Optimal Prize Allocations in Group Contests

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Abstract

We study how the group effort in contests depends on the degree of heterogeneity in ability between group members. First, we show how this analysis depends on the steepness of the cost function. Second, we provide an optimal prize allocation that maximizes the group effort relaxing the common assumption of symmetry among players. A team manager who wants to maximize her group effort faces three cases: if the marginal cost function is concave, then she should maximize the variance in ability and allocate the whole prize to the most able player; if the marginal cost function is convex and not too steep, then she should maximize the variance in ability and allocate a positive share of the prize to all group members; if the marginal cost function is convex and sufficiently steep, then she should minimize the variance in ability and allocate a positive share of the prize to all group members. Finally, we show that cooperation among players may decrease the inequality among group members.

1 Introduction

Contests are competitions in which individuals, or group of individuals, compete for a prize contributing with their effort or personal resources. In many cases contests occur between groups where the final prize, or a part of it, can be allocated in any way among winning group members. Examples may include sport competitions, lobbying, patent race and rent-seeking contests.

To the best of our knowledge, the economic literature focuses mostly on such activities assuming symmetry within groups and relies on two allocative rules: the egalitarian prize allocation and the relative effort allocation. The first one awards the prize equally among group members, while the second one follows the relative effort of members within the group. The implementation of the egalitarian rule is mainly a consequence of the symmetry assumption among agents, while the relative effort rule can be applied only for few specific cases[1]. Finally, due to the symmetry assumption, the analysis of the effect

[1] See Bandiera et al. (2011) for a specific example of fruit picking

of heterogeneity in group contests has been mainly confined to differences in group size. However, individuals in groups usually have difference incentives. More productive agents are typically rewarded with a higher share of the prize and tend to be more effective as far as the contest's outcome is concerned.

In this work we study a contest between groups whose players have heterogeneous ability in order to find answers to the following questions: how does the heterogeneity within the group affect its effort? Which prize allocation maximizes the group effort? Is this optimal allocation affected by the cooperation or lack thereof among group members?

These questions are relevant in different situations. The first includes cases in which a team manager cannot choose the compensation schemes for her group; however, she can choose whether to compete with a homogeneous group or a heterogeneous one. The second one covers such cases in which a team manager is able to allocate the prize according to players' abilities. The third question considers these situations, for instance sport competitions, in which group members cooperate.

First, our model allows us to show that allocating the whole prize to one player is the optimal solution when the marginal cost of effort is not too steep. This is because the free-riding problem is so acute that is more effective to award one player than sharing the prize among many group members. This result is in line with Baik (2008), who shows that only one player exerts positive effort when the marginal cost is constant, while all other group members do not contribute at all. Second, we show that in these situations a heterogeneous group in terms of ability has an higher probability of winning than a homogeneous one. This simply occurs because the only contribution of effort in each group is made by the player with the monopoly of the prize, that his the higher is his ability the higher the group effectiveness.

On the other hand, allocating a positive share of the prize to all group members is the optimal solution as long as the marginal cost is enough steep. Indeed, despite the free-riding problem, high ability players cannot make substantial contributions since the marginal cost increases too rapidly. It follows that, the higher is the homogeneity in both ability and prize shares the higher the group effectiveness. However, under additional assumptions, we show that this result does not always hold, and in some cases it may be overturned. Finally, assuming perfect cooperation within groups and constant elasticity cost function, we show that is always optimal to award all group members no matter how much steep is the marginal cost. Indeed, without the free-riding problem, there is no reason to not allocate the prize to all group members as long as they fully cooperate. Moreover, our results are in line with other findings in the literature of contests.

Contests between groups have been studied in a number of works and most of them are extensions of the seminal paper of Tullock (1980). Nitzan (1991a) studies a combination between the egalitarian and the relative effort rule in order to discuss possible equilibria. Nitzan (1991b), Noh (1999) and Ueda (2002) investigate the endogenous determination of the sharing rules. In their works, groups are free to choose between the egalitarian rule and relative effort one. Cubel and Sanchez-Pages (2014) show through the Atkinson index of inequality that more egalitarian groups have a higher probability of winning the contest whether the players' efforts are complementary or the cost functions steep enough. However, these papers still rely on the symmetry assumption. The closest papers to our analysis are Nitzan and Ueda (2013) and Ryvkin (2011). The first is concerned with unequal prize's distributions so as to understand whether they are disadvantageous for some individuals even if they increase the group effort. The latter considers a public-good contest between groups in order to study the sorting of players that maximizes the overall effort. In Ryvkin's framework there are two possible outcomes: if the cost functions is enough steep, then the total effort is maximized when the variance in ability across groups is minimized. On the other hand, if the cost function is moderately steep, then the minimization of the variance across groups maximizes the total effort. The results in Ryvkin (2011) are simple and clear, however he assumes that sorting players in different groups is costless. In addition to this, we do not see often the application of such policy. For instance in sport we hardly see organizers sorting players in order to balance the competition, while is more common that managers reward group members according to their abilities. In this work we explore whether the heterogeneity in ability or in prize shares within a group can enhance group effectiveness. Moreover, we provide the prize allocation that maximizes the group effort and, under some conditions, it may overturn groups' probability of winning. We study our questions considering a group contest with heterogeneous players who exert a positive effort as long as they receive a strictly positive share of the prize. In order to do this, we let the cost of effort be a convex function and the marginal cost be zero at zero effort. As we show our results are determined by the form of the cost function. The work is organized as follow. Section 2 presents the model and the uniqueness of the equilibrium. Section 3 shows the effect of heterogeneity on group effectiveness. Section 4 provides the optimal prize allocation and presents our main results providing an example with a constant elasticity function. Section 5 shows the effect of the cooperation on the optimal prize allocation. Section 6 concludes.

