x \in \operatorname{int} \mathcal{X}$ is such that*

$$\lim_{z \uparrow x} f_{i,j-1}^\phi(z) > 0 > \lim_{z \downarrow x} f_{i,j}^\phi(z),$$

*then $V_i^\phi$ is differentiable at $x$.*

The proof again uses viscosity theory, and is detailed in Appendix E. It is based on the fact that $V_i^\phi$ is by construction a viscosity subsolution of the Hamilton–Jacobi–Bellman equation of the optimisation problem of agent $i$. The conditions are equivalent to $V_i^\phi$ being also a viscosity supersolution; hence it is a viscosity solution, and as we have shown already that the viscosity solution to the HJB equation is unique, it follows that $V_i^\phi = V_i$, and consequently that $\phi_i$ is a best response.

The conditions can be interpreted. Condition a is standard. Condition b follows from the fact that the stock is a public bad, and says that there are no strategic incentives so perverse as to make the stock locally a 'good' for player $i$. Suppose this were the case and, for intuition, consider flow felicity functions without a bliss point in terms of the control variable. Player $i$ would set the maximal emission rate to grow the stock as fast as they can, at least until the value peaks. But then their flow utility will have been decreasing, as emissions have been constant but damages from the stock have been increasing.

Condition c just says that, on the edge of the state space, a player can always exit and take the associated payoff.

The remaining conditions place restrictions on the best response where the other players' dynamics are discontinuous. Condition d says that a discontinuity in value is only possible at points where the other players' strategies are discontinuous, in such a way that player $i$ is unable to control the dynamics back to the region of low stock if they ever end up on the high side of the discontinuity. Condition b then implies the value can only have a downward (not upward) discontinuity at such a point.

Condition e says that, at discontinuities at which the value is continuous, the player always gets at least the highest payoff consistent with stabilisation at that point. Note that strict concavity of utility implies that it is never optimal to stabilise with the left and right controls taking different values, while the weight is interior—so that $\lambda_i \in (0, 1)$; it is always better to play a 'pure' action, than to get a weighted sum of two actions. Thus, chattering never happens in equilibrium.

Finally, condition f follows from the fact that value is continuous. If a player's best response is to be pushed strictly towards a stabilisation point, they end up at the same point whether approaching from the left or the right, and very close to the stabilisation point the continuity of the payoffs implies that the marginal value of the stock is the same.

# Appendix

The idea of the proof of Theorem 4.3 is to show that the HJB equation of player $i$ has the player's value function as unique viscosity solution. The necessary conditions placed on player $i$'s strategy $\phi_i$ then ensure that the function $V_{i,\phi_i}$, constructed from $\phi_i$, is a viscosity solution of the HJB equation, and therefore equals the value function. This is then used to establish that $\phi_i$ is a Markovian best response.

The principal complication springs from the fact that the dynamics $g_i$ facing player $i$ are potentially discontinuous. To prove that the value function of player $i$ is the unique solution to the HJB equation, we apply the results of Barles et al. (2013, 2014).

Before this is possible, two complications have to be dealt with: in general optimal control problems with discontinuous dynamics, the HJB equation does not have unique viscosity solutions. We shall show that in our situation, uniqueness is a consequence of the stock being a public bad, that is, it follows from the assumption $(u_i)_x < 0$.

The second complication is that in general, the dynamics are not controllable at the discontinuities, but that the results of Barles et al. (2013, 2014) are derived under the assumption of controllability. We therefore have to show how these results can be extended to the non-controllable situation.

We also show that the function $V_{i,\phi_i}$ is a solution of the HJB equation assuming that all interfaces are controllable, and finally we show how the full result can be obtained if some of the interfaces fail to be controllable.

# A Singular and regular value functions are identical

In this section, we shall show first that the solution to the HJB equation with controllable interfaces is unique. Then we show that the value function is decreasing.

## A.1 Notation

As this section treats the dynamic optimisation problem of a single player, given the strategic choices of the other players, we drop the index $i$ throughout.

We recall the notation $\mathcal{I} = \bigcup_{j=1}^{J} \mathcal{I}_j$ for the set of interface points. We introduce $\bar{x}_j$ such that $\mathcal{I}_j = \{\bar{x}_j\}$, as well as

$$q^{\max}(x) = \operatorname{argmax}_{q\in\mathcal{Q}} u(x,q) \quad \text{and} \quad w^{\max}(x) = u(x, q^{\max}(x))/\rho. \tag{20}$$

If $S \subset \mathbb{R}$ is a measurable subset, $|S|$ denotes the Lebesgue measure of $S$. For continuous functions $g : \mathcal{X} \times \mathcal{Q}$, we introduce $\|g\|_\infty = \max_{(x,q)\in\mathcal{X}\times\mathcal{Q}} |g(x,q)|$.

## A.2 Auxiliary results

We begin by stating an existence result for optimal controls in our context, which is a direct corollary of Fleming and Rishel (1975, Theorem 4.1).

**Theorem A.1.** *Let $t > 0$, $S \subset \mathcal{X}$ compact, and $g : S \to \mathbb{R}$ continuous. If there exists a trajectory–control pair $(y, c)$ such that $y(0) = x$ and $y(t) \in S$, then there is also a trajectory–control pair $(y^*, c^*)$ that minimises*

$$\int_0^t \exp(-\rho s) u(y(s), c(s)) \, \mathrm{d}s + g(y(t)) \exp(-\rho t).$$

## A.3 Equality of singular and regular value

A sufficient condition for uniqueness of solutions to the HJB equation is equality of value and regular value function $V = V^{\mathrm{reg}}$ (Barles et al., 2013, 2014).

Our first result is a consequence of the assumption $u_x < 0$: if two trajectories are generated by the same emission strategy, but one has a smaller initial pollution level, then that strategy is preferable.

**Lemma A.2.** *Let $(y, c)$ and $(\tilde{y}, c)$ be two trajectory–control pairs, with respective initial points $x$ and $\tilde{x}$, that are generated by the same control schedule $c$.*

*If $\tilde{y}(t) \leq y(t)$ for all $t \geq 0$, then $U(\tilde{y}, c) \geq U(y, c)$.*

*Proof.* This follows directly from the fact that $u_i(x, q_i)$ is non-increasing in $x$. □

We do not know a priori whether a given control schedule generates a regular trajectory. The next result therefore only states that the value function is decreasing, but gives no information about the regular value function.

**Proposition A.3.** *The value function is decreasing.*

*Proof.* Take $x \in \mathcal{X}$ and $\varepsilon > 0$, and find a trajectory–control pair $(y, c)$ with initial state $x$ such that

$$U(y, c) \geq V(x) - \varepsilon.$$

Proposition 3.8 implies that for $\tilde{x} < x$ there is a trajectory–control pair $(\tilde{y}, c)$, such that $\tilde{y}(0) = \tilde{x}$.

Set $\tau = \min\{t > 0 : \tilde{y}(t) = y(t)\}$; if the set is empty, we set $\tau = \infty$. Modify $(\tilde{y}(t), c(t))$ by setting it equal to $(y(t), c(t))$ if $t \geq \tau$. The modified trajectory–control pair still satisfies $\tilde{y}(t) \leq y(t)$ for all $t \geq 0$. Lemma A.2 then implies that $U(\tilde{y}, c) \geq U(y, c)$.

Since $V(\tilde{x}) \geq U(\tilde{y}, c)$, we infer

$$V(x) - V(\tilde{x}) \leq U(y, c) + \varepsilon - U(\tilde{y}, c) \leq \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the result follows. □

To show equality of value and regular value, we are going to show that for every trajectory–control pair there is a regular trajectory–control pair that generates an outcome that is at least as good. First we show that such a regular trajectory either almost never is at its initial state, or it is there always.

**Lemma A.4.** *Let $(y, c)$ be a trajectory–control pair with initial state $x$. Then there is a trajectory–control pair $(\tilde{y}, \tilde{c})$ such that either $\tilde{y}(t) > x$ for almost all $t$, or $\tilde{y}(t) < x$ for almost all $t$, or $\tilde{y}(t) = x$ and $\tilde{c}(t)$ constant for all $t$, and $U(\tilde{y}, \tilde{c}) \geq U(y, c)$.*

The proof is given in Section F.1.

If a trajectory–control pair is always at an interface steady state where the regularity condition (13) is not satisfied, that is, at a 'pull-pull' steady state in the terminology of Barles et al. (2013), there is a second trajectory–control pair with the same control and the same initial condition such that the trajectory is always to the left of that steady state, and such that the pair has a higher total payoff. This is the heart of the following result.

**Lemma A.5.** *For every non-regular trajectory–control pair there is a regular trajectory–control pair with a higher total payoff.*

The proof consists in replacing all singular pull–pull trajectory segments by regular trajectories going to the left. The details are given in Section F.2.

Equality of value function and regular value function is now a simple corollary of the previous results.

**Proposition A.6.** *The value function $V$ and the regular value function $V^{\mathrm{reg}}$ are equal.*

*Proof.* Clearly, $V(x) \geq V^{\mathrm{reg}}(x)$ for all $x \in \mathcal{X}$. To show the opposite inequality, take $x \in \mathcal{X}$ and $\varepsilon > 0$, and find a trajectory–control pair $(y, c)$ such that $y(0) = x$ and $V(x) \leq U(y, c) + \varepsilon$. By Lemma A.5 there is a regular trajectory–control pair $(\tilde{y}, \tilde{c})$ such that $\tilde{y}(0) = x$ and $U(\tilde{y}, \tilde{c}) \geq U(y, c)$. We have that

$$V(x) \leq U(y, c) + \varepsilon \leq U(\tilde{y}, \tilde{c}) + \varepsilon \leq V^{\mathrm{reg}}(x) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude that $V(x) \leq V^{\mathrm{reg}}(x)$. $\square$

A direct consequence of Propositions A.3 and A.6 is

**Proposition A.7.** *The regular value function $V$ is decreasing.*

# B Viscosity solutions

Assume that $(y, c)$ is an optimal trajectory–control pair. There are two typical scenarios of what happens if the trajectory reaches the boundary of the state space: either it leaves the space immediately, or it never leaves. This can be understood in terms of the state-constrained value function $V^{\mathrm{sc}}$, which is the value if the trajectory is constrained to be in the state space for all time. If for all boundary points $x$ we have $V^{\mathrm{sc}}(x) > \beta(x)$, then leaving is never optimal; if however $V^{\mathrm{sc}}(x) < \beta(x)$ and leaving is possible at all, it is optimal to leave immediately. The intermediate situation is of course that $V^{\mathrm{sc}}(x) = \beta(x)$, where the player is indifferent between leaving and not leaving. In the first situation $V(x) = V^{\mathrm{sc}}(x)$; in the second $V(x) = \beta(x)$.

The value function of the optimisation problem can be characterised as the unique viscosity solution of a HJB equation with a boundary condition that reflects the two possible scenarios. The first half of the condition, $V(x) = \beta(x)$, is natural; the second half is a consequence of the fact that state constrained value functions are viscosity solutions of the HJB equation also at the boundary points.

In order to write down the HJB equation in our context, we have to introduce the Hamiltonian and the notion of viscosity solution.

For $x \in \mathcal{X}_j$ and $p \in \mathbb{R}$, introduce the *local Hamiltonian*

$$H_j(x, p) = \max_{q \in \mathcal{Q}} \left[ u(x, q) + p f_j(x, q) \right].$$

For an interface point $\bar{x}_j$, let the set of controls that stabilise $\bar{x}_j$ be

$$C_{0,j} = \{(q_{j-1}, q_j, \lambda_{j-1}, \lambda_j) \,:\, \lambda_{j-1} + \lambda_j = 1,\ \lambda_{j-1} f_{j-1}(\bar{x}_j, q_{j-1}) + \lambda_j f_j(\bar{x}_j, q_j) = 0\}.$$

The *interface Hamiltonian* is then given as

$$H_j^d(x) = \max_{(q_{j-1}, q_j, \lambda_j) \in C_{0,j}} \left[\lambda_j u(x, q_{j-1}) + (1-\lambda_j) u(x, q_j)\right].$$

We set $H_j^d(x) = -\infty$ if the set $C_{0,j}$ is empty.

The *Hamiltonian* of the optimisation problem is then

$$H(x,p) = \begin{cases} H_j(x,p), & \text{if } x \in \operatorname{int} \mathcal{X}_j, \\ H_j^d(x), & \text{if } x \in \mathcal{I}_j. \end{cases}$$

To formulate the definition of viscosity solution, we need a number of concepts. For a function $W : \mathcal{X} \to \mathbb{R}$, the *upper semi-continuous envelope* $W^*$ is the function that satisfies

$$W^*(x) = \limsup_{\substack{z \to x \\ z \in \mathcal{X}}} W(z).$$

The *lower semi-continuous envelope* $W_*$ is defined in the same way, with lim inf replacing lim sup.

The *superdifferential* $D^+V(x)$ of a bounded upper semicontinuous function $V$ defined on $\mathcal{X}$ at a point $x$ is the set

$$D^+V(x) = \left\{ p \in \mathbb{R} \,:\, \limsup_{\substack{z \to x \\ z \in \mathcal{X}}} \frac{u(z) - u(x) - p(x-z)}{|x-z|} \le 0 \right\}.$$

The *subdifferential* $D^-V(x)$ of a bounded lower semicontinuous function $V$ on $\mathcal{X}$ at $x$ is defined similarly, with sup replaced by inf and $\le$ by $\ge$ (see e.g. Bardi and Capuzzo-Dolcetta, 2008, Chapter V). We have that $p \in D^+V(x)$ if and only if there is a continuously differentiable function $\psi$ such that $\psi'(x) = p$ and $V - \psi$ restricted to $\mathcal{X}$ has a local maximum at $x$. An analogous characterisation exists for subdifferentials.

**Definition B.1.** *The function $V$ is a* **viscosity supersolution** *(respectively, a* **viscosity subsolution***) of the discontinuous HJB equation*

$$\rho V - H(x, V') = 0 \qquad \text{in } \operatorname{int} \mathcal{X}, \tag{21a}$$

$$\rho V - H(x, V') = 0 \quad \text{or} \quad V = \beta \qquad \text{on } \partial\mathcal{X}, \tag{21b}$$

*if for all $j \in \mathcal{J}$, all $x \in \operatorname{int} \mathcal{X}_j$, and all $p \in D^-V_*(x)$ (respectively, all $p \in D^+V^*(x)$), we have*

$$\rho V_*(x) - H_j(x,p) \ge 0, \tag{22a}$$

$$(\text{respectively, } \rho V^*(x) - H_j(x,p) \le 0); \tag{22b}$$

*for all $j \in \mathcal{J} \backslash \{0\}$, $x \in \mathcal{I}_j$ and all $p \in D^-V_*(x)$ (respectively, all $p \in D^+V^*(x)$), we have*

$$\rho V_*(x) - \min\{H_{j-1}(x,p), H_j(x,p)\} \ge 0, \tag{23a}$$

$$(\text{respectively, } \rho V^*(x) - \max\{H_{j-1}(x,p), H_j(x,p)\} \le 0); \tag{23b}$$

*for all $j \in \{0, J\}$, $x \in \mathcal{X}_j \cap \partial\mathcal{X}$ we have that $V$ is continuous at $x$, and for all $p \in D^-V(x)$ (respectively,* *all $p \in D^+V(x)$), we have that*

$$V(x) - \beta(x) \geq 0 \quad \text{or} \quad \rho V(x) - H_j(x,p) \geq 0, \tag{24a}$$

$$(\text{respectively, } V(x) - \beta(x) \leq 0 \quad \text{or} \quad \rho V(x) - H_j(x,p) \leq 0), \tag{24b}$$

*and if for all $j \in \mathcal{J}\backslash\{0\}$ and all $x \in \mathcal{I}_j$, we have*

$$\rho V_*(x) - H_j^d(x) \geq 0. \tag{25}$$

*Finally, $V$ is a* **viscosity solution** *of* (21) *if it is both a viscosity supersolution and a viscosity subsolution.*

Note that (25) is only a condition for being a supersolution, not a subsolution.

**Theorem B.2.** *The value function $V$ is a viscosity solution of the discontinuous HJB equation* (21).

*Proof.* As the statement is local, the proof is a combination of known results. See Proposition III.2.8 of Bardi and Capuzzo-Dolcetta (2008) for (22); Theorem 2.5 of Barles et al. (2013) for (23) and (25); and Theorem V.4.13 of Bardi and Capuzzo-Dolcetta (2008) for (24). □

# C Necessary and sufficient conditions

## C.1 The controllable case

A central assumption of Barles et al. (2014) is controllability of the dynamics at the interfaces: in our context, we have the following definition.

**Definition C.1.** *The dynamics are* **left (right) controllable** *at an interface point $\bar{x}_j$, if there is an open interval $I$ containing $0$ such that*

$$I \subset f_{j-1}(\bar{x}_j, \mathcal{Q}) \qquad \Big(I \subset f_j(\bar{x}_j, \mathcal{Q})\Big).$$

*If the dynamics are both left and right controllable at $\bar{x}_j$, they are* **controllable**.

The following result is formulated in Barles et al. (2014) for a finite horizon problem without boundary conditions, but using Theorem V.4.17 of Bardi and Capuzzo-Dolcetta (2008) it can be adapted without difficulty to our infinite horizon context with boundary conditions (cf. Barles et al., 2014, Section 6).

**Theorem C.2.** *Assume for every $j \in \{0, \ldots, J\}$, we have that*

*a. the state dynamics $f_j(x, q_j)$ are bounded and satisfy a Lipschitz condition in $x$, uniformly with respect to $q_j$;*

*b. the felicity function $u(x, q_j)$ is uniformly continuous, bounded, and concave in $q_j$;*

*c. the dynamics are controllable at every interface.*

*Then the value function $V$ is continuous and is the unique viscosity solution of the discontinuous HJB equation* (21).

In our context we have that the value function and the regular value functions are equal, and therefore the regular value function is also equal to the unique viscosity solution of the discontinuous HJB equation (21).

### C.2 The non-controllable case

If the dynamics are not controllable at $\bar{x}_j \in \mathcal{I}_j$, we introduce the closed sets $\mathcal{X}_{j,-} = \{x \in \mathcal{X} : x \leq \bar{x}_j\}$ and $\mathcal{X}_{j,+} = \{x \in \mathcal{X} : x \geq \bar{x}_j\}$, and we decompose into two coupled optimisation problems on $\mathcal{X}_{j,-}$ and $\mathcal{X}_{j,+}$ respectively. The situation for more than one non-controllable interface is easily reduced to this situation.

Following Barles et al. (2014), we introduce the state-constrained value function $V_-^{\mathrm{sc}}$ and $V_+^{\mathrm{sc}}$ for, respectively, the optimisation problems with the state restriction $y(t) \in \mathcal{X}_{j,-}$, respectively $y(t) \in \mathcal{X}_{j,+}$, that hold for all $t$.

Additionally, we introduce $V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j)$ for the state-constrained value function of the optimisation problem with the restriction $y(t) = \bar{x}_j$ for all $t$. Here only controls of the form $f_{\mathcal{I}}(x,c) = 0$ are admitted. Finally, we set $V^{\mathrm{sc}}{}_{,-,\mathcal{I},+}(x) = -\infty$ if there is no trajectory starting at $x$ that satisfies the particular state constraint.

The following result is fundamental for investigating the continuity of the value function at non-controllable interfaces. To formulate it, we introduce the notions of one-sided semi-repellers and semi-attractors.

**Definition C.3.** *An interface point $\bar{x}_j \in \mathcal{I}$ is a* **left semi-repeller** *if $f_{j-1}(\bar{x}_j, q_u) \leq 0$, and a* **right semi-repeller** *if $f_j(\bar{x}_j, q_\ell) \geq 0$.*

*Likewise, $\bar{x}_j$ is a* **left semi-attractor** *if $f_{j-1}(\bar{x}_j, q_\ell) \geq 0$, and a* **right semi-attractor** *if $f_j(\bar{x}_j, q_u) \leq 0$.*

**Proposition C.4.** *Let $\bar{x}_j \in \mathcal{I}$.*

*a.* $V(\bar{x}_j) = \max\{V_-^{\mathrm{sc}}(\bar{x}_j), V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j), V_+^{\mathrm{sc}}(\bar{x}_j)\}$.

*b. If the dynamics are left (right) controllable at $\bar{x}_j$, then $V$ is left (right) Lipschitz continuous in a left (right) neighbourhood of $\bar{x}_j$.*

*c. If $\bar{x}_j$ is a right semi-attractor, then $V$ is right continuous at $\bar{x}_j$ and*

$$V(\bar{x}_j) = \max\{V_-^{\mathrm{sc}}(\bar{x}_j), V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j)\}.$$

*d. If $\bar{x}_j$ is a left semi-attractor, then $V$ is left continuous at $\bar{x}_j$. Moreover, we have either*

$$V(\bar{x}_j) = \max\{V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j), V_+^{\mathrm{sc}}(\bar{x}_j)\}.$$

*or*

$$V(\bar{x}_j) = V_-^{\mathrm{sc}}(\bar{x}_j) > \max\{V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j), V_+^{\mathrm{sc}}(\bar{x}_j)\} \quad \text{and} \quad f_{j-1}(\bar{x}_j, q_\ell) = 0.$$

*e. If $\bar{x}_j$ is a left semi-repeller, then $V$ is left continuous at $\bar{x}_j$ and*

$$V(\bar{x}_j) = V_-^{\mathrm{sc}}(\bar{x}_j).$$

*f. The value function $V$ is not continuous at $\bar{x}_j$ if and only if $\bar{x}_j$ is a right semi-repeller and*

$$V_+^{\mathrm{sc}}(\bar{x}_j) < \max\{V_-^{\mathrm{sc}}(\bar{x}_j), V_{\mathcal{I}}^{\mathrm{sc}}(\bar{x}_j)\}.$$

The asymmetry in the result is a consequence of the fact that $u_x < 0$. The proof of this result is given in Appendix F.3.

