Borel stay-in-a-set games [1]

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Abstract. Consider an n-person stochastic game with Borel state space S, compact metric action sets $A_1, A_2, \ldots, A_n$, and law of motion q such that the integral under q of every bounded Borel measurable function depends measurably on the initial state x and continuously on the actions $(a_1, a_2, \ldots, a_n)$ of the players. If the payoff to each player i is 1 or 0 according to whether or not the stochastic process of states stays forever in a given Borel set G_i, then there is an ϵ-equilibrium for every $\epsilon > 0$.

1 Introduction.

The stochastic games treated here have n players $1, 2, \ldots, n$. The state space S is a Borel subset of a Polish space. Every player i has an action set A_i which is a compact metric space. The set $\mathcal{P}(A_i)$ of probability measures defined on the Borel subsets of A_i is given the usual weak topology and, hence, $\mathcal{P}(A_i)$ is also compact metrizable. Let $A = A_1 \times A_2 \times \cdots \times A_n$ have its product topology so that it too is compact metrizable. The law of motion q is a conditional probability distribution on S given $S \times A$ with the interpretation that, if the players choose actions $a = (a_1, a_2, \ldots, a_n) \in A$ at state $x \in S$, then $q(\cdot|x, a)$ is the conditional distribution of the next state. It is assumed that for every bounded Borel measurable function $f : S \mapsto \mathbb{R}$, the integral $\int f(y)\, q(dy|x, a)$ is, for fixed x, a continuous function of a and, for fixed a, a Borel measurable function of x.

A strategy for a player specifies how the player selects an action at each stage of play. Formally, a *strategy* σ_i for player i is a function that assigns to each partial history $p = (a^1, x_1, a^2, x_2, \ldots, x_k)$, including the empty partial history $p = \emptyset$, a probability measure $\sigma_i(p) \in \mathcal{P}(A_i)$. The measure $\sigma_i(\emptyset)$ is the distribution of player i's initial action a_i^1 and $\sigma_i(p)$ is the conditional distribution of player i's action a_i^{k+1} given $p = (a^1, x_1, a^2, x_2, \ldots, x_k)$. A function $\bar{\sigma}_i$ that assigns a strategy $\bar{\sigma}_i(x)$ to each initial state $x \in S$ is called a *family of strategies*. A strategy σ_i is *Borel* if, for $k = 1, 2, \ldots$, $\sigma_i(a^1, x_1, a^2, \ldots, x_k)$ is Borel measurable from $(A \times S)^k$ to $\mathcal{P}(A_i)$. A family of strategies $\bar{\sigma}_i$ is *Borel* if, for $k = 1, 2, \ldots$, $\bar{\sigma}_i(x)(a^1, x_1, a^2, \ldots, x_k)$ is Borel measurable from $S \times (A \times S)^k$ to $\mathcal{P}(A_i)$. All the strategies and families of strategies considered in this paper will be Borel. So we will sometimes omit the adjective "Borel."

A *Borel profile* is an n-tuple $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of Borel strategies - one for each player. A *Borel family of profiles* is an n-tuple $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n)$ of Borel families of strategies.

An initial state $x_0 = x$ and a Borel profile σ together with the law of motion q determine in the usual way a probability measure $P_{x,\sigma}$ on the Borel subsets of the infinite product space

$$H = S \times A \times S \times A \times \cdots . \tag{1.1}$$

We regard $P_{x,\sigma}$ as the distribution of the infinite sequence or play $h = (x_0, a^1, x_1, a^2, \ldots)$ with $x_0 = x$, and write $E_{x,\sigma}$ for the associated expectation operator.

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Suppose now that each player i has a bounded Borel measurable payoff function $\phi_i : H \mapsto \mathbb{R}$. Thus, if x is the initial state and each player selects a strategy σ_i, the expected payoff to player i is $E_{x,\sigma}\phi_i$, where σ is the profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

For $\epsilon \geq 0$, an *ϵ-equilibrium* at the initial state x is a profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, such that, for all $i = 1, 2, \ldots, n$,

$$E_{x,\sigma}\phi_i \geq \sup_{\mu_i} E_{x,(\sigma_1,\ldots,\sigma_{i-1},\mu_i,\sigma_{i+1},\ldots,\sigma_n)}\phi_i - \epsilon,$$

where μ_i ranges over the set of all strategies for player i. A *Nash equilibrium* is a 0-equilibrium.