2 A model of group contests

We consider N groups formed by n risk-neutral individuals. Players within groups are indexed by $ik = (i1, ...in)$. All players simultaneously and irreversibly exert an effort $x_{ik} \geq 0$, and the group effort is the linear sum of the players' effort, $X_i = \sum_{k=1}^{n} x_{ik}$. The group i's probability of winning is defined by the Tullock success function X_i/X. Defining the groups aggregate effort by $X = \sum_{i=1}^{N} X_i$, groups' probability of winning can be thought as a share of the total effort X. This ratio, denoted by σ_i, plays a key role in the subsequent analysis. That is, let

$$\sigma_i = \frac{X_i}{X}.$$

Efforts are costly and defined by $v_{ik}^{-1} g(x_{ik})$, where $0 < v_{ik} < \infty$ is a heterogeneous ability parameter. Since the higher is the v_{ik} the lower the cost of exerting effort, this simple parameter allows us to include the heterogeneity in the model.
We impose the following assumption on $g(x)$:

Assumption 1

i) $g(0) = 0$;

ii) $g'(0) = 0$;

iii) $g'(x) > 0$ for all $x > 0$;

iv) $g''(x) > 0$ for all $x > 0$.

Part (i) states that players do not bear costs when they do not exert any effort. Part (ii) states that the marginal cost of effort at $x = 0$ is zero. Part (iii) and (iv) state respectively that the effort cost function is strictly increasing and strictly convex. Finally, part (iv), in conjunction with part (iii), ensures the existence and uniqueness of an equilibrium in which all players exert a positive effort as long as they receive a strictly positive part of the prize. Moreover, since g is monotonic and continuous, it has a well defined inverse function. Assumption 1 is held throughout the paper.

Having specified the probability of winning and the cost function, we have to define the prize's characteristics. Since we are studying a winner-takes-all contest, the winning group is rewarded with a prize normalized to one while the losing groups receive zero [2]. Moreover,

[2] Another interpretation is that the group probability of winning X_i/X is the share of a perfectly divisible prize that the group wins competing in the contest.

we assume that the team manager commits in advance on the prize allocation. Thus, groups members best respond to other players' choices of effort and to the prize share that they could win. This share is defined by ϕ_{ik}, where $\sum_{k=1}^{n} \phi_{ik} = 1$.
Now we can write the expected payoff for player ik as

$$\pi_{ik} = \frac{X_i}{X}\phi_{ik} - \frac{g(x_{ik})}{v_{ik}}. \tag{1}$$

Each player ik's best response to all other players' choices of effort is given by the first-order condition associated with the maximization of π_{ik} as a function of x_{ik}, subject to $x_{ik} \geq 0$. Since (1) is strictly concave with respect to x_{ik}, the first-order condition is necessary and sufficient for the best response. It follows that the player ik's best response is

$$\frac{X_{j\neq i}}{X^2}\phi_{ik} = v_{ik}^{-1} g'(x_{ik}). \tag{2}$$

It is possible to show that there is a pure strategy Nash equilibrium in which all groups always exert a strictly positive effort. Assume that all team managers allocate the whole prize to a single player, for instance to player $i1$, i.e. $\phi_{i1} = 1$. All other players in group i face the left-hand side of Equation (2) equal to zero, and therefore their choice of effort is 0. Otherwise player $i1$, if any other player $j \neq i$ exerts a positive effort, has the left-hand side of Equation (2) positive. This implies that the right-hand side should also be positive. This, in turn, means that the best response of player $j1$ is also positive.
Suppose now that all players receive a positive share of the prize, whereby $0 < \phi_{ik} < 1$. It follows that the left-hand side is positive for all players involved in the contest, hence all of them exert a positive effort in the equilibrium. The argument is completed by the fact that all players exerting zero effort is not an equilibrium. Indeed, under the assumption $g'(0) = 0$, players always exert a positive effort except for $\phi_{ik} = 0$. Zero contributions may occur if $g'(0) > 0$. Otherwise, a strictly positive $x_{ik} > 0$ that satisfies the first-order condition always exists. Finally, since at least a player in each group exerts a positive effort in equilibrium for any prize allocation, all groups always exert a positive effort.

