Tenable Strategy Blocks and Evolutionary Stability

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Abstract

This paper analyzes relationships between tenable strategy blocks (Myerson and Weibull, 2015) and evolutionary stability concepts in finite normal-form games. A block is defined as a nonempty set of pure strategies for each player role, and a block game is a game where the strategy space is restricted to a block. An intermediate block property, between curb (Basu and Weibull, 1991) and coarse tenability, is *Nash-curb*, a block that contains all pure best replies to all Nash equilibria of the block game. Such blocks have stability properties comparable with those of equilibrium evolutionarily stable, or EES, sets (Swinkels, 1992b). In two-player games, a singelton EES set's support is in a Nash-curb block. I also show that a strategy profile in any symmetric singleton coarsely tenable (Nash-curb) block is neutrally (evolutionary) stable.

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Keywords: Settled equilibrium, tenable block, evolutionary game theory, EES sets, convention, Nash-curb

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1 Introduction

It is well-known that Nash equilibrium is not a product of only rational behavior as the concept requires restrictions on the players' expectations (see Bernheim (1984), Pearce (1984)). Thus, many game theorists advocate that a Nash equilibrium is not something players "reason" their way into playing, but should rather be viewed as something that players "learn" to play (see for example the discussion in Mailath (1998)). An early sketch of such a mechanism was proposed by Nash himself in his Ph.D thesis where he first introduced his solution concept (Nash, 1950). In the so-called Nash-mass action interpretation, there are large populations of boundedly rational individuals for every player role in the game. These individuals are recurrently and randomly drawn to play the game against each other in their respective player roles. Nash argues that in such a meta game, stable states approximate the Nash equilibria of the original game.

Restricting attention to strategic interaction in settings that the players are familiar with, either because one believes that these are the only situations where the outcome of the interaction can be reasonably predicted or because most social interaction takes place in such a setting, it is as we have seen possible to justify the Nash equilibrium concept, but also to go further and argue for more selective concepts. The latter can be formalized, just as Nash, by embedding and study the game of interest in a larger meta game. In such games, solutions to the underlaying game can either be determined by stochastic or deterministic dynamic systems (see for example Klaus Ritzberger (1995), Young (1993)) or by static requirements on stable states (such as evolutionary stable strategies Smith and Price (1973)).

Social norms or conventions develop over time and arguably play an important role in shaping peoples' behavior in strategic interaction. Hence, a greater understanding of such behavioral rules could aid in predicting the outcome of games. A subsequent question is then what is driving the creation of these conventions. Evolutionary game theorist argue that such behavioral rules can be based on concepts of adaptation, that is, the act of natural selection driving organism toward fitness maximization (usually determined by degree of reproductive success). Arguably, all social science can be reduced, at least in principle, to biology. Hence, any formalization of a social construction, such as a convention, should be attached to some form of an evolutionary foundation.

This paper compares concepts that formalize the idea of a social convention in strategic interaction as a nonempty set of pure strategies, called a block. In particular, we compare Myerson and Weibull's (2015) tenable strategy blocks with concepts of evolutionary stability to determine to what extent evolutionary interpretations of these concepts are merited. We also develop a block concept that is similar to those just mentioned but does not require the rather involved machinery of tenable blocks.

Tenable strategy blocks was introduced in an attempt to develop a theory that allows social conventions to be created endogenously within a finite normal-form game. These concepts are based on a meta game, henceforth called the consideration-set game, similar to the Nash mass-action interpretation, where there is a large population of individuals for every player role in the original game. Recurrently, a set of individuals, one from each player role, are randomly chosen to play the game against each other. Most individuals are boundedly rational in the sense that they do not consider all strategies available to them. A convention is formalized as a block consisting of all strategies that conventional individuals consider. The first definition of such a convention is a coarsely tenable block. Such a block has the property that no individual is better-off using strategies outside of the block as long as most individuals are conventional, given that the overall population play constitutes a Nash equilibrium. Any Nash equilibrium with its support in a minimal coarsely tenable block[1] is called a coarsely settled equilibrium. The authors also develop an alternative definition of a conventional where the conventional individuals only have to fare better against "rationality biased" unconventional individuals. A block with such a property is called finely tenable and any equilibrium with its support in a minimal such block is called finely settled.

The above framework shares many similarities with evolutionary game theoretical solution concepts. The canonical concept of evolutionary stability, due to Smith and Price (1973), is based on a single population of individuals recurrently playing a symmetric two-player game. All individuals in the population are pre-programmed to play a certain strategy and a strategy that is initially played by the whole population is called the incumbent strategy. Such a strategy is evolutionary stable if, for entering individuals who all play the same strategy, the incumbent strategy does better on average than does the entrant strategy, given that the population is mostly made up by the incumbents with the remaining share being the entrants.

Swinkels (1992b,a) extends the evolutionary stability concept to a set-valued notion that is defined for any finite normal-form game. In contrast with the previous concept, an equilibrium evolutionary stable, or EES, set concept uses a population of individuals for every player role. A strategy profile is robust against equilibrium entrants if there exist no entrants that uses a strategy that is a best reply to the new population mix after a small share of

[1]Here we mean minimality in the sense that there is no coarsely tenable block that is a strict subset of the block.

them already have entered. Loosely, a set of such strategy profiles is EES if the incumbent strategy profiles in the set lacks equilibrium entrant that can take the population out of the set.

The literature that has examined the properties of evolutionary stable strategies has found that such concepts imply behavior that are consistent with solution concepts based on strong ideas of rationality. For example, van Damme (1991) show that an evolutionary stable strategy is a proper Nash equilibrium (Myerson, 1978) and Swinkels (1992b) show that this is also true for an element in every EES set in the normal form of a generic extensive-form two-player game. In addition, Swinkels (1992a) show that EES sets with the property just mentioned also contain a strategic stable subset (Kohlberg and Mertens, 1986). Hauk and Hurkens (2002) show that EES sets are always consistent with forward induction whenever they exist. In contrast, while Myerson and Weibull (2015) show that every finely tenable block includes a proper equilibrium, and that every curb block (Basu and Weibull, 1991) is coarsely tenable, the properties of tenable blocks and settled equilibria are largely unexplored. For example, how does the notion of conventional blocks and evolutionary stable strategies relate to one another? Myerson and Weibull (2015) claim without a formal proof that the strategy profile generated by a symmetric coarsely tenable singleton block is neutrally stable.

To provide a formal link between tenable blocks and evolutionary stable strategies, we derive a sufficient condition for a block to be tenable. We do this by restricting the attention to so-called block games, that is, a game where the strategy space is restrict to strategies within a block. A block that includes all the pure best replies to all the Nash equilibria of the corresponding block game is called a Nash-curb block. Every such block is coarsely tenable. Due to the robustness requirements of the Nash equilibria of such blocks, these blocks tend to be big. For example, in two-player games, if a Nash-curb block only contains a pure Nash equilibrium, this equilibrium is necessarily strict. Still, the concept can be viewed as a less demanding version of curb blocks. We also establish that Nash equilibria with support in a minimal Nash-curb block generically differs from most established Nash refinements concept (although they coincide with settled equilibria).

The concept can also be given a behavioral micro foundation using the consideration-set game of Myerson and Weibull (2015). We show that in consideration-set games, a block is Nash-curb if and only if it has the property that all conventional individuals (i.e individuals only using strategies within the block) are strictly better-off than any individual using unconventional strategies, given that most players are conventional and that the population play constitutes a Nash equilibrium of the consideration-set game.

Regarding evolutionary interpretations of tenable blocks, we show that in symmetric two-player games, if a symmetric singleton block is coarsely tenable (Nash-curb) then the strategy profile in its support is neutrally (evolutionary) stable. We also investigate the link between EES sets and tenability. We show that in two-player games, every singleton EES set is in a Nash-curb block consisting of the support of the strategy profile. Hence, such a Nash equilibrium is both fully settled and regular. This shows that tenable and Nash-curb blocks can indeed be thought of as a generalization of conventions that are shaped by evolutionary forces. We show by a counterexample that this property, in contrary what is claimed by Swinkels (1992b), does not hold in games with more than two player (this is also noted by Matsui (1992)). In particular, we give an example of a pure Nash equilibrium that is in an EES set but is not a strict Nash equilibrium. In addition, we also show that this link breaks down, even in the normal form of a generic extensive-form two-player game, for Nash equilibrium components with more than one element. Although, we are able to show a somewhat stronger connection between EES sets and finely tenable blocks.

At last, we explore the connection between tenable blocks and other established block concepts. Moreover, we show that a Nash equilibrium being coarsely or finely settled do not imply that it is persistent (Kalai and Samet, 1984). We also show that it is not true that all minimal tenable and Nash-curb blocks are in minimal curb blocks.

The next section presents the notation used throughout the paper and the consideration-set game. Section 3 presents the Nash-curb concept. In section 4, we present definitions of concepts of evolutionary stability, as well as the results regarding the link between tenability and evolutionary stability. Section 5 explores the relationship between tenability and other block concepts. Finally, section 6 offers some concluding remarks.

