Dynamic adverse selection with many types*

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March 22, 2016

Abstract

I consider a large economy with adverse selection and bilateral matching where trade takes place over several periods. I provide results on welfare and on the dynamics of price offers. First, if adverse selection is severe the dynamic allocation dominates the competitive allocation but if adverse selection is mild, the result is reversed. Second, when the discount factor is high all trade takes place in the last two periods, and under certain concavity assumptions the equilibrium involves at most two prices per period.

*I am grateful to Heski Bar-Isaac, Teddy Kim, Stephan Lauermann and Thomas Wiseman as well as numerous audiences for their useful suggestions. All remaining errors are mine.

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1 Introduction

In many markets the sellers know more about their own items than the buyers. Such markets include used car markets, asset markets, or real estate markets. In many of these markets, the resulting adverse selection problem results in reduced market performance. In extreme cases, the market may break down completely as buyers are unwilling to make offers that are only accepted by sellers with low quality goods. In other cases, the amount of trade supported by simple bargaining protocols may be positive but very limited.

We set up a dynamic matching and bargaining model with a continuum of agents on each side. Buyers and sellers are randomly matched, and the buyer makes a take-it-or-leave-it offer p in each match. If the offer is not accepted, then the pair dissolves and the buyer and the seller go back to the pool to be matched with someone else in the next period until period T when trading stops. Crucially, sellers are heterogenous with respect to quality. Each seller has quality x drawn from a continuous distribution, and his cost of providing the good is $c(x)$ where function c is strictly increasing. Buyers do not observe the quality of the seller's good in any match. However, if a seller has not traded in the first k periods, then the buyer updates his beliefs and attaches a higher probability to the seller having a higher quality good as sellers with lower quality goods are more likely to have traded already (because of their having lower costs).

Our main goal is to compare the centralized (competitive) trading mechanism, and the dynamic decentralized trading game based on the welfare they provide. To highlight the main trade-offs, we benchmark these two trading mechanisms with a one-shot decentralized trading game where the uninformed buyers make the offers. There are two problems that prevent a larger amount of trade to be realized in the one-shot decentralized economy. The first problem is that each buyer makes a price offer, which is lower than his willingness to pay. The solution for this problem is to organize a centralized (competitive) market where agents are price-takers and all trade takes place at the unique competitive price. The second problem is that the one-shot mechanism restricts the extent to which sellers can be screened based on the quality of the good they own. The solution for this problem is to organize a dynamic decentralized market where buyers are able to screen the sellers: We show that with several trading rounds, sellers with low quality trade early and prices go up in later rounds, which leads to higher price offers and allows some of the high quality sellers to trade in later periods.[1]

The above two mechanisms thus both improve on the one-shot decentralized allocation but without further analysis it is unclear which of the two provides higher welfare. To answer this question, we show that welfare is higher under the centralized mechanism than in the dynamic decentralized economy if adverse selection is relatively mild, and the competitive allocation is close to being efficient. In this case the price impact of the party who makes the offer reduces equilibrium welfare, while the single-price allocation is close to being efficient. However, if adverse selection is more extreme, then the competitive allocation does not allow for a significant amount of trade at a single price but the dynamic equilibrium allocation still allows trade to occur by the above described dynamic screening process.

We also characterize the timing of trade when the discount factor is high. We show that all trade occurs in the last two periods as the discount factor approaches 1. The intuition is that if agents traded early, then by waiting for some trade to occur, the buyers could make higher profits as the composition of the sellers have improved. This undermines any outcome where any trade happens in the limit in the first $T-2$ periods, a result different from the two-type variant where trade occurs in the first and the last periods. The reason for this is that the improving composition of the sellers is better captured in our smoother continuous type model than in the two-type models studied so far.

Finally, we show that under appropriate concavity assumptions the equilibrium may be surprisingly simple when the discount factors are high. In particular, a T-period game with sufficient concavity has a single price offered by all buyers in period $T-1$, and two prices offered with positive probability in period T. No trade occurs in the first $T-2$ periods. This shows that many continuous-type specifications preserve the analytic tractability of the two-type game, while they exhibit different trade dynamics and welfare properties.

We conclude the Introduction with a brief literature review. Markets with adverse selection were first studied by the Akerlof (1970). Moreno and Wooders (2015) and Lester and Camargo

[1] A similar result was shown for the two-type case by Moreno and Wooders (2015) and Lester and Camargo (2011).

(2011) are the closest to our work.[2] They both analyze a variant of our model that allows the sellers to have two types.[3] The main novelty is that we show that the centralized economy may be more efficient than the dynamic decentralized trading allocation. Also, we identify factors that influence whether trading take place early or late when the discount factor is high, while in the two-type case some trade always takes place in the very first period. The key difference is that in our model with a continuum of types the competitive price is different from the monopsonistic price that a buyer would offer in a one-shot bargaining game, while if there are only two types the competitive and the monopsonistic prices are identical. Our paper also provides a methodological advance in the following sense. Many of the previous papers including Blouin (2003) assumed that trade can only take place at a few prespecified prices to cut down on analytical complexity. Not only that we do not need such assumptions but we also show that a relatively simple price pattern may arise even if there are a continuum of seller types allowed in a dynamic and decentralized trading game. An important paper that is similar to ours in this respect is Lauermann and Wolinsky (2011) who study different questions. Guerrieri et. al. (2010) study a market with adverse selection and search frictions where there are segmented markets, and the parties can direct their search based on the price posted in each market.

Our paper is also related to Fuchs and Skrzypacz (2014) who study a dynamic game with a single seller where in each period trade takes place at the competitive price. They show that having more periods of trade may or may not increase total welfare but clearly show that having the market open for all possible periods (before the good "expires") cannot be welfare maximizing. Janssen and Roy (2002) also studies a competitive model and show that in equilibrium trade may not take place in every intermediate periods.

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[2] An earlier literature studies the stationary equilibrium in the case where there is entry every period. See for example Kim (2011) and Moreno and Wooders (2010). These papers all provide policy implications for markets with adverse selections. See also Philippon and Skreta (2012) and Tirole (2010) for other mechanism design works in this area. Somewhat less related is Bilancini and Boncinelli (2011) who study markets with adverse selection with finitely many buyers and sellers.

[3] They study several questions that we do not consider, and the model setup is also somewhat more general on other fronts.

2 Model

2.1 Setup and equilibrium

There is an equal mass of buyers and sellers entering the market at $t = 1$. We normalize this mass to be 1. The market stays open for T periods. Each buyer demands a single unit and each sellers owns a single unit. Each period the two sides match randomly. Since there is an equal number of agents on both sides, this boils down to random bilateral matching. Then the buyer makes an offer p without observing the past history of the seller. The seller either accepts, in which case they trade and exit the market. If the seller rejects, then the buyer and the seller go back on the market for the next period and get randomly matched with an agent on the other side of the market. If T is finite, then those who did not trade at or before period T will obtain their outside options (normalized to zero).

The type of a seller $x \in [\underline{x}, \overline{x}]$ is private information to the seller, distributed according to a continuous, strictly increasing and twice differentiable distribution function F. The corresponding density is f. Buyers are homogenous and value a good of type x at x. That is, if a buyer buys a good of type x and pays p, then his utility is $x - p$. The seller's cost is $c(x) < x$ where c is a twice differentiable function of x, and $c'(x) > 0$ for all x. Trading at price p, the seller makes a profit of $p - c(x)$. Finally, both the buyers and the sellers are (risk neutral) expected utility maximizers discounting the future at a discount factor of $\delta \leq 1$.

To capture adverse selection, we assume that it leads to a loss if the buyer makes an offer that is accepted by all sellers in a static model ($T = 1$), that is

$$E(x) = \int_{\underline{x}}^{\overline{x}} f(x)x dx < c(\overline{x}). \tag{1}$$

Otherwise, the competitive allocation would allow us to reach the first best, and thus adverse selection is very mild.

A pure strategy for a buyer is a sequence of price offers made in each period if previous offers were not accepted: $(p_1, p_2, ..., p_T)$. A strategy for a seller of type x is given by cutoffs $(r_1(x), r_2(x), ..., r_T(x))$ with the interpretation that a seller of type x accepts an offer p in period t if and only if $p \geq r_t(x)$.

An equilibrium is a sequence of price offer distribution functions $(G_1, ..., G_T)$ and reservation price strategies $(r_1(x), r_2(x), ..., r_T(x))$ such that

i) Given $(G_1, ..., G_T)$ it is optimal for the sellers to accept according to $(r_1(x), r_2(x), ..., r_T(x))$

ii) Given $(r_1(x), r_2(x), ..., r_T(x))$ it is optimal for the buyers to choose any price offer in the support of G_t in period t.

We focus on monotone equilibria. An equilibrium is monotone if r_t is weakly increasing for all t, that is sellers with lower types accept an offer that a seller with a higher type that accepts. Basic incentive compatibility implies that all equilibria are monotone when $\delta < 1$. We directly select such equilibria when $\delta = 1$ to reflect that $\delta = 1$ is interpreted as the case of vanishing impatience as opposed to no impatience at all.