Proposition 2.1. *Under Assumption 1, the contest between groups has a unique Nash equilibrium in pure strategy for any prize allocation. In equilibrium, at least one player in each group exerts a positive effort, therefore all groups exert a positive effort. Moreover, the equilibrium effort x_{ik}^* satisfies the system of Equation (2) with equality.*

Proof. The following proof is an extension of Ryvkin (2011) for a perfectly divisible prize. Player ik's best response function is

$$\frac{X_{j\neq i}}{X^2} = (v_{ik}\phi_{ik})^{-1} g'(x_{ik}). \tag{3}$$

Note that the left-hand side of the Equation (5) is the same for any player k of group i. It follows that for any given effort of x_{i1} we have that $(v_{im}\phi_{im})^{-1} g'(x_{im}) = (v_{i1}\phi_{i1})^{-1} g'(x_{i1})$. Thus, the effort exerted by all im, where $m > 1$, can be uniquely determined as a share of the effort exerted by player $i1$ as

$$x_{im} = g'^{(-1)}\left(\frac{v_{im}\phi_{im}}{v_{i1}\phi_{i1}} g'(x_{i1})\right). \tag{4}$$

The group i's effort X_i can be written as

$$X_i = \alpha_i(x_{i1}) = x_{i1} + \sum_{m>1}^{n} g'^{(-1)}\left(\frac{v_{im}\phi_{im}}{v_{i1}\phi_{i1}} g'(x_{i1})\right).$$

Notice that function $\alpha_i(x_{i1})$ is strictly increasing and satisfy $\alpha_i(0) = 0$. Therefore, the contest among N groups reduces to a contest among N individuals:

$$\frac{\sum_{j\neq i} \alpha_j(x_{j1})}{\sum_{i=1}^{N} \alpha_i(x_{i1})} = (v_{ik}\phi_{ik})^{-1} g'(x_{ik}). \tag{5}$$

Let $y_i = \alpha_i(x_{i1})$ than $x_{i1} = \alpha^{-1}(y_i)$ and introduce the function $G_i'(y_i) = (v_{i1}\phi_{i1})^{-1} g'(\alpha^{-1}(y_i))$ with the initial condition $G_i'(0) = 0$. These properties uniquely define G_i as

$$G_i(y_i) = \int_0^{y_i} (v_{i1}\phi_{i1})^{-1} g'(\alpha^{-1}(t))dt.$$

Note that $G(y_i)$ is strictly increasing, strictly convex and satisfies $G(0) = G'(0) = 0$ (Part (i) and (ii) of Assumption 1). The system of Equation (5) can be written as

$$\frac{\sum_{j\neq i} y_j}{\sum_{i=1}^{N} y_i} = G_i(y_i). \tag{6}$$

The uniqueness of equilibrium follows from Theorem 3 of Cornes and Hartley (2005).

N.B. If the prize is a public good, $\phi_{i1} = \phi_{im} = 1$, or is equally shared among group members, $\phi_{i1} = \phi_{im} = 1/n$, the proof corresponds to the Appendix A in Ryvkin (2011). On the other hand, if the prize can be allocated in any way among players, we need that $\phi_{i1} > 0$; otherwise if $\phi_{i1} = 0 \Rightarrow x_{i1} = 0$ from Equation (3), $x_{im} = \infty$ from Equation (4) and $X_i = \infty$. As we see in the next section this is never the case because since $v_{i1} \geq v_{i2} \geq ... \geq v_{in}$ we have $\phi_{i1} \geq \phi_{i2} \geq ... \geq \phi_{in}$.

□

Recalling that the group effort is the linear sum of players' effort, we can write the group i's effort as

$$X_i^* = \sum_{k=1}^{n} x_{ik}^* = \sum_{k=1}^{n} f\left(\frac{1-\sigma_i^*}{X^*}(v_{ik}\phi_{ik})\right),$$

where $1-\sigma_i = X_j/X$ and $f = (g')^{-1}$.

3 Homogeneity and heterogeneity in groups

In this section we study the heterogeneity in ability within groups. In order to do this, we assume that the prize is equally shared among group members (or that it is a public good) while they can differ in their ability parameters. Thus, we can rewrite the group i's effort as

$$X_i^* = \sum_{k=1}^{n} f\left(\frac{1-\sigma_i^*}{X^*}\frac{1}{n_i}v_{ik}\right). \tag{7}$$

In order to study this question we represent the groups effort as vectors $\mathbf{x_i} = (x_{i1}, ..., x_{in})$ assuming that $\sum_{k=1}^{n} v_{ik} = \sum_{k=1}^{n} v_{jk}$. Let us recall the following theorem:

LEMMA 1 (Hardy, Littlewood and Polya). *Let* $\mathbf{x_i}$ *and* $\mathbf{x_j}$ *be two vectors in* R^n*, ordered so that* $x_{i1} \geq ... \geq x_{in}$ and $x_{j1} \geq ... \geq x_{jn}$ if
i) $\sum_{k=1}^{n} x_{ik} = \sum_{k=1}^{n} x_{jk}$ *and*
ii) $x_{j1} + ... + x_{jl} \leq x_{i1} + ... + x_{il}$*, all* $l \leq n$ *(with strict inequality for at least one* l*)*
then for any strictly concave function f *defined on* R^1*, we have* $\sum_{k=1}^{n} f(x_{ik}) > \sum_{k=1}^{n} f(x_{jk})$.

