Cheap Talk in Multi-Product Bargaining

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Abstract

I study a game in which a buyer and a seller bargain over a stock of goods. The buyer has private valuations over the seller's stock, and can communicate these through costless signalling, i.e. cheap-talk. Whilst a fully revealing, efficient equilibrium exists, standard refinements ([6],[22]) select a set of partially informative equilibria, which I characterize fully. In particular, the set contains the ex-ante buyer optimal equilibrium. The "refining down"nature of this result holds for a broad class of preferences that satisfy standard single-crossing and single-peakedness conditions, and rests on a natural trade-off faced by the buyer - *reveal* his type, in order to secure his preferred good, or *hide* his type, in order to secure a better price. As goods become less substitutable, the buyer reveals more information, whilst his share of the surplus changes non-monotonically.

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1 Introduction

In many interactions, the following natural question emerges: should I *reveal* who I am, in order to get what I want, or should I *hide* who I am, in order to get the best deal? Consider the following canonical situation. You're about to buy a carpet. There is a range of them at the store that you can see. If you don't point to the carpet you want, you might engage in largely pointless negotiations with the seller simply offering the wrong product. But if you point to the magic carpet, the seller might charge a higher price. Now consider an employee who knows that her inclinations (or ability) are particularly suited to a specific role in her organization. She might want to signal this, and yet worry that the firm would leverage this information to capture the gains for itself via an increased workload. Many more examples of such a trade-off can be found in settings ranging from real estate markets, price comparison websites, product design and procurement contracting.

Three features are essential in exploring such a tension. First, the principal must *lack commitment power*. If the seller can commit ex-ante to prices, the buyer need not fear a higher price by obscuring their preferences. Second, there must be a degree of *horizontal differentiation* in preferences. If it is commonly known that one carpet is preferred to all the others, then the seller will offer that carpet for sure, thus obviating the role of communication. Finally, the *bargaining power* cannot lie solely with either party. Under her optimal mechanism, the worker would simply demand their preferred job at the firm's marginal cost, side-stepping the subtle informational incentives captured in the tension, whilst if available to them, the firm could use a full menu of offers to separate out the worker types.

I analyze these issues in a simple bargaining game with cheap-talk. A buyer faces a seller with multiple, heterogeneous goods, over which the buyer has private valuations. They engage in the following bargaining game. First, the buyer sends a costless signal of their valuation type to the seller. The seller then makes a take-it-or-leave-it price offer for a good from his stock. Finally, the buyer may accept or reject this offer. As is often the case in games with cheap-talk, my model admits a large multiplicity of equilibria. I appeal to standard refinements to help discipline my predictions, in particular [6]'s concept of "neologism-proofness".

My main result is that, whilst a fully separating, efficient equilibrium exists, the set of neologism-proof equilibria involve only partial information revelation. This proposition is unusual in that in typical situations, refinements generally serve to generate greater separation, not less. The result turns on a fundamental property of the underlying game, namely that it is neither one of pure conflict nor pure coordination. The intuition is as follows. The buyer's gain from pooling comes through price discounts; if types pool, the seller's optimal price decreases, as he attempts to trade with a broader set of types. If these discounts are sufficient large, this gain outweighs the buyer's loss from being offered a sub-optimal good. However, if the seller's optimal price is not discounted enough, the buyer's loss from being mis-matched outweighs the price gain. The refinement selects equilibria around this turning point.

I characterize the refined set fully, proving that the ex-ante buyer-optimal equilibrium is always contained in it, and computing bounds on both buyer welfare, as well as on how informative such equilibria are. Such a characterization points to a fundamental property of the neologism-proofness; when the underlying game is neither one of pure coordination nor pure conflict - as is the case here - neologism-proofness works in favor of the sender.

The model also delivers several comparative static results; some intuitive, others less so. For instance, as goods become less substitutable, more information is revealed; there is little point buying a worthless good, even at a discount. However, the buyer's share of the surplus in his preferred equilibrium varies non-monotonically. Two countervailing forces are at play here. As the buyer's loss from product mismatch increases, information provision increases in a manner that does not perfectly offset this loss.

Literature and Contribution. That the set of neologism-proof equilibria are neither fully separating nor babbling for a wide range of parameters, even when both exist, seems a novel result. Other cheap talk models have delivered fully, or almost-fully separating equilibria. These typically involve multiple dimensions, either within the type space, action space or number of players (e.g. [1], [2], [19]). Recently, [21] exhibit cheap-talk equilibria that are not Pareto-rankable in a setting with many competing senders. A literature also exists on cheap-talk in bargaining games: see [7], [23], [9]. See [18] for an application of cheap-talk to trading in larger markets. The focus in these papers is on showing how cheap-talk can improve the equilibrium payoff set. For example, the unique neologism-proof equilibrium in [7] and [18] is separating, which is never the case in my model.

The tension driving incentives to reveal information bears some resemblance to the so-called *ratchet effect* identified in the early literature on dynamic adverse selection (see [14], [20]). [15] show how switching between skill-independent jobs can break the link between current and future incentives, thereby reducing the ratchet effect. Both their model and results are far removed from the current setting. In particular, in their setting, types are independent across jobs, rather than perfectly correlated, whilst the agent's ability to signal comes solely in the form of costly effort, rather than cheap talk.

The paper is structured as follows. Section 2 introduces the benchmark model, proving some basic properties of equilibria. Section 3 introduces various refinements, characterizing the set of neologism-proof equilibria, briefly discussing welfare and finally extending the main result to a broader set of preferences. Section 4 tests the robustness of the model to its various restrictions, and section 5 concludes with some possible extensions. All major proofs are relegated to the Appendix.

2 Model

A buyer and a seller interact. The seller has a commonly known stock of indivisible, heterogenous goods, each indexed by $V = [0, 1]$. The stock V is common knowledge. The buyer's privately known valuation for a good v is captured by some θ drawn from an ambient space Θ, which I also set as $[0, 1]$. A buyer of type θ has willingness-to-pay for good v given by $u(v, \theta)$, where $u \in \mathcal{C}^2(V \times \Theta)$. The seller's prior over Θ is given by some continuous distribution function F. The seller values all goods at 0; that is, I focus on the sub-game following any production or procurement required on the seller's part.[1]

To make the central point as cleanly as possible, I employ the following specification in the first part of the paper: $u(v, \theta) = \bar{u} - a(v - \theta)^2$, where $\bar{u}, a > 0, V = [0, 1]$ and $F \sim \mathcal{U}[0, 1]$. The uniformity assumption is made purely for tractability, whilst the quadratic-loss preference structure is standard in models of horizontal product differentiation. In Section 3.3, I show that the qualitative features of the main result holds for a far broader class of preferences satisfying single-crossing and single-peakedness assumptions.

2.1 Game

The players play the following game. First, the buyer sends a message $m \in \mathcal{M}$ to the seller, where $\mathcal{M}$ is some set large enough to reveal θ. The seller then makes an offer of some good v to the buyer at price p, which the buyer either accepts or rejects. Figure 2.1 shows the game. It is worth highlighting the restriction imposed on the seller's strategies, namely that he cannot commit to a full menu of good-dependent contracts.[2] Restricting the seller to offer only one good at a take-it-or-leave-it price is

[1] Adding small costs of production does not disrupt the analysis.

[2] Such contracts take the form $\{(q(v), p(v))\}_{v \in V}$, where $q : V \to [0, 1]$ are probabilities of trade, and $p : V \to [0, 1]$ are the associated transfers.

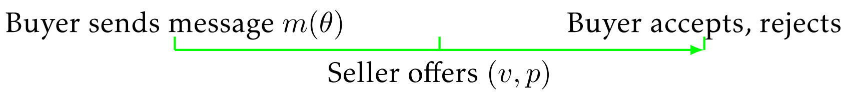

Figure 1: Game

one of many ways of restricting the seller's bargaining power, which as explained earlier is an essential component for the analysis. I discuss menus and other related issues in section 4.1.

More specifically, let $m : \theta \to \Delta(\mathcal{M})$ be the profile of messages sent by the buyer. Let $S = V \times \mathbb{R}_+$ be the space of offers - a good and a price, and let $s : \mathcal{M} \to \Delta(S)$ the seller's choice of offer, conditional on having received message m. Finally, let $x : [0, 1] \times V \times \mathbb{R}_+ \to \{0, 1\}$ be the acceptance strategies for each buyer type (where 0 indicates a rejection), given an offer (v, p).

Type θ's utility from accepting an offer (v, p) is $u(v, \theta) - p$, and hence his expected utility, given a strategy profile $\sigma = (m, s, x)$ is $\mathbb{E}_\sigma(u(v, \theta) - p)$, whilst the seller's expected profit is simply $\mathbb{E}_\sigma(p)$. Let $U(v, p, \theta) = u(v, \theta) - p$. In the event of a rejection, payoffs to both agents are zero.

I focus on *Perfect Bayesian equilibria in pure strategies*[3].

Definition 1. An equilibrium σ is a profile of message strategies m, offer strategies s and acceptance strategies x such that for each $\theta \in [0, 1]$, $m(\theta)$ solves

$$\max_{m \in \mathcal{M}} U(v(m), p(m), \theta) \tag{B1}$$

given m, $s(m) = \big(v(m), p(m)\big)$ solves

$$\max_{v,p} \int p \mathbb{I}_{x(\theta,s)=1} d\hat{\mu}(m|\theta) \tag{S}$$

where $\hat{\mu}(.|\theta)$ is derived from m by Bayes' rule, and given $s(m)$, $x(s)$ solves

$$\max_{x \in \{0,1\}} x\big(U(v(m), p(m), \theta)\big) \tag{B2}$$

2.2 The Possibility of Full Revelation

Before proceeding to a full characterization of equilibria, I state and prove the existence of a fully revealing equilibrium. Such an equilibrium is unusual in models in which Θ is one-dimensional. Indeed,

[3] I discuss mixed strategies in Section 4.2

in the leading example of [CS], full separation is not possible. How then can such an equilibrium be supported in the current context? In [CS], preference misalignment was captured by a single, one-dimensional *bias* term. Here, the misalignment is over two dimensions. The buyer and seller are *fully aligned* over the optimal choice of good - both would like to bargain after settling on the buyer's ideal good, as this maximizes joint surplus. However, they are *fully misaligned* over the optimal price. It is this multi-dimensional aspect of alignment that permits a separating equilibrium to exist.

Proposition 2. *A fully revealing equilibrium exists. Furthermore, it maximizes total welfare.*

Proof. To prove that such an equilibrium exists is straightforward. In such an equilibrium, $m(\theta)$ is injective on Θ, and hence we let θ_m be the unique member of $Theta$ such that $m(\theta_m) = m$. Then define the strategies $s(m) = (\theta_m, \bar{u})$ and $x(\theta_m, s(m)) = 1$, for all $\theta_m \in [0, 1]$. The seller achieves his maximum profit, subject to each type's IC constraint. Each type θ receives $\bar{u} - a(\theta - \theta)^2 - \bar{u} = 0$ in equilibrium. Since the signals are fully revealing, it is sufficient to consider a deviation for type θ to some message $\theta' \neq \theta$ instead. Subsequent acceptance of the seller's equilibrium offer would then yield a payoff of $\bar{u} - a(\theta' - \theta)^2 - \bar{u} < 0$, so such a deviation cannot be profitable.

