DanceSport and Power Values

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Abstract

DanceSport is a competitive form of ballroom dancing. At a DanceSport event, couples perform multiple dances in front of judges. This paper shows how a goal for a couple and the judges' evaluations of the couple's dance performances can be used to formulate a weighted simple game. We explain why couples and their coaches may consider a variety of goals. We also show how prominent power values can be used to measure the contributions of dance performances to achieving certain goals. As part of our analysis, we develop novel visual representations of the Banzhaf and Shapley-Shubik index profiles for different thresholds. In addition, we show that the "quota paradox" is relevant for DanceSport events.

1 Introduction

In addition to identifying the sports that are in the Summer and Winter Olympics, the website for the International Olympic Committee (IOC) has a list of sports governed by international federations that they recognize. DanceSport is one of the sports on the list. The website for the International Olympic Committee (IOC) also has a list of multi-sports organizations that they recognize. One of the organizations on this list is the International World Games Association (IWGA). Every four years (viz., in the years following the Summer Olympic Games), the IWGA holds "The World Games". The IWGA website states: "Competitions at the highest level in a multitude of diverse, popular and spectacular sports make up the mainstay of The World Games". DanceSport is one of the sports in The World Games.

The international federation for DanceSport that the IOC recognizes is the World DanceSport Federation (WDSF). The WDSF and other organizations sanction and regulate DanceSport events. In each round at a DanceSport event, the couples who are competing are judged on the basis of their performances in multiple dances. At most WDSF events, each couple performs the same three, four or five dances from the WDSF list of "Standard dances" (Waltz, Tango, Viennese Waltz, Foxtrot and Quickstep) or they perform the same three, four or five dances from the WDSF list of "Latin American dances" (Samba, Cha-Cha-Cha, Rumba, Paso Doble and Jive). There are also WDSF events where each couple performs ten dances (viz., all of the Standard and Latin American dances).

The following specific example of a ballroom dancing competition will be useful for motivating and illustrating what is done in this paper. In each round of the ballroom dancing competitions hosted by the internationally recognized sport organization USA Dance, the Novice Standard level three-dance event involves each couple performing the following three dances: Waltz, Foxtrot and Quickstep. There are seven judges watching and judging the couples. Each judge rates each couple in each dance by giving a score of '1' to the couple if the judge thinks the couple performed the dance well, or '0' to the couple if the judge thinks the couple did not perform the dance well. After the dances have been performed, for each couple, 1) the sum of the judges' scores for the couple are tabulated for each dance and 2) the total points for the three dances are tabulated for the couple. Accordingly, for a couple, the maximum number of points per dance is 7, and the maximum total score after three dances is 21 points. The couples are then ranked on the basis of their total points (i.e., a couple's placement is determined by the sum of their points, compared to other couples' points). A couple aims to have as many points as possible.

Juan Vasquez and Janice Vasquez compete as a couple in ballroom dancing. In 2017, they came in first in the USA Dance National DanceSport Championships in the Senior III age category of the Novice Standard level. In the semi-final round, Juan and Janice earned 5, 6, and 7 points (respectively) for Waltz, Foxtrot, and Quickstep. After learning their scores, they may have wondered about the extent to which each of their dances mattered. One of their coaches could think that their two higher-scoring dances, the Foxtrot and Quickstep, mattered the most. Another coach could think that just their performance in the Quickstep mattered the most, since it earned the highest number of points. Juan and Janice may be puzzled because they thought that all of their three dances mattered equally.

A natural question is: "Why might their coaches reach conclusions that are different from what Juan and Janice thought?". In this paper, we set out reasoning that could lead to their coaches' conclusions being different from what Juan and Janice thought.

Section 2 reviews the concept of a weighted simple game and shows how a goal for a couple and the judges' evaluations of the couple's performances in a multi-dance competition can be used to formulate a weighted simple game. It also illustrates the idea by showing in detail what the game looks like in a three-dance competition. In Section 3, we review the concept of a power value and the definitions of some prominent power values. In addition, we illustrate the definitions by showing what they look like when they are applied to a three-dance competition. Sections 4-6 analyze three-dance competitions in greater detail. Section 4 describes, synthesizes and generalizes some prior work on power values. In Section 5, we review methods for visualizing power index profiles that have been used for a variable distribution of weights. In addition, we develop a novel visual representation that is motivated by a type of question that we raise in this paper about DanceSport events. The first part of Section 6 uses power values to show why Juan and Janice might get the conflicting assessments from their coaches that were described above. The second part shows that a phenomenon known as "the quota paradox" can occur in three-dance competitions. We conclude by discussing some ways that the analysis in this article can be useful to people who compete in dance contests.

The analysis in this paper is related to the suggestion in Saari and Sieberg

(2001, pp. 242, 244 & 246), Saari (2001, pp. 9-11), and Saari (2018, Section 3.4) that equations from power values (which were originally developed to measure power in certain settings where decisions are made by voters) could potentially be used to evaluate **competitors** in a sports context. An example sketched in Saari and Sieberg (2001, p. 242) involved measuring a players contributions to professional basketball. An example sketched in Saari (2001a, pp. 10-11) indicated that a similar idea could be used to measure a players contributions to a specific basketball team. Saari (2018, pp. 80-81) sketches an example involving a professional baseball player.

The suggestion described above was made in passing in those references (en route to addressing the main topics) and the suggestion was just briefly illustrated with some short sketches of examples. Related approaches to evaluating the contributions of competitors have been been more fully developed in Hernandez-Lamoneda and Sanchez-Sanchez (2010, Section 4) (where their approach was applied to basketball players), Auer and Hiller (2015) and Hiller (2015) (where their approach was applied to soccer players) and in Cheng and Coughlin (2017, 2018) (where their approach was applied to the entrants on a figure skating team).

The references mentioned above focus on ways of evaluating the extent to which **a competitor** contributes to achieving a goal for the competitor's team (or league). An important difference between this paper and the references mentioned in the two preceding paragraphs is that this paper sets out a way of evaluating the extent to which **different aspects of the performance of a competitor** contribute to achieving a goal that the competitor has.

2 Dance competitions and weighted simple games

Consider a couple that is competing in a multiple-dance event, in which (in each round) a couple performs each type of dance one time. We will use ideas from Game Theory to set out reasoning that a coach could use to analyze the contribution of each separate dance performance toward the couple achieving certain goals. As we proceed, we will illustrate what we are doing by showing how it looks in the type of three-dance event described in Section 1.

As in Taylor and Zwicker (1999, p. 3), we will use the term *simple game* to mean a pair G = (N, $\mathcal{W}$) where N is a finite set and $\mathcal{W}$ is a collection of subsets of N which satisfies *monotonicity* (i.e., $[X \in \mathcal{W}$ and $X \subset Y \subseteq N]$ $\Rightarrow$ [Y $\in \mathcal{W}$]). The number of elements in N will be denoted by n. Simple games have been used as models of various things – including voting rules and reliability systems (Taylor and Zwicker (1999, p. 3)).

In our application, each element in N will be the couple's performance in one of the dances in the event. The dance performances will be labeled by the integers 1 (for the couple performing one dance), 2 (for the couple performing a second dance) and so on through n (for the couple performing an nth dance). In a three-dance event, $n = 3$. For the example of a three-dance event described in Section 1, N will be taken to be {1 (the couple performing the Quickstep), 2 (the couple performing the Foxtrot), 3 (the couple performing the Waltz)}.