2 Preliminaries

2.1 Notation

Denote a finite n-person normal-form game by the triplet $G = \langle N, S, u \rangle$ where $N = \{1, 2, \ldots, n\}$ is a finite number of players. For each player $i \in N$, $S_i = \{1, 2, \ldots, m_i\}$ is player i's finite and non-empty set of pure strategies. Let $S = \times_{i \in N} S_i$ be the set of pure strategy profiles. The combined payoff function u is defined as a mapping $u : S \to \mathbb{R}^n$, where $u_i(s) \in \mathbb{R}$ is player i's payoff when the pure-strategy profile $s = (s_1, \ldots, s_n) \in S$ is played, where s_i is a pure strategy for player i. The unit simplex of mixed strategies for player i is given by $\Delta(S_i) = \{x_i \in \mathbb{R}_+^{m_i} : \sum_{s_i \in S_i} x_i(s_i) = 1\}$ where $x_i(s_i)$ is the probability that strategy s_i is played given the mixed strategy x_i. Denote the mixed-strategy space by $\boxdot[S] = \times_{i \in N}\Delta(S_i)$.

The payoff function's domain is altered from S to $\boxdot[S]$ by letting $u_i(x) = \sum_{s \in S}[\prod_{j \in N} x_j(s_j)] \cdot u_i(s)$. Note that for convenience, we identify each pure strategy s_i with the degenerate mixed strategy that assigns probability 1 to the pure strategy s_i. Hence, we let $u_i(x_{-i}, s_i')$ be the utility from player i playing the degenerate mixed strategy s_i' while all other players play a mixed strategy according to $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. Denote the set of pure best-replies for player i by $\beta_i(x) = \{s_i' \in S_i : u_i(x_{-i}, s_i') \geq u_i(x_{-i}, s_i) \ \forall s_i \in S_i\}$ with $\beta(x) = \times_{i \in N}\beta_i(x)$. The set of pure strategies in the support of a mixed strategy x_i is denoted by $C_i(x_i) = \{s_i \in S_i : x_i(s_i) > 0\}$ with $C(x) = \times_{i \in N}C_i(x_i)$. For any set $X \subseteq \boxdot[S]$ we denote $\beta(X) = \cup_{x \in X}\beta(x)$ and $C(X) = \cup_{x \in X}C(x)$. A Nash equilibrium x is defined by $C(x) \subseteq \beta(x)$.

A block is any set $T = \times_{i \in N}T_i$ such that $\emptyset \neq T_i \subseteq S_i \ \forall i \in N$ and the associated block game is defined as $G^T = \langle N, T, u_{|T}\rangle$. In this notation, a block $T \subseteq S$ is curb ("closed under rational behavior") if $\beta(\boxdot[T]) \subseteq T$ and the block is minimal curb if $\beta(\boxdot[T]) = T$ (Basu and Weibull, 1991). We denote the set of all Nash equilibria of the block game G^T by $\boxdot[T]^{NE} = \{x \in \boxdot[T] : C(x) \subseteq \beta(x)\}$ with $\boxdot[S]^{NE}$ denoting the set of all Nash equilibria of the original game. The euclidean distance between $x, y \in \boxdot[S]$ is denoted by $D(x, y)$ and $D(y, X) = min_{x \in X}D(y, x)$ for a closed set $X \subset \boxdot[S]$. For any $\varepsilon > 0$, an ε-neighborhood of a set $X \subset \boxdot[S]$ is denoted $B_\varepsilon(X) = \{y : D(y, X) < \varepsilon\}$.

A symmetric two-player game is defined by $G = \langle\{1, 2\}, S, u\rangle$ with $S_1 = S_2$ and $u_2(s_1, s_2) = u_1(s_2, s_1)$ for all $(s_1, s_2) \in S$. Denote the set of pure strategies that the players have in common by $K = \{1, 2, \ldots, m\}$ and the corresponding mixed-strategy simplex by $\Delta = \{x \in \mathbb{R}^m_+ : \sum_{k \in K} x_k = 1\}$. We write $\pi(x, y)$ for the payoff to a player from playing a strategy $x \in \Delta$ when the other player plays a strategy $y \in \Delta$.

2.2 Consideration-set Games

Here we present the consideration-set game framework developed by Myerson and Weibull (2015). Give the finite game $G = \langle N, S, u\rangle$ a large population of individuals for each player role $i \in N$. One individual from each population is from time to time randomly drawn to play the game in her player role. Every individual is rational when it comes to strategic reasoning, but only considers a subset of the strategies available to her. This set, called the individual's consideration set, is equivalent to the individual's type. The type space for each player role $i \in N$ is given by $\Theta_i = \mathcal{C}(S_i)$ where $\mathcal{C}(S_i)$ is a collection of non-empty subsets $C_i \subseteq S_i$. Let μ_i define a probability distribution on $\mathcal{C}(S_i)$ where $\mu_i(C_i) \in [0, 1]$ is the probability that the individual drawn to play the role i is of the type $\theta_i = C_i \in \mathcal{C}(S_i)$. A vector $\mu = (\mu_1, \ldots, \mu_n) \in \times_{i \in N} \Delta(\mathcal{C}(S_i))$ is called a type distribution, and the draws from each population are statistically independent.

The consideration-set game is given by $G^\mu = \langle N, F, u^\mu \rangle$ and is defined by its type distribution μ. It is a game of incomplete information. A pure strategy for each player role $i \in N$ is given by a function $f_i : \mathcal{C}(S_i) \to S_i$ such that $f_i(C_i) \in C_i \ \forall C_i \in \mathcal{C}(S_i)$. The strategy set is given by $F_i \in F = \times_{i \in N} F_i$. Every mixed strategy $\tau_i \in \Delta(F_i)$ assigns each type $\theta_i \in \Theta_i$ a probability distribution over the strategies in her consideration set. The consideration-set game is connected to the underlying game because each mixed-strategy profile $\tau \in \boxdot[F]$ induces a corresponding mixed-strategy profile $\tau^\mu \in \boxdot[S]$ in G. The conditional probability distribution over the strategies in S_i, given that player $i \in N$ is of the type $\theta_i = C_i$, is denoted $\tau_{i|C_i}$. Hence, the probability that player $i \in N$ will use a pure strategy s_i given a strategy τ^μ induced by τ is

$$\tau_i^\mu(s_i) = \sum_{C_i \in \mathcal{C}(S_i)} \mu_i(C_i) \cdot \tau_{i|C_i}(s_i).$$

The payoff function in G is extended to the consideration-set game by defining $u_i^\mu : \boxdot[F] \to \mathbb{R}$ for all players $i \in N$ in G^μ with the expected payoff given by $u_i^\mu(\tau) = u_i(\tau^\mu)$. The consideration-set game is finite as G is finite, and it follows that every consideration-set game has a Nash equilibrium.

2.3 Tenable Blocks and Settled Equilibria

Two block concepts are introduced to capture the notion of a convention in recurrently played games. The first concept, called coarse tenability, define blocks that are characterized by an external stability to “unconventional individuals”, that is, individuals who considers strategies outside of the block in the consideration-set game. For a block to be such a convention, all the Nash equilibria of the consideration-set games, where almost all players only considers strategies within the block, have to give at least as high payoff to every “conventional individual” (who only considers the strategies within the block), compared to any unconventional individual. The following five definitions are due to Myerson and Weibull (2015).

Definition 1. A block T is *coarsely tenable* if there exists an $\varepsilon \in (0, 1)$ such that $T \cap \beta(\tau^\mu) \neq \emptyset$ for every type distribution μ with $\mu_i(T_i) > 1 - \varepsilon \ \forall i \in N$ and every Nash equilibrium τ of

G^μ.

To get internal stability, the attention is given to minimal such blocks (meaning that there is no proper subset of the block that is also coarsely tenable). The existence of a minimal coarsely tenable block in every game is guaranteed by the whole strategy set being trivially coarsely tenable.

Definition 2. A *coarsely settled equilibrium* is any Nash equilibrium of G that has support in some minimal coarsely tenable block.

A less demanding concept that captures the emphasis that traditional game theory puts on rationality is also developed. This concept is based on a more forgiving interpretation of a conventional block as the set of type distributions a block has to be robust against has been restricted to distributions that are "biased" towards more "rational" types.

Definition 3. For any block T and any $\varepsilon \in (0,1)$, a type distribution μ is $\varepsilon - proper$ on T if for every player $i \in N$

$$\begin{cases} (a) & \mu_i(T_i) > 1-\varepsilon, \\ (b) & \mu_i(C_i) > 0 \quad \forall C_i \in \mathcal{C}(S_i), \\ (c) & T_i \neq C_i \subset D_i \quad \Rightarrow \quad \mu_i(C_i) \leq \varepsilon \cdot \mu_i(D_i). \end{cases}$$

An ε-proper type distribution always includes unconventional players of any type. The $\varepsilon - proper$ type distribution's "rationality bias" stems from that unconventional types that consider more strategies, in terms of set inclusion, have to take up a larger share of the population.