To show that this allocation maximizes welfare, consider the social planner's ex ante problem. This can be viewed as an assignment problem. That is, the solution takes the form of a joint distribution $\pi : V \times [0, 1] \to [0, 1]$ that solves

$$\max_{\pi \in \Pi} \int u(v, \theta) d\pi$$

where Π is the set of all such distributions.[4] But

$$\max_{\pi \in \Pi} \int u(v, \theta) d\pi = \max_{\pi \in \Pi} \int \bar{u} - a(v - \theta)^2 d\pi = \min_{\pi \in \Pi} \int a(v - \theta)^2 d\pi$$

This is clearly achieved when $\pi(v, \theta) = \delta_{v\theta}$, where δ is the Kronecker delta. □

2.3 A Complete Characterization

As this is a costless signaling model, we should expect multiple equilibria to exist. This is indeed the case. Each equilibrium falls into on of three categories: partitional, semi-separating, and fully separating.

Definition 3. For $n \in \mathbb{N}$, and an interval $I = [\theta, \theta'] \subset \Theta$, we say an equilibrium σ is **n-partitional on *I*** if there exist numbers $\theta = a_0 < a_1 < \cdots < a_n = \theta'$ such that $\mu_{i+1} := \mu(.|\theta'')$ is uniform on $[a_i, a_{i+1}]$

[4]The max is well-defined by the Weierstrass extreme value theorem, since u is continuous on a compact space.

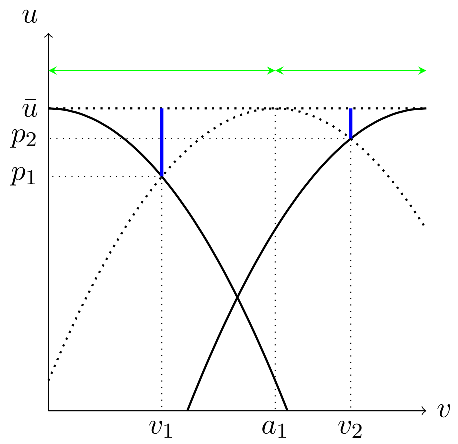

Figure 2: A 2-partitional equilibrium

for $\theta'' \in [a_i, a_{i+1}]$. Call the a_i "boundary"types. σ is **n-partitional** if it is n-partitional on Θ. Given an interval I, an equilibrium σ is **separating on *I*** if for all $\theta, \theta' \in I$, $m(\theta)$ is invertible, with $m(\theta_m) = m$, $s(m) = (\theta_m, \bar{u})$ and $x(\theta_m, s(m)) = 1$. We say an equilibrium σ is **semi-separating** if for $n, m \in \mathbb{N}$, there exists a partition $I_1, \ldots, I_n, J_1, \ldots, J_m$ of Θ such that σ is separating on $I_1, \ldots, I_n$ and partitional on $J_1, \ldots, J_n$.

Proposition 4. *1. For every $n \in \mathbb{N}$, there exists an n-partitional equilibrium.*

2. Every equilibrium is payoff-equivalent to either a partitional, semi-separating or fully separating equilibrium.

Corollary 5. *In a partitional equilibrium, it must be that each boundary type receives* 0.

Corollary 5 will be useful in the sequel, particularly in the construction of so-called self-signalling sets, and is a special case of the usual boundary indifference condition in partitional equilibria. If it were not true, the seller would effectively be leaving rents on the table.

Figure 2.3 shows an example of an equilibrium. Note that, as in [CS] without their additional monotonicity assumption (M), there are a continuum of n-partitional equilibria for each n.

3 Refined Equilibria

The large multiplicity of equilibria is an obvious obstacle on the path to making salient behavioural predictions. The modeler has at this stage a few options in order to disregard equilibria that embody seemingly implausible play. One could perform a "within-game"perturbation, for instance by modifying preferences (an exercise we perform in section 3.3) or by introducing a cost of lying (see [17]). I choose to call upon standard refinement concepts that were purpose built for games with costless signaling. In particular, I analyze the standard refinements of [3], [6] and [22]. These concepts all work with respect to a putative equilibrium, and broadly ask,"what would the seller do in response to facing some types, and would said types prefer this response to the equilibrium?"To this end, for some $G \subset \Theta$, let

$$\text{br}(G) = \arg\max_{v,p} \int_G p\mathbb{I}_{u(v,\theta)\geqslant p} d\theta$$

That is, $\text{br}(G)$ is the seller's best-response to the belief that he faces types G, given that they behave optimally at the accept/reject stage that follows and under the presumption that they indeed are in G.

The *no incentive to separate* concept, recently introduced by [3], is easily seen to have no power in the current setting. An equilibrium σ satisfies *no incentive to separate* (NITS) if $\mathbb{E}_\sigma(U(v,p,0)) \geqslant \mathbb{E}_{\text{br}(\{0\})}(U(v,p,0))$. That is, the lowest buyer type weakly prefers σ to revealing his type (if he could), and the seller best-responding to this revelation. It is immediately obvious that, in the current setting, such a refinement has no power. For if type 0 reveals himself truthfully, the seller offers good 0 at price $\bar{u}$, which is accepted. Since type 0 gets 0 in this deviation, it is clear that any incentive-compatible strategy profile satisfies NITS.

Remark 6. *All equilibria satisfy NITS.*

Next, I turn to the "neologism-proofness"concept of [6].

Definition 7. For all $\theta \in G$, we define

$$\underline{U}(\theta|G) = \min_{(v,p)\in\text{br}(G)} \mathbb{E}_{\text{br}(G)}(U(v,p,\theta))$$

$$\bar{U}(\theta|G) = \max_{(v,p)\in\text{br}(G)} \mathbb{E}_{\text{br}(G)}(U(v,p,\theta))$$

Given an equilibrium σ, a set $G \subset \Theta$ is **self-signalling** if

$$\underline{U}(\theta|G) \overset{a.s.}{>} \mathbb{E}_\sigma(U(v,p,\theta)) \ \ \forall\theta \in G$$

$$\bar{U}(\theta|G) \leqslant \mathbb{E}_\sigma(U(v,p,\theta)) \ \ \forall\theta \in \Theta \setminus G$$

σ is **neologism-proof (NP)** if no self-signalling set exists relative to it. Call the set of NP equilibria the NP set.

That is, the set G is self-signalling if precisely those types are the ones who stand to gain from making a statement that induces the seller to best-respond to the belief that $\theta \in G$. An equilibrium σ is neologism-proof if no such set exists. If the seller's best response to G is not unique, we use the convention introduced by [22] - a deviating type assesses his worst-case deviation against the putative equilibrium, whereas a non-deviating type assesses his best-case deviation. I abuse terminology and maintain the moniker "neologism-proofness". This is not to undermine the contribution of [22], but simply because my adapted definition seems closer in essence to [6].

A technical remark is needed here. I slightly adapt Farrell's original definition for finite type spaces to the current setting. I impose that at most a measure 0 of types in G are indifferent between the deviation and the equilibrium payoffs. Whilst this convention seems the most natural, how one takes a stand on this point turns out to be important for the existence of self-signalling sets. Were I to impose that the deviation be strict *for all* types in G, then all equilibria would be NP, by Corollary 5, and were I to drop the almost-sure requirement, i.e. allow a positive measure of types in G to be indifferent between the equilibrium and the deviation,[5] then the NP set would reduce in size, although it would remain non-empty.

3.1 Clearing Equilibria

To make progress towards characterizing the NP set, I study an important characteristic of equilibria, namely whether or not almost all types secure strictly positive payoff from the trade. Were this not the case, one might imagine that this positive measure of dissatisfied types could form a self-signalling set. This intuition turns out to be correct. Of course, a set of types could be dissatisfied either if they all reject the equilibrium offer made to them, or if they accept an offer that yields 0 surplus. The former case is pertinent to partitional equilibria, the latter to separating equilibria. This motivates the following definition.

Definition 8. An interval $I \subset \Theta$ is **clearing** if $x^*(\theta, \text{br}(I)) = 1$ for all $\theta \in I$. An interval is residual if it is not clearing. A partitional equilibrium σ is **clearing** if each member of the partition is clearing, and is

[5] This is the approach of [22], although their motivation for doing so was to tackle the problem of multiple best-responses to a self-signalling set.

residual if it is not clearing.

AN interval I is clearing is, faced with this set of types, the seller's optimal offer results in all types accepting. The following proposition is central in the arguments that lead to the main result. Recall that in the benchmark model under consideration, $u(v, \theta) = \bar{u} - a(v - \theta)^2$.

Proposition 9. *Given parameters $a, \bar{u}$, there exists $\Lambda(a, \bar{u}) \in (0, 1]$ such that $I = [\theta, \theta']$ is clearing if and only if $|\theta' - \theta| \leqslant \Lambda(a, \bar{u})$.*

The proposition identifies a number — a *critical width* of the game — such that all partitional equilibria that contain only intervals smaller than this width are clearing, and all partitional equilibria with at least one interval larger than this width are residual. Intuitively, as the intervals making up an equilibrium shrink, the seller's loss from not serving all types outweighs the gains from charging a higher price. Proposition 9 paves the way for a full description of clearing equilibria. We need enough intervals in the partition so that the width of each interval is below the critical threshold. In the benchmark model, the critical width is easy to calculate - it is the size of the interval served by the seller's static monopoly price, which is given by $p_M = \frac{2\bar{u}}{3}$.

Proposition 10. *There exists a finite $N^* \in \mathbb{N}$ such that for all $n \geqslant N^*$, there exists a clearing n-partitional equilibrium, and for $n < N^*$, no clearing equilibrium exists. Furthermore,*

$$N^* = \lceil \frac{1}{2} \sqrt{\frac{3a}{\bar{u}}} \rceil$$

There are many ways of identifying the lower bound N^* in Proposition 10. My approach is constructive, based on the following steps. Start with a babbling equilibrium. If the width of the entire interval $[0, 1]$ exceeds the critical width $\Lambda(a, \bar{u})$, then this babbling equilibrium is clearing. Otherwise, it is residual. Now form the babbling equilibrium with the largest residual set possible, and create a new equilibrium by having this residual set form their own interval and the seller best respond. If this best response is itself residual, repeat the previous steps to keep "filling in the holes". Such an approach has the benefit of not only proving that N^* exists, but it also constructs a special clearing equilibrium. This equilibrium, which we will refer to as the **least separating clearing (LSC) equilibrium**, is essentially the most uninformative equilibrium, subject to it being clearing. The equilibrium is essentially unique, i.e. the players ex-ante expected utilities are identical across any such equilibrium. Figure 3.1 describes the construction graphically.