The result motivates the following definition.

**Definition C.5.** *Let $\mathscr{X}$ be a covering of $\mathcal{X}$, and let $f$ be a dynamics defined on $\mathcal{X}$. The class $\mathscr{G}$ consists of functions $W : \mathcal{X} \to \mathbb{R}$ such that*

*a. $W$ is non-increasing;*

*b. $W$ is left continuous everywhere;*

*c. $W$ is continuous on* $\operatorname{int} \mathcal{X}_j$ *for all $j \in \mathfrak{J}$;*

*d. if $W$ is not continuous at $\bar{x}_j$, then this point is a right semi-repeller under $f$.*

The following proposition is a direct consequence of Proposition C.4.

**Proposition C.6.** *The value function $V$ is in the class $\mathscr{G}$.*

We shall prove an extension of Theorem C.2 to a situation where the dynamics is not necessarily controllable at interfaces. In order to this, we break up the control problem into a number of smaller problems that are related at their boundaries.

**Theorem C.7.** *The value function $V$ is the unique viscosity solution of the discontinuous HJB equation* (21) *in the class $\mathscr{G}$.*

The proof of this theorem is given in Appendix F.4.

# D Optimal Markovian controls

This section constructs the *best response map* $\mathscr{B}_i : \mathscr{S}^{N-1} \to \mathscr{S}$, which gives the strategy $\phi_i = \mathscr{B}_i(\phi_{-i})$ that is the Markovian best response of player $i$ to the strategy profile $\phi_{-i}$ of the other players. That is, given the profile $\phi_{-i}$, the best response is the feedback control that maximises the total payoff $U_i$, given the dynamics $g_i$, for every initial state.

In the construction of the best response, the value function plays a central, if auxiliary, part. The first and main step of the the construction is to show that the value function is differentiable almost everywhere. This is straightforward in the domain where the dynamics are controllable, or where the Hamilton–Jacobi equation can be written as an explicit differential equation, or again in the interior of the region where the optimal action is an extreme value of the space of actions. To show the same for the boundary of this region, which may have positive measure, calls for a number of technical arguments.

The second step of the construction improves the regularity of the value function: it first shows that the points of non-differentiability of the value function are, in fact, isolated, and correspond to so-called 'indifference' or 'Skiba' points. Also the points at which the value function is differentiable, but not analytic, are shown to be isolated. Finally, in the complement of this set, the optimal Markovian control is derived in the classical manner.

## D.1 Notations

If $\phi_{-i} \in (\mathscr{S})^{N-1}$, then the local dynamics

$$f_{i,j}(x, q_{i,j}) = f(x, \phi_{1,j}(x), \ldots, q_{i,j}, \ldots, \phi_{N,j}(x))$$

are also analytic. Having made this remark, since we focus on the optimisation problem of a single player, we drop the index $i$ throughout this section. Also, in this section we work in a fixed interval int $\mathcal{X}_j$. We therefore fix $j \in \mathcal{J}$ as well and drop this index for the sake of readability. For instance, in this section $f(x,q)$ stands for $f_{i,j}(x,q_{i,j})$, etc.

As a consequence of Assumption ($\mathrm{A}_2$), the function $P(x,p,q) = u(x,q) + pf(x,q)$ has a unique maximiser $q^*(x,p)$ in $\mathcal{Q}$.

We introduce a number of auxiliary quantities. The functions $p_\ell, p_u : \mathcal{X} \to \mathbb{R}$ are given as

$$p_\ell(x) = -u_q(x,q_\ell)/f_q(x,q_\ell), \quad p_u(x) = -u_q(x,q_u)/f_q(x,q_u).$$

In terms of these, the sets $\mathcal{P}_u$, $\mathcal{P}_\ell$, and $\mathcal{P}_I$ are defined as

$$\mathcal{P}_\ell = \{(x,p) \,:\, p \le p_\ell(x)\}, \quad \mathcal{P}_u = \{(x,p) \,:\, p \ge p_u(x)\},$$

and

$$\mathcal{P}_I = \{(x,p) \,:\, p_\ell(x) < p < p_u(x)\}.$$

The boundaries of these sets are denoted as

$$\mathcal{S}_\ell = \{(x,p) \,:\, p = p_\ell(x)\}, \quad \mathcal{S}_u = \{(x,p) \,:\, p = p_u(x)\}, \quad \mathcal{S} = \mathcal{S}_\ell \cup \mathcal{S}_u.$$

The maximiser $q^*(x,p)$ of $P(x,p,q)$ equals $q_\ell$ if $(x,p) \in \mathcal{P}_\ell$ and $q_u$ if $(x,p) \in \mathcal{P}_u$.

**Definition D.1.** *For $c \ge 0$, let*

$$\mathcal{X}_L(c) = \{x \in \operatorname{int}\mathcal{X} \,:\, H_p(x,p) < -c \textit{ if } p \le p_\ell(x) \textit{ and } H_p(x,p) > c \textit{ if } p \ge p_u(x)\}.$$

*Let $\mathcal{X}_M$ be the set of $x_0 \in \operatorname{int}\mathcal{X}$ such that the equation $w = H(x,p)$ has a unique solution $p = \chi(x,w)$ for $(x,w)$ close to $(x_0,w_0) = (x_0,\rho V(x_0))$ and $H_p(x_0,\chi(x_0,w_0)) \neq 0$.*

*The set $\mathcal{X}_N$ contains the points $x \in \operatorname{int}\mathcal{X}$ for which there is $(x,p) \in \mathcal{P}_\ell$ or $(x,p) \in \mathcal{P}_u$ such that $\rho V(x) = H(x,p)$ and $H_p(x,p) = 0$.*

The triple $\mathcal{X}_L(0)$, $\mathcal{X}_M$ and $\mathcal{X}_N$ is a partition of $\mathcal{X}$: if $x \in X_L(0)$, the expression $H(x,p)$ is non-monotonic as a function of $p$, while if $x \in \mathcal{X}_M \cup \mathcal{X}_N$ it is weakly monotonic; if $x \in \mathcal{X}_M$, the equation $w = H(x,p)$ uniquely determines $p$; if $x \in \mathcal{X}_N$, it does not.

Finally, we write the *boundary* of a set $S$ as $\partial S = \bar{S} \setminus \operatorname{int} S$.

## D.2 Differentiability of the value function in a dense set

**Proposition D.2.** *The value function $V$ is analytic on $\mathcal{X}_M \cup \operatorname{int}\mathcal{X}_N$, and it is differentiable almost everywhere on $\mathcal{X}_L(0)$.*

*Proof.* As $V$ is continuous in $\operatorname{int}\mathcal{X}$ and a supersolution of the HJB equation, for $x \in \mathcal{X}_L(c)$ and $p \in D^-V(x)$, we have

$$\rho V(x) \ge H(x,p) \ge u(x,q_u) + pf(x,q_u) \ge u(x,q_u) + cp,$$

which implies the upper bound $p \le (\rho V(x) - u(x,q_u))/c$.

As $V$ is decreasing, we have for every $p \in D^+V(x)$ that $p \le 0$ (Bardi and Capuzzo-Dolcetta, 2008, Lemma II.5.15). For $x \in \mathcal{X}_L(c)$ and $p \in D^+V(x)$, the subsolution property of $V$ and the fact that

$f(x,q)$ is increasing in $q$ imply

$$\begin{aligned}\rho V(x) \leq H(x,p) &= u(x,q^*(x,p)) + pf(x,q^*(x,p)) \\ &\leq u(x,q^*(x,p)) + pf(x,q_\ell) \leq u(x,q^*(x,p)) - cp,\end{aligned}$$

and consequently the lower bound $p \geq (\rho V(x) - u(x,q^*(x,p)))/c$.

Finally, as $|V(x)| \leq \|u\|/\rho$ for all $x$, we have

$$|p| \leq (\rho V(x) - \min\{u(x,q_u), u(x,q^*(x,p))\})/c \leq 2\|u\|_\infty/c. \tag{26}$$

Lemma II.5.15 of Bardi and Capuzzo-Dolcetta (2008) implies that $V$ is Lipschitz continuous on $\mathcal{X}_L(c)$ with Lipschitz constant $2\|u\|_\infty/c$; Rademacher's theorem (Clarke et al., 1998, Chapter 3, Corollary 4.19) subsequently ensures almost everywhere differentiability of $V$ on $\mathcal{X}_L(0)$.

Consider now $x \in \mathcal{X}_M$. The implicit function theorem implies that the solution $p = \chi(x,w)$ of $w = H(x,p)$ is analytic on $\mathcal{X}_M$.

Assume that $H_p(x,\chi(x,w)) > 0$. If $p \in D^+V(x)$, then the subsolution property of $V$ implies

$$H(x,\chi(x,\rho V(x))) = \rho V(x) \leq H(x,p)$$

and therefore $p \geq \chi(x,\rho V(x))$; similarly, the supersolution property implies $p \leq \chi(x,\rho V(x))$. Using Remark II.5.16 of Bardi and Capuzzo-Dolcetta (2008), it follows that $V(x)$ is a classical solution of $V'(x) = \chi(x,\rho V(x))$. Since $\chi$ is analytic, it follows that $V$ is an analytic classical solution of $\rho V = H(x,V')$ in $\mathcal{X}_M$.

Finally, for $x \in \operatorname{int}\mathcal{X}_N$, $\rho V(x) = u(x,q_\ell)$ or $\rho V(x) = u(x,q_u)$, and $V$ is again shown to be analytic. □

As a corollary of this result, we obtain

**Proposition D.3.** *The value function $V$ is differentiable on a dense set of points in $\mathcal{X}$.*

*Proof.* If this were not the case, there is a point $\bar{x} \in \partial\mathcal{X}_N$ such that $V$ is not differentiable for any point in an open interval $I$ with positive length containing $\bar{x}$. As $\bar{x} \in \partial\mathcal{X}_N$, there is a point $\tilde{x} \in I$ that is not element of $\mathcal{X}_N$. Hence the intersection $I \cap (\mathcal{X}_L(0) \cup \mathcal{X}_M)$ is nonempty. But this intersection is open, and therefore it contains a positive mesure subset of points where $V$ is differentiable, which is a contradiction. □

## D.3 Canonical trajectories

To show that $V$ is actually differentiable on intervals, we use the property that subderivatives are carried forward along optimal orbits by the linearised dynamics.

We need the following technical result about linearisations (see e.g. Cannarsa et al., 2015, Lemma 2.3).

**Lemma D.4.** *Let $F(t,x)$ be measurable in $t$ and continuously differentiable in $x$. For $x \in \operatorname{int}\mathcal{X}$, denote by $y(t;x)$ the solution to*

$$\dot{y}(t) = F(t,y(t)), \quad y(0) = x.$$

*Assume that for $x_0 \in \operatorname{int}\mathcal{X}$ there is $T > 0$ such that $y(t;x_0) \in \operatorname{int}\mathcal{X}$ for all $t \in [0,T]$. Let $\Phi$ be the absolutely continuous solution of the linear system*

$$\dot{\Phi}(t) = F_x(t,y(t;x_0))\Phi(t), \quad \Phi(0) = 1.$$

*Then for all $x$ in a neighbourhood of $x_0$ in $\operatorname{int}\mathcal{X}$, we have for $t \in [0, T]$*

$$y(t; x) = y(t; x_0) + \Phi(t)(x - x_0) + o_t(|x - x_0|),$$

*with $o_t(|x - x_0|)/|x - x_0| \to 0$ as $x \to x_0$, uniformly in $t$.*

The following result is closely related to the Pontryagin Maximum Principle in the finite horizon context. The difference is that we here obtain the costate equation as an initial value equation, rather than a terminal value equation. The proof is an adaptation of the proof of Theorem 3.3 in Cannarsa and Frankowska (1991) to the present infinite horizon context. Note that in the formulation, only the component $a$ of the control vector is needed.

**Proposition D.5.** *Let $(y^*, c^*)$ be an optimal trajectory–control pair with initial point $x$ such that $y^*(t) \in \operatorname{int}\mathcal{X}\backslash\partial\mathcal{X}_N$ for $t \in [0, T]$, and let $V$ be differentiable at $x$.*

*Let moreover $p^*$ be the solution of*

$$\dot{p}(t) = \rho p(t) - u_x(y^*(t), a^*(t)) - p(t) f_x(y^*(t), a^*(t)), \qquad p(0) = V'(x). \tag{27}$$

*Then for every $t \in [0, T]$ we have that*

*a. $V$ is differentiable at $y^*(t)$ and $V'(y^*(t)) = p^*(t)$;*

*b. $q = a^*(t)$ maximises $P(y^*(t), p^*(t), q) = u(y^*(t), q) + p^*(t) f(y^*(t), q)$.*

In the proof, the *lower Dini directional derivative* is used, which for a continuous function $W(x)$ is defined as $\partial^- W(x; \xi) = \liminf_{h\downarrow 0}(W(x+h\xi) - W(x))/h$. Unlike an ordinary derivative, this derivative exists for all $x$ and $\xi$. Clearly, if $W$ is differentiable at $x$, then $\partial^- W(x; \xi) = W'(x)\xi$ for all $\xi$.

*Proof.* Let $I = y^*([0, T])$ be the orbit of the optimal trajectory. As $I \subset \operatorname{int}\mathcal{X}\backslash\partial\mathcal{X}_N$, by Proposition D.2 the value function $V$ is differentiable on a full measure subset $S \subset I$. We first establish a relation between the derivatives $V'$ on different points in $S$ using a linearisation argument. Then we show that $V'$ restricted to $S$ is continuous. Finally this is shown to imply that $V'$ exists everywhere in $I$.

Take $x \in S$ and $\xi \in \mathbb{R}$, and let $y(t; x + h\xi)$ be the solution of

$$\dot{y}(t) = f(y(t), a^*(t)), \quad y(0) = x + h\xi.$$

By the optimality principle,

$$V(x + h\xi) \geq \int_0^t \exp(-\rho s) u(y(s; x + h\xi), a^*(s))\,\mathrm{d}s + V(y(t; x + h\xi)) \exp(-\rho t).$$

For the optimal pair $(y^*, c^*)$, we have

$$V(x) = \int_0^t \exp(-\rho s) u(y^*(s), a^*(s))\,\mathrm{d}s + V(y^*(t)) \exp(-\rho t). \tag{28}$$

As $V$ is differentiable at $x$, we have

$$\begin{aligned}
V'(x)\xi &= \liminf_{h\downarrow 0} (V(x+h\xi) - V(x))/h \\
&\geq \liminf_{h\downarrow 0} \Bigg( h^{-1} \int_0^t \exp(-\rho s)\Big(u(y(s; x+h\xi), a^*(s)) - u(y^*(s), a^*(s))\Big)\,\mathrm{d}s \\
&\qquad\qquad + \exp(-\rho t)\frac{V(y(t; x+h\xi)) - V(y^*(t))}{h}\Bigg) \\
&= \int_0^t \exp(-\rho s) u_x(y^*(s), a^*(s))\Phi(s)\xi\,\mathrm{d}s + \exp(-\rho t)\partial^- V(y^*(t); \Phi(t)\xi),
\end{aligned}$$

where in the last step Lemma D.4 has been used.

Selecting $t$ such that $y^*(t) \in S$, we find

$$V'(x)\xi \geq \int_0^t \exp(-\rho s) u_x(y^*(s), a^*(s))\Phi(s)\xi\,\mathrm{d}s + \exp(-\rho t)V'(y^*(t))\Phi(t)\xi.$$

Taking $\xi = 1$ and $\xi = -1$ successively yields

$$V'(x) = \int_0^t \exp(-\rho s) u_x(y^*(s), a^*(s))\Phi(s)\,\mathrm{d}s + \exp(-\rho t)V'(y^*(t))\Phi(t). \tag{29}$$

As $\Phi(t) \neq 0$ for all $t$, we can define a function $\hat{p}(t)$ by the relation

$$V'(x) = \int_0^t \exp(-\rho s) u_x(y^*(s), a^*(s))\Phi(s)\,\mathrm{d}s + \exp(-\rho t)\hat{p}(t)\Phi(t). \tag{30}$$

Differentiating this relation with respect to $t$ shows that $\hat{p}(t) = p^*(t)$ for all $t$. We then infer from (29) that $p^*(t) = V'(y^*(t))$ whenever $y^*(t) \in S$.

Take $x \in S$, and consider a sequence $x_n \in S$ such that $x_n \to x$ as $n \to \infty$. Find a sequence $t_n$ such that $x_n = y^*(t_n)$. As $[0, T]$ is compact, if necessary after passing to a subsequence we may assume that $t_n \to \tau$, and therefore $V'(x_n) = V'(y^*(t_n)) = p(t_n) \to p(\tau) = V'(x)$ as $n \to \infty$. Hence $V'$ is continuous on $S$, and can be extended to a continuous function $W$ on $I$.

As $S$ has full measure in $I$, we have for $x \in I$ that $V(x) = \int_{x_0}^x W(y)\,\mathrm{d}y$ is continuously differentiable, and hence that $V'(x)$ exists and is continuous for all $x \in I$.

Having established this, consider a trajectory–control pair such that for some $t \geq 0$ we have that $y(t) \in I$, and that both $y(t)$ and the integral $\int_0^t u(y(s), a(s))\,\mathrm{d}s$ are differentiable at $t$: this happens for a full measure set of $t$ values. Rewrite the optimality principle as

$$\begin{aligned}
0 \geq &\int_0^h \exp(-\rho(s-t)) u(y(t+s), a(t+s))\,\mathrm{d}s \\
&\qquad + V(y(t+h))(\exp(-\rho h) - 1) + (V(y(t+h)) - V(y(t)).
\end{aligned}$$

Dividing by $h$ and taking the limit $h \to 0$ yields

$$0 \geq u(y(t), a(t)) - \rho V(y(t)) + V'(y(t)) f(y(t), a(t)).$$

If the trajectory–control pair is optimal, then these two inequalities hold as equalities, which shows the last statement of the theorem. $\square$

The result of Proposition D.5 can be expressed more succinctly in the well-known form that under the assumptions of the result an optimal trajectory necessarily satisfies the *canonical equations*

$$\dot{y}(t) = H_p(y(t), p(t)), \qquad \dot{p}(t) = \rho p(t) - H_x(y(t), p(t)), \tag{31}$$

with $y(0) = x$, $p(0) = V'(x)$. As the right hand side of (31) is locally Lipschitz everywhere, and analytic if $(x, p) \notin \mathcal{S}$, the optimal state–shadow price trajectory is $C^1$ everywhere, and analytic if $(x, p) \notin \mathcal{S}$.

This motivates the following definition:

**Definition D.6.** *The map $F : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^2$ given as $F(x, p) = (H_p(x, p), \rho p - H_x(x, p))$ is the* **canonical vector field**. *A* **canonical trajectory** *is a trajectory of the canonical equations* (31). *A canonical trajectory $(y, p)$ such that $y$ is an optimal trajectory is a* **optimal canonical trajectory**.

We have that for $x \in I$ the optimal control is given by the feedback law $a^*(t) = \phi(y^*(t))$, where $\phi(x) = q^*(x, V'(x))$. This suggests that the following result is true.

**Proposition D.7.** *Let the assumptions of Proposition D.5 be fulfilled. Then $y^*(t)$ is either non-decreasing or non-increasing on* $[0, T]$.

*Proof.* Assume that $y^*$ is neither non-decreasing or non-increasing. By choosing the starting point of the trajectory, we may as well assume that $y^*(0) = y^*(T)$, while there is $0 < t_1 < T$ such that $y^*(t_1) \neq y^*(0)$.

Consider the closed region $R$ in $\mathcal{X} \times \mathbb{R}$ that has as its boundary the curve $(y^*(t), p^*(t))$ for $t \in [0, T]$ as well as the vertical straight line connecting $(y^*(0), p^*(0))$ with $(y^*(T), p^*(T))$. As $y^*(t_1) \neq y^*(0)$, the area $A$ of the region is positive. But then

$$\begin{aligned} 0 < A = \int_R \mathrm{d}p \wedge \mathrm{d}q = \int_{\partial R} p \, \mathrm{d}x = \int_0^T p^*(t) \dot{y}^*(t) \, \mathrm{d}t = \int_0^T V'(y^*(t)) \dot{y}^*(t) \, \mathrm{d}t \\ = V(y^*(T)) - V(y^*(0)) = 0, \end{aligned}$$

which is a contradiction (cf. Wagener, 2003). □

The next lemma investigates the HJB equation if the control takes a corner value. In its proof, and in several other proofs below, we shall use invariant manifold theory (Hirsch et al., 1977). In particular, we adapt the formulation of the Centre Manifold Theorem (Shub, 1987, Theorem III.7) to our context. In the formulation of this theorem, $z_\zeta$ denotes a trajectory of the vector field $G$ with initial value $z_\zeta(0) = \zeta$.