A challenging problem in the theory of stochastic games is to determine when equilibria exist. For uncountable state and action spaces, the major results available are for payoff functions that are discounted sums of daily rewards. (See Mertens and Parthasarathy (1991) and Nowak(to appear) and the references cited in these papers.)

Here we treat a very special nondiscounted case. To each player i is associated a Borel subset G_i of the state space S. The payoff function ϕ_i is the indicator function of the set $G_i^\infty \subseteq H$ defined by

$$G_i^\infty = \{(x_0, a^1, x_1, a^2, \ldots) : x_k \in G_i \text{ for all } k = 0, 1, \ldots\}. \tag{1.2}$$

Thus each player i receives a payoff of 1 if the process of states stays forever in G_i and receives 0 otherwise. A game that has payoff functions of this form and satisfies the other assumptions made above is called a *Borel stay-in-a-set game.*

Theorem 1.1. *For every n-person Borel stay-in-a-set game and every $\epsilon > 0$, there is a Borel family $\bar{\sigma}$ of profiles such that $\bar{\sigma}(x)$ is an ϵ-equilibrium at x for all $x \in S$.*

For a countable state space and finite action sets, this result was proved by Secchi and Sudderth (2001a). Here we follow the outline of their proof, but must verify that it can be carried out in a Borel measurable framework.

The next section gives a Borel treatment of finite horizon games. Section 3 treats the one-person stay-in-a-set game. The proof of Theorem 1.1 is completed in section 4. The final section has a generalization and some questions.

We conclude this section with a remark on conditioning.

Remark 1.2. *Let σ_i be a strategy and let $p = (a^1, x_1, a^2, x_2, \ldots, x_k)$ be a partial history. The* conditional strategy *$\sigma_i[p]$ is defined at each partial history $\tilde{p} = (b^1, y_1, b^2, y_2, \ldots, y_l)$ by the rule $\sigma_i[p](\tilde{p}) = \sigma_i(p\tilde{p})$ where $p\tilde{p}$ is the partial history consisting of the elements of p followed by the elements of $\tilde{p}$. If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a profile, then the* conditional profile *$\sigma[p]$ is just the profile of conditional strategies $(\sigma_1[p], \sigma_2[p], \ldots, \sigma_n[p])$. A* stopping time *$T$ is a Borel mapping from $A \times S \times A \times S \times \cdots$ to $\{0, 1, \ldots\} \cup \{\infty\}$ such that, for $k = 0, 1, \ldots$ and infinite histories $h = (a^1, x_1, a^2, x_2, \ldots)$ and $\tilde{h} = (b^1, y_1, b^2, y_2, \ldots)$, if $T(h) = k$ and h and $\tilde{h}$ agree in their first $2k$ coordinates, then $T(\tilde{h}) = k$. If $T(h) < \infty$, we define $\sigma[p_T](h) = \sigma[p_T(h)]$, where $p_T(h) = p_T(a^1, x_1, a^2, x_2, \ldots) = (a^1, x_1, a^2, x_2, \ldots, a^{T(h)}, x_{T(h)})$. Suppose that $x \in X$, σ is a profile, T is a stopping time, and $P_{x,\sigma}[T < \infty] = 1$. Then it is straightforward to verify that $P_{x_T,\sigma[p_T]}$ is a version of the conditional $P_{x,\sigma}$-distribution of $(x_T, a^{T+1}, x_{T+1}, \ldots)$ given $(a_1, x_1, \ldots, x_T)$.*

2 Borel backward induction

The objective of this section is to see that the usual backward induction argument can, under our assumptions on S, A, and q, be made Borel measurable. To do this we consider a k-day game $\Gamma_k(u)(x)$ with initial state x and terminal payoff functions $u_i : S \mapsto \mathbb{R}$, $i = 1, 2, \ldots$, that are bounded and Borel measurable. That is, for each Borel profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, the payoff to each player i is $E_{x,\sigma} u_i(x_k)$. (We could of course restrict attention to k-day strategies.)