It is fairly standard to prove that a monotone equilibrium exist for all T and $\delta < 1$.[4] To show that an equilibrium exists for $\delta = 1$ we show that there is a sequence of equilibria as $\delta \to 1$ whose limit is an equilibrium of the game with $\delta = 1$. Let $x_t(p, \delta)$ be the highest type who accepts offer p in period t and let $\widetilde{x}_t(p)$ be the pointwise limit as $\delta \to 1$.[5] If $\widetilde{x}_t$ is continuous, then the payoffs of the buyers are continuous in δ at $\delta = 1$, and existence of equilibrium for $\delta = 1$ follows directly. So, suppose that $\widetilde{x}_t$ is not continuous at some p in δ at $\delta = 1$, and let $\widetilde{x}_t^+(p) = \lim_{\varepsilon \searrow 0} \lim_{\delta \to 1} x_t(p + \varepsilon, \delta) > \widetilde{x}_t^-(p) = \lim_{\varepsilon \searrow 0} \lim_{\delta \to 1} x_t(p - \varepsilon, \delta)$. For a large enough δ it is better to offer p such that it is accepted by almost all types below $\widetilde{x}_t^+(p)$ otherwise with an arbitrarily small increase in the offer, the buyer can improve the composition of seller types accepting his offer and also increase the probability of acceptance. Therefore, for high enough δ the buyer induces all seller types below $\widetilde{x}_t^+(p)$ to accept in equilibrium, and the payoffs of the buyers converge in the sequence of equilibria as $\delta \to 1$.

[4] We can follow Hörner and Vieille (2009) to prove existence. The main difference is that we have long-lived buyers, so continuation values need to be added to their current payoffs but this poses no extra problem as continuation values are well-behaved. Given a vector $(G_1, ..., G_T)$, one can derive the seller types who are indifferent between accepting or not at a given price p in period t. Hörner and Vieille (2009) show these cutoff types to be continuous in $(G_1, ..., G_T)$ and p for an appropriate topology. Therefore, the buyer's period t belief is continuous in $(G_1, ..., G_T)$ as well. Thus best replies of buyers are upper-hemicontinuous in $(G_1, ..., G_T)$, and Glicksberg's fixed point theorem implies that an equilibrium exists. This argument fails for $\delta = 1$ because the cutoff types may not be continuous in the price offered.

[5] Since x_t is monotone in p, therefore a limit exists for an appropriate subsequence as Helly's selection theorem implies. We can just use that subsequence without of loss of generality for our proof.

To provide a formal definition of the conditions for a monotone equilibrium, we introduce further notation. Given that r_t is weakly monotone, let $r_t^{-1}(p) = \sup\{x : r_t(x) \leq p\}$ denote the highest type that accepts an offer p. In other words if an offer p is given in period t, then the set of types $[\underline{x}, r_t^{-1}(p)]$ accept that offer. Let the the buyer continuation values from period t on be denoted as V_t^B, and the continuation values of the sellers be denoted as $V_t^S(x)$. We use the convention that $V_{T+1}^B = V_{T+1}^S(x) = 0$.

Let

$$F_2(x) = \frac{\int_{\underline{x}}^{x} f(t)G_1(r_1(t))dt}{\int_{\underline{x}}^{\overline{x}} f(t)G_1(r_1(t))dt}$$

denote the probability that a seller who participates in period 2 has a type less than x. Similarly, let $F_3 = \frac{\int_{\underline{x}}^{x} f(t)G_1(r_1(t))G_2(r_2(t))dt}{\int_{\underline{x}}^{\overline{x}} f(t)G_1(r_1(t))G_2(r_2(t))dt}$ denote the distribution function for the types of the seller in period 3. For any $t > 3$, F_t can be similarly defined.

Let

$$m(x) = \frac{\int_{\underline{x}}^{x} f(t)tdt}{F(x)}$$

be the expected type of the seller given that it is not higher than x. Let $m_2(x)$ denote the expected type of the seller given that it is not higher than x and that the seller did not trade in period 1. Formally,

$$m_2(x) = \frac{\int_{\underline{x}}^{x} f(t)G_1(r_1(t))tdt}{\int_{\underline{x}}^{\overline{x}} f(t)G_1(r_1(t))dt}.$$

Similarly, $m_3(x) = \frac{\int_{\underline{x}}^{x} f(t)G_1(r_1(t))G_2(r_2(t))tdt}{\int_{\underline{x}}^{\overline{x}} f(t)G_1(r_1(t))G_2(r_2(t))dt}$ is the expected type of the seller given that it is not higher than x and that the seller did not trade in periods 1 and 2. The functions $m_t(x)$ can be defined similarly for $t = 3, 4, ..., T$. In a monotone equilibrium,

$$m \leq m_2 \leq ... \leq m_T. \tag{2}$$

The expected utility from offering price p is period t is

$$u_t(p) = F_t(r_t^{-1}(p))(m_t(r_t^{-1}(p)) - p) + (1 - F_t(r_t^{-1}(p)))\delta V_{t+1}^B \tag{3}$$

A buyer strategy G_t is optimal in period t if for all price offer p in the support of G_t it holds that

$$p \in \arg\max u_t(p). \tag{4}$$

The utility for the seller from accepting a price r in period t is $r - c(x)$, while the option value of waiting is $V_{t+1}^S(x)$. Therefore, the optimal reserve price $r_t(x)$ satisfies

$$r_t(x) - c(x) = \delta V_{t+1}^S(x). \quad (5)$$

The pair of conditions (4), (5) characterizes the equilibrium conditions i)-ii) formally.

Let $\Pr_t^S(x)$ denote the discounted equilibrium probability of trade from period t on for a seller with type x who has not traded before period t. Formally,

$$Pr_t^S(x) = G_t(r_t(x)) + \delta(1 - G_t(r_t(x)))G_{t+1}(r_{t+1}(x)) + ...$$

$$.... + \delta^{T-t}(1 - G_t(r_t(x)))(1 - G_{t+1}(r_{t+1}(x))....(1 - G_{T-1}(r_{T-1}(x))G_T(r_T(x)).$$

The following result characterizes continuation values and optimal reserve prices as functions of seller types.

Lemma 1 *For all* $t = 1, 2, ..., T$, *except for a countable subset of* $[\underline{x}, \overline{x}]$, *the following holds:*

$$V_t^{S\prime}(x) = -Pr_t^S(x)c'(x),$$

and

$$r_t'(x) = c'(x)(1 - \delta Pr_{t+1}^S(x)).$$

Proof. The first result follows from the envelope theorem directly, while the second result combines the first result and (5). Q.E.D.

2.2 Two-type benchmark

Our model is similar to Camargo and Lester (2013) and Moreno and Wooders (2015), except that there are a continuum of types instead of two. (Both of those papers are more general than our setup on other fronts.) It is useful to revisit a simple two-type version of our setup using the result of those two papers. The setup is the same as above except that $x \in \{x_L, x_H\}$ and $c(x_L) = c_L$, $c(x_H) = c_H$. The equilibrium of a $T < \infty$ model is such that each period, except the last period some buyers stay out. Some buyers make low offers accepted by only low type sellers, and some buyers make the offer c_H accepted by both seller types. In the first period offer c_H is made with zero probability.

There are two key characteristics of the equilibrium. First, as $\delta \to 1$ only the first and the last periods see trading in the limit when $T < \infty$. Second, for every $\delta > 0$ the outcome welfare dominates the competitive equilibrium, which is that only low type sellers trade (at price c_L). We show that these two predictions are changed when there are a continuum of types. We also show that the relative simplicity of the price dynamics survives under appropriate concavity assumptions, at least for the case where $\delta = 1$. In this case only a single price is offered in period $T-1$ and two prices are offered with positive probability in period T, just like in the two-type case where as ingle price is offered in period 1 and two prices are offered in the last period.

3 Main results

3.1 Equilibrium when δ is high

We focus on the case where the dynamic features of the markets are the strongest, that is when $\delta = 1$ holds. More precisely, we are interested in the limit of equilibria as $\delta \to 1$, which is also an equilibrium for the $\delta = 1$ game. We show that as long as we concentrate on equilibria for the $\delta = 1$ game that are monotone, all our results hold. Since the limits of all equilibria as $\delta \to 1$ are monotone, this also implies that all equilibria for δ high enough behaves similarly to the monotone equilibria of the $\delta = 1$ case, as characterized below. Therefore, we can interpret our results as approximately holding for the case in which $\delta < 1$ but close to 1.

We start by solving the $T = 2$ case, and then prove that the equilibrium of the two-period game remains an equilibrium when additional periods are added.

Proposition 1 *Let $\delta = 1$ and $T > 2$, and take an equilibrium $(\widetilde{G}_1, \widetilde{G}_2, \widetilde{r}_1)$ of the two-period game. It is an equilibrium of the T-period game if all buyers make non-serious offers that are rejected by all seller types in the first $T-2$ periods, and in the last two periods agents use strategies $G_{T-1} = \widetilde{G}_1$, $G_{T-2} = \widetilde{G}_2$, $r_{T-1} = \widetilde{r}_1$.*

Proof. See the Appendix.