**Theorem D.8** (Centre Manifold Theorem)**.** *Let $\bar{\zeta}$ be a fixed point of an analytic vector field $G : N \to \mathbb{R}^2$, where $N$ is a neighbourhood of $\bar{\zeta}$ in $\mathbb{R}^2$. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $DG(\bar{\zeta})$, which are such that $\lambda_1 < \lambda_2$ and $\lambda_1 \leq 0 \leq \lambda_2$, and let $E_1$ and $E_2$ be the corresponding eigenspaces. Then for every $r \geq 1$, there is a disk $D_r \subset N$ of positive radius centred at $\bar{\zeta}$, and two one-dimensional $C^r$ manifolds, $W_1$ and $W_2$, contained in $D_r$, such that for $i = 1, 2$, we have that*

*a. $W_i$ is tangent to $E_i$ at $\bar{\zeta}$;*

*b. if $\lambda_1 < 0 < \lambda_2$, then $W_i$ is defined and analytic in a disk $D_\omega \subset N$ of positive radius, centred at $\bar{\zeta}$;*

*c. $W_i$ is invariant under the flow induced by $G$, that is, restricted to $D_r$, trajectories having a point in common with $W_i$ are contained in $W_i$;*

*d. if $\lambda_1 < 0$ and $\zeta \in W_2$, then $\|z_\zeta(t) - \bar{\zeta}\|$ tends to $0$ exponentially as $t \to \infty$; similarly,if $\lambda_2 > 0$ and $\zeta \in W_2$, then $\|z_\zeta(t) - \bar{\zeta}\|$ tends to $0$ exponentially as $t \to -\infty$;*

*e. if $\lambda_1 = 0$ and $z_\zeta(t) \in D_r$ for all $t \geq 0$, then $\zeta \in W_1$; similarly, if $\lambda_2 = 0$ and $z_\zeta(t) \in D_r$ for all $t \leq 0$, then $\zeta \in W_2$.*

All these manifolds are local versions of analogous globally defined manifolds, but as we shall not use the latter, we drop the adjective 'local'. If $\lambda_1 < 0$, the manifold $W_1$ is called the stable manifold, denoted $W^{\mathrm{s}}$, and it is unique. Similarly, if $\lambda_2 > 0$, $W_2$ is called the unstable manifold, denoted $W^{\mathrm{u}}$, and it is unique as well.

Manifolds $W_1$ and $W_2$ as described under e. are called centre-stable and centre-unstable manifolds, and denoted $W^{\mathrm{cs}}$ and $W^{\mathrm{cu}}$ respectively. The eigenspace that correspond to a given invariant manifold is denoted in the same way: for instance, the centre-stable manifold $W^{\mathrm{cs}}$ is tangent to the centre-stable eigenspace $E^{\mathrm{cs}}$ at $\bar{\zeta}$.

We shall only be concerned with centre-stable manifolds. These manifolds are in general not unique and only finitely differentiable, albeit to arbitrarily high order. However, the following result of Aulbach (1986) provides unicity and analyticity of the centre-manifold in the situation that is most important for us.

**Theorem D.9.** *Let $\bar{\zeta}$ be a fixed point of an analytic vector field $G : N \to \mathbb{R}^2$, where $N$ is a neighbourhood of $\bar{\zeta}$ in $\mathbb{R}^2$. Let $\lambda_1 = 0$ and $\lambda_2 > 0$ be the eigenvalues of $DG(\bar{\zeta})$, and let $E_1$ and $E_2$ be the corresponding eigenspaces.*

*If every neighbourhood of $\bar{\zeta}$ contains a fixed point of $G$ different from $\bar{\zeta}$, then there is a disk $D \subset N$ of positive radius, centred at $\bar{\zeta}$, and a unique analytic local centre–stable manifold $W^{\mathrm{cs}} \subset D$, tangent to $E_1$, such that all points on $W^{\mathrm{cs}}$ are fixed points of $G$.*

The first application of the centre manifold theorem is the following lemma, which solves the HJB equation if the action takes one of the corner values $q_\ell$ or $q_u$.

**Lemma D.10.** *Let $f(x)$ and $u(x)$ be analytic functions, defined on an open set $O$, and let $x_0 \in O$ be such that $f(x_0) = 0$ and $f(x) \neq 0$ for $x \in O\backslash\{x_0\}$. If $f'(x_0) \leq 0$, the implicit differential equation*

$$V'(x)f(x) = \rho V(x) - u(x)$$

*has a solution $V$ such that $V(x_0) = u(x_0)/\rho$ and $V'(x_0) = u'(x_0)/(\rho - f'(x_0))$. The function $V$ is analytic in $O\backslash\{x_0\}$ and $C^\infty$ in $x_0$.*

*Proof.* Rearranging and parametering in terms of auxiliary variable $s$, the graph $(x, V(x))$ coincides with solution trajectories $(x(s), v(s))$ of the system

$$\dot{x} = f(x), \quad \dot{v} = \rho v - u(x).$$

This system has a unique steady state $(x_0, v_0)$ in $O \times \mathbb{R}$, where $v_0 = u(x_0)/\rho$ is the steady state value at $x_0$. The linearisation of the system at the steady state has the matrix

$$\begin{pmatrix} f'(x_0) & 0 \\ -u'(x_0) & \rho \end{pmatrix}$$

with eigenvalues $f'(x_0) \leq 0$ and $\rho > 0$. Theorem D.8 applies: the unstable manifold is the vertical line $x = x_0$, and a solution trajectory of the system that has a finite limit as $x \to x_0$ is necessarily located on the centre-stable manifold of the saddle.

The theorem implies that for every $r > 0$ there is a neighbourhood $I_r$ of $x_0$ such that the centre-stable manifold can be parametrised as $(x, w(x))$, such that $C^r$ if in $x \in I_r$, and that it is tangent to the stable eigenspace. From this, we infer that the value function, which is necessarily finite, is $C^r$ on $I_r$, and that its graph equals the centre-stable manifold of the saddle. Its gradient at the steady state is the inclination of the centre-stable eigenspace, which evaluates at

$$V'(x_0) = \frac{u'(x_0)}{\rho - f'(x_0)}. \tag{32}$$

As $f(x) \neq 0$ for all $x \neq x_0$, the centre-stable manifold is the graph of a function $V$ that is defined on all of $O$.

As $r$ was arbitrary, it follows that $V$ is $C^\infty$ in $x_0$. Moreover, as $V$ satisfies the differential equation $V'(x) = (\rho V(x) - u(x))/f(x)$, which has an analytic right hand side if $x \in O\backslash\{x_0\}$, the function $V$ is analytic there. □

Propositions D.5 and D.7 show that $V'(x)$ is continuous on open intervals $I$ that are orbits $y^*((0,\infty))$ of non-increasing or non-decreasing optimal trajectories.

The next result ensures that $V'(x)$ is continuous at the boundaries of these intervals.

**Proposition D.11.** *Let the assumptions of Proposition D.5 be fulfilled, and assume additionally that $y^*(t) \in \operatorname{int} \mathcal{X}\backslash\partial\mathcal{X}_N$ for all $t \geq 0$. Then $(y^*(t), p^*(t))$ converges to a limit point as $t \to \infty$, and the limit point is a steady state of the canonical vector field.*

*Proof.* Proposition D.7 implies that $y^*(t)$ is a non-increasing or a non-decreasing function in the compact set $\mathcal{X}$: hence it converges to a limit $\bar{x}$ as $t \to \infty$. Moreover, $\dot{y}^*(t) = f(y^*(t), a^*(t)) \to 0$, which implies, as $f_q(x,q) > 0$ everywhere, that $a^*(t)$ also converges to a limit $\bar{q}$. We note that $\bar{x}$ cannot be a repelling point of $\dot{x} = f(x, \bar{q})$, hence $f_x(\bar{x}, \bar{q}) \leq 0$.

If $\bar{q} \in \operatorname{int} \mathcal{Q}$, we find that $p^*(t) = -u_q(y^*(t), a^*(t))$ for $t$ sufficiently large, and hence $p^*(t) \to -u_q(\bar{x}, \bar{q})$.

If $\bar{q} \notin \operatorname{int} \mathcal{Q}$, then either $\bar{q} = q_\ell$ or $\bar{q} = q_u$. Consider first $\bar{q} = q_\ell$. Introduce

$$p_1 := \limsup_{t\to\infty} p^*(t), \quad p_2 := \liminf_{t\to\infty} p^*(t).$$

Then $p_2 \leq p_1 \leq -u_q(\bar{x}, q_\ell)$, where either $p_2$ or both $p_1$ and $p_2$ can be $-\infty$.

If $p_2 < p_1$, then for every $p \in (p_1, p_2)$ there are sequences $t_1^{(n)}$, $t_2^{(n)}$ such that $p^*(t_1^{(n)}) = p^*(t_2^{(n)}) = p$ for all $n = 1, 2, \ldots$, and such that

$$\dot{p}^*(t_1^{(n)}) = F_2(y^*(t_1^{(n)}), p^*(t_1^{(n)})) \geq 0 \quad \text{and} \quad \dot{p}^*(t_2^{(n)}) = F_2(y^*(t_2^{(n)}), p^*(t_2^{(n)})) \leq 0.$$

Taking the limit $n \to \infty$, it follows that $F_2(\bar{x}, p) = 0$ for all $p \in (p_1, p_2)$. But this is a contradiction, because $f_x(\bar{x}, q_\ell) \leq 0$ and for $p \leq p_\ell$ we have that $F_2(\bar{x}, p) = \rho p - u_x(\bar{x}, q_\ell) - p f_x(\bar{x}, q_\ell)$ is strictly increasing in $p$. We conclude that $p^*(t)$ converges to a limit $\bar{p} = p_1 = p_2$, and $(\bar{x}, \bar{p})$ is a steady state of the canonical vector field.

The argument for the situation that $\bar{q} = q_u$ is exactly parallel. □

For the next lemma, whose proof is given for the sake of completeness, see e.g. Skiba (1978, Proposition 2).

**Lemma D.12.** *Let $(y(t), p(t))$ be a canonical trajectory such that*

$$\lim_{t\to\infty} \exp(-\rho t) H(y(t), p(t)) = 0,$$

*and set $q(t) = a^*(y(t), p(t))$. Then $U(x,q) = H(y(0), p(0))/\rho$.*

*Proof.* Along a canonical trajectory

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(-\exp(-\rho t)H(y(t),p(t))\Big) = \exp(-\rho t)u(y(t),q(t)).$$

Integrating this relation yields the result. □

## D.4 Differentiability of the value function in $\partial\mathcal{X}_N$: Interior case

We turn to the differentiability of $V$ in points of $\partial\mathcal{X}_N$. As this set is closed, by the Cantor–Bendixson theorem (Carathéodory, 1927, §64), it can be written as the union of a set of isolated points $\partial\mathcal{X}_N^i$ and a perfect set $\partial\mathcal{X}_N^p$, that is, a set such that every point is the limit point of a sequence of other points in the set.

Since $\mathcal{X}$ is compact, the set $\partial\mathcal{X}_N^i$ is finite. Moreover, $V$ may fail to be differentiable at these points.

*Example.* Consider the optimal control problem with $\mathcal{X} = [-1,1]$, $\mathcal{Q} = \{0\}$, $\beta(-1) = \beta(1) = 1$, $u(x,q) = 0$ and $f(x,q) = 2\rho x$. The Hamilton function is $H(x,p) = 2\rho xp$, and the set $\mathcal{X}_N$ contains the single isolated point 0.

The Hamilton–Jacobi equation of this problem is

$$\rho V(x) = 2\rho x V'(x),$$

and its value function is $V(x) = |x|^{\frac{1}{2}}$, which is not differentiable at 0. □

First, we prove that each $\bar{x} \in \partial\mathcal{X}_N^p$ corresponds to a steady state $\bar{z} = (\bar{x},\bar{p})$ of the canonical equations that is located in $\mathcal{P}_\ell \cup \mathcal{P}_u$. If it is located in the interior of this set, the centre-stable manifold of $(\bar{x},\bar{p})$ is differentiable: as this manifold parametrises the graph of $V'$, we obtain twice differentiability of $V$ at $\bar{x}$. If the steady state is located on the boundary $\mathcal{S}$ of this set, then the canonical vector field is not differentiable at the steady state, and the discussion is more involved.

**Lemma D.13.** *If $(y,p)$ is an optimal canonical trajectory, $p(t) = V'(y(t))$ for all $t \geq 0$, and $y(t_1) \in \mathcal{X}_N$, then $\dot{y}(t_1) = 0$.*

*Proof.* The point $x_1 \equiv y(t_1)$ is in $\mathcal{X}_N$. Assume that it is such that $H_p(x_1,p) = 0$ for all $p \leq p_\ell(x_1)$, the other case being similar. Then the set of minimisers of $H(x_1,p)$ is $(-\infty, p_\ell(x_1)]$, and $\rho V(x_1) = H(x_1,p)$ if and only if $p$ is a minimiser. Since by Lemma D.12 $\rho V(x_1) = H(x_1,p(t_1))$, it follows that $p(t_1)$ is a minimiser and $\dot{y}_1(t_1) = H_p(x_1,p(t_1)) = 0$. □

A direct corollary of this result is the following.

**Lemma D.14.** *If an optimal trajectory $y$ starts in a closed interval $I$ whose boundary points are contained in $\mathcal{X}_N$, then it remains in $I$ for all time.*

*Proof.* We only have to consider the situation that $y$ is not constant and that the length of $I$ is positive. The intersection $I \cap y([0,\infty))$ of its orbit with the interval then contains a smaller open interval that does not intersect $\partial\mathcal{X}_N$. Hence it contains points at which $V$ is differentiable; we may as well assume that $V$ is differentiable at $y(0)$. Then the solution $p$ of the adjoint equation (27) is well-defined and $p(t) = V'(y(t))$ for all $t \geq 0$.

Assume that $y$ leaves $I$. Then there is $t > 0$ such that $\dot{y}(t) \neq 0$ and $y(t) \in \mathcal{X}_N$. But this is ruled out by Lemma D.13. □

**Lemma D.15.** *If $\bar{x} \in \partial\mathcal{X}_N^p$, then there is a unique value $\bar{p} = u_x(\bar{x},q_b)/\rho \in \{p : p \leq p_\ell(\bar{x})$ or $p \geq p_u(\bar{x})\}$, such that $\bar{z} = (\bar{x},\bar{p})$ is a steady state of the canonical vector field.*

*Proof.* Assume that $\bar{x}$ is such that $H_p(\bar{x}, p) = 0$ for all $p \leq p_\ell(\bar{x})$, the argument for the other situation being similar.

By perfectness of $\partial \mathcal{X}_N^p$, there is a sequence $\bar{x}_n \in \partial \mathcal{X}_N^p$ such that $\bar{x}_n \to \bar{x}$ as $n \to \infty$ and $\bar{x}_n \neq \bar{x}$ for any $n$. If necessary by selecting a subsequence, we may assume that either $x_n < x_{n+1}$ for all $n$ or $x_n > x_{n+1}$ for all $n$. Hence for every interval $I_n = [\bar{x}_n, \bar{x}_{n+1}]$ there is $x_n \in I_n$ such that $V$ is differentiable at $x_n$ and the optimal orbit starting at $x_n$ remains in $I_n$ for all $t$. By Proposition D.11, the limit of the corresponding canonical trajectory is a steady state $(\hat{x}_n, \hat{p}_n)$ of the canonical vector field, such that $\hat{x}_n \to \bar{x}$ as $n \to \infty$.

Again by passing to a subsequence, we obtain that either $\hat{p}_n$ converges, or that $\hat{p}_n \to -\infty$ as $n \to \infty$, as $\hat{p}_n \leq 0$ by Proposition A.3. In the second situation there is $n_0 > 0$ such that $(\hat{x}_n, \hat{p}_n) \in \mathcal{P}_\ell$ for all $n \geq n_0$. For those values of $n$, we have $\hat{q}_n = q^*(\hat{x}_n, \hat{p}_n) = q_\ell$ and $f(\hat{x}_n, q_\ell) = 0$, implying $f_x(\bar{x}, q_\ell) = 0$. In particular, there is $n_1 > n_0$, such that if $n > n_1$ then $|f_x(\hat{x}_n, q_\ell)| < \rho/2$ and as $n \to \infty$

$$\hat{p}_n = \frac{u_x(\hat{x}_n, q_\ell)}{\rho - f_x(\hat{x}_n, q_\ell)} \geq \frac{u_x(\hat{x}_n, q_\ell)}{\rho/2} \to 2u_x(\bar{x}, q_\ell)/\rho.$$

Hence $(\hat{x}_n, \hat{p}_n)$ has to have at least one accumulation point $(\bar{x}, \bar{p})$. Necessarily $\rho V(\bar{x}) = H(\bar{x}, \bar{p})$ and hence $\bar{p} \leq p_\ell(\bar{x})$.

Finally, as $f_x(\bar{x}, q_\ell) = 0$, we have that $\bar{p} = u_x(\bar{x}, q_\ell)/\rho$. $\square$

**Proposition D.16.** *If $\bar{z} = (\bar{x}, \bar{p}) \in \partial \mathcal{X}_N^p \times \mathbb{R}$ is a steady state of the canonical vector field, and $\bar{z} \in \operatorname{int} \mathcal{P}_\ell \cup \mathcal{P}_u$, then $V$ is $C^2$ in a neighbourhood of $\bar{x}$.*

*Proof.* Assume $\bar{z} \in \operatorname{int} \mathcal{P}_\ell$, the other case being proved similarly.

By Lemma D.15, we have that $q^*(\bar{z}) = q_\ell$ and $f_x(\bar{x}, q_\ell) = 0$. Lemma D.10 then implies the result. $\square$

### D.5 Differentiability of the value function in $\partial \mathcal{X}_N$: Boundary case

The situation that $\bar{z} = (\bar{x}, \bar{p})$ is on the boundary $\mathcal{S}$ between the interior and the corner regions is harder to analyse, as the canonical vector field fails to be differentiable here. In the following, we focus on the situation that $\bar{z} \in \mathcal{S}_\ell$; the argument for the other situation is analogous.

We introduce two $C^2$ Hamilton functions $H^I$ and $H^\ell$, which coincide with the Hamilton function $H$ on $\mathcal{P}_I$ and $\mathcal{P}_\ell$ respectively, and which induce canonical vector fields $F_I$ and $F_\ell$. In particular $H^I(x, p) = \max_q [u(x, q) + pf(x, q)]$, without the restriction of $q$ to the set $\mathcal{Q}$, and $H^\ell(x, p) = u(x, q_\ell) + pf(x, q_\ell)$. To each Hamilton function $H^k$, $k \in \{I, \ell\}$, there is associated a canonical vector field $F_k = (H_p^k, \rho p - H_x^k)$ such that $F = F_I$ on $\mathcal{P}_I$ and $F = F_\ell$ on $\mathcal{P}_\ell$. The vector field $F$ is Lipschitz continuous, but in general not differentiable, at the interface of $\mathcal{P}_I$ and $\mathcal{P}_\ell$. Finally, we introduce the derivative matrices

$$A_\ell \equiv DF_\ell(\bar{z}) = \begin{pmatrix} f_x(\bar{x}, q_\ell) & 0 \\ -u_{xx}(\bar{x}, q_\ell) & \rho - f_x(\bar{x}, q_\ell) \end{pmatrix}$$

and

$$A_I \equiv DF_I(\bar{z}) = \begin{pmatrix} H_{xp}^I(\bar{z}) & H_{pp}^I(\bar{z}) \\ -H_{xx}^I(\bar{z}) & \rho - H_{xp}^I(\bar{z}) \end{pmatrix}.$$

**Lemma D.17.** *Assume that $\bar{x} \in \partial \mathcal{X}_N^p$ and $\bar{z} = (\bar{x}, \bar{p}) \in \mathcal{S}_\ell$.*

*a.* $E_\ell^{\mathrm{u}} = \mathbb{R}(0, 1)$.

b. *Assume that the eigenvalues* $\lambda_1$, $\lambda_2$ *of* $A_I$ *satisfy* $\lambda_1 < \lambda_2$. *Then the eigenspaces of* $A_I$ *are of the form* $E_1 = \mathbb{R}(1, e_1)$ *and* $E_2 = \mathbb{R}(1, e_2)$ *with* $e_1 < e_2$.

*Proof.* The assumption implies that $f_x(\bar{x}, q_\ell) = 0$, which implies the first statement. The second statement is a consequence of the fact that $H^I_{pp}(\bar{z}) > 0$. □

We need to generalise the notions of (centre-)stable and unstable manifold to the steady state $\bar{z}$ where the vector field $F$ is not differentiable. Using Theorem D.8 as a starting point, for a disk $D$ containing $\bar{z}$, we introduce the *unstable set* $W^{\mathrm{u}}$ as the set of points $\zeta \in D$ such that the canonical trajectory $z_\zeta$ with initial point $\zeta$ has the property that $\|z_\zeta(t) - \bar{z}\|$ tends to 0 exponentially as $t \to -\infty$. Similarly, the *centre-stable set* $W^{\mathrm{cs}}$ is taken to be the set of points $\zeta \in D$ such that $z_\zeta(t) \in D$ for all $t \geq 0$.

