Evolution in potential games over connected populations

Jiabin Wu*[1] and Dai Zusai†[2]

[1]Department of Economics, University of Oregon
[2]Department of Economics, Temple University

April 15, 2016

Extended abstract (Preliminary)[1]

1 Introduction

We consider potential games played by a continuum of agents. The society of agents is divided to multiple populations. Connection between these populations is our central focus. For example, separate populations may form a network structure. Or, there may be an overlap over different communities and agents in distant communities may be connected through those who belong to the intersection of these communities. Our formulation encompasses both cases, while the focus on population-level connection simplifies formulation and analysis.

The benchmark is the united base game in which each agent is connected equally with all agents in the society, regardless of the affiliating population, and thus the division does not make any change from the game played by a single united society.

Non-uniform connection results in non-uniform equilibrium strategy distribution over populations, whose aggregate is essentially different from the one in the united base game, and exhibits multiplicity of equilibria, even if the base game has a strictly concave potential and thus only one equilibrium. To reduce multiplicity, we look at local maximizers of the potential as locally stable equilibria in deterministic evolutionary dynamics; further, at the global maximizers as stochastic stable states in stochastic evolution.

Our model of games played in connected populations provides a unified framework to study evolution of language, currency and social norm over multiple communities. Besides, according to known results on convergence of finite population evolution to large population limits, our large-population approach offers a tractable model on behavioral dynamics in social network so we can utilize conventional techniques in evolutionary dynamics.

Literature review

Our study is motivated by two streams of research on evolutionary dynamics: evolution of behavior on social network, especially by Bramoullé, Kranton, and D'Amours (2014); and, evolutionary set theory in biology, especially by Tarnita, Antal, Ohtsuki, and Nowak (2009).

*Address: PLC 515, 1285 University of Oregon, Eugene, OR 97403, U.S.A, telephone:+1-541-346-5778, e-mail: *jwu5@uoregon.edu*.

†Address: 1301 Cecil B. Moore Ave., RA 873 (004-04), Philadelphia, PA 19122, U.S.A, telephone:+1-215-204-8880, e-mail: *zusai@temple.edu*.

[1]For the most recent version, visit `http://sites.temple.edu/zusai/research/ConnectEvol/`

Evolution of behavior on social network

In economics study on evolutionary dynamics on social network, the model starts by assuming finitely many players and the connection between them is formulated as a graph in which each of them as a node in the network graph.[2] Among such papers, Bramoullé, Kranton, and D'Amours (2014) is the most relevant to our paper, as they utilize stability of the potential maximizer in best response dynamic while they consider finitely many players like other papers.

The individual-level connection has a quite high dimension; representing the connection between two players by a binary index, the whole graph requires the dimension of the square of the number of players. Typically they look at some aggregate measures borrowed from graph theory and/or eventually focus on some canonical symmetric shapes of the network graph like a complete network, a star network, or a circular network. When analyzing these canonical graphs, the nodes are explicitly or implicitly summarized into several distinguishable groups and the focus of analysis is on the connection between them.

The flexibility in connections between nodes is important when a researcher has little clue about the underlying segregated communities in the society or has interests in self-evolving social groups.[3] However, in most applied research, a researcher has some concrete ideas about how people are divided into subgroups, for example, by race, ethnicity or socioeconomic status. Then, it seems to be more natural to start the model just by looking at these groups. It hugely reduces the dimension of the model. Besides, graph-theoretic analysis and jargon are not common to the rest of economists, yet. We share the same interests on social network or connection within and over different groups of the society, but our approach directly looks at groups and employs more common techniques.

Evolutionary set theory in biology

Tarnita, Antal, Ohtsuki, and Nowak (2009) extend the graph-theoretic evolutionary model to evolutionary set theory, in which the society is divided into groups; the sets may overlap each other.[4]

It is another motivation for us, and leads to the overlapping community example. However, they are interested in the long-run outcome in co-evolutionary process of behavioral change and migration over sets, while we fix the distribution of agents over the partition of the society. While co-evolution may be suitable for genetic evolution, the fixed population may be suitable for the analysis of a particular strategic interaction in the human society. People may engage in many interactions with others and also some kind—though not all—of social segmentation would be robust and considered as a given exogenous factor in the analysis when we focus on some particular interaction at least in the medium run. For example, race can be taken as given, when we consider segregation of races over different occupations or income classes.

Also, biologists commonly adopt weak selection as the limit of stochastic evolution to select stochastically stable states. The weak selection means that agents' strategy choice becomes less sensible to the payoff change, while economists including us are interested in the limit of small noise where strategy choice becomes more rational and thus more driven by the payoff change. Further, logit dynamic is indeed the governing deterministic dynamic when the noise level is fixed; thus, we have a smooth transition from deterministic dynamic analysis to stochastic stability analysis.