The opposite holds if function f is convex. Therefore, since a unique pure strategy Nash equilibrium exists and, except for v_{ik}, all players face the same best response functions, which aggregated give Equation (7), we can apply Lemma (1) so as to state our first proposition.

Proposition 3.1. *Given a contest among groups in which the prize is equally shared among group members and* $\sum_{k=1}^{n} v_{ik} = \sum_{k=1}^{n} v_{jk}$*:*

i) if f *is convex, then the higher is the variance in ability the higher the group effort. The most heterogeneous group has the highest winning probability;*

ii) if f *is concave, then the lower is the variance in ability the higher the group effort. The most homogeneous group has the highest winning probability:*

iii) if f is linear, then the variance in ability has no effect on group effectiveness.

Proof. Recall that group i's total effort is

$$X_i^* = \sum_{k=1}^{n} f\left(\frac{1-\sigma_i^*}{X^*}\frac{1}{n_i}v_{ik}\right).$$

Consider that team i's manager replaces some of her players in order to increase the variance in ability under the condition $\sum_{k=1}^{n} v_{ik} = \overline{v}$. Now, in order to show that the group i's effort (and its probability of winning) increases, we separate the direct effect of $Var(v_i)$ from the indirect one. Thanks to Lemma 1 we have that if f is convex (concave), then:

i) $Var(v_i) \uparrow (Var(v_i) \downarrow) \Longrightarrow X_i \uparrow$;

ii) $X_i \uparrow \Longrightarrow \sigma_i \uparrow$;

iii) $X_i \uparrow \Longrightarrow X^* \uparrow$[3];

iv) (ii) and (iii) $\Longrightarrow \frac{1-\sigma_i^*}{X^*} \downarrow \Longrightarrow X_i \downarrow$.

Notice that (i) must be bigger then (iv), otherwise we have the contradiction that $X_i \uparrow \Longrightarrow \sigma_i \downarrow$.

□

Notice that Proposition (3.1) still holds under contests with a public good prize. That is, depending on the inverse function of the marginal cost $f = (g')^{-1}$, the heterogeneity (homogeneity) of a group can increase (decrease) the group's total effort. If the marginal cost is convex (concave), the well define inverse function f is concave (convex) and the homogeneous group has an higher (lower) winning probability than a heterogeneous one. Finally, it is straightforward that the degree of heterogeneity does not affect the group's effectiveness when the marginal cost is linear. In the next section, we analyse the effect of optimal prize allocations on groups' effectiveness showing why they may overturn the previous results.

[3]See Theorem (2) of Cornes and Hartley (2011). Indeed, if the total effort exerted is increasing in the number of players active in the contest, then it is also increasing in the effort exerted by a single player.

4 Optimal Prize Allocations

In this section we provide a benchmark in order to allocate the prize optimally within groups with heterogeneous players. As in the previous section, our results depend on the form of the cost function. We begin our analysis studying the symmetric case in which all players have the same ability parameter $v_{ik} = 1$. The total effort of group i is

$$X_i^* = \sum_{k=1}^{n} f\left(\frac{1-\sigma_i^*}{X^*}\phi_{ik}\right). \tag{8}$$

We can now state which prize allocation maximizes the group probability of winning. Notice that the function f plays a crucial role in such analysis. Due to Lemma 1, we can state our second proposition:

Proposition 4.1. *Given a contest among groups in which all players have the same ability parameter:*

i) if f is convex, then the higher is the variance in prize shares the higher the group effort; the group effort is maximised when a player has the monopoly of the prize;

ii) if f is concave, then the lower is the variance in prize shares the higher the group effort; the group effort is maximised when all players receive an equal share of the prize;

iii) if f is linear, then the variance in prize shares has no effect on group effectiveness.