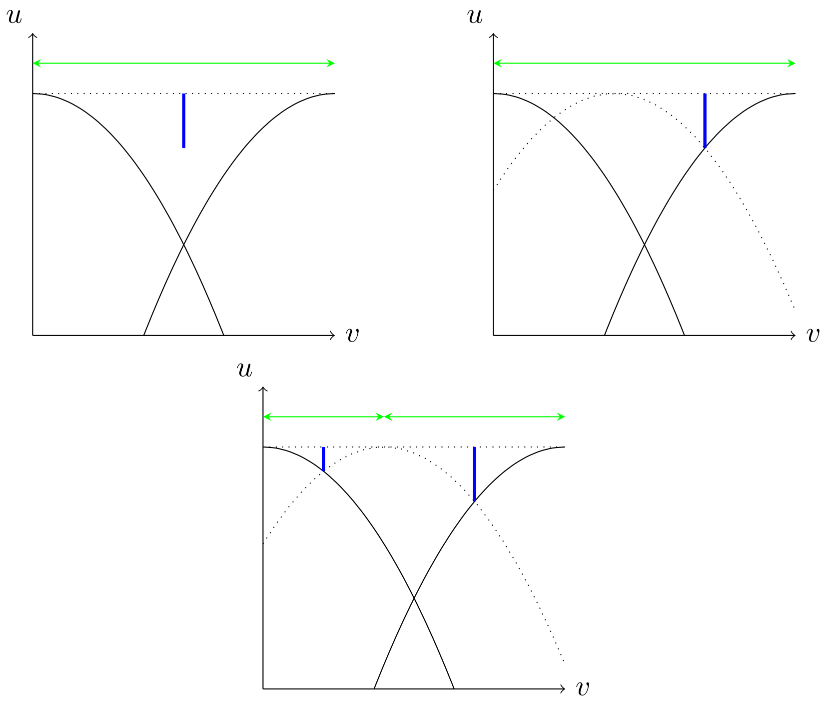

Figure 3: Constructing the LSC equilibrium

There is a good reason for focusing on clearing equilibrium and especially on the LSC equilibrium, namely that other equilibria cannot be neologism-proof.

Proposition 11. *All separating equilibria are not NP. All residual equilibria are not NP.*

In any equilibrium which admits some zone of separation, all types in that zone receive a zero payoff. Therefore any interval subset G of that zone can serve as a profitable, self-signalling set so long as the seller serves that set entirely with their subsequent pricing decision. Proposition 9 guarantees that such a G is guaranteed to exist, as long as its width is below the critical threshold $\Lambda(a, \bar{u})$. A similar logic holds for residual equilibria: simply construct a suitable self-signalling set from one of the residual zones.

Finally, I state and prove the central result of this paper, which characterizes the set of neologism-proof equilibria.

Theorem 12. *An equilibrium is neologism-proof if and only if it is clearing and* $|a_{i+2} - a_i| > \Lambda(a, \bar{u})$ *for all consecutive boundary types* a_i, a_{i+1}, a_{i+2}*. In particular, if an* n*-partitional equilibrium is neologism-proof, then* $n \in \{N^*, \ldots, 2N^* - 1\}$*. The LSC equilibrium is neologism-proof.*

Theorem 12 provides some key features of NP equilibria. Firstly, it gives bounds on interval lengths. The first bound comes directly from Proposition 9. The latter states that two consecutive intervals can't be too small when pooled together, the reason being that the pooled set would then form a self-signaling set. Combining these two bounds yields a tight bound on the number of intervals. It should be noted that for $a \leqslant \frac{4\bar{u}}{3}, N^* = 1$, and hence the (unique) babbling equilibrium is the unique NP equilibrium, and also constitutes the LSC equilibrium. That is, if goods are sufficiently substitutable, the buyer is willing to fully concede on the choice of good in favor of a large price discount. The reverse logic also holds; the more discerning is the buyer, the less willing he is to be mismatched, and hence will accept higher prices. After all, if a buyer wants only one particular item in a shop, he can't help but reveal this.

I formalize this logic with the following comparative static result, the proof of which is direct from the expression for N^* given in Proposition 10.

Corollary 13. *N^* is (weakly) increasing in a. As the buyer becomes more discerning, neologism-proof equilibria become more informative.*

3.2 Welfare

In [CS], more informative equilibria typically *ex ante* Pareto dominate less informative ones, as their game is effectively one of coordination. This is not the case here; a novelty of my model is that it is neither pure coordination nor pure conflict, a feature that that lies at the heart of the main result. Broadly speaking, below N^* equilibria of increasing informativeness Pareto dominate, since they facilitate more trade, whilst above N^*, surplus is transferred to the seller at the expense of the buyer. The following result should come then as little surprise.

Theorem 14. *The LSC equilibrium is the unique ex-ante buyer optimal equilibrium.*

Combining Theorems 14 and 12, we gain a clear insight into which equilibria are sensible for this game. They are partially informative, and are "close"to the buyer's ex-ante preferred equilibrium.

But this insight sheds light not only on this game, but a hitherto unexplored force behind neologism-proofness. In [6], the concept was motivated through a philosophical discussion of the meaning of cheap-talk messages, and their interaction with natural language that may exist outside the confines of the game's environment. My analysis points to a more mechanical feature. By nature, cheap talk endows the sender with no ability to commit to a signaling structure, unlike in the *Bayesian Persuasion* setting of [16]. As such, in equilibrium, the different sender types effectively compete with each other by taking the full, type-dependent strategy profile as given. Through the construction of self-signaling sets, the concept of neologism-proofness is based on a coalitional deviation principle that aims to mitigate this competitive force. The sender in [16] does not face this problem, and hence can secure a greater surplus than is typically achievable in cheap-talk equilibria. Hence, we might view neologism-proofness as breaching the gap between cheap talk and Bayesian Persuasion in settings that involve neither pure coordination nor pure conflict.

We conclude this section by prforming an exercise similar to that of Proposition 13; how does *ex ante* buyer welfare vary as goods become less substitutable? Focusing now on the LSC equilibrium, the answer is, perhaps surprisingly, non-monotonically.

Proposition 15. *Define*

$$W(\sigma) = \int_{\Theta} \mathbb{E}_{\sigma}(U(v, p, \theta))d\theta$$

as the map that computes the buyer's ex ante payoff for an equilibrium σ*. Let* $W_{LSC}(a)$ *denote the ex ante buyer payoff under the LSC equilibrium, given parameter values* a*. Then*

$$W_{LSC}(a) = \frac{a}{6}\left[(N^* - 1)\Lambda(a, \bar{u})^3 + \left(1 - (N^* - 1)\Lambda(a, \bar{u})\right)^3\right]$$

where $\Lambda(a, \bar{u}) = 2\sqrt{\frac{\bar{u}}{3a}}$ *and* $N^* = \lceil \frac{1}{2}\sqrt{\frac{3a}{\bar{u}}} \rceil$*. In particular,* $W_{LSC}(a)$ *is continuous and non-monotone in* a*.*

One might expect that as the buyer becomes more discerning, he is forced to reveal his preferences more precisely at the cost of facing higher prices. This intuition is only partially correct. In the LSC equilibrium, the buyer reveals precisely enough information to force the seller to charge the *same* (monopoly) price. Thus, whilst the buyer's loss from product mismatch increases, increased information provision offsets this loss. This is best understood through Figure 3.2. Note that the kinks in the welfare functions occur when N^* increments, but that the values at these kinks are identical, and hence these two countervailing forces offset each other perfectly.

3.3 Generalizing Preferences

Horizontal differentiation is one of the two canonical models of consumer preferences over multiple products. The other - *vertical differentiation* - posits that all consumers have the same ranking over the product line. Of course, in reality, a combination of the two seems the most appropriate description. Many brands offer premium and budget lines for their products, with all consumers ranking the former

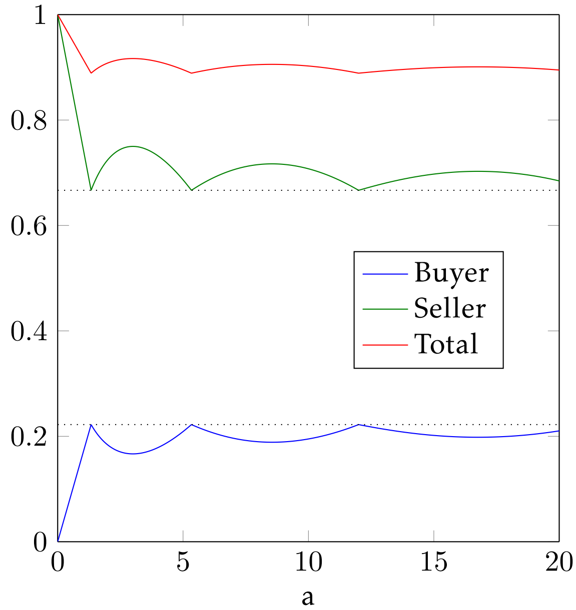

Figure 4: Buyer, seller *ex ante* welfare in the LSC equilibrium ($\bar{u} = 1$)

higher, whereas within a given line, consumers might have subjective preferences across brands, e.g. a preference for Apple over Samsung based on compatibility concerns.[6]

For a start, take the standard model of vertical preferences, given by $u(v, \theta) = v\theta$ (see for instance [24]). In this setting, the conclusion of the analysis changes dramatically; no communication occurs in equilibrium. To see this, note that for a given offer (v, p), types $\theta \geqslant \frac{p}{v}$ accept, and types $\theta < \frac{p}{v}$ reject. For a given belief μ, the seller's profit from an offer (v, p) is then $\Pi(v, p|\mu) = \int_{\frac{p}{v}}^{1} p d\mu(\theta)$. Clearly then, offering $v = 1$ weakly dominates. Suppose the equilibrium was 2-partitional. Then the seller offers v for both intervals, at prices p_0, p_1. If $p_0 < p_1$, then types in μ_1 would strictly benefit by deviating and pooling with μ_0, and vice versa if $p_1 < p_0$. A similar argument holds for any equilibrium with some separation.

Proposition 16. *With $u(v, \theta) = v\theta$, no communication occurs in equilibrium.*

That monotonicity in preferences leads to no communication is a well-known result in games with cheap-talk; Proposition 16 is simply an expression of this. A more natural exercise would be to allow for a more general $u(v, \theta)$, satisfying minimal conditions that combine both horizontal and vertical differentiation. To this end, we impose the following restrictions:

Assumption 17. (A1) $\frac{\partial u}{\partial v}|_{v=\theta} = 0$ (A2) $\frac{\partial u}{\partial \theta}|_{v=\theta} > 0$ (A3) $\frac{\partial^2 u}{\partial v^2} < 0$ (A4) $\frac{\partial^2 u}{\partial \theta \partial v} \geqslant 0$ (A5) $u(\theta, \theta) > 0$

[6]See [4] for preferences that combine vertical and horizontal differentiation.