2 The model

2.1 Single-population base game

Let us first define the **base game**, before introducing the structure into the population. Imagine a continuum of agents in a large *society*. The total mass is assumed to be one. Each agent makes a choice from action set $\mathcal{A} := \{1, \cdots, a\}$. That is, the action set is common for all the agents. The payoff function is $\mathbf{F} : \Delta^A \to \mathbb{R}^A$. Here Δ^A is a simplex on $\mathbb{R}^A$, i.e., $\Delta^A := \{\mathbf{x} \in \mathbb{R}^A_+ \mid \sum_{a=1}^A x_a = 1\}$. We extend the domain of $\mathbf{F}$ to $\mathbb{R}^A$. The base game is thus defined by $(\mathcal{A}, \mathbf{F})$.

[2] See Jackson and Zenou (2014) for the survey in this literature in economics.

[3] Staudigl (2011) studies co-evolutionary process of behavior and link formation/destruction in potential games.

[4] Nowak, Tarnita, and Antal (2009) provide an extensive survey on this theory.

For now, we assume *linearity* of the payoff function: $\mathbf{F}$ is given by $\mathbf{F}(\mathbf{x}) = \mathbf{\Pi x}$ with an $A \times A$ matrix $\mathbf{\Pi}$. For example, this can be interpreted as random matching in a symmetric normal-form game $\mathbf{\Pi}$. The (i,j)-th cell Π_{ij} is the payoff from action i against action j.

We further assume *symmetry* of the payoff matrix. So the base game is a common interest game Sandholm (2010). Then the base population game has a potential $f : \Delta^A \to \mathbb{R}$: $f(\mathbf{x}) = 0.5\mathbf{x} \cdot \Pi\mathbf{x}$.

2.2 Structured game over connected populations

We extend the single-population base game to a multi-population game, where the payoff is determined from a weighted sum of each population action distribution. The weight that an agent in population p puts on population q's state measures the degree of connection between the two populations.

Population set

First, we define the populations. The society is divided into P populations: $\mathcal{P} := \{1, \dots, P\}$. p is reserved as a representative index for a population. Denote by m^p the mass of population p. Assume that the total mass is one, i.e., $\sum_{p\in\mathcal{P}} m^p = 1$ Denote by $\mathbf{x}^p \in m^p\Delta^A =: \mathcal{X}^p$ the distribution of actions in population p, or in short, **the state** of population p: $x_a^p \in [0, m^p]$ is the *mass* of action-a players in population p. The whole composition $\mathbf{x}^\mathcal{P} \in \mathbb{R}^{AP}$ defined by

$$\mathbf{x}^\mathcal{P} := \begin{pmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^P \end{pmatrix}$$

is called the social state. Denote by $\mathcal{X}^\mathcal{P}$ the set of feasible social states, i.e., $\mathcal{X}^\mathcal{P} := \times_{p\in\mathcal{X}} \mathcal{X}^p = \times_{p\in\mathcal{X}} m^p\Delta^A$. (Note that it is not just an AP-dimensional simplex $\Delta^{AP} \subset \mathbb{R}^{AP}$.)

Connection between populations

The connection between populations is represented by a $P \times P$ matrix $\mathbf{G}$. Let g_{pq} be on the p-th row and q-th column in matrix G, and $\vec{g}_p \in \Delta^P$ be the p-th row vector of $\mathbf{G}$:

$$\mathbf{G} = \begin{pmatrix} g_{11} & \cdots & g_{1P} \\ \vdots & \ddots & \vdots \\ g_{P1} & \cdots & g_{PP} \end{pmatrix}, \qquad \vec{g}_p = (g_{p1}, \dots, g_{pP}) \text{ for each } p \in \mathcal{P}.$$

As we formulate below, an agent evaluates the payoff of each action based on the weighted sum of the action distribution in each population. Agents in different populations may have different weights. Here, g_{pq} is the weight on population q for an agent in population p; that measures the impact of strategy choices of agents in q on the payoff of agents in p. If the linear payoff is obtained as the expected payoff in random matching, g_{pq} can be interpreted as the probability that an agent in population p to be matched with *each* agent in population q.

