Weak Robust (Virtual) Implementation

Chih-Chun Yang*

Institute of Economics, Academia Sinica, Taipei 115, Taiwan

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Abstract

We provide a characterization of (virtual) implementation in iterated elimination of weakly dominated strategies (IEWDS). In the interdependent-value environment with single-crossing preferned proposed by Bergmann and Morris (ERS, 2009), a social choice function is implementable in IEWDS only if it satisfies "partial" strict ex post incentive compatibility and BM's (2009) contraction property. They are also sufficient for implementation in the direct mechanism. A social choice function is virtually implementable in IEWDS only if it is ex post incentive compatible and strategic measurable in IEWDS. Under an economic condition, they are also sufficient for virtual implementation.

Keywords: Iterative admissibility, iterated elimination of weakly dominated strategies, implementation, virtaul implementation.

JEL classification: C72, D80.

1 Introduction

IImplementation theory characterizes, given a social choice function (SCF), when and how to design a mechanism such that whose equilibrium outcomes achieved by asymmetrically informed agents are consistent with the SCF. In particular, the Bayesian incentive compatibility constraints characterize whether a SCF can be implemented in Bayesian equilibrium.

*Tel:+886-2-2782 2791 ext. 211. Fax:+886-2-2785 3946. E-mail: cyang16@econ.sinica.edu.tw.

The Bayesian incentive compatibility analysis uses Bayesian Nash equilibrium as a solution concept and usually assumes common knowledge of a common prior on agents' types. The common prior assumption may be too strong to be realistic in many economically important environments. It says a common prior must be known to a social planner, who may not be able to observe such details about the agents' beliefs. In the spirit of the "Wilson doctrine" (Wilson, 1987), an implementation should be robust to different assumptions about what agents do or do not know about other agents' types. Hence, the literature turns its focus on finding detail-free mechanisms in order to avoid this dependence of common prior assumption. In particular, Bergmann and Morris (henceforth, BM 2009a) proposed "*robust implementation*": The idea is to try to avoid making any restriction of the agents' beliefs and higher order beliefs about others' types in mechanism design. Robust implementation requires that every equilibrium *on every type space* achieves an outcome coinciding with the SCF outcome. This is equivalent to require a mechanism under which the outcome of iteratively eliminating strictly dominated strategies (IESDS) coincides with the SCF outcomes *for every type profile*.

The motivation of this paper is to carefully re-examine mechanism design problems when the agents are "cautious", without making other substantial restrictions about the agents' beliefs and higher-order beliefs about other agents' payoff types. More precisely, we investigate when and how a social choice function is implementable when asymmetrically informed agents are rational and "cautious" and that is *common weak assumption*, i.e., the agents are in the states of "*rationality and common weak assumption of rationality*" (henceforth, RCWAR) (Yang, 2015), which is a epistemic characterization of *iterated elimination of weakly dominated strategies (IEWDS)*. RCWAR requires that each agent has a (lexicographic) belief with full support, which is what we mean by *cautiousness*, plays a best response to it, which is what we mean by *rationality*, and it is *common weak assumption* that all the agents are rational and cautious.

BM (2009a) provided a characterization for robust implementation by strict ex post incentive compatibility (strict EPIC) and contraction property. In many cases, robust implementation may be too demanding and hence, BM (2009b) further considered "robust virtual implementation", which only requires that the outcome of IESDS in a mechanism coincides the SCF outcomes with a probability arbitrarily close to 1. BM (2009b) provided a characterization for robust virtual implementation by ex post incentive compatibility (EPIC) and robust measurability.

In this paper, we follow closely BM's (2009ab) robust (virtual) implementation approach and further investigate when and how a social choice function is robustly (virtually) implementable in our environment of cautious agents. Due to the subtle difference between IESDS and IEWDS, our extension to implementation with respect to RCWAR requires a specially careful analysis.

We provide a full characterization of our notion of (virtual) implementation in

IEWDS, called "*weak robust (virtual) implementation*. We first propose the notion of *partial strict EPIC*, which is weaker than BM's (2009a) *strict EPIC*, and the notion of *weak robust measurability*, which is weaker than BM's (2009b) *robust measurability*. For *weak robust implementation*, *partial strict EPIC* and *contraction property* are sufficient conditions for the *direct* mechanism. If the SCF is responsive, then *partial strict EPIC* and *contraction property* are also necessary for weak robust implementation in *every* mechanism. Hence, similar to BM (2009a), complex mechanisms can't outperform the direct mechanism. For *weak robust virtual implementation*, *EPIC* and *weak robust measurability* are necessary in every mechanism. Under BM's (2009b) uniform economic property, they are also sufficient for *weak robust virtual implementation* in a indirect mechanism.

