Evolution in Coordination Games with Cheap Talk

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Abstract

We consider a special case of cheap talk in a coordination game with a finite message space in the sense that each allowed decision rule assigns every action in the base game to at least one message, which eliminates the result that a player plays the same base game action regardless of the message received. In this setup, we show that the continuous-time best-response dynamic can converge to a Nash equilibrium with the smallest positive payoff in the base game payoff matrix only from a set of initial points with Lebesgue measure 0. Unlike a standard coordination game, the equilibrium stability result is not independent of the payoffs corresponding to the unused actions in the base game any more. We characterize the proposed initial point selection criteria for Nash equilibrium and compare this solution concept with ESS, NSS, asymptotic stability and Lyapunov stability under the best-response dynamic. We also show that from the same initial point, the best-response dynamic and the replicator dynamic may converge to different equilibrium outcomes in this class of cheap-talk games.

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1 Introduction

Finite doubly symmetric games are well studied in economic theory. We restrict our attention in this paper to one of its simplest models, coordination games, namely distinct positive payoffs along the payoff matrix diagonal and 0 otherwise. Then, as long as both players play the same strategy, they are in a strict Nash equilibrium, which satisfies almost all solution concepts such as perfection, properness, and ESS. If, however, they are currently at the undesired Nash equilibrium with the smallest payoff along the matrix diagonal, then they are "locked" in that Nash equilibrium forever: not only neither player would like to deviate from the current strategy, but also stuck there even if all other payoffs along the matrix diagonal increase.

This observation is somehow contradictory to our intuition and experience. Firstly, how can they be "locked" playing this poor equilibrium if both players agree that both of them will be better off in any other pure strategy Nash equilibrium? Secondly, if they are at the poor equilibrium but the game environment changes significantly for all other strategies, say, resulted from technology advance, then how can they stay isolated and not benefit from it?

The development in evolutionary game theory in 1990s shows that preplay communication may help players reach Pareto optimal outcome; see the review in Weibull (1995), Chapter 2.6. Before the coordination game is played, which is called the base game henceforth, each player simultaneously sends the other player one of finitely many costless and fully observable messages. Each player has a decision rule that assigns an action in the base game for each possibly received message. We call this meta game as a cheap-talk game, and each pure strategy in this cheap-talk game specifies a message to be sent and a decision rule. The payoff in the cheap-talk game is simply the payoff in the base game, where each player picks an action according to the response of her decision rule to the other player's message. As both the number of actions in the base game and the number of messages are finite, a cheap-talk game is essentially a doubly symmetric game. The standard population game model can be applied here. That is, there is a large population of individuals, each of them characterized by a pure strategy. The proportion of individuals playing a given strategy may be viewed as the probability of that pure strategy being used in the mixed strategy. The payoff to each individual is evaluated by the average payoff when it plays against every other individuals exactly once in the population; the population payoff is the average of all individual payoffs.

There has been a substantial literature discussing evolutionary settings that lead to the long-run outcome of Pareto efficiency. For example, Schlag (1993) argues that for 2×2 coordination games, there exists an asymptotically stable set with payoffs arbitrarily close to the Pareto optimal payoff if there are sufficiently many messages. Kim and Sobel (1995) show that if stochastic drift in population distribution is allowed, then the long-run result must be Pareto optimal. Robson (1990) and Wärneryd (1991) observe that mutants may use a currently unsent message to recognize each other and play the "good" action, while matching the expected action being played by the majority of the population. In that way, the mutants will successfully invade the population, and the population payoff will increase. The message that mutant sent is called a "secret handshake".

On the one hand, the existence of unused messages in the whole population may be too strong under some game situations. On the other hand, if a player simply plays the same action regardless of the message received, which is allowed in a cheap-talk game, then this cheap-talk game strategy reduces to a strategy in the coordination base game. In this paper, we would like to study a model in which each player uses cheap talk seriously: there are sufficiently many types of messages compared with the number of actions in the base game; given any action a, a decision rule must specify at least one message μ such that if μ is received, then the player will play action a. We call such a game a serious cheap-talk game.

Such decision rule structure is inspired by the mechanism behind a coordination game. That is, no pure strategy is dominated by another one, and your optimal strategy is the same strategy as the one you anticipate what your opponent will play. During the long-term interaction between individuals in the population, the message received gives an individual some belief on the probability of which action the opponent would pick. If you expect very few individuals would choose one particular action under any circumstance in the future, regardless of the message sent, then all other individuals also have the same belief. In that case, one can safely eliminate that action from the base game and concentrate on the simplified version, which is still a serious cheap talk game.

Given any population distribution in a cheap-talk game with m messages and n actions and payoffs bounded by b in the base game, a best-response serious cheap talk strategy gives the payoff in the $(n-1)b/m$-neighborhood of the payoff generated by a standard best-response cheap-talk strategy (not necessarily a serious cheap-talk strategy). To see this, observe that if the standard best-response strategy requires the same action regardless of the message received, then the best-response serious cheap talk strategy will assign all other $n-1$ actions to the $n-1$

messages sent from the least $n-1$ proportions in the population. Therefore, when the number of messages is large, the best-response correspondence with respect to classic cheap talk strategies can be approximated by the one using serious cheap talk strategies.

In contrast to the current literature focusing on ESS, we also study the game outcomes under evolutionary dynamics. In particular, we consider the best-response dynamic and the replicator dynamic. As shown in Weibull (1995), the fundamental theorem of natural selection applies to all doubly symmetric games under the replicator dynamic: the dynamic induces a monotonic increase in the population payoff over time, which further implies that the dynamic converges to a Nash equilibrium component. Similar results hold for the best response dynamic. However, this does not mean that the population payoff necessarily increases to the Pareto optimal payoff in the long run. For serious cheap-talk games, the long-run result of the best-response dynamic cannot be "too bad". We find that it can approach to the poor equilibrium component, i.e., the equilibrium component with the worst payoff along the diagonal of the base game payoff matrix, only from a set of initial points with Lebesgue measure 0.

Let us consider an example of 2×2 base game with a large number of messages.Suppose that the initial population distribution is an interior state but most of the individuals have a decision rule that plays poor action for most of the messages received. The best-response serious cheap talk strategy would, of course, use the poor action dominantly, and just leave one message, say μ, to the good action in the way that μ is the one in all messages that makes the smallest cost by not using the poor action. (For instance, the proportion of individuals sending μ is the smallest.) Note that every individual in the population reasons like this, and the population moves towards the agreed desired state with poor payoff in the best-response dynamic. However, it cannot converges to it in the end. Since after sufficiently long time, the reserved message μ to the good action is well aligned in the population. Individuals begin to realize that they could profit by sending message μ and playing the good action against the majority of individuals. Whether they play the good action against those who sends μ depends on the initial state. For simplicity, we assume they play poor action to μ. Such individuals will receive good payoffs when meeting ones from the majority and poor payoffs when meeting ones who send μ and have the same reasoning. It is worthwhile to send μ from this time on, and the population proportion sending μ is increasing to $1/2$ of the population. Thus, the population payoff converges to $1/2$(good payoff+poor payoff); see Example 1 in Section 4.1 for more details. The only cases that the best-response dynamic can converge to the poor equilibrium component is that the initial distribution is in symmetry, i.e., each individuals is indifferent of associating the poor action to at least two messages, or the initial state itself is in the poor equilibrium component.

We propose a new Nash equilibrium refinement based on the set of initial points:an equilibrium that can be approached in the best-response dynamic from a set of initial points with Lebesgue measure greater than 0. For a serious cheap talk game, we can characterize an equilibrium s satisfying this initial point criteria. If s is a pure strategy, then it must be Pareto optimal. If s is a mixed strategy not Pareto optimal, then for each supporting strategy h in s, the payoff of h against any other supporting strategy in s is higher than the one of h against itself. In other words, individuals can discriminate the individuals sending the same message from the ones sending a different message.

This characterization condition reminds us the NSS characterization of 2×2 coordination game by Banerjee and Weibull (2000). In fact, the set of population payoffs of the equilibria satisfying the initial point criteria is the same as the one of NSS, except the poor equilibrium payoff. Indeed, the poor equilibrium corresponds to the case that players ignore messages and stuck with the poor action. This may be the result that players would like to avoid. On the other hand, our criteria is different from NSS. We will show two examples that an equilibrium satisfying one criteria but not the other one. For serious cheap talk game, we will give the relationship among all related solution concepts: ESS, asymptotic stability, Lyapunov stability, initial point criteria, and NSS.