Lemma 2.1. *For each $k = 1, 2, \ldots$ there is a Borel family $\bar{\sigma}^k$ of profiles such that $\bar{\sigma}^k(x)$ is a Nash equilibrium for $\Gamma_k(u)(x)$ for every $x \in S$.*

Proof. Consider first the case $k = 1$. For each $x \in S$, the one-day game $\Gamma_1(u)(x)$ is just a one-shot game with compact action sets $A_1, A_2, \ldots, A_n$ and payoffs

$$w_i(x, a) = w_i(x, a_1, a_2, \ldots, a_n) = \int u_i(x_1)\, q(dx_1|x, a).$$

By our assumptions about q, the functions $w_1, w_2, \ldots, w_n$ are, for fixed x, continuous on the compact set A. So, by a standard result (Theorem I.4.1, Mertens et al (1994)), there is a Nash equilibrium $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \mathcal{P}(A_n)$ for $\Gamma_1(u)(x)$. To complete the argument for the case $k = 1$, we must make a selection $\bar{\mu}(x)$ from the set $\mathcal{N}_1(u)(x)$ of Nash equilibria for $\Gamma_1(u)(x)$ in such a way that the function $\bar{\mu}$ is Borel. To see that a Borel selection is possible, notice that the graph $Gr(\mathcal{N}_1(u))$ is the collection of all (x, μ) such that, for all $i = 1, 2, \ldots, n$, and all $\nu \in \mathcal{P}(A_i)$,

$$\begin{aligned} &\int \cdots \int w_i(x, a)\, \mu_1(da_1) \cdots \mu_n(da_n) \geq \\ &\qquad \int \cdots \int w_i(x, a)\, \mu_1(da_1) \cdots \mu_{i-1}(da_{i-1}) \nu(da_i) \mu_{i+1}(da_{i+1}) \cdots \mu_n(da_n). \end{aligned}$$

Because each space A_i is separable, the same is true of $\mathcal{P}(A_i)$. Thus the inequality above holds for all $\nu \in \mathcal{P}(A_i)$ if and only if it holds for all ν belonging to a countable dense subset of $\mathcal{P}(A_i)$. It follows easily that $Gr(\mathcal{N}_1(u))$ is a Borel subset of $S \times \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \mathcal{P}(A_n)$. We have already seen that every x-section, namely $\mathcal{N}_1(u)(x)$, is nonempty. Furthermore, it is straightforward to check that $\mathcal{N}_1(u)(x)$ is a closed and, therefore, compact subset of the compact set $\mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \mathcal{P}(A_n)$. It now follows from the Kunugui-Novikov selection theorem (cf. section 5.7.1 of Srivastava (1998)) that there is a Borel selector $\bar{\mu}$ such that $\bar{\mu}(x) \in \mathcal{N}_1(u)(x)$ for all $x \in S$. This completes the proof for $k = 1$.

Assume now the inductive hypothesis that the result holds for some $k \geq 1$, and let $\bar{\sigma}^k$ be a Borel family of Nash equilibria for the k-day games $\Gamma_k(u)(\cdot)$. Let

$$v_i(x) = E_{x, \bar{\sigma}^k(x)} u(x_k), \qquad x \in S, i = 1, 2, \ldots, n.$$

It follows from our assumptions on q and the Borel measurability of $\bar{\sigma}^k$ that the v_i are also Borel. Hence, by the case $k = 1$, there is a Borel family of profiles $\bar{\mu}$ for the one-day games $\Gamma_1(v)(\cdot)$. Define $\bar{\sigma}^{k+1}(x) = (\bar{\sigma}_1^{k+1}(x), \cdots, \bar{\sigma}_n^{k+1}(x))$ by letting, for every $x \in S$ and $i = 1, 2, \ldots, n$, the strategy $\bar{\sigma}_i^{k+1}(x)$ begin with the mixed action $\bar{\mu}_i(x)$ and then continue with $\bar{\sigma}_i^k(x_1)$ given that the next state is x_1. Then $\bar{\sigma}^{k+1}(x)$ is clearly Borel in x. A standard backward induction shows that $\bar{\sigma}^{k+1}(x)$ is a Nash equilibrium for $\Gamma_{k+1}(u)(x)$ for every x.