Let the social state be $\mathbf{X}$ and consider an agent in population p. The total weight of action-a players in population q is $g_{pq}x_a^q$. The aggregate total weight of action-a players in the whole society for an agent in p is given by $\tilde{x}_a^p := \sum_{q\in\mathcal{P}} g_{pq}x_a^q$. We can interpret this as the observation of action a for an agent in population p. As a result, the distribution of *observed actions* for an agent in population p is $\tilde{\mathbf{x}}^p := \sum_{q\in\mathcal{P}} g_{pq}\mathbf{x}^q$. Note that[5]

$$\tilde{\mathbf{x}}^p = \sum_{q\in\mathcal{P}} g_{pq}\mathbf{I}^A\mathbf{x}^q = \underbrace{[g_{p1}\mathbf{I}^A \ \cdots \ g_{pP}\mathbf{I}^A]}_{A\times AP} \underbrace{\begin{pmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^P \end{pmatrix}}_{AP\times 1} = [\vec{g}^p \otimes \mathbf{I}^A]\mathbf{x}^\mathcal{P} \in \mathbb{R}^{AP},$$

[5]Here $\mathbf{I}^d$ is a $d \times d$ identity matrix.

and this may not add up to one ($\tilde{\mathbf{x}}^p$ may not be a probability vector) unless $\sum_{q\in\mathcal{P}} g_{pq} = 1$. Thus, the payoff vector for population p is obtained as

$$\mathbf{F}(\tilde{\mathbf{x}}^p) = \mathbf{\Pi}\tilde{\mathbf{x}}^p = \mathbf{\Pi}[\vec{g}^p \otimes \mathbf{I}^A]\mathbf{x}^{\mathcal{P}} =: \tilde{\mathbf{F}}^p(\mathbf{x}^{\mathcal{P}}).$$

The total number of observations for an agent in population p is $\sum_{a\in\mathcal{A}}\sum_{q\in\mathcal{P}} g_{pq}x_p^q = \sum_{q\in\mathcal{P}} g_{pq}m^p$, which may not be equal to one if $\sum_{q\in\mathcal{P}} g_{pq} \neq 1$. Under the assumption of a linear payoff function in the base game, this has only a scaling effect on the payoff vector. That is, with $\mathbf{x}^{\mathcal{P}}$ fixed, doubling the total number of observations uniformly doubles the payoff from each action but does not change the payoff ranking over the actions.

As every agent is assumed to have the same base payoff *function* $\mathbf{F}$, the difference in the payoff ranking over actions across different populations comes from the difference in the groups of opponent players due to the biased observations.

The structured game

Now under biased connection between populations, agents evaluate the payoff based on biased observations. The payoff function $\tilde{\mathbf{F}} : \mathcal{X}^{\mathcal{P}} \to \mathbb{R}^{AP}$ is thus obtained as[6]

$$\begin{aligned}
\tilde{\mathbf{F}}^p(\mathbf{x}^{\mathcal{P}}) &:= \begin{pmatrix} \tilde{\mathbf{F}}^1(\mathbf{x}^{\mathcal{P}}) \\ \vdots \\ \tilde{\mathbf{F}}^P(\mathbf{x}^{\mathcal{P}}) \end{pmatrix} = \begin{pmatrix} \mathbf{\Pi}[\vec{g}^1 \otimes \mathbf{I}^A]\mathbf{x}^{\mathcal{P}} \\ \vdots \\ \mathbf{\Pi}[\vec{g}^P \otimes \mathbf{I}^A]\mathbf{x}^{\mathcal{P}} \end{pmatrix} = \begin{pmatrix} \mathbf{\Pi}[\vec{g}^1 \otimes \mathbf{I}^A] \\ \vdots \\ \mathbf{\Pi}[\vec{g}^P \otimes \mathbf{I}^A] \end{pmatrix} \mathbf{x}^{\mathcal{P}} \\
&= \underbrace{\begin{pmatrix} \mathbf{\Pi} & \mathbf{O}^A & \cdots & \mathbf{O}^A \\ \mathbf{O}^A & \mathbf{\Pi} & \cdots & \mathbf{O}^A \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{O}^A & \mathbf{O}^A & \cdots & \mathbf{\Pi} \end{pmatrix}}_{AP\times AP} \underbrace{\begin{pmatrix} g_{11}\mathbf{I}^A & g_{12}\mathbf{I}^A & \cdots & g_{1,P}\mathbf{I}^A \\ g_{21}\mathbf{I}^A & g_{22}\mathbf{I}^A & \cdots & g_{2,P}\mathbf{I}^A \\ \vdots & \vdots & \ddots & \vdots \\ g_{P1}\mathbf{I}^A & g_{P2}\mathbf{I}^A & \cdots & g_{P,P}\mathbf{I}^A \end{pmatrix}}_{AP\times AP} \underbrace{\mathbf{x}^{\mathcal{P}}}_{AP\times 1} \\
&= [\mathbf{I}^P \otimes \mathbf{\Pi}]\,[\mathbf{G} \otimes \mathbf{I}^A]\mathbf{x}^{\mathcal{P}} = [\mathbf{\Pi} \otimes \mathbf{G}]\mathbf{x}^{\mathcal{P}} \in \mathbb{R}^{AP}.
\end{aligned}$$

We call this game the **structured game**.