An important feature of our initial point criteria is that such selected stable states are sensitive to the background payoffs in the base game. As we shall see in Theorem 4.11, the population payoffs of this solution concept has a lower bound related to the Pareto optimal payoff. If we increase that payoff to much larger than any other positive payoff in the payoff matrix, then individuals would simply check which supporting strategy has the biggest proportion in the current population distribution and switch to that strategy's message that corresponds to the action associated to the Pareto optimal payoff, even if this message might cause mismatch with all individuals of a different strategy. In our initial point solution concept, stable equilibria are no longer isolated from any background payoff in the base game.

Last but not least, we would like to study the replicator dynamic in serious cheap talk games and compare the stability results with the ones of best-response dynamics. We know from Thomas (1985) and Bomze and Weibull (1995) that each neutrally stable outcome corresponds to a Lyapunov stable population state under the replicator dynamics in a doubly symmetric game. The example that some state is NSS but not Lyapunov stable under the best-response dynamic then shows that the best-response dynamic and the replicator dynamic may have different solution trajectories and approach different equilibrium components from the same initial point. For the technical analysis on replicator dynamic, we can apply the Akin transformation and convert the replicator dynamic to the gradient system for a serious cheap-talk game.

2 Model

Given a mixed strategy x, the set of pure strategies that is assigned positive probabilities by x is called *the support of* x, and it is denoted by $C(x)$.

Consider a symmetric coordination game with payoff matrix

$$\mathrm{V} = \begin{pmatrix} v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_n \end{pmatrix}$$

with

$$v_1 > v_2 > ... > v_n > 0 \tag{2.1}$$

at the diagonal and all the other entries 0. Let G be the associated base game with pure strategy set $A = \{1, ..., n\}$ and payoff function u, and let M be a finite set of messages. A decision rule is a function $f : M \to A$ which says that if the opponent's message is $\mu \in M$, then use pure strategy $f(\mu)$, and so on. Let F be the set of all such functions. A pure strategy in the associated cheap-talk game G_M is a pair $(\mu, f) \in H := M \times F$, and the payoff to a pure-strategy profile $((\mu, f), (\nu, g))$ is $u[f(\nu), g(\mu)]$. We call the base-game strategies as actions in the cheap-talk game. Given a pure strategy h, we denote the associated message by $m(h)$.

Restriction in this paper: given the game G_M, every decision rule f in F has the property that for any action $a \in A$,

$$\exists \mu \in M, \text{ s.t. } f(\mu) = a. \tag{2.2}$$

Thus, the players must use cheap talk seriously: no player can ignore messages sent from the opponent and just use some action uniformly. To this end, suppose

$$|M| \geq n. \tag{2.3}$$

For each pure strategy $h = (\mu, f)$ in the cheap-talk game, let x_h be the population share of individuals programmed to play h. A population state is a vector $\vec{x} = (x_h)_h$, i.e., a point in the unit simplex $\Delta_M := \Delta(H)$ of mixed strategies in the cheap-talk game. (We may leave out the arrow of a vector if there is no ambiguity.) Given a population state $x \in \Delta_M$, the payoff of any pure strategy $h = (\mu, f)$ in G_M is $u_M(h, x)$, the average payoff in the population is $u_M(x, x)$, the set of best-response strategy is

$$BR(x) = \underset{h \in \Delta(H)}{\operatorname{argmax}}\ u_M(h, x).$$

Dynamics:

1. Best-response dynamics

$$\dot{x} \in BR(x) - x \tag{2.4}$$

   Since the best-response correspondence is upper hemicontinuous with closed and convex values, a solution trajectory through any given initial state $x_0 \in H$ exists, though it is not necessarily unique; see, e.g., Aubin and Cellina (1984). Moreover, the solution trajectory is Lipschitz continuous, since $\dot{x}$ is uniformly bounded. Given a best-response trajectory $(x(t))_{t \geq 0}$, define $b(t) := \dot{x}(t) + x(t)$ whenever $x(t)$ is differentiable at t, otherwise let $b(t)$ be any element in $BR(x(t))$. Therefore, $b(t) \in BR(x(t))$ for all $t \geq 0$.

2. Replicator dynamics

$$\forall h \in H,\ \dot{x}_h = u_M(h - x, x)x_h.$$

   Since this derivative is a polynomial in the population shares, the system of differential equations has a unique solution through any initial state $x_0 \in H$, by the Picard-Linderlof theorem.

3 Gradient System and Replicator Dynamics

We first consider a general quadratic function $w(\vec{x}) = w(x_1, ..., x_n)$ where the domain is a connected closed set $Q \in \mathbb{R}^n$.

Lemma 3.1. *From any initial point $x(0)$ in Q, consider the dynamic $\dot{x} = \nabla(w)$. In this dynamic, when x is at boundary of Q and $\nabla w(x)$ is not towards Q, i.e.,*

$$\forall \lambda > 0,\ x + \lambda \nabla w(x) \notin Q,$$

we let $\dot{x} = 0$. Suppose that ∇w is a unique vector most of the time.

1. *Each dynamic trajectory will converge to a single point $\vec{p}$ in Q.*

2. *The dynamic converges to $\vec{p}$ only from a set of initial points with Lebesgue measure 0 in Q if and only if $\vec{p}$ is a saddle point.*

3. *If $\vec{p}$ is a boundary point but not a saddle point in Q, then $\vec{p}$ is a local maximum in Q.*

4. *If the dynamic can converge to an interior point $\vec{p}$ from a set of initial points with Lebesgue measure greater than 0, then $\vec{p}$ is a global maximum point in Q.*

5. *If $\vec{p}$ is a boundary point but neither a saddle point in Q nor the initial point $x(0)$, and if $\vec{p}$ is a convex combination of a set of points $p_1, ..., p_l$ with $l < n$, then $\vec{p}$ is a the global maximum point in the intersection of Q and the subspace spanned by $\vec{p}_1, ..., \vec{p}_l$.*

Note that the gradient system may reach the boundary of Q in finite time.

Proof. 1. Firstly, note that the quadratic form w is continuously differentiable. Given any dynamic trajectory $(x(t))_{t \geq 0}$, we observe that $\frac{dw}{dt} \geq 0$ for all $t \geq 0$. Thus, $\vec{x}(t)$ converges to some subset in Q. Since ∇w is unique most of the time, we may further infer that $\vec{x}(t)$ converges to a single point in Q.

2. We explicitly put w as

$$w = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \right) + \left(\sum_{i=1}^{n} b_i x_i \right) + c$$

where a_{ij}, b_i and c are all in $\mathbb{R}$. We may also write w as

$$w = \vec{x}^T A \vec{x} + \vec{b}^T \vec{x} + c \tag{3.1}$$

where A is an $n \times n$ matrix with entries a_{ij} and $\vec{b}$ is the column vector with entries b_i. We then obtain the Hessian matrix H of w. Note that it is a symmetric matrix, and hence all eigenvalues are real, and the eigenvectors corresponding to distinct eigenvalues are orthogonal. Denote by S^- (or S^+) the subspace spanned by all the eigenvectors corresponding to a negative (or positive) eigenvalue for H. We denote by $\vec{y}_+$ an eigenvector corresponding to a positive eigenvector λ_{y_+}.

Note that if there exists a saddle point in Q, then both S^- and S^+ exist, and the Lebesgue measure of both these two sets is zero. Furthermore, $\vec{b}$ is in the image of A with domain Q, and the saddle point is at $-\frac{1}{2}\vec{v}$ where $\vec{v}$ is some solution to $A\vec{v} = \vec{b}$.

