A Model of Trust Building with Anonymous Re-match

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Abstract

We develop a repeated lender-borrower model with anonymous re-match (that is, once an ongoing relationship is terminated players are rematched with new partners and prior histories are unobservable). We propose an equilibrium refinement based on two assumptions: (a) default implies termination of the current relationship; (b) in a given relationship, a better history (i.e., uniformly higher loan levels, all of which are repaid) implies weakly higher continuation values for both parties. We show that, under these conditions, if the discount factor and the probability of re-match are large enough, then the loan size is strictly increasing over time along the equilibrium path. As such, this paper helps explain gradualism in long-term relationships, especially credit relations.

Keywords: gradualism, trust building, moral hazard, social equilibrium, credit relation
JEL classification: C72, C73, D82, D86

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1 Introduction

Gradualism, or "starting small", is often observed in long-term relationships. It is reflected in the increasing level of interactions over time between parties in a relationship. For example, the credit line granted by a credit card company to a specific customer normally increases over time as more and more on-time repayments are made; Antràs and Foley (2011) document the pattern of financing terms of a single U.S.-based exporter in the poultry industry, and find that the amount of trade credit the exporter grants to its trading partners increases with the length of their relationships.

This paper focuses on one specific driving force of gradualism, the possibility of anonymous re-match. We present a repeated lender-borrower model with anonymous re-match, in which a lender and a borrower interact in a society with a group of lenders and a group of buyers. In a given relationship, at each period the lender first decides the size of the loan provided to the borrower; then the borrower decides whether or not to repay; finally the lender and the borrower decide whether or not to continue their relationship. If a relationship is terminated, each party becomes unmatched; in the next period, they will be anonymously matched with a new partner with some exogenous probability. The re-match is anonymous in the sense that the agents' actions in previous relationships are unobservable in the current relationship. This captures the idea that it is costly to acquire past history information of the other party in a new relationship. All agents in this model have a same discount factor.

We focus on equilibria in which the same strategies are used in every relationship; that is, every new relationship is just a restart of the first relationship. We call such a strategy profile a *social equilibrium*. Such equilibria are studied by Datta (1996) in a setting similar to ours with a focus on the maximal (i.e. efficient) ones, and by Ghosh and Ray (1996) in a setting with a simultaneous stage game and incomplete information about players' types. We further restrict our attention to *orthodox social equilibria*, which are social equilibria such that: (i) an ongoing relationship is terminated on default; (ii) a better loan-repayment history is followed by weakly higher continuation values for the lender and the borrower. Our main result is that, if the discount factor and the probability of re-match are larger than some thresholds, the size of loans along the equilibrium path of any (non-trivial) orthodox social equilibrium is strictly increasing over time.

In the definition of orthodox social equilibrium, the first restriction that the relationship is terminated once default happens is standard in the literature (for example, see Datta (1996) and Kranton (1996)). The second restriction is motivated by the following idea. In credit relations, repayment is viewed as a better action than default, as default directly hurts the lender and causes the current relationship to end. In addition, a larger loan is better than a smaller one, in the sense that given repayment happens, a larger loan leads to higher per-period payoffs for both parties. At the beginning of a given date t,[1] if there are two histories of actions such that neither entails default and the loan sizes in one history are weakly larger than those in the other at any time prior to t, then it can be said that the loan-repayment history with larger loans is (weakly) better than

[1] The beginning of date t is the first subperiod of t when the lender decides how much to lend. A history at that node consists of all the actions (loan sizes, repayment decisions and continuation decisions) from date 0 to $t-1$.

the other. The second restriction in defining an orthodox equilibrium says that the continuation values of the players following a better history should be weakly higher; otherwise, if they are strictly lower, agents can be viewed as punishing each other by using strictly lower continuation values to discourage themselves from going to a better loan-repayment history, which is generally not reasonable in credit relations.

One important implication of our refinement is that in any orthodox social equilibrium, the lender will make the borrower's no-deviation (i.e. no-default) constraint bind at all dates. This is because if it is not binding at some date, i.e. the borrower strictly prefers not to default, then the lender can increase the loan at that date by a little without inducing default. The reason the borrower does not default on the deviated loan is that she knows that if she repays, she will enjoy a weakly better continuation value than before, because as long as she does so the new history is better than the one on path;[2] since her no-deviation constraint is not binding on path, it is still satisfied at a slightly higher loan followed by a weakly higher continuation value on repayment. But if a slightly higher loan does not induce default, the lender will also enjoy a weakly higher continuation value (together with a strictly higher current payoff) because a better loan-repayment history will be achieved. Therefore the lender would have an incentive to increase the loan size as long as the borrower's no-deviation constraint is not binding, which implies that those constraints must be binding at all dates on path.

Given the above implication, we can explain the intuition for our main result which says that the equilibrium loan sequence in any orthodox social equilibrium is strictly increasing. On the one hand, when past histories are unobservable, the high re-match probability undermines the punishment power of the threat of terminating a relationship; so in order to induce repayment, there has to be some additional cost of starting a new relationship. This additional cost is reflected in the fact that the value of a relationship is increasing in its length, so that restarting a new relationship is worse than staying in the current relationship. On the other hand, the lender knows that the longer the borrower has stayed in a given relationship, the higher the value of this relationship to the borrower; since the value of becoming unmatched is constant, the cost inflicted upon the borrower by terminating the current relationship becomes larger as time goes on. This implies that the short-term temptation of default that the borrower can overcome increases over time. Then, since the lender has an incentive to exploit the borrower's no-deviation constraints, the size of loans she offers also increases over time.

The rest of the paper is organized as follows. Section 2 sets up the model and states the main results; Section 3 discusses the intuition for our results, the connection of our work with Datta (1996), and the case when the discount factor and re-match probability are low. Section 4 concludes. All proofs are in the Appendix.

Related Literature. Gradualism in repeated interactions has been studied by a number of papers in the literature. There are at least two main strands of explanations. One strand combines moral hazard with incomplete information, in which there are multiple types (usually reflected in

[2] Note that these two histories only differ in the last period and the new history has larger loan at that period.

levels of patience) in one side or both sides of a relationship. The reason for starting small in such environments is that, when the history of cooperation is longer, the probability that the other party is of the "good" type is higher, so that the optimal level of interaction increases over time. If they ever defect, the relationship is terminated. Ghosh and Ray (1996) study a repeated game with both incomplete information (2 types) and re-match, where the stage game is simultaneous with incentives similar to a prisoners' dilemma; they characterize the equilibrium sequence of cooperation levels to be "starting small" for the first period and reaching the efficient level from the second period on. Waston (1999, 2002) studies a general model of long-term relationships with incomplete information, and characterizes the level of interactions to be increasing over time under certain refinements. Rauch and Watson (2003) apply similar analysis to an environment where a developed-country buyer has incomplete information about the ability of a less-developed-country supplier to fill large orders, so that their relationship starts with a trial period of small orders. Kartal et al. (2015) studies a repeated trust game with two types of "receivers" (without re-match), and finds a similar increasing pattern of trust levels. Araujo and Ornelas (2007) and Antràs and Foley (2011) apply such arguments to explain similar patterns observed in the use of trade credit.

Another strand of explanations focuses on the environment with only moral hazard. Kranton (1996) considers a repeated game with a prisoners' dilemma-like simultaneous stage game, and has a same characterization of the *efficient* equilibrium path as Ghosh and Ray (1996), i.e. "starting small" only for the first period. Datta (1996) considers a repeated lender-borrower game very similar to ours, and proves that in the maximal (i.e. *efficient*) social equilibrium without default on path, the value of a relationship is non-decreasing over time. Ray (2002) studies a general repeated moral hazard problem without re-match or incomplete information, and proves a similar result of "starting small" for *efficient* equilibria.