If the payoff function of the base game is linear, then so is the payoff function of the structured game with the $AP \times AP$ payoff matrix $\tilde{\mathbf{\Pi}} := \mathbf{\Pi} \otimes \mathbf{G}$.

Further, if the payoff matrix $\tilde{\Pi}$ is symmetric, the game has a potential, say $\tilde{f} : \mathcal{X}^{\mathcal{P}} \to \mathbb{R}$ given by

$$\tilde{f}(\mathbf{x}^{\mathcal{P}}) = 0.5\mathbf{x}^{\mathcal{P}} \cdot \tilde{\mathbf{\Pi}}\mathbf{x}^{\mathcal{P}}.$$

$\tilde{\Pi}$ is symmetric if the base payoff matrix Π and the network structure matrix $\mathbf{G}$ are both symmetric. The symmetry of $\mathbf{G}$ means $g_{pq} = g_{qp}$; in the context of random matching, the probability for an agent in population p to meet each agent in population q is equal to that for an agent in q to meet each agent in p. Definiteness of $\tilde{\mathbf{\Pi}}$ is also decomposed to those of $\mathbf{\Pi}$ and $\mathbf{G}$.

2.3 Equilibrium and evolutionary dynamics

In the large population setting, a Nash equilibrium is defined as the social state in which almost no agent in each population has no incentive to change the action. So, the social state $\mathbf{x}^{\mathcal{P}}$ is a Nash equilibrium, if $F_a^p(\mathbf{x}^{\mathcal{P}}) \geq F_b^p(\mathbf{x}^{\mathcal{P}})$ for any $b \in \mathcal{A}$ whenever $x_a^p > 0$.

As we see in examples, the structured game is more likely to have multiple equilibria, even if the base game has a unique equilibrium. So it is of practical and theoretical interests to analyze evolutionary dynamics toward equilibria. The potential function has a tight relation with stability in evolutionary dynamics, both deterministic and stochastic. We will utilize these results.

[6]Here $\mathbf{O}^d$ is a $d \times d$ zero matrix.

- (Sandholm, 2001)[7] If and only if Nash equilibrium $\mathbf{x}_*^{\mathcal{P}}$ is an isolated local maximizer of the potential function $\tilde{f}^{\mathcal{P}}$, it is asymptotically stable under the best response dynamic, or any deterministic evolutionary dynamics that satisfies Nash stationarity and positive correlation.
- (Blume, 1995)[8] If Nash equilibrium $\mathbf{x}_*^{\mathcal{P}}$ is a global maximizer of the potential function $\tilde{f}^{\mathcal{P}}$, it is stochastically stable under logit dynamic in the double limit of large population and small noise.

We should note deterministic dynamics in large population have solid rigorous connection as the limit of finite-player stochastic evolutionary process: see Sandholm (2010, Ch. 10).

3 Canonical types of connection structure

3.1 Multiple affiliations

Consider a situation in which each agent belongs to multiple affiliations and they influence each other through socialization in the affiliations: it is considered in Tarnita and Nowak, though they endogenize the affiliation structure. Such a situation fits well with our model.

From affiliations to populations

- A community is a subset of the population. Let $\mathcal{C}$ be the set of communities (taken as given in our model) and $c \in \mathcal{C}$ be the representative index of a community. An agent may belong to multiple communities. (Depending on complexity in notation, we may exclude the agents who do not belong to any affiliation.)
- Now we define the set of populations $\mathcal{P}$ as the power set over $\mathcal{C}$ except an empty set. That is, a population p is defined as a subset of $\mathcal{C}$.
  - Say $\mathcal{C} = \{c_1, c_2, c_3\}$. For example, a subset $\{c_1, c_2\} \subset \mathcal{C}$ is a population. We interpret it as the set of agents who belongs to both communities c_1 and c_2. Then, $\mathcal{P} = 2^{\mathcal{C}}$ and thus $P = 2^3 = 8$. Let this subset abbreviate as p_{11}. With similar notation, we can write $\mathcal{P} = \{\emptyset, p_1, p_2, p_3, p_{12}, p_{13}, p_{23}, p_{123}\}$.
  - Define a binary variable to represent the affiliation as

$$B_c^p = \begin{cases} 1 & \text{if } c \in p, \text{ i.e., agents in } p \text{ belong to community } c, \\ 0 & \text{if } c \notin p, \text{ i.e., agents in } p \text{ do not belong to community } c \end{cases}$$

    for each $p \in \mathcal{P}, c \in \mathcal{C}$.