Given a vector y, $H\vec{y}$ describes how the value of the first derivative of w changes as one moves in the direction $\vec{y}$. (Note that H is constant, and hence $H\vec{y}$ does not depend on the current position.) At a solution vector $\vec{v}$, the directional derivative

$$\vec{y}_+^T \nabla w(\vec{v}) = \vec{v}^T H \vec{y}_+ + \vec{y}_+^T \vec{b} = 0 \tag{3.2}$$

is 0 in the direction of any eigenvector $\vec{y}_+$. By the definition of eigenvector,

$$\vec{y}_+^T H \vec{y}_+ = \vec{y}_+^T (\lambda_{y_+} \vec{y}_+) > 0.$$

Denote the space spanned by all eigenvectors corresponding to a positive eigenvector except y_+ by S^*. Thus, for a vector $\vec{x}$ decomposed as

$$\vec{x} = \vec{v} + (a\vec{y}_+) + (b\vec{x}_{S^*}) + (c\vec{x}_{S^-}) \tag{3.3}$$

where $a > 0$,

$$\begin{aligned}
\vec{y}_+^T \nabla w(\vec{x}) &= (\vec{v} + (a\vec{y}_+) + (b\vec{x}_{S^*}) + (c\vec{x}_{S^-}))^T H(\vec{y}_+) + \vec{y}_+^T \vec{b} \\
&= a\vec{y}_+^T (\lambda_{y_+} \vec{y}_+) \\
&> 0,
\end{aligned}$$

by (3.2) and the property that all the eigenvectors are orthogonal to each other. Similarly,

$$-\vec{y}_+^T \nabla w(\vec{x}) > 0$$

for a vector $\vec{x}$ with parameter $a < 0$. We have thus shown that the projection of w onto the subspace S^+ is a convex function with the global minimum attained at the projection of solutions $\vec{v}$ onto S^+.

Given a set S with positive Lebesgue measure, for any vector $\vec{x} \in S \setminus S^-$, $(\vec{x} - \vec{v})^T (\vec{y}_+ - \vec{v}) \neq 0$.

(a) If $(\vec{x} - \vec{v})^T (\vec{y}_+ - \vec{v}) > 0$, then $a > 0$ in (3.3) and hence $\vec{y}_+^T \nabla w(\vec{x}) > 0$. Therefore, w increases in the direction of $\vec{y}_+$ at $\vec{x}$.

(b) If $(\vec{x} - \vec{v})^T (\vec{y}_+ - \vec{v}) < 0$, then $a < 0$ in (3.3) and hence $-\vec{y}_+^T \nabla w(\vec{x}) > 0$. Therefore, w increases in the direction of $-\vec{y}_+$ at $\vec{x}$.

In both these two cases, w increases as $x(t)$ moves away from S^-. Hence, the dynamic $\dot{x} = \nabla(w)$ cannot converge to $\vec{p} \in S^-$ from an initial point $\vec{x} \notin S^-$.

3. If $\vec{p}$ is a stationary point but not a saddle point, then clearly $\vec{p}$ is a local maximum point.

   If $\vec{p}$ is not a stationary point, then $\vec{y}^T \nabla w(\vec{p}) \leq 0$ in any direction $\vec{y}$ in the tangent cone of D at $\vec{p}$, by the definition of the dynamic and the continuity of w. Thus, $\vec{p}$ is a local maximum in D.

4. We can immediately see that $\vec{p}$ is a stationary point. By the argument to 2, we can reach the desired conclusion.

5. This is a corollary of 3 and 4.

□

Replicator dynamic: by Theorem 7.1.6 in Sandholm (2010), after transforming the replicator dynamic for u from $\text{int}(X)$ to $\text{int}(\mathcal{X})$ by the Akin transformation H, the resulting dynamic is the gradient system for the transported potential function $\bar{u} \circ H^{-1}$, where $\bar{u}(x) = u(x, x)$ can be taken as the potential function in $\Delta(H)$. For the replicator dynamic in a doubly symmetric game, we can thus apply Lemma 3.1 for the gradient system in $\text{int}(\mathcal{X})$.

4 Best-response dynamics

In a cheap-talk game G_M, we use the notation x^μ and $x^{\mu\nu}$ as defined in Section 4.2.2 in Weibull (1995). Let x^μ denote the population share of individuals of sender type μ, i.e., x^μ is the sum of all x_h such that $h = (\mu, f)$ for some $f \in F$. Let $x_i^{\mu\nu}$ be the share of individuals in subpopulation μ who take action i when meeting an individual of type ν. Thus, the associated vector $x^{\mu\nu} = (x_i^{\mu\nu})_{i \in A}$ is a point on the unit simplex Δ of the base game.

4.1 Examples

We now show an example how the best-response dynamic evolves in a cheap-talk game whose base-game has only two actions.

Example 1: Consider a cheap-talk game associated to a base game with 2 actions. Suppose that $|M|$ is sufficiently big, and we consider a generic initial point $x(0)$ of the best-response dynamic such that

$$\forall \mu, \nu \in M, \; x_1^{\mu\nu}(0) < \frac{v_2}{v_1 + v_2}. \tag{4.1}$$

(Note that a generic point $x(0)$ is an interior point in $\Delta(H)$.) Then, $BR(x(0))$ is a singleton $\{\bar{h} = (m(\bar{h}), f_{\bar{h}}))\}$ with $f_{\bar{h}}(m) = 2$ for all but one message in M (due to the constraint (2.2)). Denote that special message by $\bar{m}$, so $f_{\bar{h}}(\bar{m}) = 1$.

Case I: $\bar{m} \neq m(\bar{h})$. By the genericity condition, we now show that there is a period of time $[0, t_1]$ such that $BR(x(t))$ is the singleton $\{\bar{h}\}$ for all t with $0 \leq t \leq t_1$ (t_1 to be determined later). To be specific, we consider a strategy h with $f_h(m(\bar{h})) = f_{\bar{h}}(m(h)) = 2$, so $m(h) \neq \bar{m}$. At any time t with $0 < t \leq t_1$, if $BR(x(t')) = \{\bar{h}\}$ for all t' with $0 \leq t' < t$, then $x(t)$ may be viewed as a convex combination of $\bar{h}$ and $x(0)$, by Lemma A.1. It then follows that

$$u_M(h, \bar{h}) = u_M(\bar{h}, \bar{h}) = v_2$$

and

$$u_M(\bar{h}, x(0)) > u_M(h, x(0)),$$

from the definition of $BR(x(0))$. Thus, any such pure strategy h with $m(h) \neq \bar{m}$ cannot be the next best-response strategy after $\bar{h}$ in the dynamic.

On the other hand, the best-response dynamic cannot converge to $\bar{h}$. To see this, suppose that at some time t in the dynamic, $x(t)$ is close to the pure strategy $\bar{h}$. We then consider some strategy $\hat{h} = (\bar{m}, f_{\hat{h}})$ with $f_{\hat{h}}(m(\bar{h})) = 1$. Then $u_M(\hat{h}, x(t))$ is close to v_1. Thus, in the best-response dynamic $(x(t))_{t \geq 0}$, there exists a time $\hat{t}$ and a strategy $\hat{h}$ such that $BR(x(\hat{t})) = \{\bar{h}, \hat{h}\}$. By Lemma A.1 and (4.1), $f_{\hat{h}}(m) = 2$ for all $m \neq m(\bar{h})$. We may further infer that t_1 can be put as $\hat{t}$ and $BR(x(\hat{t})) = \{\bar{h}, \hat{h}\}$ for all $t \geq t_1$. We then observe that the dynamic converges to a point p with $p_{\bar{h}} = p_{\hat{h}} = 1/2$ (by Corollary 4.6 to be proved later).

In summary, $p = (\bar{h}/2, \hat{h}/2)$ is an asymptotically stable equilibrium with

$$u_M(\hat{h}, \bar{h}) = v_1, \;\; u_M(\hat{h}, \hat{h}) = u_M(\bar{h}, \bar{h}) = v_2, \;\text{and}\; u_M(p, p) = \frac{v_1 + v_2}{2}.$$

Case II: $\bar{m} = m(\bar{h})$. The dynamic then converges to a pure strategy Nash equilibrium $\bar{h}$ and $u_M(\bar{h}, \bar{h}) = v_1$.

Example 2: The equilibrium $p = (\bar{h}/2, \hat{h}/2)$ in Case 1 above is NSS but not Lyapunov stable under the best-response dynamic, e.g., when $|M| = 3$. To see this, consider a strategy $\tilde{h}$ such that $m(\tilde{h}) \neq m(\bar{h})$ or $m(\hat{h})$ and

$$f_{\tilde{h}}(m(\tilde{h})) = 2 \text{ and } f_{\tilde{h}}(m(\bar{h})) = f_{\tilde{h}}(m(\hat{h})) = 1.$$

Note that the strategy $p' = (1 - \epsilon)p + \epsilon\tilde{h}$ can be arbitrarily close to p by the choice of ϵ. Let us consider the best-response dynamic with $x(0) = p'$. Since $u_M(\tilde{u}, \bar{u}) = u_M(\tilde{u}, \hat{u}) = 0$, $b(0) \neq \tilde{h}$.

