Shubik's Dollar Auction with Spiteful Players

Marcin Waniek[1], Long Tran-Tranh[2], Tomasz Michalak[1,3], and Nicholas R. Jennings[2]

[1]Institute of Informatics, University of Warsaw
[2]Electronics and Computer Science Department, University of Southampton
[3]Department of Computer Science, University of Oxford

Abstract

The dollar auction is a simple auction model used to analyse the dynamics of conflict escalation. The well-know work by O'Neill [11] provide solution of the setting with two players interested only in profit. However, the situation changes when participants are driven by other motives.

In this paper, we analyse the course of an auction when participating players are spiteful, *i.e.*, they are motivated not only by their own profit, but also by the desire to hurt the opponent. We investigate this model both for the complete information setting, and for the situation where one player does not know the spitefulness level of her opponent. Our results give us insight into the possible effects of meanness onto conflict escalation.

1 Introduction

More often than not, conflict situations require from all the opposing parties to commit considerably resources which are then unrecoverable for those who were unlucky to win the stake. Such sunk costs occur, for instance, when lobbyist compete for a public contract [5], oligopolists engage in the R&D race [2], or military powers engage in the arm race [10]. When the conflict progresses the resources of all parties deplete but it is typically only the winner who is able to recover them, and this sometimes only partially.

Shubik [13] proposed a simple yet powerful model to study such conflict situations. In his so-called *dollar auction*, two bidders i and j compete for a dollar bill. There is no reserve price but it is required that any new bid increases the current best offer by a pre-defined increment (*e.g.*, 5 cents) or its multiplicity. Similarly to an English auction, the highest bidder wins the prize, but, unlike in the English auction, both the winner and the loser have to pay their bids to the auctioneer. For instance, if the auction stops with the bids of \$.35 and \$.40, then the auctioneer receives $\$.35 + \$.40$, and the dollar is awarded to the highest bidder.

One might argue that it is best not to participate in the above all-pay auction. Similarly, one might argue that in real life it is also best to avoid potential conflict situations. Unfortunately, while such an approach should certainly be given a serious consideration, it is not always possible to escape from conflicts. For instance, despite genuine efforts of some countries and organizations, the arms races were at a center of the 20th century international politics. Also, despite the detente end of the post-Cold War era, the 21st century arms race accelerates and seems to be here to stay in the decades to come [10].

The explanation of why it is difficult to avoid conflicts can be found in the dollar auction. In particular, the possibility that a player may refrain from bidding creates a clear incentive for the other player to bid and to become a winner at a very low price. For instance, if no other competitor invest in R&D, an oligopolistic company has an incentive to make a small investment and dominate the market. However, since such a situation can be life-threatening, the competitors have no choice but to bid and the conflict escalations begins.

To illustrate this point, let us assume that the auction has started with player i bidding \$.05, and player j raising the price to \$.10. Player i faces the following dilemma: withdraw from the auction and lose \$.05

with certainty, or increase the bid to \$.15 with the hope of gaining \$.85. Since the same reasoning holds at any stage during the auction, the bidding may continue well past the bill of \$1.00 to be won. While past this point the bidders can only seek to minimize losses, they are still incentivized to increase their bids rather than drop out and lose everything.

The above dollar auction has become an influential model of conflict escalation. It makes for a great class-play for management students [6] but, more importantly, it offers insight into the dynamics of such processes as international conflicts, arms races, investment decisions or human relations, just to name a few. Any such situation may escalate to irrational levels despite the fact that, locally, every single participant makes a rational decision. Accordingly, in experiments with the dollar auction a dollar bill is sold for considerably more than a dollar [13, 7].

The "paradox of escalation" [13] occurring in the dollar auction can be explained by that a rational strategy for this game is, unfortunately, far from obvious. Matter-of-factly, the optimal solution remained unknown for fifteen years after the publication of Shubik's original work. Such a solution was finally offered by O'Neill [11] who proved that, assuming pure strategies and finite budgets of players, there exists the unique first bid (smaller than a dollar) that guarantees winning the dollar. The exact amount of such a "golden" bid is a non-trivial function of the stake, the budgets, and the minimum allowable increment. In our example, if players i and j have equal budgets of \$2.50 each, the first player, who has the first chance to move, should bid \$0.60. If her opponent is rational, she should leave the game with no prize and with no losses.

O'Neill's result mean that the conflict in the dollar auction should not escalate. There should be always a single winner—a player who has a chance to move first. Hence, the escalation observed in real-life experiments is due to some factors not present in the model. Given this, O'Neill's results were revisited by Leininger [8] who showed that the escalation occurs when we allow for mixed strategies.[1] Next, Demange [4] proved the same if there is some uncertainty about the strength of the players.

In this paper, we reconsider O'Neill's results in pure equilibria from a different perspective. Following recent literature on spiteful bidders [1, 12, 14], we ask the question whether the escalation in the dollar auction may actually be caused by the meanness of the participants who are happy to loose some of the invested resources as long as the opposing party also does so.[2] In other words, do we allow ourselves to be dragged along to the abyss because we know that we will not go there alone.

Reagan's policy to accelerate the USA-USSR arms race can be seen as the prime example of such a situation. It is widely believed that it was a blunt move aimed at winning the Cold War by destabilizing Soviet economy that was not flexible enough to tackle the technological challenge [16]. Similarly, the recent policy of Saudi Arabia of excessive oil production is aimed to hurt major market competitors. In both cases, an important element of one player utility was the other player disutility—the phenomenon modelled in the literature as the spiteful utility function.

Our preliminary results on spite in the dollar auction [15] indicate that it can lead to conflict escalations. However, those results were obtained only for a restrictive setting in which a spiteful player challenged a non-spiteful one, and the non-spiteful player did not suspect the meanness of his opponent, meaning that he followed the strategy proposed by O'Neill [11]. In other words, only one player behaved strategically.

In what follows, we extend the above setting by analysing the dollar auction with both players acting strategically. In particular, we consider the *complete knowledge* case, when both players know each other spite coefficients. In this setting we study different levels of players' spitefulness: *weakly* and *strongly spiteful* players as well as the players of extreme spitefulness called *malicious players*, for which we derive the optimal bidding strategy for each player.

Furthermore, we extend the classic analysis by considering not only the dollar auctions starting from the very beginning (when bids of both players are $(0, 0)$) but also auctions that start in any state (not necessarily $(0, 0)$). This corresponds to the situations when players face the conflict escalation situation but they were not the ones who started it, *e.g.*, new governments and the arms race. Our objective here is to identify

[1] We note that the result very similar to Leininger [8] were obtained more recently by Dekel et al. [3].

[2] A spiteful bidder, contrary to the common assumption of self-interest, maximizes a convex combination of his own profit and the opponent's loss. See Section 2 for more details.

the set of equilibrium states (x, y) from which if the auction is started, the optimal decision would be to stop increasing the bid. These equilibria represent the states from where further conflict escalation is not expected.

While the main focus of the present paper is on the complete knowledge case, we also investigate the *incomplete knowledge* case. Although a standard way to investigate this case is to apply the Bayesian framework, we argue that due to the difficult corner cases of the problem (see Section 3), it is very difficult to provide analytical solution. Nevertheless, we show that under some restrictions, dollar auctions with incomplete knowledge becomes well behaved such that we do not need to rely on the Bayesian framework in order to derive the optimal or near-optimal strategies. In particular, we show that in the case of asymmetric knowledge, *i.e.*, one player has the full information and is truthful to her type of spitefulness, while the other has not, the latter can be oblivious about the type of spitefulness of the opponent. Thus, she does not need to maintain her (Bayesian) belief distribution about the opponent's type.