Properties (A1) and (A3) ensure that u is single-peaked and strictly concave in v. (A2) captures the idea that, although it is not true that all types share the same ranking, it is the case that higher v products entail greater joint surplus. (A4) is a standard sufficient condition for single-crossing, and (A5) ensures the problem is non-trivial.

The following result demonstrates the fragility of the fully separating equilibrium to even the smallest vertical perturbation. By (A2) the gain for type θ to deviate marginally to the left is first-order, whereas by (A1) the loss is second-order. Specifically, in a fully separating equilibrium, it must be that $s(v,p|\theta) = \big(\theta, u(\theta,\theta)\big)$ - by Assumption 17, $v = \theta$ maximizes $u(v,\theta)$. Suppose type θ considers a deviation to $\theta - \epsilon$, for some small $\epsilon > 0$. His payoff from accepting is then $u(\theta-\epsilon,\theta) - u(\theta-\epsilon,\theta-\epsilon)$. But for small ϵ,

$$u(\theta-\epsilon,\theta) - u(\theta-\epsilon,\theta-\epsilon) \approx \epsilon u_\theta(\theta,\theta) > 0$$

by (A2).

Proposition 18. *If $u(v,\theta)$ satisfies Assumption 17, then a fully separating equilibrium cannot exist. Furthermore, there exists an M^* such that all partitional equilibria have at most M^* intervals.*

Much like in [CS], M^* can be determined as the solution to a recursive difference equation. In this case, if a_{i+1} is a boundary type, then the solution to the difference equation

$$u(a_i,a_i) = u(a_i,a_{i+1})$$

forms a lower bound for how close a_i can be to a_{i+1} in equilibrium. Figure 3.3 shows this construction graphically.

Finally, does the qualitative feature of Theorem 12 hold in this general setting? Consider the case when $u(v,\theta) = ax - (v-\theta)^2$.[7] As $a \to \infty$, no solution to the equation $u(a_i,a_i) = u(a_i,a_{i+1})$ exists, preferences become fully vertical, and hence no communication occurs. It is vacuously false then that the most informative equilibria are refined away. This limit clearly embodies a convergence towards Proposition 16.

Instead, consider the function $f(\theta) = u(\theta,\theta)$ that describes the locus of local maxima of the function

[7] It is easily verified that this function satisfies Assumption 17.

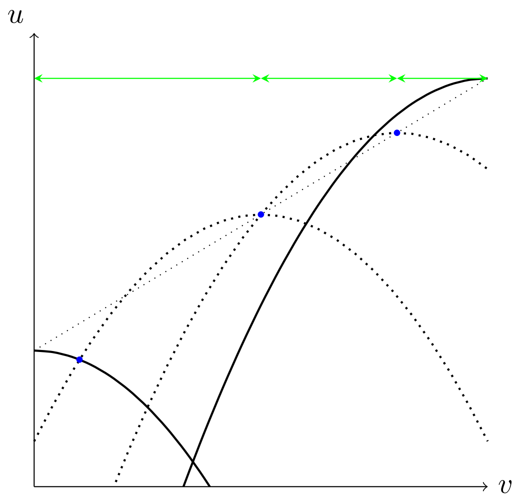

Figure 5: An M* equilibrium ($u(v,\theta) = 3 + \frac{3}{5}\theta - \frac{1}{5}(v-\theta)^2$)

$u(v,.)$, and suppose $a > 0$ is such that[8]

$$a = \max_{\theta \in [0,1]} f'(\theta)$$

In the benchmark case, $f(\theta) = \bar{u}$ and $a = 0$. As a becomes small, the first-order gain that prevents full separation starts to shrink, thus allowing finer partitions to emerge in equilibrium. Denoting this dependence as $M^*(a)$, we obtain the following generalization of Theorem 12.

Proposition 19. *Suppose $u(v,\theta)$ satisfies Assumption 17, and $a > 0$ is defined as above. Then as $a \to 0$, $M^*(a) \to \infty$. Furthermore, there exists δ such that for all $a < \delta$, any $M^*(a)$-equilibrium is not NP.*

As in the benchmark case, the proof relies on strict concavity of $u(v,.)$ to ensure only boundary types are indifferent between the putative equilibrium and the candidate self-signaling set.

4 Discussion

4.1 Menus

Having the seller restricted to making an offer on just one good restricts his bargaining power. Allowing the seller access to richer menus of offers enables him to separate types, rather than relying on cheap talk to do so. Types are now forced to sort themselves by accepting different offers.

To see this, suppose the seller can offer a complete menu of take-it-or-leave-it offers $\{(v,p)\}_{v \in V}$.[9] Then,

[8]Note that since f' is continuous on a compact set, the max here is well-defined.
[9]This is the optimal mechanism for the seller, as he secures full surplus extraction.

for any belief, an optimal menu for the seller is simply $\{(v, \bar{u})\}_{v \in V}$. All types accept the offer pertaining to their ideal good, and receive 0, much as in the fully separating equilibrium of the benchmark model. Such a situation would best model competitive retailers, e.g. supermarkets, who clearly have the ability to commit to a menu of prices on all goods in their stock. Indeed, buyers rarely hesitate when telling a shop assistant what they are looking for.

A more moderate approach would be to ask what changes if the seller can offer two-good bundles. Building on the intuition gained through Section 3.2, whereas the buyer finds it optimal to pool in unequal intervals (broadly speaking, *ex ante* buyer welfare is convex in interval widths), the seller finds it optimal to separate types into equal groups. This intuition carries through with two-good bundles.

The seller's strategies now take form $s : \mathcal{M} \to (V \times \mathbb{R}_+)^2$, whereas the buyer's accept/reject strategies are now $x : [0,1] \times (v \times \mathbb{R}_+)^2 \to \{0, 1, 2\}$, where $x = 1, 2$ means the buyer accepts offer $1, 2$ respectively. Single-crossing again ensures that the strategy profile x partitions types. However, if the seller can ensure that the boundary type receives 0, he will be extracting as much surplus as possible.

Definition 20. Take a partitional equilibrium σ with intervals $M_1, \dots, M_n$. A two-good bundle is partitional if for each $M_i = [\theta_i, \theta_{i+1}]$, there exists a type $\theta_{m,i}$ such that $u(v_{i,1}, \theta_{m,i}) - p_{i,1} = u(v_{i,2}, \theta_{m,i}) - p_{i,2} = 0$. A partitional two-good bundle is equal if $\theta_{m,i} = \frac{\theta_{i+1} + \theta_i}{2}$.

Proposition 21. *In a partitional equilibrium, the seller's optimal bundle is equal.*

Proof. Take interval M_i, and suppose the seller's bundle is $(\mathbf{v}, \mathbf{p}) = \{(v_{i,1}, p_{i,1}), (v_{i,2}, p_{i,2})\}$. Then single crossing ensures that there exists a type $\theta_{m,i}$ such that $x(\theta, \mathbf{v}, \mathbf{p}) \in \{0, 1\}$ for all $\theta \in [\theta_i, \theta_{m,i}] := \Theta_{i,1}$, $x(\theta, \mathbf{v}, \mathbf{p}) \in \{0, 2\}$ for all $\theta \in [\theta_{m,i}, \theta_{i+1}] := \Theta_{i,2}$. Hence, the seller's expected profit on M_i is

$$\Pi(\mathbf{v}, \mathbf{p} | M_i) = \sum_j p_{i,j} \int \mathbb{I}_{x(\theta, v_{i,j}, p_{i,j}) = 1} \mathbb{I}_{\theta \in \Theta_{i,j}} d\theta$$

We claim that the IC constraint of type $\theta_{i,m}$ must bind under $(\mathbf{v}, \mathbf{p})$, that is $U(\mathbf{v}, \mathbf{p}, \theta_{i,m}) = 0$. Suppose not, i.e. $U(\mathbf{v}, \mathbf{p}, \theta_{i,m}) = \epsilon > 0$. Consider a bundle $(\mathbf{v}, \mathbf{p}')$, where $\mathbf{p}' = (p_{i,1} + \delta, p_{i,2} + \delta)$, for some $\delta > 0$. By continuity of $U(., \theta)$, for sufficiently small δ, $U(\mathbf{v}, \mathbf{p}', \theta_{i,m}) = \frac{\epsilon}{2}$, and hence by monotonicity of $U(\mathbf{v}, \mathbf{p}, .)$, the IC constraints of all types remain slack. Thus, the bundle $\mathbf{v}, \mathbf{p}'$ is a strict improvement for the seller.

Hence, we can characterize the optimal bundle by cut-off type $\theta_{i,m}$. The seller's expected profit is given by

$$\Pi(\theta_{i,m}|M_i) = (\theta_{m,i} - \theta_i)\big(\bar{u} - a\big(\frac{\theta_{m,i} - \theta_i}{2}\big)^2\big) + (\theta_{i+1} - \theta_{m,i})\big(\bar{u} - a\big(\frac{\theta_{i+1} - \theta_{m,i}}{2}\big)^2\big)$$

It is easily verified that $\frac{\partial^2 \Pi(\theta_{i,m}|M_i)}{\partial \theta_{m,i}^2} < 0$, hence the FOC for the seller's problem is sufficient. The FOC implies

$$\big(\bar{u} - a\big(\frac{\theta^*_{m,i} - \theta_i}{2}\big)^2\big) - \frac{a\theta^*_{m,i}(\theta_{m,i} - \theta_i)}{2} + \big(\bar{u} - a\big(\frac{\theta_{i+1} - \theta^*_{m,i}}{2}\big)^2\big) - \frac{a\theta^*_{m,i}(\theta_{i+1} - \theta^*_{m,i})}{2} = 0$$
$$\theta^*_{m,i} = \frac{\theta_{i+1} - \theta_i}{2}$$

□

Just as before, any separating or residual equilibrium will not be NP, since we can use Proposition 21 to find a clearing self-signaling set. Some of the most informative partitional equilibria will also be ruled out. Following the same construction of the LSC equilibrium - which remains the buyer optimal equilibrium - as in Proposition 10, I would expect N^* to be smaller with two-bundles, as the seller can clear a larger portion of the market. Fully characterizing the new NP set is reserved for future work. However, in general, the NP set is likely to expand - deviations are less appealing now, since the seller can use the menu to extract more surplus, making self-signaling sets harder to construct.

4.2 Mixed Strategies

Thus far, I have restricted our attention to equilibria in pure strategies. Allowing the seller to randomize has a role to play in the existence of self-signaling sets. Suppose $a > \frac{4\bar{u}}{3}$. Then the babbling equilibrium is residual, and thus (v^*, p^*) is optimal for the seller, for $v^* \in V^*$, for some interval V^*. In pure strategies, all such equilibria are residual, and hence not NP. However, if we allow the seller to uniformly randomize over V^*, then this equilibrium is, strictly speaking, NP. In expectation, only the two boundary types receive 0, and hence any deviating group would render some types strictly worse off. As such, we might expect the following result to hold.