Denote the strategy $h_1 := (m(\bar{h}), f_{h_1})$ such that

$$\begin{cases} f_{h_1}(m(\bar{h})) = 2 \\ f_{h_1}(m(\hat{h})) = 1 \\ f_{h_1}(m(\tilde{h})) = 1, \end{cases}$$

and $h_2 := (m(\hat{h}), f_{h_1})$ such that

$$\begin{cases} f_{h_2}(m(\bar{h})) = 1 \\ f_{h_2}(m(\hat{h})) = 2 \\ f_{h_2}(m(\tilde{h})) = 1. \end{cases}$$

Then, $\bar{h} \neq h_1$ and $\hat{h} \neq h_2$. We can further infer that if ϵ is sufficiently small, then there exists some time $\tilde{t} > 0$ such that $b(t) = (h_1/2, h_2/2)$ for all $t < \tilde{t}$ and the distance between p and $x(\tilde{t})$ is bigger than ϵ. Thus, p is not Lyapunov stable.

We can also give an example that a Lyapunov stable equilibrium is not asymptotically stable.

Example 3: consider the game G_M in Example 2 (so $|M| = 3$), and the strategies h_3 and h_4 such that

$$\begin{cases} m(h_3) \neq m(h_4) \\ u_M(h_3, h_4) = u_M(h_3, h_3) = u_M(h_4, h_4) = v_1 \\ u_M(h_3, h) = u_M(h_4, h) = v_2 \ \forall h \text{ with } m(h) \neq m(h_3) \text{ or } m(h_4). \end{cases}$$

Then $p = (h_3/2, h_4/2)$ is a Lyapunov stable equilibrium, but not asymptotically stable, as $b(t)$ can always be taken as the strategy h_3 in the best-response dynamic starting from $x(0) = p$.

It is also possible that the dynamic converges to an NSS equilibrium with more than 2 supporting strategies.

Example 4: In the case of $|M| > 2$ in the game G_M in Example 1, consider a point p with $C(p) = \{h_1, h_2, h_3\}$ and the property

$$\begin{cases} p_{h_1} = p_{h_2} = p_{h_3} = 1/3 \\ f_{h_i}(m(h_j)) = 1 \ \forall i \in \{1,2,3\}, \ \forall j \in \{1,2,3\} \setminus \{i\} \\ f_{h_i}(m) = 2, \ \forall i \in \{1,2,3\}, \ \forall m \notin (\{m(h_1), m(h_2), m(h_3)\} \setminus \{m(h_i)\}). \end{cases}$$

In a cheap-talk game, there is some equilibrium p that is NSS but there is no best-response dynamic that can converge to p from a set of initial points with Lebesgue measure greater than 0.

Example 5: Consider a cheap-talk game G_M of $M = \{m_1, \dots m_6\}$ associated to the base game G with 3 actions such that $v_2 > v_1/3 + 2v_3/3$. Consider the strategies $h_1 = (m_1, f_{h_1})$, $h_2 = (m_2, f_{h_2})$ and $h_3 = (m_3, f_{h_3})$ such that

$$\begin{cases} m(h_i) \neq m(h_j) \ \forall i, j \in \{1, 2, 3\} \text{ with } i \neq j \\ f_{h_i}(m_{h_j}) = 2 \ \forall i, j \in \{1, 2, 3\} \\ f_{h_i}(m_{i+3}) = 1, \ f_{h_j}(m_{i+3}) = f_{h_l}(m_{i+3}) = 3 \ \forall i, j, l \in \{1, 2, 3\}, \text{ with } i \neq j \text{ or } l. \end{cases}$$

Then $p = (h_1/3, h_2/3, h_3/3)$ is an equilibrium and NSS, but cannot be reached by the best-response dynamic from a set of initial points with Lebesgue measure greater than 0, by Lemma 4.4 (to be proved later).

Not every Nash equilibrium in G_M is NSS.

Example 6: In G_M in Example 1, consider the equilibrium $p = (h_1/2, h_2/2)$ such that $f_{h_1}(m(h_1)) = f_{h_2}(m(h_2)) = 1$ and $f_{h_i}(m) = 2$ for all $m \neq m(h_i)$, for both $i \in \{1, 2\}$. Then, $u_M(h_1, h_1) = u_M(h_2, h_2) = v_1$ and $u_M(h_1, h_2) = v_2$. Hence, $u_M(p, p) = (v_1 + v_2)/2$. However, this equilibrium is not NSS. This is because for a small enough $\epsilon > 0$,

$$u_M(p, \epsilon h_1 + (1 - \epsilon)p) < u_M(h_1, \epsilon h_1 + (1 - \epsilon)p).$$

Example 7: We give an example of a symmetric equilibrium in G_M with population payoff less than v_2. In G_M in Example 1, suppose that for some integer $q > 2$,

$$\frac{v_1}{q} < \frac{(q-1)v_2}{q}.$$

Consider the point p such that $|M_p| = q$, where M_p is the set of messages sent by strategies in p. We further require that $p_1^{\mu\nu} = 1/q$ for all pairs of messages μ, ν in M_p (possibly $\mu = \nu$). We observe that such p is an equilibrium in G_M, and

$$u_M(p, p) = \frac{1}{q}\left((q-1)\frac{(q-1)v_2}{q} + \frac{v_1}{q}\right) < \frac{(q-1)v_2}{q}.$$

However, this equilibrium is not asymptotically stable. Under a generic perturbation from p to p', one supporting strategy h in $C(p)$ is the strict best-response strategy to p', and hence the dynamic moves towards the pure strategy h. The dynamic then evolves similarly as in Example 1 in this section.

4.2 Preliminary results

Consider a chap-talk game G_M and define $\bar{u}(x) := u(x, x)$ as the average population payoff of x. By Corollary B.2, we find that the best-response dynamic is not necessarily the same as the gradient system in $\mathcal{G}_{\mathcal{M}}$. However, we have the following lemma for best-response dynamics.

Lemma 4.1. *Consider a best-response dynamic $(x(t))_{t\geq 0}$ for a cheap-talk game G_M. Given a point p, suppose that the best-response dynamic can converge to this point p from a set of Lebesgue measure greater than 0, then*

$$\bar{u}_M(p) = \max_{y \in \Delta(C(p))} \bar{u}_M(y). \tag{4.2}$$

Proof. When p is a pure strategy, there is nothing to prove. From now on, we suppose that $C(p)$ contains at least two elements.

Assume that the desired conclusion is not true. Then p is a saddle point for $\bar{u}_M$ in the space spanned by $C(p)$. (Note that if p is a local maximum point in the interior of $C(p)$, then p must be a global maximum point in $C(p)$.)

Step 1: For every pair of two pure strategies h_i and h_j in $C(p)$ there exists a hyperplane $S^{i,j} \ni p$ in which any point is indifferent between h_i and h_j. (Hyperplanes are defined in $\mathbb{R}^{|H|-1}$.) That is,

$$u_M(h_i, x) = u_M(h_j, x), \ \forall x \in S^{i,j};$$

Note that this condition is equivalent to

$$(h_i - x)^T \nabla \bar{u}_M(x) = (h_j - x)^T \nabla \bar{u}_M(x),$$

as seen in Corollary B.2. It then follows that p is the intersection of all hyperplanes $S^{i,j}$ with both i and j in $C(p)$.

Step 2: The hyperplanes partition $\Delta(H)$. From each partitioning set, the trajectory can move to one neighbor partitioning set. The flow is one-way across any boundary between two partitioning sets, if the dynamic can cross this boundary.

Case I: the dynamic leads to a cycle of all the partitioning sets which contain p. Recall that the dynamic converges to the point p. Then, since $\bar{u}_M$ is a quadratic function, p must be a global maximum point, not a saddle point. Contradiction.

Case II: these sets give a partial order chain. So, in the end, the trajectory converges to the boundary of some partitioning sets. The analysis below follows this case.

Step 3: As p is a saddle point for $\bar{u}_M$ in the space spanned by $C(p)$, there exist two pure strategies, say h_1 and h_2 in $C(p)$, such that the hyperplane $S^{1,2}$ separates h_1 and h_2. Furthermore, if a point x is on the same side of h_1 with respect to $S^{1,2}$, then $u_M(h_1, x) \geq u_M(h_2, x)$; otherwise $u_M(h_1, x) \leq u_M(h_2, x)$.