The rest of the paper is organized as follows. We discuss some related literature in this introduction. Section 2 consider BM's (2009b) environment and provides a characterization of weak robust virtual implementation. Section 3 considers BM's (2009a) environment and provides a characterization of weak robust implementation.[1] Section 4 provides concluding remarks about possible extensions.

1.1 Related Literature.

Bergemann and Morris (2009b) characterized robust virtual implementation by ex post incentive compatibility (EPIC) and robust measurability. One important result from their analysis is that, every static mechanism cannot robustly virtually implement non-constant social choice functions if preferences are sufficiently interdependent. To overcome this negative result for static mechanisms, Müller (2015) further considered dynamic mechanisms and impose a stronger epistemic condition for agents: "*rationality and common strong belief of rationality*" (henceforth, RCSBR), which is a characterization for EFR (Battigalli and Siniscalchi, 2002). Müller's (2015) implementation with respect to RCSBR shows, by using a slightly stronger solution concept, there is a SCF, which can't be robustly virtually implemented in any (direct or indirect) static mechanisms, implementable in a dynamic mechanism regardless of the level of preference interdependence.

Müller (2015) proposed *strong dynamic robust measurability* and *dynamic robust measurability*, both are weaker than BM's (2009b) robust measurability. Coupled with EPIC, Müller (2015) proved they are sufficient and necessary, respectively, for robust virtual implementation in dynamic mechanisms. Note that Müller's (2015) motivation for using EFR is epistemic: Agents are in the states of "*rationality and common strong*

[1] Note that in Yang (2016), the strategy set is finite while in BM (2009a), types and messages are assumed to be compact, which are possibiliy infinite.

belief of rationality" (henceforth, RCSBR) (Battigalli and Siniscalchi, 2002) in the dynamic mechanism he constructed. Our approach can be viewed as the counterpart in static mechanism design problems.

The intuition behind our approach for weak robust virtual implementation is simple. The relationship of EFR and IEWDS has made clear in the literature (Brandenburger and Dekel, 1987; Battigalli and Siniscalchi, 2003): In generic games, EFR and IEWDS are outcome equivalent. We now further extend this result to implementation theory literature. In fact, based on the known relationship of EFR and IEWDS, we show, even in static mechanism, we can overcome BM's (2009b) negative result by considering a "slightly" stronger solution concept,

2 Weak Robust Virtual Implementation

We consider BM's (2009b) environment. The model setting is as follows.

Let I be a finite set of agents. Agent i's **payoff type** is $\theta_i \in \Theta_i$, a finite. Let $\theta = (\theta_1, ..., \theta_I) \in \Theta_1 \times \cdots \times \Theta_I = \Theta$. Let X be a finite set of deterministic outcomes and $\Delta(X)$ be the lottery space generated by X. Each agent i has a expected utility function $u_i : \Delta(X) \times \Theta \to \mathbf{R}$. A social choice function is a function $f : \Theta \to \Delta(X)$.

Let $\mathcal{M} = (M_1, ..., M_I, g(\cdot))$ be a **mechanism**, where M_i is a compact set of messages available to i and $g : M \to \Delta(X)$ is the outcome function. If $M_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$, then we call $\mathcal{M}$ a direct mechanism.

Given a mechanism $\mathcal{M}$, a correspondence $S = (S_1, ..., S_I)$ is a *message profile*, where each $S_i : \Theta_i \to 2^{\Theta_i}/\emptyset$. A message profile S in the direct mechanism is called a *report profile* if for each i and θ_i, $\theta_i \in S_i(\theta_i)$.

Bergemann and Morris (2009a) considered implementation in iterative elimination of never-best response, which is outcome equivalent to IESDS.

IESDS. Let $S_i^0(\theta_i) = M_i$ for each i and θ_i. Inductively, define

$$S_i^{k+1}(\theta_i) \equiv \left\{ \begin{array}{c} m_i \in M_i : \exists \lambda_i \in \Delta(M_{-i} \times \Theta_{-i}) \text{ s.t.} \\ (1)\ \lambda_i\left(\left\{m_{-i}, \theta_{-i} : m_j \in S_j^k(\theta_j), \forall j \neq i\right\}\right) = 1 \\ (2)\ \int u_i(g(m_i, m_{-i}), (\theta_i, \theta_{-i}))\, d\lambda_i \geq \\ \int u_i(g(m_i', m_{-i}), (\theta_i, \theta_{-i}))\, d\lambda_i, \forall m_i' \in M_i \end{array} \right\}.$$

Denote the limit set $S_i^{\mathcal{M}}(\theta_i) \equiv \cap_{k \geq 0} S_i^k(\theta_i)$, for each i and θ_i.