$\square$

Remark 2.2. *The assumption made in the lemma of a terminal payoff $u_i(x_k)$ for each player i is not necessary. The lemma remains true if each player has a payoff function that is a Borel measurable function of the entire history up to time k. This more general form of the lemma is easily reduced to that already proved by defining new states*

$$\tilde{x}_k = (x_0, a^1, x_1, a^2, \ldots, x_k)$$

and introducing a new law of motion

$$\tilde{q}(\tilde{x}_{k+1}|\tilde{x}_k, a^{k+1}) = q(x_{k+1}|x_k, a^{k+1}).$$

This is a standard technique sometimes called the "partial history trick." (See Secchi and Sudderth (2001b).)

3 The one-person game

We assume in this section that there is a single player and we omit the subscript "1". Thus G is written for G_1, A for A_1, etc.

This one-person, stay-in-a-set game can be viewed as a gambling problem or as a negative dynamic programming problem in which the player loses 1 unit when the process of states first exits from G. Indeed, the game of this section is a special case of those treated in Maitra et al (1991). However, the result needed below, namely that there exists a Borel measurable family of optimal strategies, is not proved in this reference. So we will prove it here. Closely related results are in Schäl (1975).

Let v be the value function for the game defined by

$$v(x) = \sup_{\sigma} P_{x,\sigma}(G^\infty), \qquad x \in S,$$

and, for $n = 1, 2, \ldots$, let v_n be the value function for the n-day game defined by

$$v_n(x) = \sup_{\sigma} P_{x,\sigma}(G^n), \qquad x \in S,$$

where

$$G^n = \{h = (x_0, a^1, x_1, a^2, \ldots) \, : \, x_i \in G, \, i = 0, 1, \ldots, n\}.$$

Note that $v(x) = v_n(x) = 0$ for all $x \notin G$.

Let $f : S \mapsto A$ be Borel measurable. Then f determines a (deterministic) *Borel family of stationary strategies* f^∞, where $f^\infty(x)$ is the strategy at initial state x that uses action $f(y)$ whenever the current state is y.

Lemma 3.1. *(i) The n-day value functions v_n are pointwise nonincreasing and converge pointwise to v.*

(ii) The v_n are Borel measurable, and, hence, so is v.

(iii) There is a Borel family f^∞ of optimal stationary strategies so that

$$v(x) = P_{x,f^\infty(x)}(G^\infty), \qquad x \in S.$$

Proof. Since $G^n \supseteq G^{n+1}$, it is trivial that $v_n(x) \geq v_{n+1}(x)$ for all $x \in S, n \geq 1$. Let $w(x) = \lim_n v_n(x)$, $x \in S$. We will show below that $v = w$. Obviously, both functions are identically zero on the complement of G. So we will be mainly concerned with $x \in G$.

Define an operator R on bounded Borel measurable functions $\phi : S \mapsto \mathbb{R}$ by the rule

$$(R\phi)(x) = \begin{cases} \sup_a \int_G \phi(x_1)\, q(dx_1|x,a), & x \in G \\ 0, & x \notin G. \end{cases}$$

By our assumptions on q, $\int_G \phi(x_1)\, q(dx_1|x,a)$ is a continuous function of a for fixed x. So the supremum above can be taken over a countable dense set of a's. Hence, $R\phi$ is Borel measurable and bounded. Furthermore, the backward induction algorithm can be written as

$$v_1 = R1,\ v_n = Rv_{n-1} \qquad n \geq 2.$$

It follows that the v_n are Borel measurable, and, hence, so is w.