This paper contributes to the literature by presenting a lender-borrower model in a pure moral hazard environment with random re-match, and characterizing the equilibrium loan sequence to be strictly increasing in *any* orthodox social equilibrium. It abstracts from incomplete information or type uncertainty. Compared to Kranton (1996) and Ghosh and Ray (1996), the stage game in our model is sequential rather than simultaneous; in addition, we characterize the equilibrium level of interactions (loan sizes) to be strictly increasing over time, rather than constant from the second period on. Compared to Ray (2002), the driving force of our results is the possibility of rematch, and we do not impose efficiency; indeed, our results apply to all orthodox social equilibria in our model, not only the efficient ones. Our model is closest to Datta (1996). In terms of model setup, we additionally consider varying the probability of re-match from 0 to 1, whereas Datta (1996) only considers the case where such probability is 1; in addition, we allow for more general payoff functions while Datta (1996) only considers linear ones. In terms of results, we do not impose efficiency and focus on a reasonable refinement, for which we can characterize the equilibrium loan sequence in any orthodox social equilibrium to be strictly increasing. In contrast, Datta (1996) only shows that in the maximal (i.e. efficient) social equilibrium, the *value* of a relationship (defined as the discounted sum of loans from the current period on) is non-decreasing over time, while being silent about the pattern of per-period level of interactions which is where gradualism is usually observed.

2 Model

2.1 Model Setup

Consider a lender and a borrower in a society with a group of lenders and a group of borrowers. Both are infinitely lived. Time is discrete and starts from 0. The lender's and the borrower's utility functions are the discounted sums of their expected per-period payoffs. Specifically, let δ be the common discount factor and let $\mathbf{y}^L = \{y_0^L, y_1^L, ...\}$ be the sequence of expected per-period payoffs to the lender. The utility function of the lender at time t is:

$$V_t^L(\mathbf{y}^L) = \Sigma_{i=0}^{\infty}\delta^i y_{t+i}^L. \tag{1}$$

Similarly, let $\mathbf{y}^B = \{y_0^B, y_1^B, ...\}$ be the sequence of expected per-period payoffs to the borrower and define:

$$V_t^B(\mathbf{y}^B) = \Sigma_{i=0}^{\infty}\delta^i y_{t+i}^B. \tag{2}$$

The stage game takes the following form. Each period t is divided into 3 subperiods.[3] At t^0, the lender chooses the size of the loan, $L_t \in [0, L^*]$, granted to the borrower. At t^1, the borrower chooses whether to repay or default. At t^2, the lender and the borrower simultaneously choose whether or not to continue the relationship. The relationship continues only if both of them choose so.

If the relationship continues, in the next period they repeat the stage game as described; if it is terminated, each party enters the next period as an unmatched lender/borrower. In each period, an unmatched agent will be anonymously matched with a new partner with an exogenous probability $\lambda \in [0, 1]$. If matched, she starts the stage game with the new partner in this period;[4] otherwise, she earns a payoff of 0 in this period and enters the next period as an unmatched agent.

The history of agents' actions in the *current* relationship is common knowledge to the borrower and the lender forming this relationship; histories of all past relationships of any party are unobservable,[5] which means that newly matched partners effectively restart from the very beginning of the game.

The per-period payoffs are determined by the loan size offered by the lender and the repayment

[3] Note that t is understood as the date of the *current* relationship; as explained later, it is sufficient to study one specific relationship, since we focus on equilibria where every new relationship is a restart of the same relationship.

[4] Once matched, a previously unmatched lender (borrower) still plays the role of a lender (borrower) in the new relationship.

[5] Note that this is equivalent to saying that the histories of all past relationships of the *other* party are unobservable, and the agents also do not base their decisions on their own history of actions in past relationships, which in principle are observable to themselves.

decision of the borrower. Specifically, for a relationship in its period t,

$$y_t^L = \begin{cases} R(L_t), & \text{if repayment happens;} \\ -L_t, & \text{if default happens;} \end{cases} \tag{3}$$

$$y_t^B = \begin{cases} C(L_t), & \text{if repayment happens;} \\ D(L_t), & \text{if default happens.} \end{cases} \tag{4}$$

Assumption 1. *$R(0) = C(0) = D(0) = 0$; $R(\cdot), C(\cdot), D(\cdot)$ are continuous and strictly increasing; $\Delta(L) \equiv D(L) - C(L) > 0, \forall L \in (0, L^*]$; $\Delta(\cdot)$ is strictly increasing; $0 < \inf_{L\in(0,L^*]} \frac{C(L)}{D(L)} \leq \sup_{L\in(0,L^*]} \frac{C(L)}{D(L)} < 1$.*

Assumption 1 requires that the borrower's payoffs from both default and repayment increase with loan size; in addition, the gains from default, $\Delta(L)$, are positive and also increase with loan size, capturing the borrower's moral hazard of default. For the lender, if repayment occurs, a larger loan generates a higher per-period payoff; meanwhile, if default occurs, the lender bears a cost that increases with loan size. Note that L_t in general can be viewed as the level of trust offered by the lender. The higher the level of trust, the higher the payoffs for both parties given cooperation (repayment); however, a higher L_t also leads to a higher temptation to defect (default), as captured by the assumption that $\Delta(L) \equiv D(L) - C(L)$ is strictly increasing.

Notice that in the unique subgame perfect Nash equilibrium of the stage game (without termination decision), the lender chooses a loan size of 0 and the borrower always defaults. In the repeated game setting, we would like to focus on the equilibria in which the strategies depend only on the history of the current relationship, and the same strategy profile is played by all the agents in all relationships. That is, what agents do in a new relationship is an exact repetition of any prior relationship. Given our focus, it is without loss to look at the game of a lender and a borrower in a single relationship while taking as given the continuation game (payoffs) on terminating the current relationship, which itself is determined in equilibrium.

2.2 Orthodox Social Equilibrium

We now define an equilibrium concept for our game. Let $L_t \in [0, L^*]$ be the loan size chosen at t^0; let $d_t \in \{0, 1\}$ denote the borrower's default decision at t^1, s.t. $d_t = 1$ iff default happens at t; let $f_t \in \{0, 1\}$ and $g_t \in \{0, 1\}$ denote the lender's and the borrower's decisions on continuing the relationship at t^2, s.t. $f_t g_t = 0$ iff their relationship is terminated at t. Denote by a_t the outcome of these decisions at period t; that is, $a_t = \{L_t, d_t, f_t, g_t\}$. The history at each node is denoted by:

$$\begin{aligned} h(t^0) &= \{a_0, a_1, ...a_{t-1}\}, \text{ where } h(0^0) = \emptyset \\ h(t^1) &= h(t^0) \cup \{L_t\} \\ h(t^2) &= h(t^1) \cup \{d_t\} \end{aligned}$$

Let $H(t^i)$ be the collection of all possible histories at t^i, for $i = 0, 1, 2$.

A strategy of the lender $l = \{l_0, l_1, ...\}$ consists of a sequence of decision rules that maps each information set to her decision at that node. Specifically, $l_t = (\tilde{L}_{t^0}, \tilde{f}_{t^2})$, where $\tilde{L}_{t^0} : H(t^0) \to [0, L^*]$ and $\tilde{f}_{t^2} : H(t^2) \to \{0, 1\}$, s.t. $L_t = \tilde{L}_{t^0}[h(t^0)]$ and $f_t = \tilde{f}_{t^2}[h(t^2)]$. Similarly, a strategy of the borrower $b = \{b_0, b_1, ...\}$ is defined as $b_t = (\tilde{d}_{t^1}, \tilde{g}_{t^2})$, where $\tilde{d}_{t^1} : H(t^1) \to \{0, 1\}$ and $\tilde{g}_{t^2} : H(t^2) \to \{0, 1\}$, s.t. $d_t = \tilde{d}_{t^1}[h(t^1)]$ and $g_t = \tilde{g}_{t^2}[h(t^2)]$. Notice that given a strategy profile $\{l, b\}$, we are not yet able to compute the payoff of each agent, if according to $\{l, b\}$ a relationship is terminated at some point of time. This is because the continuation values after termination of a relationship depend on equilibrium payoffs, but we have not solved for them; therefore, the equilibrium concept involves a fixed point between re–match values and equilibrium payoffs.