The remainder of this paper is organized as follows. In the next section, we introduce necessary notations and definitions. In Section 3, we analyse the complete information setting and in Section 4 the incomplete information one. Conclusions follow.

2 Preliminaries

In this section, we formally describe the dollar auction model, the concept of spitefulness and regret, as well as the different types of spiteful players.

2.1 The Dollar Auction

Two players, $N = \{1, 2\}$, participate in an auction in which the winner receives the stake $s \in \mathbb{N}$. Every bid made during the auction must be a multiplicity of a minimal bid increment Δ. Without the loss of generality, in what follows we assume that $\Delta = 1$.

Each player has a certain budget at her disposal. We denote them b_1 and b_2, respectively,[3] and, unless stated otherwise, we assume that both are equal, *i.e.*, $b_1 = b_2 = b$ and that $b > s > \Delta$. We will often refer to players as i and j when their exact identities are irrelevant.

The players make bids in turns. The starting player is determined randomly at the beginning of the auction. She can either make the first bid or pass. If she decides the latter, then the other player can either make the first bid or pass. If neither of the players decides to make the first bid, the auction ends without giving the stake to anyone.

Once the first bid has been made, any player making the bid can either make a bid higher than the opponent, or pass. This time, however, the turn is not offered to the opponent and the auction ends. The auction also ends when one of the players makes the bid that her opponent cannot top (in a setting with equal budgets this would mean bidding the entire budget). The stake is then given to the higher bidder. However, since this is an all-pay auction, both players have to pay their final bids.

Let us denote by x_i and x_j the final bids of the players such that $x_i + x_j > 0$. The profit of player i, denoted p_i, is either

$$p_i = \begin{cases} s - x_i & \text{if } x_i > x_j \text{, or} \\ -x_i & \text{if } x_i < x_j. \end{cases}$$

The dollar auction can be conveniently presented at a graph in which nodes represent states of the auction (they correspond, in particular, to a pair of bids that have been made by both players) and edges outgoing from a node represent bids that the players can make. We call it a *dollar auction graph*, or simply a graph.

A sample dollar auction graph for the auction with budget $b = 7$ and for player 1 starting the auction is shown in Figure 1. The initial state is marked with the letter S. All blue nodes are the states in which the bid of player 2 is higher than the bid of player 1. Hence, in the blue nodes, player 1 makes a decision

[3] While in his original work, Shubik [13] considered the auction with no finite budget, the subsequent literature found this assumption unrealistic.

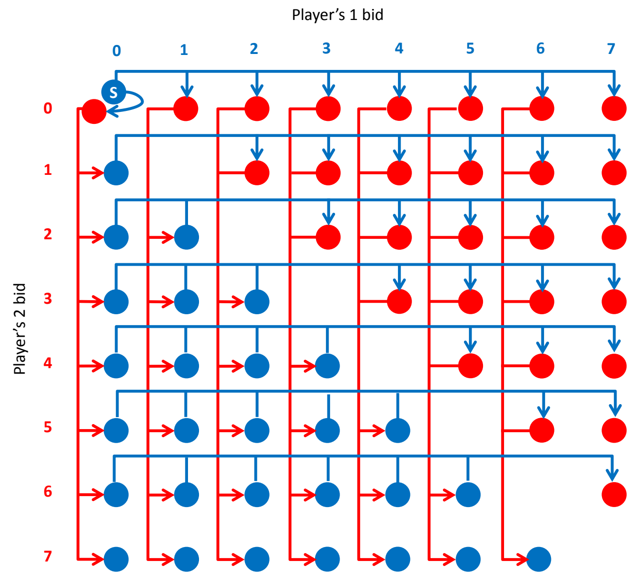


Figure 1: Example of the dollar auction graph, with player 1 (blue) as starting player.

about her next move. If she passes then player 2 wins the auction. Otherwise, any bid available to player 1 is represented with blue edge outgoing from the given node. At the same token, the red nodes in the graph correspond to the states where player 2 has to either pass or make her bid, and the red edges correspond to the bids available.

We use X_i to denote the set of states of the auction in which player i can make the bid, *i.e.*, $X_i = \{(x_i, x_j) \in \{0, \dots, b-1\} \times \{0, \dots, b-1\} : x_j > x_i\} \cup \{(0,0), (-1,0)\}$, where x_i is the last bid of the player currently choosing her bid and x_j is the last bid of her opponent. State $(-1, 0)$ represents the decision to be made when starting the auction and state $(0, 0)$ represents the decision to be made when given the chance to start by the opponent.

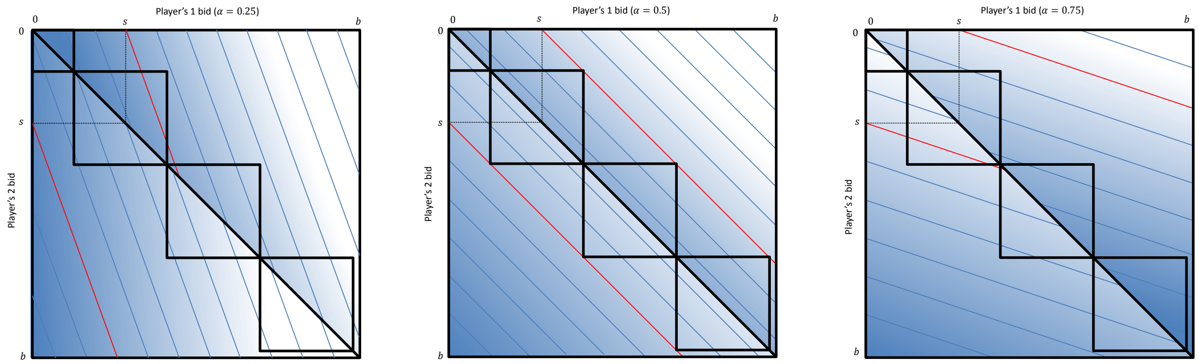
Having described the graphical representation of an auction, we move on to formally defining the set of all possible strategies. We will denote this set by $\mathcal{S}$. Following O'Neill [11], we assume that all the strategies are deterministic. In particular, the strategy of player i is a function $f : X_i \to \{0, \dots, b\}$, where value $f(x_i, x_j)$ represents the bid to be made by the player in state (x_i, x_j). For valid strategies we have $f(x_i, x_j) \geq x_j$, where $f(-1, 0) = 0$ represents the decision to let the opponent move first when the player starts, and $f(x_i, x_j) = x_j$ represents the decision to pass in all other cases.

The solution concept we use, following the work of O'Neill [11], is subgame perfect equilibria. Moreover, we assume (again, following O'Neill [11]) that players are risk-averse. Given a choice of two strategies holding the same utility outcome, the player will choose the strategy with lower cost.

2.2 Spitefulness and Types of Spiteful Players

Spitefulness, as introduced by Brandt et al. [1], is a desire of a player to hurt her opponents, and this possibly by incurring some cost. A spiteful player i is characterised by a *spite coefficient* $\alpha_i \in [0, 1]$ which represents the weight that player i attributes to own profit in relation to the opponent profit. The higher the spite coefficient, the more the player i is interested in minimizing the profit of her opponent and less in maximizing own profit. The utility function of a spiteful player is:

$$u_i = (1 - \alpha_i)p_i - \alpha_i p_j,$$



(a) Weakly-spiteful player with $\alpha_1 = 0.25$

(b) Strongly-spiteful player with $\alpha_1 = 0.5$

(c) Strongly-spiteful player with $\alpha_1 = 0.75$

Note: The blue lines represent the same utility, the red lines represents zero utility. The darker the color, the higher the utility.