Now fix $x \in G$ and consider the functions

$$\psi_n(a) = \int_G v_{n-1}(x_1)\, q(dx_1|x,a),\ \psi(a) = \int_G w(x_1)\, q(dx_1|x,a),\ a \in A.$$

By our assumptions on q, the functions ψ, ψ_n, $n \geq 2$ are continuous on the compact set A. By the monotone convergence theorem, ψ_n decreases to ψ as $n \to \infty$. By Dini's theorem (Kelley (1955), p. 239), ψ_n converges uniformly to ψ. Using the continuity of ψ_n and the compactness of A, we also have

$$v_n(x) = \sup_a \psi_n(a) = \psi_n(a_n),$$

for some $a_n \in A$ and every n. Let a_{n_k} be a convergent subsequence of a_n and let $a* = \lim_k a_{n_k}$. Then

$$w(x) = \lim \psi_{n_k}(a_{n_k}) = \psi(a*) = \int_G w(x_1)\, q(dx_1|x,a*) \leq (Rw)(x).$$

On the other hand, for every n, $v_{n+1}(x) = (Rv_n)(x) \geq (Rw)(x)$ and, hence, $w(x) \geq (Rw)(x)$.

Thus w solves the optimality equation $\phi = R\phi$. To see that $v = w$, first notice that $v \leq v_n$ because $G^\infty \subseteq G^n$. Therefore, $v \leq \lim v_n = w$.

Next use again the Kunugui-Novikov selection theorem to find a Borel mapping $f : S \mapsto A$ such that for all $x \in G$,

$$w(x) = \int_G w(x_1)\, q(dx_1|x,f(x)) \leq \int_G 1\, q(dx_1|x,f(x)) = P_{x,f^\infty(x)}(G^1),$$

and f is equal to some arbitrary fixed action on the complement of G. An argument by induction shows that, for all x and n,

$$w(x) \leq P_{x,f^\infty(x)}(G^n) \to P_{x,f^\infty(x)}(G^\infty) \leq v(x).$$

We conclude that $w = v$ and $f^\infty(x)$ is optimal for every x.

□

Corollary 3.2. *For every $\epsilon > 0$, there is a Borel function $N_\epsilon : S \mapsto \{1, 2, \ldots\}$ such that*

$$v_{N_\epsilon(x)}(x) \leq v(x) + \epsilon$$

for all $x \in S$.

Proof. Let $N_\epsilon(x) = \inf\{n : v_n(x) \leq v(x) + \epsilon\}$. □

4 The proof of Theorem 1.1

The proof is by induction on the number of players n, and is essentially a Borel version of the proof of Theorem 1.2 in Secchi and Sudderth (2001a).

For $n = 1$, the theorem is immediate from Lemma 3.1(iii). So assume $n \geq 2$, and that the theorem holds for every (n-1)-person Borel stay-in-a-set game. Fix $\epsilon > 0$ and consider an n-person Borel stay-in-a-set game as defined in section 1. It suffices to find a Borel family $\bar{\sigma}$ of ϵ-equilibria for this n-person game.

We consider three cases based on the position of the initial state $x_0 = x$.

Case 1. $x \notin \bigcap_{i=1}^n G_i$.

Suppose that $x \notin G_i$ for some player i. Then the payoff to player i will be zero for every profile , and every strategy for player i will be optimal versus the strategies of the other players. To construct a Borel family $\bar{\mu}_i$ of profiles such that $\bar{\mu}_i(x)$ is an $\epsilon/2$-equilibrium for every $x \notin G_i$, fix an action $a_i^* \in A_i$ for player i and consider the (n-1)-player stay-in-a-set game on S with sets $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_n$ and law of motion

$$\tilde{q}(\cdot|x, (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)) = q(\cdot|x, (a_1, \ldots, a_{i-1}, a_i^*, a_{i+1}, \ldots, a_n)).$$

By the inductive hypothesis, this game has a Borel family $\bar{\nu}_i$ of $\epsilon/2$-equilibria. Let $\bar{\mu}_i$ be the family of profiles for the original game obtained from $\bar{\nu}_i$ by adding at each $x \in S$ the strategy for player i that always plays action a_i^*. Then $\bar{\mu}_i$ is Borel and $\bar{\mu}_i(x)$ is an $\epsilon/2$-equilibrium if $x \notin G_i$.