We will have a subset of a couple's dance performances be in $\mathcal{W}$ if and only if the points earned by the performances in that subset can achieve a certain goal. Each goal that we will consider will be specified by a positive number of points. That number will be called a *threshold*. Achieving the goal means earning at least as many points as the threshold. So, in particular, a subset of a couple's dance performances will achieve the goal if and only if the total number of points earned by the performances in that subset is at least as large as the threshold. In addition, each goal that we will consider will be one that is achieved by the total number of points avaiable from all three dance performances. For instance: suppose that, in the example in Section 1, the goal is to get at least 12 points. Then the subsets in $\mathcal{W}$ are {1, 2, 3}, {1, 2}, {1, 3}.

As in Taylor and Zwicker (1999, p. 5), saying that a simple game G = (N, $\mathcal{W}$) is *weighted* will mean there exists a vector of n real numbers, $(w_1, \ldots, w_n)$, together with a real number, q, such that

$$S \in \mathcal{W} \Leftrightarrow \sum_{i \in S} w_i \geq q. \tag{1}$$

As in Taylor and Zwicker (1999), we will refer to the entries in the vector $(w_1, \ldots, w_n)$ as *weights* for the corresponding elements in N and we will call q a *quota*. Also as in Taylor and Zwicker (1999), for a given simple game G, any specific example of weights and a quota where (1) is satisfied will be said to *realize* G as a weighted game.

For our model, the quota will be whatever threshold is being considered. In addition, we will have the number of points from a couple's performance of a particular type of dance be the weight for that performance. For instance, in our running example, $q = 12$, $w_1 = 7$, $w_2 = 6$ and $w_3 = 5$.

For our model, the sum of the weights for a particular subset of a couple's dance performances will be the total number of points earned by the performances in the subset. By definition, a subset of N will be in $\mathcal{W}$ if and only if the points earned by the performances in that subset can achieve the goal being considered. Therefore a subset of N will be in $\mathcal{W}$ precisely when the sum of the weights of the performances in the subset is greater than or equal to q. Therefore the weights and quota we are using realize our model of a round in a multiple-dance event as a weighted game.

3 Dance competitions and power values

We assume that the coach is only concerned with whether a particular goal that he is considering is achieved. We also assume that achieving the goal is valued more highly than not achieving it. For a simple game G = (N, $\mathcal{W}$), Felsenthal and Machover (1998, p. 16), call the function

$$v(S) = \begin{cases} 1 & \text{if S} \in \mathcal{W} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

the *characteristic function* of the game. Since v will assign 1 to a subset that achieves the goal and will assign 0 when a subset doesn't achieve it, the coach's approach to evaluating a couple's dance performances can be represented by the game's characteristic function. Accordingly, as in Felsenthal and Machover (1998, p. 16), for any $S \subseteq N$ we will refer to $v(S)$ as the *worth* of S.

For the weighted simple game defined in the previous section, the characteristic function can be written as

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{if } \sum_{i \in S} w_i < q. \end{cases} \tag{3}$$

In our example: $w_1 + w_2 + w_3 > 12$, so $v(\{1, 2, 3\}) = 1$; $w_1 + w_2 > 12$, so $v(\{1, 2\}) = 1$; $w_1 + w_3 = 12$, so $v(\{1, 3\}) = 1$; $w_2 + w_3 < 12$, so $v(\{2, 3\}) = 0$; $w_1 < 12$, so $v(\{1\}) = 0$; $w_2 < 12$ so $v(\{2\}) = 0$; $w_3 < 12$, so $v(\{3\}) = 0$; $v(\emptyset) = 0$.

Saari (2018, Section 3.4) describes a method that can be used to assess the overall contribution that a particular $j \in N$ makes in a simple game. The method involves the following two steps.

First: Consider a particular $S \subseteq N$. For any $j \in N$ that is not in S,

$$v(S \cup \{j\}) - v(S) \tag{4}$$

is the change in the worth when j is added to S. As in Felsenthal and Machover (1998, p. 181), for any $j \in N$ and $S \subseteq N \backslash j$, we will refer to (4) as the *marginal contribution* that j would make by being added to S. In our example: For the set S whose only element is their performance in the Foxtrot, their performance in the Waltz doesn't contribute anything if it is added to S – but the marginal contribution from their performance in the Quickstep being added to S is 1.

Second: The overall contribution of a particular $j \in N$ is measured as follows. For each $S \subseteq N \backslash j$, multiply (4) by a non-negative coefficient λ_S and then add the resulting terms – obtaining

$$p_j = \sum_{\{S \subseteq N \backslash j\}} (\lambda_S)(v(S \cup \{j\}) - v(S)). \tag{5}$$

Saari (2018, Section 3.4) calls p_j the *power value* of j.

Consider $j = 1$ in our example. The subsets of $N \backslash 1$ are $\{2, 3\}$, $\{2\}$, $\{3\}$, and $\emptyset$. In addition, $v(S \cup \{j\}) - v(S)) = 1$ if and only if S is not empty. So

$$p_1 = \lambda_{\{2,3\}} + \lambda_{\{2\}} + \lambda_{\{3\}}. \tag{6}$$

Significantly, (5) leaves open what Saari (2018, p. 81) describes as "the crucial choice of the λ_S coefficients".

One prominent approach to choosing the coefficients (which is based on Banzhaf (1965)) has the measurement of j's overall contribution be the number of subsets of N where v is 0, but when j is added to the subset, v increases to 1. Felsenthal and Machover (1998, p. 39) call this number the *Banzhaf score* for j. We will denote j's Banzhaf score by $BS(j)$. j's Banzhaf score is given by (5) when

$$\lambda_S = 1 \tag{7}$$

for each subset of $N \backslash j$. In our example, (6) and (7) imply $BS(1) = 3$. Similar reasoning establishes that $BS(2) = BS(3) = 1$.

Felsenthal and Machover (1998, p. 39) use the term *index* when the measurements of the overall contributions for the elements in N always add up to one. Having an index which preserves the relative sizes of the numbers assigned to the elements of N by (5) when the coefficients in (7) are used implies that

$$\lambda_S = \frac{1}{\sum_{i \in N} BS(i)} \tag{8}$$

for each subset S of $N \backslash j$ When these coefficients are used, we have what Felsenthal and Machover (1998, p. 39) called the *Banzhaf index* for j. We will denote j's Banzhaf index by $BI(j)$. When the Banzhaf index is being used, $(BI(1), \ldots, BI(n))$ will be called the *Banzhaf index profile*. In our example, $BS(1)+BS(2)+BS(3) = 5$ and $(BI(1), BI(2), BI(3)) = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$.

Another prominent approach (which is based on Shapley and Shubik (1954)) focuses on the orders in which the weights for the elements in N could potentially be added up. In our example, the points from the Foxtrot could be considered first, then the points from the Quickstep could be added in, followed by the points from the Waltz being added to the total (with, of course, five other orders also being possible). Consider a particular order in which the weights for the elements in N could potentially be added up. $j \in N$ is said to be *pivotal* for the order when (i) the set of elements that come before j in the order is a set that is not in $\mathcal{W}$ and (ii) when j is added to the set, the enlarged set is in $\mathcal{W}$. In other words, $j \in N$ is pivotal if it is located at a place in the order where it "tips the balance" in favor of the threshold being met. In our example, if just the points from the couple performing the Foxtrot are considered, they don't meet the 12 point threshold. But if the points from the Quickstep are added in, they do meet the threshold. So the Quickstep is pivotal for the order where the Foxtrot is first, the Quickstep is second and the Waltz is third.