For any best-response solution trajectory $(x(t))_{t\geq 0}$, suppose that at any time t, both h_1 and h_2 are not in $b(t)$, then by Lemma A.1, the relative proportion $h_1(t)/h_2(t)$ does not change at time t. If one of the two strategies h_1 and h_2 is in $b(t)$, and if $x(t)$ is on the same side of h_1 with respect to $S^{1,2}$, then by the above argument, $x(t)$ is moving towards h_1 and $h_1(t)/h_2(t)$ increases. Similarly, $h_1(t)/h_2(t)$ decreases when $x(t)$ is on the same side of h_2 with respect to $S^{1,2}$.

We know that the dynamic converges to $\Delta(C(p))$. Suppose that the dynamic crosses $S^{1,2}$ when moving towards some other pure strategy, say h_3. From (2.1) in the base game G, $S^{1,2}$ is not the same as $S^{1,3}$ or $S^{2,3}$. Hence, the dynamic will not stop at $S^{1,2}$ but continue towards h_3 for some time until it reaches another boundary. Recall in Case II in Step 2, there are only finitely many partitioning sets. So, the dynamic almost surely converges to h_1 or h_2, or a boundary point without support h_1 or h_2.

Indeed, the dynamic $(x_t)_{t\geq 0}$ can converge to p only if it starts from some boundary, e.g., $S^{1,2}$, or anywhere which may lead to somewhere in an (extended) line connecting some pure strategies and p (so that the dynamic finally converges to p). All these sets have measure 0, as there are only finitely many hyperplanes partitioning $\Delta(H)$. Under all other circumstances, the dynamic will converge to some boundary where at least h_1 or h_2 is not a component. □

Proposition 4.2 in Weibull (1995) also applies in the context of best-response dynamics.

Lemma 4.2. *Suppose that p is a stationary point in (2.4). If $x^\mu, x^\nu > 0$, then $(x^{\mu\nu}, x^{\nu\mu})$ is a Nash equilibrium in the base game G.*

Hence, if there are only two pure strategies (μ, f) and (ν, g) in the support of p, then the action $f(\nu) = g(\mu)$ in the base game G.

We now show a sharper result than Lemma 4.2 when a Nash equilibrium is reached from a set of initial points with Lebesgue measure greater than 0.

Lemma 4.3. *Suppose that the best-response dynamic $(x(t))_{t\geq 0}$ can converge to a point p from a set of initial points with Lebesgue measure greater than 0, then for any two messages μ and ν which have been used by a pure strategy in the support of p, $p^{\mu\nu} = p^{\nu\mu}$ is a pure strategy in G and hence $u(p^{\mu\nu}, p^{\nu\mu}) \in \{v_1, \ldots, v_n\}$.*

Note that Lemma 4.1 does not show that a mixed-strategy $p^{\mu\nu}$ or $p^{\nu\mu}$ cannot be asymptotically stable or be a local maximum and hence cannot be reached from a set of initial points with Lebesgue measure greater than 0. (Recall a (doubly symmetric) hawk-dove game.) The following proof follows the implicit property that the projection of the game G_M onto $(X^{\mu\nu}, X^{\nu\mu})$ is a coordination game.

Proof. Given the two messages μ and ν, define the space

$$X(\mu,\nu) := X^{\mu\nu} \times X^{\nu\mu} = \Delta(A) \times \Delta(A).$$

That is, each $x \in \Delta_M$ has a projection $(x^{\mu\nu}, x^{\nu\mu})$ onto $X(\mu,\nu)$. We have thus constructed a "projection game" $G(\mu,\nu)$ in $X(\mu,\nu)$ under the payoff function: $u(i,j) = 0$ if $i \neq j$ and $u(i,i) = v_i$ for all $i \in \{1, ..., n\}$. Note that the payoff in this projection game only depends on $x^{\mu\nu}$ and $x^{\nu\mu}$ but not other components of x.

Given time t and the chosen best-response strategy

$$b(t) = \sum_{i=1}^{|H|} \lambda_h(t) h \in BR(x(t)),$$

for every pure strategy h_i, define

$$\lambda^{\mu}(t) = \sum_{i=1}^{|H|} \mathbb{1}_{\mu=m(h_i)} \lambda_{h_i}(t)$$

and

$$\lambda^{\nu}(t) = \sum_{i=1}^{|H|} \mathbb{1}_{\nu=m(h_i)} \lambda_{h_i}(t).$$

By the condition that the two messages μ and ν have been used for infinitely long time in the trajectory $(x(t))_{t\geq 0}$, we have

$$\int_0^\infty (\lambda^{\mu}(t))\, dt = \infty$$

and

$$\int_0^\infty (\lambda^{\nu}(t))\, dt = \infty.$$

When $\dot{u}_M$ exists at any time $t > 0$, we consider the following four cases.

1. Both messages μ and ν have not been used in any pure strategy in $C(x(t))$. Then $\dot{u} = 0$ in $G(\mu,\nu)$ at this time t.

2. Only message μ has been used in some pure strategy in $C(x(t))$. Then by Lemma A.1, $\dot{x}^{\nu\mu} = 0$, and hence $\dot{u} > 0$ in $G(\mu,\nu)$ at this time t.

3. Only message ν has been used in some pure strategy in $C(x(t))$. Similarly to case 2, $\dot{u} > 0$ in $G(\mu, \nu)$ at this time t.

4. Both messages μ and ν have been used in any pure strategy in $C(x(t))$. We can apply Lemma C.1 here.

If $(x(0)^{\mu\nu}, x(0)^{\nu\mu})$ is not a saddle point for u, which can only happen with Lebesgue measure 0, in the projection game $G(\mu, \nu)$, then u is always increasing in the trajectory $x(t)_{t\geq 0}$, and $(x(t)^{\mu\nu}, x(t)^{\nu\mu})$ converges to a pure-strategy Nash equilibrium in this coordination game $G(\mu, \nu)$. Thus both $(x(t)^{\mu\nu}$ and $x(t)^{\nu\mu})$ converge to the same pure strategy in A in the game $G(\mu, \nu)$. □

Lemma 4.4. *Suppose that S is a connected set of Nash equilibria in G_M, and suppose that the best-response dynamic $(x_t)_{t\geq 0}$ can converge to S from a set of initial points with Lebesgue measure greater than 0. We further assume that there exists a point $y \in S$ such that $u_M(y, y) < v_1$.*

1. *Then the best-response dynamic $(x_t)_{t\geq 0}$ in fact converges to one specific point in S.*

2. *Moreover, one cannot find two pure strategies $h, \bar{h} \in C(p)$ such that*

$$u_M(h, h') = u_M(\bar{h}, h') \ \forall h' \in C(p). \tag{4.3}$$

Proof. From Lemma B.1, it follows that the best-response dynamic converges to a connected set of Nash equilibria which has the same population payoff.

Denote $H_S := \{h \in C(p) : p \in S\}$. Given the set M_S of messages used by strategies in S, i.e.,

$$M_S = \{\mu \in M : \exists h \in H_S \text{ s.t. } \mu = m(h)\},$$

from Lemma 4.3, it follows that for any strategy $h \in H_S$, $f_h(m)$ must be a fixed action in the base game G, for all messages in $m \in M_S$.

In the game G_M, if the the global maximum point of $\bar{u}_M$ in $\Delta(H)$ is an interior point. Then, it is unique and there is no saddle points for $\bar{u}_M$ in $\Delta(H)$. From Lemma B.1 and the property of the quadratic function $\bar{u}_M$, it follows that the dynamic cannot converge to any interior non-global-maximum point from a set of initial points with measure greater than 0. Thus, it converges to either the (unique) interior global maximum point (if it exists) or some boundary points. We discuss the boundary case below.

For every pair (h_i, h_j) of strategies in H_S, there exists a hyperplane S^{h_i,h_j} in which any point is indifferent between h_i and h_j. (Hyperplanes are defined in $\mathbb{R}^{|H|-1}$.) That is

$$u_M(x, h_i) = u_M(x, h_j),\ \forall x \in S^{h_i,h_j}.$$

These hyperplanes partitions $\Delta(H)$. By the structure of the cheap-talk game G_M and (2.1), no two hyperplanes are identical in $\Delta(H)$.