We shall prove that to either side of the line $x = \bar{x}$, the centre-stable set $W^{\mathrm{cs}}$ coincides either with $W^{\mathrm{cs}}_I$ or $W^{\mathrm{cs}}_\ell$. Lemma D.17 implies that both these invariant manifolds can be parametrised as the graphs of functions $p = w^{\mathrm{cs}}_I(x)$ or $p = w^{\mathrm{cs}}_\ell(x)$, which are $C^r$, for $r \geq 1$, in a sufficiently small neighbourhood of $\bar{x}$.

The interface $\mathcal{S}_\ell$ between $\mathcal{P}_I$ and $\mathcal{P}_\ell$ is parametrised as $p = p_\ell(x)$. Let $B$ denote the tangent line to $\mathcal{S}$ at $\bar{z}$. Assume that an eigenspace of either $A_\ell$ or $A_I$ coincides with $B$. The fact that $F_I(z) = F_\ell(z)$ if $z \in \mathcal{S}$ implies that if $\zeta \in B$, then also $A_I\zeta = A_\ell\zeta$. Hence if $B$ is an eigenspace of $A_I$ with a certain eigenvalue, then it is also an eigenspace of $A_\ell$ with the same eigenvalue. In particular, if $f_x(\bar{x}, q_\ell) = 0$, then $E^{\mathrm{u}}_\ell = \mathbb{R}(0, 1)$ cannot coincide with $B$, and hence $B$ can only be the centre-stable eigenspace of $A_\ell$. It then necessarily is also the centre-stable eigenspace of $A_I$. In that situation, we have the result that to either side of $\bar{x}$, the centre-stable set coincides with one of $W^{\mathrm{cs}}_I$ or $W^{\mathrm{cs}}_\ell$, or the three sets $W^{\mathrm{cs}}_I$, $W^{\mathrm{cs}}_\ell$ and $\mathcal{S}_\ell$ all coincide.

First we need an auxiliary result. If $D$ is a disk centred at a point $\bar{z}$, let $D_+ = D \cap \{x > \bar{x}\}$ and $D_- = D \cap \{x < \bar{x}\}$, and let $D_{I,+}$, $D_{I,-}$, $D_{\ell,+}$ and $D_{\ell,-}$ be defined as $D_{I,+} = D_+ \cap \overline{\mathcal{P}_I}$ etc. Moreover, the symbol $O(f_1(z), \ldots, f_m(z))$ denotes a analytic function $g(z)$ such that

$$|g(z)| \leq C_1|f_1(z)| + \ldots + C_m|f_m(z)|$$

for some constants $C_1, \ldots, C_m > 0$ and $z$ close to 0.

**Lemma D.18.** *Assume that* $\bar{z} = (\bar{x}, \bar{p})$ *is a steady state of the canonical vector field such that* $\bar{p} = p_\ell(\bar{x})$, *the eigenvalues of* $A_k$ *satisfy* $\lambda^{\mathrm{cs}}_k \leq 0 < \lambda^{\mathrm{u}}_k$ *for* $k \in \{\ell, I\}$, *and* $B = E^{\mathrm{cs}}_\ell = E^{\mathrm{cs}}_I$. *Let* $D$ *be a disk centred at* $\bar{z}$ *with positive radius.*

a. *If* $F$ *restricted to* $\mathcal{S}$ *points into* $D_{I,+}$, *then* $W^{\mathrm{cs}}_\ell$ *and* $W^{\mathrm{cs}}_I$ *restricted to* $D_+$ *are contained in* $D_{\ell,+}$;

b. *if* $F$ *restricted to* $\mathcal{S}$ *points into* $D_{\ell,+}$ *then* $W^{\mathrm{cs}}_\ell$ *and* $W^{\mathrm{cs}}_I$ *restricted to* $D_+$ *are contained in* $D_{I,+}$.

*The analogous statements with* $+$ *replaced by* $-$ *hold true as well.*

*Proof.* We only consider the first statement with respect to $W^{\mathrm{cs}}_\ell$, the others being similar. On $\mathcal{S}$, $F_\ell = F$.

Introduce local coordinates $\zeta = (\xi, \eta)$ by setting $(x, p) = (\bar{x} + \xi, p_\ell(x) + \eta)$. In these coordinates, the set $\mathcal{S}_\ell$ is parametrised as $\eta = 0$, and the invariant manifold $W^{\mathrm{cs}}_\ell$, takes the form $\eta = w(\xi) \equiv w^{\mathrm{cs}}_\ell(\bar{x}+\xi) - p_\ell(\bar{x}+\xi)$. As $B = E^{\mathrm{cs}}_\ell = \mathbb{R}(1, 0)$ in the local coordinates, we have that $F_1(\zeta) = \lambda_1\xi + O(\|\zeta\|^2)$ and $w(0) = w'(0) = 0$.

If all derivatives of $F_2(\xi, 0)$, with respect to $\xi$ and evaluated at $\xi = 0$, equal 0, then $F_2(\xi, 0) = 0$ for all $\xi$ because of analyticity. It follows that $\mathcal{S}_\ell$ is invariant under $F$ and $W^{\mathrm{cs}}_\ell = \mathcal{S}_\ell$.

If not all derivatives vanish, there is a smallest integer $a > 0$ and a constant $c_a > 0$ such that $F_2(\xi, 0) = c_a \xi^a + O(\xi^{a+1})$. Assume that $D$ is sufficiently small that the invariant manifold $W_\ell^{\mathrm{cs}}$ can be parametrised by $C^{a+1}$ functions $w_\ell^{\mathrm{cs}}$; then also the function $w$ is $C^{a+1}$.

The vector $n(\xi) = (-w'(\xi), 1)$ is normal to $W_\ell^{\mathrm{cs}}$ at $(\xi, w(\xi))$. Using it, the invariance condition reads as

$$F(\xi, w(\xi)) \cdot n(\xi) = -w'(\xi) F_1(\xi, w(\xi)) + F_2(\xi, w(\xi)) = 0.$$

Using that $w(\xi) = O(\xi^2)$, this implies first that $F_2(\xi, 0) = 0 + O(\xi^2)$, hence that $a \geq 2$ and $F_2(\xi, \eta) = \lambda_2 \eta + O(\|\zeta\|^2)$.

Substitute $w(\xi) = c_b \xi^b + O(\xi^{b+1})$ in the invariance condition to obtain

$$0 = (\lambda_2 - b\lambda_1) c_b \xi^b + c_a \xi^a + O(\xi^{a+1}, \xi^{b+1}).$$

If $b < a$, then $c_b = 0$; if $b > a$, the condition cannot be satisfied. Hence $b = a$ and $c_b = c_a/(b\lambda_1 - \lambda_2)$. But as $\lambda_1 \leq 0$ and $\lambda_2 > 0$, it follows that $c_b < 0$ and $W_\ell^{\mathrm{cs}}$ is contained in $D_{\ell,+}$. ☐

**Proposition D.19.** *Assume that $\bar{x} \in \partial \mathcal{X}_N^p$, $\bar{p} = p_\ell(\bar{x})$, and $B = E_\ell^{\mathrm{cs}} = E_I^{\mathrm{cs}}$. Let $r \geq 1$ be given. Then there is a neighbourhood $N$ of $\bar{x}$ in $\mathcal{X}$, and a continuous function $w^{\mathrm{cs}} : N \to \mathbb{R}$, such that locally around $\bar{z}$*

$$W^{\mathrm{cs}} = \{(x, p) \,:\, p = w^{\mathrm{cs}}(x)\}$$

*and the following alternative is true.*

*a. Either $w^{\mathrm{cs}}$ is $C^r$ for $x \neq \bar{x}$ and $w^{\mathrm{cs}}(x) = p_\ell(x)$ has the single solution $x = \bar{x}$ in $N$;*

*b. or $w^{\mathrm{cs}}$ is analytic and $w^{\mathrm{cs}}(x) = p_\ell(x)$ for all $x \in N$.*

*Proof.* Let $\psi(x) = F(x, p_\ell(x)) \cdot n_\ell(x)$, where $n_\ell$ is the vector normal to $\mathcal{S}_\ell$ at $(x, p_\ell(x))$ and pointing into $\mathcal{P}_I$. Then $\psi$ is real analytic and $\psi(x) = 0$ if and only if $F$ is tangent to $\mathcal{S}_\ell$ at $(x, p_\ell(x))$.

Consider first the situation that in every neighbourhood of $\bar{x}$, there are infinitely many zeros of $\psi$. Then, as $\psi$ is analytic, it has to be identically zero. This implies that $\mathcal{S}_\ell$ is invariant under $F$, and as $\mathcal{S}_\ell$ is tangent to both $W_\ell^{\mathrm{cs}}$ and $W_I^{\mathrm{cs}}$ at $\bar{x}$, it is equal to both these manifolds. Hence $\mathcal{S}_\ell$ equals $W^{\mathrm{cs}}$, and the second case of the alternative is true.

If $\bar{x}$ is an isolated zero of $\psi$, then there is a neighbourhood $N$ of $\bar{x}$ such that $\psi(x) \neq 0$ for all $x \in N$ such that $x \neq \bar{x}$.

Consider first the region $x > \bar{x}$. If $\psi(x) > 0$ there, then Lemma D.18 implies that $W_\ell^{\mathrm{cs}}$ and $W_I^{\mathrm{cs}}$ are contained in $\mathcal{P}_\ell$, and $W^{\mathrm{cs}} = W_\ell^{\mathrm{cs}}$; if $\psi(x) < 0$ for $x > \bar{x}$, then similarly $W^{\mathrm{cs}} = W_I^{\mathrm{cs}}$ there. The argument for $x < \bar{x}$ is similar. This completes the proof. ☐

The next result gives the geometry of $W^{\mathrm{cs}}$ is $B$ is not equal to an eigenspace of either $A_I$ or $A_\ell$.

**Proposition D.20.** *If $B$ does not coincide with an eigenspace of either $A_I$ or $A_\ell$, then there are continuous functions $w^{\mathrm{u}}, w^{\mathrm{cs}} : N \to \mathbb{R}$, defined on an open neighbourhood $N$ of $\bar{x}$ and analytic except at $\bar{x}$, such that $(x, w^{\mathrm{u}}(x))$ parametrises $W^{\mathrm{u}}$ and $(x, w^{\mathrm{cs}}(x))$ parametrises $W^{\mathrm{cs}}$.*

The proof of this proposition is given in Section F.5. The main difficulty is excluding the case that $W_\ell^{\mathrm{cs}}$ and $W_I^{\mathrm{cs}}$ are both contained in either of the half planes $x \leq \bar{x}$ or $x \geq \bar{x}$.

**Lemma D.21.** *If for a dense set $S$ of points $x$ in a right (left) neighbourhood of $\bar{x}$, we have that $V$ is differentiable and $(x, V'(x)) \in W^{\mathrm{cs}}$, then $V$ is right (left) differentiable at $\bar{x}$.*

*Proof.* This is an immediate consequence of Propositions D.19 and D.20, as $V'(x) = w(x)$ on a dense set and $w$ is a continuous function. $\square$

We show an auxiliary result about the limits of sequences of points of differentiability of $V$.

**Lemma D.22.** *Let $(y,p)$ be a canonical trajectory such that for every $\tau > 0$ the trajectory $y_\tau(t) = y(\tau + t)$ is optimal and $p(\tau + t) = V'(y_\tau(t))$ for all $t \geq 0$. Then $y$ is an optimal trajectory.*

*Proof.* For every $\tau > 0$, we have by Lemma D.12 that $U(y_\tau) = H(y(\tau), p(\tau))/\rho = V(y(\tau))$. By continuity of $V$, this relation still holds if $\tau \to 0$. $\square$

**Lemma D.23.** *Let $\delta > 0$ and let $I$ be either $(\bar{x} - \delta, \bar{x})$ or $(\bar{x}, \bar{x} + \delta)$. Assume there is a dense subset $D \subset I$, such that for all $x \in D$ for which $V$ is differentiable at $x$, the optimal trajectory $y_x$ starting at $x$ satisfies $|y_x(t_x) - \bar{x}| = \delta$ for some $t_x > 0$. Then $(x, V'(x)) \in W^{\mathrm{u}}$ for all $x \in I$.*

*Proof.* We only give the proof for $I = (\bar{x}, \bar{x} + \delta)$. For $x \in I$ such that $V$ is differentiable at $x$, let $(y(t;x), p(t;x))$ be the canonical trajectory starting at $(x, V'(x))$.

The trajectories $y_x$ are time translates of each other, as all pass through the same point $(\bar{x}+\delta, V'(\bar{x}+\delta))$. By definition of $t_x$, we have $y_{x_2}(t) = y_{x_1}(t + (t_{x_1} - t_{x_2}))$. In particular, $V$ is differentiable for all $x \in I$, and $\dot{y}_x(0) > 0$ for all such $x$.

Fix $x \in I$ and consider the canonical orbit $(y(t;x), p(t;x))$ for $t < 0$. Then either $y_x(t_0) = \bar{x}$ for some $t_0 < 0$, or $y(t;x) > \bar{x}$ for all $t$.

In the former situation, Lemma D.22 implies that $y(t) = y_x(t_0 + t)$ is optimal. Set $p(t) = p_x(t_0 + t)$. Then $p(0) \neq \bar{p}$, for $(\bar{x}, \bar{p})$ is a steady state of the canonical vector field. If $p(0) < \bar{p}$, then the canonical trajectory $(y,p)$ is on the unstable manifold $W^{\mathrm{u}}_\ell$, which implies $y(t) = \bar{x}$ for all $t > 0$, which is impossible. Hence $p(0) > \bar{p}$. But then $(y(0), p(0)) \in \mathcal{P}_I$ and $H(y(0), p(0)) > H(\bar{x}, \bar{p})$, which, by Lemma D.12 implies that $\bar{x} \notin \mathcal{X}_N$, contrary to assumption. We conclude that it is not possible that $y_x(t_0) = \bar{x}$ for some $t_0 < 0$.

If $y(t;x) \downarrow \bar{x}$ as $t \to -\infty$, we must have either $p(t;x) \to -\infty$ or $p(t;x) \to \bar{p}$. In the former situation, there is $\tau$ such that $(y(t;x), p(t;x)) \in \mathcal{P}_\ell$ for all $t \leq \tau$.

Let $\bar{x} < x_1 < \min\{y_x(\tau), \bar{x} + \delta\}$ such that $0 \leq F_1(x,p) = f_x(x, q_\ell) < \rho/2$ for all $(x,p)$ such that $\bar{x} < x < x_1$. Let $p_1$ be such that

$$F_2(x,p) = u_x(x, q_\ell) + (\rho - f_x(x, q_\ell))p < 0$$

if $\bar{x} < x < x_1$ and $p < p_1$.

Assume that $p_x(t_1) < p_1$ for some $t_1 < \tau$. The set

$$\{(x,p) \,:\, \bar{x} < x < x_1,\, p \geq p_x(t_1)\}$$

is backward invariant: as $(y_x(t_1), p_x(t_1))$ is contained in the set, $(y(t;x), p(t;x))$ is contained in the same set for all $t < t_1$, and it is impossible that $p(t;x) \to -\infty$.

Hence $p(t;x) \to \bar{p}$ as $t \to -\infty$, which implies that the canonical trajectory $(y_x, p_x)$ is contained in $W^{\mathrm{u}}$. Consequently $V'(x) = w^{\mathrm{u}}(x)$ for $x \in I$ and $V'(x) \to \bar{p}$ as $x \downarrow \bar{x}$. $\square$

We can now show the main result of this subsection.

**Proposition D.24.** *If $\bar{x} \in \partial\mathcal{X}^p_N$ and $\bar{p} = p_\ell(\bar{x})$, then $V$ is $C^1$ at $\bar{x}$.*

*Proof.* We show that $V$ is differentiable from the right; the argument for differentiability from the left is entirely analogous.

There are two main situations to consider: either there is a right neighbourhood $I = (\bar{x}, \bar{x} + \delta)$ with $\delta > 0$ such that $\mathcal{X}_N \cap I$ is empty, or for every $\delta > 0$ the intersection consists of infinitely many points accumulating on 0.

Assume first that for some $I$ we have that $\mathcal{X}_N$ and $I$ have no points in common. There are three possibilities.

First, there is $x_1 \in I$ such that $V$ is differentiable at $x_1$ and the optimal trajectory $y$ starting at this point converges to $\bar{x}$. Then the canonical trajectory $(y(t), V'(y(t))$ converges to $(\bar{x}, \bar{p})$, hence it is contained in $W^{\mathrm{cs}}$ and $V' = w^{\mathrm{cs}}$ is continuous by Lemma D.21.

Second, every optimal trajectory $y_x$ starting at a point of differentiability $x \in I$ satisfies $y_x(t_x) = \bar{x} + \delta$ for some $t_x > 0$: then $(x, V'(x)) \in W^{\mathrm{u}}$ and $V' = w^{\mathrm{u}}$ is continuous by Lemma D.23.

Third, no optimal trajectory $y$ starting at a point of differentiability in $I$ converges to $\bar{x}$, but some of them,and hence infinitely many, remain in $I$. Their limit points correspond to steady states of the canonical vector field: hence there are infinitely many steady states in every neighbourhood of $\bar{z}$. By the Aulbach theorem, the centre-stable manifold of $\bar{z}$ is analytic and only consists of steady states. Again, we see that $V' = w^{\mathrm{cs}}$ is continuous at $\bar{x}$.

Consider now the situation that $\mathcal{X}_N \cap I$ contains infinitely many points that accumulate on $\bar{x} = 0$. If necessary by decreasing $\delta$, we may assume that $\delta \in \mathcal{X}_N$.

Take $x \in I$, and consider the optimal trajectory $y$ with initial point $x$. By Lemma D.14, it cannot leave $I$; hence it accumulates on a steady state $\bar{x}_1$. Again, in every neighbourhood of $\bar{z}$ there are infinitely many steady states, and the same argument as before shows that $V'$ is continuous at $\bar{x}$. □

## D.6 Existence of Markovian best response

In this section, we show first that $V$ is analytic except at a finite number of points, and, second, that $(x, V'(x))$ is not in $\mathcal{S}$ except again for a finite number of points.

This allows us to define the piecewise analytic strategy $\phi(x) = q^*(x, V'(x))$ and show that $\phi$ is a Markovian best response.

Combining Propositions D.2, D.16 and D.24 yields:

**Proposition D.25.** *The value function $V$ is continuously differentiable on $\mathcal{X}_M \cup \mathcal{X}_N$ and it is differentiable almost everywhere on $\mathcal{X}$.*

A point of non-differentiability is an *indifference point*, that is, an initial point to two optimal trajectories:

**Proposition D.26.** *If $V$ is not differentiable at $x_0$, then there are $p_0^{(1)} < p_0^{(2)}$ such that $H_p(x_0, p_0^{(1)}) < 0 < H_p(x_0, p_0^{(2)})$ and both $(x_0, p_0^{(1)})$ and $(x_0, p_0^{(2)})$ are initial point to an optimal canonical trajectory.*

*Proof.* Take a point $x_0 \in \mathcal{X}$ at which $V$ is not differentiable. By Proposition D.25, there is $c > 0$ such that $x_0$ is a point in the open set $\mathcal{X}_L(c)$. Recall from Equation (26) that for all $x \in \mathcal{X}_L(c)$ such that $V$ is differentiable at $x$, we have $|V'(x)| \leq 2\|u\|_\infty / c$. Then there is a convergent sequence $x_n \to x_0$ and a number $p_0$, such that $z_0 \equiv (x_0, p_0) \in \mathcal{P}_I$, for all $n$ the function $V$ is differentiable at $x_n$, and $z_n \equiv (x_n, V'(x_n)) \to z_0$. If necessary by passing to a subsequence, we may assume that all $x_n - x_0$ have the same sign.

If $H_p(z_0) = 0$, then $p_0$ minimises $p \mapsto H_p(x_0, p)$. Every other convergent sequence $\tilde{x}_n \to x_0$ has the property that $V'(\tilde{x}_n) \to p_0$, making $V'$ continuous at $x_0$, contrary to the choice of $x_0$.

Hence $H_p(z_0) \neq 0$, and we take $\varepsilon > 0$ such that $H_p(z) \neq 0$ for all $z$ such that $\|z - z_0\| < \varepsilon$. Let $n_0 > 0$ be such that $\|z_n - z_0\| < \varepsilon$ if $n > n_0$, and let $(y_n, p_n)$ be the canonical trajectory starting in $z_n$. If $y_n(t) = x_0$ for some $t > 0$, then $V$ would be differentiable at $x_0$, contrary to assumption. Hence $x_n - x_0$ and $\dot{y}_n(0)$ have the same sign for all $n$. If necessary by increasing $n_0$, we have that for every $m > 0$ there is $t_{n,m}$ such that $y_{n+m}(t_{n,m}) = y_n(0)$. But then $z_{n+m}(t_{n,m}) = z_n(0)$, and the trajectories are identical up to a time translation.