One way in which this approach has been used has been to have the measurement of j's overall contribution be the number of possible orders where j tips the balance. Felsenthal and Machover (1998, p. 196) call this number the *Shapley-Shubik score* for j. We will denote j's Shapley-Shubik score by $SS(j)$.

There are coefficients that can be used in (5) which will have the sum in (5) be j's Shapley-Shubik score. Consider a particular $j \in N$ and $S \subseteq N \backslash j$. To begin with, for any order where (i) the elements that come before j in the order are the elements in S and (ii) the elements that come after j in the order are the elements in $N\backslash(S\cup\{j\})$. we will have $v((S\cup\{j\})-v(S))=1$ if j is pivotal for the order and $v((S\cup\{j\})-v(S))=0$ if j is not pivotal for the order. Therefore $v((S\cup\{j\})-v(S))$ times (the number of orders where (i) and (ii). both hold) is equal to the number of orders where (i) and (ii) both hold **and** j tips the balance. Therefore, if for each subset S of $N\backslash j$, we have $\lambda_S =$ (the number of orders where (i) and (ii) both hold) then the sum in (5) will be j's Shapley-Shubik score.

In our example, $n=3$. So, for any $j\in N$, a set in $N\backslash j$ will have no more than two elements. If $|S|=2$, then $\lambda_S=2$; If $|S|=1$, then $\lambda_S=1$; If $|S|=0$, then $\lambda_S=2$. In our example, Equation (6) and these values for the coefficients imply $SS(1)=4$. Similar reasoning establishes that $SS(2)=SS(3)=1$.

One can also specify an index which preserves the relative sizes of the Shapley-Shubik scores. In particular, this can be done by having the coefficient assigned to a subset S of $N\backslash j$ be $\lambda_S =$ (the number of orders where (i) and (ii) both hold) / (n!). In our example: If $|S|=2$, then $\lambda_S=\frac{2}{6}$; If $|S|=1$, then $\lambda_S=\frac{1}{6}$; If $|S|=0$, then $\lambda_S=\frac{2}{6}$. When these coefficients are used, we have what Felsenthal and Machover (1998, p. 196) called a *Shapley-Shubik index*. We will denote j's Shapley-Shubik index by $SSI(j)$. When the Shapley-Shubik index is used, the measurement of the overall contribution of a particular $j\in N$ is the fraction of the possible orders where j "tips the balance" (in our application, that will be the fraction of the possible orders where a particular dance performance gets the couple to go from not achieving the goal under consideration to achieving it). When $SSI(j)$ is written by using (5) along with the coefficients specified above, it is clear that (like the Banzhaf measure) j's Shapley-Shubik index is an average marginal contribution – although, in this case, it is the average with respect to the orders in which the coefficients for the elements in N could potentially be added up. When the Shapley-Shubik index is being used, $(SSI(1),\,\ldots,\,SSI(n))$ will be called the *Shapley-Shubik index profile*. In our example, n! $=6$ and $(SSI(1),SSI(2),SSI(3))=(\frac{2}{3},\frac{1}{6},\frac{1}{6})$.

Felsenthal and Machover (1998: p. 211) point out that the Banzhaf and the Shapley-Shubik indices "are by far the most important and widely used indices".

4 Comparison of Banzhaf and Shapley-Shubik power index profiles when $n = 3$

In the preceding sections, we used a three-dance competition to illustrate what the weighted simple game in Section 2 looks like and to illustrate what the definitions of some prominent power values mean when they are applied to that weighted simple game.

In this section, we state conditions for the power index profiles that can result from quotas that could be of interest when $n = 3$. We will denote $w_1 + w_2 + w_3$ by w. We will assume that q is less than or equal to w.

Weighted simple games with $n = 3$ have been analyzed in voting contexts by Fischer and Schotter (1976, Section II, Appendix B and Appendix C), Schotter (1981, Section 2), Berg and Holler (1986, pp. 424-425), Leech (2003, Section 4), Haines and Jones (2005), Jones (2009), Kirstein (2010) and elsewhere. Haines and Jones (2005, p. 144) observe that, when a weighted simple game is used as a model of a voting rule, it is typically assumed that q is greater than half of the sum of the voters' weights. The reason is that those quotas avoid the possibility of contradictory alternatives passing. Drawing on results in Haines and Jones (2005) and Kirstein (2010), conditions for the resulting power index profiles with $n = 3$ are reported in the first subsection.

After a dance competition has concluded, the dancers' points are known quantities and hence are fixed. At that stage, any quota in $(0, w]$ could be of interest to a couple. So, in the second subsection, we extend the results from the first subsection to identify conditions for power index profiles for quotas which are less than or equal to half of w.

Haines and Jones (2005) and Kirstein (2010) both assume that $w_i \geq 0$ holds for each $i \in \{1, 2, 3\}$ and that $w > 0$. We will use these assumptions. For any three-player weighted simple game, one can (as in Kirstein (2010, p. 6)) assign index numbers to the elements in N so that their weights have the following relationship: $w_1 \geq w_2 \geq w_3$. We will assume that these inequalities hold throughout subsection 4.1 through 5.2.

4.1 Conditions for power index profiles when $q > \frac{1}{2}w$

Haines and Jones (2005) identified conditions for the Shapley-Shubik index profiles that can occur when $n = 3$ and $q > \frac{1}{2}w$. In a related paper, Kirstein (2010) identified conditions for the Banzhaf index profiles that can occur when $n = 3$ and $q > \frac{1}{2}w$.

We will describe the analyses of weighted simple games with $n = 3$ that are in Haines and Jones (2005) and Kirstein (2010). We will then state conclusions that follow from combining their results.

As was shown in Haines and Jones (2005), in the weighted simple games with $n = 3$, there are four distinct possible Shapley-Shubik power index profiles: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, $(\frac{1}{2}, \frac{1}{2}, 0)$, and $(1, 0, 0)$. For each of these possible Shapley-Shubik power index profiles, Haines and Jones (2005) identified conditions on q and the values of w_1, w_2 and w_3 under which the profile occurs.

As was shown in Kirstein (2010), in the weighted simple games with $n = 3$, there are four distinct possible Banzhaf power index profiles: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$, $(\frac{1}{2}, \frac{1}{2}, 0)$, and $(1, 0, 0)$. For each of these possible Banzhaf power index profiles, Kirstein (2010) identified conditions on q and the values of w_1, w_2 and w_3 under which the profile occurs.

By using the conditions for the Shapley-Shubik power index profiles identified by Haines and Jones (2005) along with the separate conditions for the Banzhaf power index profiles identified by Kirstein (2010) we get the following four cases:

1. If $w_1 + w_2 < q$ or $w_1 < q \leq w_2 + w_3$, then $(BI(1),\ BI(2),\ BI(3)) = (SSI(1),\ SSI(2),\ SSI(3)) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

2. If $w_1 + w_3 < q \leq w_1 + w_2$, then $(BI(1),\ BI(2),\ BI(3)) = (SSI(1), SSI(2),\ SSI(3)) = (\frac{1}{2}, \frac{1}{2}, 0)$.

3. If $w_2 + w_3 < q \leq w_1 + w_3$ or $w_1 < q \leq w_1 + w_3$, then $(BI(1),\ BI(2), BI(3)) = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ and the $SSI(2),\ SSI(3)) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.