Definition 4. A block T is *finely tenable* if there exists an $\varepsilon \in (0,1)$ such that $T \cap \beta(\tau^\mu) \neq \emptyset$ for every type distribution μ that is $\varepsilon - proper$ on T and every Nash equilibrium τ of G^μ.

As the set of type distributions that a finely tenable block has to be robust against constitute a subset of the set a coarsely counterpart has to be robust against, it follows that every coarsely tenable block is finely tenable. It follows from the above definition that every consideration-set game with an $\varepsilon - proper$ type distribution projects $\varepsilon - proper$ strategy profiles to the underlying game. Hence, every finely tenable block contains the support of a proper equilibrium.[2]

[2]For any $\varepsilon > 0$, a strategy profile x in the interior of $\boxdot[S]$ is ε-proper if $u_i(x_{-i}, s_i) < u_i(x_{-i}, s_i') \Rightarrow x_i(s_i) < \varepsilon \cdot x_i(s_i')$ for all $i \in N$ and all $s_i, s_i' \in S_i$. A proper equilibrium is any limit of some sequence of ε-proper strategy profiles (Myerson, 1978).

Definition 5. A *finely settled equilibrium* is any proper equilibrium that has support in some finely tenable block.

A Nasg equilibrium is *fully settled* if it is both coarsely and finely settled. Such an equilibrium is guaranteed to exist in every game. In the section below, we analyze two games to demonstrate the properties of tenable blocks and settled equilibria.

2.4 Games 1 and 2

	C	D
A	$1,1$	$0,0$
B	$0,0$	$1,1$

Game 1

	E	F	G	H
A	$1,1$	$1,1$	$2,-2$	$-2,2$
B	$1,1$	$1,1$	$-2,2$	$2,-2$
C	$-2,2$	$2,-2$	$1,1$	$1,1$
D	$2,-2$	$-2,2$	$1,1$	$1,1$

Game 2

Game 1 is a simple coordination game with three Nash equilibria, two pure and one completely mixed. The pure Nash equilibria have support in $T^1 = \{A\} \times \{C\}$ and $T^2 = \{B\} \times \{D\}$ respectively. Both blocks are coarsely and finely tenable. The totally mixed-strategy Nash equilibrium is rejected as its support is in $T^3 = \{A, B\} \times \{C, D\}$ which contains the other two blocks.

The second game, proposed by Myerson and Weibull (2015), is an extension of the first game where the extensive form of the game has both (A, C) and (B, D) replaced by a zero-sum subgame. The reduced normal form is given in Game 2 above. The set of Nash equilibria of this game consists of three components: $\Theta = \{(pA + (1-p)B, qE + (1-q)F) : p, q \in [1/4, 3/4]\}$, $\Omega = \{(pC + (1-p)D, qG + (1-q)H) : p, q \in [1/4, 3/4]\}$, and $\Gamma = \{x^*\}$, where x^* is the uniform randomization over each strategy set. The two blocks $T^1 = \{A, B\} \times \{E, F\}$ and $T^2 = \{C, D\} \times \{G, H\}$ are both minimal finely tenable and respectively include the support of the Θ and Ω component. The finely settled equilibria are hence the proper equilibria in the Θ and Ω components given by the strategy profiles $(0.5A + 0.5B, 0.5E + 0.5F) \in \Theta$ and $(0.5C + 0.5D, 0.5G + 0.5H) \in \Omega$. Thus, the concept rejects the x^* Nash equilibrium.

The whole strategy set S is the only minimal coarsely tenable block in the game. To see this, note that if all individuals in player role i in the consideration-set game only consider T_i^1 for all $i \in \{1, 2\}$ (i.e. $\mu(T_i) = 1, \forall i \in N$), then all strategy profiles with support in that block are Nash equilibria. Let τ^μ be the induced strategy profile by a Nash equilibrium $\tau \in \boxdot[T^1] \setminus \Omega$. It must have better replies outside of the block as it is not a Nash equilibrium in the original game. For all blocks $T' \subset S$, the consideration-set game with type distribution $\mu(T_i') = 1, \forall i \in N$ includes a Nash equilibrium that is not Nash in the original game. If for example, the strategy H is not in $T_2' = S_2 \backslash \{H\}$, then the strategy profile $y = (A, 0.5E{+}0.5F)$ is a Nash equilibrium in the consideration-set game where everyone is conventional and only consider strategies in $S_1 \times T_2'$. As $\{H\}$ is the unique best reply to y for player two, the block is not coarsely tenable. The remaining cases can be verified in the same manner.

We see that both finely settled equilibria are fully settled and that every Nash equilibrium of the game is coarsely settled. The question is whether coarsely or finely settled equilibrium make the most reasonable selection in this game. It is possible to argue that the features of the game does not change when the strategies in Game 1, giving zero to both players, are replaced by the zero-sum game, which is an argument in favor for the selection made by finely settled equilibrium. Myerson and Weibull (2015, Proposition 5 p.960) show that this is a generally feature of the concept; every finely settled equilibrium is invariant against the injection of a subgame with a unique totally mixed payoff-equivalent equilibrium.

3 Nash-curb Blocks and Tenability

Here we formalize a block notion that is based on the properties of the set of Nash equilibria of the corresponding block game. We show that equivalent robustness properties can be derived using the consideration-set games framework, hence the block can be interpret as a potential convention. A block T is called *Nash-curb* if all the strategy profiles in G that are Nash equilibria of the block game G^T have all their pure best replies in the block.

Definition 6. A block $T \subseteq S$ is Nash-curb if $\beta(\boxdot[T]^{NE}) \subseteq T$.

The first thing to note about such a block is that the set of Nash equilibria in the block game coincides with the set of Nash equilibria with support in the block in the original game. It is easy to see that every curb block is a Nash-curb block, but a more subtle observation is that every Nash-curb block is coarsely tenable. In finite games, $\beta(x)$ is always an entire sub-simplex of the mixed-strategy space implying that $\beta(X)$ is a union of such sub-simplexes. Hence, due to the upper-hemi continuity of the best-reply correspondence, the set of best replies to X in the payoff perturbed game $\tilde{G}$, denoted $\tilde{\beta}(X)$, is such that $\tilde{\beta}(X) \subseteq \beta(X)$. The consideration-set games can be viewed as a block game where the conventional block is the block game's strategy space and where unconventional individuals are interpreted as perturbations of the game's payoff function. As a Nash-curb block includes all the best replies to the corresponding block game's set of Nash equilibria, all the best replies to the set of Nash equilibria in the payoff perturbed block game are still going to be within the block. Hence, every Nash-curb block is coarsely tenable.

To give the concept a micro behavioral foundation, we show that a block is Nash-curb if and only if, in consideration-set games, the block is a potential convention such that if most individuals are conventional, they do better than any unconventional individual using strategies outside of the block, given that the population play constitutes a Nash equilibrium.

Proposition 1. A block T is *Nash-curb* if and only if there exists an $\varepsilon \in (0,1)$ such that $\beta(\tau^\mu) \subseteq T$ for every type distribution μ with $\mu_i(T_i) > 1-\varepsilon$ $\forall i \in N$ and every Nash equilibrium τ of G^μ.

Proof. ($\Longrightarrow$) The consideration-set games with $\mu_i(T_i) > 1-\varepsilon$ $\forall i \in N$ for an $\varepsilon \in (0,1)$ can be viewed as a block game G^T where the payoff function $u \in \mathbb{R}^{|N||T|}$ is perturbed by the ε-share of players for whom $C_i \neq T_i$. We define $\rho = \times_{i\in N}\rho_i$ with $\rho_i : \mathcal{C}(S_i)\backslash T_i \to \Delta(S_i)$ given by $\rho_{i|C_i} \in \Delta(C_i)$ and $\rho_i = \sum_{C_i \in \mathcal{C}(S_i)\backslash T_i} \mu_i(C_i)\cdot \rho_{i|C_i}$. The mixed-strategy profile ρ is interpreted as a perturbation of the payoff function. From the upper-hemi continuity of the best-reply correspondence, we can pick an $\varepsilon > 0$ small enough so that all Nash equilibria of the perturbed game are close to the Nash equilibria in the original block game. The best replies to such strategy profiles also have to be best replies to the Nash equilibria in the unperturbed block game. Hence, all induced strategy profiles from Nash equilibria of the consideration-set games with a small share of unconventional individuals have their best replies within the Nash-curb block.