We consider implementation in IEWDS. Let $\Delta^{+}(M_{-i}\times\Theta_{-i})$ denote the probability measures overs $M_{-i}\times\Theta_{-i}$ with full support.

IEWDS. Let $W_i^0(\theta_i)=M_i$ for each i and θ_i. Inductively, define

$$W_i^{k+1}(\theta_i)\equiv\left\{\begin{array}{c} m_i\in M_i:\exists\lambda_i\in\Delta^{+}(M_{-i}\times\Theta_{-i})\ \text{s.t.} \\ (1)\ \lambda_i\left(\left\{m_{-i},\theta_{-i}:m_j\in W_j^k(\theta_j),\forall j\neq i\right\}\right)=1 \\ (2)\ \displaystyle\int u_i\left(g(m_i,m_{-i}),(\theta_i,\theta_{-i})\right)d\lambda_i\geq \\ \displaystyle\int u_i\left(g(m_i',m_{-i}),(\theta_i,\theta_{-i})\right)d\lambda_i,\forall m_i'\in M_i \end{array}\right\}.$$

Denote the limit set $W_i^{\mathcal{M}}(\theta_i)\equiv\cap_{k\geq 0}W_i^k(\theta_i)$, for each i and θ_i.

Definition 1. (Bergemann and Morris, 2009a) Social choice function f is robustly implemented using $\mathcal{M}$ if $m\in\beta^{\mathcal{M}}(\theta)$ implies $g(m)=f(\theta)$.

For each $p,q\in\Delta(X)$, denote $\|p-q\|$ by the rectilinear norm, i.e.,

$$\|p-q\|\equiv\sum_{x\in X}|p(x)-q(x)|.$$

Definition 2. (BM 2009b) **Robust ϵ-Implementation**. The mechanism $\mathcal{M}$ *robustly ϵ-implements* the social choice function f if

$$m\in S^{\mathcal{M}}(\theta)\ \text{implies}\ \|g(m)-f(\theta)\|\leq\epsilon.$$

The social choice function f is robustly ϵ-implementable if there exists a mechanism $\mathcal{M}$ that robustly ϵ-implements f.

Definition 3. (BM 2009b) **Robust Virtual Implementation**. The social choice function f is *robustly virtually implementable* if, for every $\epsilon>0$, f is robustly ϵ-implementable.

Now, we are ready to define our notion of **Weak Robust Virtual Implementation, i.e., Virtual Implementation with respect to RCWAR,** which is an epistemic characterization of IEWDS in finite games.

Definition 4. Weak ϵ-Implementation in IEWDS. The mechanism $\mathcal{M}$ *weak robustly ϵ-implements* the social choice function f if

$$m\in W^{\mathcal{M}}(\theta)\ \text{implies}\ \|g(m)-f(\theta)\|\leq\epsilon.$$

The social choice function f is weak robustly ϵ-implementable if there exists a mechanism $\mathcal{M}$ that weak robustly ϵ-implements f.

Definition 5. (BM 2009b) **Weak Robust Virtual Implementation**. The social choice function f is *weak robustly virtually implementable* if, for every $\epsilon > 0$, f is weak robustly ϵ-implementable.

Definition 6. (BM 2009b) **EPIC**. The social choice f satisfies *ex post incentive compatibility (EPIC)* if, for each i, θ_i, θ_{-i}, and θ_i',

$$u_i\left(f\left(\theta_i,\theta_{-i}\right),\left(\theta_i,\theta_{-i}\right)\right) \geq u_i\left(f\left(\theta_i',\theta_{-i}\right),\left(\theta_i,\theta_{-i}\right)\right).$$

Strategically Indistinguishable. We review BM's (2009b) notion of strategically indistinguishability, which we call *S strategical indistinguishability*, and propose our notion of *W strategically indistinguishability* a follows.

Definition 7. Types θ_i and θ_i' are *S-strategically indistinguishable if* $S_i^{\mathcal{M}}\left(\theta_i\right) \cap S_i^{\mathcal{M}}\left(\theta_i'\right) \neq \emptyset$ for every $\mathcal{M}$. Types θ_i and θ_i' are *W-strategically indistinguishable if* $W_i^{\mathcal{M}}\left(\theta_i\right) \cap W_i^{\mathcal{M}}\left(\theta_i'\right) \neq \emptyset$ for every $\mathcal{M}$.

Definition 8. (BM 2009b) **Robust Measurability**. The social choice f is *robust measurable* if whenever two types θ_i and θ_i' are *S-strategically indistinguishable*, then $f\left(\theta_i,\theta_{-i}\right) = f\left(\theta_i',\theta_{-i}\right)$ for every θ_{-i}.