Let $\bar{V}^L$ and $\bar{V}^B$ be the re-match values (i.e. values of a newly-matched relation) for the lender and the borrower, respectively. Note that the continuation values of an unmatched lender and borrower are given by $\lambda' \bar{V}^L$ and $\lambda' \bar{V}^B$, where $\lambda' = \frac{\lambda}{1-(1-\lambda)\delta}$.[6] Given a strategy profile (l, b), we are able to trace out a sequence of decisions (on path), $\{a_t\}$, where $a_t = \{L_t, d_t, f_t, g_t\}$. Let $T(l, b)$ be the date at which the relationship is terminated according to (l, b), i.e. $f_T g_T = 0$, and $\forall t < T, f_t g_t = 1$. From (1) and (2), we can write:

$$V_t^L(l, b, \bar{V}^L, \bar{V}^B) = \sum_{i=0}^{T(l,b)-t} \delta^i [(1 - d_{t+i}) R(L_{t+i}) - d_{t+i} L_{t+i}] + \delta^{T(l,b)-t+1} \lambda' \bar{V}^L, \quad (5)$$

$$V_t^B(l, b, \bar{V}^L, \bar{V}^B) = \sum_{i=0}^{T(l,b)-t} \delta^i [(1 - d_{t+i}) C(L_{t+i}) + d_{t+i} D(L_{t+i})] + \delta^{T(l,b)-t+1} \lambda' \bar{V}^B. \quad (6)$$

In (5), a lender's payoff is the discounted sum of her payoff at each date, which is $R(L_{t+i})$ if repayment happens at $t + i$ and $-L_{t+i}$ otherwise; in addition, at $T(l, b)$ when the relationship is terminated, she will also get the discounted continuation value. In (6), a borrower's payoff has the same structure as that of a lender, where the borrower gets $C(L_{t+i})$ if repayment happens at $t + i$ and $D(L_{t+i})$ otherwise, plus the discounted continuation value when the relationship is terminated.

Definition 1. *A* ***social equilibrium*** *consists of a strategy profile* (l, b) *and re-match values* $(\bar{V}^L, \bar{V}^B)$*, such that*
(i) Given $\bar{V}^L$ *and* $\bar{V}^B$*,* l *and* b *are perfect best responses to each other;*
(ii) $\bar{V}^L = V_0^L(l, b, \bar{V}^L, \bar{V}^B)$*,* $\bar{V}^B = V_0^B(l, b, \bar{V}^L, \bar{V}^B)$*.*

Part (i) of Definition 1 is just the standard requirement of subgame perfection, while part (ii) captures our fixed point requirement for re–match values. We call it social equilibrium because part (ii) implicitly assume that every pair in the society plays such a strategy profile in every relationship.

We further restrict our attention to orthodox social equilibria, which are social equilibria such that: (i) an ongoing relationship is terminated on default; (ii) a better loan-repayment history is

[6] To see this, let $\tilde{V}^B$ be the continuation payoff to the unmatched borrower. We have $\tilde{V}^B = \lambda \bar{V}^B + (1 - \lambda)\delta \tilde{V}^B$, so that $\tilde{V}^B = \frac{\lambda}{1-(1-\lambda)\delta} \bar{V}^B$.

followed by weakly higher continuation values for the lender and the borrower. Formally, given a strategy profile (l, b), let $V_t^L : H(t^0) \to \mathbb{R}$ and $V_t^B : H(t^0) \to \mathbb{R}$ be the induced continuation value functions for the lender and the borrower at the beginning of each period, which are maps from the set of histories at t^0 to real numbers.

Definition 2. *A social equilibrium strategy profile* (l, b) *is* ***orthodox****, if*
(i) $\tilde{f}_{t^2}[h(t^2)] = 0$*, for all* t *and all* $h(t^2) \in H(t^2)$ *s.t.* $d_\tau = 1, \exists \tau \leq t$*;*
(ii) $V_t^L[h'(t^0)] \geq V_t^L[h(t^0)]$ *and* $V_t^B[h'(t^0)] \geq V_t^B[h(t^0)]$*, for all* t *and for all* $h(t^0), h'(t^0) \in H(t_0)$ *s.t.* $\forall \tau < t$*,* $L'_\tau \geq L_\tau$*,* $d'_\tau = d_\tau = 0$ *and* $f'_\tau g'_\tau = f_\tau g_\tau = 1$*.*

Notice that part (i) of Definition 2 requires that the lender terminates the relationship whenever default has happened. This requirement is standard in the literature (see also Datta (1996), Kranton (1996), Ghosh and Ray (1996), etc). Part (ii) of Definition 2 says that at the beginning of any period t, if we consider two histories such that the borrower is making repayments at all dates in both of them, and the loan sizes in one history are weakly larger than those in the other at all dates prior to t, then the continuation values for the players following the history with larger loans should be weakly higher.[7] Note that a history of larger loans without default is "better", in the sense that offering larger loans potentially benefits both parties given repayment, and the borrower indeed repays so that such benefits are achieved. Part (ii) of Definition 2 is then motivated by the idea that the agents should not punish each other by using strictly lower continuation values to discourage themselves from going to a better loan-repayment history.

2.3 The Structure of Orthodox Social Equilibrium

Note first that a trivial (orthodox) social equilibrium always exists, in which the lender always offers a loan size of 0 and the borrower defaults on any positive loan. Also observe that in any non-trivial social equilibrium, the relationship is never terminated on path, because if it is terminated at date t of a relationship, then at that last period (of the relationship) the borrower will default on any positive loan; as a result, the loan size at date t must be 0. But then, at the end of the second-to-last period $t-1$, each agent has an incentive to terminate the relationship, because both of them would prefer getting the values of an unmatched agent right away (which is positive because the equilibrium is non-trivial), rather than waiting for another period with a payoff of 0 and then getting such values. But this is a contradiction to the optimality of continuing the relationship at the end of $t-1$. One immediate implication is that in any orthodox social equilibrium, the borrower never defaults on path, because a relationship will be terminated on default by definition of an orthodox social equilibrium.

Now we state the main results of this paper.

[7] To be sure, we only make comparisons at subperiod t^0, where a history $h(t^0)$ consists of a full set of actions (loan size, default decision, continuation decisions) for each period from 0 to $t-1$. We do not make comparisons or impose such restriction at other subperiods.

Proposition 1. *Under Assumption 1, $\exists \delta^*, \lambda^*_\delta \in [0,1)$ s.t. when $\delta > \delta^*$ and $\lambda > \lambda^*_\delta$, the loan sequence $\{L_t\}$ on the equilibrium path of any non-trivial orthodox social equilibrium is strictly increasing.*

Proposition 2. *Under Assumption 1, when $\delta > \delta^*$ and $\lambda > \lambda^*_\delta$, a non-trivial orthodox social equilibrium exists.*

3 Discussion

3.1 An Intuition for Proposition 1 with Linear Payoff Functions

To understand the intuition behind Proposition 1, consider the case where payoff functions are linear in loan size as follows:[8]

$$y^L_t = \begin{cases} (1-\alpha)L_t, & \text{if repayment happens;} \\ -L_t, & \text{if default happens;} \end{cases}$$
$$y^B_t = \begin{cases} \alpha L_t, & \text{if repayment happens;} \\ L_t, & \text{if default happens.} \end{cases}$$

Let $\mathbf{L} = \{L_t\}$ be the sequence of loans on the equilibrium path of some non-trivial orthodox social equilibrium. As we observed earlier, a relationship is not terminated in any non-trivial social equilibrium. Therefore we can write the lender's and the borrower's values at each date as:

$$V^L_t[(1-\alpha)\mathbf{L}] = \Sigma^{\infty}_{i=0}\delta^i(1-\alpha)L_{t+i},$$
$$V^B_t(\alpha\mathbf{L}) = \Sigma^{\infty}_{i=0}\delta^i\alpha L_{t+i}.$$

Note that the borrower's no-deviation (no-default) constraints are: $\forall t$,

$$(1-\alpha)L_t \leq \delta[V^B_{t+1}(\alpha\mathbf{L}) - \lambda' V^B_0(\alpha\mathbf{L})], \tag{7}$$

where the LHS is the current period gain from default, and the RHS is the future cost of default which is the difference between the value of continuing the relationship and the value of terminating the relationship.

[8] This can be interpreted as a model of trade credit. The lender (as an exporter) cannot directly face the consumers in another country and has to sell the product to a borrower (as an importer). The borrower does not consume the good but resells them to the consumers and collects the revenue. Assume that the financing cost to the borrower is prohibitive so that it is impossible to require the borrower to pay any amount of money before it receives and resells the good. Thus in each period, the lender first ships the good of value L_t to the borrower; the borrower resells the good, collects the revenue and decides whether or not to pay back $(1-\alpha)L_t$. If repayment happens, the borrower gets αL_t and the lender gets $(1-\alpha)L_t$; if default happens, the borrower keeps all the revenue L_t, while certain additional cost is incurred to the lender so that she gets $-L_t$.