Proof. Recall that the group i's total effort is given by

$$X_i^* = \sum_{k=1}^{n} f\left(\frac{1-\sigma_i^*}{X^*}\phi_{ik}\right).$$

Assume that all players have the same ability parameter $v_{ik} = 1$ and that the prize is equally shared among group members. Furthermore, the manager of team i allocates the prize increasing (decreasing) $Var(\phi_{ik})$ s.t. $\sum_{k=1}^{n}\phi_{ik} = 1$. In order to show that the group i's effort is increasing in $Var(\phi_{ik})$, we separate the direct effect of $Var(\phi_{ik})$ from the indirect one. Because of Lemma 1, we have that if f is convex (concave) then:

i) $Var(\phi_{ik}) \uparrow (Var(\phi_{ik}) \downarrow) \Longrightarrow X_i \uparrow$;

ii) $X_i \uparrow \Longrightarrow \sigma_i \uparrow$;

iii) $X_i \uparrow \Longrightarrow X^* \uparrow$;

iv) (ii) and (iii) $\Longrightarrow \frac{1-\sigma_i^*}{X^*} \downarrow \Longrightarrow X_i \downarrow$.

Notice that (i) must be bigger than (iv), otherwise we have the contradiction that $X_i \uparrow \Longrightarrow \sigma_i \downarrow$.

□

Allocating the whole prize to a player is the optimal solution as long as the function f is convex. Moreover, due to the symmetry assumption, it does not matter which group member has the monopoly of the prize. As a result, if all team managers apply this optimal allocation, the contest between N groups is equivalent to a contest between N individuals. Notice that allocating the whole prize to a single player is equivalent to reducing the number of active players in the contest for a given group. As a matter of fact, if a player does not receive a positive share of the prize, than he does not exert any effort. Under our assumptions we can shed light on what is know as the group size paradox. Olson (1965) argues that the free-riding problem makes smaller groups more effective than larger ones. However, Esteban and Ray (2001) shows that this is not true when the marginal cost function is convex. What we show here is that Olson statement is not even true if the prize is optimal allocated and the marginal cost function is concave. This occurs because there is only an active player in the group that receives the whole prize, leading the group to be as much effective as a singleton.
On the other hand, if the best response function is concave and players are symmetric, then the equal prize allocation is also the optimal solution. Thus, we are able to extend the Proposition (1-2)[4] of Esteban and Ray (2001) as follows:

Proposition 4.2. *Given a contest between groups of symmetric individuals where the pure private good prize is optimally allocated among group members we have that:*

i) if f is convex or linear the group's probability of winning does not depend on the size of the group, moreover, larger groups have equal probability of winning as smaller ones.

ii) if f is concave the group's probability of winning is strictly increasing in group size, moreover, larger groups have higher probability of winning than smaller groups.

This statement is straightforward and allows us to stress the importance of a optimal prize allocation in the study of group contests. Nevertheless, what is important to underline

[4]See also "The anti-Olson theorem" in Nitzan and Ueda (2013) that together with the Proposition (1-2) in Estaban and Ray (2001) shed light on the group size paradox

is that allocating the whole prize to a single player and competing with a singleton are not equivalent strategies. Consider for instance a prize composed both by private and public characteristics. The private part should be allocated to a single player. However, due to the public part, all other members are active in equilibrium and exert positive effort.

4.1 A discussion about fairness

In the previous section we have seen that if the marginal cost of effort is concave, then the team manager allocates the whole prize to a player so as to maximize the group effort. However, in some cases this policy is not implementable due to a lack of fairness. For instance, it may occur that in some public environments the prize must be equally shared among group members. This restriction is also justified by the fact that group members have the same abilities. Our concern here is the following: which is the additional cost that a team manager has to bear in order to guarantee fairness and the same effectiveness of a group that is optimally allocating the prize? The study of this question is straightforward since we have already found the unique equilibrium and the group total effort under the optimal prize allocation.

The total effort of any group $j \neq i$ that is allocating the prize optimally is given by $X_j = x_{j1} = f(\frac{1-\sigma_j^*}{X^*})$. On the other hand, the total effort exerted by the group i, which is equally sharing the prize among group members, is

$$X_i = nx_{ik} = f(\frac{1-\sigma_i^*}{X^*}\frac{1}{n})n,$$

where $\sigma_i < \sigma_j$ for every $n > 1$. In this analysis we are looking for the additional prize defined by t_i, which allows group i to be as effective as any other group j that is maximizing its effort allocating the whole prize to a single player. It is straightforward that t_i is given by solving the equation $X_i(t_i) = X_j$. Hence, rewriting group i's total effort as

$$X_i = f(\frac{1-\sigma_i^*}{X^*}\frac{1+t_i}{n})n,$$

setting it equal to X_j and solving for t_i we have that

$$t_i = \frac{n}{f^{-1}(X_j)} f^{-1}\left(\frac{X_j}{n}\right) - 1. \tag{9}$$

Where $X_j = f(\frac{1-\sigma_j}{X^*})$. It is possible to notice that t_i is equal to zero if and only if $n = 1$. Since the prize is normalized to 1, t_i can be thought as the percentage of the additional

prize that a team manager has to pay in order to guarantee fairness and make her group as effective as a group that applies the optimal allocation.
For concreteness consider $g(x) = x^\alpha$ where $1 < \alpha < 2$. Equation (9) is simply $t_i = n^{2-\alpha} - 1$. The additional cost t_i is increasing on the number of group members and decreasing on the steepness of the marginal cost.