($\Longleftarrow$) If $s \in \boxdot[T]^{NE}$ is such that for an $i \in N$, $\beta_i(s_i) \cap (S_i \setminus T_i) \neq \emptyset$ we have a contradiction as if we set $\mu_i(T_i) = 1$ $\forall i \in N$ then $s = \tau^\mu$ where τ^μ is induced from a Nash equilibrium τ in the consideration-set game. $\square$

Thus, the Nash-curb concept is a more demanding version of coarse tenability as unconventional individuals have to be strictly worse off compared to conventional individuals. As with most block concepts, we restrict attention to minimal Nash-curb blocks. In Game 4 below, we will show that, unlike a curb block T that is minimal if and only if $\beta(\boxdot[T]) = T$, a minimal Nash-curb block T' is sometimes such that $\beta(\boxdot[T']^{NE}) \subset T'$.

The whole strategy space of a game is trivially a Nash-curb block implying existence of such a block in every finite game. As shown by Myerson and Weibull (2015), every absorbing block[3] is coarsely tenable, but we show in Game 3 below that this does not imply that the block is Nash-curb. Similar to the curb concept, either a whole Nash equilibrium component is in a Nash-curb block, or the whole component is outside of the block. In the game below, we give two examples where coarsely tenable and Nash-curb blocks differ.

3.1 Games 3 and 4

	D	E
A	$2,2$	$0,0$
B	$2,0$	$0,2$
C	$0,0$	$3,3$

Game 3

Both games have nongeneric features and illustrates the difference between Nash-curb and coarsely tenable blocks. In Game 3, the unique minimal Nash-curb block is $T^1 = \{C\} \times \{E\}$ whereas the two minimal coarsely tenable blocks are T^1 and $T^2 = \{A\} \times \{D\}$. The game has two Nash equilibrium components. The first one is the pure Nash equilibrium with support in T^1, and the second is the component Θ that has support in the whole strategy set and includes the pure strategy profile with support in T^2. The Θ component includes a proper Nash equilibrium with support in T^2.[4] If T^2 is viewed as convention, there is no drawback to be an unconventional individual in player role one that only considers strategy B. The reason is that such individuals will also get a payoff of 2 if individuals in player role two only considers strategy D. If the share of individuals only considering strategy B becomes higher than 1/2, the best reply to the induced strategy profile for individuals in player role two is to play strategy E. Hence, the block T^2 could arguably be deemed to be an unstable convention.

[3] A block $T \subseteq S$ is absorbing if $X = \boxdot[T]$ such that $\beta(B_{\bar{\varepsilon}}(x)) \cap T \neq \emptyset$ for all $x \in X$ for an $\bar{\varepsilon} > 0$ (Kalai and Samet, 1984).

[4] It is interesting to note that the component is not essential. A set of Nash equilibria is essential if for every slightly payoff perturbed game, the set has an equilibrium close to the component (Kohlberg and Mertens, 1986; Wu and Jiang, 1962). Also, notice that T^2 is an absorbing block.

	E	F	G
A	$0,2$	$2,2$	$0,0$
B	$2,0$	$2,1$	$1,1$
C	$3,0$	$0,0$	$1,-1$
D	$4,0$	$0,5$	$0,0$

Game 4

Game 4 has a unique Nash equilibrium component which is the L-shaped component where player one is indifferent between strategy A and B when player two plays F, and when player two is indifferent between strategy F and G when player one plays B. Here the only minimal Nash-curb block is the whole strategy set. The strategy D is not a best reply to any Nash equilibrium in the game, but still needs to be in the block as the block game $T^1 = \{A, B, C\} \times S_2$ includes the Nash equilibrium component $\{(C, pE + (1-p)F) : p \in [2/3, 1]\}$. The unique minimal coarsely tenable block of this game is $\{B\} \times \{F\}$. This shows that a block T with the property that $X = \boxdot[T] \cap \boxdot[S]^{NE}$ and $\beta(X) \subseteq T$ not necessarily is Nash-curb. For example, the block T^1 has this property but is neither coarsely tenable nor Nash-curb. One advantage with Nash-curb blocks is that the properties of the Nash equilibria of the Nash-curb block game can, in most cases, be analyzed without paying any attention to the original game. Although this property, as we have seen, implies that minimal such blocks tend to be large.

4 Evolutionary Stability

4.1 Evolutionary Stability Concepts

Another concept, that just like Nash's mass-action interpretation uses a multi-population framework, stems from evolutionary game theory. Swinkels' 1992b set-valued solution concept, called Equilibrium Evolutionary Stability, is defined for any finite normal-form game. Just like in the consideration-set game, from time to time, n individuals, one for each player population, are randomly drawn to play the game against each other. Every individual is "pre-programmed" to play a certain mixed strategy when playing the game. The distribution of individuals playing a certain strategy in every population determines what can be seen as a mixed strategy profile that an individual drawn to play the game will face.

Assume that initially, all individuals in the same player role i play the same strategy $x_i \in \Delta(S_i)$. Imagine that a small share of individuals enter every player role, where all entering individuals in the same player role plays the same strategy $y_i \in \Delta(S_i)$ (not necessarily different from x_i for each $i \in N$). Hence, the new population strategy profile, $z = (1-\varepsilon)x + \varepsilon y$, consists of a convex combination of mostly the incumbent strategy profile but also a small share, $\varepsilon > 0$, of the entrant strategy profile. The incumbent strategy profile is robust against equilibrium entrants if there exists no entrants using a strategy profile $y \neq x$ that is a best reply to the new population strategy profile z.

This robustness criterion is based on the notion that the individuals are endowed with some form of rationality in the sense that they can foresee how well they will do in a certain population. Hence, individuals would never enter a population if they would not do as well as any other individual in the new environment. This concept is extended to sets of incumbents; a set of strategy profiles is equilibrium evolutionary stable, or EES, if the set is a minimal set of Nash equilibria such that no small share of equilibrium entrants can take the population out of the set. That is, there exists no incumbent strategy profile in the set that has an equilibrium entrant strategy profile that, when a small share of equilibrium entrants has entered, the new population strategy profile is not in the set.

Definition 7. A non-empty and closed set $X \subseteq \boxdot[S]^{NE}$ is an *equilibrium evolutionary stable set* (EES set) if it is minimal with respect to the following property: There is an $\bar{\varepsilon} \in (0,1)$ such that if $x \in X$, $C(y) \subseteq \beta[(1-\varepsilon)x + \varepsilon y]$ and $\varepsilon \in (0, \bar{\varepsilon})$, then $(1-\varepsilon)x + \varepsilon y \in X$.

Swinkels (1992b, Theorem 3) shows that every EES set is a Nash equilibrium component. Moreover, if an EES set is a singleton or is of the normal form of a generic extensive-form two-player game, it contains a proper equilibrium and a strategically stable subset (Swinkels 1992b Theorem 5, and Swinkels 1992a Theorem 1). A drawback with the concept is that many games lack EES sets.

The canonical concept of evolutionary game theory was developed by Smith and Price (1973) and Maynard Smith (1982). The setting is similar to the EES concept, but the framework is less involved. The evolutionary stable strategy concept, or ESS, is defined for finite and symmetric two-player normal-form games. In this setting, recurrently, two individuals from a single large population are randomly matched to play the game against each other.

Here the incumbents consists of a single population of individuals programmed to play the same strategy. Correspondingly, the entrants are individuals programmed to play a strategy that differs from the incumbents'. The incumbent strategy is evolutionary stable if it does better than any entrant strategy, when the population consists mostly of the incumbents with the remaining share consisting of the entrants.

Definition 8. $x \in \Delta$ is a *evolutionary stable strategy* (ESS) if for every $y \in \Delta$ there exist some $\bar{\varepsilon} \in (0, 1)$ such that

$$\pi[x, (1 - \varepsilon)x + \varepsilon y] > \pi[y, (1 - \varepsilon)x + \varepsilon y]$$

holds for all $\varepsilon \in (0, \bar{\varepsilon})$.

A weaker version of evolutionary stability is that of neutral stability. A strategy $x \in \Delta$ is a *neutrally stable*, or NSS, if Definition 8 holds with a weak inequality. A necessary condition for a strategy profile to be NSS is for it to be a symmetric Nash equilibrium.

4.2 Games 1 and 2 Revisited

In Game 1 above, the three evolutionary concepts just mentioned coincide. The two pure-strategy Nash equilibria are both NSS, ESS, and singleton EES sets. Thus, all three concepts reject the totally mixed-strategy profile. Due to the single-population framework of evolutionary stable strategies, we get the opposite result if we switch the order of player two's strategies. In the new game, the totally mixed strategy is the only ESS and NSS whereas the predictions of the other concepts we have mentioned stays the same. A framework that uses a different population for each player role allows the individuals in each player role to specialize on different strategies, whereas in a single population framework, the individuals cannot determine in which position they are going to play. Thus, asymmetric Nash equilibria can never be evolutionary stable.