One implication of part (ii) of Definition 2 is that in any orthodox social equilibrium, (7) must hold at equality in each period, as long as $L_t < L^*$. That is, the lender has an incentive to increase the loan size as much as possible so that in each period, either the borrower's no-deviation constraint is binding, or the loan size reaches its maximum L^*. To see this, assume that in some period t (7) holds at strict inequality and $L_t < L^*$. Then in subperiod t^0 on the equilibrium path, the lender can consider deviating to $L'_t = L_t + \varepsilon < L^*$ s.t.

$$(1-\alpha)L'_t < \delta[V^B_{t+1}(\alpha\mathbf{L}) - \lambda' V^B_0(\alpha\mathbf{L})]. \tag{8}$$

Is this deviation profitable? Note first that such a deviation will not induce default in period t, because in subperiod t^1 when the borrower is deciding whether or not to repay, she will find:

$$(1-\alpha)L'_t < \delta[V'^B_{t+1} - \lambda' V^B_0(\alpha\mathbf{L})], \tag{9}$$

where V'^B_{t+1} is the borrower's continuation value at $(t+1)^0$ (i.e. the beginning of date $t+1$) following the new history with her repayment for a larger loan L'_t. (9) holds because of (8) and $V'^B_{t+1} \geq V^B_{t+1}(\alpha\mathbf{L})$, as the loan sizes in history $\{L_0, L_1, ...L_{t-1}, L'_t\}$ are weakly higher than those in history $\{L_0, L_1, ...L_{t-1}, L_t\}$ at all dates and both histories do not entail default. As a result, deviating to L'_t would not induce default.

Knowing this, the lender's total payoff from time t by offering L'_t will be $(1-\alpha)L'_t + V'^L_{t+1}$, which is strictly larger than $(1-\alpha)L_t + V^L_{t+1}[(1-\alpha)\mathbf{L}]$, because $L'_t > L_t$ and $V'^L_{t+1} \geq V^L_{t+1}[(1-\alpha)\mathbf{L}]$ as the lender's value following a better loan-repayment history is also weakly higher. But this just implies that deviating to L'_t is profitable for the lender, a contradiction to $\{L_t\}$ being on the equilibrium path of some (non-trivial) orthodox social equilibrium.

Therefore, when agents know that they are not punishing each other to discourage themselves from going to a better loan-repayment history (i.e. (ii) of Definition 2 is satisfied), the lender will have an incentive to increase the loan size as much as possible, so that in each period either the borrower's no-deviation constraint is binding, or L_t reaches its maximum L^*. It turns out that the latter case can be ruled out when λ (the probability of re-match) is high, so the borrower's no-deviation constraint is binding at each date.

Now we can explain the intuition for our main result, which says that when the discount factor and the probability of re-match are high, the equilibrium loan sequence in any orthodox social equilibrium is strictly increasing. On one hand, when past histories are unobservable, the high re-match probability undermines the punishment power of the threat of terminating a relationship; so in order to induce repayment, there has to be some additional cost of starting a new relationship, which here is reflected in the fact that V^B_t is strictly increasing over time. On the other hand, such a pattern of V^B_t implies that the cost inflicted upon the borrower by terminating the current relationship is increasing over time, because the value of becoming unmatched is constant while V^B_t grows. This means that the short-term temptation of default that the borrower can endure increases over time. Since the lender has an incentive to make the borrower's no-deviation constraint bind in each period, she will finally offer a strictly increasing sequence of loans.

3.2 Connection with Datta (1996)

Datta (1996) studies a model very similar to ours, where the payoff functions are assumed to be linear (as in the current section) and re-match probability is 1. He focuses on the efficient equilibrium within the class of social equilibria that entail no default on path, which he called *maximal social equilibrium*.[9] His main result is that the value sequences (of the lender and the borrower) in any maximal social equilibrium are non-decreasing. Note that this is a weaker characterization because it is silent about the pattern of equilibrium loan sizes. In fact, in the linear case, there are maximal social equilibria that are not orthodox; there are orthodox social equilibria that are not maximal; and there is a unique loan sequence that is both orthodox and maximal.

To illustrate these cases, consider the following parametrization.

$$\delta = 0.8, \alpha = 0.5, \lambda = 1, L^* = 100$$

Under those parameters, we can find a maximal social equilibrium that is not orthodox (we only list the loan sizes on equilibrium path):[10]

$$\{L_t\} = \{35, 25, 100, 100, 100, ...\}$$

Note that the loan sequence above is non-monotone. Also, we can find an orthodox social equilibrium that is not maximal:

$$\{L_t\} = \{18.8, 30.5, 37.8, 42.4, 45.2, ...\}$$

[9] Note that with linear payoff functions, the ratio between the lender's and the borrower's values at each date is $\frac{(1-\alpha)}{\alpha}$ in any social equilibrium without default on path. Therefore, efficiency here is equivalent to maximizing the borrower's value at time 0.

[10] To see $\{L_t\} = \{35, 25, 100, 100, 100, ...\}$ is maximal (i.e. efficient among the class of social equilibria that entail no default on path), note first that for any social equilibrium without default on path, V_0^B is bounded by V_0^{*B}, where V_0^{*B} solves $(1-\alpha)L^* = \delta(\frac{\alpha L^*}{1-\delta} - \lambda' V_0^{*B})$. This is because if there is a social equilibrium without default s.t. its loan sequence on path is $\{L'_t\}$ and $V_0'^B > V_0^{*B}$, without loss we have $\limsup L'_t = L^*$ due to linearity of payoff functions. But then, for L'_t close enough to L^*, we have $(1-\alpha)L'_t > \delta(\frac{\alpha L'_t}{1-\delta} - \lambda' V_0'^B)$, a contradiction to the requirement that the borrower does not default on path. It can be checked that $\{L_t\} = \{35, 25, 100, 100, 100, ...\}$ achieves V_0^{*B} and satisfies the borrower's incentive constraints under our parametrization, so it is maximal.

In addition, there is a unique loan sequence that is both orthodox and maximal:[11]

$$\{L_t\} = \{37.5, 60.9, 75.6, 84.7, 90.5, ...\}$$

3.3 When Re-match Probability/Discount Factor is Low

One may wonder about the structure of orthodox social equilibria when the re-match probability or the discount factor is smaller than their thresholds. It turns out that if we assume linear payoff functions, we can have a complete characterization.

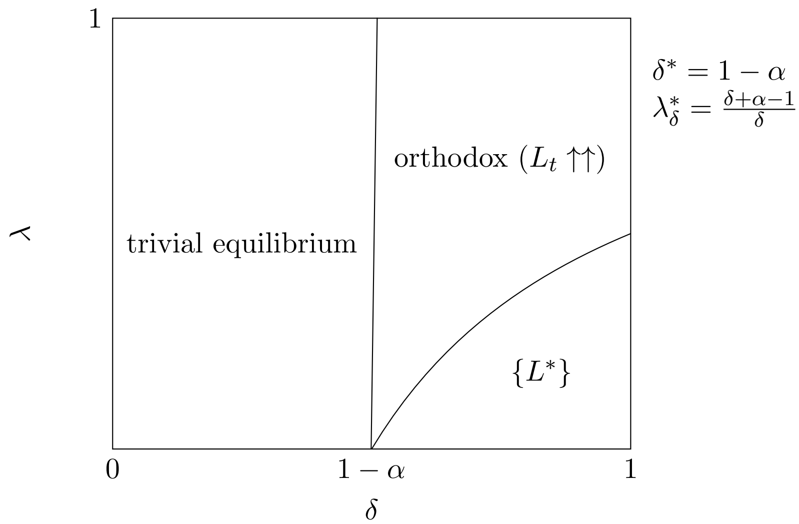

Figure 1

As illustrated in Figure 1, when $\delta < 1-\alpha$, only the trivial equilibrium ($L_t = 0, \forall t$) exists. This is standard in repeated games even without re-match: when agents do not care enough about the future, they can only play the unique SPE of the stage game. When the agents are patient and the re-match probability is low, the threat of terminating a relationship is strong enough to sustain a maximum loan level from the beginning. This is because re-match in this case is so unlikely that losing the current relationship is nearly as bad as being thrown out of the market completely. Finally, notice that the threshold λ^*_δ increases with δ. This is intuitive because when the agents become more patient, it is easier to sustain $\{L^*\}$, as the future cost of terminating a relationship is higher for a more patient borrower.