To complete this case, define the family $\bar{\sigma}$ on the complement of $\bigcap_{i=1}^n G_i$ by setting

$$\bar{\sigma}(x) = \begin{cases} \bar{\nu}_1(x), & \text{if } x \notin G_1 \\ \bar{\nu}_i(x), & \text{if } x \in \bigcap_{j=1}^{i-1} G_j,\ x \notin G_i, i = 2, \ldots, n. \end{cases}$$

Then $\bar{\sigma}(x)$ is an $\epsilon/2$-equilibrium for every $x \notin \bigcap_{i=1}^n G_i$ and $\bar{\sigma}$ is Borel on this set.

For use below, let

$$\psi_i(x) = P_{x,\bar{\sigma}(x)}(G_i^\infty), \qquad x \notin \bigcap_{i=1}^n G_i,$$

be the payoff to each player i from the profile $\bar{\sigma}(x)$.

For the next case, define the set

$$G^\epsilon = \{x \in \bigcap_{i=1}^n G_i : \text{there exists a profile } \tau \text{ such that } P_{x,\tau}(\bigcap_{i=1}^n G_i^\infty) \geq 1 - \epsilon/2\}.$$

Case 2. $x \in G^\epsilon$.

Note that $\bigcap_{i=1}^n G_i^\infty = (\bigcap_{i=1}^n G_i)^\infty$ and consider the one-person stay-in-a-set game with $G = \bigcap_{i=1}^n G_i$, action set $A = A_1 \times \cdots \times A_n$, and the same law of motion q. Let f^∞ be the optimal stationary family of Lemma 3.1(iii). Then f^∞ can be regarded as a Borel family of profiles in the n-person game. Define $\bar{\sigma}(x)$ to be the profile corresponding to $f^\infty(x)$ for $x \in G^\epsilon$. Clearly, $\bar{\sigma}(x)$ is an $\epsilon/2$-equilibrium for all $x \in G^\epsilon$.

Notice also that, if v is the value function for the one-person game, then

$$G^\epsilon = \{x \in \bigcap_{i=1}^n G_i : v(x) \geq 1 - \epsilon/2\}.$$

So, by Lemma 3.1, G^ϵ is a Borel set.

As in case 1, set

$$\psi_i(x) = P_{x,\bar{\sigma}(x)}(G_i^\infty), \qquad x \in G^\epsilon.$$

Now set $F^\epsilon = \bigcap_{i=1}^n G_i - G^\epsilon$.

Case 3. $x \in F^\epsilon$.

Define the Borel measurable stopping time T on histories $h = (a^1, x_1, a^2, \ldots)$ by

$$T(h) = \inf\{k \geq 1 \,:\, x_k \notin F^\epsilon\}.$$

Lemma 4.1. *For all $x \in F^\epsilon$ and all profiles τ, $P_{x,\tau}[T < \infty] = 1$.*

Proof. Same as for Lemma 3.1 in Secchi and Sudderth (2001a). □

Consider the one-person game with $G = F^\epsilon$, action set A, and law of motion q. The lemma says that, if v is the value function for this one-person game, then $v(x) = 0$ for all $x \in F^\epsilon$. The corresponding N-day one-person game has value function

$$v_N(x) = \sup_\sigma P_{x,\sigma}[T > N].$$

Set $N(x) = N_{\epsilon/4}(x)$ where $N_{\epsilon/4}$ is the function given by Corollary 3.2. Then, for all $x \in F^\epsilon$ and profiles τ,

$$P_{x,\tau}[T \leq N(x)] \geq 1 - \epsilon/4. \tag{4.1}$$

For $x \in F^\epsilon$ and each positive integer N, consider the game $\Gamma_N(u)(x)$ of section 2, where the terminal payoff to to player i is defined by

$$u_i(y) = \begin{cases} \psi_i(y), & \text{if } y \notin F^\epsilon, \\ 0, & \text{if } y \in F^\epsilon\,; i = 1.2. \ldots, n, \end{cases}$$

and the law of motion is

$$\tilde{q}(\cdot|y,a) = \begin{cases} q(\cdot|y,a), & y \in F^\epsilon, \\ \delta(y), & y \notin F^\epsilon. \end{cases}$$