Game 2 above has two EES sets, Θ and Ω. To determine why, we only need to look at the strategy profiles on the boundary of each component (as they are the only strategy profiles with best replies outside of the support of the set). For example, look at the boundary element $\alpha = (\frac{1}{4}A + \frac{3}{4}B, \frac{1}{4}E + \frac{3}{4}F)$ in Θ. As we will see below, the only relevant strategies in this situation are the best replies to α. The block of such best replies is $\beta(\alpha) = \{A, B, D\} \times \{E, F, G\}$. Looking at the block game $G^{\beta(\alpha)}$, all the Nash equilibria of the game are in the component $\Lambda = \{(pA + (1 - p)B, qE + (1 - q)F) : p, q \in [1/4, 1]\}$. Examining all the convex combinations of α and elements in Λ sufficiently close to α, we see that all are inside of Θ. Hence, all equilibrium entrants are already in Θ. Repeating this exercise for all elements on the boundary of Θ leads us to conclude that Θ is an EES set.

All the symmetric elements in Θ and Ω are NSS. To see this, notice that no small share of entrants can get a higher payoff than one. In this game there exists no ESS. The reason is that all symmetric Nash equilibria, except the totally mixed, are in a symmetric Nash equilibrium component with more than one element. As all the strategies in the game are best replies to the totally mixed Nash equilibrium, it cannot be an ESS.

4.3 Characterization of EES sets

We here discuss a related property of closed sets of Nash equilibria that is similar to the property in Definition 7. For convenience, we call the property inherited by an EES set property (P). A closed set of Nash equilibria has property (P') if $y \in \boxdot[\beta(x)]^{NE}$ for a $x \in X$, then there exists an $\bar{\varepsilon} \in (0,1)$ such that $(1-\varepsilon)x + \varepsilon y \in X$ for all $\varepsilon \in (0, \bar{\varepsilon})$. Swinkels (1992b) claims that every EES set has property (P'). Matsui (1992) shows that this is not true for games with more than two players. The claim is indeed true for the special case of two-player games.

Lemma 2. In a two-player game, a minimal closed set of Nash equilibria X has property (P') if it has property (P).

Proof. If X is an EES set and $y \in \boxdot[\beta(x)]^{NE}$ for a $x \in X$ then $(1-\bar{\varepsilon})u_i(x_{-i}, y_i) + \bar{\varepsilon}u_i(y_{-i}, y_i) \geq (1-\bar{\varepsilon})u_i(x_{-i}, x_i') + \bar{\varepsilon}u_i(y_{-i}, x_i')$ for all $x_i' \in \Delta(S_i)$, all $i \in \{1,2\}$, and a small $\bar{\varepsilon} \in (0,1)$. This implies that $C(y) \subseteq \beta((1-\varepsilon)x + \varepsilon y)$ for all $\varepsilon \in (0, \bar{\varepsilon})$ as y is a best reply to x, and among the strategies that are best replies against x, y is a best reply to itself. □

This relationship goes in the other direction for a singleton EES set or a EES set of the normal form of a generic extensive-form two-player game.

Lemma 3. If either a minimal closed set of Nash equilibria X with property (P) is a singleton or is of the normal form of a generic extensive-form two-player game, then it has property (P').

Proof. As shown by Swinkels (1992b, Lemmas 7-10), for a EES set X let $z \in B_{\bar{\varepsilon}}(X)$ for a small $\bar{\varepsilon} \in (0,1)$ where $x \in X$ and $y \in \boxdot[S]$, then $C(y) \subset \beta(z) \implies z \in X$. If $y \in \boxdot[\beta(x)]^{NE}$ for a $x \in X$, then y is both a best reply to itself and x. Hence for a small $\delta > 0$, $C(y) \subseteq \beta(z)$ for all $(1-\delta)x + \delta y = z$ implying that $(1-\varepsilon)x + \varepsilon y \in X$ for all $\varepsilon \in (0, min\{\bar{\varepsilon}, \delta\})$.

□

This shows that in many two-player games, the two properties coincide. This is important as it shows an intimate relationship between EES sets in two-player games, and the properties of the block of best replies to the elements in the set. It also links the concept with a dynamic process leading to the modified concept of *cyclically stable sets*, or CSS (Matsui, 1992), originally developed by Gilboa and Matsui (1991). In the upcoming section we will introduce another counterexample showing that this does not hold for $n \geq 3$ player games.

4.4 Game 5

Swinkels (1992b, Lemma 4) claim that every pure EES set is a strict Nash equilibrium. The game below, which is a generalization of a game due to van Damme (1991, Fig 3.4.1.), shows that this is false. It also highlight a problematic selection by many solution concept that many block concepts solve.

E

	C	D
A	$0,0,0$	$2,0,0$
B	$0,0,2$	$0,2,0$

F

	C	D
A	$0,1,0$	$0,0,1$
B	$1,0,0$	$\lambda_1,\lambda_2,\lambda_3$

Game 5

Game 5 has up to three Nash equilibria depending on the value of $\lambda_i \in \mathbb{R}, i \in \{1,2,3\}$. For example if $\lambda_i = 3\ \forall i \in \{1,2,3\}$, there are three Nash equilibria, two of which are pure and one completely mixed. They are $x = (A, C, E)$, $x' = (B, D, F)$ which is strict, and the completely mixed strategy profile $x'' = (\frac{15}{16}A + \frac{1}{16}B, \frac{9}{10}C + \frac{1}{10}D, \frac{6}{7}E + \frac{1}{7}F)$.

Obviously, the pure Nash equilibrium x always exists independent of the value of the payoff from (B, D, F). We claim that x is a singleton EES set for all values of the λ-constants. The block of best replies to x is clearly the whole game, hence this game also contradicts the claim that a singleton EES set is the only Nash equilibrium in the block game of its best replies, that is, it does not have property (P') (see Swinkels 1992b, p.317).

To see this, first note that all players have two strategies making it possible to represent the mixed-strategy space by the unit cube. We give the (A, C, E)-equilibrium the coordinates $x = (1,1,1)$ and the (B, D, F) is defined as the origin $x' = (0,0,0)$. Thus, the mixed-strategy space is given by $\boxdot[S] = [0,1]^3$. For a strategy profile y to be an equilibrium entrant for an $\bar{\varepsilon} \in (0,1)$, it has to be the best reply to the strategy profile $z = (1-\varepsilon)x + \varepsilon y$ for all players and for all $\varepsilon \in (0, \bar{\varepsilon})$. As player i's unit-simplex of mixed strategies is given by $\Delta(S_i) = [0,1]$, the scalar $u_i = 1 - \varepsilon(1 - y_i)$ can be interpreted as the probability that player i will play strategy x_i and $1 - u_i = \varepsilon(1 - y_i)$ the probability that she will play strategy x_i'.

We are going to show that there exists no equilibrium entrant to x by first showing that a strategy profile $y \in [0, 1)^3$ cannot be a best reply to z when ε is small (note that $y_i = 0 = x_i'$). After that we argue why the remaining strategy profiles are not equilibrium entrants.

We start by deriving the condition for having x_i' as a best reply to z (i.e the cases when $y \in [0, 1)^3$ might be a best reply).

- x_1' is a best reply to z if $2u_3(1 - u_2) \leq u_2(1 - u_3) + \lambda_1(1 - u_2)(1 - u_3)$
- x_2' is a best reply to z if $u_1(1 - u_3) \leq 2u_3(1 - u_1) + \lambda_2(1 - u_1)(1 - u_3)$
- x_3' is a best reply to z if $2u_2(1 - u_1) \leq u_1(1 - u_2) + \lambda_3(1 - u_1)(1 - u_2)$

Hence, we have three inequalities that have to hold for all $\varepsilon \in (0, \bar{\varepsilon})$. If we combine the inequalities for player one and two, we get

$$2u_3(1-u_2) \leq (u_2+\lambda_1(1-u_2))(1-u_3) \leq \frac{u_2 + \lambda_1(1 - u_2)}{u_1}[(2u_3+\lambda_2(1-u_3))(1-u_1)]. \quad (1)$$

Combining this with the inequality from player three, we get

$$2u_3(1 - u_2) \leq \frac{(u_2 + \lambda_1(1 - u_2))(2u_3 + \lambda_2(1 - u_3))}{2u_1u_2}[(u_1 + \lambda_3(1 - u_1))(1 - u_2)]. \quad (2)$$

Thus we have

$$4u_1u_2u_3 \leq [u_2 + \lambda_1(1 - u_2)][2u_3 + \lambda_2(1 - u_3)][u_1 + \lambda_3(1 - u_1)]. \quad (3)$$

For a small $\varepsilon > 0$, then u_i is close to 1 implying that the left-hand side of (3) approaches 4 whereas the right-hand side approaches 2. This shows that no such y can invade x.