Figure 2: Utility of a spiteful player 1 for different values of the spite coefficient.

where p_i and p_j are the profits of players i and j, respectively. Given this, the utility of a spiteful player i in an auction that ended with bids x_i and x_j such that $x_i + x_j > 0$ is:

$$u_i(x_i, x_j) = \begin{cases} \alpha_i x_j + (1-\alpha_i)(s - x_i) & \text{if } x_i > x_j, \\ \alpha_i(x_j - s) - (1-\alpha_i)x_i & \text{if } x_i < x_j. \end{cases}$$

When $\alpha = 0$, player i only cares about her own profit, i.e., she is a *non-spiteful* player. We call a player with the spite coefficient $0 < \alpha < \frac{1}{2}$ a *weakly spiteful player*. Conversely, we call a player with the spite coefficient $\alpha \geq \frac{1}{2}$ a *strongly spiteful player*. A player with a spite coefficient $\alpha = 1$ is called a *malicious* player. She is only interested in minimizing the profit of her opponent.

It is interesting to map utility to the nodes in the dollar auction graph. Figure 2 presents such a mapping for players with different values of spite coefficient. The figure indicates that while weakly spiteful players are generally more interested in ending the auction in states with low sum of the bids, strongly spiteful players prefer states with higher sum of the bids. Indeed, our analysis will show that the optimal strategies of weakly and strongly spiteful players are significantly different.

2.3 Regret

In settings with uncertainty it is common to evaluate different strategies based on regret [9]. Let $U(f)$ be the utility gained from using in auction strategy f and let $\hat{f}$ be an optimal strategy in the given auction. Regret $R(f)$ from using strategy f is the different between utility for the optimal strategy and utility for strategy f:

$$R(f) = U(\hat{f}) - U(f).$$

Having presented the basic concepts of our setting, we move to the analysis of optimal strategies of different kinds of players during the auction.

3 Complete Information Setting

In this section we assume that both players participating in the dollar auction have complete information of the opponent, *i.e.*, they know each other spite coefficients. First, we describe the optimal strategies for each combination of players' types. Next, we consider alternative starting states of the auction.

3.1 Optimal Strategies

We start with introducing the concept of the *maximal preserving increase* which will be used later on in the description of optimal strategies. We then derive the maximal preserving increase for each type of a player (i.e., her spite coefficient).

Definition 1 (Maximal preserving increase). *The maximal preserving increase δ_i of a player i with spite coefficient α_i is the maximal increase of her bid such that her utility increases, i.e.,* $\delta_i = \text{argmax}_\delta \forall_{\substack{(x_i,x_j)\in X_i:\\ 0<x_j-x_i<\delta}} u_i(\min(x_i+\delta, b), x_j) > u_i(x_i, x_j)$.

Theorem 1. *The value of the maximal preserving increase of a player i with the spite coefficient α_i is:*

$$\delta_i = \min(\lceil \frac{s}{1-\alpha_i} - 1\rceil, b).$$

Proof. From the definition, the utility of player i in a state (x_i, x_j) where she is able to make a bid (*i.e.*, $(x_i, x_j) \in X_i$) is:

$$u_i(x_i, x_j) = \alpha_i(x_j - s) - (1 - \alpha_i)x_i.$$

By definition, the state (x_i, x_j) is winning for j. Furthermore, her utility after raising the bid by δ and transferring to the winning state $(x_i + \delta, x_j)$ is:

$$u_i(x_i + \delta, x_j) = \alpha_i x_j + (1 - \alpha_i)(s - x_i - \delta).$$

Let us now consider when $u_i(x_i + \delta, x_j) > u_i(x_i, x_j)$, *i.e.*, when it is more profitable to bid. After using the definition of a spiteful player utility presented in Section 2.2 and solving this inequality we obtain the following condition:

$$(1 - \alpha_i)\delta < s.$$

However, according to the rules of the dollar auction, any bid has to be a multiplicity of the minimal bid increment, *i.e.*, $\Delta = 1$. The bid increase of $\frac{s}{1-\alpha_i}$ would give exactly the same utility (*i.e.*, player i could pass with the same result); hence, to guarantee the increase in utility, the value $\lceil \frac{s}{1-\alpha_i} - 1\rceil$ is needed. Furthermore, for a malicious player i (*i.e.*, $\alpha_i = 1$) we need an upper bound; hence, $\min(\lceil \frac{s}{1-\alpha_i} - 1\rceil, b)$. □

We now describe a strategy which we then show to be the optimal one for a strongly-spiteful player in every setting.

Definition 2 (Malicious strategy). *The malicious strategy of a player i in a dollar auction is:*

$$\hat{f}(x_i, x_j) = \begin{cases} x_j + 1 & \text{if } x_j < b - s, \\ b & \text{otherwise.} \end{cases}$$

Next, we consider the dollar auction between two strongly spiteful players.

Theorem 2. *Let us assume that both players participating in the dollar auction are strongly spiteful, i.e., $\alpha_i, \alpha_j \geq \frac{1}{2}$. It is optimal for both of them to use the malicious strategy.*

The following two lemmas will be used in the proof.

Lemma 1. *If the dollar auction is in the state (x_i, x_j) such that $x_i \geq b - s$ and $x_j \geq b - s$ then it is optimal for the player who makes a bid to bid b (which ends the auction).*

Proof. Consider a player i with spite coefficient α_i. Out of all states such that $x_i \geq b - s$, $x_j \geq b - s$ and j wins the auction, i achieves the highest possible utility when:

- j pays the highest possible price of b in order to get the stake; and
- i makes the lowest possible bid of $b - s$.

Hence, the highest possible utility of i in the losing states under consideration is:

$$\hat{u}_i = \alpha_i(b-s) + (1-\alpha_i)(s-b) = (2\alpha_i - 1)(b-s).$$

Out of all states such that $x_i \geq b-s$, $x_j \geq b-s$ and i wins the auction, i achieves the lowest possible utility when:

- i pays the highest possible price of b in order to get the stake; and
- j makes the lowest possible bid of $b-s$.

Hence, the lowest possible utility of i in considered winning states is

$$\hat{U}_i = \alpha_i(b-s) + (1-\alpha_i)(s-b) = (2\alpha_i - 1)(b-s).$$

Therefore, the highest possible utility of any player i in the losing states under consideration is equal to the lowest possible utility of any player i in the winning states under consideration. From this, we conclude that ending the auction whenever possible in a winning state with a bid of b is always better than giving the opponent an opportunity to achieve greater utility by ending the game in her winning state. □

Lemma 2. *In an auction between two strongly spiteful players ending in state (x_i, x_j), we have that if $u_i(x_i, x_j) > (2\alpha_i - 1)(b-s)$ then $u_j(x_i, x_j) < (2\alpha_j - 1)(b-s)$.*

Proof. First, we focus on the case in which $x_i < b-s \vee x_j < b-s$. Assume by contradictory that $u_i(x_i, x_j) > (2\alpha_i - 1)(b-s)$ and $u_j(x_i, x_j) \geq (2\alpha_j - 1)(b-s)$. In that case by adding both inequalities we have:

$$u_i(x_i, x_j) + u_j(x_i, x_j) > 2(\alpha_i + \alpha_j - 1)(b-s).$$

Without the loss of generality assume that $x_i > x_j$ (the proof for the symmetric case is analogous). We now know that $x_j < b-s$. After expanding the utility formulas we get that:

$$(\alpha_i + \alpha_j - 1)(x_i + x_j - 2b + s) > 0.$$

The same can be rewritten as:

$$(\alpha_i + \alpha_j - 1)((x_i - b) + (x_j - (b-s)) > 0.$$

However, since $x_i \leq b$, $x_j < b-s$ and $\alpha_i + \alpha_j - 1 \geq 0$ (as they are both strongly spiteful players) then the left hand side of the inequality cannot be positive. Therefore, the assumption that $u_i(x_i, x_j) > (2\alpha_i - 1)(b-s)$ and $u_j(x_i, x_j) \geq (2\alpha_j - 1)(b-s)$ has to be false.