Therefore, an increasing sequence of loans will arise when agents are sufficiently patient while

[11] $\{L_t\} = \{37.5, 60.9, 75.6, 84.7, 90.5, ...\}$ is constructed using the algorithm proposed in the proof of Proposition 2 by setting the limit $L = L^*$. To see $\{L_t\}$ is the unique loan sequence that is both orthodox and maximal, note first that by construction it can be supported by an orthodox social equilibrium using the strategy profile proposed in the proof of Proposition 2. Note also that by construction $(1-\alpha)L_t = \delta(\frac{\alpha L_t}{1-\delta} - \lambda V_0^B), \forall t$. Since $L_t \to L^*$, we have $(1-\alpha)L^* = \delta(\frac{\alpha L^*}{1-\delta} - \lambda V_0^B)$; therefore $V_0^B = V_0^{*B}$, where V_0^{*B} is defined in footnote 10. This implies that such orthodox social equilibrium is also maximal. Finally, the uniqueness of $\{L_t\}$ follows from the fact that it is the unique loan sequence that satisfies (10) and converges to L^* (see the proof of Proposition 2), and any other such sequence converging to $L < L^*$ generates a lower V_0^B.

the re-match probability is also high; the former ensures that the agents are willing to have some positive level interactions instead of only playing the trivial equilibrium, while the latter creates the need for "starting small" because $\{L^*\}$ would directly result in "hit and run".

4 Conclusion

This paper studies a lender-borrower model in a pure moral hazard environment with anonymous re-match. The main result states that as long as the discount factor and the probability of re-match are larger than certain thresholds, the size of loans along the equilibrium path of any orthodox social equilibrium is strictly increasing over time. This characterization gives a formal argument that qualifies the possibility of anonymous re-match as a driving force of gradualism in long-term relationships. Re-match is not the only reason for gradualism, and type uncertainty, among other things, is also a reasonable and important driving force for such phenomenon. However, given that in reality it is indeed costly to acquire the past history information of the other party in a new relationship, this paper provides insights from one specific aspect into understanding gradualism in long-term relationships, especially credit relations.

Appendix

We present the proofs of Proposition 1 and 2 in this Appendix. Throughout Appendix, we assume Assumption 1 hold.

Lemma 1. *Let* (l, b) *be an orthodox social equilibrium, and* $\mathbf{L} = \{L_t\}$ *be the sequence of loan sizes on its equilibrium path. We have* $\forall t$*:*

$$\begin{aligned} L_t = {} & \max L \\ s.t. \;\; & L \leq L^*, \\ & \Delta(L) \leq \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]. \end{aligned} \tag{10}$$

where $V_t^B(\mathbf{L}) = \Sigma_{i=0}^{\infty} \delta^i C(L_{t+i})$*.*

Proof. Note that we have remarked in Section 2.3 that in any orthodox social equilibrium, a relationship is never terminated and there is no default on path, therefore we can write the borrower's continuation value on path at the beginning of each period as $V_t^B(\mathbf{L}) = \Sigma_{i=0}^{\infty} \delta^i C(L_{t+i})$.

Now we prove (10) by contradiction. Let (l, b) be an orthodox social equilibrium s.t. (10) does not hold. Let t be the first period that (10) fails. Then either $\Delta(L) > \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$, or $L_t < L^*$ and $\Delta(L) < \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$.

If $\Delta(L) > \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$, since there is no default on equilibrium path, the borrower will be better off by a one-shot deviation to default at time t, because the LHS is the current benefit from default (remember that $\Delta(L) \equiv D(L) - C(L)$), while the RHS is the present value of future cost of default which is the difference between the value of continuing the relationship by repaying and the value of terminating the relationship by defaulting. This is a contradiction to (l, b) being mutual perfect best responses.

If $L_t < L^*$ and $\Delta(L) < \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$, choose L'_t s.t. $L_t < L'_t \leq L^*$ and $\Delta(L'_t) < \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$. Denote by V'^B_{t+1} and V'^L_{t+1} of the continuation values for the borrower and the lender at $(t+1)^0$ following history $\{L_1, L_2, ..., L_{t-1}, L'_t\}$ (without default). By the fact that (l, b) is a social equilibrium, at time t with history $\{L_1, L_2, ..., L_{t-1}\}$ (without default), the lender should not find it profitable to offer L'_t. This implies that either L'_t does not induce default and the lender gets weakly worse off, i.e.[12]

$$R(L'_t) + V'^L_{t+1} \leq R(L_t) + V_{t+1}^L(\mathbf{L}), \tag{11}$$

or L'_t induces default, which means that from the borrower's viewpoint,

$$\Delta(L'_t) \geq \delta[V'^B_{t+1} - \lambda' V_0^B(\mathbf{L})]. \tag{12}$$

Since $L'_t > L_t$ and R is strictly increasing, we know that (11) implies $V'^L_{t+1} < V_{t+1}^L(\mathbf{L})$. Also, since by construction $\Delta(L'_t) \leq \delta[V_{t+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$, we know that (12) implies $V'^B_{t+1} < V_{t+1}^B(\mathbf{L})$.

[12] Note that if repayment happens, both agents will not terminate the relationship at t^2, because $V'^L_{t+1} \geq V_{t+1}^L(\mathbf{L}) > \lambda' V_0^L(\mathbf{L})$ and $V'^B_{t+1} \geq V_{t+1}^B(\mathbf{L}) > \lambda' V_0^B(\mathbf{L})$.

Therefore, either $V'^L_{t+1} < V^L_{t+1}(\mathbf{L})$ or $V'^B_{t+1} < V^B_{t+1}(\mathbf{L})$. However, note that the loan sizes history $\{L_1, L_2, ..., L_{t-1}, L'_t\}$ (without default) are weakly larger than $\{L_1, L_2, ..., L_{t-1}, L_t\}$ (without default) at all dates, thus we have reached a contradiction to part (ii) of Definition 2 of orthodox social equilibrium. □

Lemma 2. *If $\{L_t\}$ satisfies:*

$$\Delta(L_t) = \delta[V^B_{t+1}(\mathbf{L}) - \lambda' V^B_0(\mathbf{L})], \forall t \geq 0, \tag{13}$$

where $V^B_t(\mathbf{L}) = \Sigma_{i=0}^{\infty}\delta^i C(L_{t+i})$, then $\{L_t\}$ is strictly monotonic or constant.

Proof. We first show that $\{L_t\}$ is either weakly increasing or strictly decreasing. Take any $\{L_t\}$ satisfying (13) and not weakly increasing. From (13), we have:

$$\Delta(L_t) - \Delta(L_{t-1}) = \delta\Sigma_{\tau=t}^{\infty}\delta^{\tau-t}[C(L_{\tau+1}) - C(L_\tau)], \forall t \geq 1. \tag{14}$$

As $\{L_t\}$ is not weakly increasing, $\exists t, L_{t+1} < L_t$. Since Δ and C are strictly increasing, (14) implies there must be infinitely many t, s.t. $L_{t+1} < L_t$. We consider the following 2 cases.

Case 1: $\exists T$, s.t. $\forall t \geq T, L_{t+1} < L_t$. That is, eventually $\{L_t\}$ becomes strictly decreasing. We claim that in this case, $\{L_t\}$ must be a strictly decreasing sequence (from time 0). To see this, notice that since $\forall t \geq T, L_{t+1} < L_t$, we know from (14) that $L_T - L_{T-1} < 0$, i.e. $L_T < L_{T-1}$, because Δ and C are strictly increasing. Apply this step all the way back to $t = 1$ to obtain $L_t > L_{t-1}, \forall t \geq 1$. So $\{L_t\}$ is a strictly decreasing sequence.