The main step of the proof, Lemma 3 in the Appendix, shows that in the equilibrium of the two-period game buyers are indifferent between staying out or making an offer in period 1, in fact a positive fraction stays out in period 1 in equilibrium. Therefore, the buyers gain their utility by participating in the last period. Therefore, if there are more than two periods it is intuitive that the equilibrium of the two period game remains an equilibrium together with all buyers making non-acceptable offers in periods $1, 2, ..., T-2$.

The next result shows that in the T period game there are no other equilibria than the ones identified above, that is trade is delayed until the last two periods.

Proposition 2 *When $\delta = 1$ and $T > 2$, then there is no equilibrium with a positive measure of trade in the first $T-2$ periods, that is, $G_T(r_T(x)) = 1$ for all $t \leq T-2$ and $x > \underline{x}$.*

Proof. See the Appendix.

The proof exploits two features of the equilibrium. First, by (2),

$$m \leq m_2 \leq ... \leq m_T.$$

Second, by (5), and $V_1^S \geq V_2^S \geq ..... \geq V_T^S$,[6]

$$r_1 \geq r_2 \geq ... \geq r_T.$$

Both of these effects imply that buyers are better off waiting, and thus early trade cannot occur in equilibrium.

Moreno and Wooders (2015) show that if there are two types, then trade vanishes except in the first and the last periods, a quite different result. The proof of Proposition 2 identifies a profitable local deviation where types just above the types who already accepted in previous periods are poached by the buyers in the next period. In the two-type model there is no such profitable local deviation - because there are no types just above the low type. In this respect the two-type model behaves in a non-generic way.

[6] The time pattern of outside options follows from $\delta = 1$.

3.2 Welfare comparisons

In the two-type model it is well known that welfare is higher in the decentralized dynamic trading model than in the centralized (one-shot) economy as captured by the competitive equilibrium. In this Section, we show that this is not the case in our model, and provide a discussion of the main trade-offs involved.

The competitive equilibrium price is the highest price p^* such that break-even occurs if that price is offered. To formally define this price, recall that $m(y) = E[x : x \leq y] = \frac{\int_{\underline{x}}^{y} f(x)x dx}{F(y)}$ is the expected type of the seller given that it is less than y. Then $p^* = c(x^*)$ where x^* is the highest value that satisfies

$$m(x^*) = c(x^*).$$

We show that if almost all seller types transact in a competitive equilibrium, then the equilibrium of our model offers less than full trading and has thus lower welfare. Formally, take a sequence of economies with cost functions and distribution functions that converge uniformly, $\{c_k, F_k\}_{k=1,2,...} \rightarrow (\widetilde{c}, \widetilde{F})$, and assume that

(i) $\lim_{k\rightarrow\infty} m_k(x) - c_k(x) > 0$ for all $x < \overline{x}$;

(ii) for all k, $m_k(\overline{x}) - c_k(\overline{x}) < 0$.

Under (i), and (ii), the competitive outcome converges to full trade, that is $x_k^* \rightarrow \overline{x}$, and

$$\lim_{k\rightarrow\infty} (m_k(\overline{x}) - c_k(\overline{x})) = 0. \tag{6}$$

Proposition 3 *Take a sequence of economies such that $\{c_k, F_k\}_{k=1,2,...} \rightarrow (\widetilde{c}, \widetilde{F})$ uniformly, and (i), (ii) hold. The competitive equilibrium features full trading in the limit, while any equilibrium of the dynamic market features less than full trading in the limit for any T if $\delta = 1$; that is, there is a type $\widehat{x} < \overline{x}$ such that for all $x > \widehat{x}$, $\lim_{k\rightarrow\infty} Pr_{1,k}^S(x) < 1$.*

Proof. Given Proposition 2, it is sufficient to consider the case in which $T = 2$. Suppose that the limit of the dynamic game features an equilibrium with full trade, that is, for all $x < \overline{x}$, $\lim_{k\rightarrow\infty} Pr_{1,k}^S(x) = 1$.

Step 0. I first show that in a full-trade equilibrium each seller with type $x < \overline{x}$ trades at an expected price $e_k(x)$ such that $\lim_{k\to\infty} e_k(x) - \lim_{k\to\infty} c_k(\overline{x}) = 0$.

Given Lemma 1, in a full-trade equilibrium it holds that

$$\lim_{k\to\infty} (V_{1,k}^{S\prime}(x) - c_k'(x)) = 0.$$

But in a full-trade equilibrium, it also holds for all $x < \overline{x}$ that

$$\lim_{k\to\infty} V_{1,k}^{S}(x) = \lim_{k\to\infty} e_k(x) - \lim_{k\to\infty} c_k(x).$$

The last two equations yield that $\lim_{k\to\infty} e_k'(x) = 0$ for all $x < \overline{x}$, and thus for some $e \in \mathbb{R}$,

$$\lim_{k\to\infty} e_k(x) = e, \ \forall x < \overline{x}$$

Given, that all types $x < \overline{x}$ trade with a probability converging to 1, therefore individual rationality implies that $e = \lim_{k\to\infty} c_k(\overline{x})$.

Step 1. Buyers obtain zero expected utility in the limit, that is, $\lim_{k\to\infty} V_{1,k}^{B} = 0$.

This follows from Step 0 directly, as trades take place at expected prices close to $\lim_{k\to\infty} c_k(\overline{x})$ in the limit, which provides the break-even point for the buyers under (6). Formally, let $w_{1,k}$ denote the mean utility of the participating buyers,

$$w_{1,k} = \int_{\underline{x}}^{\overline{x}} Pr_{1,k}^{S}(x)x dx - \int_{\underline{x}}^{\overline{x}} Pr_{1,k}^{S}(x)e_k(x)dx.$$

Under the assumption of full trade, the result of Step 0, and ((6)), we obtain that

$$\lim_{k\to\infty} w_{1,k} = \lim_{k\to\infty} (m_k(\overline{x}) - c_k(\overline{x})) = 0.$$

All buyers obtain the same expected utility, that is, for all k,

$$V_{1,k}^{B} = w_{1,k}.$$

Therefore, the last two formulas imply the statement of Step 1.

Step 2. Because for each k there is adverse selection, there is an $x_k < \overline{x}$ such that $G_{1,k}(r_{1,k}(x)) = 0$.

Step 3. By Step 2, we can define the expected type of the seller $x_{2,k}$ who stays on the market for the second period,

$$x_{2,k} = \frac{\int_{\underline{x}}^{\overline{x}} f(t)G_{1,k}(r_{1,k}(t))tdt}{\int_{\underline{x}}^{\overline{x}} f(t)G_{1,k}(r_{1,k}(t))dt}.$$

Case 1. Suppose that $x_{2,k} \to \overline{x}$. Then staying out in period 1 and offering a price $c_k(\overline{x})$ in period 2 would mean that the buyer trades in period 2 for sure and receives a good with an expected quality converging to $\overline{x}$. Therefore, this buyer makes a profit of $V_{1,k}^B \to \overline{x} - c(\overline{x}) > 0$, contradicting Step 1.

Case 2. Suppose that $\liminf_{k\to\infty} x_{2,k} < \overline{x}$. Then there is a subsequence of k for which there is a value $x' < \overline{x}$ such that the probability that a seller has a type less than equal to x' has a positive limit. Formally,

$$\limsup_{k\to\infty} F_{2,k}(x') = \limsup_{k\to\infty} \frac{\int_{\underline{x}}^{x'} f(t)G_{1,k}(r_{1,k}(t))dt}{\int_{\underline{x}}^{\overline{x}} f(t)G_{1,k}(r_{1,k}(t))dt} > 0$$

holds for some $x' < \overline{x}$. Moreover, the utility conditional on trade is $\limsup_{k\to\infty} m_{2,k}(x') - c_k(x') \geq \lim_{k\to\infty} m_k(x') - c_k(x') > 0$, where the first inequality follows from the fact that lower types accept more offers than higher types in a monotone equilibrium, and thus $m_2 \geq m$. Therefore, using (3), the utility from staying out in period 1 and offering $c(x')$ in period 2 satisfies

$$\limsup_{k\to\infty} V_{1,k}^B = \limsup_{k\to\infty} F_{2,k}(x') \limsup_{k\to\infty} (m_{2,k}(x') - c_k(x')) > 0.$$

However, this contradicts Step 1, and concludes the proof. Q.E.D.

It is also interesting to compare the outcome of our dynamic game ($T > 1$) with the standard lemon problem where $T = 1$. Moreno and Wooders (2015) show that in the two-type case there is more trade when $T > 1$ than when $T = 1$. This seems plausible with many types as well. Given that trade in periods before the last period improves the composition of the remaining seller types, it should be profitable in the last period to offer prices that are higher than the static monopsony price. Therefore, even if all the trade before the last period was ignored, the amount of trade would still be higher than in the one-period model. In the next Section, we prove this result under further assumptions as part of Claim 2.