Construction of the structure matrix

- Assume that an agent observes the behavior of another agent once in each community that both of them belong to. That is, we assume that g_{pq} is proportional to the number of communities shared over population p and population q and the mass of population q.
  - As an agent in p sees another agent in q in more communities, the social influence between each other becomes stronger.
  - More specifically, we assume g_{pq} is equal to the expected number of observations for an agent in p with each agent in q, i.e.,

$$g_{pq} = \sum_{c \in C} B_c^p B_c^q.$$

    The total number of observations for an agent in p is $\sum_{q \in \mathcal{P}} g_{pq} = \sum_{q \in \mathcal{P}} \sum_{c \in C} B_c^p B_c^q$.

[7] See also Sandholm (2010, Sec. 8.2).
[8] See also Alós-Ferrer and Netzer (2010) and (Sandholm, 2010, Sec. 12.2).

Generalization

- *Heterogeneous degree of with-in influence.* We can have different frequency of observations within each community. Say, an agent in community c has an observation with another agent in this community with probability δ_c.[9] Then, g_{pq} is modified as

$$g_{pq} = \sum_{c \in C} B_c^p \delta_c B_c^q.$$

- *Heterogeneous degree of affiliation to a community.* In the above community-based construction, a community is the basic unit of the society.[10] But, if we see a population as the basic unit of the society, it is possible that each population may have different degree of participation in each community; there is a continuous spectrum of degree of participation. To represent such an idea, we can just let B_c^p take a value from $[0, 1]$, not only from $\{0, 1\}$.

Alternative construction of the structure matrix

- Alternatively, we can assume that g_{pq} take a binary value, either 1 or 0; $g_{pq} = 1$ if p and q has at least one community to share, and $g_{pq} = 0$ if p and q has no common community. That is,

$$g_{pq} = \max\{B_c^p B_c^q \mid c \in \mathcal{C}\}.$$

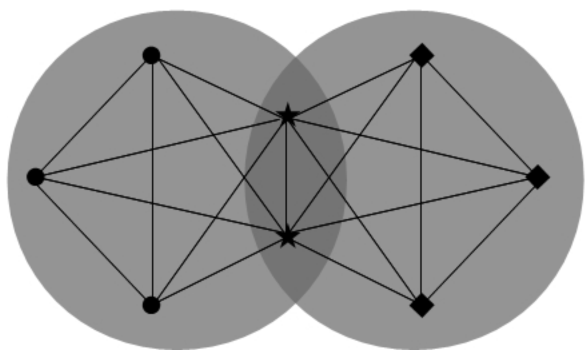

Figure 1: Each shadowed circle represents a community. According to the affiliation to communities, an agent is divided into three populations. An agent is represented by a node and his belonging population is symbolized by the marker on the node: population 1 for circles, 2 for stars and 3 for squares. Each agent is connected only with agents in the same community. Here we have $3 + 2 + 3$ agents. Our model can be considered as the limit where the population becomes a continuum while keeping the ratio of sizes between these three populations at $3 : 2 : 3$ and also the connection structure $\vec{g}^1 = (1, 1, 0), \vec{g}^2 = (1, 1, 1), \vec{g}^3 = (0, 1, 1)$.

3.2 Bipolar connection

In the above community example, agents who belong to the same population must have interactions—more than anyone else. But, in a bipolar case, it is not the case: agents interact only with the ones on a different side and agents on the same side do not interact at all.

This case cannot be captured by the above construction of $\mathcal{P}$ and $\mathbf{G}$. But our general model can capture this bipolar case by setting $g_{pp} = 0$ and $g_{pq} = 1$ $(p \neq q)$. This can be generalized as follows.

- Consider two populations $\mathcal{P} = \{1, 2\}$.

[9] Even if they do not encounter in this community, they may see in a different community.

[10] Then, it would ease the analysis if we could write q_{pq} somehow in terms of $\mathcal{C}$.

- Agents in population p has interaction only with the ones in the other population $q \neq p$. That is, $g_{11} = g_{22} = 1 - g$ and $g_{12} = g_{21} = g$. So,

$$\mathbf{G} = \begin{pmatrix} 1-g & g \\ g & 1-g \end{pmatrix}$$

Notice that $\mathbf{G}$ is positive definite if and only if $g \in [0, 0.5)$ and negative semidefinite if $g = 1$; it is indefinite if $g \in (0.5, 1)$.