Claim 22. *Allowing the seller to mix, if an equilibrium is NP then it is clearing with $n \in \{1, \ldots, 2N^* - 1\}$.*

Note that the invalidity of residual self-signaling sets still holds, thanks to the extension afforded by [22] that is built in to Definition 7.

I am not convinced by the equilibrium described above. Every pure strategy in the support of this mixed strategy would constitute an equilibrium, and hence the buyer is quite optimistic in assuming the seller would adopt this strategy. Indeed, this is what lead [22] to formulate their “prepare for the worst”conditions C1’ and C2’.

Relaxing the assumption of pure strategies in messages leads to issues with tractability. Allowing types to perform uniform randomization has no effect, since under pure strategies, conditional distributions are uniform. The key condition is the concavity of the seller’s expected profit function depends, which depends on the conditional distributions induced by the buyer’s message profile. As such, were I to allow more complex randomization on the part of the buyer, it may be that the seller’s unique best response would involve a mixed strategy.

5 Conclusion

This paper presented a model of bargaining with communication in which a buyer faces the following, ubiquitous question; do I reveal my preferences, in order to get the good I most desire, or do I hide my preferences, to protect myself against price discrimination? I derived a number of results in line with this tension: sensible equilibria are partially informative, while as the buyer becomes more discerning, he reveals more about his type.

The simplicity of the model allows for a number of interesting extensions. For instance, incorporating multiple rounds of bargaining into the game might produce a more realistic extensive form. I posit that one affect would be to rule out residual equilibria, borrowing intuition from the extensive literature surrounding the Coase conjecture (see [13],[8]).[10] In line with applications to search, one could consider a model in which the seller does not have access a priori to the set V, but must search for goods. Think of a real estate agent searching for sellers. If the agent works on commission, then agent and sellers are effectively vertically integrated, and the model’s structure applies. If one assumes the agent increases their chances of finding a good with a wider search radius, this would naturally rule out full separation. Other extensions might include multiple competing sellers who themselves have idiosyncratic costs of production. Such interesting topics are saved for future work.

[10] As the number of offers become infinite, and period lengths shrink, the seller is forced down to marginal cost.

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A Proofs

A.1 Proof of Proposition 4

i) Take $n \in \mathbb{N}$. Pick an arbitrary sequence $0 = \theta_0 < \theta_1 < \cdots < \theta_{n-1} = 1$, and set $\mu_{i+1} = \mu(.|\theta)$ uniform on $[\theta_i, \theta_{i+1}]$ for $\theta \in [\theta_i, \theta_{i+1}]$. Henceforth, let $[a, b]$ also denote the uniform distribution on the interval $[a, b]$. Consider the seller's problem, conditional on the belief $\mathcal{U}[\theta_i, \theta_{i+1}]$. If he offers good v at price p, his expected profit is

$$\Pi(v, p|\mu_{i+1}) = \frac{p}{\theta_{i+1} - \theta_i} \int \mathbb{I}_{u(v,\theta) \geqslant p} d\theta$$

We have that $u(v, \theta) \geqslant p$ holds for types such that

$$\max\{\theta_i, v - \left(\frac{\bar{u} - p}{a}\right)^{\frac{1}{2}}\} \leqslant \min\{\theta_{i+1}, v + \left(\frac{\bar{u} - p}{a}\right)^{\frac{1}{2}}\}$$

and hence the seller's expected profit is given by

$$\Pi(v, p|\mu_{i+1}) = \frac{p}{\theta_{i+1} - \theta_i} \int_{g_l(v,p)}^{g_u(v,p)} d\theta = \frac{p\big(g_u(v, p) - g_l(v, p)\big)}{\theta_{i+1} - \theta_i}$$

where $g_l(v, p) = \max\{\theta_i, v - \left(\frac{\bar{u}-p}{a}\right)^{\frac{1}{2}}\}$, and $g_u(v, p) = \min\{\theta_{i+1}, v + \left(\frac{\bar{u}-p}{a}\right)^{\frac{1}{2}}\}$. Simple analysis shows that the function $\Pi(v, p|\mu)$ is either weakly or strictly concave in v, depending on $|\theta_{i+1} - \theta_i|$, and strictly concave in p. Hence, p^* is unique in equilibrium, whereas V^* may either be a singleton or a closed interval.

Suppose (v^*, p^*) constitute the seller's equilibrium strategy. We claim that

$$p^*_{i+1} \geq \max\{u(v^*_{i+1}, \theta_i), u(v^*_{i+1}, \theta_{i+1}), 0\}$$

To see this, suppose not, and wlog suppose $p^*_{i+1} < u(v^*_{i+1}, \theta_i)$. By the single-crossing property, there exists a type θ_m such that $p^*_{i+1} = u(v^*_{i+1}, \theta_m)$ and $p^*_{i+1} > u(v^*_{i+1}, \theta)$ for all $\theta \in (\theta_m, \theta_{i+1}]$. Hence, we have that $g_l(v^*_{i+1}, p^*_{i+1}) = \theta_i, g_u(v^*_{i+1}, p^*_{i+1}) = \theta_m$, and so equilibrium profits are

$$\Pi^* = \frac{p^*_{i+1}(\theta_m - \theta_i)}{\theta_{i+1} - \theta_i}$$

Now consider the profile $(v^*_{i+1} + \epsilon, p^*_{i+1})$. By continuity, we can find a sufficiently small ϵ such that

$p^*_{i+1} < u(v^*_{i+1} + \epsilon, \theta_i)$ and $p^*_{i+1} < u(v^*_{i+1} + \epsilon, \theta_m)$. Hence by continuity again, there exists a type $\theta_n > \theta_m$ such that $p^*_{i+1} = u(v^*_{i+1} + \epsilon, \theta_n)$. Profits are now given by

$$\Pi(v^*_{i+1} + \epsilon, p^*_{i+1}) = \frac{p^*_{i+1}(\theta_n - \theta_i)}{\theta_{i+1} - \theta_i} > \frac{p^*_{i+1}(\theta_m - \theta_i)}{\theta_{i+1} - \theta_i} = \Pi^*$$

a contradiction.

We construct the strategy s. One any interval $[\theta_i, \theta_{i+1}]$, we have that $\Pi(v, 0) = \Pi(v, \bar{u}) = 0$,

By the claim, we can restrict attention to $p^*_{i+1} \geq \max\{u(v^*_{i+1}, \theta_i), u(v^*_{i+1}, \theta_{i+1})\}$. On this range, $\Pi(., p)$ is strictly concave and hence there exists a unique maximal $p^*(v) \in \arg\max \Pi(v, p)$. Next, fix p. Since $\Pi(v, p^*(v))$ is linear in v, it may be that several v solve $\arg\max_v \Pi(v, p^*(v))$. Pick one such v^*, and now set $s(\theta_i, \theta_{i+1}) = (v^*_{i+1}, p^*_{i+1}(v^*))$.

Finally, we check that no types have an incentive to deviate. Consider type $\theta_m \in [\theta_i, \theta_{i+1}]$, for some $i \in \{0, \ldots, n\}$. If θ_m deviates to the group $[\theta_{i+1}, \theta_{i+2}]$, he faces an offer (v^*_{i+2}, p^*_{i+2}) such that $u(v^*_{i+2}, \theta_m) < p^*_{i+2}$. This follows from the earlier claim, which established that $u(v^*_{i+1}, \theta_m) \leqslant p^*_{i+2}$, and hence the statement follows by continuity.

ii) I need to show that non-partitional equilibria, i.e. equilibria where the supports of the conditional distributions might overlap on a positive measure of types, either cannot exist, or have a payoff-equivalent partitional form.

Let $\hat{\mu}_1, \hat{\mu}_2$ be conditionals in σ such that there exists a positive measure set $\Theta' \in \Theta$ such that $\theta' \in \text{supp}(\hat{\mu}_1(.|m)), \theta' \in \text{supp}(\hat{\mu}_2(.|m'))$ for all $\theta' \in \Theta'$. Thus types θ' are indifferent between pooling with types supp$(\hat{\mu}_1)$ or with types supp$(\hat{\mu}_2)$. Denote the seller's offers given these beliefs $(v_1, p_1), (v_2, p_2)$ respectively. Then for all $\theta' \in \Theta'$, we have that $U(v_1, p_1, \theta') = U(v_2, p_2, \theta')$. If it is the case that for all $\theta' \in \Theta', U(v_1, p_1, \theta') = U(v_2, p_2, \theta') = 0$, then we can construct an equivalent partitional equilibrium through a procedure demonstrated by Figure A.1.

So suppose there exists $\theta' \in \Theta'$ such that $U(v_1, p_1, \theta') = U(v_2, p_2, \theta') > 0$. By continuity, there exists $\delta > 0$ such that $U(v_1, p_1, \theta') = U(v_2, p_2, \theta') > 0$ for all $\theta \in \mathcal{O}_\delta(\theta') := \{\theta || \theta - \theta'| < \delta\}$. Take $\theta'' \in \mathcal{O}_\delta(\theta')$

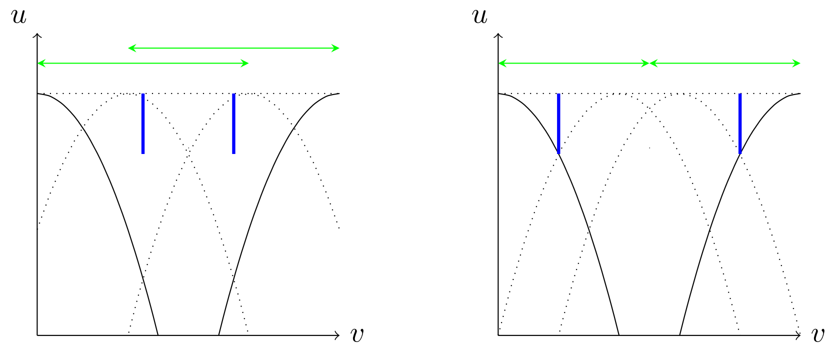


Figure 6: Finding an equivalent partitional equilibrium

and set $\epsilon > 0$ such that $\theta'' = \theta' + \epsilon$. Then

$$\begin{aligned} u(v,\theta'') - u(v_2,\theta'') &= p_1 - p_2 \\ \Rightarrow u(v_1,\theta'+\epsilon) - u(v_2,\theta'+\epsilon) &= u(v_1,\theta') - u(v_2,\theta') \\ u(v,\theta') - u(v_2,\theta') + \epsilon\big[u_\theta(v,\theta') - u_\theta(v_2,\theta')\big] &= u(v_1,\theta') - u(v_2,\theta') \\ \epsilon\big[\underbrace{u_\theta(v,\theta') - u_\theta(v_2,\theta')}_{\neq 0 \because u_{vv}<0}\big] &= 0 \end{aligned}$$

a contradiction.