The above discussion confirms that while the static equilibrium typically features less trade than the dynamic equilibrium but the competitive (centralized) outcome can be even better than the outcome of the dynamic trading game. The main trade-off between the competitive and dynamic outcomes is as follows. On one hand, a dynamic model allows more screening and thus more trade than any static outcome including the centralized competitive equilibrium. On the other hand, the competitive equilibrium features zero profit and thus lower prices than the equilibrium of the dynamic trading game. This enhances trading volumes. The comparison between the competitive and the dynamic equilibria are ambiguous in general. If the competitive equilibrium allocation is close to full trade, then it dominates the dynamic outcome. However, if the distribution is more similar to a two-type distribution (as studied by Moreno and Wooders (2015) and Lester and Camargo (2011)), then the dynamic outcome dominates the competitive allocation as there is not much trade in the competitive allocation itself. From another angle, in the two-type model the monopsonistic price is equal to the competitive price and thus the trade-off between the dynamic allocation and the competitive allocation is trivially in favor of the dynamic allocation.

We now provide a set of sufficient conditions under which the competitive allocation is constrained efficient, that is it maximizes total surplus in the class of all incentive compatible and individually rational allocations that balance budget across all agents. In this case the competitive allocation dominates the welfare of the dynamic trading game as well *regardless of the discount factor* δ. Let x_m denote the highest seller type who trades in the static economy where buyers make (monopsony) offers.[7]

Proposition 4 *Assume that* $\frac{f(x-c)}{Fc'}$ *is strictly decreasing on* $[x_m, \overline{x}]$. *Then the competitive allocation is welfare maximizing in the class of all incentive compatible and individually rational allocations that balance budget across all agents.*

Proof. Let $g(x)$ be the probability that type x of the seller trades. Basic incentive compatibility implies that g is monotone decreasing. The utility of the seller is $v(x)$. The envelope theorem implies that $v'(x) = c'(x)g(x)$. It is easy to show that the highest type makes a zero profit in the optimal allocation. Also, the buyers make zero profits in the optimal allocation.

[7] Formally, $x_m = \arg\max\{F(m-c)\}$. In case of multiple maximizers, take x_m to be the highest value in this set.

Therefore, total welfare can be written as

$$\int_{\underline{x}}^{\overline{x}} f(x)v(x)dx = \int_{\underline{x}}^{\overline{x}} F(x)g(x)c'(x)dx, \tag{7}$$

upon using $v(\overline{x}) = 0$ and performing integration by parts. The buyers binding participation constraint can be written as

$$\int_{\underline{x}}^{\overline{x}} f(x)g(x)(x - c(x))dx = \int_{\underline{x}}^{\overline{x}} F(x)g(x)c'(x)dx. \tag{8}$$

Then the problem is to maximize (7) with respect to (8). After setting up the Lagrangian and taking a derivative with respect to $g(x)$ we obtain

$$L_g = F(x)c'(x) + \lambda[f(x)(x - c(x)) - F(x)c'(x)].$$

Under our assumptions, $\frac{L_g}{F(x)c'(x)} = 1-\lambda+\lambda\frac{f(x)(x-c(x))}{F(x)c'(x)}$ is decreasing in x and thus L_g is positive if and only if x is smaller than a threshold. Therefore, the pointwise optimum calls for a bang-bang solution where $g(x) = 1$ for all $x \leq \widetilde{x}$ and $g(x) = 0$ otherwise. Given the buyers' participation constraint this boils down to implementing the competitive allocation. Q.E.D.

Theorem 2 in Fuchs and Skrzypacz (2014) proves a closely related result for a dynamic trading environment. They consider an environment where the competitive price is set every period, and show that if $\frac{f(x-c)}{Fc'}$ is decreasing and a further assumption holds, opening the market only once at the beginning (and a prespecified end time T, if any) is welfare optimal compared to any other arrangement where markets are open more often. They allow the case where there is no end time ($T = \infty$ in their notation), which is identical to our setting. Our result is not a consequence of their Theorem 2: Our approach allows the allocation to be implemented by arbitrary mechanisms that satisfy only incentive compatibility and budget balance (which include the set of dynamic games considered by Fuchs and Skrzypacz (2014)).[8]

An easy-to-interpret sufficient condition on the primitives is stated in the next Corollary:

[8] Also, we only assume the condition for prices above the monopsony price ($x > x_m$), while they assume it for all values of x.

Corollary 1 *Assume that m'/c' is weakly decreasing. Then the competitive allocation is welfare maximizing in the class of all incentive compatible and individually rational allocations that balance budget across all agents.*

Proof. See the Appendix.

The assumption that m'/c' is decreasing is easy to interpret. Let

$$\widetilde{m}(c_0) = E[x : c(x) \leq c_0]$$

denote the expected use value if the cost is less than c_0. If the monopsonist makes a price offer c_0, his profit *conditional on acceptance* is $\widetilde{m}(c_0) - c_0$ in the one shot game. The assumption that m'/c' is weakly decreasing is equivalent to the profit conditional on acceptance being concave in the price offered.[9]

Corollary 1 states that if the profit conditional on acceptance is concave in the price offered, then the competitive allocation is welfare maximizing in the set of feasible mechanisms. This result is interesting in itself, as the literature has not studied such conditions before. Moreover, the result suggests that the important object is the expected profit of the uninformed agents conditional on their offers being accepted, and not the expected profit itself.

It is interesting to relate these results to Samuelson (1984) who also studies the welfare optimal mechanism in a similar two-agent game. He shows - and his result applies in our setting as well - that the optimal allocation can be implemented as a mechanism where the buyers offer a single price or randomize between two price offers. The above result provides conditions under which a single price offer (the competitive allocation in our large economy) is welfare maximizing.

[9] Take function $m(c^{-1}(z)) = \widetilde{m}(z)$. It holds that $\widetilde{m}'(z) = m'(c^{-1}(z))(c^{-1})'(z) = \frac{m'(c^{-1}(z))}{c'(c^{-1}(z))}$. Letting $x = c^{-1}(z)$, we have $\widetilde{m}'(z) = \frac{m'(x)}{c'(x)}$. If $\frac{m'(x)}{c'(x)}$, then given that $z = c(x)$ is an increasing function of x, it follows that $\widetilde{m}'(z)$ is decreasing in z, which is just concavity of $\widetilde{m}$.

4 Equilibrium under concavity

4.1 Assumptions and analysis

The main assumption of this Section is that m'/c' is strictly decreasing. As we explained above, this assumption is equivalent to the profit of the monopsonist conditional on acceptance being concave in the offer made. The reason why the assumption is on the profit conditional on acceptance is because it is optimal to stay out in the first $T-1$ periods, and thus the profit conditional on acceptance has to be equal to the profit achievable in the last period, a fixed equilibrium quantity. Therefore, any offer made in the first period must maximize the profit of the buyer conditional on acceptance. We also make the standard assumption that $c'' \geq 0$, which is only needed for the argument for period 2 price offers and may be substantially weakened. We show that it is also sufficient for our result to assume $c' \geq 1$ instead of $c'' \geq 0$.[10]

Under these assumptions, we show that there are only a few prices offered if the discount factor is high and the number of periods are finite. To study this question, we analyze the simplest case where $\delta = 1$. Given Corollary 1, and the discussion afterwards, it is sufficient to study the $T = 2$ case because in the first $T-2$ there is no trade in the T-period game in equilibrium.

We start the analysis by a useful result:

Lemma 2 *Under our assumptions, the static monopsony profit $U_s = F(m-c)$ is single-peaked in the cutoff type x: there exists x_m such that $U_s'(x) < 0$ if $x > x_m$; and $U_s'(x) > 0$ if $x < x_m$.*

Proof. See the Appendix.

By Lemma 3 in the Appendix, buyers are indifferent between trading or not in period 1. Recall, that $r_1(x)$ denotes the price that makes a seller of type x indifferent between accepting and not. Then buyer indifference between trading or staying out in period 1 implies that for all cutoff level $\overline{x}_1$ that is induced in equilibrium in period 1, it must hold that

$$m(\overline{x}_1) - r_1(\overline{x}_1) = V_2^B. \tag{9}$$

[10] The assumption that $c' \geq 1$ is also standard as it refers to a case where adverse selection is severe, see Fuchs and Skrzypacz (2014) for details.

Also,

$$\text{for all } x,\, m(x) - r_1(x) \leq V_2^B. \tag{10}$$

At any point x where r_1 is differentiable, (9) and (10) imply that

$$m'(\overline{x}_1) - r_1'(\overline{x}_1) = 0 \tag{11}$$

for all cutoff $\overline{x}_1$ used in equilibrium.