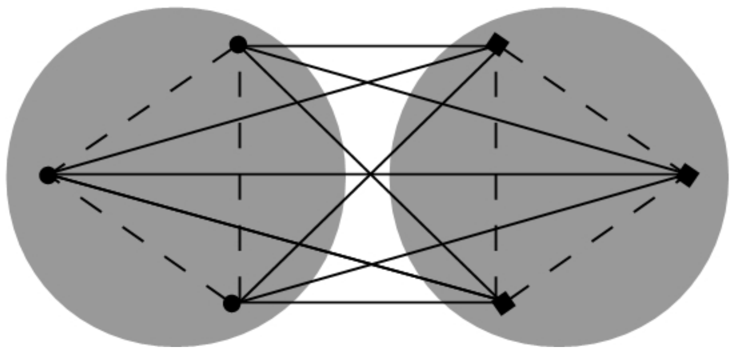

Figure 2: Each shadowed circle represents a population: population 1 on left and population 2 on right. The connection between agents may vary by their belong populations. The weight on the connection is $1 - g$ if it is connected by a dotted line; the weight is g if it is a solid line. Again, our model can be the large population limit of such network structure: $g_{pp} = 1 - g$ and $g_{pq} = g$ whenever $p \neq q$.

4 Fundamental analysis

To utilize the potential function, we focus on a stochastic stable state. In stochastic evolution,[11] stochastic stability is equivalent to the maximum of the potential function. So, instead of directly solving an equilibrium or tracking a dynamic, we can just consider the maximization of the potential function over the state space.

The existence of the maximum of $\tilde{f}^{\mathcal{P}}$ over $\mathcal{X}^{\mathcal{P}}$ is guaranteed because $\tilde{f}^{\mathcal{P}}$ is continuous and $\mathcal{X}^{\mathcal{P}}$ is nonempty and compact because it is a P-dimensional Cartesian product of nonempty and compact sets $m^p \Delta^A$.

Strictly concave potential function

Theorem 1. *Consider a structured game with continuous and strictly concave potential function $\tilde{f}^{\mathcal{P}}$. Then,*

1. *A local maximum must be the global maximum.*
2. *There exists a unique maximum of the potential function.*
3. *If $\tilde{f}$ is continuously differentiable, the solution of Karash-Kuhm-Tucker condition is the unique maximum of the potential.*

These are straightforward applications of well-known theorems on concave optimization: see Sundaram (1996, Sec.7.3).

[11] Specifically, it is justified by double limits under direct exponential protocols or large population limit under imitative exponential protocols with committed agents: see Sandholm (2010, Sec.12.4).

Strictly convex potential function

Call a social state **monomorphic** if each population's mass is concentrated on one particular action $a \in \mathcal{A}$: in each population $p \in \mathcal{P}$, there is an action $a^p \in \mathcal{A}$ such that $x^p_{a^p} = m^p$ and $x^p_a = 0$ for any $a \neq a^p$. Denote by $\mathcal{E}^{\mathcal{P}} \subset \mathcal{X}^{\mathcal{P}}$ the set of monomorphic social states. Let $E^A := \{\mathbf{e} \in \Delta^A \mid \exists i \in \mathcal{A}\ e_i = 1 \text{ and } e_j = 0 \text{ for any } j \neq i\}$. Then, $\mathcal{E}^{\mathcal{P}} = \times_{p\in\mathcal{P}} m^p E^A$.

Theorem 2. *Consider a structured game with continuous and strictly convex potential function $\tilde{f}^{\mathcal{P}}$. Then, the maximum of the potential function exists and must be in $\mathcal{E}^{\mathcal{P}}$.*

According to this theorem, even if the base game has only a completely mixed strategy Nash equilibrium, i.e, $\mathbf{x} \in \Delta^A \setminus E^A$, the stochastic stable state in the structured game is monomorphic; possibly, different populations have different monomorphic population states, concentrating on different actions.

One technical implication of the theorem is that, once the potential function $\tilde{f}^{\mathcal{P}}$ is numerically defined, we can find the stochastic stable state by calculating the value of potential at each monomorphic state (only finitely many, A^P).

5 Example: Binary anti-coordination game

5.1 Base game

To illustrate how the connection structure makes a different landscape on equilibrium outcome, here we focus on a binary anti-coordination game.[12] The action set is set to $\mathcal{A} = \{I, O\}$.