Lemma D.22 then implies that the canonical trajectory starting at $(x_0, p_0)$ is optimal, and the sign of $H_p(x_0, p_0)$ is equal to $x_n - x_0$.

Taking two sequences converging to $x_0$, one from above, one from below, then completes the proof. □

Then we show that there can only be finitely many indifference points.

**Proposition D.27.** *The value function fails to be differentiable at only finitely many points.*

*Proof.* If the proposition were not true, the set of points at which $V$ is not differentiable has an accumulation point. Let $x_n \in \mathcal{X}$ be a monotone convergent sequence $x_n \to \bar{x}$ of points at which $V$ is not differentiable. We consider the situation that the sequence is increasing, $x_1 < x_2 < \ldots$; the other case is argued similarly. As $x_n$ is initial point to an increasing optimal trajectory, and $x_{n+1}$ initial point to a decreasing one, every interval $(x_n, x_{n+1})$ contains a steady state $\bar{x}_n$ at which $V$ is differentiable. Also $\bar{x}_n \to \bar{x}$ as $n \to \infty$.

If necessary by passing to a subsequence, we find $\bar{p}$ such that $(\bar{x}_n, V'(\bar{x}_n)) \to \bar{z} \equiv (\bar{x}, \bar{p})$. By continuity of the canonical vector field $\bar{z}$ is also a steady state. Moreover, if $n$ is sufficiently large, $(\bar{x}_n, V'(\bar{x}_n))$ is contained in the centre-stable manifold of $\bar{z}$. Hence this manifold contains infinitely many steady states, and by the Aulbach theorem, it is parametrised by the graph of a real analytic function $w(x)$, which satisfies $w(x) = V'(x)$ close to $\bar{x}$. But the centre-stable manifold also contains the initial points $x_n$ if $n$ is sufficiently large, implying that $V$ is differentiable at $x_n$, and leading to a contradiction. □

The next proposition shows that the value function is actually piecewise analytic. For it, we first need the following auxiliary result

**Lemma D.28.** *An optimal canonical trajectory cannot contain a point $\bar{z}$ with $F_1(\bar{z}) = 0$ and $F_2(\bar{z}) \neq 0$.*

*Proof.* Assume that such a point $\bar{z}$ can be part of an optimal canonical trajectory $(y^*, p^*)$; we may as well assume that $\bar{z}$ is the initial point. The function $H(\bar{x}, p)$ is minimal at $p = \bar{p} = p^*(0)$. It is therefore not possible, by Proposition D.26, that $V$ fails to be differentiable at $\bar{z}$.

Set $\bar{p} = V'(\bar{x})$ and $\bar{z} = (\bar{x}, \bar{p})$; then $\dot{y}^*(0) = F_1(\bar{z}) = 0$. If $\dot{p}^*(0) = F_2(\bar{z}) \neq 0$, then the canonical trajectory through $\bar{z}$ can be parametrised as a curve $x = X(p)$, where $X$ is the solution to the differential equation

$$X'(p) = F_1(X(p), p)/F_2(X(p), p), \qquad X(\bar{p}) = \bar{x}.$$

We have that $X'(\bar{p}) = 0$. Differentiating and substituting $p = \bar{p}$ yields

$$X''(\bar{p}) = (F_1)_p(\bar{z})/F_2(\bar{z}) = H_{pp}(\bar{z})/F_2(\bar{z}) \equiv c \neq 0.$$

Consider the case that $c > 0$, the other being similar. We have that

$$X(p) = \bar{x} + \frac{c}{2}(p - \bar{p})^2 + (p - \bar{p})^3 R(p),$$

with $R(p) \to 0$ as $p \to \bar{p}$. In particular, there is $t_1 > 0$ such that

$$y^*(t_1) = X(p^*(t_1)) > \bar{x}.$$

Choose a sequence $x_n \to \bar{x}$ such that $x_n < \bar{x}$ for all $n$, and let $\gamma_n = (y_n, p_n)$ be the optimal canonical trajectory starting at $(x_n, V'(x_n))$. Then $\gamma_n(0) < \bar{x}$ and, as $\gamma_n(t_1) \to (y^*(t_1), p^*(t_1))$, we have that $y_n(t_1) > \bar{x}$ if $n$ is sufficiently large. But then there is $0 < t_2 < t_1$ such that $y_n(t_2) = \bar{x}$ and $p_n(t_2) > \bar{p}$, which implies, since $H(\bar{x}, p)$ is minimal at $p = \bar{p}$, that

$$\rho V(\bar{x}) = H(\bar{x}, p_n(t_2)) > H(\bar{x}, \bar{p}) = \rho V(\bar{x}),$$

which is a contradiction. □

**Proposition D.29.** *The value function $V$ function fails to be analytic at only finitely many points.*

*Proof.* Proposition D.5 implies that if $(y^*, p^*)$ is an optimal canonical trajectory starting at $z = (x, V'(x))$, then $V''(y^*(t)) = \dot{p}^*(t)/\dot{y}^*(t)$. Hence $V'$ can only fail to be differentiable at $x$ if the optimal trajectory starting at $z$ satisfies $\dot{y}^*(0) = 0$.

By Lemma D.28, the former situation can only occur if $z$ is a steady state of the canonical vector field. Assume that it occurs at infinitely many points. Then there is a convergent sequence $z_n = (x_n, p_n) \to \bar{z}$ of steady states: for $n$ sufficiently large $z_n$ is contained in the centre-stable manifold of $\bar{z}$, which is analytic by the Aulbach theorem, contradicting that $V'$ is not differentiable at $x_n$. Hence $V'$ fails to be differentiable at only finitely many points.

Since the canonical vector field is not differentiable at $\mathcal{S}$, the third derivative $V'''(x)$ can fail to exist if $z \in \mathcal{S}$. This can occur only if $z$ is a switching point, that is, in every neighbourhood of $x$, there are points $x_1$ and $x_2$ such that $(x_1, V'(x_1)) \in \mathcal{P}_I$ and $(x_2, V'(x_2)) \in \mathcal{P}_\ell \cup \mathcal{P}_u$

Assume therefore that there are infinitely many switching points in $\mathcal{S}_\ell$ — the other case being similar — accumulating on $(\bar{x}, \bar{z}) = \bar{z} \in \mathcal{S}_\ell$. If $f(\bar{x}, q_\ell) \neq 0$, then there is an optimal canonical trajectory $\gamma(t)$ passing through infinitely many switching points. Consider the function $\psi(x) = F(x, p_\ell(x)) \cdot n_\ell(x)$, where $n_\ell$ is the vector normal to $\mathcal{S}_\ell$ at $(x, p_\ell(x))$ and pointing into $\mathcal{P}_I$. Let $x_1$ and $x_2$ be two consecutive switching states. Then either $\psi(x_1)$ and $\psi(x_2)$ are nonzero and have opposite signs, or either of them is zero. We conclude that in the closed interval $[x_1, x_2]$ whose boundary points are the switching states the function $\psi$ has a zero, and hence that $\psi$ has infinitely many zeros. Analyticity implies that $\psi$ is identically zero, hence that $\mathcal{S}_\ell$ is invariant. This contradicts the existence of infinitely many switching states.

If $V'$ is differentiable at $x_0$ and $z_0 = (x_0, V'(x_0))$ is not a steady state or a switching state, the canonical trajectory $(y(t), p(t))$ starting at $z$ is analytic in $t$ and satisfies $y'(t) \neq 0$. Hence we can solve $x = y(t)$ as $t = y^{-1}(x)$, locally around $x_0$, and obtain $V'(x) = p(t) = p(y^{-1}(x))$ as an analytic function. □

**Theorem D.30.** *The strategy $\phi$ given as*

$$\phi(x) = q^*(x, V'(x))$$

*for all $x$ such that $V$ is differentiable at $x$, and defined arbitrarily in all other points, is a Markovian best response in $\mathcal{S}$.*

*Proof.* Let $V$ be differentiable at a point $x$, and let $(y^*, c^*)$ be an optimal trajectory–control pair with initial point $x$. Let $q^*(x, p)$ denote the unique maximiser $u(x, q) + pf(x, q)$. Then proposition D.5 implies that

$$a^*(t) = q^*(y^*(t), V'(y^*(t)))$$

for all $t \geq 0$. This implies the first compatibility condition (16). The second compatibility condition (17) then holds automatically. As $V'$ is analytic except at finitely many points, so is $\phi$. □

# E Proofs of Proposition 4.2 and Theorem 4.3

*Proof of Proposition 4.2.* Introduce

$$f_{i,j}^{\phi}(x) = f(x, \phi_{i,j}(x), \phi_{-i,j}(x)) \quad \text{and} \quad \mu_{i,j}^{\phi}(x) = |f_{i,j}^{\phi}(x)|/Z_{i,j}^{\phi}(x),$$

where $Z_{i,j}(x)$ is such that for $x = \bar{x}_j$ we have that $\mu_{i,j-1}^{\phi}(x) + \mu_{i,j}^{\phi}(x) = 1$.

If player $i$ plays strategy $\phi_i$ in response to the strategy profile $\phi_{-i}$, the dynamics take the form

$$\begin{aligned} \dot{y}(t) &= f_{i,j}^{\phi}(y(t)) && y(t) \in \mathfrak{X}_j, \\ \dot{y}(t) &= \mu_{i,j-1}^{\phi}(y(t)) f_{i,j-1}^{\phi}(y(t)) + \mu_{i,j}^{\phi}(y(t)) f_{i,j}^{\phi}(y(t)), && y(t) \in \mathfrak{I}_j. \end{aligned}$$

Given the trajectory $y$ with initial state $x$, the payoff is

$$V_i^{\phi}(x) = \int_0^{\infty} \exp(-\rho t) u_i(y(t), \phi_i(y(t))) \, \mathrm{d}t.$$

It is clear that $V_i^{\phi}$ is bounded, as $u_i$ is bounded and $\rho > 0$.

If $x \in \operatorname{int} \mathfrak{X}_j$ satisfies $f_{i,j}^{\phi}(x) = 0$, then $V_i^{\phi}(x) = u_i(x, \phi_{i,j}(x))/\rho$. For $x \in \operatorname{int} \mathfrak{X}_j$ such that $f_{i,j}^{\phi}(x) \neq 0$, we have

$$\begin{aligned} V_i^{\phi}(y(t)) - V_i^{\phi}(x) &= V_i^{\phi}(x + t f_{i,j}^{\phi}(x) + o(t)) - V_i^{\phi}(x) \\ &= (\exp(\rho t) - 1) V(x) - \exp(\rho t) \int_0^t \exp(-\rho s) u_i(y(s), \phi_{i,j}(s)) \, \mathrm{d}s, \end{aligned}$$

which on dividing by $t$ and taking the limit $t \to 0$ yields

$$\left(V_i^{\phi}\right)'(x) f_{i,j}^{\phi}(x) = \rho V_i^{\phi}(x) - u_i(x, \phi_{i,j}(x)). \tag{33}$$

This implies that $V_i^{\phi}$ is real analytic whenever $x \in \operatorname{int} \mathfrak{X}_j$ and $f_{i,j}^{\phi}(x) \neq 0$, as it is the solution of an ordinary differential equation with real analytic right hand side.

If there are infinitely many distinct points $x_n \in \operatorname{int} \mathfrak{X}_j$ such that $f_{i,j}^{\phi}(x_n) = 0$, then by real analyticity we have $f_{i,j}^{\phi}(x) = 0$ for all $x \in \operatorname{int} \mathfrak{X}_j$ and $V_i^{\phi}(x) = u_i(x, \phi_{i,j}(x))/\rho$ is analytic on the whole interval.

Hence $V_i^{\phi}$ can fail to be real analytic for at most finitely many points. □

If we introduce

$$H_{i,j}^{\phi}(x, p) = u_i(x, \phi_{i,j}(x)) + p f_{i,j}^{\phi}(x),$$

then Equation (33) can be reformulated as the HJB equation

$$\rho V_i^{\phi}(x) = H_{i,j}^{\phi}\left(x, (V_i^{\phi})'(x)\right)$$

whenever $x \in \mathfrak{X}_j$.

*Proof of Theorem 4.3.* To prove the theorem, we have to show that Conditions b.–a. imply that $V_i^\phi$ satisfies (22), (23), (24) and (25). From this it then follows that $V_i^\phi = V_i$, and hence that $\phi_i$ is a best response.

Conversely, we have to show that if $\phi_i$ is a best response, then Conditions b.–a. hold true.

**Notations** We recall the notations $q^*_{i,j}(x,p)$ for the maximiser of $q_i \mapsto u_i(x,q_i)+pf_{i,j}(x,q_i)$, $H_{i,j}(x,p) = u_i(x, q^*_{i,j}(x,p)) + pf_{i,j}(x, q^*_{i,j}(x,p))$, as well as

$$p_{i,j,b}(x) = -\frac{(u_i)_{q_i}(x,q_b)}{(f_{i,j})_{q_i}(x,q_b)}$$

for $b \in \{\ell, u\}$. For a given $x$, we write the left and right limits of $(V_i^\phi)'$ at $x$ as

$$p_- = \lim_{z\uparrow x}(V_i^\phi)'(z) \quad \text{and} \quad p_+ = \lim_{z\downarrow x}(V_i^\phi)'(z).$$

We here allow the possibility that the limits take the values $-\infty$ or $\infty$.

### E.1 Sufficiency of the conditions

Assume that Conditions b.–a. hold.

**Subdifferentials and superdifferentials** For any point $x \in \mathcal{X}$ where $V_i^\phi$ is continuous, if $p_- < p_+$, then $D^- V_i^\phi(x) = [p_-, p_+]$ and $D^+ V_i^\phi(x) = \emptyset$; similarly, if $p_+ < p_-$, then $D^- V_i^\phi(x) = \emptyset$ and $D^+ V_i^\phi(x) = [p_+, p_-]$; finally if $p_- = p_+ = p$, then $D^- V_i^\phi(x) = D^+ V_i^\phi(x) = \{p\}$. The final situation occurs if and only if $V_i^\phi$ is differentiable at $x$.

Let $x \in \mathcal{X}$ be a point at which $V_i^\phi$ is not continuous. Condition b. implies then that

$$D^-(V_i^\phi)_*(x) = (-\infty, p_+] \quad \text{and} \quad D^+(V_i^\phi)^*(x) = (-\infty, p_-].$$

**Interior of $\mathcal{X}_j$** Take first $x \in \operatorname{int} X_j$: then $V_i^\phi$ is continuous at $x$.

If $V_i^\phi$ is differentiable at $x$, set $p = (V_i^\phi)'(x)$: by condition a. of the theorem we have that

$$u_i(x, \phi_i(x)) + pf_{i,j}(x, \phi_i(x)) = H_{i,j}(x,p) = \rho V_i^\phi(x) \tag{34}$$

where the last equality holds as $V_i^\phi$ is the value function of strategy $\phi_i$ to player $i$.

If $V_i^\phi$ is not differentiable at $x$, then $p_- \neq p_+$. Since $\phi_i$ is continuous at $x$, we have that $\phi_i(x) = q^*_{i,j}(x,p_-) = q^*_{i,j}(x,p_+)$, and therefore either $p_-, p_+ \leq p_{i,j,\ell}(x)$ or $p_-, p_+ \geq p_{i,j,u}(x)$. Take $p \in D^- V_i^\phi(x) \cup D^+ V_i^\phi(x)$: one of the two sets is empty. Then $q^*_{i,j}(x,p) = \phi_i(x) = q_b$, with $b \in \{\ell, u\}$. By Proposition 4.2 we have $f_{i,j}(x, q_b) = 0$ and $\rho V_i^\phi(x) = u_i(x, q_b)$.

It follows that

$$\rho V_i^\phi(x) = u_i(x,q_b) + pf_{i,j}(x,q_b) = u_i(x, q^*_{i,j}(x,p)) + pf_{i,j}(x, q^*_{i,j}(x,p)) = H_{i,j}(x,p). \tag{35}$$

Equations (34) and (35) together show (22).

**Interface points at which $V_i^\phi$ is continuous** Consider $x \in \mathcal{I}_j$, that is, $x = \bar{x}_j$ with $j \in \{1, \dots, N\}$, such that $V_i^\phi$ is continuous at $x$.

Assume that $D^-(V_i^\phi)(x)$ is nonempty. Writing $H_-(p) = H_{i,j-1}(x,p)$ and $H_+(p) = H_{i,j}(x,p)$, we have to show that

$$\rho V_i^\phi(x) \geq \min\{H_-(p), H_+(p)\}$$

for all $p \in D^-(V_i^\phi)(x)$. By continuity, $\rho V_i^\phi(x) = H_-(p_-) = H_+(p_+)$. Assume there is a point $\hat{p} \in (p_-, p_+)$ such that

$$H_-(p_-) = \rho V_i^\phi(x) < H_-(\hat{p}) \quad \text{and} \quad H_+(p_+) = \rho V_i^\phi(x) < H_+(\hat{p}).$$

By convexity of $H_-$ and $H_+$, it follows that $\hat{f}_- := (H_-)_p(\hat{p}) > 0$ and $\hat{f}_+ := (H_+)_p(\hat{p}) < 0$. Hence there are $\lambda_-, \lambda_+ > 0$ such that $\lambda_- + \lambda_+ = 1$ and $\lambda_- \hat{f}_- + \lambda_+ \hat{f}_+ = 0$. Set $u_- = u_{i,j-1}(x, q^*_{i,j-1}(x,\hat{p}))$ and $u_+ = u_{i,j}(x, q^*_{i,j}(x,\hat{p}))$. Condition e. then implies that

$$\begin{aligned}
\rho V_i^\phi(x) &\geq \lambda_- u_- + \lambda_+ u_+ = \lambda_- u_- + \lambda_+ u_+ + \hat{p}(\lambda_- \hat{f}_- + \lambda_+ \hat{f}_+) \\
&= \lambda_- H_-(\hat{p}) + \lambda_+ H_+(\hat{p}) \geq \min\{H_-(\hat{p}), H_+(\hat{p})\} > \rho V_i^\phi(x),
\end{aligned}$$

a contradiction, which proves (23a).

Next, assume that $D^+V_i^\phi(x)$ is nonempty. We now have to show that

$$\rho V_i^\phi(x) \leq \max\{H_-(p), H_+(p)\}$$

for all $p \in D^+(V_i^\phi)(x) = [p_+, p_-]$. Assume, as before, that the relation does not hold for some $\hat{p} \in (p_+, p_-)$, that is

$$H_-(p_-) = \rho V_i^\phi(x) > H_-(\hat{p}) \quad \text{and} \quad H_+(p_+) = \rho V_i^\phi(x) > H_+(\hat{p}).$$

Convexity now implies that $f_- := (H_-)_p(p_-) > 0$ and $f_+ := (H_+)_p(p_+) < 0$. By Condition f., this implies that $V_i^\phi$ is differentiable at $x$, and hence $p_- = p_+$ and $\rho V_i^\phi(x) = H_-(p) = H_+(p)$ for all $p \in D^+V_i^\phi(x)$. Hence (23b) holds.

**Interface points at which $V_i^\phi$ is not continuous** The next situation to consider is $x \in \mathcal{I}_j$ with $j \in \{1, \dots, N\}$, such that $V_i^\phi$ is not continuous at $x$.

To show that (23a) holds in this case, take $p \in D^-(V_i^\phi)_*$ and assume that $\rho(V_i^\phi)_*(x) < \min\{H_-(p), H_+(p)\}$. Then we have in particular that $p < p_+$ and $H_+(p_+) = \rho(V_i^\phi)_*(x) < H_+(p)$. Convexity of $H_+$ implies that $(H_+)_p(p) < 0$. This however contradicts Condition d., which implies that $(H_+)_p(p) \geq 0$ for all $p$.

Turning to (23b), take $p \in D^+(V_i^\phi)^*$ and assume that $\rho(V_i^\phi)^*(x) > \max\{H_-(p), H_+(p)\}$. This implies $p < p_-$ and $H_-(p_-) = \rho(V_i^\phi)^*(x) > H_-(p)$. Again invoking convexity, this time of $H_-$, we obtain that $(H_-)_p(p_-) > 0$. Again, this is incompatible with condition d.

**Boundary points** We only consider the situation that $x = \bar{x}_0$, the other being entirely analogous. At this boundary point, we have that $D^+V_i^\phi(x) = [p_+, \infty)$ and $D^-V_i^\phi(x) = (-\infty, p_+]$. We write $H(p)$ for $H_{i,0}(x,p)$.

It follows from condition d that $V_i^\phi$ is continuous at $x$. To prove (24a), assume that $V_i^\phi(x) - \beta(x) < 0$. Then there is no action that can take the dynamics out of $\mathcal{X}$, which implies that $f(x, q^*(x,p)) =$

$H_p(p) \geq 0$ for all $p$. Since $\rho V(x) - H(p_+) = 0$ by continuity, it follows that $\rho V(x) - H(p) \geq 0$ for all $p \in (-\infty, p_+] = D^- V_i^\phi(x)$, which implies (24a).

To show (24b), assume that $V_i^\phi(x) - \beta(x) > 0$. Then necessarily $f(x, q^*(x, p_+)) = H_p(p_+) \geq 0$, and, by convexity of $H(p)$, it follows that $H(p) \geq H(p_+)$ for all $p > p_+$, implying that $\rho V(x) - H(p) \leq \rho V(x) - H(p_+) = 0$ for all $p \in D^+ V_i^\phi(x)$.