Now assume that $x_i \geq b-s \wedge x_j \geq b-s$. In the proof of Lemma 1, we show that $(2\alpha_i - 1)(b-s)$ is the minimal utility in the winning states and the maximal utility in the losing states for player i. Therefore higher utility can be achieved by i only in the winning states. However, those are the losing states for j, so by symmetric argument the maximal utility in those states for her is $(2\alpha_j - 1)(b-s)$. From Lemma 1, we also know that for $x_i \geq b-s \wedge x_j \geq b-s$ the auction can end only with either $x_i = b$ or $x_j = b$. Therefore, the utility of player i higher than $(2\alpha_i - 1)(b-s)$ can be only achieved for $x_i = b$ and $x_j > b-s$. However, player j would never bid higher than $b-s$ but lower than b. □

We are now ready to present the proof of Theorem 2.

of Theorem 2. If player i uses the malicious strategy, then a strongly spiteful player j never has the incentive to pass, as her maximal preserving increase δ_j is always higher than 2 (which is the increase needed to be made in order to continue the auction when i uses the malicious strategy). If player i follows the malicious strategy and her opponent does not pass at any point, she gets utility $(2\alpha_i - 1)(b-s)$ at the end of the auction, as from Lemma 1 continuing the auction past the bid of $b-s$ would not bring her higher utility.

Assume that player i can end an auction against j with utility higher than $(2\alpha_i - 1)(b-s)$ in some state (x_i, x_j). If (x_i, x_j) is winning for player i then player j will not pass in this state since by Lemma 2 she gets there lower utility than achieved by continuing with malicious strategy. If (x_i, x_j) is losing for player i then the auction can get to that state only by the move of player j. However, j would not make such a move, since by Lemma 2 she gets lower utility than achieved by using malicious strategy. Therefore, there is no state where i can end the auction and get higher utility than by using malicious strategy.

The analogous analysis holds for a strongly spiteful player j. □

We now move on to the case of an auction between a strongly spiteful and a weakly spiteful player.

Theorem 3. *Assume that out of players participating in a dollar auction, player i is strongly spiteful, i.e., $\alpha_i \geq \frac{1}{2}$, and player j is weakly spiteful, i.e., $\alpha_j < \frac{1}{2}$. It is optimal for player i to use the malicious strategy. It is optimal for player j to use a malicious strategy if $b < \frac{1-\alpha_j}{1-2\alpha_j}s - \frac{\alpha_j}{1-2\alpha_j}$ and pass whenever given the chance to bid otherwise.*

Proof. Since $\alpha_i > \alpha_j$, then for the maximal preserving increases the following holds $\delta_i \geq \delta_j$. Therefore, player j will never raise her bid by a value so high that player i would not be able to outbid her without getting a lower utility. Hence, if player j continues to bid during the auction, player i can get the same result as against the strongly spiteful player.

However, unlike the strongly spiteful player i, the weakly spiteful player j may decide to pass. Outbidding by higher value than 1 may cause the necessary increase of player j to exceed her maximal preserving increase and end the auction prematurely, while strongly spiteful player i is interested in prolonging the auction. Due to Lemma 1 it is still optimal for her to end the auction when value of the bid reaches $b - s$. Therefore, the malicious strategy is still optimal for the strongly spiteful player.

Since player i uses the malicious strategy, the auction can either end in one of the states $(b, b-s)$, $(b-s, b)$ or in a state winning for i, where i outbids j by 1. The utility of player j in states $(b, b-s)$ and $(b-s, b)$ is:

$$u_j(b, b-s) = u_j(b-s, b) = (2\alpha_j - 1)(b-s).$$

The utility of player j in state $(x_j, x_j + 1)$ winning for i is:

$$u_j(x_j, x_j + 1) = (2\alpha_j - 1)x_j - \alpha_j(s-1).$$

Since for a weakly spiteful player $2\alpha_j - 1 < 0$, the utility is maximal for $x_j = 0$. Therefore, it is optimal for player j to end the auction either in states $(b, b-s)$ or $(b-s, b)$ (she can accomplish that by following the malicious strategy) or in state $(0, 1)$ (she can accomplish that by passing when given a chance to).

Finally, expanding utilities and solving the inequality we have that $u_j(b, b-s) > u_j(0, 1)$ when:

$$b < \frac{1-\alpha_j}{1-2\alpha_j}s - \frac{\alpha_j}{1-2\alpha_j},$$

which concludes the proof. □

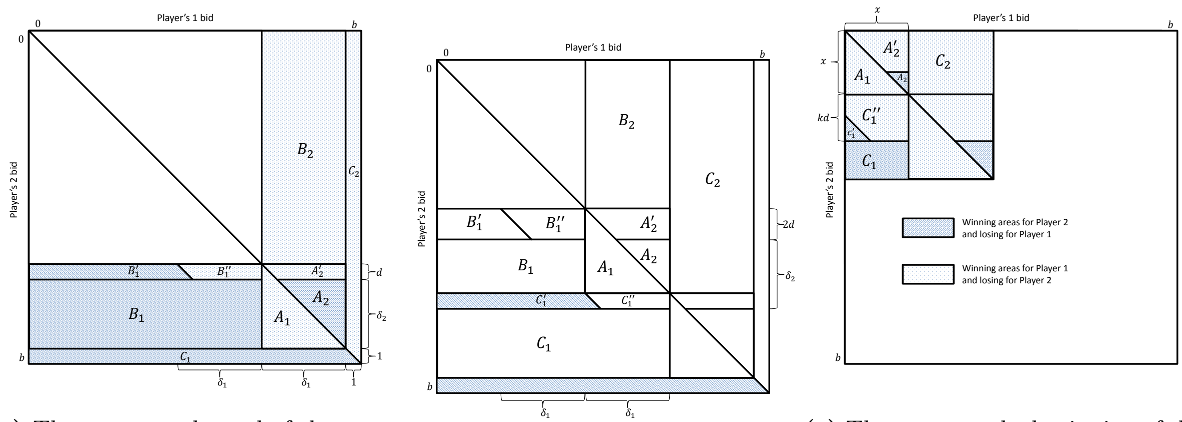
Finally, we describe the optimal strategies in a dollar auction between two weakly spiteful players.

Theorem 4. *Assume that both players participating in the dollar auction are weakly spiteful and spite coefficient of player i is not lower than that of player j, i.e., $\alpha_j \leq \alpha_i < \frac{1}{2}$.*

Let $d = \delta_i - \delta_j$, where δ_i and δ_j are the maximal preserving increases of players i and j, respectively. Also, let $k = \lfloor \frac{b-1}{\delta_i} \rfloor$ and $x = (b-1) \bmod \delta_i + 1$, i.e., $k\delta_i + x = b$, where $0 < x \leq \delta_i$. Let $w = 1$ if $x > \delta_j - kd$, and let $w = x$ otherwise. Finally, let $W = \min(x + kd, \delta_i)$.

If player i makes the first bid (either by a random roll or by the opponent passing her the first bid) and $w < s$, she should bid w. Otherwise she should pass.

If player j makes the first bid and $W < \frac{s-\alpha_j w}{1-\alpha_j}$ and $w < s$, she should bid W. Otherwise she should pass.



(a) The states at the end of the auction

(b) Backtracking and expanding

(c) The states at the beginning of the auction

Figure 3: Winning and losing areas during different moments of the auction.