4. If $w_2 + w_3 < q \leq w_1$, then $(BI(1),\ BI(2),\ BI(3)) = (SSI(1),\ SSI(2), SSI(3)) = (1, 0, 0)$.

Among other things, the conditions stated in this subsection make it clear that, when $n = 3$ and q must be greater than half of the sum of the weights (and less than or equal to w), the Banzhaf and Shapley-Shubik power index profiles have the following qualitative features:

- There are only four possible **pairs** of Banzhaf and Shapley-Shubik power index profiles

- There is a one-to-one correspondence between the set of possible Banzhaf power index profiles and the set of possible Shapley-Shubik power index profiles (i.e., each Banzhaf power index profile has a unique, corresponding Shapley-Shubik power index profile and vice versa).

- When the numbers aren't exactly the same, $(BI(1), BI(2), BI(3)) = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ and $SSI(2), SSI(3)) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.

4.2 Conditions for power index profiles when $q \leq \frac{1}{2}w$

Extending the results of Haines and Jones (2005) and Kirstein (2010), we identify the following additional conditions for the Banzhaf and Shapley-Shubik power index profiles:

1. If $0 < q \leq w_3$, then $(BI(1), BI(2), BI(3)) = (SSI(1), SSI(2), SSI(3)) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. I.e., when the threshold is lower than the lowest of the three weights, all of the weights have equal contribution.

2. If $w_3 < q \leq w_1$, then $(BI(1), BI(2), BI(3)) = (SSI(1), SSI(2), SSI(3)) = (\frac{1}{2}, \frac{1}{2}, 0)$. I.e., when the threshold is higher than the lowest of the three weights, only the two higher weights contribute and those contributions are equal.

3. If (a) $w_1 < w_2 + w_3$ and $w_2 < q < w_1$ or (b) $w_1 > w_2 + w_3$ and $w_2 < q < w_2 + w_3$, then $(BI(1), BI(2), BI(3)) = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ and $SSI(2), SSI(3)) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$

4. If $w_2 + w_3 < q \leq w_1$, then $(BI(1), BI(2), BI(3)) = (SSI(1), SSI(2), SSI(3)) = (1, 0, 0)$. I.e., when the threshold exceeds the sum of the lower two weights and the threshold is lower than the highest weight, then highest weight has full contribution.

Significantly, the conditions stated in this subsection imply that, when q is simply any positive number that is less than or equal to w, the qualitative features described at the end of the previous subsection continue to hold.

5 Visualizations of conditions for power index profiles

There has been important work which focuses on the power index profiles that result from various distributions of weights. In what follows, the first subsection will describe the method for visualizing power index profiles for a variable distribution of weights which is in Kirstein (2010). In the next subsection, we describe an alternative approach that is motivated by the type of question about DanceSport events that was raised in the previous sections and that we address in the subsequent sections. The third subsection describes a second established way of visualizing power index profiles for a variable distribution of weights. We also compare and contrast these different visualizations of conditions for power index profiles.

5.1 An established way of visualizing power index profiles for a variable distribution of weights

This subsection describes the visualization of power index profiles for a variable distribution of weights that is in Kirstein (2010). This visualization is based on a geometric representation for the weights. For that representation, the weights and the quota are normalized. More specifically: For any initial (w'_1, w'_2, w'_3) and q', using the sum $w' = w'_1 + w'_2 + w'_3$, he sets $w_i = w'_i/w'$ for each $i \in \{1, 2, 3\}$ and he sets $q = q'/w'$.

Kirstein (2010, pp. 8-9) illustrates the Banzhaf power index profiles in two-dimensional graphs, where the normalized weight w_1 is on a horizontal axis and the normalized weight w_2 is on a vertical axis. So, under the assumptions that he made, 1) on the vertical axis he has the possible normalized weights for an element of N whose weight is greater than or equal to each other element's weight and 2) on the horizontal axis he has the possible normalized weights for a second element whose weight is greater than or equal to the remaining element's weight. These two axes are enough for his purposes because $w_3 = 1 - (w_1 + w_2)$. Along each axis, the corresponding normalized weight has a minimum of 0 and a maximum of 1. There are three inequalities which the ordered pairs (w_1, w_2) must satisfy: 1) $w_1 \geq w_2$ (since this was an initial assumption about the relative orderings of w_1 and w_2), 2) $w_1 \leq -w_2 + 1$ (since each ordered pairs (w_1, w_2) must be on or below the line connecting (1,0) and (0,1)), and 3) $w_1 \geq 1 - 2w_2$ (which can be derived from the fact that $w_2 \geq w_3 = 1 - w_1 - w_2$). The set of ordered pairs where these three inequalities are satisfied is a triangular region in $\mathbb{R}^2$, with the vertices being $(0, 1)$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{3}, \frac{1}{3})$.

Having finished our description of the geometric representation for the weights that was used by Kirstein (2010), we now turn to the way that conditions for power index profiles are visualized in that paper. In particular, the visualization works as follows: for a given quota q, the triangular region described above is partitioned into sub-regions in which different Banzhaf power index profiles occur.

5.2 Visualizing power index profiles for a variable quota

In this subsection, we provide a different approach to illustrating the conditions under which the possible power index profiles are relevant. In the figures in Kirstein (2010), q is fixed and the weights are treated as varying quantities. In our approach, we will treat q as a varying quantity and have the weights of the three elements in N be fixed.

Additionally, while Kirstein (2010) solely illustrated Banzhaf power index profiles, we will illustrate both Banzhaf and Shapley-Shubik power index profiles together (taking advantage of the conclusions stated in the previous section).

The Banzhaf and Shapley-Shubik power index profiles can change, depending on whether there are 'strictly greater than' or 'equal to' relationships among the quantities w_1, w_2, and w_3. As a consequence, we have created separate figures to illustrate the following four cases to separate out the possible ways in which the three quantities' can have 'greater than or equal to' relationships with each other: $w_1 > w_2 > w_3$ (see Figure 1), $w_1 = w_2 > w_3$ (see Figure 2), $w_1 > w_2 = w_3$ (see Figure 3) and $w_1 = w_2 = w_3$ (see Figure 4).

We chose a number line representation in order to clearly show where the quantities w_1, w_2, w_3, $w_1 + w_3$, $w_2 + w_3$, and $w_1 + w_2$ lie in relation to one another. Each point labeled on a number line represents a distinct quantity, and the number line shows lower values to the left of higher values. In all cases, $w_1 \leq w_1 + w_3 \leq w_1 + w_2 \leq w$.

When w_1 and w_2 are not equal (see Figures 1 and 3), there are three possibilities for the relative ordering of w_1 and $w_2 + w_3$: $w_1 < w_2 + w_3$ (see the top number line), $w_1 = w_2 + w_3$ (see the middle number line), and $w_1 > w_2 + w_3$ (see the bottom number line). In the middle number line of