The cases we have left to check is when one or more players use strategy x_i with probability 1 (i.e. $u_i = 1$). If for example $u_3 = 1$, then according to Player one's optimality constraint $2(1 - u_2) \leq 0$ if player one is prefer to play a strategy with $y_1 \in [0, 1)$. This in turn implies that $0 \geq (1 - u_1)$ for strategy x_2 to be a best reply for player two, hence this only holds for $x = y$. Thus, if one player plays x_i and the other player plays x_j' with positive probability, x_k' is the only best reply for the last player. If both $u_1 = 1$ and $u_3 = 1$, then player three's optimality constraint implies that u_2 has to be equal to 1. Going through all the cases we end up a the same conclusion, we have thus covered all strategy profiles in

$y \in [0.1]^3$ that is not equal to x and have shown for a small $\varepsilon > 0$ none can enter x. This leads us to conclude that x is an EES set.

Due to the independence between the payoff of the (B, D, F) outcome and x being an EES set, we note that what is important for a set to be EES is not what happens in the block game of best replies, it is rather the payoff to the other players when one player deviates to a best reply (these notions coincide in two-player games). The reason is that, when drawn to play the game, the probability for an individual to run into two or more equilibrium entrants is very small compared to most other interactions when the share of entrants is small.

The game highlights a problem with most solution concepts in games with more than two players. Due to conflicting interest of the other players, seemingly unreasonable Nash equilibria might selected by very demanding solution concepts. For example, if $\lambda_i > 0\ \forall i \in \{1, 2, 3\}$, the Nash equilibrium x is the only strategy profile in the whole mixed-strategy space that gives all players their minimal payoff. In contrast, many block concepts, like coarsely tenable and curb blocks all selects the now strict Nash equilibrium x'. On the other hand, if $\lambda_i = 0$ for all players, the game only contains the two pure Nash equilibria x and x'. In that case, the x equilibrium seems like the better choice of the two. In that game, the only coarsely tenable and Nash-curb block is the whole strategy space. It is possible to argue that none of the Nash equilibria in that game is very focal, hence the failure to uniquely select one of them might not be a drawback of the concepts. In the upcoming section, we will derive relationships between tenable blocks and evolutionary stable strategies. This includes showing that the block $\{A\} \times \{B\} \times \{C\}$ is finely tenable.

5 Evolutionary Interpretations of Tenability

As highlighted by Myerson and Weibull (2015), tenable blocks can be given evolutionary interpretations. Think of a game where there are large populations of individuals in each player role i. Every once in a while, an individual from every player role are picked at random to subsequently play the game against each other. Assume that all individuals initially only consider the strategies in a block T of the game. Then we can think of entrants as a small share of individuals for every player role $i \in N$ playing a strategy equivalent to x. These entrants is equivalent to a strategy profile x induced by a set of unconventional individuals with type $C_i \in \mathcal{C}(S_i) \setminus T_i$.

Viewed from an evolutionary perspective, the requirement for a block to be coarsely tenable can be interpreted as no entrants can fare better than the incumbents in the same player role, assuming the incumbents responds rationally given their consideration sets. Observe that there is no requirement that the entrants act rationally.[5] The evolutionary interpretation of Nash-curb blocks relates to coarsely tenable blocks as ESS relates to NSS; it requires the incumbents do better than any entrants.

Similarly to coarse tenability, the requirement for a block to be finely tenable could correspondingly be viewed as no entrants that are "ε-proper", in any player role i, can fare better than the incumbent, if the incumbent respond rationally given its consideration sets. An ε-proper entrant is defined as entrants programmed to play a strategy profile that is ε-proper given the new population mix.[6] The requirement is similar to the requirement of robustness against equilibrium entrants that an EES set is endowed with. In the upcoming sections we provide formal results regarding concepts of evolutionary stability and tenability.

5.1 Evolutionary Stability and Tenability

We start by proving Myerson and Weibull's (2015) claim that in every symmetric two-player game, if a symmetric singleton block is coarsely tenable, then the strategy with support in the block is a NSS.

Proposition 4. In a finite and symmetric two-player game, if a block $T = \{t\}^2$ is coarsely tenable, then $t \in \Delta$ is a neutrally stable strategy.

Proof. Let G be a symmetric two-player game and let $\{t\}^2 \subset S$ be a coarsely tenable singleton block. Then by Definition 1, there exists an $\bar{\varepsilon} \in (0,1)$ such that

$$u_i(\tau_j^\mu, t) = \max_{k \in K} u_i(\tau_j^\mu, k) \; \forall i, j \in \{1,2\}, i \neq j \tag{4}$$

for every type distribution μ with $\mu_i(\{t\}) > 1 - \bar{\varepsilon}$ $\forall i \in \{1,2\}$ and every Nash equilibrium τ of G^μ. This implies that (4) holds for every type distribution with $\mu_i(\{t\}) = 1 - \varepsilon$ for all $\varepsilon \in [0, \bar{\varepsilon})$ and for all $i \in \{1,2\}$. As the consideration set for an individual with $C_i \neq \{t\}$ can be any set $C_i \in \mathcal{C}(K) \setminus \{t\}$ depending on the type distribution, we know that

[5] As with any given type distribution $\mu_i(T) > 1 - \bar{\varepsilon}$, all the possible strategies τ^μ induced by an equilibrium τ is of the form $\tau_i^\mu = (1-\varepsilon)t_i + \varepsilon x_i$ for all $i \in N$ where $\varepsilon \in (0, \bar{\varepsilon})$, $t_i \in \Delta(T_i)$, and $x_i \in \Delta(S_i)$. Whether a best reply to τ_i^μ is in T_i does not depend on the combination of types that induced x_i, what matters is the "expected" strategy's position in the unit simplex $\Delta(S_i)$.

[6] Myerson and Weibull (2015) show that every induced strategy τ^μ from an ε-proper type distribution is an ε-proper strategy profile if T is a finely tenable block (Proposition 2). If we remove the assumption that T is finely tenable, the ε-fraction of unconventional individuals still induce an ε-proper strategy profile, but the combined strategy profile, including the strategies used by conventional individuals, might not be ε-proper.

in particular, (4) holds for all $\tau_j^\mu = \tau_i^\mu$ of the form $\tau_j^\mu = (1-\varepsilon)t + \varepsilon x$ for any $x \in \Delta$. Using this observation, if we rewrite (4) in the notation of symmetric two-player games, we get $\pi[t, (1-\varepsilon)t + \varepsilon x] \geq \pi[k, (1-\varepsilon)t + \varepsilon x]\ \forall k \in K,\ \forall x \in \Delta,\ \forall \varepsilon \in (0, \bar{\varepsilon})$. It is immediate that the inequality implies that t is a neutrally stable strategy.

□

The intuition behind the result is that in a singleton coarsely tenable block, the conventional individuals are precisely as rational as any individual that is pre-programmed to play the strategy in support of the block. Hence, as described above, the robustness properties of the two concepts are virtually identical with the exception that NSS only have to be stable against symmetric entrants. We note that Proposition 4 has a slightly stronger corresponding result for Nash-curb blocks. In a symmetric two-player game, if a symmetric Nash-curb block T only includes a pure Nash equilibrium, then the equilibrium is an ESS. This follows from the fact that if a pure Nash equilibrium is alone in a Nash-curb block in such a game, then it is necessarily a strict equilibrium (see Proposition 5 below), and the Nash equilibrium in the block is necessarily symmetric as all symmetric games include a symmetric Nash equilibrium, hence all symmetric block games do too.

The relationship between coarsely tenable and Nash-curb blocks can be used to link tenability with EES sets. In two-player games, a singleton EES set's support is a minimal Nash-curb block. Hence every such Nash equilibrium is fully settled and regular.[7]

Proposition 5. In a two-player game, a singleton EES set $\{x\}$ is in a Nash-curb block $C(x)$ and is a regular equilibrium.

Proof. Assume that $\{x\}$ is an EES set, then from Lemma 2 we have that x is a Nash equilibrium and x is the only Nash equilibrium of the block game $T = \beta(x)$, otherwise x would not be a singleton. Thus, $T = \beta(x)$ must be a Nash-curb block. As x is the only Nash equilibrium of the block game T, it is an Essential equilibrium as all games contains an Essential equilibrium component (Kohlberg and Mertens, 1986). This component must be essential in the original game too. An isolated essential equilibrium in a bimatrix game is regular (van Damme (1991) Theorem 3.4.4) and such equilibrium is quasi-strict, hence $C(x) = \beta(x)$. □

The intuition behind this result is that if a singleton Nash equilibrium do not have any equilibrium entrants, there are no strategy profiles that are a best reply to the equilibrium

[7]In a two-player game, every quasi-strict (i.e. $\beta(x) = C(x)$ for a Nash equilibrium x) and isolated Nash equilibrium is regular (van Damme, 1991, Corollary 3.4.2).

that conventional individuals would ever settled down on.

As most nontrivial extensive-form games generates a nongeneric normal form, we also need to consider Nash equilibrium components that include more than one strategy profile. A straightforward inquiry is: If X is an EES set with more than one element, can we say something about the tenability of the support or best replies of X? Below we provide a negative answer to the question by giving an example of a game where the support of an EES set is not in a minimal tenable nor Nash-curb block.