Given that $\delta = 1$ and $Pr_2^S(x) = 1 - G_2(c(x))$, Lemma 1 yields that if $r_1'(x)$ exists, then

$$r_1'(x) = c'(x)G_2(c(x)). \tag{12}$$

For point where $G_2(c(x))$ is not continuous (jumps up), r_1' does not exist: Both the right-hand and the left-hand derivatives exist, the right-hand derivative being strictly larger. In particular, the right-hand derivative satisfies

$$\tau(x) = c'(x)G_2(c(x)). \tag{13}$$

Claim 1 *In equilibrium a unique cutoff level $\overline{x}_1$ is used, that is, all buyers offer the same price r_1 (or stay out) in period 1.*

Proof. By (13), it holds that

$$\alpha = \frac{m' - \tau}{c'} = \frac{m'}{c'} - G_2(c).$$

Because m'/c' is strictly decreasing and $G_2(c)$ is increasing, it follows that α is strictly decreasing.

Let

$$\widetilde{x} = \sup_{x \in [\underline{x}, \overline{x}]} \{x : \alpha(x) \geq 0\}.$$

I show that $\widetilde{x} \in (\underline{x}, \overline{x})$. Otherwise, suppose that $\alpha < 0$ for all $x \in [\underline{x}, \overline{x}]$. Then all buyers would stay out in period 1, which contradicts Lemma 3 in the Appendix. Then suppose that $\alpha > 0$ for all $x \in [\underline{x}, \overline{x}]$. But this contradicts with the assumption of adverse selection, which implies that $m(\overline{x}) - r_1(\overline{x}) \leq m(\overline{x}) - c(\overline{x}) < 0$ and thus $m - r_1$ must become negatively sloped on some interval (x_1, x_2). At almost all points on (x_1, x_2), $\alpha < 0$ as $\alpha = \frac{m' - r_1'}{c'}$ everywhere except for at most countably many points.

Finally, $\widetilde{x} \in (\underline{x}, \overline{x})$ is the unique maximizer of $m(x) - r_1(x)$ as α is positive at any $x < \widetilde{x}$ and α is negative at any $x > \widetilde{x}$, and $\alpha = \frac{m' - r_1'}{c'}$ everywhere except for at most countably many points. Q.E.D.

Given Lemma 2, the optimal offer is unique because the first period profit function is single-peaked in the cutoff type chosen. Next we show that in period 2 there are exactly two prices used in equilibrium.

Claim 2 *In period 2, buyers mix between offering two prices: the static monopsony price* $c(x_m)$ *and* $c(\overline{x}_2)$ *where* $\overline{x}_2 > x_m$.

Proof. See the Appendix.

The proof shows that the second-period profit of buyers is single-peaked in the cutoff type on $[\underline{x}, \overline{x}_1]$ and also on $[\overline{x}_1, \overline{x}]$ and thus there are at most two reserve prices used in equilibrium in period 2. This is the only result of the Section that relies on $c'' \geq 0$ (or $c' \geq 1$).

4.2 Explicit characterization

Let $\overline{x}_1$ be the highest trading seller in period 1 at a price p_1. In period 2, buyers mix by offering the static monopsony price $c(x_m)$ and offering $c(\overline{x}_2)$. Let α_1 be the probability of making the offer p_1 in period 1 (otherwise staying out), and α_2 the probability of making the low offer in period 2 (that is the monopsony offer).

Then there are five variables $(p_1, \overline{x}_1, \overline{x}_2, \alpha_1, \alpha_2)$ and we provide five (necessary and sufficient) conditions for a monotone equilibrium. Buyer indifference between period 1 offer p_1 or staying out requires that the buyer is indifferent between his offer being accepted or not in period 1. If the offer is accepted his payoff is $m(\overline{x}_1) - p_1$, and upon rejection his continuation payoff is $U_2(x_m)$ where function U_2 is as introduced in (38). Therefore, this condition boils down to

$$m(\overline{x}_1) - p_1 = U_2(x_m). \tag{14}$$

Buyer indifference in second period between low and high price yields

$$U_2(\overline{x}_2) = U_2(x_m). \tag{15}$$

Seller indifference between accepting in period 1 or not requires that $p_1 - c(\overline{x}_1) = (1-\alpha_2)(c(\overline{x}_2) - c(\overline{x}_1))$ or

$$p_1 = \alpha_2 c(\overline{x}_1) + (1-\alpha_2)c(\overline{x}_2). \tag{16}$$

Noting, that $\overline{x}_1 \neq \overline{x}_2, x_m$ we obtain that r_1 is differentiable at $\overline{x}_2$. Therefore, buyer incentives for not offering slightly different price in period 1 requires $m'(\overline{x}_1) = r_1'(\overline{x}_1)$ by (33). Therefore, $m'(\overline{x}_1) = r_1'(\overline{x}_1)$ where $r_1(x) = \alpha_2 c(x) + (1-\alpha_2)c(\overline{x}_2)$, so $r_1' = \alpha_2 c'$ and thus

$$m'(\overline{x}_1) = \alpha_2 c'(\overline{x}_1). \tag{17}$$

Finally, buyer incentives for $c(\overline{x}_2)$ to be optimal in period 2 is simply either

$$U_2'(\overline{x}_2) = 0. \tag{18}$$

or

$$U_2'(\overline{x}) \geq 0, \tag{19}$$

depending on whether $\overline{x}_2 < \overline{x}$ or $\overline{x}_2 = \overline{x}$.

Proposition 5 *There exists an equilibrium of the form characterized by formulas (14) - (19).*

Proof. Equilibrium existence and the results above (Claims 1 and 2) imply that a solution exists and is an equilibrium for the game. Q.E.D

The fact that the equilibrium is so simple - one price in period 1 and two prices in period 2 - is a consequence of the concavity of function $\widetilde{m}$. This assumption holds for a wide range of specifications, and is easy to interpret. The relative simplicity of the equilibrium opens up the possibility for future extensions of this simple model.

5 Discussion

To study the robustness of our results, it is interesting to consider an infinite horizon version of our model where end-date effects are absent. Hörner and Vieille (2009) consider a model in which a single long-lived seller interacts with a fixed sequence of short-lived buyers in an infinite horizon game with discounting. Depending on whether the previous offers are observed or not, they consider two cases. The case of private offers, where offers are not observed by subsequent buyers, is similar to our model. The fact that their model features a single seller, while our model has a large economy with a continuum of sellers does not invalidate their analysis for our game because in the two models buyers form similar beliefs upon being called to make an offer in periods $t = 2, 3....$ Therefore, when extending our model to infinite horizon, the results of Hörner and Vieille (2009) are relevant for our analysis. The main complication in using the results of Hörner and Vieille (2009) is that our buyers are long-lived, unlike the buyers of Hörner and Vieille (2009). Therefore, when buyers make their offers, the buyers of our model need take their continuation values into account. Take an equilibrium of the HV game with private offers, $(G_1, r_1, G_2, r_2...)$ for a fixed level of discounting δ. Given the above discussion, if that equilibrium is such that the utility of buyers $t = 2, 3, ...$ is zero, that is, $V_t^B = 0$, then $(G_1, r_1, G_2, r_2...)$ is an equilibrium of our economy as well, with the suitable reinterpretation of the meaning of the strategies. However, HV show that there are equilibria where $V_t^B > 0$ for large values of t, so the equivalence between the two models break down, and our discussion has to stay at an informal level.

There are two results in HV that together suggest that there is no convergence (as $\delta \to 1$) to the competitive allocation in the infinite horizon version of our model. First, Propositions 4 and 5 in HV imply that the sum of the discounted utilities of the buyers converge to zero in every equilibrium of the HV game. Therefore, one may conjecture, that the equilibria in the HV game have similar properties as the equilibria of the infinite horizon version of our model when δ is large; see the discussion in the previous paragraph. Second, in their Proposition 3, HV show that under the assumption of adverse selection (1), it holds that as $\delta \to 1$ almost all types of the seller have positive discounted probability of trading. This is a markedly different allocation from the competitive allocation in which only types below a cutoff trade with probability one, while types above the cutoff do not trade with a positive probability at all. The welfare consequences of our conjecture of non-convergence would be straightforward. Suppose that m'/c' is decreasing. Then the competitive allocation is welfare maximizing in the set of feasible allocations, and thus the outcome of the infinite horizon trading game would provide a lower welfare than the centralized, competitive allocation as $\delta \to 1$.

6 Conclusion

We study a dynamic model of adverse selection in a decentralized economy. We show that depending on the extent of adverse selection the (static) centralized allocation may or may not welfare dominate the dynamic decentralized allocation. The paper highlights the key trade-offs and mechanisms at work, and discusses the intuition behind those results. We also show that when the discount rate is high, trade is deferred until the very end of the game. Finally, we show that the equilibrium is strikingly simple for a wide range of economies that satisfy certain concavity properties.