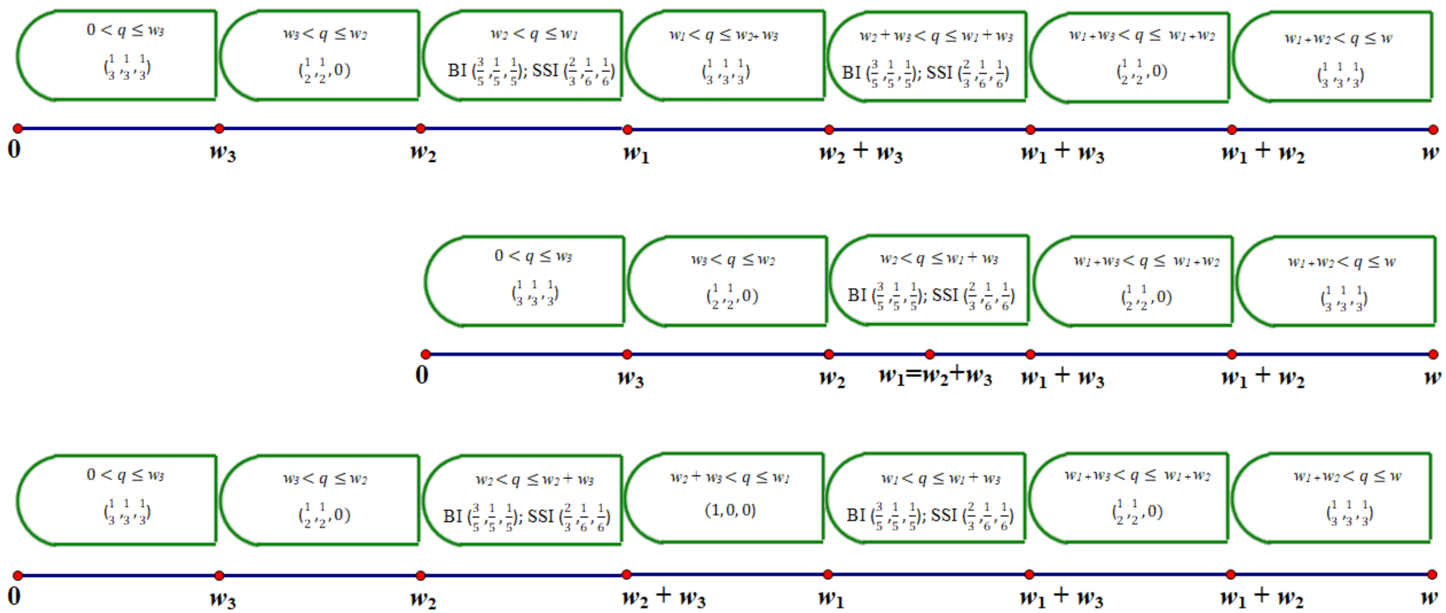

Figure 1: $w_1 > w_2 > w_3$

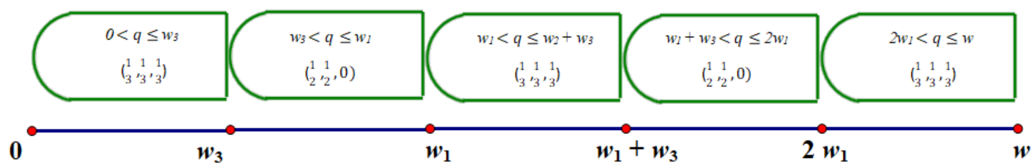

Figure 2: $w_1 = w_2 > w_3$

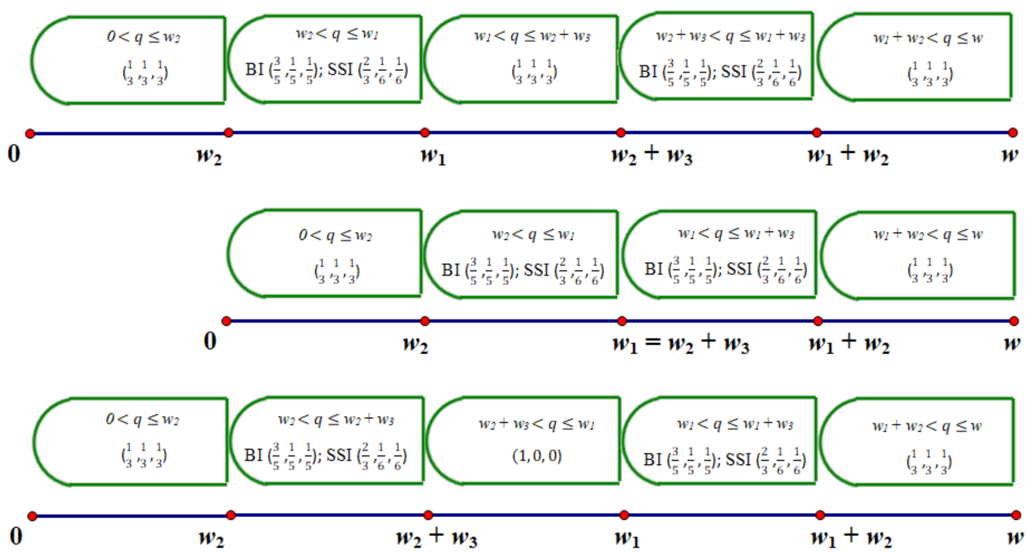

Figure 3: $w_1 > w_2 = w_3$

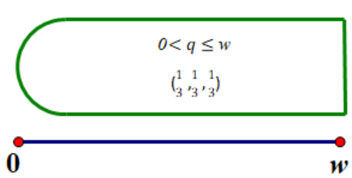

Figure 4: $w_1 = w_2 = w_3$

Figure 3, where $w_1 = w_2 + w_3$ and $w_2 = w_3$, we note that $w_1 = \frac{1}{2}(w)$ and $w_2 = w_3 = \frac{1}{4}(w)$.

In Figure 2 (where $w_1 = w_2$), the quantity $w_1 + w_2$ is equivalently labeled as $2w_1$.

When two or more of the three quantities w_1, w_2, and w_3 are equal (see Figures 2, 3, and 4), there are fewer than four possible Banzhaf and Shapley-Shubik power index profiles that occur. This is because, if $w_i = w_j$, then $BI(i) = BI(j)$ and $SSI(i) = SSI(j)$. In Figure 2 (where $w_1 = w_2$), the Banzhaf and Shapley-Shubik power index profile of (1,0,0) is not relevant because this would incorrectly imply that $BI(1) = 1$ but $BI(2) = 0$. Similarly, the Banzhaf power index profile $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ and its corresponding Shapley-Shubik power index profile $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$ are not relevant. In Figure 3 (where $w_1 > w_2 = w_3$), the Banzhaf and Shapley-Shubik power index profile of $(\frac{1}{2}, \frac{1}{2}, 0)$ cannot occur. In Figure 4 (where $w_1 = w_2 = w_3$), the only Banzhaf and Shapley-Shubik power index profile that occurs is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

For each relevant interval of possible values of q, the shape above it reflects the fact that it is a left half-open interval.

5.3 The number lines vs. Kirstein's triangles

When the weights and the quota are normalized, the possible weights and quotas correspond to the points that are in the right triangular prism which is the Cartesian product of a) the triangular region for the possible weights in $\mathbb{R}^2$ that is in Kirstein (2010), and b) the half-open line segment $(0, 1]$. This prism in $\mathbb{R}^3$ has the following features:
1) each base is like the triangular region for the possible weights in Kirstein (2010) (in the sense that the w_1 and w_2 coordinates in the base are the same as the w_1 and w_2 coordinates in the triangular regions in Kirstein's figures),
2) the third dimension for the prism corresponds to the possible values for q,
3) the prism is a convex set,
4) the bottom base is open (since $q > 0$), and
5) the top base is closed (since $q \leq 1$).
This type of prism is shown in Figure 5.

Figures 1-4 are one-dimensional representations of the power index profiles that occur at points in the prism. More specifically, Figures 1-4 show which power index profiles occur if you hold the weights constant and allow the quota to vary (or, in terms of Figure 5, if you drop a vertical line [that is, a line parallel to the q axis] through the prism).