We consider any pair of pure strategies $(h, \bar{h})$ in H_S, and study the times when the dynamic trajectory crosses the corresponding hyperplane $S^{h,\bar{h}}$, if it ever crosses $S^{h,\bar{h}}$. Recall that $S^{h,\bar{h}}$ is not identical to any other hyperplane. Hence, when the conjunction of all three conditions below satisfies,

1. the trajectory crosses the hyperplane $S^{h,\bar{h}}$;
2. the crossing point is not at the intersection of two hyperplanes;
3. the dynamic is moving towards some other pure strategy, say $\hat{h}$;

the dynamic will not stop at $S^{h,\bar{h}}$ but continue towards $\hat{h}$ for some time until it reaches another hyperplane or it converges to $\hat{h}$. This property applies to all pairs of pure strategies $(h, \bar{h})$ in H_S.

As the trajectory will finally converge to S on the boundary of $\Delta(H)$, there is some time t' such that from t' on, the trajectory never moves to any other strategy not in H_S. W.l.o.g., we assume that t' is the minimum time with such property. If (4.3) holds, then the dynamic trajectory must moves along $S^{h,\bar{h}}$. However, the hyperplane $S^{h,\bar{h}}$ must include the edge $(h, \bar{h})$. Thus, except starting from a set of initial points with measure 0, (at $S^{h,\bar{h}}$ or every crossing at $S^{h,\bar{h}}$ is not also in another hyperplane) the trajectory can only cross $S^{h,\bar{h}}$ finitely many times when moving to a third strategy. (The union of partitioning sets, generated by all $S^{h,h'}$ for all pairs of strategies h and h' in H, containing p is a subset of the basin of attraction to p.) If the trajectory is moving to either h or $\bar{h}$, then, because of the special position of $S^{h,\bar{h}}$, the trajectory will not cross $S^{h,\bar{h}}$.

At time t', $x(t')$ is at the hyperplane corresponding to one strategy in H_S and one not in H_S, hence $x(t') \notin S^{h,\bar{h}}$. Which of h and $\bar{h}$ is a better response strategy to $x(t')$ depends on the relative proportion of all pure strategies not in H_S. Note that after time t', the relative proportion of all pure strategies not in H_S will not change (by Lemma B.1) and hence the trajectory will stay at the same half space.

Therefore, except from a set of initial points with Lebesgue measure 0, (i.e., starting in $S^{h,\bar{h}}$ or along some well-calculated line through the intersection of some pair of hyperplanes) $x(t')$ is not in $S^{h,\bar{h}}$. This holds for all pairs of strategies in H_S.

Hence, if the dynamic converges to the boundary of $\Delta(H)$, then the convergence has to be one of the isolated points that (4.3) does not hold. $\square$

From now on, we only consider a G_M with a generic base game G.

Lemma 4.5. *1. Suppose p is a Nash equilibrium point in G_M with only two supporting strategies $h_1 = (\mu, f)$ and $h_2 = (\nu, g)$. We further suppose that $u_M(p, p) < v_1$. Then $f(\nu) = g(\mu)$.*

2. We further suppose that if a best-response trajectory $(x_t)_{t\geq 0}$ can converge to point p from a set of initial points with Lebesgue measure greater than 0.

Then

$$u(f(\nu), g(\mu)) > \max\{u(f(\mu), f(\mu)), u(g(\nu), g(\nu))\}. \tag{4.4}$$

Proof. 1. It follows from Lemma 4.2 and the coordination property of the base game G that $f(\nu) = g(\mu)$.

2. It follows from Lemma 4.1 that

$$u(f(\nu), g(\mu)) \geq \max\{u(f(\mu), f(\mu)), u(g(\nu), g(\nu))\}.$$

If

$$u(f(\nu), g(\mu)) = u(f(\mu), f(\mu)) > u(g(\nu), g(\nu))$$

or

$$u(f(\nu), g(\mu)) = u(f(\nu), f(\mu)) > u(g(\mu), g(\mu)),$$

then p is not a Nash equilibrium point. Another case

$$u(f(\nu), g(\mu)) = u(f(\nu), f(\mu)) = u(g(\mu), g(\mu))$$

contradicts Lemma 4.4.3.

$\square$

Corollary 4.6. *Under the condition of Lemma 4.5.2,*

$$p_{h_1} = \frac{u_3 - u_2}{2u_3 - u_2 - u_1}$$

where

$$\begin{aligned} u_1 &= u_M(h_1, h_1) = u(f(\mu), f(\mu)), \\ u_2 &= u_M(h_2, h_2) = u(g(\nu), g(\nu)), \text{ and} \\ u_3 &= u_M(h_1, h_2) = u(f(\nu), g(\mu)). \end{aligned}$$

In particular, when $u_1 = u_2 < u_3$, $p_{h_1} = p_{h_2} = 1/2$ and

$$u_M(p, p) = \frac{u_1 + u_3}{2}.$$

Proof. This follows from the fact that p is a Nash equilibrium. □

Lemma 4.7. *Suppose that p is a Nash equilibrium point in G_M with three supporting strategies h_1, h_2, and h_3. We further suppose that $u_M(p,p) < v_1$. If the best-response dynamics $(x_t)_{t\geq 0}$ can converge to point p from a set of initial points with Lebesgue measure greater than 0, then for every pair of pure strategies h_i and h_j in $C(p)$*

$$u_M(h_i, h_j) > \max\{u_M(h_i, h_i), u_M(h_j, h_j)\}. \tag{4.5}$$

Proof. Assume that (4.5) does not hold, then we can show that one of the following 3 cases must hold.

1. There exists a strategy h_1 such that h_1 dominates another pure strategy, say h_2, i.e.,
$$\begin{cases} u_M(h_1, h_1) \geq u_M(h_2, h_1) \\ u_M(h_1, h_2) \geq u_M(h_2, h_2) \\ u_M(h_1, h_3) \geq u_M(h_2, h_3). \end{cases}$$
Then $u_M(h_1, h_1) \geq u_M(p, p)$. We then reach a contradiction by Lemma 4.4.

2. There exists a strategy h_1 such that
$$u_M(h_1, h_1) \geq \max\{u_M(h_1, h_2), u_M(h_1, h_3)\}.$$
Then Lemma 4.5 leads to a contradiction, when we view the h_2 and h_3 as a combined strategy.

3. There exists a pair of strategies h_1 and h_2 such that
$$u_M(h_1, h_2) \leq \min\{u_M(h_1, h_1)), u_M(h_2, h_2)\}.$$
Then, the hyperplane $S^{1,2}$ must (weakly) separate h_1 and h_2; see the notation $S^{1,2}$ in the proof of Lemma 4.1. As both h_1 and h_2 are in the support of p, the best response dynamic cannot converge to p from a set of initial points with Lebesgue measure greater than 0, by Step 3 in the proof of Lemma 4.1.

□

Comment: It thus shows that Case I in Step 2 in the proof of Lemma 4.1 cannot hold when $|C(p)| = 3$.

Lemma 4.8. *Suppose that p is a Nash equilibrium in G_M. We further suppose that $u_M(p,p) < v_1$. If the best-response dynamics $(x_t)_{t\geq 0}$ can converge to point p from a set of initial points with Lebesgue measure greater than 0, then for every pair of distinct supporting strategies $h, h' \in C(p)$,*

$$u_M(h,h') > \max\{u_M(h,h), u_M(h',h')\}. \tag{4.6}$$

Proof. Note that he case of $|H| = 2$ follows Lemma 4.5.2 and the case of $|H| = 3$ follows Lemma 4.7.

Assume that (4.6) does not hold for a game G_M with $|H| > 3$. To be specific, suppose that p' is an asymptotically stable Nash equilibrium which has two strategies $h_1, h_2 \in C'(p')$ such that the property (4.6) does not hold. Then we think of a three-strategy game G_M with pure strategies, h_1, h_2, and

$$h_3 = \sum_{h \in C(p') \setminus (\{h_1\} \cup \{h_2\})} p_h h.$$

That is, we may view all strategies other than h_1 or h_2 as a combined strategy. Now apply Lemma 4.7 and we reach a contradiction. □

Comment: It thus shows that Case I in Step 2 in the proof of Lemma 4.1 cannot hold.