Proof. Our proof is the extension of the proof presented by O'Neill [11] for the case of two non-spiteful players. Consider the graph-based representation of the dollar auction, discussed in Section 2.1 and illustrated in Figure 1.

We call node (x_1, x_2) a *winning node* for player i if, by starting a game from this node, player i is guaranteed to eventually win the stake s without having to raise her bid by more than δ_i. Note that it is irrational for player i to raise her bid by more than δ_i, as, from the definition of the maximal preserving increase, this would not bring her any higher utility and it would be better for her to pass. Finally, recall that if one player bids all her budget, the game ends.

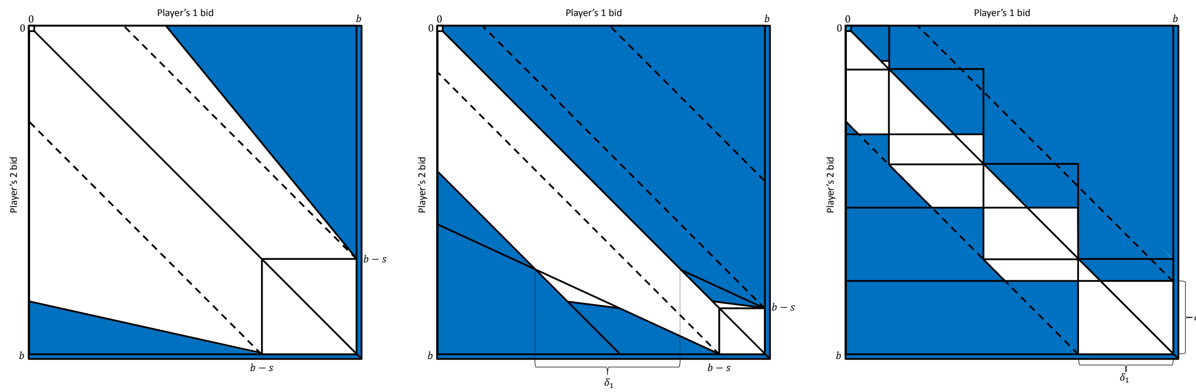
We will now analyse each node of the graph to determine whether it is a winning node for player i or for player j. Figure 3a presents states where at least one bid is close to the whole budget. All figures assume that player 1 is player i, *i.e.*, the player with δ_i no lower than δ_j. The width of areas C_1 and C_2 is one node, while the width of areas B_2 and A_1 is δ_i nodes each. The height of areas B_1 and A_2 is δ_j nodes, while the height of areas B'_1, B''_1 and A'_2 is d nodes each.

Any node in C_1 is a winning one for player 2, since player 1 cannot bid more than b and her only choice is to pass. An analogous analysis holds for C_2. Nodes in A_1 are winning for player 1, since she can end the auction and get the stake by raising her bid by δ_1 or less. Analogical analysis holds for A_2. For nodes in areas B_2 and A'_2 any valid bid increments of 2 are either higher than δ_2 or lead to a state winning for 1. Therefore they are winning for player 1. Analogous analysis holds for areas B_1 and B'_1. that are winning for player 2. Finally, nodes in area B''_1 are winning for 1 because she can make a move from them to a winning area A'_2.

We now analyse states that are gradually closer to $(0, 0)$, as shown in Figure 3b. Previously analysed areas now play the role of areas C_1, C'_1, C''_1 and C_2. The height of areas B_1 and A_2 is $\delta_j - d$ nodes each, as player 2 can make a move to C_1 (and avoid area C''_1) in order to get to a winning state. Therefore, B_1 and A_2 are winning areas for player 2. For the same reason the height of areas B'_1, B''_1 and A'_2 is now $2d$ nodes. Otherwise the previous analysis holds.

We can repeat this process until we reach an area with dimensions that are less or equal than $\delta_i \times \delta_i$, as illustrated in Figure 3c. Width (as well as height) of a left to analyse area is $b \ mod \ \delta_i$ nodes or δ_i nodes if this value is zero. This conditional value can be expressed as $x = (b-1) \ mod \ \delta_i + 1$, as defined in the theorem. In this case, previously analysed area C_2 is winning for player 1, while previously analysed areas C'_1 and C_1 are winning for player 2.

Now, let us analyse what are the possible initial bids of player 1. The state corresponding to the lowest possible bid of player 1, *i.e.*, the state $(1, 0)$ is winning for 1 only if area A'_2 is of a positive height. Height of area A'_2 is $x - (\delta_j - kd)$, as area A_2 contains states where player 2 can get to area C_1 by making bid



(a) Two strongly spiteful players, with $\alpha_1 > \alpha_2 > \frac{1}{2}$. **(b)** A strongly spiteful player 1 and a weakly spiteful player 2. **(c)** Two weakly spiteful players, with $\frac{1}{2} > \alpha_1 > \alpha_2$.

Figure 4: Examples of equilibrium states in a dollar auction (equilibrium states marked with blue color).

increment of at most δ_j. In other case, *i.e.*, when area A'_2 is non-existing, player 1 has to make a bid of x in order to get to a winning state of hers. That gives us the initial bid of $w = 1$ if $x > \delta_j - kd$, and $w = x$ otherwise.

Let us consider now player 2. In order to get to a winning state of hers, she needs to move to area C_1 (the bid of $x + kd$) or to area C'_1 (the bid of δ_i), whichever is smaller. That gives us the initial bid of $W = \min(x + kd, \delta_1)$. Alternatively, if she passes, the player 1 will make the bid of w. To maximize her utility, she should let player 1 do that only when $u_2(0, w) \geq u_2(W, 0)$. After expanding the utilities we get that this is true for:

$$(1 - \alpha_j)W \geq s - \alpha_j w.$$

Therefore, if this is the case, player 2 should let player 1 move first.

It might seem that this is the end of analysis, *i.e.*, the auction should end in either state $(w, 0)$ or $(0, W)$. However, if $w \geq s$ both of those states hold non-positive utility for both players. Therefore, it is optimal for them to never bid and let the auction end without conclusion. If any player would decide to leave this forced stalemate and make a bid lower than s, the other would bid her maximal preserving increase and bring her opponent to negative utility. □

Having described optimal strategies for all types of players, we move to describing the concept of equilibrium states.

3.2 Equilibrium States

We now propose a slightly different setting. Let us assume, that the dollar auction does not start in state $(0, 0)$, but rather in some other state (x_1, x_2), where $x_1 + x_2 > 0$. Such setting might model a confrontation, where one player has advantage at the beginning or was forced into unfavourable situation, for example a new government that has to take part in a conflict started by their predecessors. In order to model such situations we propose the concept of equilibria states.

Definition 3 (Equilibrium state)**.** *We call dollar the auction state* (x_1, x_2)*, such that* $x_1 + x_2 > 0$*, an equilibrium state when player who can make a bid in that state (i.e., player* i *such that* $x_i < x_j$*) decides to pass, knowing her opponent's spite coefficient. We consider state* $(0, 0)$ *an equilibrium state when both players decide to pass in that state.*

In other words, an equilibrium state is the state of the auction where a player that can make a bid decides not to do so and finish the auction, either because she is satisfied with the outcome or because she does not wish to escalate the conflict. First, we solve the case of an auction between two strongly spiteful players.

Theorem 5. *Consider a dollar auction between two strongly spiteful players, i.e.,* $\alpha_1, \alpha_2 \geq \frac{1}{2}$.

State (x_i, x_j) *such that* $x_i + x_j > 0$ *and* $x_i < x_j$ *is an equilibrium state if and only if* $x_i(1 - \alpha_i) \leq \alpha_i(x_j - s) + (1 - 2\alpha_i)(b - s)$ *or* $x_j = b$.