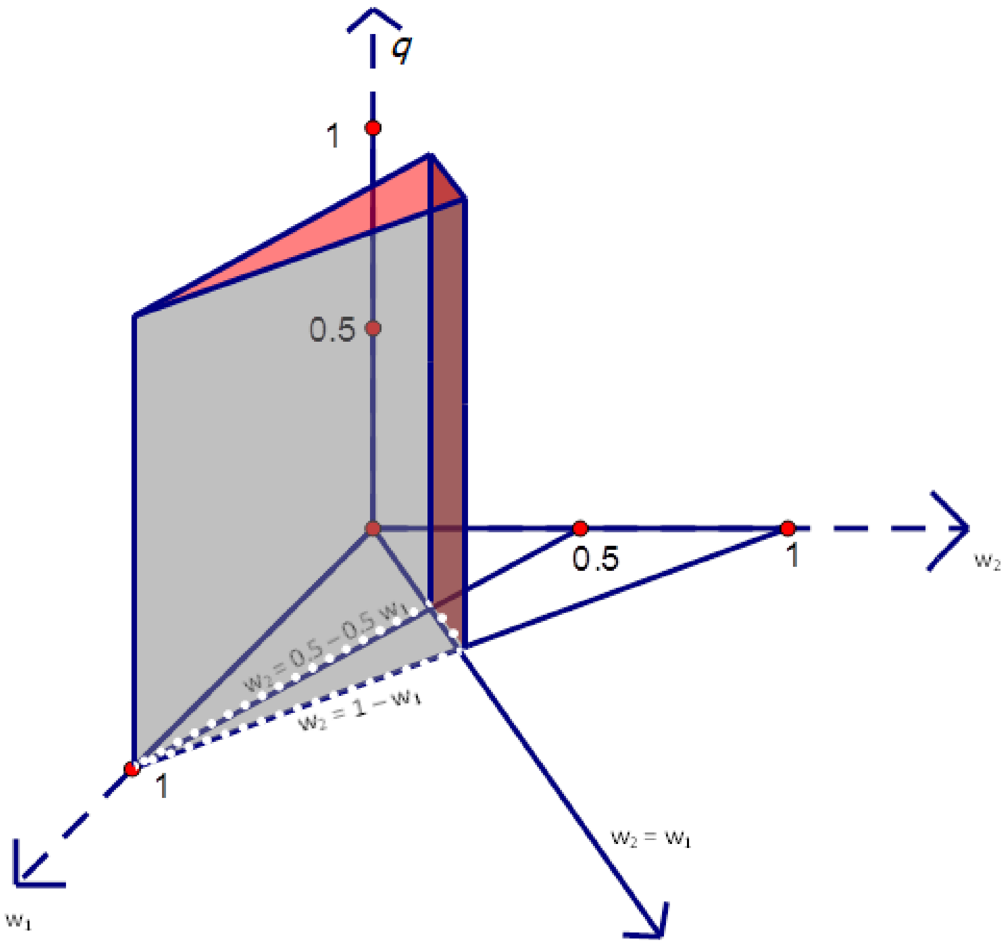

Figure 5: A prism of possible inputs for a power index

The triangles in Kirstein (2010) (viz., in Figure 1 and Figure 2 in that paper) show which power index profiles occur if $q > \frac{1}{2}w$ and you hold q constant and allow the weights to vary (or, in terms of Figure 5, if you take a slice of the prism by passing a horizontal plane [that is, a plane parallel to the (w_1, w_2) plane] through the prism).

5.4 A second established way of visualizing power index profiles for a variable distribution of weights

Next, we will describe a method for visualizing power index profiles for a variable distribution of weights that is in Fischer and Schotter (1976, Appendix B and Appendix C) and has also been used in subsequent sources – including Berg and Holler (1976, pp. 424-425), Leech (2003), Haines and Jones (2005) and Jones (2009). We will then discuss the relation between this visualization of power index profiles and the one we have in Figures 1-4.

As in Kirstein (2010), this visualization of power index profiles is based on a geometric representation for the weights. Also as in Kirstein (2010), the weights and the quota are normalized.

In contrast with Kirstein (2010), the weights are not required to satisfy $w_1 \geq w_2 \geq w_3$. Also in contrast with Kirstein (2010), the possible normalized weights are treated as points in (w_1, w_2, w_3) space. More specifically, the possible normalized weights are represented as points on the unit 2-simplex which is the intersection of 1) the plane $w_1 + w_2 + w_3 = 1$ and 2) the nonnegative octant where $w_i \geq 0$ for each $i \in \{1, 2, 3\}$. This forms a triangular region in $\mathbb{R}^3$, where the coordinates for any given point in the triangular region are the barycentric coordinates with respect to the standard basis vectors for $\mathbb{R}^3$ (that is, with respect to $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$).

In Fischer and Schotter (1976) and subsequent sources, the simplex is drawn as a planar figure – using a "normal view" of the plane $w_1 + w_2 + w_3 = 1$ (that is, the view in which the direction of sight is perpendicular to that plane). Accordingly, their diagrams are two-dimensional – although, of course, each point in the planar figure has three coordinates (so, for instance, the vertices of the triangle in their two-dimensional diagrams are labelled as $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$).

As in Kirstein (2010), the figures in Fischer and Schotter (1976) and subsequent sources have q fixed and treat the weights as varying quantities. Using an approach that is analogous to the representation in Kirstein (2010) of different Banzhaf power index profiles for a given quota q, Fischer and Schotter (1976) and subsequent sources partition their simplex into sub-regions in which different power index profiles occur. The simplex is specifically partitioned into sub-regions in which different Shapley-Shubik power index profiles occur.

The simplex described above is in the three-dimensional (w_1, w_2, w_3) space. However, the following observations apply. Let S denote the simplex in Fischer and Schotter (1976) and subsequent sources. Let P denote the (w_1, w_2) plane in (w_1, w_2, w_3) space (that is, the set of points in (w_1, w_2, w_3) space where $w_3 = 0$). The function $T_1 : S \rightarrow P$ where $T_1(w_1, w_2, w_3) = w_1(1, 0, 0) + w_2(0, 1, 0)$ for each $(w_1, w_2, w_3) \in S$ is the *orthogonal projection function* from the simplex in Fischer and Schotter (1976) and subsequent sources to the (w_1, w_2) plane. [Note: For each s in S that is in the (w_1, w_2) plane, this function simply maps from the point to itself; For each s in S that is not in the (w_1, w_2) plane, if you draw a line (passing through s) that is perpendicular to the (w_1, w_2) plane, this line will intersect the (w_1, w_2) plane at $T_1(s)$.] The image of T_1 is called the *orthogonal projection* of S. T_1 is a one-to-one function. So the orthogonal projection of S is also a geometric representation of the possible values of the normalized weights – with any results about a point $T_1(s)$ in the projection being equivalent to corresponding results about s.

Since the orthogonal projection of S is entirely in the (w_1, w_2) plane in (w_1, w_2, w_3) space, the image of the function $T_2 : S \rightarrow P$ where $T_2(w_1, w_2, w_3) = w_1(1, 0) + w_2(0, 1)$ for each $(w_1, w_2, w_3) \in S$ (which is an image that is in the two-dimensional (w_1, w_2) space) will similarly be a geometric representation of the possible values of the normalized weights – with any results about a point, $T_2(s)$, also being equivalent to corresponding results about s.