7 Appendix

Proof Proposition 1:

Proof. We start the proof with a Lemma:

Lemma 3 *Let $\delta = 1$ and $T = 2$. In the first period, a fraction of the buyers $\alpha \in (0, 1)$ make offers that are rejected by all seller types.*

Proof of the Lemma. First, we show that $\alpha > 0$. Let y be defined as

$$y = \sup\{x : G_2(r_2(x)) = 0\}, \tag{20}$$

and

$$y_1 = \sup\{x : G_1(r_1(x)) = 0\}. \tag{21}$$

Step 1. In this step, we show that $y > y_1 \geq \underline{x}$ and

$$V_2^B = F_2(y)(m_2(y) - c(y) > 0. \tag{22}$$

Let $U_2(x) = F_2(x)(m_2(x) - c(x)$. In period 2, buyers solve (3) for $t = 2$:

$$V_2^B = \max_x U_2(x) = \max_x F_2(x)(m_2(x) - c(x).$$

By construction, $U_2(\underline{x}) = 0$.

For all $\varepsilon > 0$, it holds that

$$F_2(y_1 + \varepsilon) > 0, \tag{23}$$

and by (21),

$$m_2(y_1 + \varepsilon) = \frac{\int_{\underline{x}}^{y_1+\varepsilon} f(t)G_1(r_1(t))tdt}{\int_{\underline{x}}^{y_1+\varepsilon} f(t)G_1(r_1(t))dt} = \frac{\int_{y_1}^{y_1+\varepsilon} f(t)G_1(r_1(t))tdt}{\int_{y_1}^{y_1+\varepsilon} f(t)G_1(r_1(t))dt} \geq y_1. \tag{24}$$

Also, $y_1 > c(y_1)$, by assumption. Then (24) implies that for $\varepsilon > 0$ small enough,

$$m_2(y_1 + \varepsilon) > c(y_1 + \varepsilon). \tag{25}$$

Therefore, by (3), (23), and (25),

$$U_2(y_1 + \varepsilon) > 0,$$

and thus, setting a cutoff of y_1 or less cannot be optimal in period 2. Therefore, $y > y_1 \geq \underline{x}$ and $V_2^B > 0$. By (20), and by the continuity of U_2 it follows that

$$V_2^B = U_2(y) = F_2(y)(m_2(y) - c(y).$$

Step 2. All seller types $x < y$ have the same reservation price in period 1 by (12), and (20); that is,

$$\forall x < y, r_1(x) = r_1(\underline{x}). \tag{26}$$

Step 3. Suppose that $\alpha = 0$, that is, all buyers make offers that are accepted by some types in period 1. Then by (26), all offers are actually accepted by all types up to type y. Consequently,

$$\alpha = 0 \Longrightarrow F_2(y) = 0. \tag{27}$$

Then (22) and (27) yield a contradiction; and thus, $\alpha > 0$ must hold.

Suppose now that $\alpha = 1$, that is, there is no trade in period 1. Then all buyers offer the monopsony price $c(x_m)$ in period 2. Then offering $c(x_m + \varepsilon)$ in period 1 is a profitable deviation for ε small enough, as the buyer now has two chances for trading, and the price achieved remains almost the same if ε is small. Thus the proof of the Lemma is complete. Q.E.D.

Now, we are ready for rest of the proof for Proposition 1:

Step 1. First, I show that if the buyers use strategies $G_{T-1} = \widetilde{G}_1$, $G_{T-2} = \widetilde{G}_2$, then the optimal reservation price strategies of the sellers satisfy $r_t(x) \geq \widetilde{r}_1(x)$ for all $t = 1, 2, ..., T-2$.

By (5), and the fact that all offers before period $T-1$ are not profitable to accept we obtain that

$$r_t(x) = c(x) + V_{t+1}^S(x). \tag{28}$$

Also, by (5)

$$\widetilde{r}_1(x) = c(x) + \widetilde{V}_2^S(x). \tag{29}$$

Given that all other agents' strategies satisfy $G_{T-1} = \widetilde{G}_1$, $G_{T-2} = \widetilde{G}_2$, $r_{T-1} = \widetilde{r}_1$, it follows that $\widetilde{V}_2^S(x) = V_{T-1}^S(x)$. Moreover, for all $t = 1, 2, ..., T-2$,

$$V_{t+1}^S(x) \geq \widetilde{V}_2^S(x) \tag{30}$$

as the seller can always just reject the offer in period $T-1$ (and all offers between periods $t+1$ and $T-2$), and then he is in the same situation in terms of continuation values as in the second period of the two-period game. Then (28), (29) and (30) imply the claim of Step 1.

Step 2. By Lemma 3, it is optimal in the two-period game for the buyers to abstain in period 1, and make an optimal offer in the last (second) period. The utility from having an offer $\widetilde{r}_1(x)$ accepted in the two-period game in period 1 is $m(x) - \widetilde{r}_1(x)$. The utility from rejection (or abstention) is $\widetilde{V}_2^B$. Since abstention is optimal in period 1 of the two-period game, for all $x \in [\underline{x}, \overline{x}]$,

$$m(x) - \widetilde{r}_1(x) \leq \widetilde{V}_2^B. \tag{31}$$

Step 3. By playing according to the purported equilibrium in the T-period game, buyers make a profit of $\widetilde{V}_2^B$ because they all abstain in periods $1, 2, ..., T-2$ and (by Lemma 3) they can obtain their optimal profits by also abstaining in period $T-1$.

Step 4. It is thus sufficient to show that a higher payoff than $\widetilde{V}_2^B$ cannot be achieved by the buyers when T is larger than 2.

The utility from trading with types x and below in any period $t = 1, 2, ..., T-2$ is $m(x) - r_t(x)$. Using Step 1 and (31), for all $x \in [\underline{x}, \overline{x}]$,

$$m(x) - r_t(x) \leq m(x) - \widetilde{r}_1(x) \leq \widetilde{V}_2^B.$$

Therefore, there is no profitable deviation in periods $1, 2, ..., T-2$. Strategies are optimal in periods $T-1, T-2$ by construction. Q.E.D.

Proof of Proposition 2:

Proof. Assume that there is trade in period 1, and we show that in period 2 there is a profitable deviation for the buyers. The argument can be extended in a straightforward manner (extending Lemma 3 in the process) to show that when $\delta = 1$ and $T > 2$, then there is no trade in equilibrium in the first $T-2$ periods.

Step 1. Using the proof of Lemma 3, it follows that a positive fraction of buyers abstain in periods $1, 2, ..., T-1$. Given this, for all $x \in [\underline{x}, \overline{x}]$ it holds that $G_1(r_1(x)) > 0$. Recall that $m_t(x)$ denotes the expected type of the seller given that it is not higher than x and that the seller did not trade in period 1. Consequently, it holds that for all $\widetilde{x}$ such that $r_t(\widetilde{x})$ is an optimal offer in period $t = 1, 2, ..., T-1$,

$$m_t(\widetilde{x}) - r_t(\widetilde{x}) = V_T^B.$$

Moreover, for all $x \in [\underline{x}, \overline{x}]$,

$$m_t(x) - r_t(x) \leq V_T^B. \tag{32}$$

Using, the last two displays it follows that if $r_t(\widetilde{x})$ is an optimal reserve price in period t and r_t is differentiable at $\widetilde{x}$, then

$$m_t'(\widetilde{x}) - r_t'(\widetilde{x}) = 0. \tag{33}$$

Step 3. Let $x_{\min} = \inf\{x : G_1(r_1(x)) > G_1(r_1(\underline{x}))\}$. Such an $x_{\min} < \overline{x}$ exists by the assumption of adverse selection (1), which implies that no offers are accepted by all seller types in period 1. Let $x' \in [x_{\min}, \overline{x})$ be defined as $x' = \sup\{x : r_1(x) = r_1(x_{\min})\}$. Then given the monotonicity of sellers' acceptance strategies we obtain that the composition of the sellers improves over time:

$$\text{for all } x > x_{\min},\ m_2(x) > m(x). \tag{34}$$

Step 4. By the definition of x' it is optimal for the buyers to make an offer $r_1(x')$ in period 1, which together with Step 1 yields that

$$m(x') - r_1(x') = V_T^B. \tag{35}$$

Step 5. It holds for all $x \in [\underline{x}, \overline{x}]$ that

$$r_2(x) \leq r_1(x), \tag{36}$$

because of (5) and the fact that $V_3^S(x) \leq V_2^S(x)$ when $\delta = 1$.

Step 6. We show that $m_t(x' + \varepsilon) - r_t(x' + \varepsilon) > V_T^B$ for some $t = 2, 3, ..., T-1$, which contradicts (32), and concludes the proof.

Case 1. Suppose that G_1 has no atom at $r_1(x_{\min})$.

In this case, there exists an $\varepsilon > 0$ such that for all $x \in (x', x' + \varepsilon)$ it is optimal to offer price $r_1(x)$ in period 1. By Step 1, for all $x \in (x', x' + \varepsilon)$ we have that $m(x) - r_1(x) = V_T^B$. Then (34) and (36) imply together with $m(x) - r_1(x) = V_T^B$ that

$$m_2(x) - r_2(x) > m(x) - r_1(x) = V_T^B,$$

so Step 6 is complete for Case 1 with $t = 2$.