5.2 Example 6

In Game 6 below, we will show that there exist convex EES sets in two-player games (implying that they contain a strategic stable subset) that include no coarsely or finely settled equilibrium. Game 6 is Game 2 with the strategies $\mathcal{E}$ and $\mathcal{F}$ added for player one and $\mathcal{I}$ and $\mathcal{J}$ added for player two. Although the game now contains nine Nash equilibrium components, we are only going to focus on two. The components we are interested in are $\Theta = \{(pA + (1-p)B, qE + (1-q)F) : p, q \in [1/4, 3/4]\}$, which is the same component as in the old game, and the strict Nash equilibrium $\sigma = (\mathcal{F}, \mathcal{J})$. The Θ component is still an EES set as the argument for Game 2 still applies. We will show that the block $T^* = \{A, B\} \times \{E, F\}$, that contains the support of the Θ component, is neither in any minimal tenable nor Nash-curb block.

	E	F	G	H	$\mathcal{I}$	$\mathcal{J}$
A	$1,1$	$1,1$	$-2,2$	$2,-2$	$0,3$	$-4,-8$
B	$1,1$	$1,1$	$2,-2$	$-2,2$	$-4,-8$	$0,3$
C	$2,-2$	$-2,2$	$1,1$	$1,1$	$5,5$	$5,5$
D	$-2,2$	$2,-2$	$1,1$	$1,1$	$5,5$	$5,5$
$\mathcal{E}$	$3,0$	$-8,-4$	$5,5$	$5,5$	$6,6$	$7,7$
$\mathcal{F}$	$-8,-4$	$3,0$	$5,5$	$5,5$	$7,7$	$8,8$

Game 6

A Nash equilibrium in a consideration-set game with an ε-proper type distribution on T always induce a strategy profile of the form

$$\tau = (1 - \varepsilon - \varepsilon^2 K_S)x + \varepsilon y + \varepsilon^2 K_S z,$$

where $x \in \boxdot[T]$, $C(y) \subseteq \beta(\tau)$, and z is in the interior of $\boxdot[S]$. The reason is that the share of individuals that considers the whole strategy space, here represented by the strategy profile y, must be $1/\varepsilon$-times larger than any other share of types. In addition, every type always has positive probability to exist which implies that the induced strategy profile has to be completely mixed. The constant K_S depends on the strategy set of each player and is bounded.[8]

We argue that in the consideration-set game with an ε-proper type distribution on T^*, a Nash equilibrium is given by

$$\tau = (1 - \varepsilon - \varepsilon^2 K_S)(A, E) + \varepsilon(\mathcal{E}, \mathcal{I}) + \varepsilon^2 K_S z.$$

For a small $\varepsilon > 0$, the conventional individuals in player one's position have no incentive to deviate as strategy A gives the payoff $(1 - \varepsilon - \varepsilon^2 K_S) + 3\varepsilon + \varepsilon^2 K_S z_A$ which is clearly larger than $(1 - \varepsilon - \varepsilon^2 K_S) - 4\varepsilon + \varepsilon^2 K_S z_B$ which is given by playing B. For the fully rational individuals in player role one, the unique best reply to the conventional individuals' strategy is $\mathcal{E}$. By symmetry, the same argument works for the individuals in player two's role. As the best replies to the induced strategy profile is outside of the block, the block cannot be coarsely or finely tenable.

In the consideration-set game with an ε-proper type distribution on $\{A, B, C, D\} \times \{E, F\}$, the argument is similar to the previous one. The difference is that the conventional individuals in player role two cannot play a pure strategy as then C or D is a best reply for the conventional individuals in player role one. We argue that

$$\tau = (1 - \varepsilon - \varepsilon^2 K_S)[A, E(1 - q) + Fq] + \varepsilon[A, \mathcal{I}] + \varepsilon^2 K_S z$$

for some $q \in (1/4 - \varepsilon, 3/4 + \varepsilon)$ is a Nash equilibrium in the consideration-set game. All conventional individuals in player role two play a mixed strategy and the fully rational individuals play $\mathcal{I}$, therefore conventional and fully rational individuals in player role one prefers A. The few unconventional individuals left determine the size of q.

The arguments above work for all consideration-set games where the block of interest is $T \subset T^{old} = \{A, B, C, D\} \times \{E, F, G, H\}$. There is always going to exist a Nash equilibrium were some of the players considering the whole strategy space will play one of the new

[8]The weight K_S can be seen as the maximum weight given to strategies by unconventional individuals who do not consider her entire strategy set. It is defined by: $K_S \equiv \underset{i \in N}{max} \sum_{j=1}^{|\mathcal{C}(S_i)|} \varepsilon^{j-1} L_{S_i,j}$. The constant $L_{S_i,j}$ depends linearly on the number of strategies for each player i. As the game is finite the sum above is clearly bounded.

strategies. Hence, the best replies to the induced strategy is outside the block for at least one of the players. The block T^{old} can clearly not be finely tenable because of the Nash equilibrium

$$\tau = (1-\varepsilon-\varepsilon^2 K_S)[C(1-p)+Dp, G(1-q)+Hq]+\varepsilon[\mathcal{E}(1-r)+r\mathcal{F}, \mathcal{I}(1-s)+\mathcal{J}s]+\varepsilon^2 K_S z$$

for some $p, q \in [1/4, 3/4]$. Hence, if a block that includes $T^* = \{A, B\} \times \{E, F\}$ is to be coarsely or finely tenable it has to include both $\mathcal{F}$ and $\mathcal{J}$, but as $T^\sigma = \{\mathcal{F}\} \times \{\mathcal{J}\}$ is a minimal coarsely and finely tenable block, no minimal such block exists.

Notice that the game is symmetric so it is possible to apply the evolutionary stable strategy concepts. All the symmetric strategy profiles in the EES set are neutrally stable as well as the σ equilibrium which is also an ESS. As in the earlier examples, this is not a robust feature of an ESS, as if we switch the order of player two's strategies we have a symmetric game with two ESS strategies, the strict Nash equilibrium and an equilibrium that is rejected by the tenability and Nash-curb concepts.

It is clear that it is not the difference in stability properties that is driving the result in the game above, but rather that these properties are assigned to different objects (blocks versus subsets of the mixed-strategy space). What is not clear is which of the two interpretations, evolutionary stability or tenability, selects the most reasonable Nash equilibria in this situation. The answer undoubtedly depends on which context the game is viewed in.

An argument against the evolutionary framework is that the concept does not take into account the finiteness of the normal-form game being analyzed; when all individuals are pre-programmed to play a certain mixed strategy, what is being analyzed is arguably the mixed extension of the finite game, not the finite game itself. The argument is based on the following reasoning. In the game above, if we assume that the individuals playing the strategies within the Θ component can choose between the strategies in the block, if there is a period of time without any entrants, the individuals have no incentives to not play any of the pure strategy profiles within the block. In such a situation, there is no reason as to why individuals playing any other strategies would not be able to enter the population.

A rebuttal to this reasoning is to argue that the individuals are not rational enough to be able to choose the strategies available to them. Hence, they pick at random with their strategy being the expected strategy from this random choice. This line of reasoning is hard to combine with the EES concept. The concept extends the rationality of the individuals in the evolutionary setting by a small degree, but the direction of the rationality increase seem to misdirected. Before giving the individuals the ability to predict the outcome of entering a population or communicate with each other, which is two of the arguments Swinkels (1992b) uses to justify his concept, a more natural extension is to give the individuals the ability to slightly change their behavior in the game. Such an extension could lead towards a notion similar to the tenability.

5.3 Blocks Robust Against Equilibrium Entrants

Despite the negative result above, it is still possible to draw some connections between non-singleton EES sets and tenability. We will prove a more general result using an alternative notion of a stable block that highlights its connection with EES sets. Swinkels (1992a) calls a closed set uniformly robust to equilibrium entrant, or UREE, if it has the following property: There is an neighborhood $B_{\bar{\varepsilon}}(X)$ such that if $x \in X$, $z = (1-\varepsilon)x + \varepsilon y$, $C(y) \subseteq \beta(z)$, and $z \in B_{\bar{\varepsilon}}$, then $z \in X$. Swinkels (1992a) shows that if an EES set is a singleton or a EES set of the normal form of a generic extensive-form two-player game, then it is an UREE set. We show that, in block form, an UREE set is finely tenable.

Proposition 6. If a block T is such that $\boxdot[T]$ is an UREE set, then T is finely tenable.