4.2 Optimal Prize Allocations with Heterogeneous Players

In this section we study the optimal allocation among heterogeneous players. Recall that the total group effort is given by

$$X_i^* = \sum_{k=1}^{n} f\left(\frac{1-\sigma_i^*}{X^*}(v_{ik}\phi_{ik})\right).$$

Since the sum of strictly convex functions is strictly convex, and the sum of strictly concave functions is a strictly concave, we can state our next proposition.

Proposition 4.3. *Given a contest among groups where players are heterogeneous in their ability parameters we have that:*

i) If f is convex or linear, then the optimal allocation is to award the whole prize to the player with the highest ability parameter, furthermore, the higher is the variance in ability the higher the group effort.

ii) If f is concave, then there exists a unique prize allocation defined as ϕ_{ik}^* that maximizes the group probability of winning. Under ϕ_{ik}^* all players receive a positive share of the prize.

Proof. Recall that the total group effort is given by

$$X_i^* = \sum_{k=1}^{n} f\left(\frac{1-\sigma_i^*}{X^*}(v_{ik}\phi_{ik})\right).$$

Moreover, $(1-\sigma_i)/X^*$ is the same for all group members and the maximization of X_i is constrained to $\sum_{k=1}^{n} \phi_{ik} = 1$. The maximization of the group i's effort is given by the following Lagrangean problem:

$$L = \sum_{k=1}^{n} f\left(\frac{1-\sigma_i^*}{X^*}(v_{ik}\phi_{ik})\right) + \lambda(1 - \sum_{k=1}^{n} \phi_{ik}).$$

By differentiating with respect to ϕ_{ik} and λ, we have that

$$f'\left(\frac{1-\sigma_i^*}{X^*}(v_{i1}\phi_{i1})\right)v_{i1} = f'\left(\frac{1-\sigma_i^*}{X^*}(v_{im}\phi_{im})\right)v_{im}, \tag{10}$$

where $m > 1$. Recall that $v_{i1} \geq v_{im}$ and, since f is convex, $f' > 0$ and $f'' > 0$. Accordingly, the left-hand side of Equation (10) is always grater (except for $\phi_{ik} = 0$) than the right-hand side. Thus, there is a corner solution in which the group effort is maximized when the whole prize is allocated to player $i1$. The same holds if f' is a linear function.

On the other hand, when f is concave and the constrain is linear, the Lagrangean equation is concave. Hence, there is one optimal solution ϕ_{ik}^*, which is a stationary point of the Lagrangean equation, moreover, it is possible to notice that since $f' > 0$ and $f'' < 0$, the higher is the ability parameter of a player the higher the prize share he receives. Rearranging Equation (10) as

$$f'\left(\frac{1-\sigma_i^*}{X^*}(v_{i1}\phi_{i1})\right)/f'\left(\frac{1-\sigma_i^*}{X^*}(v_{im}\phi_{im})\right) = v_{im}/v_{i1},$$

it is possible to notice that the right-hand side is equal or smaller than one since $v_{i1} \geq v_{im}$. Furthermore, because f' is decreasing in ϕ_{ik}, players with higher ability parameter always receive a higher prize share.

□

The first part of the proposition is straightforward. When we have a linear maximization the whole prize should be allocated to the player with the highest ability parameter in order to maximize the the group effort. The same holds for a sum of convex functions. In addition, the competition between N groups reduces to a competition between the most able individuals in each group. Moreover, because the only ability parameter that matters is the one of the the most able player, the most heterogeneous group has the highest probability of winning. Unfortunately, if f is concave, we are only able to provide an example that allows us to understand how the prize allocation can overturn a group probability of winning, without supplying a more general statement.

4.3 Example with constant elasticity functions

Let consider the case $g(x) = x^\alpha$, which for $\alpha > 1$ satisfies Assumption 1. The total group effort can be written as

$$X_i = \sum_{k=1}^{n}\left(\frac{1-\sigma_i^*}{X^*}v_{ik}\phi_{ik}\right)^{\frac{1}{\alpha-1}}. \tag{11}$$

When $\alpha > 2$ exists a unique optimal prize allocation given by the first derivative of X_i with respect to ϕ_{ik} subjected to $\sum_{k=1}^{n} \phi_{ik} = 1$. After some tedious calculation we have that it is given by

$$\phi_{ik}^* = \frac{v_{ik}^{\frac{1}{\alpha-2}}}{\sum_{k=1}^{n} v_{ik}^{\frac{1}{\alpha-2}}}. \tag{12}$$

Proof. In order to maximize Equation (11) with respect to ϕ_{ik}, we have to solve the following Lagrangean problem:

$$L = \sum_{k=1}^{n} \left(\frac{1-\sigma_i^*}{X^*} (v_{ik}\phi_{ik}) \right)^{\frac{1}{\alpha-1}} + \lambda(1 - \sum_{k=1}^{n} \phi_{ik}).$$