To make $\tilde{\Pi}$ symmetric, the payoff matrix $\mathbf{\Pi}$ is assumed to be symmetric, i.e., $\Pi_{IO} = \Pi_{OI}$. Then, without *further* loss of generality, we can assume $\Pi_{IO} = \Pi_{OI} = 0$. Now the payoff matrix is

$$\mathbf{\Pi} := \begin{pmatrix} \Pi_{II} & \Pi_{IO} \\ \Pi_{OI} & \Pi_{IO} \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix}.$$

We focus on an anti-coordination game: $a, b > 0$. Again, without further loss of generality, assume $a + b = 1$. The base game one unique mixed strategy Nash equilibrium: $x_I = b \in (0, 1)$.

5.2 Bipolar connection

First consider the example of the bipolar connection, as introduced before. For simplicity, assume $a = b$ and $m^1 = m^2 = 0.5$. Then, we find the following results.

Theorem 3. *i) If $g < 0.5$, then the monomorphic replication of the base game equilibrium $\mathbf{x}_I := (x^1_I, x^2_I) = (0.25, 0.25)$ is the only equilibrium. It is globally asymptotically stable in deterministic dynamics.*

ii) Consider the cases of $g > 0.5$, including the bipolar case $g = 1$. The monomorphic replication $\mathbf{x}_I = (0.25, 0.25)$ is still an equilibrium but not stable. There are two more equilibria, $\mathbf{x}_I = (0.5, 0)$ and $\mathbf{x}_I = (0, 0.5)$; they are locally stable.

These results indicate that biased observations affects the predictions of the evolutionary dynamic. When g is sufficiently small, heterogeneity in each population is achieved. When g is sufficiently larger, polarization arises.

5.3 Overlapping communities

Every one in the society is assumed to belong to at least one of two communities, $\mathcal{C} = \{c_1, c_2\}$, possibly both. Thus, the society is divided to three populations: Population 1 is the population of agents who belong only to community c_1. Population 2 is that of those who belong to both community c_1 and community c_2. Population 3 is that of those who belong only to community c_2.

For simplicity of analysis, we assume $m^1 = m^3 =: \tilde{m} \in (0, 0.5)$; then, $m^2 = 1 - 2\tilde{m} \in (0, 1)$.

[12] Bramoullé (2007) studies potential games played by finitely many players on a graph-theoretic network.

- If the weight is *proportional* to the number of communities that the two agents share, then the structure matrix $\mathbf{G}$ is

$$\mathbf{G} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

  This is positive semidefinite.[13] Thus, the payoff matrix $\tilde{\mathbf{\Pi}}$ of the structured game is negative semidefinite and thus the potential function $\tilde{f}^{\mathcal{P}}$ is concave.

- On the other hand, assume that the weight between populations does not increase with the number of shared communities. More specifically, let $g_{pq} = 1$ whenever p and q share at least one community, and $g_{pq} = 0$ if they do not share any community; g_{pq} does not become greater than 1 even if p and q share more than one communities. In this example of overlapping two communities, it implies $g_{22} = 1$. Then, the structure matrix $\mathbf{G}$ is

$$\mathbf{G} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

  This matrix is indefinite; so is the payoff matrix $\tilde{\mathbf{\Pi}}$ of the structured game. Thus, the potential function $\tilde{f}^{\mathcal{P}}$ is neither concave nor convex. Therefore, the global potential maximizer of $\tilde{f}^{\mathcal{P}}$ is not immediately found from the potential function of the base game f. Henceforth, we work on this $\mathbf{G}$.

The payoff matrix of the structured game is

$$\tilde{\mathbf{\Pi}} = \mathbf{\Pi} \otimes \mathbf{G} = \begin{pmatrix} \mathbf{\Pi} & \mathbf{\Pi} & 0 \\ \mathbf{\Pi} & \mathbf{\Pi} & \mathbf{\Pi} \\ 0 & \mathbf{\Pi} & \mathbf{\Pi} \end{pmatrix}.$$

The potential function is

$$\tilde{f}^{\mathcal{P}}(\mathbf{x}^{\mathcal{P}}) = 0.5\mathbf{x}^{\mathcal{P}} \cdot \tilde{\mathbf{\Pi}}\mathbf{x}^{\mathcal{P}} = 0.5\left(\mathbf{x}^1 \cdot \mathbf{\Pi}\mathbf{x}^1 + 2\mathbf{x}^1 \cdot \mathbf{\Pi}\mathbf{x}^2 + \mathbf{x}^2 \cdot \mathbf{\Pi}\mathbf{x}^2 + 2\mathbf{x}^2 \cdot \mathbf{\Pi}\mathbf{x}^3 + \mathbf{x}^3 \cdot \mathbf{\Pi}\mathbf{x}^3\right),$$

where $\mathbf{x}^p = (x_I^p, x_O^p)$.

The equilibria can be found and selected as follows.