State $(0, 0)$ *is never an equilibrium state.*

Proof. From Lemma 2 we know that player j will not pass in any state where utility of player i is higher than in states $(b-s, b)$ and $(b, b-s)$, but she would rather continue with malicious strategy. Therefore player i should pass in state (x_i, x_j) only if $u_i(x_i, x_j) \geq u_i(b - s, b)$. After expanding utilities from definition and solving the inequality we get that:

$$x_i(1 - \alpha_i) \leq \alpha_i(x_j - s) + (1 - 2\alpha_i)(b - s).$$

Analogical analysis holds for player j.

As shown in the previous section, it is optimal for strongly spiteful player to use malicious strategy when auction starts in state $(0, 0)$. Therefore it is never an equilibrium state. □

Figure 4a presents an example of equilibrium states in an auction between two strongly spiteful players. The higher the spite coefficient, the smaller the equilibria area corresponding to the states where player should not make a bid.

Now we move to a case of an auction between a weakly and a strongly spiteful player.

Theorem 6. *Consider a dollar auction between a strongly spiteful player* i *and a weakly spiteful player* j*, i.e.,* $\alpha_i \geq \frac{1}{2} > \alpha_j$.

State (x_i, x_j) *such that* $x_i + x_j > 0$ *and* $x_i < x_j$ *is an equilibrium state if and only if* $x_j \geq x_i + \delta_i$ *or* $x_i(1 - \alpha_i) \leq \alpha_i(x_j - s) + (1 - 2\alpha_i)(b - s) \wedge \alpha_j(x_i + \delta_i) < (1 - \alpha_j)x_j + (2\alpha_j - 1)b + (1 - \alpha_j)s$ *or* $x_j = b$.

State (x_i, x_j) *such that* $x_i + x_j > 0$ *and* $x_j < x_i$ *is an equilibrium state if and only if* $x_j(1 - \alpha_j) \leq \alpha_j(x_i - s) + (1 - 2\alpha_j)(b - s)$ *or* $x_i = b$.

State $(0, 0)$ *is never an equilibrium state.*

Proof. From the perspective of a weakly spiteful player j, if she continues the auction, player i will never pass before reaching states $(b - s.b)$ or $(b, b - s)$, as in the proof of Theorem 3. Therefore player j should pass in state (x_1, x_2) if her utility is at least the same as in $(b - s.b)$ and $(b, b - s)$. As shown in the proof of Theorem 4a, this is equivalent to:

$$x_j(1 - \alpha_j) \leq \alpha_j(x_i - s) + (1 - 2\alpha_j)(b - s).$$

The same is generally true for a strongly spiteful player i. However, in that case it might happen, that i can make a bid lesser or equal to δ_i (thus increasing her utility) and still get to an equilibrium state, *i.e.*, a state that j passes in. States where i can make such a bid are those where $u_j(x_i + \delta_i, x_j) \geq u_j(b - s, b)$ and $x_j < x_i + \delta_i$. After expanding utilities in the first condition and solving the inequality we get:

$$\alpha_j(x_i + \delta_i) \geq (1 - \alpha_j)x_j + (2\alpha_j - 1)b + (1 - \alpha_j)s.$$

As shown in Theorem 3, it is optimal for a strongly spiteful player to use malicious strategy when auction starts in state $(0, 0)$. Therefore it is never an equilibrium state. □

Figure 4b presents an example of the equilibrium states in an auction between a strongly spiteful player 1 and a weakly spiteful player 2. White the triangular area under the diagonal corresponds to the states in which player 1 can make a move to states where she gets a higher utility than in states $(b-s.b)$ and $(b, b-s)$, but her opponent still passes.

We now move to the case where both players are weakly spiteful.

Theorem 7. *Assume that both players participating in the dollar auction are weakly spiteful and spite coefficient of player* i *is not lower than that of player* j*, i.e.,* $\frac{1}{2} > \alpha_i \geq \alpha_j$.

Let $d = \delta_i - \delta_j$, where δ_i and δ_j are the maximal preserving increases of players i and j respectively. Also, let $k = \lfloor \frac{b-1}{\delta_i} \rfloor$ and $x = (b-1) \bmod \delta_i + 1$, i.e., $k\delta_i + x = b$, where $0 < x \leq \delta_i$. Let $w = 1$ if $x > \delta_j - kd$, and let $w = x$ otherwise. Finally, let $W = \min(x + kd, \delta_1)$.

State (x_i, x_j) such that $x_i + x_j > 0$ and $x_i < x_j$ is an equilibrium state if and only if $x_j = b$ or $x_j \geq x_i + \delta_i$ or $\exists_{l \in \{0,\ldots,k-1\}} x_i \leq x + l\delta_i \wedge x_j \geq x + l\delta_i + (k-l)d$.

State (x_i, x_j) such that $x_i + x_j > 0$ and $x_j < x_i$ is an equilibrium state if and only if $x_i = b$ or $x_i \geq x_j + \delta_j$ or $x_j \leq x - (\delta_j - kd)$ or $\exists_{l \in \{0,\ldots,k-1\}} x_i \geq x + l\delta_i \wedge x_j \leq x + l\delta_i + (k-l)d$.

State $(0, 0)$ is an equilibrium state if and only if $w \geq s$.

Proof. State (x_i, x_j) such that $x_i + x_j > 0$ is an equilibrium state if and only if it is a losing state for a player that can make a bid in it. Such states are described by the formulas in the theorem. Categorization of all states as losing and winning for both players is described in the proof of Theorem 4.

State $(0, 0)$ is an equilibrium state when optimal strategies of both players are to pass in their first move, *i.e.*, when $w \geq s$, as proved in Theorem 4. □

Figure 4c presents an example of equilibrium states in an auction between two weakly spiteful players.

Having solved the case of a dollar auction with complete information, we move to the setting where one player does not know the spite coefficient of her opponent.

4 Incomplete Information Setting

Given the results for the case of complete information, we now turn to discuss a number of results we have derived for the incomplete knowledge case. As mentioned earlier in Section 1, we do not provide a comprehensive Bayesian analysis for this case, due to the difficulty of handling the different cases described in the previous section. However, we show that under some more restricted settings, we can derive optimal or near optimal strategies such that the player can be oblivious to her opponent's spitefulness type (i.e., the opponent's spitefulness coefficient). In particular, we consider the asymmetric information case, where one has a full knowledge of her opponent's (spitefulness) type, whereas the other has not. More formally, we analyse a dollar auction between two players i and j with spite coefficients α_i and α_j, and we assume that while player j knows the value of α_i, player i does not know the value of α_j in advance.

We consider two different settings, one where player j still uses the strategy optimal in the complete information setting described in Section 3 and one where she may deviate from it to achieve higher utility.

4.1 Truthful Opponent

We first describe an optimal strategy of a player i that takes part in an auction against truthful opponent.

Definition 4 (Truthful player)**.** *A truthful player j knowing her opponent's spite coefficient α_i uses optimal strategy from a complete information setting, even when player i does not know the value of α_j.*

The concept of an truthful player j can be interpreted in a way that while she knows the spite coefficient of i, she is not sure whether i knows the value of α_j or not. For now, we assume that player j is truthful, and player i, while the does not have the full information, she does know that j is truthful. We first investigate the case when player i is a strongly spiteful player.

Theorem 8. *If i is a strongly spiteful player playing against an truthful opponent j and i does not know the value of α_j, it is optimal for her to use the malicious strategy.*

Proof. The malicious strategy is optimal for strongly spiteful player against every player in a complete information setting, as described in Theorems 2 and 3. □

We now move to the case of player i being a weakly spiteful player. We will divide our analysis into two cases—when player i makes first bid and when she bids second—beginning with the latter.