To ensure *EPIC* and *robust measurability* are sufficient for *robustly virtually implementation*, BM (2009b) required a restriction of the domain of preference as follows. First, let $\overline{y}$ be the central lottery assigning equal probability on each of the pure outcomes X. BM's (2009b) *uniform economic property* essentially says that the agents' preferences are sufficiently diversified.

Definition 9. (BM 2009b) Uniform Economic Property.The *uniform economic property* is satisfied if there exists a profile of lotteries $\left(z_i\right)_i^I$ such that, for each i and θ, $u_i\left(z_i,\theta\right) > u_i\left(\overline{y},\theta\right)$ and $u_i\left(\overline{y},\theta\right) \geq u_j\left(z,\theta\right)$ for all $j \neq i$.

We review BM's (2009b) main theorem here.

Theorem 0. (BM 2009b) If a SCF f is *robustly virtually implementable*, then f is *ex post incentive compatible* and *robustly measurable*. Under the *uniform economic property*, if f satisfies *EPIC* and *robust measurability*, then f is *robustly virtually implementable*.

Main Result 1.

Theorem 1. If a SCF f is *weakly robustly virtually implementable*, then f is *EPIC* and *weakly robustly measurable*. Under the *uniform economic property*, if a SCF f satisfies *EPIC* and *weakly robust measurability*, then f is *weakly robustly virtually implementable*.

Remark 1. Note that, as BM (2009b) indicated, uniform economic property is not the weakest restriction to ensure that *EPIC* and *robust measurability* are sufficient for robustly virtually implementation. Hence, we may consider the counterpart in our environment to characterize *weak robust virtual implementation.*

3 Weak Robust Implementation

We consider BM's (2009a) environment. The model setting is as follows.

Let I be a finite set of agents. Agent i's **payoff type** is $\theta_i \in \Theta_i$, a compact subset of the real line. Let $\theta = (\theta_1, ..., \theta_I) \in \Theta_1 \times \cdots \times \Theta_I = \Theta$. Let X be a compact set of deterministic outcomes and $\Delta(X)$ be the lottery space generated by X. Each agent i has a expected utility function $u_i : \Delta(X) \times \Theta \to \mathbf{R}$. A social choice function is a function $f : \Theta \to \Delta(X)$.

We follow BM's assumption and impose more restrictions to the model.

Existence of a Monotone Aggregator. (BM 2009a) For each agent i, there is a monotonic aggregator $h_i :\to \mathbf{R}$ such that we may rewrite the utility function as follows: For each $y \in \Delta(X)$

$$u_i(y, \theta) \equiv \upsilon_i(y, h_i(\theta)),$$

where h_i is strict increasing in θ_i and continuous and $\upsilon_i : \Delta(X) \times \mathbf{R} \to \mathbf{R}$ is continuous in h_i.

Strict Single Crossing. (BM 2009a) The utility function $\upsilon_i(y, \phi)$ satisfies Strict Single Crossing (SSC) if for all $\phi < \phi' < \phi''$ and $y, y' \in \Delta(X)$

$$\upsilon_i(y, \phi) > \upsilon_i(y', \phi) \text{ and } \upsilon_i(y, \phi') = \upsilon_i(y', \phi') \text{ implies}$$

$$\upsilon_i(y, \phi'') < \upsilon_i(y', \phi'').$$

Definition 10. Partial Strict EPIC. A SCF f satisfies Partial Strict EPIC if it satisfies EPIC and for all for each i, $\theta_i \neq \theta_i'$, there is $\theta_{-i} \in \Theta_{-i}$ such that

$$u_i(f(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})) > u_i(f(\theta_i', \theta_{-i}), (\theta_i, \theta_{-i})).$$

Theorem 2. If a SCF f satisfies *partial strict EPIC* and *contraction property*, then f is *weakly robustly implementable* in the *direct* mechanism. If a SCF f is responsive and *weakly robustly implementable*, then f satisfies *partial strict EPIC* and *contraction property.*

4 Conclusion

The benefit of this project is to clarify a subtle issue in mechanism design literature. BM (2009b) claimed that, in terms of implementation, there is no difference between IESDS and IEWDS. We show, IESDS and IEWDS, as different solution concepts, do have different implication in mechanism design problems.

Our analysis provides additional insight to the issue of robust implementation: There are many situations where dynamic mechanisms are not feasible, physically or economically. Moreover, in many situations, agents are cautious in the sense that they must restrict their strategies in IEWDS. Müller (2015) showed that even with strong dependence of preference, a social planner still may "robustly virtually implement" a SCF if there is RCSBR in the dynamic mechanism. We then show, even when dynamic mechanisms are not feasible, as long as agents in our static mechanism are in RCWAR, our static mechanism can "robustly virtually implement" a SCF, too. This finding is new and complementary to Müller's (2015) result.

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