The geometric representation that we just obtained is in (w_1, w_2) space (rather than, as with the orthogonal projection, being in (w_1, w_2, w_3) space). The figures in Kirstein (2010) are also in (w_1, w_2) space – although, unlike the figures in Kirstein (2010), the geometric representation that we just obtained contains every point in the triangular region with the vertices $(1, 0)$, $(0, 1)$, and $(0, 0)$. However, we have already observed that, for any weighted simple game where $n = 3$, one can (as in Kirstein (2010, p. 6) assign index numbers so that their weights have the following relationship: $w_1 \geq w_2 \geq w_3$. What's more, when that assumption is made, the only relevant normalized weights (in the geometric representation that we obtained in (w_1, w_2) space) are the ones that are contained in the triangular region with the vertices $(0, 1)$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{3}, \frac{1}{3})$. So we end up with a triangular region which is like the bases in the prism in Figure 5. As a consequence, statements that are similar to the ones that we previously made about Figure 5 apply to the type of prism that is obtained if we 1) obtain the bases by applying T_2 to the simplex that is used in Fischer and Schotter (1976) and subsequent sources, 2) assign index numbers so that $w_1 \geq w_2 \geq w_3$ and 3) have the third dimension correspond to the possible values for q.

6 Applying indices to the context of ballroom dancers' scores

The first part of this section provides a rationale for considering different thresholds and shows why Juan Vasquez and Janice Vasquez might get the conflicting assessments from their coaches that were described in the introduction. The second part of this section draws attention to what Jones (2009) calls the 'quota paradox' and demonstrates that certain instances in the ballroom dancing context exemplify this paradox.

6.1 Varying goals with fixed numbers of points earned for ballroom dancers' scores

The number lines in Figures 1, 2, and 3 of the previous section can help couples determine the contribution of a specific performance to achieving a particular goal. Once the points earned per dance are known, couples can find the corresponding number lines that relate to their points earned. Once couples decide upon what thresholds they would like to consider, couples can find where the thresholds lie on the number line and determine the Banzahf and Shapley-Shubik indices that result from the points and thresholds under consideration.

The results of the competitors in the semi-final round at the 2017 USA Dance National DanceSport Championships in the Senior III age category of the Novice Standard level (which was described in the introduction) are shown in Table 1.

Juan and Janice earned 18 points across their three dances: Waltz ($w_3 = 5$ points), Foxtrot ($w_2 = 6$ points), and Quickstep ($w_1 = 7$ points). In this

Table 1: 2017 USA Dance Nationals Senior III Novice Standard semi-final event

Placement	Couple	Waltz	Foxtrot	Quickstep	Total points
1	Juan and Janice	5	6	7	18
2	Christos and Susan	7	6	4	17
3	Thomas and Maureen	5	5	6	16
4	Gregory and Susan	6	5	3	14
5	John and Etsuko	5	3	4	12
6	Terrance and Candace	4	3	4	11
7	Pg and Jeanne	3	4	2	9
8	Terry and Sandra	3	2	3	8
9	Donald and Fannia	1	3	3	7
10	Clement and Lilian	0	2	3	5
11	Carl and Mimi	2	1	2	5
12	Timothy and Deborah	1	2	1	4

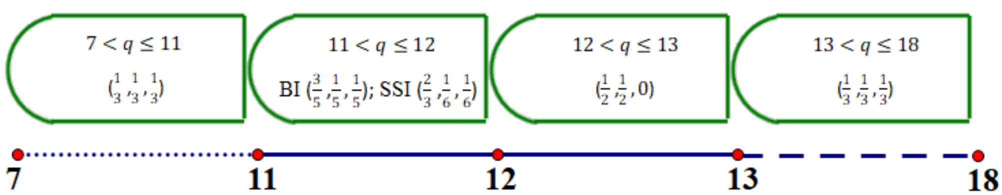

Figure 6: Possible thresholds for Juan and Janice and their corresponding indices

context, $w_2 + w_3 > w_1$ and the top number line in Figure 1 corresponds to the situation. For the points earned by Juan and Janice, the specific number line that corresponds is Figure **??**.

Looking back, Juan and Janice may want to determine the contributions that each of their dances made towards important goals. Depending on their goals, the contributions of each dance towards the goals could differ. Below, we provide examples of four different thresholds that might be sensible, provide the corresponding Banzhaf and Shapley-Shubik index profiles for these thresholds, and relate these to the number line representation.

One possible threshold that Juan and Janice might consider is the number of points needed to qualify for the final round. Upon examination of Table

1, that threshold would be 10 points because the seventh-placing couple of Pg and Jeanne earned 9 points and any couple that earned more than 9 points would qualify for the final round. Both the Banzhaf and Shapley-Shubik index profiles are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for the threshold of 10 points. That is, each of Juan's and Janice's dances contributed equally towards helping them advance to the final round.

Juan and Janice might not just want to consider how their three dances helped them to marginally qualify for the final round. A higher threshold is 12 points, the number of points earned by the fifth-placing couple John and Etsuko. In this situation, the Banzhaf index profile is $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ and the Shapley-Shubik index profile is $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. Both indices agree that Juan's and Janice's Quickstep contributed twice as much as either their Waltz or their Foxtrot towards achieving at least fifth place.

Another possible threshold is 13 points, one point higher than the number of points earned by the fifth-placing couple. If 13 points is the goal, then the Banzhaf and Shapley-Shubik index profiles are both $(\frac{1}{2}, \frac{1}{2}, 0)$. This indicates that the 7 points earned from their Quickstep and the 6 points earned from their Foxtrot both contributed equally towards meeting this goal, whereas the points earned from their Waltz did not contribute at all.

Juan and Janice might also want to know how each of their dances contributed to them placing within the top three couples in this round. If that is the case, then their threshold would be 16 points, the number of points that the third-placing couple of Thomas and Maureen earned. The Banzhaf and Shapley-Shubik index profiles for the threshold of 16 points are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ because points from all three dances are equally needed to meet the goal.

Table 2 summarizes the results described above.

Table 2: Possible thresholds for Juan and Janice interpretations

q (threshold)	Interval in which q lies	Interpretation
10	$7 < q \leq 11$	Qualify for final round
12	$11 < q \leq 12$	Tie for fifth place
13	$12 < q \leq 13$	One point higher than fifth place
16	$13 < q \leq 18$	Tie for third place

Next, we provide a summary of two other scenarios from the same dance competition where varying the thresholds may be sensible. The two scenarios are the possible thresholds that couples Terrance and Candace (who finished

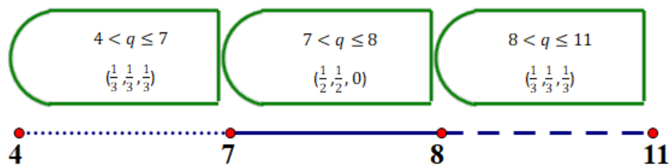

Figure 7: Possible thresholds for Terrance and Candace and their corresponding indices

Table 3: Terrance and Candace's possible thresholds and interpretations

q (threshold)	Interval in which q lies	Interpretation
6	$4 < q \leq 7$	Tie for tenth place
8	$7 < q \leq 8$	Tie for eighth place
10	$8 < q \leq 11$	Qualify for final round

in sixth place) and Thomas and Maureen (who finished in third place) might consider for the contributions of their dances in the same semi-final round in which Juan and Janice competed.

Terrance and Candace earned 4 points on each of their Waltz and Quickstep (so $w_1 = w_2 = 4$), and they earned 3 points on their Foxtrot (so $w_3 = 3$). The points earned are in the relationship $w_1 = w_2 > w_3$, and this corresponds to Figure 2. For the points earned by Terrance and Candace, the specific number line that corresponds is Figure **??**. In Table 3, we provide some of Terrance and Candice's possible thresholds and their interpretations.

Thomas and Maureen earned 6 points on their Quickstep (so $w_1 = 6$), and 5 points on each of their Waltz and Foxtrot (so $w_2 = w_3 = 5$). For the points that Thomas and Maureen earned, $w_1 > w_2 = w_3$, and this corresponds to Figure 3. The specific number line that corresponds is depicted in Figure **??**. In Table 3, we provide some of Thomas and Maureen's possible thresholds and their interpretations.