Thus, we have that

$$f'_{i1}(.)v_{i1} = f'_{im}(.)v_{im}.$$

We can rewrite the prize of any player $im \neq i1$ as a share of the prize received by player $i1$ as

$$\phi_{im} = \phi_{i1} \left(\frac{v_{im}}{v_{i1}} \right)^{\frac{1}{\alpha-2}}.$$

Replacing it in the constraint we get

$$\phi_{i1} + \sum_{m=2}^{n} \phi_{i1} \left(\frac{v_{im}}{v_{i1}} \right)^{\frac{1}{\alpha-2}} = 1;$$

a simple rearrangement gives us

$$\phi_{i1} = \frac{v_{i1}^{\frac{1}{\alpha-2}}}{\sum_{k=1}^{n} v_{ik}^{\frac{1}{\alpha-2}}}.$$

Hence, for every im, we have $\phi_{ik}^* = \frac{v_{ik}^{\frac{1}{\alpha-2}}}{\sum_{k=1}^{n} v_{ik}^{\frac{1}{\alpha-2}}}$. □

Replacing ϕ_{ik}^* into Equation (11) and rearranging, we have that the group i's effort is

$$X_i = \sum_{k=1}^{n} \left(\frac{1-\sigma_i^*}{X^*} v_{ik} \right)^{\frac{1}{\alpha-2}}.$$

Proposition 4.4. *Given a contest among groups in which players have heterogeneous ability parameters and the prize is optimally allocated:*

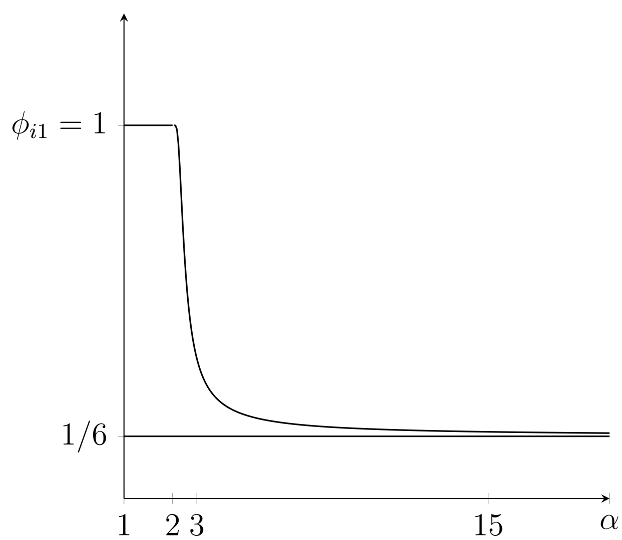
i) if $2 < \alpha < 3$, then the higher is the variance in ability the higher the group effort;

ii) if $\alpha > 3$, then the lower is the variance in ability the higher the group effort;

iii) if $\alpha = 3$ the variance in ability has no effect on group effectiveness.

It follows that once that the price is optimally allocated if $\alpha > 3$, then a homogeneous group has a higher winning probability than heterogeneous ones, whereas for $1 < \alpha < 3$ the opposite holds. However, this is not a intuitive results, since in Proposition (3.1) we stated that if f is concave ($\alpha > 2$), a homogeneous group outperforms a heterogeneous one under the equal prize allocation. However, as long as the prize is optimally allocated, Proposition (3.1) is no longer valid. For instance, consider a competition where the cost function can be represented by x^{α}, where $2 < \alpha < 3$. If a team manager is not able to allocate the prize because it is a public good or because it must be equally shared among group members, Proposition (3.1) tells us which group is the most effective. The most homogeneous group has the highest probability of winning. On the other hand, if the prize can be allocated optimally, Proposition (3.1) is overturned. Indeed the group with the highest variance in ability is the most effective one.

It is also possible to notice from Equation (12) that the share of the prize that players receive depends on the parameter α. Furthermore, if $\alpha \to \infty$, then $\phi_{ik} \to 1/n$. For instance, the share of the prize allocated to the most able player in a group composed of 6 players, where $v_{i1} = 3$ and $v_{mk} = 1$, is



Although for $\alpha = 1$ the cost function does not respect Assumption 1, Baik (2008) shows that only the player with the highest valuation for the prize, which is equivalent to the player with the highest ability parameter, exerts positive effort in the equilibrium. It follows that, a team manager who allocates the whole prize to the most able player maximizes the group effort.

5 Does Cooperation crowd out inequality?

In a cooperative model, agents decide how much effort to exert simultaneously and cooperatively. In order to investigate cooperative contests, we assume that players consider the cooperation as the most fruitful individual strategy. We are not concerned with why this occurs. Indeed, the cooperation may be a consequence of organizational schemes directed to monitor agents' behavior, or it may be the winning strategy as in sport competitions. In all these cases, free-riding is less acute. However, even if we know that players cooperate, our assumption remains "ad hoc".