This concludes the proof of the sufficiency statement.

## E.2 Necessity of the conditions

To prove the necessity of Conditions b.–a., assume that $\phi_i = \mathscr{B}(\phi_{-i})$, which implies that $V_i^\phi = V_i$ and that $V_i$ is a viscosity solution of (21).

Condition b. follows from Proposition A.3.

We show Condition c., for $x = \bar{x}_0$, the other case being analogous. Set $H(p) = H_{j,0}(x, p)$. If $V(x) - \beta(x) > 0$, then (24b) implies that $\rho V(x) - H(p) \leq 0$ for all $p \geq p_+$. As $\rho V(x) - H(p_+) = 0$, it follows that $H(p) \geq H(p_+)$ for all $p \geq p_+$, and hence $H_p(p_+) = f_{i,0}^\phi(x) \geq 0$.

If $V(x) - \beta(x) < 0$, by (24a) we have that $\rho V(x) - H(p) \geq 0$ for all $p \leq p_+$, which implies as before that $H_p(p_+) = f_{i,0}^\phi(x) \geq 0$.

Condition d.

If $V_i$ fails to be continuous at $x$, then $x = \bar{x}_j$ for some $j$ and $f_{i,j}(x, q_\ell) \geq 0$ according to Theorem C.7. It therefore remains to be shown that $f_{i,j-1}^\phi(x) = (H_{i,j-1})_p(x, p) \geq 0$.

Proposition C.4, point f. implies that $V_i^*(x) = \max\{V_{i,-}^{\mathrm{sc}}(x), V_{i,\jmath}^{\mathrm{sc}}(x)\}$.

If $V_i^*(x) = V_{i,\jmath}^{\mathrm{sc}}(x)$, then according to (25), we have $\rho V_{i,*}(x) \geq H_\jmath^d(x) = \rho V_{i,\jmath}^{\mathrm{sc}}(x) = \rho V_i^*(x)$, and $V_i$ is actually continuous at $x$, which is ruled out by hypothesis.

So assume that $V_i^*(x) = V_{i,-}^{\mathrm{sc}}(x)$, then for all $p \in D^+ V_i^*(x) = (-\infty, p_-]$ we have that

$$H_{i,j-1}(x, p_-) = \rho V_i^*(x) \leq H_{i,j-1}(x, p),$$

and $(H_{i,j-1})_p(x, p_-) = f_i^\phi(x) \leq 0$, which had to be proved.

Condition e. is a direct consequence of (25).

To show Condition f., assume that $x$ is a strong push–push steady state. We introduce the notations $f_- = \lim_{z \uparrow x} f_{i,j-1}^\phi(z)$ and $f_+ = \lim_{z \downarrow x} f_{i,j}^\phi$, as well as $H_-(p) = H_{i,j-1}(x, p)$ and $H_+(p) = H_{i,j}(x, p)$, $u_- = u_i(x, q_{i,j-1}^*(x, p_-))$ and $u_+ = u_i(x, q_{i,j}^*(x, p_+))$, and $q_-^*(p) = q_{i,j-1}^*(x, p)$ and $q_+^*(p) = q_{i,j}^*(x, p)$.

We have $f_- > 0 > f_+$. Let $\lambda_-, \lambda_+ \in (0, 1)$ be such that $\lambda_- + \lambda_+ = 1$ and $\lambda_- f_- + \lambda_+ f_+ = 0$. Then $\rho V(x) = \lambda_- u_- + \lambda_+ u_+$. We also have $\rho V(x) = H_-(p_-) = H_+(p_+)$. Combining these equalities, we see that

$$\begin{aligned} 0 &= \rho V(x) - (\lambda_+ u_+ + \lambda_- u_-) = \lambda_- H_-(p_-) + \lambda_+ H_+(p_+) - (\lambda_+ u_+ + \lambda_- u_-) \\ &= \lambda_- p_- f_- + \lambda_+ p_+ f_+ = \lambda_-(p_- - p_+) f_-. \end{aligned}$$

As $\lambda_- \neq 0$ and $f_- \neq 0$, we infer that $p_- = p_+ = p^*$ and $V_i^\phi$ is differentiable at $x$, proving Condition f.

If $x \in \operatorname{int} \mathcal{X}_j$ and $V_i$ is differentiable at $x$, then $D^- V_i(x) = D^+ V_i(x) = \{V_i'(x)\}$, and (22a) and (22b) imply that $\rho V_i(x) = H_{i,j}(x, V_i'(x))$. Moreover, since $V_i = V_i^\phi$, we also have that $V_i'(x) = (V_i^\phi)'(x) =: p$ and $H_{i,j}(x, p) = H_{i,j}^\phi(x, p)$, which is equivalent to

$$u_i(x, \phi_{i,j}(x)) + p f(x, \phi_{i,j}(x), \phi_{-i,j}(x)) = \max_{q_i} \left( u_i(x, q_i) + p f(x, q_i, \phi_{-i,j}(x)) \right),$$

which implies Condition a.

This completes the proof of Theorem 4.3. □

# F Proofs of results

## F.1 Proof of Lemma A.4

*Proof.* Let $\sigma$ be the Borel measure on $[0,\infty)$ defined by

$$\sigma([a,b]) = \int_a^b \exp(-\rho t)\,\mathrm{d}t = (\exp(-\rho a) - \exp(-\rho b))/\rho.$$

The set $\{t \geq 0 \,:\, y(t) \neq x\}$ can be written as the union of at most countably many intervals $I_k = (a_k, b_k)$ such that $\sigma(I_k) > 0$ and $y(a_k) = y(b_k) = x$, where $k = 1, 2, \dots, K$, and where $K = \infty$ is allowed. Let $I_0 = [0,\infty)\setminus \bigcup_{k=1}^{K} I_k$: this set is measurable, possibly of measure 0.

For $0 \leq k \leq K$ such that $\sigma(I_k) > 0$, introduce

$$v_k \equiv \frac{1}{\sigma(I_k)} \int_{I_k} v(y(t), c(t)) \exp(-\rho t)\,\mathrm{d}t,$$

and set $v_0 = 0$ if $\sigma(I_0) = 0$. Then

$$U(y,c) = \int_0^\infty v(y(t), c(t)) \exp(-\rho t)\,\mathrm{d}t = \sum_{k=0}^{K} v_k \sigma(I_k).$$

If $v_k < \rho U(y,c)$ for all $k$ such that $\sigma(I_k) > 0$, then necessarily

$$U(y,c) = \sum_{k=0}^{K} v_k \sigma(I_k) < \rho U(y,c) \sum_{k=0}^{K} \sigma(I_k) = \rho U(y,c)\sigma([0,\infty)) = U(y,c),$$

which is a contradiction. Hence $v_k \geq \rho U(y,c)$ and $\sigma(I_k) > 0$ for some $k$.

Assume first that $k > 0$ and $y(t) > x$ for $t \in I_k$. Set $\Delta = b_k - a_k > 0$ and construct a trajectory-control pair by setting for $\ell = 0, 1, 2, \dots$

$$(\tilde{y}(t), \tilde{c}(t)) = (y(a_k + t - \ell\Delta), c(a_k + t - \ell\Delta)), \qquad \text{if} \quad \ell\Delta \leq t < (\ell+1)\Delta$$

We have

$$\begin{aligned}
U(\tilde{y}, \tilde{c}) &= \int_0^\infty v(\tilde{y}(t), \tilde{c}(t)) \exp(-\rho t)\,\mathrm{d}t \\
&= \sum_{\ell=0}^{\infty} \int_{\ell\Delta}^{(\ell+1)\Delta} v(y(a_k + t - \ell\Delta), c(a_k + t - \ell\Delta)) \exp(-\rho t)\,\mathrm{d}t \\
&= \exp(\rho a_k) \int_{a_k}^{b_k} v(y(s), c(s)) \exp(-\rho s)\,\mathrm{d}s \sum_{\ell=0}^{\infty} \exp(-\rho\ell\Delta) \\
&= \frac{1 - \exp(-\rho\Delta)}{\rho} v_k \frac{1}{1 - \exp(-\rho\Delta)} = v_k/\rho \geq U(y,c).
\end{aligned}$$

Moreover $\tilde{y}(t) > x$ for almost all $t \geq 0$. Hence we have constructed the required trajectory.

The argument for the situation that $y(t) < x$ for $t \in I_k$ is entirely analogous.

Finally assume that $\sigma(I_0) > 0$ and $v_0 \geq \rho U(y,c)$. Then the set $C_0$ of constant action–weight pairs $\kappa = (q, \lambda)$ such that $g(x,\kappa) = 0$ is non-empty. As $C_0$ is compact, there is a maximiser $\bar{\kappa}$ of $v(x,\kappa)$ restricted to $C_0$. Let $(\tilde{y}, \tilde{c})$ be the trajectory–action pair $\tilde{y}(t) = x$, $\tilde{c}(t) = \kappa$ for all $t$. Then $v(y,c) \leq v(\tilde{y}, \tilde{c})$ for all $t \in I_0$, and

$$\begin{aligned}\rho U(y,c) = v_0 &\leq \frac{1}{\sigma(I_0)} \int_{I_0} v(\tilde{y}(t), \tilde{c}(t)) \exp(-\rho t)\,\mathrm{d}t = \frac{1}{\sigma(I_0)} \int_{I_0} v(x, \bar{\kappa}) \exp(-\rho t)\,\mathrm{d}t \\ &= v(x, \bar{\kappa}) = \rho U(\tilde{y}, \tilde{c}),\end{aligned}$$

completing the construction of the trajectory also in this situation. □

## F.2 Proof of Lemma A.5

*Proof.* Let $\mathcal{I} = \bigcup_j \mathcal{I}_j = \{\bar{x}_1, \ldots, \bar{x}_J\}$ be the union of all interface points, and let $\Delta = \min_{j \neq k} |\bar{x}_j - \bar{x}_k|$. Let $M > 0$ be such that $|f_j(x,q)| \leq M$ for all $j$ and all $(x,q) \in \mathcal{X}_j \times \mathcal{Q}$. Introduce moreover for an arbitrary trajectory–control pair $(y,c)$ the set

$$S_j(y,c) \equiv \{t \geq 0 \,:\, y(t) = \bar{x}_j,\ f_{j-1}(y(t), a_{j-1}(t)) < 0\,,\ f_j(y(t), a_j(t)) > 0\}.$$

If $(y,c)$ is not regular, the union $\bigcup_j S_j(y,c)$ has positive Lebesgue measure.

Let $(y^{(0)}, c^{(0)})$ be a non-regular trajectory–control pair. For $\ell = 1, 2, \ldots$, we shall inductively construct a sequence $(y^{(\ell)}, c^{(\ell)})$ of trajectory–control pairs, such that $S_j(y^{(\ell)}, c^{(\ell)}) \cap [0, \Delta/(\ell M))$ has measure zero for every $j$ and $U(y^{(\ell+1)}, c^{(\ell+1)}) \geq U(y^{(\ell)}, c^{(\ell)})$ for all $\ell \geq 0$.

Assume that $(y^{(\ell)}, c^{(\ell)})$ has already been constructed. Let

$$\tau = \inf\{t_1 \geq 0 \,:\, S_j(y^{(\ell)}, c^{(\ell)}) \cap [0, t_1] \text{ has positive measure for some } j\}.$$

If $\tau > (\ell+1)M/\Delta$, then we set $(y^{(\ell+1)}, c^{(\ell+1)}) = (y^{(\ell)}, c^{(\ell)})$.

If not, then $y^{(\ell)}(\tau) = \bar{x}_j$ for some $j$. Introduce

$$(y_\tau(t), c_\tau(t)) = (y^{(\ell)}(t-\tau), c^{(\ell)}(t-\tau)).$$

By Lemma A.4, there is a trajectory–control pair $(y, c_y)$ such that either $y(t) < \bar{x}_j$ for almost all $t \geq 0$, or $y(t) > \bar{x}_j$ for almost all $t \geq 0$, or $y(t) = \bar{x}_j$ for all $t \geq 0$, and $U(y, c_y) \geq U(y_\tau, c_\tau)$. In the first two cases, $y(t) \notin \mathcal{I} \backslash \{\bar{x}_j\}$ for all $0 \leq t < \Delta/M$, as for those values of $t$ we have $|y(t) - y(0)| \leq Mt < \Delta$. In this case we set

$$(y^{(\ell+1)}(t), c^{(\ell+1)}(t)) = \begin{cases} (y^{(\ell)}(t), c^{(\ell)}(t)), & \text{for } 0 \leq t < \tau, \text{ and} \\ (y(t-\tau), c_y(t-\tau)), & \text{for } t \geq \tau. \end{cases} \tag{36}$$

Then

$$\begin{aligned} U(y^{(\ell+1)}, c^{(\ell+1)}) &= \int_0^\tau v(y^{(\ell)}(t), c^{(\ell)}(t)) \exp(-\rho t)\,\mathrm{d}t + \exp(-\rho\tau) U(y, c_y) \\ &\geq \int_0^\tau v(y^{(\ell)}(t), c^{(\ell)}(t)) \exp(-\rho t)\,\mathrm{d}t + \exp(-\rho\tau) U(y^{(\ell)}, c^{(\ell)}) \\ &= U(y^{(\ell)}, c^{(\ell)}). \end{aligned}$$

In the third case, according to Lemma A.4, we may assume that $(y, c_y)$ is generated by a constant control $c_y(t) = (q, \lambda)$ for all $t \geq 0$. If $(y, c_y)$ is a regular trajectory, then we set $(y^{(\ell+1)}, c^{(\ell+1)})$ as in

(36) and this furnishes the desired $(\tilde{x}, \tilde{c})$. If $(y, c_y)$ is singular, then in particular $f_{j-1}(\bar{x}_j, q_{j-1}) < 0$. Consider the trajectory–control pair $(z, c_y)$ that satisfies $z(0) = \bar{x}_j$ and $\dot{z}(t) = f_{j-1}(z(t), q_{j-1})$ for $0 \leq t < M/\Delta$. As before, we have that $z(t) \notin \mathcal{I}$ for $0 < t < M/\Delta$ and, as $f_{j-1}(\bar{x}_j, q_{j-1}) < 0$, we have that $z(t) < \bar{x}_j = y(t)$ for all $t > 0$. By Lemma A.2, it follows that $U(z, c_y) > U(y, c_y)$. Setting

$$(y^{(\ell+1)}(t), c^{(\ell+1)}(t)) = \begin{cases} (y^{(\ell)}(t), c^{(\ell)}(t)), & \text{for } 0 \leq t < \tau, \\ (z(t-\tau), c_y(t-\tau)) & \text{for } t \geq \tau \end{cases} \tag{37}$$

and noting that also in this case $U(y^{(\ell+1)}, c^{(\ell+1)}) \geq U(y^{(\ell)}, c^{(\ell)})$ finishes the inductive step.

The induction either breaks off at the $\ell$'th step and produces a regular trajectory, as indicated, or it continuous indefinitely. In the latter case, we set

$$(\tilde{y}(t), \tilde{c}(t)) = \lim_{\ell \to \infty} (y^{(\ell)}(t), c^{(\ell)}(t)).$$

Then $S_j(\tilde{y}, \tilde{c})$ has measure zero for all $j$ and $\tilde{y}$ is regular also in this case. □

### F.3 Proof of Proposition C.4

Recall the definitions of $q^{\max}$ and $w^{\max}$ from Equation (20).

**Lemma F.1.** *Let $\bar{x} \in \mathcal{I}$.*

*a. If $V^{\text{sc}}_{\mathcal{I}}(\bar{x}) > \max\{V^{\text{sc}}_{-}(\bar{x}), V^{\text{sc}}_{+}(\bar{x})\}$, then $V^{\text{sc}}_{\mathcal{I}}(\bar{x}) = w^{\max}(\bar{x})$.*

*b. $V^{\text{sc}}_{+}(\bar{x}) \leq w^{\max}(\bar{x})$.*

*Proof.* The first statement is a direct consequence of Lemma **??**: note that the payoff with $f^- = 0$ or $f^+ = 0$ can be achieved exactly with the appropriate state constrained solution.

For the second, it is sufficient to note that $\rho w^{\max}(\bar{x}) = u(\bar{x}, q^{\max}(\bar{x})) > v(y(t), c(t))$ for any trajectory–control pair $(y, c)$ that satisfies $y(t) \in \mathcal{X}_+$ for all $t \geq 0$. □

*Proof of Proposition C.4.*

(a). This is a direct corollary of Lemma A.4.

(b). Assume that the dynamics are right controllable at $\bar{x}$: the argument for left controllability is exactly similar.

By controllability and continuity of $f_+$, there are $\delta, \varepsilon > 0$ such that $[-\varepsilon, \varepsilon] \subset f_+(\bar{x}, \mathcal{Q})$ for all $\bar{x} \leq x \leq \bar{x} + \delta$. Take $x_1, x_2 \in [\bar{x}, \bar{x} + \delta]$ as well as $\sigma \in \{-\varepsilon, \varepsilon\}$ such that $y(t) = x_1 + \sigma t$ satisfies $y(0) = x_1$ and $y(\tau) = x_2$ if $\tau = |x_2 - x_1|/\varepsilon$.

As $|\dot{y}(t)| = \varepsilon$ and $y(t) \in [\bar{x}, \bar{x} + \delta]$ for $0 \leq t \leq \tau$ there is $a(t)$ such that $\dot{y}(t) = f(y(t), a(t))$ for all $0 \leq t \leq \tau$. Then

$$V(x_1) \geq \int_0^\tau u(y(t), a(t)) \exp(-\rho t) \, \mathrm{d}t + V(x_2) \exp(-\rho \tau).$$

As $|V(x)| \leq \|u\|_\infty / \rho$ for all $x$, we obtain

$$V(x_1) - V(x_2) \geq -(\exp(-\rho\tau) - 1)\|u\|_\infty/\rho - \|u\|_\infty \tau \geq -2\|u\|_\infty \tau.$$

Interchanging the roles of $x_1$ and $x_2$, and using the definition of $\tau$, then gives

$$|V(x_1) - V(x_2)| \leq \frac{2\|u\|_\infty}{\varepsilon} |x_1 - x_2|$$

(c). Let $\bar{x}$ be a right semi-attractor: then $f_+(\bar{x}, q_u) \leq 0$. Take $\varepsilon > 0$. We shall consider initial conditions $x > \bar{x}$ that are close to $\bar{x}$: the precise degree of proximity will be specified below.

For each $x$, find a regular trajectory–control pair $(y, c)$, where $c(t) = (a_-(t), a_+(t), \mu(t))$, such that $y(0) = x$ and

$$V(x) \leq \int_0^\infty v(y(t), c(t)) \exp(-\rho t) \, \mathrm{d}t + \varepsilon/2.$$

According to Lemma **??**, we can assume that $a_-(t) = a_+(t) = a(t)$ and $v(y(t), c(t)) = u(y(t), a(t))$ for all $t$.

Introduce $\tau = \inf\{t \geq 0 : y(t) \notin \operatorname{int} \mathcal{X}_+\}$ and let $T$ be the unique solution of $\exp(-\rho T)\|u\|_\infty/\rho = \varepsilon/2$ if that solution is positive; otherwise, set $T = 0$. Let moreover $L_f > 0$ be such that $f_+(z, q) \leq L_f(z - \bar{x})$ for all $z \in \mathcal{X}_+$.

From the Gronwall inequality we obtain the inequality $0 \leq y(t) - \bar{x} \leq \exp(L_f t)(x - \bar{x})$ for all $0 \leq t \leq \tau$.

The function $w^{\max}$ is Lipschitz continuous; denote its Lipschitz constant by $L_w$. We now specify that the initial state $x$ should satisfy $0 < x - \bar{x} < \varepsilon/(2\rho L_w T \exp(L_f T))$.

Choosing $x(0) = \bar{x}$ and $q(t) = q^{\max}(y(t))$ implies, first, that $y(t) \leq \bar{x}$ for all $t$, and second, that $V(\bar{x}) \geq w^{\max}(\bar{x})$. It then follows for $\tau > T$ that

$$\begin{aligned}
V(x) &\leq \int_0^T u(y(t), a(t)) \exp(-\rho t) \, \mathrm{d}t + \varepsilon/2 \leq \rho \int_0^T w^{\max}(y(t)) \exp(-\rho t) \, \mathrm{d}t + \varepsilon/2 \\
&\leq \rho \int_0^T \Big( w^{\max}(\bar{x}) + L_w(y(t) - \bar{x}) \Big) \exp(-\rho t) \, \mathrm{d}t + \varepsilon/2 \\
&\leq V(\bar{x}) + \int_0^T \rho L_w \exp(L_f t)(x - \bar{x}) \exp(-\rho t) \, \mathrm{d}t + \varepsilon/2 \leq V(\bar{x}) + \varepsilon.
\end{aligned}$$

This shows right upper semi-continuity of $V$ at 0.