Case 2: $\nexists T$, s.t. $\forall t \geq T, L_{t+1} < L_t$. That is, $\{L_t\}$ never becomes strictly decreasing after any T. Now let $t_1 \geq 1$ be the smallest time index, s.t. $L_{t_1} < L_{t_1-1}$. By the assumption in Case 2, $\exists t'_1 \geq t_1$, s.t. $L_{t'_1} < L_{t'_1-1}$ but $L_{t'_1+1} \geq L_{t'_1}$. Now consider $L_{t'_1}$. By (14), we have:

$$0 > \Delta(L_{t'_1}) - \Delta(L_{t'_1-1}) = \delta\{[C(L_{t'_1+1}) - C(L_{t'_1})] + \delta[C(L_{t'_1+2}) - C(L_{t'_1+1})] + ...\}, \tag{15}$$

$$0 \leq \Delta(L_{t'_1+1}) - \Delta(L_{t'_1}) = \delta\{[C(L_{t'_1+2}) - C(L_{t'_1+1})] + \delta[C(L_{t'_1+3}) - C(L_{t'_1+2})] + ...\}. \tag{16}$$

Let:

$$\begin{aligned} K &= (1-\delta)\{[C(L_{t'_1+1}) - C(L_{t'_1})] + \delta[C(L_{t'_1+2}) - C(L_{t'_1+1})] + ...\} \\ &= (1-\delta)\Sigma_{\tau=t'_1}^{\infty}\delta^{\tau-t'_1}[C(L_{\tau+1}) - C(L_\tau)] \\ &< 0, \end{aligned} \tag{17}$$

where the inequality follows directly from (15). We claim that $(1-\delta)\{[C(L_{t'_1+2}) - C(L_{t'_1+1})] + \delta[C(L_{t'_1+3}) - C(L_{t'_1+2})] + ...\} < K < 0$, which is a contradiction to (16). To see this, assume that

$(1-\delta)\{[C(L_{t_1'+2}) - C(L_{t_1'+1})] + \delta[C(L_{t_1'+3}) - C(L_{t_1'+2})] + ...\} \geq K$. Then:

$$\begin{aligned} K &= (1-\delta)\{[C(L_{t_1'+1}) - C(L_{t_1'})] + \delta[C(L_{t_1'+2}) - C(L_{t_1'+1})] + ...\} \\ &\geq (1-\delta)\{[C(L_{t_1'+1}) - C(L_{t_1'})] + \frac{\delta K}{1-\delta}\} \\ &> (1-\delta)(K + \frac{\delta K}{1-\delta}) \\ &= K, \end{aligned}$$

where the second line follows from the assumption we just made and the third line follows from $C(L_{t_1'+1}) - C(L_{t_1'}) \geq 0 > K$. So Case 2 is not possible.

Combining Case 1 and 2, we conclude that any $\{L_t\}$ satisfying (13) must be either weakly increasing or strictly decreasing.

Now we show that in the case that $\{L_t\}$ is weakly increasing, it must be either strictly increasing or constant. Take any $\{L_t\}$ satisfying (13) and weakly increasing. If it is not strictly increasing, then $\exists$ a smallest t, call it t_2, s.t. $L_{t_2} = L_{t_2-1}$. According to (14), since $\{L_t\}$ is weakly increasing, RHS of (14) at $t = t_2$ is 0 only if $L_{t+1} = L_t, \forall t \geq t_2$. Therefore. $\{L_t\}$ is constant from $t_2 - 1$. Now we show that $t_2 = 1$. Assume not, i.e. $t_2 \geq 2$. Since $\{L_t\}$ is weakly increasing and t_2 is the smallest t, s.t. $L_t = L_{t-1}$, we have $L_{t_2-1} > L_{t_2-2}$. According to (14), we should have:

$$\Delta(L_{t_2-1}) - \Delta(L_{t_2-2}) = \delta\Sigma_{\tau=t_2-1}^{\infty}\delta^{\tau-(t_2-1)}[C(L_{\tau+1}) - C(L_\tau)]. \tag{18}$$

Notice that RHS of (18) is 0 because we have already proved that $\{L_t\}$ is constant from $t_2 - 1$, whereas LHS of (18) is positive. So (18) cannot hold, a contradiction. Therefore, $t_2 = 1$, which implies that $\{L_t\}$ has to be a constant sequence, if it is weakly increasing but not strictly increasing.

Therefore, we conclude that for any $\{L_t\}$ satisfying (13), it is strictly monotonic or constant.

□

Lemma 3. *For any orthodox social equilibrium, the loan sequence $\{L_t\}$ on its equilibrium path, must satisfy one and only one of the following 4 properties:*
(i) it is strictly increasing;
(ii) it is constant;
(iii) it is strictly decreasing;
(iv) it is constant at L^* until some T and then becomes strictly decreasing.

Proof. By Lemma 1, we already know that $\{L_t\}$ satisfies (10). Consider the following 2 cases:

Case 1: $L_t < L^*, \forall t \geq 0$, i.e. the 1st constraint in (10) never binds, which implies that $\{L_t\}$ satisfies (13). By Lemma 2, $\{L_t\}$ satiesfies (i), (ii) or (iii) in Lemma 3.

Case 2: $L_t = L^*$, for some $t \geq 0$. Note first that if $L_T = L^*$, then $L_{T-1} = L^*$, which implies $L_t = L^*, \forall t \leq T$. To see this, consider time $T - 1$. The RHS of the 2nd constraint in (10) at time

$T-1$ is: $\delta[V_T^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})]$. Notice that:

$$\begin{aligned} V_T^B(\mathbf{L}) &= \Sigma_{i=0}^{\infty}\delta^i C(L_{T+i}) \\ &= C(L_T) + \delta V_{T+1}^B(\mathbf{L}) \\ &= C(L^*) + \delta V_{T+1}^B(\mathbf{L}) \\ &\geq V_{T+1}^B(\mathbf{L}), \end{aligned}$$

where the last line follows from $V_{T+1}^B(\mathbf{L}) \leq \frac{C(L^*)}{1-\delta}$, as C is strictly increasing. Then we have:

$$\delta[V_T^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] \geq \delta[V_{T+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] \geq \Delta(L^*). \tag{19}$$

(19) implies that at time $T-1$, L^* is the solution to (10). Apply this argument all the way back to $t=0$ to obtain that $L_t = L^*, \forall t \leq T$.

Now let t_3 be the period s.t. $L_t = L^*, \forall t \leq t_3$ and $L_t < L^*, \forall t \geq t_3 + 1$. If such a t_3 does not exist, then $\{L_t\}$ is a constant sequence at L^*, which satisfies (ii) in Lemma 3, so we're done. If t_3 exists, we claim that $L_{t+1} > L_t, \forall t \geq t_3$.

To see this, notice first that the 2nd constraint in (10) is binding for all $t \geq t_3 + 1$, since $L_t < L^*, \forall t \geq t_3 + 1$. Then by Lemma 2, $\{L_t\}$ from $t = t_3 + 1$ has to be strictly increasing, or strictly decreasing or constant. We now show that it must be strictly decreasing. Assume not, then it must be strictly increasing or constant from $t_3 + 1$. By definition of $V_t^B(\mathbf{L})$, we have $V_{t_3+2}^B(\mathbf{L}) \geq V_{t_3+1}^B(\mathbf{L})$, which implies:

$$\delta[V_{t_3+2}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] \geq \delta[V_{t_3+1}^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] \geq \Delta(L^*), \tag{20}$$

where the last inequality follows from $L_{t_3} = L^*$. But this implies that at time $t_3 + 1$, L^* is the solution to (10), a contradiction to $\{L_t\}$ being on equilibrium path and $L_{t_3+1} < L^*$. Therefore, $\{L_t\}$ has to be strictly decreasing from $t_3 + 1$, meaning that it satisfies (iv) in Lemma 3. □

Proof of Proposition 1. Let $\bar{\alpha} = \sup_{L\in(0,L^*]} \frac{C(L)}{D(L)}$ and $\underline{\alpha} = \inf_{L\in(0,L^*]} \frac{C(L)}{D(L)}$. Under Assumption 1, we have $0 < \underline{\alpha} \leq \bar{\alpha} < 1$. Now define $\delta^* \equiv 1 - \underline{\alpha}$ and $\lambda_\delta^* \equiv \frac{\bar{\alpha}+\delta-1}{\delta}$. We will show that such δ^* and λ_δ^* work.