(Here $\delta(y)$ is point mass at y.) We have changed the law of motion from what it was in section 2, but will continue to denote this game by $\Gamma_N(u)(x)$. Notice that the new law of motion essentially stops each path at the time T of first exit from F^ϵ. Hence, the expectation operator $\tilde{E}$ corresponding to $\tilde{q}$ is related to that of the original law of motion by the equality

$$\tilde{E}_{x,\sigma} f(x_N) = E_{x,\sigma} f(x_{T \wedge N}), \tag{4.2}$$

for bounded Borel $f : S \mapsto \mathbb{R}$, where $T \wedge N$ is the minimum of T and N. By Lemma 2.1, there is a Borel family $\bar{\mu}_N$ of profiles such that $\bar{\mu}_N(x)$ is a Nash equilibrium for the game $\Gamma_N(u)(x)$ for every $x \in F^\epsilon$.

Returning to the Borel stay-in-a-set game, for each $x \in F^\epsilon$ such that $N(x) = N$, we define the profile $\bar{\sigma}(x)$ to agree with $\bar{\mu}_N(x)$ up to time T and, then, at time T, switch to $\bar{\sigma}(x_T)$. (Note that $x_T \notin F^\epsilon$ and so $\bar{\sigma}(x_T)$ is available from either Case 1 or Case 2.)

The family of profiles $\bar{\sigma}$ is now defined for all $x \in S$. It is Borel by construction and, by Cases 1 and 2, $\bar{\sigma}(x)$ is an $\epsilon/2$-equilibrium for $x \notin F^\epsilon$. So it only remains to show that $\bar{\sigma}(x)$ is an ϵ- equilibrium for $x \in F^\epsilon$.

Fix $x \in F^\epsilon$ and assume $N(x) = N$. Fix also a player i and an arbitrary strategy μ_i for player i. Let τ be the profile obtained from $\bar{\sigma}(x) = (\bar{\sigma}(x)_1, \ldots, \bar{\sigma}(x)_i, \ldots, \bar{\sigma}(x)_n)$ by substituting μ_i for $\bar{\sigma}(x)_i$. It suffices to show that

$$P_{x,\bar{\sigma}(x)}(G_i^\infty) \geq P_{x,\tau}(G_i^\infty) - \epsilon. \tag{4.3}$$

Calculate as follows:

$$\begin{aligned}
P_{x,\bar{\sigma}(x)}(G_i^\infty) &= E_{x,\bar{\sigma}(x)}(P_{x_T,\bar{\sigma}(x_T)}(G_i^\infty) 1_{G_i}(x_T)) \\
&= E_{x,\bar{\sigma}(x)}(\psi_i(x_T)) \\
&= E_{x,\bar{\sigma}(x)}(u_i(x_T)) \\
&= E_{x,\bar{\mu}_N(x)}(u_i(x_T)) \\
&\geq E_{x,\bar{\mu}_N(x)}(u_i(x_{T\wedge N})) - \epsilon/4 \\
&= \tilde{E}_{x,\bar{\mu}_N(x)}(u_i(x_N)) - \epsilon/4 \\
&\geq \tilde{E}_{x,\tau}(u_i(x_N)) - \epsilon/4 \\
&= E_{x,\tau}(u_i(x_{T\wedge N})) - \epsilon/4 \\
&\geq E_{x,\tau}(u_i(x_T)) - \epsilon/2 \\
&= E_{x,\tau}(\psi_i(x_T)) - \epsilon/2 \\
&= E_{x,\tau}(P_{x_T,\bar{\sigma}(x_T)}(G_i^\infty)) - \epsilon/2 \\
&= E_{x,\tau}(P_{x_T,\bar{\sigma}(x)[p_T]}(G_i^\infty)) - \epsilon/2 \\
&\geq E_{x,\tau}(P_{x_T,\tau[p_T]}(G_i^\infty)) - \epsilon \\
&= P_{x,\tau}(G_i^\infty) - \epsilon
\end{aligned} \tag{4.4}$$