Lemma 4.9. *For a best-response trajectory from a genetic interior point, if the trajectory converges to a pure strategy $p = (\mu, f)$ in G_m, then $u(f(\mu), f(\mu)) = v_1$.*

Proof. We prove it by contradiction. Suppose that $u(f(\mu), f(\mu)) < v_1$, and that $f(\nu) = 1$ for some message ν. Then for any strategy $h \in H$ which sends message ν, $u_M(h,p) > u_M(p,p)$ and hence p cannot be a Nash equilibrium. □

4.3 General results

We denote by f_h the decision function of a pure strategy h in G_M. Suppose a Nash equilibrium point p has the support $C(p) = \{h_1, ..., h_l\}$, and we define

$$u_{\min}(p) = \min_{1 \leq i \leq j \leq l} u(f_{h_i}(m(h_j)), f_{h_j}(m(h_i)))$$

and

$$u_{\max}(p) = \max_{1 \leq i \leq j \leq l} u(f_{h_i}(m(h_j)), f_{h_j}(m(h_i))).$$

Again, we only consider a G_M with a generic base game G here.

Theorem 4.10. *If the best-response dynamic can converge to a Nash equilibrium p from a set of initial points with Lebesgue measure greater than 0, then*

$$u_M(p,p) \geq \frac{u_{\min}(p) + u_{\max}(p)}{2}.$$

Proof. We find a pair of strategies (h_1, h_2) such that $u_M(h_1, h_2) = u_{\max}(p)$. By Corollary 4.6,

$$\begin{aligned} u(p,p) &\geq \max_{\lambda\in[0,1]} u(\lambda h_1 + (1-\lambda)h_2, \lambda h_1 + (1-\lambda)h_2). \\ &= u(h_1/2 + h_2/2, h_1/2 + h_2/2) \\ &\geq \frac{u_{\max}(p) + \min\{u_M(h_1,h_1), u_M(h_2,h_2)\}}{2}. \end{aligned}$$

□

Recall that there are n actions in the base game G.

Theorem 4.11. *Suppose that a best-response trajectory can converge to a point p with $|C(p)| = i$ from a set of initial points with Lebesgue measure greater than 0, then*

$$u_M(p,p) \geq \max\{\frac{v_1 + (i-1)v_n}{i}, \frac{v_n + (i-1)v_{n-1}}{i}\}.$$

Proof. Recall Lemma 4.3.

1. One can observe that the structure of p with the lowest payoff is such that for every pair $h, h' \in C(p)$, $u_M(h,h) = v_n$ and $u_M(h,h') = v_{n-1}$. Then,
$$u_M(p,p) \geq \frac{v_n + (i-1)v_{n-1}}{i}.$$

2. To against the intrusion from a pure strategy $\bar{h}$ which has $u_M(\bar{h}, h) = v_1$ to the strategy h in $C(p)$ which has the maximum p_h, it requires
$$u_M(p,p) \geq \frac{v_1 + (i-1)v_n}{i}.$$

□

5 Stability under two dynamics

5.1 Best-response dynamics

Consider the best-response dynamic $(x(t))_{t\geq 0}$ in a cheap-talk game G_M. Recall $\Delta = \Delta(H)$ and $\mathrm{Leb}(S)$ denotes the Lebesgue measure of the set S. Define the subsets in Δ:

$S_1 := \{p : p \text{ is a Nash equilibrium in } G_M\}$,
$S_2 := \{p : p \text{ is NSS}\}$,
$S_3 := \{p : \exists S^0 \subseteq \Delta \text{ with Leb}(S^0) > 0 \ s.t. \ \forall x(0) \in S^0,\ \exists (x(t))_{t\geq 0} \text{ and } x(t) \to p\}$,
$S_4 := \{p : p \text{ is Lyapunov stable}\}$,
$S_5 := \{p : p \text{ is asymptotically stable}\}$,
$S_6 := \{p : p \text{ is ESS}\}$.

Lemma 5.1. *If $|C(p)| < |M|$ and $p \in \Delta$ is Lyapunov stable, then $u_M(p,p) = v_1$ and $|M| = 2$.*

Proof. Denote the set of messages used by one $h \in C(p)$ by M_p. Suppose that $|M| > 2$ and we denote one message not in M_p by μ.

Recall that A is the set of actions in the base game G. Now consider a pure strategy $h = (\mu, f_h)$ such that for any message $m \in M_p$

$$f_h \in \operatorname*{argmax}_{a \in A} d(f_h(m), p^{m\mu}),$$

where d is the Euclidean distance in Δ. We can then find some $\bar{\epsilon} > 0$ such that for any positive $\epsilon < \bar{\epsilon}$, the best-response dynamic $(x(t))_{t\geq 0}$ with $x(0) = \epsilon h + (1-\epsilon)p$ will converge to some point with distance more than $\bar{\epsilon}$ to p. □

Theorem 5.2. *In a cheap-talk game G_M,*

$$S_6 = S_5 \subsetneq S_4 \subsetneq S_2 \subsetneq S_1.$$

and

$$S_6 = S_5 \subsetneq S_4 \subsetneq S_3 \subsetneq S_1.$$

Proof. We show the following relationships.

- $S_6 = S_5$. Consider any p in S_5. Recall that $\bar{u}_M$ is a quadratic function, and thus
$$S_5 = \{\text{strict local maximum points for } \bar{u}_M\}.$$
On the other hand, every point in S_5 is obviously ESS. An ESS point p must be a local maximum point. If it is not a strictly maximum point, then, there exists a strategy $h \in H$ and $\epsilon > 0$ such that $u_M(p, p') = u_M(p', p')$ where $p' = \epsilon h + (1-\epsilon)p$, which is contradictory to the definition of ESS.

- $S_5 \subseteq S_4$. This is trivial.

- $S_5 \neq S_4$. See Example 3 in Section 4.1.
- $S_4 \subseteq S_3$. Consider any $p \in S_4$.

  Case I: p is a pure strategy and $|M| = 2$. Then, from Lemma 5.1, it follows that $p \in S_3$.

  Case II: $|M| > 2$. Then, by the argument in Lemma 5.1 and Example 3 in Section 4.1, we can infer that $|C(p)| = |M|$. So each support strategy in $C(p)$ sends a different message. By the constraint (2.2), $u_M(p,p) < v_1$.

  The cheap-talk game G_M reduces to a standard coordination game with the action set $\{a^{\mu\nu}\}_{\mu,\nu \in M}$ and each $a^{\mu\nu} \in \{1, \ldots, n\}$. By the strict payoff order $v_i > v_j$ for all i and j with $1 \leq i < j \leq n$, we may infer that $S_4 = S_5 = S_6$. Moreover, every p in S_4 is a strict local maximum point and hence p is also in S_3.
- $S_4 \neq S_3$, by Example 2 in Section 4.1.
- $S_4 \subseteq S_2$, similarly to the argument to $S_4 \subseteq S_3$.
- $S_4 \neq S_2$. See Example 2 in Section 4.1.
- $S_3 \subseteq S_1$ and $S_2 \subseteq S_1$. Trivial by the definition of NE and Lemma B.1.
- $S_2 \neq S_1$. See Example 6 in Section 4.1.
- $S_3 \neq S_1$, by Example 7 in Section 4.1 and Lemma 4.1.

□

Lemma 5.3. *Given a cheap-talk game G_M,*

1. *if $p \in S_3$, then p is not necessarily in S_2;*
2. *if $p \in S_2$, then p is not necessarily in S_3.*

Proof. 1. Consider a cheap-talk game G_M with $M = \{m_1, m_2, m_3, m_4\}$, associated to the base game G with $|A| = 3$. Consider strategies

- $h_1 = (m_1, f_{h_1})$ with $f_{h_1}(m_1) = 2$, $f_{h_1}(m_2) = f_{h_1}(m_3) = 1$, $f_{h_1}(m_4) = 3$.
- $h_2 = (m_2, f_{h_2})$ with $f_{h_2}(m_1) = 1$, $f_{h_2}(m_2) = f_{h_1}(m_3) = 2$, $f_{h_1}(m_4) = 3$.
- $h_3 = (m_3, f_{h_3})$ with $f_{h_3}(m_1) = 1$, $f_{h_3}(m_2) = 2$, $f_{h_3}(m_3) = f_{h_1}(m_4) = 3$.
- $h_4 = (m_3, f_{h_4})$ with $f_{h_4}(m_1) = f_{h_4}(m_3) = 1$, $f_{h_4}(m_2) = 2$, $f_{h_3}(m_4) = 3$.