Theorem 9. *Consider an auction between a weakly spiteful player i, who does not know the value of α_j, and an truthful player j. Assume that j starts the auction.*

If the first bid of j is different than 1 or $b \geq \frac{1-\alpha_i}{1-2\alpha_i}s - \frac{\alpha_i}{1-2\alpha_i}$, player i knows the optimal strategy. Otherwise, the optimal strategy is either to pass (with expected regret $\Pr[\alpha_j \geq \frac{1}{2}]((2\alpha_i - 1)b + (1-2\alpha_i)s - \alpha_i)$) or to use the malicious strategy (with expected regret $\Pr[\alpha_j < \frac{1}{2}](2 - \alpha_i(\delta_j + 2))$).

Proof. Theorems 3 and 4 indicate that when opponent passes in the first move, it is optimal for i to also pass if $w \geq s$ and bid w otherwise (it is true when $\alpha_i > \alpha_j$, so knowing α_i player i can compute the value of w). The truthful player j makes a bid other than 1, only when she is weakly spiteful player and in all cases it is optimal for i to pass then.

The only case when passing is not the optimal response for player j making the first bid, is when j is strongly spiteful and $b < \frac{1-\alpha_i}{1-2\alpha_i}s - \frac{\alpha_i}{1-2\alpha_i}$. However, it is impossible for player i to distinguish this case from when j is weakly spiteful and $w = 1$ or $W = 1$.

If player i passes instead of using the malicious strategy against a strongly spiteful opponent, the auction ends in state $(0, 1)$, rather than in state $(b - s, b)$ or $(b, b - s)$. Her regret is then equal to:

$$u_i(b - s, b) - u_i(0, 1) = (2\alpha_i - 1)b + (1 - 2\alpha_i)s - \alpha_i.$$

In case the opponent is in fact weakly spiteful, the regret for passing is 0, as it is the optimal strategy.

If player i makes the first move of a malicious strategy instead of passing against a weakly spiteful opponent, she ends the auction in state $(2, \delta_j + 1)$, instead of $(0, 1)$. It is because after second bid of j (that will be raising her bid by δ_j), player i recognizes the situation and can pass. Her regret is then equal to:

$$u_i(0, 1) - u_i(2, \delta_j + 1) = 2 - \alpha_i(\delta_j + 2).$$

In case the opponent is in fact strongly spiteful, i can continue with the malicious strategy, getting the regret of 0, as it is the optimal strategy. □

We now move to the case where player i makes the first bid.

Theorem 10. *Consider an auction between a weakly spiteful player i, who does not know the value of α_j, and an truthful player j. Assume that i starts the auction.*

Let w and W be defined as in Theorem 4, i.e., let w be the optimal initial bid of a more spiteful player and let W be the optimal initial bid of a less spiteful player, and let $y = (1 - \alpha_i)W - s + \alpha_i w$.

Finally, for $z \in \mathbb{N}$ let $\tilde{R}(z) = \Pr[\alpha_i < \alpha_j < \frac{1}{2} \wedge W > z \wedge y < 0](s + (1 - \alpha_i)(z - W) - \alpha_i w) + \Pr[\alpha_i < \alpha_j < \frac{1}{2} \wedge w > z \wedge y \geq 0](1 - \alpha_i)z$ and let $\hat{R}(z) = \tilde{R}(z)$ if $(1 - 2\alpha_i)b < (1 - \alpha_i)s - \alpha_i$ and $\hat{R}(z) = \tilde{R}(z) + \Pr[\alpha_j \geq \frac{1}{2}](1 - 2\alpha_i)z$ otherwise.

The expected regret of an opening bid of $v = (b-1) \bmod \delta_i + 1$ such that $v > 1$ and $v < s$ is $\hat{R}(v) + \Pr[\alpha_i \geq \alpha_j \wedge w = 1](1 - \alpha_i)(1 - v)$. If $v \geq s$ it is never optimal to bid v.

The expected regret of an opening bid of 1 is $\hat{R}(1) + \Pr[\alpha_i \geq \alpha_j \wedge w > 1](s + (1 - \alpha_i)(1 - w) - \alpha_i W)$.

The expected regret of passing is $\Pr[\alpha_j < \frac{1}{2} \wedge \delta_i \geq \delta_j \wedge w \leq s \wedge (1 - \alpha_i)W < s - \alpha_i w](s - (1 - \alpha_i)w - \alpha_i W) + \Pr[\alpha_j < \frac{1}{2} \wedge \delta_i < \delta_j \wedge w \leq s \wedge (1 - \alpha_i)W < s - \alpha_i w](s - (1 - \alpha_i)W - \alpha_i w)$.

Any other opening bids are irrational.

Proof. First, analyse bidding $v = (b - 1)$ *mod* $\delta_i + 1$ or $v = 1$ as the first move, *i.e.*, a situation where player i assumes that player j is weakly spiteful and that $\alpha_i \geq \alpha_j$. If the opponent is strongly spiteful and player i should have passed then the auction will end in state $(v, v + 1)$, rather than in state $(0, 1)$. The regret is then:

$$u_i(0, 1) - u_i(v, v + 1)1 = (1 - 2\alpha_i)v.$$

If the opponent is weakly spiteful, but $\alpha_i < \alpha_j$ and player i should have bid W then the auction will end in state (v, w) (as the opponent will use her optimal strategy), rather than in state $(W, 0)$. The regret is then:

$$u_i(W, 0) - u_i(v, w) = s + (1 - \alpha_i)(v - W) - \alpha_i w.$$

If the opponent is weakly spiteful, but $\alpha_i < \alpha_j$ and player i should have passed then the auction will end in state (v, w) (as the opponent will use her optimal strategy), rather than in state $(0, w)$. The regret is then:

$$u_i(0, w) - u_i(v, w) = (1 - \alpha_i)v.$$

If $v = (b - 1)$ *mod* $\delta_i + 1$, the opponent is weakly spiteful and $\alpha_i \geq \alpha_j$, but the optimal bid of player i is actually 1 then the auction will end in state $(v, 0)$ rather than in state $(1, 0)$. The regret is then:

$$u_i(v, 0) - u_i(1, 0) = (1 - \alpha_i)(1 - v).$$

If $v = 1$, the opponent is weakly spiteful and $\alpha_i \geq \alpha_j$, but the optimal bid of player i is actually $w = (b - 1)$ *mod* $\delta_i + 1 > 1$ then the auction will end in state $(1, W)$ rather than in state $(w, 0)$. The regret is then:

$$u_i(w, 0) - u_i(1, W) = s + (1 - \alpha_i)(1 - w) - \alpha_i W.$$

We now analyse passing as the first move. If the opponent is strongly spiteful, the regret is 0, as even if the malicious strategy is optimal for player i, it can still be executed. If the opponent is weakly spiteful and $\alpha_i \geq \alpha_j$ then the auction will end in state $(0, W)$ (as the opponent will use her optimal strategy) rather than in state $(w, 0)$ for the optimal strategy of player i. The regret is then:

$$u_i(w, 0) - u_i(0, W) = s - (1 - \alpha_i)w - \alpha_i W.$$

If the opponent is weakly spiteful and $\alpha_i < \alpha_j$ then the auction will end in state $(0, w)$ (as the opponent will use her optimal strategy) rather than in state $(W, 0)$ for the optimal strategy of player i. The regret is then:

$$u_i(W, 0) - u_i(0, w) = s - (1 - \alpha_i)W - \alpha_i w.$$

The only other opening move of an optimal strategy is the malicious strategy used against strongly spiteful player in case that $(1 - 2\alpha_i)b < (1 - \alpha_i)s - \alpha_i$. However, the effect of such first move is the same when the first bid is v. Since strongly spiteful player uses the malicious strategy, the auction will end in state $(b - s, b)$ or $(b, b - s)$. □

Having dealt with the case of an truthful opponent, we move to the description of possible ways to deceive an unsuspecting opponent.