Corollary 23. *In any partitional equilibrium, the seller's expected profit function* $\Pi(v,p|\mu_i)$ *is strictly concave in* p*, and piecewise-linear and concave in* v*.*

A.2 Proof of Corollary 5

First, we prove that there exists $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$, the seller's best response to belief $[\theta-\epsilon,\theta+\epsilon]$ involves a price p such that the constraint $p \geqslant \max u(v,\theta-\epsilon), u(v,\theta+\epsilon)$. For suppose not. Then it must be that $g_l(v,p) = v - \big(\frac{\bar{u}-p}{a}\big)^{\frac{1}{2}}$ and $g_u(v,p) = v + \big(\frac{\bar{u}-p}{a}\big)^{\frac{1}{2}}$, and hence the seller's profit is $\Pi(v,p) = 2p\big(\frac{\bar{u}-p}{a}\big)^{\frac{1}{2}}$. Since $p < \bar{u}$, this profit is bounded away from $\bar{u}$. Note also that it is independent of ϵ.

Now take the following strategy for the seller: $v = \theta, p = u(\theta,\theta-\epsilon) = u(\theta,\theta+\epsilon) = \bar{u} - a\epsilon^2$. This strategy satisfies all types IC constraints. Furthermore, since such a strategy binds both types $\theta-\epsilon$ and $\theta+\epsilon$ IC constraints, the seller's profit under this strategy is $\Pi_1(v,p) = p = \bar{u} - a\epsilon^2$. Clearly, for sufficiently small ϵ, $\Pi_1(v,p) > \Pi(v,p)$, a contradiction. This proves the claim. The proposition follows immediately.

A.3 Proof of Proposition 9

Without loss, we take $v^* = \frac{\theta+\theta'}{2}$. By Corollary 5, if p doesn't bind, then the FOC is sufficient and p solves $\frac{\partial \Pi(p)}{\partial p} = 0$, where $\Pi(p) = 2p\big(\frac{\bar{u}-p}{a}\big)^{\frac{1}{2}}$. Hence

$$\begin{aligned}\Pi'(p) = 0 \Rightarrow (\bar{u}-p)^{\frac{1}{2}} &= \frac{1}{2}p(\bar{u}-p)^{-\frac{1}{2}}\\ (\bar{u}-p) &= \frac{1}{2}p\\ p &= \frac{2\bar{u}}{3}\end{aligned}$$

If p does bind, the p satisfies $p = u(\frac{\theta+\theta'}{2}, \theta - \frac{\theta+\theta'}{2}) = u(\frac{\theta+\theta'}{2}, \theta' - \frac{\theta+\theta'}{2})$. By continuity of $\Pi(p)$, it must be that the value $\Lambda(a, \bar{u})$ solves

$$\begin{aligned}\bar{u} - a\big(\frac{\Lambda(a,\bar{u})}{2}\big)^2 &= \frac{2\bar{u}}{3}\\ \Lambda(a,\bar{u}) &= 2\sqrt{\frac{\bar{u}}{3a}}\end{aligned}$$

A.4 Proof of Proposition 10

For a strategy profile σ, let

$$\Theta_1(\sigma) = \{\theta \in \Theta | x(\theta, s) = 1\}$$

$$\Theta_0(\sigma) = \{\theta \in \Theta | x(\theta, s) = 0\}$$

Call $\Theta_1(\sigma)$ the acceptance set, and $\Theta_0(\sigma)$ the rejection set. A partitional equilibrium σ^* is **clearing** if the acceptance set is of full Lebesgue measure, and is **residual** if it is not clearing.

The steps constituting the construction are as follows.

Step 1 Take the profile σ in which $\mu(.|\theta)$ is uniform over $\mathcal{M}$, and $s(m) = (v^*, p^*)$ solves the seller's problem S. Then σ is a babbling equilibrium. If σ is clearing, we are done. To see this, note that we can form an n-partitional equilibrium from σ as follows. Take types $[0, \epsilon]$ for some $\epsilon > 0$, and form $n-1$ subdivisions. By Proposition 9, we have that as $\epsilon \to 0$, for each $i \in 0, \ldots, n-1$, the seller's best response is clearing.

Step 2 If σ is not clearing, then by arguments in Proposition 4, there exists a closed, convex set $V_0^* =$

$[v_0^l, v_0^r]$ of goods such that $s(m) = (v^*, p^*)$ form part of a babbling equilibrium for all $v^* \in V_0^*$. Wlog, take $v_0^* = v_0^r$, and construct the babbling equilibrium σ_0 with $s_0(m) = (v_0^r, p_0^*)$, where $p_0^* = p^* = \frac{2\bar{u}}{3}$, since σ was residual. Then σ_0 is residual, with $\Theta_0(\sigma_0)$ and $\Theta_1(\sigma_0)$ convex and partitioning Θ.

Step 3 Consider the seller's best response the belief $\Theta_0(\sigma_0)$. Again, by Proposition 4, the price p^* is uniquely defined, whilst v^* is defined up to a convex set $V_1^* = [v_1^l, v_1^r]$. If v^* is unique, then form the strategy profile σ_1 by setting $\mu_1 = \Theta_0(\sigma_0), \mu_2 = \Theta_1(\sigma_0), (v_1^*, p_1^*) = (v_0^r, \frac{2\bar{u}}{3}), (v_2^*, p_2^*) = (v^*, p^*)$. By construction, σ_1 is a clearing 2-partitional equilibrium, and the same argument as in step 1 completes the proof.

Step 4 If v^* is not unique, then take $v_1^* = v_1^r$ and $p_1^* = \frac{2\bar{u}}{3}$, and construct the strategy profile σ_1 such that $\mu_1 = \Theta_0(\sigma_0), \mu_2 = \Theta_1(\sigma_0), (v_1^*, p_1^*) = (v_0^r, \frac{2\bar{u}}{3}), (v_2^*, p_2^*) = (v_1^r, \frac{2\bar{u}}{3})$. Again, $\Theta_0(\sigma_1)$ and $\Theta_1(\sigma_1)$ are convex and partition Θ. Repeat steps 3 and 4.

Proposition 9 ensures this process ends after finitely many iterations, since for some finite N^*, the maximal width of any interval induced by this algorithm will be smaller than $\Lambda(a, \bar{u})$. To calculate N^* explicitly, note that the set $\Theta_1(\sigma_0)$ constructed by step 2 has width $\Lambda(a, \bar{u})$, since it is minimally clearing. Hence, the equilibrium constructed by the algorithm has a clear structure - $N^* - 1$ intervals are of width $\Lambda(a, \bar{u})$, and the boundary interval has width strictly less than $\Lambda(a, \bar{u})$. Hence,

$$N^* = \lceil \frac{1}{\Lambda(a, \bar{u})} \rceil = \lceil \frac{1}{2}\sqrt{\frac{3a}{\bar{u}}} \rceil$$

A.5 Proof of Theorem 12

I prove the result with a sequence of claims. The main idea is to find necessary and sufficient conditions for a set G to be self-signaling.

Proposition 24. *Take a clearing n-equilibrium σ with boundary types $0 = a_0 < \cdots < a_n = 1$. If $br\big([a_i, a_{i+2}]\big)$ is clearing for some $i \in \{0, \ldots, n-2\}$, then $[a_i, a_{i+2}]$ is self-signaling and hence σ is not NP.*

Proof. It suffices to show that almost all types in $[a_i, a_{i+2}]$ strictly gain from the deviation when $br([a_i, a_{i+2}])$ is clearing. By hypothesis, under σ, $br([a_i, a_{i+1}])$ and $br([a_{i+1}, a_{i+2}])$ are both clearing, and hence $v_i = \frac{a_i + a_{i+1}}{2}, p_i = \bar{u} - a\big(\frac{a_{i+1} - a_i}{2}\big)^2$ and $v_{i+1} = \frac{a_{i+1} + a_{i+2}}{2}, p_i = \bar{u} - a\big(\frac{a_{i+2} - a_{i+1}}{2}\big)^2$. Furthermore, if $br([a_i, a_{i+2}])$ is clearing, then the seller's unique best response is $v' = \frac{a_i + a_{i+2}}{2}, p_i = \bar{u} - a\big(\frac{a_{i+2} - a_i}{2}\big)^2$.

First, take $\theta \in (a_i, a_{i+1})$. Under σ, type θ's payoff is

$$\bar{u} - a\big(\frac{a_i + a_{i+1}}{2} - \theta\big)^2 - \bar{u} + a\big(\frac{a_{i+1} - a_i}{2}\big)^2 = a\big[\big(\frac{a_{i+1} - a_i}{2}\big)^2 - \big(\frac{a_i + a_{i+1}}{2} - \theta\big)^2\big] \tag{1}$$

whereas under $br[a_i, a_{i+2}]$, θ receives

$$a\big[\big(\frac{a_{i+2} - a_i}{2}\big)^2 - \big(\frac{a_i + a_{i+2}}{2} - \theta\big)^2\big] \tag{2}$$

Subtracting 1 from 2, we have type $\theta's$ gain from deviating

$$a\big[\big(\frac{a_{i+2} - a_i}{2}\big)^2 - \big(\frac{a_{i+1} - a_i}{2}\big)^2\big] - a\big[\big(\frac{a_{i+2} + a_i}{2} - \theta\big)^2 - \big(\frac{a_{i+1} + a_i}{2} - \theta\big)^2\big] \tag{2}$$

The first term is strictly positive since $a_i < a_{i+1} < a_{i+2}$. Define the function f as

$$f(\theta) = a\big[\big(\frac{a_{i+2} + a_i}{2} - \theta\big)^2 - \big(\frac{a_{i+1} + a_i}{2} - \theta\big)^2\big]$$

Then

$$f'(\theta) = 2\big(\frac{a_i + a_{i+1}}{2} - \theta\big) - 2\big(\frac{a_i + a_{i+2}}{2} - \theta\big) = \frac{a_{i+1} - a_{i+2}}{2} < 0$$

Since f is linear it must be that $\sup_{\theta \in [a_i, a_{i+1}]} f(\theta) = a_i$. Hence f is maximized at a_i, and so

$$\begin{aligned}
&a\big[\big(\frac{a_{i+2} - a_i}{2}\big)^2 - \big(\frac{a_{i+1} - a_i}{2}\big)^2\big] - a\big[\big(\frac{a_{i+2} + a_i}{2} - \theta\big)^2 - \big(\frac{a_{i+1} + a_i}{2} - \theta\big)^2\big] \\
&> a\big[\big(\frac{a_{i+2} - a_i}{2}\big)^2 - \big(\frac{a_{i+1} - a_i}{2}\big)^2\big] - a\big[\big(\frac{a_{i+2} - a_i}{2}\big)^2 - \big(\frac{a_{i+1} - a_i}{2}\big)^2\big] \\
&\geqslant 0
\end{aligned}$$

That is, all types $\theta \in (a_i, a_{i+1})$ strictly benefit from the deviation to $[a_i, a_{i+2}]$, with type a_i indifferent. By a symmetric argument, all types $\theta \in (a_{i+1}, a_{i+2})$ strictly gain, with type a_{i+2} indifferent. That is, $[a_i, a_{i+2}]$ is a self-signaling set.