Case 2. Suppose that G_1 has an atom at $r_1(x_{\min}) = r_1(x')$.

i) If $r_2(x') < r_1(x')$, then $m_2(x') \geq m(x')$ implies that $m_2(x') - r_2(x') > m(x') - r_1(x') = V_T^B$.

ii) Therefore, we only need to consider the case where $r_2(x') = r_1(x')$ and $m_2(x') = m(x')$ otherwise (32) is violated and the proof is complete.

iii) An iterative argument implies that we only need to consider the case where for all $t = 2, 3, ..., T-1$ it holds that $r_t(x') = r_1(x')$ and $m_t(x') = m(x')$ otherwise the period t version of (32) is violated and abstaining is not optimal in period t, violating Step 1.

iv) The condition that $m_t(x') = m(x')$ for all $t = 2, 3, ..., T-1$ implies that no prices lower than $r_t(x') = r_1(x')$ are offered with positive probability in periods $t = 1, 2, 3, ..., T-2$.

v) The condition that $r_t(x') = r_1(x')$ for all $t = 2, 3, ..., T-1$ implies that no prices above than $r_t(x') = r_1(x')$ are offered with positive probability in periods $t = 2, 3, ..., T-1$. By construction, no price lower than $r_1(x')$ is offered in period 1.

vi) We show that in period T it is not optimal for the buyers to offer a price that induces exactly types on $[\underline{x}, x']$ to accept.

If $x' \leq x_m$, the monopsony price, then this follows directly as it can only be best reply to choose a cutoff type of above x_m in period T. If $x' > x_m$, then this follows by noticing that given iv) and v) all price offers in periods $t = 1, 2, ..., T-2$ are exactly at $r_t(x')$. Therefore, the profit function at period T has a kink in it at the cutoff type x', which makes the right-hand derivative larger than the left-hand derivative. This prevents the second-order condition to hold at $x = x'$, and thus the claim of point vi) follows.

vii) Given the result of vi), it follows that r_{T-1} is differentiable at x'.

To see this, note that for all x

$$r_{T-1}(x) - c(x) = V_T^S(x).$$

Letting $\Pr_T^S(x)$ denote the trading probability of a seller with type x in period T, the envelope theorem implies (see Lemma 1) that at all continuity points of $\Pr_T^S(x)$, $V_T^{S\prime}(x) = -c'(x)\Pr_T^S(x)$, and thus $r'_{T-1}(x) = c'(x)(1 - \Pr_T^S(x))$. Given vi), $\Pr_T^S(x)$ is continuous at $x = x'$, and thus the statement in vii) follows.

viii) By point iii) and Step 4, $m_{T-1}(x') - r_{T-1}(x') = V_T^B$. By Step 1, for all $x < x'$, it holds that $m_{T-1}(x) - r_{T-1}(x) \leq V_T^B$. Therefore, the left-hand derivative of $m_{T-1} - r_{T-1}$ is non-negative at x'.

ix) Under the assumption of Case 2 (and thus points iv) and v)), it follows that the right-hand derivative of m_{T-1} is larger than the left-hand derivative at $x = x'$. Therefore, given point vii), the right-hand derivative of $m_{T-1} - r_{T-1}$ is larger than the left-hand derivative at $x = x'$. But the left-hand derivative is non-negative, so the right-hand derivative is strictly positive. Therefore, by point viii), $m_{T-1}(x' + \varepsilon) - r_{T-1}(x' + \varepsilon) > V_T^B$ for some $\varepsilon > 0$ small enough. This, however, contradicts Step 1 and concludes the proof of Proposition 2. Q.E.D.

Proof of Corollary 1:

Proof. We only need to show that for all $x < x^* < y$ it holds that

$$\frac{f(x)(x-c(x))}{F(x)c'(x)} \geq \frac{f(x^*)(x^*-c(x^*))}{F(x^*)c'(x^*)} \geq \frac{f(y)(y-c(y))}{F(y)c'(y)}, \tag{37}$$

because choosing the the multiplier such that $1-\lambda+\lambda\frac{f(x^*)(x^*-c(x^*))}{F(x^*)c'(x^*)}=0$ we have that $L_g(y) \leq 0 \leq L_g(x)$.

To show (37), note that under m'/c' being decreasing implies that $m(x) > c(x)$ for all $x < x^*$.[11] Thus

$$\frac{f(x)(x-c(x))}{F(x)c'(x)} > \frac{f(x)(x-m(x))}{F(x)c'(x)} = \frac{m'(x)}{c'(x)} \geq \frac{m'(x^*)}{c'(x^*)} =$$
$$= \frac{f(x^*)(x^*-m(x^*))}{F(x^*)c'(x^*)} = \frac{f(x^*)(x^*-c(x^*))}{F(x^*)c'(x^*)}.$$

The case of $y > x^*$ requires a symmetric argument. Q.E.D.

Proof of Lemma 2:

Proof. The static profit function is

$$U_s(x) = F(x)(m(x)-c(x)) = \int_{\underline{x}}^{x} f(t)tdt - c(x)F(x).$$

Taking a derivative, we obtain that the buyers' static profit function has the derivative

$$U_s'(x) = f(x-c) - Fc'.$$

Under our assumption that m'/c' is strictly decreasing, Corollary 1 implies that $f(x-c)/Fc'$ is strictly decreasing. Therefore, $U_s'(x) = Fc'(\frac{f(x-c)}{Fc'}-1)$ changes sign only at one point (at $x = x_m$) where $\frac{f(x-c)}{Fc'} = 1$. Q.E.D.

Proof of Claim 2:

Proof. Let α_1 be the probability of making the offer p_1 in period 1 (otherwise the buyer stays out). Using Claim 1, we obtain that the mass of sellers left on the market for period 2 are as follows:

$$f_2(x) = (1-\alpha_1)f(x) \text{ for } x \leq \overline{x}_1$$

[11] Inequality $m(x) < c(x)$ would imply that for some $z \in (x, x^*)$ it holds that $m'(z) > c'(z)$. But due to $m(\underline{x}) > c(\underline{x})$ and $m(x) < c(x)$ there must be some $w < x < z$ such that $m'(w) < c'(w)$, which then contradicts with our assumption that m'/c' is decreasing.

and

$$f_2(x) = f(x) \text{ for } x \geq \overline{x}_1,$$

because all seller types less than $\overline{x}_1$ accept in period 1 with probability α_1 (an offer of p_1), while all other seller types never accept.

The problem of the buyer in period 2 is to maximize

$$U_2(x) = \int_{\underline{x}}^{x} f_2(t)tdx - c(x)F_2(x), \tag{38}$$

and thus

$$\forall x \leq \overline{x}_1,\ U_2 = (1-\alpha_1)F(x)(m(x) - c(x)). \tag{39}$$

Using that for all $x > \overline{x}_1$, $F_2(x) = F(x) - \alpha_1 F(\overline{x}_1)$, the profit at $x > \overline{x}_1$ is equal to

$$U_2(x) = (F(x) - \alpha_1 F(\overline{x}_1))(m_2(x) - c(x)) = \int_{\underline{x}}^{t} f_2(x)xdx - (F(x) - \alpha_1 F(\overline{x}_1))c(x). \tag{40}$$

The derivative at any $x > \overline{x}_1$ becomes

$$U_2' = f_2(x)(x - c(x)) - F_2(x)c'(x) = \tag{41}$$

$$f(x-c) - (F - \alpha_1 F(\overline{x}_1))c'.$$

Step 1: $\overline{x}_1 > x_m$ must hold in equilibrium.

Step 1.1: We show that

$$\overline{x}_1 \leq x_m \Longrightarrow \overline{x}_2 = c^{-1}(p_2) > \overline{x}_1.$$

In period 2, any optimal choice satisfies

$$p_2 > c(\overline{x}_1).$$

To see this, note that the static monopsony problem is single-peaked by Lemma 2. For all $x \leq \overline{x}_1$, the profit function in period 2 is proportional to the static profit function by (39). Therefore, given that the static profit function is single-peaked by Lemma 2, the static profit function must be strictly increasing for all values $x \leq \overline{x}_1 \leq x_m$. Thus for all $x \leq \overline{x}_1$,

$$U_2(\overline{x}_1) \geq U_2(x).$$

Moreover, by (41) the right-hand derivative of U_2 is

$$U_2'(\overline{x}_1) = f(x-c) - (F - \alpha_1 F(\overline{x}_1))c'$$

$$> f(x-c) - Fc' \geq 0,$$

where the second inequality follows because the static profit function is increasing for values less than $\overline{x}_1$. Thus U_2 is increasing in a neighborhood $[\overline{x}_1, \overline{x}_1 + \varepsilon]$, and thus for some small $\varepsilon > 0$, it holds for all $x \leq \overline{x}_1$ that

$$U_2(\overline{x}_1 + \varepsilon) > U_2(x).$$

Step 1.2: By Step 1.1, the optimal cutoff type in period two satisfies $\overline{x}_2 = c^{-1}(p_2) > \overline{x}_1$ if $\overline{x}_1 \leq x_m$. Therefore, there exists a type $\widehat{x} > \overline{x}_1$ such that all types smaller than $\widehat{x}$ trade for sure in period 2. Thus, $G_2(c(x)) = 0$ for all $x < \widehat{x}$, and $r_1'(x) = 0$ (see (12) above). Therefore, $\overline{x}_1$ cannot be the equilibrium cutoff type in period 1 as the offer $r_1(\overline{x}_1)$ is accepted by all types up to type $\widehat{x}$, a contradiction.