Proof. Let $\delta \in (0, \varepsilon)$ and let τ denote an Nash equilibrium in a consideration-set game with a type distribution ε-proper on T. As shown above, τ is of the form $\tau = (1 - \delta - \delta^2 K_S)x + \delta y + \delta^2 K_S z$ where $x \in \boxdot[T]$, $C(y) \subseteq \beta(\tau)$, and z is in the interior of $\boxdot[S]$. We show that for a strategy of the form $\tau = (1-\delta)x + \delta y$ where $C(y) \subseteq \beta(\tau)$ as before, there exists no equilibrium with y not in $B_\varepsilon(X)$. Then, we view $\delta^2 K_S z$ as a perturbation of the payoff function (as K_S is bounded we can make δ so small that $\delta^2 K_S z$ is arbitrary compared to δy), because the best-reply correspondence is upper hemi-continuous, the UREE feature still holds. Thus, we have to have $y \in B_\varepsilon(X)$. For a small enough $\delta > 0$ we know that $\tau = (1-\delta)x + \delta y \in B_\varepsilon(X)$ as show by Lemma **??**. Assume that all the best replies to τ_i^μ are outside of T_i for some $i \in N$. Then $C_i(y_i) \cap T_i = \emptyset$, but as $B_\varepsilon(X)$ has the property in the proposition we know that there exist no strategy profile $z = (1-\delta)x + \delta y$ such that $C(y) \subseteq \beta(z)$. This is still true in the payoff perturbed game G_ε, as the perturbed best-reply correspondence, denoted $\beta_\varepsilon(X)$, is such that $\beta_\varepsilon(X) \subseteq \beta(X)$ for a small payoff perturbation. □

The proposition above implies that in a n-player game, including Game 5, the block of the support of a pure strategy EES set is finely tenable. We also see that property (P) implies robustness to ε-proper entrants.

6 Relationships Between Block Concepts

Absorbing retracts[9] and Curb blocks are two concepts that have similar robustness properties as tenable and Nash-curb blocks. Myerson and Weibull (2015) show that every absorbing retract in block form and every curb block is coarsely tenable, and we noted above that every curb block but not every absorbing block is Nash-curb. It is not true, as claimed by other authors (see Herings et al., 2016 and Kah and Walzl 2015), that settled equilibria is a selection of persistent equilibria nor is the tenability concepts a refinement of the curb concept. We show in Example 6 below that there exist coarsely tenable and Nash-curb blocks that are not in any persistent retract nor in any minimal curb block.

	D	E	F
A	$1,1$	$2,-2$	$-2,2$
B	$1,1$	$-2,2$	$2,-2$
C	$0,0$	$3,3$	$3,3$

Game 7

Game 7 has three Nash equilibrium components. They are given by $\Theta = \{(pA + (1-p)B, D) : p \in [1/4, 3/4]\}$, $\Omega = \{(C, qE + (1-q)F) : q \in [0,1]\}$ and the $\Gamma = \{\gamma\}$ which consists of the completely mixed strategy γ. The blocks $T^{\alpha} = \{A, B\} \times \{D, E, F\}$ and $T^{\beta} = \{C\} \times \{E, F\}$ are both minimal coarsely tenable and Nash-curb as C is never a best response for a Nash equilibrium in the Θ component and the Ω component does not have any best replies outside of its support. Focusing on persistent retracts, we see that Θ is not a persistent equilibrium component. To see this, assume that the Θ component is in an absorbing retract. The strategy combination that is a convex combination of player two's strategies D and E has a unique best reply for player one in A, no matter how small the share of strategy E is. Thus, the pure strategy A has to be included in the persistent retract. But the unique best reply for player two to A is F . Hence, the mixed-strategy with player two playing F with probability of 1 has to be included in the persistent retract, which in turn implies that the block T^{β} also has to be included. It easy to see that T^{β} is a persistent

[9] A retract $X = \times_{i \in N} X_i$ is a closed, convex and nonempty subset of mixed strategy profiles. It is absorbing if it has a neighborhood $B_{\bar{\varepsilon}}$ such that for all players, the retract contains a best reply to all strategy profiles in $B_{\bar{\varepsilon}}$. A persistent retract is any minimal absorbing retract and a persistent equilibrium is any Nash equilibrium within such set (Kalai and Samet, 1984).

retract, implying that T^{α} cannot be. Thus, Θ is not in a persistent retract. The same reasoning works for the unique minimal curb block T^{β}.

It is not entirely clear which solution concept picks the most reasonable set of self-enforcing Nash equilibria in the game in Figure 6. The unique persistent retract selects the set of Pareto-optimal equilibria, but the reasoning behind the exclusion of the Θ component seems peculiar. If players are used to the Θ equilibrium being played, there is no compelling reason why out of equilibrium outcomes should be robust to outside strategies as they are never used, even if they sometimes are best replies to the equilibrium being played. There are many real world examples of stable, but Pareto-inefficient, equilibria stemming from path dependency.[10]

It is worth pointing out that, in generic normal-form games, Nash-curb and coarsely tenable (as well as finely tenable) blocks coincide. To see this, notice that in generic game all Nash equilibria are regular, that is quasi-strict and isolated. Hence, the best replies to a Nash equilibrium is always in its support. If a block is coarsely tenable then the block includes all the best replies to the Nash equilibria with support in the block.

Related block concepts include p-best response sets by Tercieux (2006) and minimal prep sets by Voorneveld (2004). A p-best response set is a block that contains all the best replies to all beliefs that put at least a probability p on the strategies in the block. This concept has been linked with curb and absorbing blocks (see Myerson and Weibull (2015) and Klaus Ritzberger (1995)). A minimal prep set is a block that contains at least one best reply for each player to every strategy profile with its support in the block.[11] We notice that this concept is not enough to imply coarse or fine tenability nor do Nash-curb imply that the block is a prep set either. Voorneveld (2005) shows that generically curb blocks, prep sets, and persistent retracts coincide. In contrast, Myerson and Weibull (2015) show that those concepts do not generically coincide with tenable blocks (see for example the game in Table 7 from Myerson 1997).

7 Concluding Remarks

In this paper, we analyze the relationship between tenable strategy blocks and evolutionary stability concepts. Although, the concepts are defined on different objects, they are similar enough to allow for evolutionary interpretations of tenability. When focusing on symmetric

[10]For example, this paper is written on a QWERTY-keyboard where the placement of the keys is certainly not optimal.

[11]A block T is a prep set if $\beta(x) \cap T \neq \emptyset$ for all $x \in \boxdot[T]$.

two-player games, we saw that a singleton symmetric coarsely tenable block is the support of a neutrally stable strategy. Comparing tenability with EES sets, we found that in two-player games, a singleton EES set's support is a Nash-curb block, hence such Nash equilibria are fully settled and regular. We also saw that the concepts drifts apart when the Nash equilibrium component contains more than one element or in games with more than two players. A conjecture is that all EES sets have their support in a Nash-curb block in generic normal-form games.

A potential application of the Nash-curb block concept is to capture forward induction reasoning. The set of Nash equilibria of a Nash-curb block has similar properties in the original game as it has in the block game, hence we conjecture that the index of equilibria in a Nash-crub block should always add up to +1.[12] Hauk and Hurkens (2002) show that EES sets always adhere to forward induction when existing. This should also true for equilibria in minimal Nash-curb blocks when the outside option is an EES set. The type of games that Hauk and Hurkens (2002) consider are two-player games where the first player can either choose to play a subgame that has a generic reduced normal form, or to choose an outside option which ends the game. The game has two Nash equilibrium components, the outside option and the subgame regular equilibrium that gives a higher payoff to the first player than the outside option. Since, if the subgame Nash equilibrium is an EES set, its support is a minimal Nash-curb block, the other Nash equilibrium cannot be in a minimal Nash-curb block as that would imply that the equilibria in the game have a combined index of +2.

Further exploring the properties of Nash-curb blocks in a dynamic sense is also an avenue for future research. An open question pertains to the existence of a stochastic or deterministic dynamic that would lead to minimal Nash-curb blocks (or tenable blocks for that matter). For example, Hurkens (1995) develop a dynamic learning process that lead players to play strategies within a minimal curb set and Kets and Voorneveld (2008) provides an adaptive process that eventually settles on a prep set. This suggests that a similar process could lead to a minimal Nash-curb or tenable block.

Finally, we looked at the relationship between settled and persistent equilibria. We showed that neither of the notions imply the other. An interesting observation is that there are Nash-curb blocks that do not include a persistent equilibrium nor is in any minimal curb block. It shows that concepts that demands properties of the environment of Nash equilibria may differ between the block game of a Nash-curb block and the original game. On the other

[12] For references to index theory see Ritzberger (1994, 2002) and for the definition of the forward induction game discussed, see Hauk and Hurkens (2002).

hand, all refinements with existence in all games that solely focus on the properties of (sets of) Nash equilibria (and what happens to them in perturbed games) exist in the original game if they exist in a Nash-curb block game.

A potential use of the Nash-curb concept is to single out stable games within a game, and then to apply preferred Nash equilibrium refinement, such as the notion of proper equilibrium, to single out stable equilibria within the smaller games. For example, a well-known drawback of the proper equilibrium concept is that it is not invariant to the addition of strictly dominated strategies. In contrast, applying the concept in the block game of a minimal Nash-curb block would solve this. A potential problem with this practice is that strategies that are weakly dominated in the original game could stop being that in a minimal Nash-curb block game.

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