□

Proposition 25. *Take a clearing n-equilibrium σ with boundary types $0 = a_0 < \cdots < a_n = 1$. If $G \subset \Theta$ is self-signaling wrt σ, then $G = [a_j, a_{k+1}]$, for some $j \in \{1, \ldots, n\}$, $k > j$.*

Proof. Clearly $G = [a_i, a_{i+1}]$ cannot be self-signaling, since σ is an equilibrium. By single-crossing, we may then restrict our attention to interval sets $G = [b, c], b < c$. Suppose b is not a boundary type, i.e. $b \in (a_i, a_{i+1})$ for some i. For $\epsilon > 0$ and define

$$\mathcal{O}_\epsilon(b) = \{\theta \in [a_i, a_{i+1}] | |\theta - b| < \epsilon\}$$
$$U(\mathcal{O}_\epsilon^-(b); \sigma) = \inf_{\theta \in \mathcal{O}_\epsilon(b)} \mathbb{E}_\sigma(U(v, p, \theta))$$
$$U(\mathcal{O}_\epsilon^+(b); \sigma) = \sup_{\theta \in \mathcal{O}_\epsilon(b)} \mathbb{E}_\sigma(U(v, p, \theta))$$

For sufficiently small ϵ, we have that $\mathcal{O}_\epsilon(b) \subset (a_i, a_{i+1})$ and $U(\mathcal{O}_\epsilon^-(b); \sigma) = \delta(\epsilon) > 0$. Fix such an ϵ. By Corollary 5, we have that $U(b; br(G)) = 0$, and hence by continuity of U, there exists $\epsilon^* > 0$ such that $U(\mathcal{O}_{\epsilon^*}^+(b); br(G)) < \delta(\epsilon)$. That is, the positive measure of types $\mathcal{O}_{\epsilon^*}(b)$ are strictly worse off from the deviation. An identical argument shows that c must also be a boundary type. □

Proposition 26. *σ is NP if and only if σ is clearing and for all $G = [a_i, a_{i+2}]$, $br(G)$ is residual. Hence, if σ is babbling, then σ is NP if and only if it is clearing.*

Proof. For the first part, note that the $\Rightarrow$ implication follows directly from the contrapositive statement of Proposition 24. For the $\Leftarrow$ implication, note first that if $br([a_i, a_{i+2}])$ is residual, then by Proposition 9 so too is $br([a_i, a_{i+k}])$ for all $k > 2$. Now take any G such that $br(G)$ is residual. We claim this cannot be a self-signaling set. To this end, fix an offer $(v, p) \in br(G)$. Since $br(G)$ is residual, there exists an open set $\mathcal{O}(v, p) = \{\theta \in G | E_{br(G)}(U(v, p, \theta)) = 0\}$, and hence $\mathcal{O}(v, p)$ cannot strictly gain from the deviation. Since $(v, p) \in br(G)$ was arbitrary, G cannot be self-signaling. The implication then follows from 25. □

Theorem 27. *An equilibrium is NP if and only if it is clearing and $[a_i, a_{i+2}] > \Lambda(a, \bar{u})$ for all consecutive boundary types a_i, a_{i+1}, a_{i+2}. In particular, if an n-partitional equilibrium is NP, then $n \in \{N^*, \ldots, 2N^* - 1\}$.*

Proof. The bounds are now immediate from Proposition 26. For the last part, a simple counting argument shows that the upper bound on the number of intervals is given by $\lfloor \frac{1-\Lambda(a,\bar{u})}{\Lambda(a,\bar{u})/2} \rfloor + 2$. Applying Hermite's

Identity (see [10]), we have that

$$\begin{aligned}\lfloor\frac{1-\Lambda(a,\bar{u})}{\Lambda(a,\bar{u})/2}\rfloor+2 &= \lfloor\frac{2}{\Lambda(a,\bar{u})}\rfloor \\ &= \lfloor\frac{1}{\Lambda(a,\bar{u})}\rfloor+\lfloor\frac{2}{\Lambda(a,\bar{u})}+\frac{1}{2}\rfloor \\ &= N^*-1+N^* \\ &= 2N^*-1\end{aligned}$$

□

A.6 Proof of Theorem 14

Towards a proof of Theorem 14, we introduce some notation. Let E_n^c be the set of clearing, n-partitional equilibria, and E_n^R similarly for residual equilibria. Losing sub/superscripts implies the natural superset e.g. E^c are the clearing equilibria, whilst E is the set of all equilibria. Let $W : E \to \mathbb{R}$ map equilibria to their ex-ante buyer welfare. That is, for an equilibrium σ,

$$W(\sigma) = \int_\Theta \mathbb{E}_\sigma(U(v,p,\theta))d\theta$$

Proposition 28. *Any equilibrium that is not NP is ex-ante buyer dominated in E.*

Proof. This follows directly from the definition of an NP equilibrium, since if G is self-signaling for σ, then the profile σ' defined as $\sigma'|_{\Theta\setminus G} = \sigma, \sigma'|_G = br(G)$ constitutes an equilibrium, as argued in the proof of Proposition 10. □

Note that in the set of clearing equilibria, each σ is uniquely defined by the boundary types $0 = a_0, \dots, a_n = 1$, since the seller best-responses are uniquely defined. Let $M(\sigma)$ denote the set of intervals defined by σ. That is $M(\sigma) = (M_1, \dots, M_n) = (a_1 - a_0, \dots, a_n - a_{n-1})$

Proposition 29. *If $\sigma = (a_1, \dots, a_{n-1}), \sigma' = (b_1, \dots, b_{m-1})$ are two clearing equilibria such that there exists a permutation $\tau : \{1, \dots, n\} \to \{1, \dots, n\}$ with $\tau(M_1, \dots, M_n) = (N_1, \dots, N_n)$, then $W(\sigma) = W(\sigma')$. Furthermore, W is convex in $(M_1, \dots, M_n)$.*

Proof. Take a clearing n-equilibrium σ. Then

$$\begin{aligned}
W(\sigma) &= \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} U(v,p,\theta)d\theta \\
&= \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \bar{u} - a\big(\frac{a_i + a_{i-1}}{2} - \theta\big)^2 - \bar{u} + a\big(\frac{a_i - a_{i-1}}{2}\big)^2 d\theta \\
&= \sum_{i=1}^{n} a \int_{a_{i-1}}^{a_i} \big(\frac{a_i - a_{i-1}}{2}\big)^2 - \big(\frac{a_i + a_{i-1}}{2} - \theta\big)^2 d\theta \\
&= a \sum_{i=1}^{n} \big[\big(\frac{a_i - a_{i-1}}{2}\big)^2\theta - \frac{1}{3}\big(\frac{a_i + a_{i-1}}{2} - \theta\big)^3\big]_{a_{i-1}}^{a_i} \\
&= \frac{a}{6} \sum_{i=1}^{n} (a_i - a_{i-1})^3 \\
&= \frac{a}{6} \sum_{i=1}^{n} M_i^3
\end{aligned}$$

W is clearly then invariant over equilibria with identical intervals, regardless of their order. Since $(M_1, \dots, M_n)$ are such that $M_i \in [0,1]$ for all i, and $\sum_{i=1}^{n} M_i = 1$, we may consider W as a function $W : \Delta_n \to \mathbb{R}_+$, where Δ_n is the n-dimensional simplex. Since Δ_n is convex, a sufficient condition for W to be a convex function is for its Hessian

$$\mathbf{D}^2 W = \frac{a}{6} \begin{pmatrix} 6M_1 & 0 & \dots & 0 \\ 0 & 6M_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 6M_n \end{pmatrix}$$

to be positive semi-definite. But $|\mathbf{D}^2 W - x\mathbf{I}| = \frac{a}{6}\prod_{i=1}^{n}(6M_i - x)$, hence $\mathbf{D}^2 W$ has all positive eigenvalues and hence the condition is met. □

To prove the proposition, note that by the proof of Proposition 10, any equilibrium σ that is not LSC is such that there exist at least two intervals $M_i, M_j \in M(\sigma)$ such that $M_i \leqslant M_j < \Lambda(a, \bar{u})$, where the first inequality is without loss. Construct a profile σ' such that $(M_1, \dots, M_i, \dots, M_j, \dots, M_n) = (M_1, \dots, M_i - \epsilon, \dots, M_j + \epsilon, \dots, M_n')$. For sufficiently small ϵ, σ' is a clearing n-equilibrium. Since W is convex in $(M_1, \dots, M_n)$, it must be that $W(\sigma') > W(\sigma)$.

A.7 Proof of Proposition 15

In the LSC equilibrium, $N^* - 1$ intervals have width $\Lambda(a, \bar{u})$, with a residual interval having width $1 - (N^* - 1)\Lambda(a, \bar{u})$. Hence

$$\begin{aligned}
W_{LSC}(a) &= \frac{a}{6}\sum_{i=1}^{n} M_i^3 \\
&= \frac{a}{6}\Big[\sum_{i=1}^{N^*-1} \Lambda(a,\bar{u})^3 + (1-(N^*-1)\Lambda(a,\bar{u}))^3\Big] \\
&= \frac{a}{6}\big[(N^*-1)\Lambda(a,\bar{u})^3 + (1-(N^*-1)\Lambda(a,\bar{u}))^3\big]
\end{aligned}$$

To prove non-monotonicity, I first argue that if a is such that $1/\Lambda(a, \bar{u}) \in \mathbb{N}$, then $W_{LSC}(a) = \frac{2\bar{u}}{9}$. At such values of a, all intervals have width $\Lambda(a, \bar{u})$, and the seller charges a price $\frac{2\bar{u}}{3}$. Let $1/\Lambda(a, \bar{u}) = N$. Then $\frac{1}{2}\sqrt{\frac{3a}{\bar{u}}} = N$ and so $a = \frac{4N^2\bar{u}}{3}$. Hence

$$\begin{aligned}
W_{LSC}(a) &= W_{LSC}(\frac{4N^2\bar{u}}{3}) \\
&= \sum_{i=1}^{N}\Big[\int_0^{\frac{1}{N}} \bar{u} - \frac{4N^2\bar{u}}{3}\big(\frac{1}{2N}-\theta\big)^2 d\theta\Big] - \frac{2\bar{u}}{3} \\
&= N\Big[\frac{\bar{u}}{N} - \frac{2a}{3}\big(\frac{1}{8N^3}\big)\Big] - \frac{2\bar{u}}{3} \\
&= \frac{2\bar{u}}{9}
\end{aligned}$$