Step 2: In period 2 at least two prices are offered with positive probabilities.

Suppose only one price $p_2 = c(\overline{x}_2)$ was offered in equilibrium by all buyers in period 2. The second period price has to be weakly lower than the first period price otherwise no seller would accept in equilibrium in period 1, contradicting Lemma 3. Thus $p_1 = c(\overline{x}_1)$ and $p_2 = c(\overline{x}_2)$ need to hold with $\overline{x}_2 \leq \overline{x}_1$.

Case 1. Suppose that $\overline{x}_2 < \overline{x}_1$.

Using (39), the first order condition of the buyer's decision problem in period 2 boils down to the static problem, and thus $\overline{x}_2 = x_m$. Since $p_2 = c(\overline{x}_2)$, it holds that $r_1(x) = c(x)$ for all $x > \overline{x}_2$. Therefore, r_1 is differentiable at $\overline{x}_1$ and the first-order condition in period 1 (see (33)) implies that $m'(\overline{x}_1) - c'(\overline{x}_1) = 0$. Since m'/c' is decreasing, it follows that $m'(\overline{x}_2)/c'(\overline{x}_2) > m'(\overline{x}_1)/c'(\overline{x}_1) = 1$. But by construction, $m'(\overline{x}_2)/c'(\overline{x}_2) = m'(x_m)/c'(x_m) < 1$, a contradiction.

Case 2. Suppose that $\overline{x}_2 = \overline{x}_1$.

As we argued in Step 1, the profit function U_2 has a kink upward at $\overline{x}_1$ as the left-hand derivative equals to $f(x-c) - Fc'$, and the right-hand derivative equals $f(x-c) - (F - \alpha_1 F(\overline{x}_1))c'$.

Therefore, the local second-order condition is violated and $\overline{x}_2 = \overline{x}_1$ cannot be the optimal cutoff in period 2.

Step 3: In period 2 at most two prices are offered.

By (39), for all $x \leq \overline{x}_1$, U_2 simplifies to $(1 - \alpha_1)F(x)(m(x) - c(x))$. This expression is proportional to the static monopoly profit, which is single-peaked and thus has the unique maximizer x_m by Lemma 2. We only need to show that there is a unique maximum for U_2 on $[\overline{x}_1, \overline{x}]$ as well to conclude that there at most two prices offered in period 2. The following Lemma is established below:

Lemma 4 *There is a unique maximum for U_2 on $[\overline{x}_1, \overline{x}]$.*

Step 4: There are exactly two prices offered in equilibrium and there are two local maximizers, x_m and $\overline{x}_2 > \overline{x}_1$, which concludes the proof of Claim 2. Q. E. D.

Lemma 4: *There is a unique maximum for U_2 on $[\overline{x}_1, \overline{x}]$.*

Proof. At any point $x \geq \overline{x}_1$ it holds that

$$\frac{U_2'(x)}{(F - \alpha_1 F(\overline{x}_1))c'} = \frac{f(x-c)}{(F - \alpha_1 F(\overline{x}_1))c'} - 1 = z(x) - 1, \tag{42}$$

where

$$z(x) = \frac{f(x-c)}{(F - \alpha_1 F(\overline{x}_1))c'} = \frac{f(x-c)}{Fc'} \frac{F}{F - \alpha_1 F(\overline{x}_1)}.$$

Let $x \in (\overline{x}_1, x^*)$, and let $\alpha = \frac{x-c}{x-m}$. Then let

$$\beta = \frac{f(x-c)}{Fc'} = \frac{m'(x-c)}{c'(x-m)} = \frac{m'}{c'}\alpha.$$

Step 1. First, we show that β is decreasing on $[x_m, x^*]$. Then $z = \beta \frac{F}{F - \alpha_1 F(\overline{x}_1)}$ is decreasing and U_2 is increasing, decreasing or has a single peak on $(\overline{x}_1, x^*]$ by (41).

Upon taking a derivative, we obtain that

$$\alpha' \overset{sgn}{=} (1 - c')(x - m) - (1 - m')(x - c) = (c - m) + (m'(x - c) - c'(x - m)) = \gamma. \tag{43}$$

Under m'/c' being decreasing and $x < x^*$ we have that (see the proof of Corollary 1)

$$m(x) > c(x). \tag{44}$$

Taking a derivative yields

$$\beta' = \left(\frac{f(x-c)}{Fc'}\right)' = \left(\frac{m'}{c'}\right)' \alpha + \frac{m'}{c'}\alpha'. \tag{45}$$

At $x = x_m$ it holds that $m'(x-c) - c'(x-m) = 0$ and thus $\alpha'(x_m) < 0$ by (43) and (44). Then under our assumption that m'/c' is decreasing, $\beta'(x_m) < 0$ holds by (45). Also, by construction $\beta(x_m) = 1$. Take the smallest value $x \in (\overline{x}_m, x^*)$ such that $\beta'(x) = 0$. Then it must hold by (45) that $\alpha'(x) \geq 0$. Then (43) and (44) imply that $m'(x-c) - c'(x-m) > 0$, which implies that $\beta(x) > 1$. But since β is decreasing on $[x_m, x)$ and $\beta(x_m) = 1$, this yields a contradiction. Therefore, β is decreasing on $[x_m, x^*]$, and Step 1 is complete.

Step 2.

Case 1: $U_2'(x^*) \leq 0$

Then using (37) and that $F/(F - \alpha_1 F(\overline{x}_1))$ is decreasing we have that $U_2'(x^*) \leq 0$ implies that $z(y) < z(x^*)$. Therefore, (42) implies that $U_2'(y) \leq 0$ for all $y > x^*$. Thus in this case there is a unique maximizer on $(\overline{x}_1, \overline{x})$, which is on the interval $(\overline{x}_1, x^*)$.

Case 2: $U_2'(x^*) > 0$

Then again (37) and the fact that $F/(F - \alpha_1 F(\overline{x}_1))$ is decreasing implies that for all $x \in (\overline{x}_1, x^*)$ we have that $U_2'(x) > 0$. Then al the maximizers must be above x^*. Also, by (43) and $m(x^*) = c(x^*)$,

$$\alpha'(x^*) \overset{sgn}{=} \beta(x^*) - 1 < 0.$$

Then using that m'/c' is decreasing, we have that $m''c' \leq c''m'$. This implies that

$$\gamma' = \big((c-m) + (m'(x-c) - c'(x-m))\big)' = m''(x-c) - c''(x-m) \leq$$

$$\leq c''(\frac{m'}{c'}(x-c) - (x-m)) \overset{sgn}{=} c''(\frac{f(x-c)}{Fc'} - 1) = c''(\beta - 1).$$

Take a point $x > x^*$ such that $\alpha'(x) = 0$. Formula (37) implies that for $x > x^*$, $\beta(x) < \beta(x^*) < 1$. If $c'' \geq 0$, then $\gamma'(x) < 0$ and thus $\alpha''(x) < 0$.[12] Since $\alpha'(x^*) < 0$, the smallest such point $x > x^*$ where $\alpha'(x) = 0$ must have $\alpha''(x) \geq 0$, a contradiction. Therefore, α is decreasing on $[x^*, \overline{x}]$, and thus by (45), β is also decreasing, which clearly implies that z is decreasing on

[12]This holds because $\gamma(x) = 0$ and $\gamma'(x) < 0$ implies that $\gamma(x - \varepsilon) > 0$, and thus by (43) it holds that $\alpha'(x-\varepsilon) > 0$, which implies $\alpha''(x) < 0$.

$[x^*, \overline{x}]$. Therefore, by (42), U_2 is single-peaked on $[x^*, \overline{x}]$, which means that there is a unique local maximum on $[x^*, \overline{x}]$. But we have already argued above, that all the maximizers must be above x^*, which then establishes the result for the case in which $c'' \geq 0$.

Now suppose that instead of $c'' \geq 0$, we have that $c'(x) \geq 1$ for all $x > x^*$. Then

$$\gamma \overset{sgn}{=} \frac{c-m}{x-m} + c'(\beta - 1) = (1 - \frac{x-c}{x-m}) + c'(\beta - 1) =$$

$$= 1 - \beta \frac{c'}{m'} + c'(\beta - 1).$$

Given that $\beta \in (0,1)$ by (37), $\frac{c'}{m'} \geq 1$ for all $x > x^*$, and $c' \geq 1$, we obtain that $\gamma < 0$. Thus $\alpha' < 0$ and $\beta' < 0$, which concludes our proof using the same steps as above. Q.E.D.

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