As we already defined, $\lambda' = \frac{\lambda}{1-(1-\lambda)\delta}$. It can be checked that $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$, iff $\lambda > \frac{\bar{\alpha}+\delta-1}{\delta}$.

Call a sequence $\{L_t\}$ an equilibrium loan sequence if it satisfies (10) for all t. We first that show any non-trivial equilibrium loan sequence $\{L_t\}$ cannot be a constant sequence. Assume it is, i.e. $0 < L_t = \tilde{L} \leq L^*, \forall t \geq 0$. Then we directly know that $V_t^B(\mathbf{L}) = \frac{C(\tilde{L})}{1-\delta}, \forall t$. Now we check the 2nd constraint in (10):

$$\begin{aligned}\delta[V_t^B(\mathbf{L}) - \lambda' V_0^B(\mathbf{L})] &= \delta(\frac{C(\tilde{L})}{1-\delta} - \lambda' \frac{C(\tilde{L})}{1-\delta}) \\ &< \delta(1 - \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}})(\frac{C(\tilde{L})}{1-\delta}) \\ &= \delta\frac{(1-\bar{\alpha})(1-\delta)}{\delta\bar{\alpha}}(\frac{C(\tilde{L})}{1-\delta}) \\ &= \frac{(1-\bar{\alpha})}{\bar{\alpha}}C(\tilde{L}) \\ &\leq D(\tilde{L}) - C(\tilde{L}),\end{aligned}$$

where the second line (strictly inequality) follows from the condition $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$, and the last line follows from $\bar{\alpha} = \sup_{L\in[0,L^*]} \frac{C(L)}{D(L)} \geq \frac{C(\tilde{L})}{D(\tilde{L})}$. This implies that (10) is not satisfied (at any t), a contradiction to $\{L_t\}$ being an equilibrium loan sequence. Therefore $\{L_t\}$ cannot be a constant sequence when $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$.

By Lemma 3 and the result above, we know that $\{L_t\}$ is convergent, and (13) is eventually satisfied by $\{L_t\}$ after some T; that is, $\exists T$, s.t. (13) is satisfied $\forall t \geq T$.[13] Let $\bar{L}$ be the limit of $\{L_t\}$. By definition of $V_t^B(\mathbf{L})$, it converges to $\frac{C(\bar{L})}{1-\delta}$. As (13) is eventually satisfied, we must have:

$$D(\bar{L}) - C(\bar{L}) = \delta[\frac{C(\bar{L})}{1-\delta} - \lambda' V_0^B(\mathbf{L})], \tag{21}$$

which gives us:

$$\begin{aligned}V_0^B(\mathbf{L}) &= \frac{\delta C(\bar{L}) - (1-\delta)[D(\bar{L}) - C(\bar{L})]}{(1-\delta)\delta\lambda'} \\ &\leq \frac{\delta - (1-\delta)(\frac{1}{\bar{\alpha}} - 1)}{(1-\delta)\delta\lambda'}C(\bar{L}) \\ &< \frac{C(\bar{L})}{1-\delta},\end{aligned} \tag{22}$$

where the second line follows from $\frac{D(\bar{L})}{C(\bar{L})} \geq \frac{1}{\bar{\alpha}}$, and the third line is obtained by using $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$. Among the 4 possible properties in Lemma 3 one of which $\{L_t\}$ has to satisfy, only if $\{L_t\}$ is strictly increasing will (22) hold. Therefore we conclude that when $\lambda' > \frac{\bar{\alpha}+\delta-1}{\delta\bar{\alpha}}$, the size of loans along the equilibrium path of any non-trivial orthodox social equilibrium must be strictly increasing. □

Proof of Proposition 2. We keep the definition of δ^* and λ^*_δ, i.e. $\delta^* \equiv 1 - \underline{\alpha}$ and $\lambda^*_\delta \equiv \frac{\bar{\alpha}+\delta-1}{\delta}$, where $\bar{\alpha} = \sup_{L\in(0,L^*]} \frac{C(L)}{D(L)}$ and $\underline{\alpha} = \inf_{L\in(0,L^*]} \frac{C(L)}{D(L)}$.

We first establish the existence of an equilibrium loan sequence, i.e. the sequence that satisfies

[13] This is because, the only case where (13) is not necessarily eventually satisfied (i.e. the 2nd constraint of (10) is not eventually binding) is that $\{L_t\}$ is constant at L^*, which has just been ruled out.

(10). Define:

$$\tilde{D}(L) = \begin{cases} D(L), & \text{if } L \in [0, L^*]; \\ D(L^*) + (L - L^*), & \text{if } L > L^*; \end{cases}$$

$$\tilde{C}(L) = \begin{cases} C(L), & \text{if } L \in [0, L^*]; \\ C(L^*) + \frac{C(L^*)}{D(L^*)}(L - L^*), & \text{if } L > L^*. \end{cases}$$

Notice that by construction, $\tilde{D}$ and $\tilde{C}$ are continuous extensions of D and C, respectively; both are strictly increasing and unbounded above, with $\inf_{L>0} \frac{\tilde{C}(L)}{\tilde{D}(L)} = \inf_{L\in[0,L^*]} \frac{C(L)}{D(L)} = \underline{\alpha}$ and $\sup_{L>0} \frac{\tilde{C}(L)}{\tilde{D}(L)} = \sup_{L\in[0,L^*]} \frac{C(L)}{D(L)} = \bar{\alpha}$.

As implied by Lemma 3, any equilibrium loan sequence converges. Given this property, pick any $\bar{L} \in [0, L^*]$, we can construct as follows a unique equilibrium loan sequence $\{L_t\}$ converging to $\bar{L}$:

$$V_0^B = \frac{\frac{C(\bar{L})}{1-\delta} - D(\bar{L})}{\delta\lambda'}, \tag{23}$$

$$L_t = \tilde{D}^{-1}(V_t^B - \delta\lambda' V_0^B), \tag{24}$$

$$V_{t+1}^B = \frac{V_t^B - \tilde{C}(L_t)}{\delta}. \tag{25}$$

We first show that $\{L_t, V_t^B\}$ constructed above is well-defined. It is enough to show that $V_t^B - \delta\lambda' V_0^B > 0$ for all t, because then by the fact that $\tilde{D}$ is continuous, strictly increasing and unbounded we know L_t is well-defined. To see $V_t^B - \delta\lambda' V_0^B > 0$, note that (24) implies that $\{V_t^B\}$ satisfies:

$$\tilde{D}(L_t) + \delta\lambda' V_0 = V_t^B.$$

Combining with (25) we have:

$$\frac{\tilde{D}(L_t)}{\tilde{C}(L_t)}(V_t^B - \delta V_{t+1}^B) + \delta\lambda' V_0^B = V_t^B.$$

Rearranging we get:

$$V_{t+1}^B = \frac{1 - \frac{\tilde{C}(L_t)}{\tilde{D}(L_t)}}{\delta} V_t^B + \frac{\tilde{C}(L_t)}{\tilde{D}(L_t)} \lambda' V_0^B. \tag{26}$$

By (23) and the assumption that $\delta \geq 1 - \inf \frac{C(L)}{D(L)}$, we first have $V_0^B > 0$, so that $V_0^B - \lambda'\delta V_0^B > 0$. Then by induction on (26), we have $V_t^B > \delta\lambda' V_0^B$ for all t. Therefore $\{L_t, V_t^B\}$ constructed in (23) through (25) is well-defined.

Now we show that $\{V_t^B\}$ is bounded. Consider another sequence $\{\tilde{V}_t\}$ s.t. $\tilde{V}_0 = V_0^B$ and

$$\tilde{V}_{t+1} = \frac{1 - \underline{\alpha}}{\delta} \tilde{V}_t + \tilde{V}_0. \tag{27}$$

Since for all t, $1 > \frac{\tilde{C}(L_t)}{\tilde{D}(L_t)} \geq \underline{\alpha} \equiv \inf \frac{\tilde{C}(L)}{\tilde{D}(L)}$, we know from (26) and (27) that $V_t^B \leq \tilde{V}_t, \forall t$. Note also that the solution to $\{\tilde{V}_t\}$ is:

$$\tilde{V}_t = \frac{\delta}{\underline{\alpha} + \delta - 1} V_0 - \frac{1 - \underline{\alpha}}{\underline{\alpha} + \delta - 1} (\frac{1 - \underline{\alpha}}{\delta})^t, \tag{28}$$

which is bounded because $\frac{1-\underline{\alpha}}{\delta} < 1$ by Assumption 1. Then we know that $\{V_t^B\}$ is also bounded.