The first equality above uses the facts that $T < \infty$ a.s. $(P_{x,\bar{\sigma}(x)})$, that a version of the conditional $P_{x,\bar{\sigma}(x)}$-distribution given the pre-T σ-field $\mathcal{F}_T$ is $P_{x_T,\bar{\sigma}(x_T)}$, and that the $p_T(h)$-section of G_i^∞ is G_i^∞ or $\emptyset$ according as $x_T \in G_i$ or not because $F^\epsilon \subseteq G_i^\infty$. The second and third equalities depend on the definitions of ψ_i and u_i, respectively. The fourth equality is by virtue of the fact that $\bar{\sigma}(x)$ and $\bar{\mu}_N(x)$ agree up to time T. The first inequality follows from (4.1). The next equality is by (4.2). The second inequality is a consequence of the fact that $\bar{\mu}_N(x)$ is a Nash equilibrium in $\Gamma_N(u)(x)$. The equality following is again by (4.2). The third inequality is by (4.1). The final inequality uses the fact that $\bar{\sigma}(x_T) = \bar{\sigma}(x)[p_T]$ is an $\epsilon/2$-equilibrium in the stay-in-a-set game starting at x_T and the notation $P_{x_T,\tau[p_T]}$ from Remark 1.2 for the conditional $P_{x,\tau}$-distribution given $\mathcal{F}_T$.

5 A generalization

Prior to this section we have worked with the product topology on the space H of (1.1) where S is assigned the topology under which it is a Borel subset of a Polish space and A has the topology under which it is compact metrizable. It is now convenient to introduce a second topology on H, namely, the product topology when S and A are assigned the discrete topology. In this section, the term "Borel" will always refer to the first topology, while the terms "closed," "continuous," and "upper semi-continuous" will refer to the second.

For example, if G_i is a Borel subset of S with its original topology, then the set G_i^∞ of (1.2) is Borel in the first topology and closed in the second. We will call it a Borel, closed set. The indicator function of the set G_i^∞ is Borel, upper semi-continuous.

For another example, consider the discounted reward function

$$\phi_i(h) = \phi_i(x_0, a^1, x_1, a^2, \ldots) = \sum_{k=0}^{\infty} \beta^k r_i(x_k, a^{k+1}) \tag{5.1}$$

where $0 < \beta < 1$ and $r_i : S \times A \mapsto \mathbb{R}$ is bounded Borel for the original topologies on S and A. The function ϕ_i is Borel, continuous.

Here is a generalization of Theorem 1.1. (We continue to make the assumptions on S, A, and q described in section 1.)

Theorem 5.1. *Suppose that each player i for $i = 1, 2, \ldots, n$, has a bounded Borel, upper semi-continuous payoff function. Then, for every $\epsilon > 0$, there is a Borel family $\bar{\sigma}$ of profiles such that $\bar{\sigma}(x)$ is an ϵ-equilibrium at x for every $x \in S$.*

In the special case that S is countable and $A_1, A_2, \ldots, A_n$ are finite, this theorem coincides with Theorem 1.1 of Secchi and Sudderth (2001b). Indeed, our Theorem 5.1 can be proved by going through the steps of their proof and verifying that they can be made Borel measurable. With one exception, all of this is straightforward and analogous to our proof of Theorem 1.1.

The exception concerns the adaptation of Lemma 3.6 from Secchi and Sudderth (2001b) to the present setting. In that lemma it is shown that, with the aid of the partial history trick, an arbitrary closed set $C_i \subseteq H$ can be reduced to a set G_i^∞ as in (1.2). However, it is not clear that the set G_i^∞ is Borel. Luckily it was proved in Lemma 6.2 of Maitra et al (1991) that G_i can be chosen to be a Borel subset of the state space.

Remark 5.2. *Suppose that the payoff functions in Theorem 5.1 are Borel, continuous. With this stonger hypothesis, perhaps the conclusion can be strengthened to say that there exists a Borel family of Nash equilibria. This is true for discounted payoffs (Mertens and Parthasarathy, 1991) or if S is finite and $A_1, A_2, \ldots, A_n$ are finite (cf. 5.2 of Secchi and Sudderth, 2001b). Indeed, we have no counterexample to the existence of Nash equilibria even when the payoffs are only Borel, upper semi-continuous as in the theorem.*

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