I now prove that, over regions where N^* is constant, $W_{LSC}(a)$ is strictly convex. Clearly, in these regions, $W_{LSC}(a)$ is differentiable, and hence it suffices to show that $\frac{\partial^2 W_{LSC}}{\partial a^2} > 0$. Define $W_1(x) = (N^* - 1)x^3 + (1 - (N^* - 1)x)^3$. Then

$$\begin{aligned}
W_{LSC}(a) &= \frac{a}{3}W_1(\Lambda(a)) \\
W'_{LSC}(a) &= W'_1(\Lambda(a))\big(\Lambda'(a)\frac{a}{3}\big) + W_1(\Lambda(a)) \\
W''_{LSC}(a) &= \Lambda'(a)\frac{a}{3}W''(\Lambda(a))\Lambda'(a) + W'_1(\Lambda(a))\big(\Lambda''(a)\frac{a}{3} + \frac{\Lambda'(a)}{3}\big) + W'_1(\Lambda(a))\Lambda'(a) \\
&= \frac{\Lambda'(a)^2 W''_1(\Gamma(a))a}{3} + W_1(\Lambda(a))\big(\Lambda''(a)\frac{a}{3} + \frac{4}{3}\Lambda'(a)\big)
\end{aligned}$$

By direct calculation, it suffices to show that $\Lambda''(a)\frac{a}{3} + \frac{4}{3}\Lambda'(a) < 0$. But

$$\begin{aligned}\Lambda''(a)\frac{a}{3} + \frac{4}{3}\Lambda'(a) &= (\frac{3}{\bar{u}})^2(\frac{3a}{\bar{u}})^{\frac{-5}{2}}\frac{a}{3} - \frac{4}{3}\frac{3}{\bar{u}}(\frac{3a}{\bar{u}})^{-\frac{3}{2}} \\ &= \frac{a^{-\frac{3}{2}}}{u^{-\frac{1}{2}}}(3^{-\frac{5}{2}} - 4(3^{-\frac{3}{2}})) \\ &< 0\end{aligned}$$

as required.

A.8 Proof of Proposition 18

The first part is immediate from the preceding paragraph. For the remainder, we proceed with some propositions.

First, for $\theta \in [0,1]$, let $b(\theta) \in [0,1] \setminus \{\theta\}$ solve $u(b,b) = u(b,\theta)$ (if no solution to the equation exists in $[0,1]$, set $b(\theta) = 0$). Then the function $b : [0,1] \to [0,1]$ is well-defined, and furthermore $b(\theta) \in [0,\theta)$ by (A2).

Define $\lambda : [0,1] \to \mathbb{R}_+$ by $\lambda(x) = x - b(x)$.

Proposition 30. *Given a partitional equilibrium σ, if $a_{i+1} - a_i < \lambda(a_{i+1})$, then $U(v_i, p_i, a_{i+1}) > 0$ and $U(v_i, p_i, a_i) = 0$.*

Proof. The last equality follows from the usual arguments forcing the lowest type IC to bind. Towards a contradiction, suppose $U(v_i, p_i, a_{i+1}) = 0$, i.e. $p_i = u(v_i, a_{i+1})$. Then there exists $v_m \in [a_i, a_{i+1})$ such that $x(\theta, v_i, p_i) = 0 \forall \theta \in [a_i, v_m], x(\theta, v_i, p_i) = 1 \forall \theta \in [v_m, a_{i+1}]$, i.e. v_m forms a cut-off type. Then the seller's profit is given by

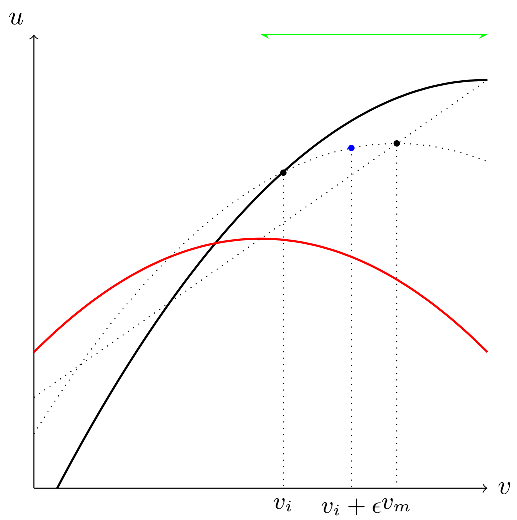
$$\Pi(v_i, p_i) = \frac{p_i(a_{i+1} - v_m)}{a_{i+1} - a_i} = \frac{u(v_i, a_{i+1})(a_{i+1} - v_m)}{a_{i+1} - a_i}$$

Consider instead the offer $(v', p') = (v_i + \epsilon, u(v_i + \epsilon, v_m)$, for some small $\epsilon > 0$. By definition, v_m solves $(u(v_i, a_{i+1}) = u(v_i, v_m)$. Hence, for sufficiently small ϵ, $v' < v_m$, thus the acceptance set is the same as the original offer (v_i, p_i). The seller's profit becomes

$$\begin{aligned}\Pi((v', p') &= \frac{u(v_i+\epsilon, v_m)(a_{i+1}-v_m)}{a_{i+1}-a_i} \\ &\approx \frac{\big(u(v_i, v_m)+\epsilon u_v(v_i, v_m)\big)(a_{i+1}-v_m)}{a_{i+1}-a_i} \\ &= \frac{u(v_i, v_m)(a_{i+1}-v_m)}{a_{i+1}-a_i} + \frac{\epsilon u_v(v_i, v_m)(a_{i+1}-v_m)}{a_{i+1}-a_i} \\ &> \Pi(v_i, p_i)\end{aligned}$$

by (A2). Hence, (v', p') constitutes a profitable deviation for the seller.

Set $\lambda(f) = \inf_{\theta\in\Theta}(\lambda(\theta)$. This is clearly well-defined, since $\lambda([0,1]) \subset [0,1]$.



□

Finally, to prove the proposition, suppose no such M^* exists. Then for large enough M, there must exist two consecutive intervals $[a_i, a_{i+1}], [a_{i+1}, a_{i+2}]$ such that $a_{i+j} - a_{i+j-1} < \lambda(f)\ \forall j = 1, 2$. By Proposition 30, type a_{i+1} receives positive surplus by pooling with $[a_i, a_{i+1}]$, but zero surplus by pooling with $[a_{i+1}, a_{i+2}]$, a contradiction.

A.9 Proof of Proposition 19

For sufficiently small a, (A5) allows us to restrict attention to clearing equilibria. Using the definitions of the functions $b(\theta)$ and $\lambda(\theta)$ as in the proof of Proposition 18, we proceed in steps.

Proposition 31. *Take a boundary type a_i. As $a \to 0$, $b(a_i) \to a_i$.*

Proof. Let $a_{i-1} = b(a_i)$. Applying the Mean Value Theorem to $f(\theta)$ on the closed interval $[a_{i-1}, a_i]$, there exists $y \in (a_{i-1}, a_i)$ such that $f(a_i) - f(a_{i-1}) = f'(y)(a_i - a_{i-1})$. Hence

$$\begin{aligned}
|u(a_i, a_{i-1}) - u(a_{i-1}, a_{i-1})| &< |u(a_i, a_i) - u(a_{i-1}, a_{i-1})| \\
&= |f(a_i) - f(a_{i-1})| \\
&= |f'(y)||a_i - a_{i-1}| \\
&\leqslant \max_{\theta \in [0,1]} f'(\theta)|a_i - a_{i-1}| \\
&= a|a_i - a_{i-1}| \\
&\leqslant a \\
&\to 0
\end{aligned}$$

as $a \to 0$. The first inequality follows from (A1). The result follows from continuity of $u(., \theta)$. □

In particular, the Mean value bound constructed in the proof of Proposition 31 says that the minimum interval width in an M^*-equilibrium scales with a, and hence $M^* \to \infty$ as $a \to 0$.

To prove the remainder of the proposition, take an M^*-equilibrium, and a sequence of boundary types a_i, a_{i+1}, a_{i+2}. I will show that the set $G = [a_i, a_{i+2}]$ forms a self-signaling set. Single-crossing (A4) ensures that types outside $[a_i, a_{i+2}]$ prefer the equilibrium, given it and G are clearing. By (A3), it is sufficient consider local IC constraints around the boundary types a_i, a_{i+2}.

First consider type $a_i + \epsilon$, for some small $\epsilon > 0$. In equilibrium, this type receives payoff $u(a_i, a_i + \epsilon) - u(a_i, a_i)$. Denote this $U_\sigma(a_i + \epsilon)$. Under the self-signaling deviation G, this type receives $u(v, a_i + \epsilon) - u(v, a_i)$, for some v; by (A4), we know that $a_i < v < a_{i+2}$, so let $\delta_1, \delta_2 > 0$ be such that $v = a_i + \delta_2 = a_{i+2} - \delta_2$. Then

$$\begin{aligned}
u(v, a_i + \epsilon) - u(v, a_i) &= u(a_i + \delta_1, a_i + \epsilon) - u(a_i + \delta_1, a_i) \\
&\approx u(a_i, a_i + \epsilon) + \delta_1 u_v(a_i, a_i + \epsilon) - \big[u(a_i, a_i) + \delta_1 u_v(a_i, a_i)\big] \\
&= U_\sigma(a_i + \epsilon) + \delta_1 \big[\underbrace{u_v(a_i, a_i + \epsilon)}_{>0} - \underbrace{u_v(a_i, a_i)}_{=0}\big] \\
&> U_\sigma(a_i + \epsilon)
\end{aligned}$$

Note that Proposition 31 validates the first-order approximation. Similarly, the payoff for type $a_{i+2} - \epsilon$

in equilibrium is $u(a_{i+1}, a_{i+2} - \epsilon) - u(a_{i+1}, a_{i+1}) = U_\sigma(a_{i+2} - \epsilon)$, whilst under G,

$$\begin{aligned} u(v, a_{i+2} + \epsilon) - u(v, a_{i+2}) &= u(a_{i+2} - \delta_2, a_{i+2} - \epsilon) - u(a_{i+2} - \delta_2, a_{i+2}) \\ &\approx u(a_{i+2}, a_{i+2} - \epsilon) - \delta_2 u_v(a_{i+2}, a_{i+2} - \epsilon) - \big[u(a_{i+2}, a_{i+2}) - \delta_2 u_v(a_{i+2}, a_{i+2})\big] \\ &\geqslant U_\sigma(a_{i+2} - \epsilon) - \delta_2 \big[\underbrace{u_v(a_{i+2}, a_{i+2} - \epsilon)}_{<0} - \underbrace{u_v(a_{i+2}, a_{i+2})}_{=0}\big] \\ &> U_\sigma(a_{i+2} - \epsilon) \end{aligned}$$

where the third inequality holds by (A4), since $u(a_{i+2}, a_{i+2} - \epsilon) - u(a_{i+2}, a_{i+2}) > u(a_{i+1}, a_{i+2} - \epsilon) - u(a_{i+1}, a_{i+1}) = U_\sigma(a_{i+2} - \epsilon)$. Hence only boundary types are left indifferent under the deviation G, and so G is self-signaling.