The point $p = (h_1/2, h_2/2)$ is an equilibrium, and we can show $p \in S_3$. For a sufficiently small $\epsilon > 0$, take

$$x(0) = \frac{(1-\epsilon)h_1}{2} + \frac{(1-\epsilon)h_2}{2} + \epsilon h_3.$$

Then, the best-response dynamic $(x(t))_{t\geq 0}$ from this initial point converges to p. To see this, we firstly observe that

$$\begin{cases} u_M(h_1, x(0)) = \frac{(1-\epsilon)v_1}{2} + \frac{(1-\epsilon)v_2}{2} + \epsilon v_1 \\ u_M(h_2, x(0)) = \frac{(1-\epsilon)v_1}{2} + \frac{(1-\epsilon)v_2}{2} + \epsilon v_2 \\ u_M(h_3, x(0)) = \frac{(1-\epsilon)v_1}{2} + \frac{(1-\epsilon)v_2}{2} + \epsilon v_3, \end{cases}$$

and for all $i = 1, 2, 3$,

$$h_i \in \underset{h: m_i = m(h)}{\text{argmax}} \ u_M(h, x_0),$$

and for all h with $m(h) = m_4$, $u_M(h, x(0)) \leq v_3$. Thus, there exists a best-response dynamic $(x(t))_{t\geq 0}$ and a time t_1 such that for all t with $0 \leq t \leq t_1$, $b(t) = h_1$. However, this best-response trajectory cannot converge to h_1, by Lemma B.1. So, w.l.o.g., for all $t > t_1$, $h_2 \in C(b(t)$. Moreover, this dynamic must converge to p. We observe that for a sufficiently small $\epsilon' > 0$, if the initial point $x(0)'$ is in the ϵ'-neighborhood of the $x(0)$ above and with the property

$$\forall i = \{1, 2, 3, 4\} \ \forall j \in \{1, 2, 3\}, \ x(0)_j^{m_4 m_i} \in (1/3 - \epsilon', 1/3 + \epsilon'),$$

then the best-response dynamic also converges to p.

However, p is not NSS, for the mutation strategy $h = (m_3, f_h)$ s.t. $f_h(m_1) = 1$, $f_h(m_2) = 2$, $f_h(m_3) = 1$ and $f_h(m_4) = 3$.

2. See Example 5 in Section 4.1.

□

Comment: One can also talk about the ES set or asymptotically stable set (they are the same) in G_M. For instance, if S is an ES and $|M| > |M_S|$, then for any $p \in S$, $u_M(p, p) = v_1$. (Proof to be completed later, by Shlag's argument.)

We now characterize the initial point criteria, i.e., S_3, in a cheap-talk game G_M.

Theorem 5.4. *Given a cheap-talk game G_M, a Nash equilibrium $x \in S_3$ if and only if either $u_M(x,x) = v_1$ or for every pair of distinct supporting strategies $h_1, h_2, \in C(x)$,*

$$u_M(h_1, h_2) > \max\{u_M(h_1, h_1), u_M(h_2, h_2)\}.$$

Proof. In the case that $u_M(x,x) = v_1$, there is nothing to prove.

Now we consider the case that $u_M(x,x) < v_1$. We can prove "only if" by Lemma 4.8. For the direction "if", given any supporting strategy $h \in C(x)$, we can find two sufficiently small parameters ϵ and ϵ' with $\epsilon' << \epsilon$ such that from any initial point

$$y \in \bigcup_{y \in \Delta_M} \{(1 - \epsilon - \epsilon')x + \epsilon h + \epsilon' y\}$$

the best-response dynamic converges to x. □

5.2 Replicator dynamics

Note that the trajectory of best-response dynamic and replicator dynamic may be different from the same initial point in G_M. Take $x(0) = p'$ in Example 2 in Section 4.1 for instance. For best-response dynamic, the trajectory will not convert back to p. However, as p is NSS in G_M, by $S_4 = S_2$, the replicator dynamic will converge to some point very close to p.

Appendix A Preliminary results

Consider a best-response dynamic in any normal-form game.

Lemma A.1. *In a best-response solution trajectory $(x(t))_{t \geq 0}$ from an interior point $x(0)$, if during a time period (t_1, t_2), two pure strategy*

$$g, k \notin BR(x(t)), \ \forall t \in (t_1, t_2),$$

then

$$\frac{x_g(t_2)}{x_k(t_2)} = \frac{x_g(t_1)}{x_k(t_1)}.$$

Proof. This is straightforward, or see the paper of best-response dynamics in extensive-form games. □

Appendix B Best-response Dynamics in Doubly Symmetric Games

Consider a general doubly symmetric game $\mathcal{G}$ in the simplex $X = \Delta(H)$ with payoff function u.

Lemma B.1. *Given any best response dynamic $(x(t))_{t\geq 0}$ in $\mathcal{G}$,*

$$\dot{u}(x, x) = 2(u(b, x) - u(x, x)) \geq 0$$

where $b \in BR(x)$.

Proof. Firstly note

$$u(x, x) = \sum_{h,h' \in H} u(h, h') x_h x_{h'}. \tag{B.1}$$

We may then infer that

$$\dot{u}(x, x) = \sum_{h,h' \in H} (\dot{x}_h u(h, h') x_{h'} + x_h u(h, h') \dot{x}_{h'}) = 2 \sum_{h,h' \in H} (\dot{x}_h u(h, h') x_{h'}).$$

Without loss of generality, we assume that $BR(x) = b$ is a singleton. Then, it follows from (2.4) that

$$\dot{u}(x, x) = 2 \left(\sum_{h,h' \in H} (-x_h u(h, h') x_{h'}) + \sum_{h' \in H} (u(b, h') x_{h'}) \right),$$

and hence by (B.1) again,

$$\dot{u}(x, x) = 2 \left(u(b, x) - u(x, x) \right).$$

□

We define $\bar{u}(x) := u(x, x)$.

Corollary B.2. *For any time $t \geq 0$ in a best-response trajectory $(x(t))_{t\geq 0}$ in $\mathcal{G}$,*

$$BR(x(t)) = \underset{y \in \Delta(H)}{\text{argmax}} \ (y - x(t))^T \nabla \bar{u}(x_t).$$

Appendix C A Generalized Best-response Dynamic in a Coordination Game

Consider a 2-player coordination game G with payoff matrix V in Section 2. As $u^1(x^1, x^2) = u^2(x^1, x^2)$ for every mixed strategy profile (x^1, x^2), we do not need the superscript for the payoff function u. For each player $i = 1$ or 2, we define the best response set at state x as

$$BR^i(x) = \operatorname*{argmax}_{y^i \in \Delta(A)} u(y^i, x^{-i}).$$

Given a deterministic system $(\lambda^i(t))_{i\in\{1,2\}}$ where for both i

$$\forall t \geq 0,\ 0 < \lambda^i(t) < 1 \text{ and } \int_0^\infty \lambda^i(t)dt = \infty. \tag{C.1}$$

We define a generalization of the best-response dynamic as

$$\dot{x}^i(t) \in \lambda^i(t)\left(BR^i(x(t)) - x^i(t)\right) \tag{C.2}$$

for both players $i \in \{1, 2\}$. Again, there exists a solution trajectory $(x(t))_{t\geq 0}$ from any initial point $x(0) \in \Delta(A) \times \Delta(A)$, and the derivative $\dot{x}$ exists except at most countably many times.

Lemma C.1. *From every initial point $x(0)$, the dynamic in the form of (C.2) converges to the set of Nash equilibria in G. If $x(t)$ is not a Nash equilibrium and $\dot{x}(t)$ exists, then $\dot{u}(t) > 0$.*

Proof. Suppose that at time t, $\dot{x}$ exists and

$$\forall i \in \{1, 2\},\ \dot{x}^i = \lambda^i(t)\left(b^i(t) - x^i(t)\right),$$

where $b(t) \in BR(x(t))$. Thus, by the linearity of u,

$$\begin{aligned}
\dot{u} &= x^{1T}\nabla u(x) + x^{2T}\nabla u(x) \\
&= u(\dot{x}^1, x^2) + u(x^1, \dot{x}^2) \\
&= \lambda^i\left(u(b^i, x^2) - u(x^1, x^2)\right) + \lambda^j\left(u(x^i, b^2) - u(x^1, x^2)\right) \\
&> 0,
\end{aligned}$$

when (x^1, x^2) is not a Nash equilibrium. By (C.1) and the fact that u is bounded, we can conclude that $x(t)$ converges to the set of Nash equilibria.

□

By the standard result in coordination games, we have the following corollary.

Corollary C.2. *If the trajectory of (C.2) can converge to a Nash equilibrium p in G from a set of initial points with Lebesgue measure greater than 0, then p is a pure-strategy Nash equilibrium, i.e., $p^1 = p^2$.*

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