4.2 Deceiving Opponent

We now consider an auction between player i, who does not know the value of her opponent's spite coefficient α_j, and player j, who knows the value of α_i. Moreover, we assume that player i considers her opponent truthful, while in fact, player j can deviate from strategy optimal in a complete information case in order to achieve higher utility. In addition, we assume that when player i observes behaviour that is not consistent with any optimal strategy of an truthful player, she passes to avoid further damage. We will now analyse how player j can use assumptions made by player i against her. We begin with the case of strongly spiteful player i.

Theorem 11. *If player i is strongly spiteful, then the deceiving player j can gain nothing by deviating from strategy optimal in the complete information case.*

Proof. Malicious strategy is optimal for player i in all cases against truthful opponent, as shown in Theorem 8. Therefore she can use it without considering the actions of player j. □

We now move to the case of weakly spiteful player i. First we consider an auction against a strongly spiteful player j.

Theorem 12. *Let w be defined as in Theorem 7, i.e., let it be the optimal initial bid of a more spiteful player in an auction with a complete information between two weakly spiteful players. If $(1-2\alpha_i)b \geq (1-\alpha_i)s-\alpha_i$, $w < s$ and strongly spiteful player j starts the auction, then she can increase her utility by passing at the beginning. Otherwise, she can not increase her utility by deviating from the malicious strategy.*

Proof. If player i starts the auction, then from Theorem 10 we know that she either uses malicious strategy or makes only one bid and then either passes or expects her opponent to pass. Either way, any actions of player j other than following the malicious strategy can only hurt her (force her to pay more in order to acquire the stake).

If $(1-2\alpha_i)b < (1-\alpha_i)s-\alpha_i$ then player i uses the malicious strategy, which again cannot be changed by the actions of player j.

Therefore any gain of utility from deceiving is possible only when player j starts the auction and player i does not respond with malicious strategy. However, any bid of j higher than 1 results in i passing, therefore bringing her lower utility than just bidding 1.

However, if player j passes, than player i assumes that she is playing against another weakly spiteful player with lower spite coefficient than hers, as any other type of player would not pass in the first move. Then, if only her optimal initial bid w in this situation is lower than s, she will make it. Player j can then respond with bidding $w+1$, which finishes the auction as player i realizes her opponent is not truthful. Strongly spiteful player j achieves higher utility in state $(w+1, w)$ than in state $(1, 0)$, where the auction would normally end. □

Finally, we consider an auction between two weakly spiteful players.

Theorem 13. *Assume that player j is weakly spiteful. If either $(1-2\alpha_i)b \geq (1-\alpha_i)s-\alpha_i$ or $(1-2\alpha_i)b < (1-\alpha_i)s-\alpha_i$ and player i chooses to pass, then player j can increase her utility by bidding 1 if she starts the auction and overbidding player i by 1 if she bids second. Otherwise she cannot increase her utility by deceiving player i.*

Proof. If $(1-2\alpha_i)b < (1-\alpha_i)s-\alpha_i$ and player i chooses the malicious strategy, then it is optimal for player j to pass.

Otherwise, the reaction of player i to any bid made by player j is passing, as she either considers her opponent a strongly spiteful player using the malicious strategy, or a weakly spiteful player with higher spite coefficient. This way player j can win the auction by overbidding her opponent by 1, rather than letting her win and get the stake. □

5 Conclusions

In our work we investigated the problem of a dollar auction between two spiteful players. We described the optimal strategy for each type of player in a complete information setting, *i.e.*, when each player knows the value of spite coefficient of her opponent. Our results indicate, that while the optimal strategy of a strongly spiteful player (*i.e.*, a player with value of spite coefficient higher or equal to $\frac{1}{2}$) remains the same in most settings, the optimal strategy of a weakly spiteful player highly depends on the available information. Interestingly, it is possible in an auction between two weakly players that the optimal strategy for both players is to pass, leaving the auction unresolved, rather than even start the conflict. We showed which of the initial states are stable, *i.e.*, an auction started in one of them ends without a single bid being made. Again, those equilibrium points proved to be highly dependent on the levels of spitefulness of both players. Finally, we analysed a special case of the incomplete information setting, where one of the players did not know the spite coefficient of her opponent. As it turns out, under some assumptions, a player can learn the type of her opponent by observing her bids.

As a potential future work, we intend to analyse the general incomplete information setting where none of the players has any knowledge about their opponent's spitefulness. This problem is not straightforward, as the theoretical investigation of equilibrium concepts in the Bayesian framework might require novel analytical tools. In particular, as argued earlier, due to the difficult case conditions, it is not trivial how to combine them within the Bayesian framework in order to derive solution concepts. An alternative feasible way is to use online learning theory to provide convergence to such solutions. However, to our best knowledge, such approached has only been successfully applied to repeated games, and not their sequential counterpart, such as the dollar auctions.

References

[1] F. Brandt, T. Sandholm, and Y. Shoham. Spiteful bidding in sealed-bid auctions. In *Computing and Markets*, 2005.

[2] P. Dasgupta. The theory of technological competition. In *New developments in the analysis of market structure*. 1986.

[3] E. Dekel, M. O. Jackson, and A. Wolinsky. Jump bidding and budget constraints in all-pay auctions and wars of attrition. Technical report, Discussion paper//Center for Mathematical Studies in Economics and Management Science, 2007.

[4] G. Demange. Rational escalation. *Annales d'conomie et de Statistique*, (25/26):pp. 227–249, 1992. ISSN 0769489X. URL `http://www.jstor.org/stable/20075865`.

[5] H. Fang. Lottery versus all-pay auction models of lobbying. *Public Choice*, 112(3-4):351–371, 2002.

[6] M. James. http://www.michaeljamesonmoney.com/2009/10/tricky-auction.html. Technical report, 2009.

[7] J. H. Kagel and D. Levin. Auctions: a survey of experimental research, 1995–2008. 2008.

[8] W. Leininger. Escalation and cooperation in conflict situations the dollar auction revisited. *Journal of Conflict Resolution*, 33(2):231–254, 1989.

[9] G. Loomes and R. Sugden. Regret theory: An alternative theory of rational choice under uncertainty. *The economic journal*, pages 805–824, 1982.

[10] T. Mahnken, J. Maiolo, D. Stevenson, et al. *Arms Races in International Politics: From the Nineteenth to the Twenty-First Century*. Oxford University Press, 2016.

[11] B. O'Neill. International escalation and the dollar auction. *Journal of Conflict Resolution*, 30(1):33–50, 1986.

[12] A. Sharma and T. Sandholm. Asymmetric spite in auctions. In *AAAI*, 2010.

[13] M. Shubik. The dollar auction game: A paradox in noncooperative behavior and escalation. *Journal of Conflict Resolution*, pages 109–111, 1971.

[14] P. Tang and T. Sandholm. Optimal auctions for spiteful bidders. 2012.

[15] M. Waniek, A. Niescieruk, T. Michalak, and T. Rahwan. Spiteful bidding in the dollar auction. In *Proceedings of the 24th International Conference on Artificial Intelligence*, pages 667–673. AAAI Press, 2015.

[16] D. Wirls. *Irrational Security: The Politics of Defense from Reagan to Obama*. JHU Press, 2010.