Table 4: Thomas and Maureen's possible thresholds and interpretations

q (threshold)	Interval in which q lies	Interpretation
10	$6 < q \leq 10$	Qualify for final round
11	$10 < q \leq 11$	Tie for sixth place
12	$8 < q \leq 16$	Tie for fifth place

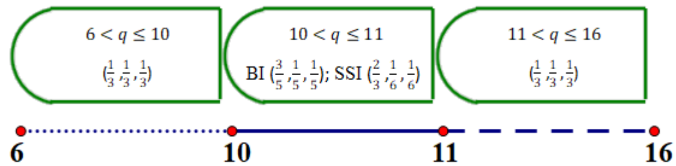

Figure 8: Possible thresholds for Thomas and Maureen and their corresponding indices

6.2 The quota paradox

Jones (2009, p. 110) observes that one might expect that lowering the quota for a weighted simple game would benefit the element of N that has the highest weight (i.e., it would increase the power value of that element). He then points out that decreasing the quota actually does **not** always benefit the element of N that has the highest weight! Jones (2009, p. 110) calls this the *quota paradox*. Jones (2009, pp. 110-111) illustrated the quota paradox with an example with $n = 3$ where lowering the quota lowered the Shapley-Shubik index for the element of N that has the highest weight.

Figures 1-3 make it clear that, in a simple weighted game where $n = 3$ and either the Banzhaf index or the Shapley-Shubik index is being used, there are several situations in which lowering the quota will lower the index number for the element of N that has the highest weight.

1. In Figure 1 (where $w_1 > w_2 > w_3$):

    - When the quota is lowered from being in $(w_2 + w_3, w_1 + w_3]$ to being in $(w_1, w_2 + w_3]$, or from being in $(w_2, w_1]$ to being in $(0, w_3]$, the Shapley-Shubik index of the highest weight decreases from $\frac{2}{3}$ to $\frac{1}{3}$, and its Banzhaf index decreases from $\frac{3}{5}$ to $\frac{1}{3}$.
    - When the quota is lowered from being in $(w_1 + w_3, w_1 + w_2]$ to being in $(w_1, w_2 + w_3]$, or from being in $(w_3, w_2]$ to being in $(0, w_3]$, the Shapley-Shubik and Banzhaf indices decrease from $\frac{1}{2}$ to $\frac{1}{3}$.
    - When the quota is lowered from being in $(w_2 + w_3, w_1]$ to being in $(w_2, w_2 + w_3]$, the Shapley-Shubik index of the highest weight decreases from 1 to $\frac{2}{3}$, and the Banzhaf index of the highest weight decreases from 1 to $\frac{3}{5}$. When the quota is further lowered to being in $(w_3, w_2]$, both indices of the highest weight decrease to $\frac{1}{2}$. When

the quota is even further lowered to being in $(0, w_3]$, both indices of the highest weight decrease to $\frac{1}{3}$.

2. In Figure 2 (where $w_1 = w_2 > w_3$): When the quota is lowered from being in $(w_1 + w_3, 2w_1]$ to being in $(w_1, w_1 + w_3]$, or from being in $(w_3, w_1]$ to being in $(0, w_3]$, the Shapley-Shubik and Banzahf indices of the highest weight both decrease from $\frac{1}{2}$ to $\frac{1}{3}$.

3. In Figure 3 (where $w_1 > w_2 = w_3$): When the quota is lowered from being in $(w_2 + w_3, w_1 + w_3]$ to being in $(w_1, w_2 + w_3]$, or from being in $(w_2, w_1]$ to being in $(0, w_2]$, the Shapley-Shubik index of the highest weight decreases from $\frac{2}{3}$ to $\frac{1}{3}$, and the Banzhaf index of the highest weight decreases from $\frac{3}{5}$ to $\frac{1}{3}$.

Applying the initial observation in the discussion of the quota paradox in Jones (2009) to dance competitions: If the number of points assigned to each dance performance is fixed and the goal changes, one might expect that lowering the goal will benefit the highest scoring dance. However, the first situation described above appears in our ballroom dancing context when Juan and Janice's quota is initially within $(11, 12]$ and is then lowered to being within $(7, 11]$ (for instance, by going from the q for tying for fifth place to the q for qualifying for the final round). The second situation described above appears when Juan and Janice's quota is initially within $(12, 13]$ and is then lowered to being within $(7, 11]$ (for instance, by going from the q for being one point higher than fifth place to the q for qualifying for the final round). The third situation does not appear within our examples.

The fourth situation occurs with Thomas and Maureen's scores when the quota is initially within $(10, 11]$ and is then lowered to being within $(6, 10]$ (for instance, by going from the q for tying for sixth place to the q for qualifying for the final round). What's more, the fifth situation occurs with Terrance and Candace's scores when the quota is initially within $(7, 8]$ and is then lowered to being within $(4, 7]$ (for instance, by going from the q for tying for eighth place to the q for tying for tenth place).

7 Conclusion: How might competitors use this information?

In this paper, we provide geometric representations to show what the sets of Banzhaf and Shapley-Shubik indices are when the number of players in a game is three and when there is a range of possible thresholds. Conrad provides an online calculator for Banzhaf and Shapley-Shubik index computations for simple weighted games with any number of players. It is useful to know how index calculations will vary based on thresholds selected. The index calculations help competitors determine, relative to a particular goal, how much each of their three dances contributed towards their goal. Below, we mention two possible interpretations or reactions to knowledge of the index calculations in the context of three-dance competitions.

1) Competitors may use information from the index calculations during the time between the semi-final round and the final round. Within the competition schedule, there are typically there are approximately 15-30 minutes in between each round during which time competitors have Internet access to the results from the previous round, and they have access to a practice floor space where they can consult with their coaches. Knowledge of the indices can assist in making decisions for the final round. The judges remain the same between rounds for the same event, and a subset (approximately half) of the couples advance from the semi-final to the final round.

Competitive couples typically have more than one version of each dance routine, with the difficulty level of a version being inversely related to the consistency of the couple's performance of the routine. A more difficult routine may be slightly less consistent in quality, but may be worth trying in the final round if it could help the team towards its goal for the final round. Or, if the team already performed a more difficult routine in the semi-final round but it did not help the couple reach its goal, the couple may consider using an easier routine that is more consistent for the final round.

For example: Without using index calculations or considering a goal, Juan and Janice might consider just changing their lowest-scoring dance routine (Waltz, which scored 5 points) between the semi-final round and the final round. If Juan and Janice's personal goal was in the range of (11, 12], they know that their index profile is $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ and their two lowest-scoring dance routines of Waltz and Foxtrot contributed equally towards their goal (not just their lowest-scoring routine). They can choose to change their routines in either of the two lower-contributing dances, or both of these dances. [Obviously, if the couple made a mistake during one of the lower-scoring dances, the first thing to consider doing is to fix the mistake, rather than changing the version of the routine].

2) Competitors may use information from the index calculations to help them decide which of their dances to perform in a subsequent exhibition opportunity. Typically competitors like to perform their "best" dance, but how should they determine which of their dances were the "best"?

For example: Without using index calculations or considering a goal, Juan and Janice might only consider performing their highest-scoring dance routine (the one that scored 7 points) in the semi-final round. If Juan and Janice's goal was in the range of (12, 13], both of their two higher-scoring dances of Foxtrot and Quickstep contributed equally towards their goal, so they could consider performing either of the two dances.

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