If $\tau \leq T$, then by the mean value theorem we can find $\vartheta(t) \in (0, 1)$ such that

$$\begin{aligned}
V(x) &\leq \int_0^\tau u(y(t), a(t)) \exp(-\rho t) \, \mathrm{d}t + V(\bar{x}) \exp(-\rho\tau) + \varepsilon/2 \\
&\leq \int_0^\tau \Big( u(\bar{x}, a(t)) + u_x(\bar{x} + \vartheta(t)(y(t) - \bar{x}), a(t))(y(t) - \bar{x}) \Big) \exp(-\rho t) \, \mathrm{d}t \\
&\qquad + V(\bar{x}) \exp(-\rho\tau) + \varepsilon/2 \\
&\leq \int_0^\tau \rho w^{\max}(\bar{x}) \exp(-\rho t) \, \mathrm{d}t + \rho L_w \tau \exp(L_f \tau)(x - \bar{x}) + V(\bar{x}) \exp(-\rho\tau) + \varepsilon/2 \\
&\leq V(\bar{x}) + \varepsilon,
\end{aligned}$$

again implying right upper semi-continuity.

Proposition IV.3.4 of Bardi and Capuzzo-Dolcetta (2008) implies that the value function is also lower semi-continuous at $\bar{x}$. This then establishes right continuity.

Finally, if there is a regular trajectory starting at $\bar{x}$ and remaining in $\mathcal{X}_+$ for all $t \geq 0$, it must be equal to $y(t) = \bar{x}$. Hence $V_+^{\mathrm{sc}}(\bar{x}) \leq V_{\mathcal{J}}^{\mathrm{sc}}(\bar{x})$, which shows the second part of the statement.

(d). Let $\bar{x}$ be a left semi-attractor.

If $f_-(\bar{x}, q_\ell) > 0$, then there are $m > 0$ and $\delta > 0$ such that $f_-(z, q) > m$ for all $z \in [\bar{x} - \delta, \bar{x}]$. Fix $x \in [\bar{x} - \delta, \bar{x}]$. Then for any regular trajectory–control pair $(y, c)$ with $y(0) = x$ there is $0 < \tau < |x - \bar{x}|/m$ such that $y(t) < \bar{x}$ for all $0 < t < \tau$ and $y(\tau) = \bar{x}$. But then

$$\begin{aligned} V(x) - V(\bar{x}) &= \sup \left( \int_0^\tau v(y(t), c(t)) \exp(-\rho t) \, \mathrm{d}t + (\exp(-\rho\tau) - 1) V(\bar{x}) \right) \\ &\leq 2\|u\|_\infty \tau = \frac{2\|u\|_\infty}{m} |x - \bar{x}|, \end{aligned}$$

which shows that $V$ is left upper semi-continuous at $\bar{x}$. Proposition IV.3.4 of Bardi and Capuzzo-Dolcetta (2008) again ensures left continuity.

If $f_-(\bar{x}, q_\ell) = 0$, take $\varepsilon > 0$ and $x < \bar{x}$. Let $T$ be the unique solution of $\exp(-\rho T)\|u\|_\infty/\rho = \varepsilon/2$ if that solution is positive; otherwise, set $T = 0$. Let $L_f > 0$ be such that $|f_-(z, q_\ell)| \leq L_f |z - \bar{x}|$ for all $z \in \mathcal{X}_-$. Let $(y, c)$ be a regular trajectory–control pair, with $y(0) = x$, such that

$$V(x) \leq \int_0^T v(y(t), c(t)) \exp(-\rho t) \, \mathrm{d}t + \varepsilon/4.$$

Set $\tau = \inf\{t \geq 0 \,:\, y(t) \in \mathcal{X}_+\}$. Again using the Gronwall inequality, we have that $-\exp(L_f t)|x - \bar{x}| \leq y(t) - \bar{x} \leq 0$ for all $0 \leq t \leq \tau$.

Let $\theta = \min\{\tau, T\}$. Take $\eta > 0$ and form the partition $T_1 \cup T_2 \cup T_3 \cup T_4$ of the interval $[0, \theta]$, where $T_1 = \{t \,:\, \dot{y}(t) > \eta\}$, $T_2 = \{t \,:\, 0 \leq \dot{y}(t) \leq \eta\}$, $T_3 = \{t \,:\, \dot{y}(t) < 0\}$, and $T_4 = \{t \,:\, y \text{ is not differentiable at } t\}$. Note that $T_4$ is a set of measure zero.

Clearly

$$y(\theta) - x = \int_0^\theta \dot{y}(t) \, \mathrm{d}t = \int_{T_1} + \int_{T_2} + \int_{T_3} \dot{y}(t) \, \mathrm{d}t.$$

Writing

$$\int_{T_1} \dot{y}(t) \, \mathrm{d}t = y(\theta) - x - \int_{T_2} \dot{y}(t) \, \mathrm{d}t - \int_{T_3} \dot{y}(t) \, \mathrm{d}t,$$

we infer, since $-\exp(L_f\theta)|x - \bar{x}| \leq y(\theta) - \bar{x} \leq 0$, that the measure $|T_1|$ of the first partitioning set satisfies

$$\begin{aligned} \eta|T_1| &\leq |y(\theta) - x| - \int_{T_3} \dot{y}(t) \, \mathrm{d}t \leq |y(\theta) - \bar{x}| + |\bar{x} - x| - \int_{T_3} f(y(t), q_\ell) \, \mathrm{d}t \\ &\leq (1 + \exp(L_f\theta))|x - \bar{x}| + \int_{T_3} L_f |y(t) - \bar{x}| \, \mathrm{d}t \\ &\leq (1 + \exp(L_f\theta) + L_f\theta \exp(L_f\theta)) \, |x - \bar{x}| =: C_1 |x - \bar{x}|. \end{aligned}$$

Consequently, the integral of the discounted flow payoff evaluated over $T_1$ is bounded by

$$\int_{T_1} u(y(t), c(t)) \exp(-\rho t) \, \mathrm{d}t \leq \|u\|_\infty |T_1| \leq \frac{C_1 \|u\|_\infty}{\eta} |x - \bar{x}|.$$

Let $L_u$ be such that $|u(z, q) - u(\bar{x}, q_\ell)| \leq L_u(|z - \bar{x}| + |q - q_\ell|)$ for all $(z, q)$. Then

$$\begin{aligned} \{t \,:\, \dot{y}(t) \leq \eta\} &= \{t \,:\, g(y(t), c(t)) \leq \eta\} = \{t \,:\, f(y(t), q_\ell) \leq \eta\} \\ &\subset \{t \,:\, q(t) - q_\ell \leq \eta + L_f |y(t) - \bar{x}|\} \\ &\subset \Big\{t \,:\, |u(y(t), q(t)) - u(\bar{x}, q_\ell)| \leq L_u(|y(t) - \bar{x}| + \eta + L_f |y(t) - \bar{x}|)\Big\}. \end{aligned}$$

This implies

$$\int_{T_2\cup T_3} u(y(t),q(t))\exp(-\rho t)\,\mathrm{d}t \le \int_0^\theta \Big(u(\bar{x},q_\ell)+L_u(1+L_f)|y(t)-\bar{x}|+\eta\Big)\exp(-\rho t)\,\mathrm{d}t$$
$$\le (1-\exp(-\rho\theta))u(\bar{x},q_\ell)/\rho + C_2|x-\bar{x}|+T\eta,$$

where $C_2 = L_u(1+L_f)\exp(L_f T)$.

Combining these estimates yields

$$\begin{aligned} V(x) &\le \int_{T_1}+\int_{T_2}+\int_{T_3} u(y(t),q(t))\,\mathrm{d}t + \exp(-\rho\theta)V(y(\theta))+\varepsilon/4 \\ &\le (1-\exp(-\rho\theta))u(\bar{x},q_\ell)/\rho + \exp(-\rho\theta)V(y(\theta)) \\ &\quad + \varepsilon/4 + \frac{C_1\|u\|_\infty}{\eta}|x-\bar{x}| + C_2|x-\bar{x}| + T\eta. \end{aligned}$$

Choose $\eta = \varepsilon/(12T)$ and $|x-\bar{x}| < \min\{\varepsilon^2/(12C_1\|u\|_\infty T), \varepsilon/(12C_2)\}$, and recalling that $V_-^{\mathrm{sc}}(\bar{x}) = u(\bar{x},q_\ell)/\rho \le V(\bar{x})$ to obtain that

$$V(x) \le (1-\exp(-\rho\theta))V(\bar{x}) + \exp(-\rho\theta)V(y(\theta)) + \varepsilon/2.$$

If $\theta = T$, then $\exp(-\rho\theta)V(y(\theta)) \le \varepsilon/2$ and

$$V(x) \le V(\bar{x}) + \varepsilon,$$

showing that $V$ is left upper semi-continuous at $\bar{x}$. If $\theta = \tau$, then $V(y(\theta)) = V(\bar{x})$ and

$$V(x) \le V(\bar{x}) + \varepsilon/2,$$

again showing that $V$ is left upper semi-continuous at $\bar{x}$. As lower semi-continuity is assured, it follows that $V$ is left continuous at $\bar{x}$.

(e). If $\bar{x}$ is a left semi-repeller, let $x(t)$ be a regular trajectory with associated control $c(t)$ such that $x(0)=\bar{x}$, $(q_{j-1}(t),q_j(t),\mu_j(t)) = (q^{\max}(\bar{x}),q^{\max}(\bar{x}),1)$ and $x(t)\in\mathfrak{X}_-$ for all $t\ge 0$. It is clear that such a trajectory exists. Then $V_-^{\mathrm{sc}}(\bar{x}) \ge U(x,c) \ge w^{\max}(\bar{x})$. As $V(\bar{x}) = \max\{V_-^{\mathrm{sc}}(\bar{x}), V_+^{\mathrm{sc}}(\bar{x})\}$, we have that $V$ is left continuous at $\bar{x}$.

(f). We enumerate the possibilities.

A. Consider first the situation that $\bar{x}$ is a left semi-repeller. Then no trajectory starting in $\operatorname{int}\mathfrak{X}_-$ can enter $\mathfrak{X}_+$, and the optimisation problem in $\mathfrak{X}_-$ is independent of that of $\mathfrak{X}_+$. Hence $V(x) = V_-^{\mathrm{sc}}(x)$ for all $x\in\operatorname{int}\mathfrak{X}_-$.

There are three possibilities for the right hand side behaviour. If $\bar{x}$ is also a right semi-repeller, then by a similar argument $V(x) = V_+^{\mathrm{sc}}(x)$ for all $x\in\operatorname{int}\mathfrak{X}_+$, and $V_-^{\mathrm{sc}}(\bar{x}) \ge w^{\max}(\bar{x}) \ge V_+^{\mathrm{sc}}(\bar{x})$. This implies that

$$V(\bar{x}) = V_-^{\mathrm{sc}}(\bar{x}),$$

and that $V$ is continuous at $\bar{x}$ if and only if $V_-^{\mathrm{sc}}(\bar{x}) = V_+^{\mathrm{sc}}(\bar{x}) = w^{\max}(\bar{x})$.

If the dynamics are right controllable at $\bar{x}$, or $\bar{x}$ is a right semi-attractor, then $V$ is also right continuous at $\bar{x}$, hence continuous, and $V_-^{\mathrm{sc}}$ furnishes a boundary value for the right hand problem: that is, the value function necessarily satisfies $V_-^{\mathrm{sc}}(\bar{x}) = V(\bar{x})$.

B. Next, consider the situation that the dynamics are left controllable at $\bar{x}$ and that $V$ is therefore left continuous at $\bar{x}$. If $\bar{x}$ is a right semi-repeller, then $V$ is continuous at $\bar{x}$ if and only if $V(\bar{x}) = V_+^{\mathrm{sc}}(\bar{x})$; if the dynamics are also right controllable at $\bar{x}$, they are not non-controllable at $\bar{x}$; finally, if $\bar{x}$ is a right semi-attractor, then $V$ is right continuous and hence continuous at $\bar{x}$.

C. Finally, let $\bar{x}$ be a left semi-attractor. Then $V$ is left continuous at $\bar{x}$.

If $\bar{x}$ is a right semi-attractor, or if the dynamics are right controllable at $\bar{x}$, then $V$ is also right continuous at $\bar{x}$, hence continuous. In the former situation we additionally have $V(\bar{x}) = w^{\max}(\bar{x})$.

If $\bar{x}$ is a right semi-repeller, then we need to make another distinction. If $f_-(\bar{x}, q_\ell) > 0$, then $V(\bar{x}) = V_+^{\mathrm{sc}}(\bar{x})$, and $V$ is continuous at $\bar{x}$. If however $f_-(\bar{x}, q_\ell) = 0$, then $V(\bar{x}) = \max\{V_-^{\mathrm{sc}}(\bar{x}), V_+^{\mathrm{sc}}(\bar{x})\}$ and $V$ is continuous if and only if $V_-^{\mathrm{sc}}(\bar{x}) = V_+^{\mathrm{sc}}(\bar{x})$.

□

## F.4 Proof of Theorem C.7

Theorem C.7 is an immediate consequence of the following comparison result.

**Proposition F.2.** *Let $v \in \mathscr{G}$ and $w \in \mathscr{G}$ be respectively a supersolution and a subsolution of* (21)*. Then $v(x) \geq w(x)$ for all $x \in \mathcal{X}$ where $v$ and $w$ are continuous.*

*Proof of Theorem C.7 from Proposition F.2.* The argument is standard. Let $V$ be the value function and $v$ another viscosity solution of (21). Let $x$ be such that $V$ and $v$ are continuous at $x$. As $V$ is a subsolution and $v$ a supersolution, $V(x) \leq v(x)$; as $v$ is a subsolution and $V$ a supersolution, $v(x) \leq V(x)$, implying that $v(x) = V(x)$.

The only points where $v$ and $V$ can be discontinuous are interface points, at which both $v$ and $V$ are left continuous. Hence, by continuity, $v(x) = V(x)$ even at these points. □

We shall first prove Proposition F.2 for the case that $v$ and $w$ are continuous. Then we indicate how the argumentation is modified if either of these functions is discontinuous.

The proof in the continuous case is a generalisation of the argument of Barles et al. (2013) to the situation of non-controllable interfaces. We need a number of technical results. These are stated in a form that allows them to be used also in the case that either the subsolution or the supersolution are discontinuous.

First, we formulate superoptimality and suboptimality principles for our context. Let $\bar{x}$ be an interface point, and introduce $\mathcal{X}_- = \{x \in \mathcal{X} : x \leq \bar{x}\}$ and $\mathcal{X}_+ = \{x \in \mathcal{X} : x \geq \bar{x}\}$. For $i \in \{-, +\}$, we write $\theta_i(x) = \min\{t \geq 0 : y(t; x) \notin \operatorname{int} \mathcal{X}_i\}$ for the exit time from $\mathcal{X}_i$ when starting at $x$, and we denote by $a \wedge b$ the minimum of $a, b \in \mathbb{R}$.

**Lemma F.3.** *Let $v : \mathcal{X} \to \mathbb{R}$ be a supersolution and $w : \mathcal{X} \to \mathbb{R}$ a subsolution of* (21)*, such that $v, w \in \mathscr{G}$. Then there are $a < \bar{x}$ and $t_0 > 0$ such that for all $0 \leq t \leq t_0$ and all $x \in I_- = [a, \bar{x}]$, we have for $\theta = \theta_-(x)$*

$$v(x) \geq \sup_c \left( \int_0^{t \wedge \theta} u(y(s; x), c(s)) \exp(-\rho s) \, \mathrm{d}s + \exp(-\rho(t \wedge \theta)) v(y(t \wedge \theta, x)) \right) \tag{38}$$

*and*

$$w(x) \leq \sup_c \left( \int_0^{t \wedge \theta} u(y(s; x), c(s)) \exp(-\rho s) \, \mathrm{d}s + \exp(-\rho(t \wedge \theta)) w(y(t \wedge \theta, x)) \right). \tag{39}$$

*If $v$ or $w$ are, respectively, continuous at $\bar{x}$, there is also $b > \bar{x}$ such that for all $x \in I_+ = [\bar{x}, b]$ and all $0 < t < t_0$, the respective inequalities* (38) *and* (39) *hold with $\theta$ replaced by $\theta_+(x)$.*

*Proof.* As the hypothesis implies that $v$ and $w$ restricted to $I_-$ and $I_+$ are continuous, and as $y(s;x)$ cannot reach an interface point different from $\bar{x}$ if $t_0 > 0$ is sufficiently small, equation (38) is implied by Bardi and Capuzzo-Dolcetta (2008, Corollary IV.3.14), and (39) is implied by Bardi and Capuzzo-Dolcetta (2008, Remark IV.3.16). □

Being a subsolution $w$ only gives information for points where the superdifferential $D^+w(x)$ is non-empty. This is not necessarily the case at the interface point, but, as the set of points with non-empty superdifferential is dense, it may give information for a sequence of points that converges to the interface. The next lemma states inferences about $w$ in both situations: if the first statement does not hold, the superdifferential is empty at the interface, and the second statement holds.

**Lemma F.4.** *Let $w \in \mathscr{G}$ be a subsolution of (21).*

*Then either of the following two statements holds.*

*A.* $\rho w(\bar{x}) - H^d(\bar{x}) \leq 0$

*B.* *(i) If $w$ is continuous at $\bar{x}$, then there is a constant $\eta > 0$, an index $i \in \{-,+\}$, and a sequence $x_k \to \bar{x}$ such that $x_k \in \mathcal{X}_i$ for all $k$, $w(x_k) \to w(\bar{x})$ as $k \to \infty$, and for each $k$ there is a trajectory–control pair $(y_k, c_k)$ such that $y_k(0) = x_k$, $y_k(t) \in \mathcal{X}_i$ for all $t \in [0, \eta]$ and*

$$w(x_k) \leq \int_0^\eta u(y_k(t), c_k(t)) \exp(-\rho t)\,\mathrm{d}t + w(y_k(\eta)) \exp(-\rho\eta) \tag{40}$$

*(ii) If $w$ is not continuous at $\bar{x}$, the same statement holds but with the sequence satisfying $x_k \in \mathcal{X}_-$ for all $k$.*

*Proof.* Introduce

$$\phi(\delta) := \rho w(\bar{x}) - \max\{H_-(\bar{x}, \delta), H_+(\bar{x}, \delta)\}.$$

Consider first the situation that $\phi(\delta) \leq 0$ for all $\delta$. Then, as $\phi(\delta)$ is an affine function of $\delta$ if $\delta < p_\ell(\bar{x})$ or $\delta > p_u(\bar{x})$, it follows that a maximiser $\bar{\delta}$ of $\phi$ exists.

The function $\phi$ is concave and maximal at $\bar{\delta}$, hence 0 is an element of the subgradient of $-\phi(\bar{\delta})$, which is the closed convex hull of the derivatives $(H_-)_p(\bar{x}, \bar{\delta})$ and $(H_+)_p(\bar{x}, \bar{\delta})$ (see Aubin, 1993, Corollary 4.4). Using the fact that $(H_j)_p(x, p) = f(x, q^*(x, p))$, and setting $\bar{q}^* = q^*(\bar{x}, \bar{\delta})$, this implies that there is $0 \leq \lambda \leq 1$ such that

$$\lambda f_-(\bar{x}, \bar{q}^*) + (1 - \lambda) f_+(\bar{x}, \bar{q}^*) = 0 \tag{41}$$

and $(\bar{q}^*, \bar{q}^*, \lambda) \in C_0$, where $C_0$ is the set of controls stabilising $\bar{x}$.

If the maximum in $\phi(\bar{\delta})$ is realised by respectively $H_-$ or $H_+$, but not by both, we have that $\lambda = 1$ or $\lambda = 0$. Therefore, if $0 < \lambda < 1$, then $H_-(\bar{x}, \bar{\delta}) = H_+(\bar{x}, \bar{\delta})$, and it follows that

$$\begin{aligned} 0 &\geq \phi(\bar{\delta}) \\ &= \rho w(\bar{x}) - (\lambda H_-(\bar{x}, \bar{\delta}) + (1 - \lambda) H_+(\bar{x}, \bar{\delta})) \\ &= \rho w(\bar{x}) - \lambda u(\bar{x}, \bar{q}^*) - (1 - \lambda) u(\bar{x}, \bar{q}^*) - \bar{\delta}(\lambda f_-(\bar{x}, \bar{q}^*) + (1 - \lambda) f_+(\bar{x}, \bar{q}^*)) \\ &\geq \rho w(\bar{x}) - H^d(\bar{x}), \end{aligned}$$

where in the last inequality we used (41) and the definition of $H^d$. This shows A.

Consider now the second situation, that there is $\bar{\delta}$ such that $\phi(\bar{\delta}) > 0$. Let $\varepsilon > 0$ and set

$$\psi_\varepsilon(x) = w(\bar{x}) + \bar{\delta}(x - \bar{x}) + \frac{(x - \bar{x})^2}{2\varepsilon^2}.$$

Now $\bar{x}$ cannot maximise $w - \psi_\varepsilon$ for any $\varepsilon > 0$, for if it did, $\psi'_\varepsilon(\bar{x}) = \bar{\delta} \in D^+ w(\bar{x})$, which would imply, as $w$ is a subsolution, that $\phi(\bar{\delta}) \leq 0$.