Now we claim that $\{L_t, V_t^B\}$ satisfies: $\forall t \geq 0$,

$$\tilde{D}(L_t) - \tilde{C}(L_t) = \delta[V_{t+1}^B - \lambda' V_0^B], \tag{29}$$

$$V_t^B = \Sigma_{i=0}^{\infty} \delta^i \tilde{C}(L_{t+i}). \tag{30}$$

It can be checked that (29) is obtained by substituting (25) into (24), and (30) is obtained by expanding (25) recursively and using the boundedness of $\{V_t^B\}$.

By applying Lemma 2 to (29) and (30),[14] we know that $\{L_t\}$ are monotonic. Since $\tilde{C}$ are strictly increasing, V_t^B by (30) is monotonic. Because $\{V_t^B\}$ is also bounded, as we have just shown, $\{V_t^B\}$ is convergent. Then by (29) again, $\{L_t\}$ is also convergent. But then, by construction of V_0^B in (23), we have $L_t \to \bar{L}$.

Because $\{L_t\}$ can only be strictly increasing, strictly decreasing or constant (by Lemma 2), then using the condition $\lambda > \lambda_\delta^* \equiv \frac{\bar{\alpha}+\delta-1}{\delta}$ and by the same deduction as in (22), we know that L_t strictly increases to $\bar{L} \leq L^*$. This in turn implies that $L_t \in [0, L^*]$ for all t, and by definition of $\tilde{D}$ and $\tilde{C}$ as well as (29) and (30), we have:

$$\Delta(L_t) = \delta[V_{t+1}^B - \lambda' V_0^B], \tag{31}$$

$$V_t^B = \Sigma_{i=0}^{\infty} \delta^i C(L_{t+i}). \tag{32}$$

Therefore, $\{L_t\}$ satisfies (10), meaning that it is an equilibrium loan sequence converging to $\bar{L}$.

Finally we show the existence of an orthodox social equilibrium by construction. Let $\mathbf{L}^* = \{L_t^*\}$ be the (unique) equilibrium loan sequence converging to $\bar{L} = L^*$. We construct (l, b) as follows:

[14] Note that the proof of Lemma 2 only use the conditions that D, C and $D - C$ are strictly increasing, which hold for $\tilde{D}$, $\tilde{C}$ and $\tilde{D} - \tilde{C}$ here.

$l = \{l_0, l_1, ...\}, b = \{b_0, b_1, ...\}$, where $\forall t, l_t = (\tilde{L}_{t^0}, \tilde{f}_{t^2}), b_t = (\tilde{d}_{t^1}, \tilde{g}_{t^2})$, in which:

$$\tilde{L}_{t^0}[h(t^0)] = \begin{cases} L_t^*, & \text{if } d_\tau = 0, \forall \tau < t; \\ 0, & \text{otherwise;} \end{cases} \tag{33}$$

$$\tilde{d}_{t^1}[h(t^1)] = \begin{cases} 0, & \text{if } \Delta(L_t) \leq \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)] \text{ and } d_\tau = 0, \forall \tau < t; \\ 1, & \text{otherwise;} \end{cases} \tag{34}$$

$$\tilde{f}_{t^2}[h(t^2)] = \begin{cases} 1, & \text{if } d_\tau = 0, \forall \tau \leq t; \\ 0, & \text{otherwise;} \end{cases} \tag{35}$$

$$\tilde{g}_{t^2}[h(t^2)] = \begin{cases} 1, & \text{if } d_\tau = 0, \forall \tau \leq t; \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

Now we claim that the constructed (l, b) is an orthodox social equilibrium. By construction of $\tilde{L}_{t^0}$ in (33), part (ii) of Definition 2 is satisfied, because at the beginning of a given date the continuation values are always the same as long as there is no default. By construction of $\tilde{f}_{t^2}$ in (35), part (i) of Definition 2 are satisfied. It remains to be shown that (l, b) is a social equilibrium, i.e. given that the re-match values being V_0^L and V_0^B, (l, b) are mutual perfect best responses.

We check for one-shot deviation at any history.

First consider $h(t^0)$. At history $h(t^0)$ s.t. $d_\tau = 0, \forall \tau < t$, i.e. there is no default before t, if the lender sets $L_t > L_t^*$, we know from (34) that this will induce default. But then by (35), this will result in the termination of the current relationship. So by setting $L_t > L_t^*$, the lender's payoff changes from $V_t^L(\mathbf{L}^*)$ to $\delta\lambda' V_0^L(\mathbf{L}^*)$, where $V_t^L(\mathbf{L}^*) = \Sigma_{i=0}^{\infty} \delta^i R(L_{t+i})$. As $\{L_t^*\}$ is strictly increasing, $V_t^L(\mathbf{L}^*) > V_0^L(\mathbf{L}^*) > \delta\lambda' V_0^L(\mathbf{L}^*)$, so this deviation is not profitable. If the lender sets $L_t < L_t^*$, based on (l, b) the borrower will not default and the loan sequence $\{L_t^*\}$ will be restored from next period; so this deviation just lowers the lender's current payoff from $R(L_t^*)$ to $R(L_t)$ while keeping future payoff constant, which is not profitable.

At history $h(t^0)$ s.t. $d_\tau = 1, \exists \tau < t$, i.e. there exists default before t but the relationship is not yet terminated, we know from (34) and (35) that no matter what the lender offers, the borrower will default and the relationship will be terminated at the end of this period. Then offering anything larger than 0 will not be a profitable deviation for the lender.

Now consider $h(t^1)$. At history $h(t^1)$ s.t. $\Delta(L_t) \leq \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)]$ and $d_\tau = 0, \forall \tau < t$, a one-shot deviation to default will result in the termination of the current relationship, which is not profitable for the borrower exactly because $\Delta(L_t) \leq \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)]$.

At history $h(t^1)$ s.t. $\Delta(L_t) > \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)]$ or $d_\tau = 1, \exists \tau < t$, i.e. either the extra payoff from default is higher than its cost or there exists default record before t (or both), if the borrower chooses not to default at such $h(t^1)$, then either she can only get $C(L_t) + \delta[V_{t+1}^B(\mathbf{L}^*) - \lambda' V_0^B(\mathbf{L}^*)]$ instead of $D(L_t)$, or the relationship is terminated after this period anyway so she gets $C(L_t)$ instead of $D(L_t)$. In either case, this one-shot deviation is not profitable.

Finally consider $h(t^2)$. At history $h(t^2)$ s.t. $d_\tau = 0, \forall \tau \leq t$, i.e. there is no default at or before

t, if the lender one-shot deviates to terminating the relationship, she will get $R(L_t^*) + \delta\lambda' V_0^L(\mathbf{L}^*)$ instead of $R(L_t^*) + \delta V_{t+1}^L(\mathbf{L}^*)$, which is not profitable because $\{V_t^L(\mathbf{L}^*)\}$ is increasing in t. Similarly, if the borrower one-shot deviates to terminating the relationship, she will get $C(L_t^*) + \delta\lambda' V_0^B(\mathbf{L}^*)$ instead of $C(L_t^*) + \delta V_{t+1}^B(\mathbf{L}^*)$, which is not profitable because $\{V_t^B(\mathbf{L}^*)\}$ is increasing in t.

At history $h(t^2)$ s.t. $d_\tau = 1, \exists \tau < t$, i.e. there exists default at or before t but the relationship is not yet terminated, if the lender or the borrower one-shot deviates to continuing the relationship, the relationship will still be terminated because the other party will do so according to her equilibrium strategy. So such one-shot deviation will not change anything, which is not profitable.

Therefore, at no history can we find a profitable one-shot deviation for any player, so (l, b) are mutual perfect best responses. $\square$

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