Theorem 4. *Consider a binary anti-coordination game in the overlapping communities, as specified above.*

i) In each $\tilde{m} \in (0, 0.5)$ and $b \in (0, 0.5]$, we have three equilibria.

Type 1 $\mathbf{x}_I := (x_I^1, x_I^2, x_I^3) = (0, b, 0)$ *if* $\tilde{m} < (1-b)/2$, $(0, 1-2\tilde{m}, 0)$ *if* $(1-b)/2 \le \tilde{m} \le (1-b)/(2-b)$; $((2-b)\tilde{m} - 1 + b, 1 - 2\tilde{m})$ *otherwise. In this type of equilibria, action a is taken mostly at the intersection of the two communities, population 2.*

Type 2 $\mathbf{x}_I = (b\tilde{m}, b(1-2\tilde{m}), b\tilde{m})$, *i.e., the monomorphic replication of the base game equilibrium.*

Type 3 $\mathbf{x}_I := (x_I^1, x_I^2, x_I^3) = (\tilde{m}, b - 2\tilde{m}, \tilde{m})$ *if* $\tilde{m} < b/2$, $(\tilde{m}, 0, \tilde{m})$ *if* $b/2 \le \tilde{m} \le b/(1+b)$; $(b(1-\tilde{m}), 0, b(1-\tilde{m}))$ *otherwise. In this type of equilibria, action b is taken mostly at the intersection of the two communities, population 2.*

ii) The interior monomorphic equilibrium (type 2) is a saddle point of the potential function. Thus, it is unstable in deterministic evolutionary dynamics.

iii) With b fixed, there is a threshold value $\bar{m} \in (0, 0.5)$ where the unique global potential maximizer is type-1 equilibrium if $\tilde{m} < \bar{m}$ and it is type 3 if $\tilde{m} > \bar{m}$. So either one boundary equilibrium is the only stochastically stable state in logit dynamic. Yet, the unique equilibrium exhibits non-monomorphic action distribution over different populations in the society.

[13] Notice that $\deg \mathbf{G} = 0$, while the first and second leading principal minors are both positive. It does not change even if the weight on community 1 $g_{12} = g_{21}$ and the weight on community 2 $g_{23} = g_{32}$ are generalized to arbitrary constants between 0 and 1, as long as the weight on the intersection of these two communities g_{22} is set to the sum of these two constants.

6 Concluding remarks

Our research is aimed at proposing a general framework to study population games over connected groups and showing importance of bringing the connection structure into population games. To show its tractability and also clear-cut implications of the connection structure, we focus on two canonical examples with a rather simple two-strategy game in this current draft. But we believe that our framework unifies dynamic study on intermediaries over different communities, such as language and money, and dynamic integration/segregation of social norms.[14]

References

ALÓS-FERRER, C., AND N. NETZER (2010): "The logit-response dynamics," *Games and Economic Behavior*, 68(2), 413–427.

BLUME, L. E. (1995): "Population games," in *The Economy as an Evolving Complex System II*, ed. by S. D. B. Arthur, and D. Lane, pp. 425–460. Santa Fe Institute, Addison-Wesley, Reading, MA.

BRAMOULLÉ, Y. (2007): "Anti-coordination and social interactions," *Games and Economic Behavior*, 58(1), 30–49.

BRAMOULLÉ, Y., R. KRANTON, AND M. D'AMOURS (2014): "Strategic Interaction and Networks," *American Economic Review*, 104(3), 898–930.

JACKSON, M. O., AND Y. ZENOU (2014): *Games on networks*vol. 4, chap. 3, pp. 95–163. Elsevier.

MATSUYAMA, K., N. KIYOTAKI, AND A. MATSUI (1993): "Toward a Theory of International Currency," *The Review of Economic Studies*, 60(2), 283.

NOWAK, M. A., C. E. TARNITA, AND T. ANTAL (2009): "Evolutionary dynamics in structured populations," *Philosophical Transactions of the Royal Society B: Biological Sciences*, 365(1537), 19–30.

SANDHOLM, W. H. (2001): "Potential Games with Continuous Player Sets," *Journal of Economic Theory*, 97(1), 81–108.

——— (2010): *Population games and evolutionary dynamics*. MIT Press.

STAUDIGL, M. (2011): "Potential games in volatile environments," *Games and Economic Behavior*, 72(1), 271–287.

TARNITA, C. E., T. ANTAL, H. OHTSUKI, AND M. A. NOWAK (2009): "Evolutionary dynamics in set structured populations," *Proceedings of the National Academy of Sciences*, 106(21), 8601–8604.

[14]Matsuyama, Kiyotaki, and Matsui (1993) is a classical study on evolution of international currency, shedding light on integration—or overlap in our word—of two countries.