For every $\varepsilon > 0$ let $x_\varepsilon$ denote a maximiser of $w - \psi_\varepsilon$. Then $x_\varepsilon \neq \bar{x}$ and $0 = w(\bar{x}) - \psi_\varepsilon(\bar{x}) \leq w(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon)$, which implies that

$$\begin{aligned}
&(x_\varepsilon - \bar{x})^2 + 2\varepsilon^2 \bar{\delta}(x_\varepsilon - \bar{x}) \leq 2\varepsilon^2 (w(x_\varepsilon) - w(\bar{x})) \leq 4\varepsilon^2 \|w\|_\infty, \\
&((x_\varepsilon - \bar{x}) + \varepsilon^2 \bar{\delta})^2 \leq \varepsilon^2 \left(4\|w\|_\infty + \varepsilon^2 \bar{\delta}^2\right),
\end{aligned}$$

and finally $|x_\varepsilon - \bar{x}| \leq C\varepsilon$, where $C = (4\|w\|_\infty + \varepsilon^2\bar{\delta}^2)^{\frac{1}{2}} + \varepsilon\bar{\delta}$. So $x_\varepsilon \to \bar{x}$ as $\varepsilon \to 0$. In particular, if $\varepsilon > 0$ is sufficiently small, it is in a neighbourhood of $\bar{x}$ that only contains the single interface point $\bar{x}$.

We can say more about $x_\varepsilon$ if $w$ is discontinuous at $\bar{x}$. As $w$ is left continuous, there is $\sigma > 0$ such that $\lim_{x \downarrow \bar{x}} w(x) = w(\bar{x}) - \sigma$. Since $w$ is non-increasing, for $\bar{x} < x \leq \bar{x} + C\varepsilon$ we have that

$$w(x) - \psi_\varepsilon(x) \leq -\sigma - \bar{\delta}(x - \bar{x}) - \frac{(x - \bar{x})^2}{2\varepsilon^2} \leq -\sigma + C|\bar{\delta}|\varepsilon < 0$$

if $\varepsilon > 0$ is sufficiently small. Since $w(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) \geq 0$, it follows that $x_\varepsilon \leq \bar{x}$ for those values of $\varepsilon > 0$.

We select a sequence $\varepsilon_k > 0$ such that $\varepsilon_k \to 0$, an index $i \in \{-, +\}$, and a sequence of maximisers $x_k$ of $w - \psi_{\varepsilon_k}$ such that $x_k \in \mathcal{X}_i$ for all $k$; by the previous remark, $x_k \in \mathcal{X}_-$ for all $k$ if $w$ is discontinuous at $\bar{x}$. By Lemma F.3 there is $t_1 > 0$ such that for every $k > 0$ we have

$$w(x_k) \leq \sup_c \left( \int_0^{t_1 \wedge \theta_k} u(y(s; x_k), c(s)) \exp(-\rho s)\, \mathrm{d}s + \exp(-\rho(t \wedge \theta_k)) w(y(t_1 \wedge \theta_k, x_k)) \right), \tag{42}$$

where we have introduced $\theta_k := \theta_i(x_k)$.

If the alternative B holds, we are done. So assume that it does not hold. Then $\theta_k \to 0$ as $k \to \infty$ and $t_1 \wedge \theta_k = \theta_k$ for $k$ sufficiently large.

For every $k$, let $(y_k, c_k)$ be a trajectory–control pair that realises the supremum on the right hand side of (42). Note that $y_k(\theta_k; x_k) = \bar{x}$. From the fact that $w(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) \geq 0$, we derive $w(x_k) \geq \psi_{\varepsilon_k}(x_k) \geq w(\bar{x}) + \bar{\delta}(x_k - \bar{x})$. Combining this with (42) then yields

$$\begin{aligned}
0 &\leq \int_0^{\theta_k} u(y_k(s; x_k), c_k(s)) \exp(-\rho s)\, \mathrm{d}s + (\exp(-\rho\theta_k) - 1) w(\bar{x}) - \bar{\delta}(x_k - \bar{x}) \\
&= \int_0^{\theta_k} u(y_k(s; x_k), c_k(s)) \exp(-\rho s)\, \mathrm{d}s + \bar{\delta} \int_0^{\theta_k} f(y_k(s; x_k), c_k)\, \mathrm{d}s + (\exp(-\rho\theta_k) - 1) w(\bar{x}) \\
&\leq \int_0^{\theta_k} \max_{q \in \mathcal{Q}} \left[ u(y_k(s; x_k), q) \exp(-\rho s) + \bar{\delta} f(y_k(s; x_k), q) \right]\, \mathrm{d}s + (\exp(-\rho\theta_k) - 1) w(\bar{x}).
\end{aligned}$$

Dividing by $\theta_k$ and taking the limit $k \to \infty$ then yields

$$0 \leq H_i(\bar{x}, \bar{\delta}) - \rho w(\bar{x})$$

which implies that

$$\phi(\bar{\delta}) = \rho w(\bar{x}) - \max\{H_-(\bar{x}, \bar{\delta}), H_+(\bar{x}, \bar{\delta})\} \leq 0,$$

contradicting the choice of $\bar{\delta}$. □

We now have enough results to settle the continuous case.

**Lemma F.5.** *Let $v \in \mathscr{G}$ be a continuous viscosity supersolution, and $w \in \mathscr{G}$ a continuous viscosity subsolution of the HJB equation* (21)*. Then $v \geq w$ in $\mathcal{X}$.*

*Proof of Lemma F.5.* The continuous function $w - v$ takes on the compact set $\mathcal{X}$ a maximum $M$ at a point $\bar{x}$. Assume that $M > 0$, as otherwise the lemma is proved.

If $\bar{x}$ is neither an interface point nor a boundary point of $\mathcal{X}$, the proof uses the classical "doubling of variables" technique, (see Bardi and Capuzzo-Dolcetta, 2008, Theorem II.3.1), using the subsolution property of $w$ and the supersolution property of $v$, to derive a contradiction.

If $\bar{x} \in \partial\mathcal{X}$, then the boundary condition (24) implies that either $w(\bar{x}) \leq \beta(\bar{x}) \leq v(\bar{x})$, contradicting $M > 0$, or that either of the conditions $\rho w(\bar{x}) - H_j(\bar{x}, p) \leq 0$ for all $p \in D^+w(\bar{x})$ or $\rho v(\bar{x}) - H_j(\bar{x}, p) \geq 0$ for all $p \in D^-w(\bar{x})$ hold. The argumentation proceeds then as in the proof of Bardi and Capuzzo-Dolcetta (Theorem V.4.17 2008).

Hence we only have to consider the situation that $\bar{x}$ is an interface point. According to Lemma F.4, one of two alternatives can obtain. If alternative A is true, then

$$\rho w(\bar{x}) \leq H^d(\bar{x}) \leq \rho v(\bar{x}),$$

where the second inequality is implied by (25). This implies that $w(\bar{x}) - v(\bar{x}) = M \leq 0$, a contradiction.

Hence alternative B holds. According to this alternative, there is $\eta > 0$ and $i \in \{-, +\}$, and a sequence $x_k \to \bar{x}$, such that $x_k \in \mathcal{X}_i$ for all $k$ and there is a trajectory–control pair $(y_k, c_k)$ such that $y_k(0) = x_k$, $y_k(t) \in \mathcal{X}_i$ for all $t \in [0, \eta]$, and

$$w(x_k) \leq \int_0^\eta u(y_k(t), c_k(t)) \exp(-\rho t)\, \mathrm{d}t + w(y_k(\eta)) \exp(-\rho\eta). \tag{43}$$

Moreover, from (38) we obtain that for $k$ sufficiently large

$$v(x_k) \geq \int_0^\eta u(y_k(t), c_k(t)) \exp(-\rho t)\, \mathrm{d}t + v(y_k(\eta)) \exp(-\rho\eta). \tag{44}$$

Combining (43) and (44) yields

$$w(x_k) - v(x_k) \leq (w(y_k(\eta)) - v(y_k(\eta))) \exp(-\rho\eta) \leq M \exp(-\rho\eta).$$

Taking the limit $k \to \infty$ yields then $M \leq M \exp(-\rho\eta) < M$, again a contradiction. We conclude that necessarily $M \leq 0$. □

*Proof of Proposition F.2.* Let $D$ be the set of points at which either $v$ or $w$ is discontinuous. Then $D$ is a subset of the set of interface points, hence it is a finite set of isolated points. The proof proceeds by induction on the cardinality of $D$.

If $D$ is the empty set, the result is a consequence of Lemma F.5. Assume therefore that the result is proved if $D$ contains less than $n$ points.

Let $\bar{x} = \bar{x}_j$ be the smallest interface point in $D$, and set, as before, $\mathcal{X}_- = \{x \in \mathcal{X} : x \leq \bar{x}\}$ and $\mathcal{X}_+ = \{x \in \mathcal{X} : x \geq \bar{x}\}$. We shall show that the problem restricted to $\mathcal{X}_+$ is decoupled from the problem on $\mathcal{X}_-$; that is, we shall first show the comparison property $v \geq w$ on $\mathcal{X}_+$, which is then used to show the comparison property generally.

In order to apply the induction hypothesis on $v$ and $w$ restricted to $\mathcal{X}_+$, we have to show that the continuous extensions $\bar{v}$ and $\bar{w}$ of respectively $v$ and $w$ from $\mathcal{X}_+\backslash\{\bar{x}\}$ to $\mathcal{X}_+$ are respectively a supersolution and a subsolution on $\mathcal{X}_+$. □

**Lemma F.6.** *The functions $\bar{v}$ and $\bar{w}$ are respectively a supersolution and a subsolution of* (21) *on $\mathcal{X}_+$.*

*Proof.* We prove the lemma for the subsolution case; the supersolution case is similar.

By hypothesis, the subsolution property holds for all $x \in \mathcal{X}_+\backslash\{\bar{x}\}$. Assuming that the statement of the lemma is false, let $\psi$ be a $C^1$ function such that, firstly, $\psi(\bar{x}) = \bar{w}(\bar{x})$, secondly $\bar{w}(x) - \psi(x)$ restricted to $\mathcal{X}_+$ is maximal at $\bar{x}$, and finally

$$\rho\bar{w}(\bar{x}) - H_+(\bar{x}, \psi'(\bar{x})) > 0, \tag{45}$$

where $H_+ = H_j$. Introduce

$$\Delta(y) = \bar{w}(\bar{x} + y) - \psi(\bar{x} + y) - y^2.$$

Then $\Delta$ is continuous, maximal at $y = 0$, and $\Delta(0) = 0$. Continuity implies that for every $n > 0$ there is $\xi_n > 0$ such that $\Delta(\xi_n) > -1/n$. On the other hand, if $y \geq 2/\sqrt{n}$, then

$$\Delta(y) \leq -y^2 \leq -4/n.$$

It follows that $0 < \xi_n < 2/\sqrt{n}$.

Set $\varepsilon_n = \xi_n/(2n)$. The function

$$\Delta(y) - \varepsilon_n/y = \bar{w}(\bar{x} + y) - \big(\psi(\bar{x} + y) + y^2 + \varepsilon_n/y\big)$$

satisfies $\Delta(\xi_n) - \varepsilon_n/\xi_n \geq -3/n$ and $\Delta(y) - \varepsilon_n/y \leq -4/n$ if $y \geq 2/\sqrt{n}$. Hence it takes its maximum at a point $0 < y_n < 2/\sqrt{n}$, and, setting $x_n = \bar{x} + y_n$

$$p_n = \psi'(x_n) + 2y_n - \varepsilon_n/y_n^2 \in D^+\bar{w}(x_n).$$

We note that in particular, since $\Delta(y_n) - \varepsilon_n/y_n \geq \Delta(\xi_n) - \varepsilon_n/\xi_n \geq -3/n$, that $\varepsilon_n/y_n \leq \Delta(y_n) + 3/n \leq 3/n$, and hence that

$$p_n y_n = \psi'(x_n)y_n + 2y_n^2 - \varepsilon_n/y_n \to 0 \tag{46}$$

as $n \to \infty$. The function $\bar{w}$ is a subsolution in the point $x_n$, hence

$$\rho\bar{w}(x_n) - H_+(x_n, p_n) \leq 0. \tag{47}$$

Since $v, w \in \mathscr{G}$ and either function is discontinuous at $\bar{x}$, this point is a right semi-repeller under $f$. In particular, this implies that

$$(H_+)_p(\bar{x}, p) = f_+(\bar{x}, p) \geq 0 \tag{48}$$

for all $p$. Writing $q_n = q^*(x_n, p_n)$, we therefore have, for some $0 < \theta_n^{(1)}, \theta_n^{(2)} < 1$, that

$$\begin{aligned} H_+(x_n, p_n) &= u(\bar{x}, q_n) + u_x(\bar{x} + \theta_n^{(1)} y_n, q_n) y_n + p_n(f_+(\bar{x}, q_n) + (f_+)_x(\bar{x} + \theta_n^{(2)} y_n, q_n) y_n) \\ &= H_+(\bar{x}, p_n) + r_n \\ &\leq H_+(\bar{x}, \psi'(x_n) + 2y_n) + r_n. \end{aligned}$$

In the second equality, we have set $r_n = u_x(\bar{x} + \theta_n^{(1)} y_n, q_n) y_n + (f_+)_x(\bar{x} + \theta_n^{(2)} y_n, q_n) p_n y_n$; in the inequality, we have used that $p_n \leq \psi'(x_n) + 2y_n$ as well as (48). Since $u_x$ and $(f_+)_x$ are bounded, using (46) we obtain that $r_n \to 0$ as $n \to \infty$. Equation (47) then implies

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} \big(\rho \bar{w}(x_n) - H_+(x_n, p_n)\big) \\ &\geq \limsup_{n\to\infty} \big(\rho \bar{w}(x_n) - H_+(\bar{x}, \psi'(x_n) + 2y_n) - r_n\big) = \rho w(\bar{x}) - H_+(\bar{x}, \psi'(\bar{x})), \end{aligned}$$

contradicting (45). □

*Continuation of proof of Proposition F.2.* Lemma F.6 combined with the induction hypothesis applied to $\mathcal{X}_+$ rather than $\mathcal{X}$ then implies that $\bar{w} \leq \bar{v}$ on $\mathcal{X}_+$, which implies that $w \leq v$ on $\mathcal{X}_+ \backslash \{\bar{x}\}$.

On the compact set $\mathcal{X}_-$, the functions $v$ and $w$ are continuous. If $M = \max_{x \in \mathcal{X}_-}(w(x) - v(x))$ satisfies $M \leq 0$, the theorem is proved; so assume that $M > 0$, and that it is taken in a point $\hat{x} \in X_-$. If $\hat{x}$ is not an interface point, we apply again doubling of variables. That leaves the situation that $\hat{x}$ is an interface point. If $\hat{x} \notin D$, the argument goes as in the proof of Lemma F.5.

So assume that $\hat{x} \in D$, which implies that $\hat{x} = \bar{x}$. Since $w \leq v$ on $\mathcal{X}_+ \backslash \{\bar{x}\}$, and $w(\bar{x}) = v(\bar{x}) + M > v(\bar{x})$, it follows that $w$ is discontinuous at $\bar{x}$.

We follow the same argumentation as in the proof of Lemma F.5. If alternative A of Lemma F.4 holds, we derive a contradiction as before. If alternative B holds, as $w$ is discontinuous at $\bar{x}$, there is $\eta > 0$ and a sequence $x_k \to \bar{x}$ with $x_k \in \mathcal{X}_-$ for all $k$, such that (43) holds. Since both $v$ and $w$ are left continuous at $\bar{x}$, the argumentation of Lemma F.5 again yields a contradiction. □

### F.5 Proof of Proposition D.20

*Proof.* We need to show that it is impossible that $W_\ell^{\mathrm{cs}}$ and $W_I^{\mathrm{cs}}$ are both contained in the closed half plane $x \leq \bar{x}$, or both in $x \geq \bar{x}$. As neither of them can be tangent to $(0, 1)$, because of Lemma D.17, the result then follows.

So assume $W_I^{\mathrm{cs}}$ and $W_\ell^{\mathrm{cs}}$ are both contained in $x \leq \bar{x}$. We argue that $W_I^{\mathrm{u}}$ is then contained in the same half-plane. Clearly, it is sufficient to show the analogous relation for the corresponding eigen-halflines, that is, the intersections of $E_I^{\mathrm{cs}}$ and $E_I^{\mathrm{u}}$ with the half plane $B_I$ bounded by $B$ and containing $(0, 1)$, and the intersection of $E_\ell^{\mathrm{cs}}$ with the half plane $B_\ell$ bounded by $B$ and containing $(0, -1)$.

The solution flow $z_\zeta$ of the linear dynamics $\dot{z} = Az$, $z_\zeta(0) = \zeta$ maps multiples of initial vectors to proportional multiples: $z_{c\zeta}(t) = cz_\zeta(t)$ if $c \in \mathbb{R}$, and hence half lines with vertices at 0 to other such half lines. We can find the half line dynamics by setting $z_\zeta(t) = r(t)(\cos\theta(t), \sin\theta(t)) \equiv r(t)\omega_1(\theta(t))$, introducing $\omega_2(\theta(t)) \equiv \dot{\omega}_1(\theta(t)) = (-\sin\theta(t), \cos\theta(t))$ and computing

$$\dot{r}\omega_1(\theta) + r\omega_2(\theta)\dot{\theta} = \dot{z}_\zeta = Az_\zeta = rA\omega_1(\theta).$$

Using that $\omega_1$ and $\omega_2$ are orthogonal, premultiplication with $\omega_2^\top$ yields the half line dynamics

$$\dot{\theta} = \omega_2(\theta)^\top A \omega_1(\theta).$$

An eigen half line is, by definition, a steady state of $\psi$. Let $\lambda_1, \lambda_2$ be the eigenvalues of $A$, and let $\theta_1$ and $\theta_2$ be such that $\omega_1(\theta_j)$ is an eigenvector associated to $\lambda_j$, $j = 1, 2$. If $\lambda_1 < \lambda_2$, then it is easily verified that $\theta_1$ is a repelling steady state of the half line dynamics, and $\theta_2$ an attracting steady state.

Let $\theta_b \in (-\pi/2, \pi/2)$ be such that $\omega_1(\theta_b) \in B$. Let $\psi$ denote the half line dynamics induced by the canonical vector field $F$ at $\bar{z}$: that is,

$$\psi(\theta) = \omega_2(\theta)^\top A_I \omega_1(\theta)$$

if $\theta_b \le \theta < \theta_b + \pi$,

$$\psi(\theta) = \omega_2(\theta)^\top A_\ell \omega_1(\theta)$$

if $\theta_b + \pi \le \theta < \theta_b + 2\pi$, and $\psi$ is extended periodically for other values of $\theta$. Since $A_I \omega_1(\theta_b) = A_\ell \omega_1(\theta_b)$, $\psi$ is Lipschitz continuous.

The dynamics $\psi$ has, counted up to multiples of $2\pi$, four steady states, $\theta_k^j$, $j \in \{\text{cs}, \text{u}\}$, $k \in \{\ell, I\}$, corresponding to the four eigen half lines $E_k^j$. Lemma D.17 implies that $\theta_\ell^{\text{u}} = 3\pi/2$. Moreover, each two repelling $\theta_1 < \theta_2$ steady states of $\psi$ have to be separated by an attracting steady state $\theta_3$. To see this, note that for $\varepsilon > 0$ sufficiently small we have that $\psi(\theta_1 + \varepsilon) > 0$ and $\psi(\theta_2 - \varepsilon) < 0$, and apply the intermediate value theorem. In particular, if $\pi/2 < \theta_I^{\text{cs}} < \theta_\ell^{\text{cs}} < 3\pi/2 = \theta_\ell^{\text{u}}$, then necessarily $\theta_I^{\text{u}} \in (\theta_I^{\text{cs}}, \theta_\ell^{\text{cs}})$. This shows our claim that $W_I^{\text{u}}$ is also contained in the half plane $x \le \bar{x}$.

We now have to show that it is not possible for this situation to occur.

If there is $\delta > 0$ such that $V$ is differentiable for all $x \in I_\delta = (\bar{x}, \bar{x} + \delta)$ and the optimal trajectory $y_x$ starting at $x$ satisfies $y_x(t_x) = \bar{x} + \delta$ for some $t_x > 0$, then by Lemma D.23 we have that $(x, V'(x)) \in W^{\text{u}}$. But this is impossible, as $W^{\text{u}}$ is contained in $x \le \bar{x}$.

Hence there is $\delta > 0$ such that for all $x \in I_\delta$ such that $V$ is differentiable at $x$, the optimal canonical trajectory $y$ starting at $x$ does not satisfy $y(t) = \bar{x} + \delta$ for some $t > 0$. Moreover, it cannot satisfy $y(t) = \bar{x}$ for some $t > 0$, and therefore it has to converge to a steady state $\hat{x}$. By Proposition D.11, $V$ is differentiable at $\hat{x}$ and $(\hat{x}, V'(\hat{x}))$ is a steady state of the canonical vector field. As $x \to \bar{x}$, $(\hat{x}, V'(\hat{x}))$ to $(\bar{x}, \bar{p})$, and $(\hat{x}, V'(\hat{x})) \in W^{\text{cs}}$ for $\hat{x}$ sufficiently close to $\bar{x}$. But this contradicts the fact that $W^{\text{cs}}$ is contained in the region $x \le \bar{x}$. □

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