In a cooperative model, the player ik's expected payoff is

$$\pi_{ik} = \frac{X_i}{X}\phi_{ik} - g(X_i s_{ik}).$$

Defining $X_i s_{ik} = x_{ik}$, the player ik's effort is rewritten as the share s_{ik} of his group effort X_i. For instance, under the symmetry assumption, $s_{ik} = 1/n$.

In a cooperative contest, player ik's choice of effort is given by the maximization of π_{ik} as a function of X_i. That is,

$$g'(X_i s_{ik}) s_{ik} = \frac{X_{j \neq i}}{X^2}\phi_{ik}. \tag{13}$$

Under Assumption 1, Equation (13) is necessary and sufficient for maximization.

Proposition 5.1. *Under Assumption 1, the contests between groups with cooperative players has a unique Nash equilibrium in pure strategies for any prize allocation. In equilibrium, at least one player in each group exerts positive effort, therefore all groups exert positive effort. Moreover, the equilibrium effort x^*_{ik} satisfies the system of Equation (13) with equality.*

We can rearrange Equation (13) as

$$g'(x_{ik})x_{ik} = (1 - \sigma_i)\sigma_i\phi_{ik}.$$

Notice that given Assumption 1, function $g'(x_{ik})x_{ik}$ is monotonically increasing, continuous and it has a well defined inverse function that we define as β. The players ik's best response is

$$x_{ik} = \beta(1 - \sigma_i)\sigma_i\phi_{ik}. \tag{14}$$

In non-cooperative contests, we have shown that, if the marginal cost function is concave, the group effort is maximized when a players has the whole prize . We are here in a different environment, where the preceding propositions may be not valid. Indeed, we show how the cooperation among players strongly influences the optimal prize allocation, leading to different results.

Let us aggregate Equation (14) as

$$X_i = \sum_{k=1}^{n} x_{ik} = \sum_{k=1}^{n} \beta(1 - \sigma_i)\sigma_i\phi_{ik}$$

Proposition 5.2. *Given a cooperative contest between groups with symmetric players:*

i) if β is convex, then the higher is the variance in prize shares the higher the group effort; the total group effort is maximized when one player has the monopoly of the prize:

ii) if β is concave, then the lower is the variance in prize shares the higher the group effort; the total group effort is maximized when all players receive an equal share of the prize:

iii) if β is linear, then the variance in prize shares has no effect on group effectiveness.

Considering again the cost function $g(x_{ik}) = x^\alpha$ we have that parts (i) and (iii) of Proposition 4.5 never occur. Indeed, we have that $g'(x_{ik})x_{ik} = x^\alpha$. As a result, its inverse function is always concave, which in turn implies that the equal prize allocation always maximizes group effectiveness. For more general functions that satisfy Assumption 1 $g'(x_{ik})x_{ik}$ is concave, and then β is convex, when $k = x_{ik}g'''(x_{ik})/g''(x_{ik}) < -2$. Notice that for the non cooperative case the condition to allocate the whole prize to a single player is $g'''(x_{ik}) < 0$.

6 Conclusions

The main goal of this work is to explore the heterogeneity in ability and in prize allocations in contests between groups. Our results provide a clear solution that depends on the form of the marginal cost function and, in turn, on the form of the best response function. In situations where the prize is a public good, or where it is equally shared among group members, the highest variance in ability maximizes the group effectiveness if the marginal cost of effort is concave. Otherwise, the lowest variance in ability maximizes the group effort if the marginal cost of effort is convex. Our results find in part application in sports, where the prize of a tournament can be approximated by a public good. In such cases, if the cost of effort does not increase too rapidly, the manager can form a very heterogeneous team so as to increase group effectiveness. On the other hand, a more balanced team has a higher probability of winning when the cost of effort increases rapidly enough. We provide also the optimal prize allocation under the assumption of symmetry among players. In this framework, the results are in line with our findings about the heterogeneity. That is, a team manager has to allocate the whole prize to one player in order to maximize the group effort when the marginal cost is concave. Vice versa, if the marginal cost is convex, the equal prize allocation is the optimal solution. In addition to this, we discuss an alternative solutions for those environments that can not apply the optimal rule. In the model we include the heterogeneity in ability within groups so as to provide an optimal prize allocation under a more real setting. We found that if the marginal cost of effort are concave the optimal allocation awards the whole prize to the most able player, leading all the other members inactive and turning the group contest between N groups into a competition between N individuals. However, this solution does not exclude the cases in which the prize may be composed also by a public part. Indeed, under this condition, also other group members exert a positive effort due to the public part of the prize. On the other hand, if the marginal costs of effort are convex and players differ in their ability parameters, all group members receive a positive share of the prize. In the latter case, the prize shares assigned to the players is increasing in their ability parameter and decreasing in the steepness of the marginal cost function. These findings can be related to sales. For instance, retail firms usually set up contests which reward the shop achieving higher sales, and they also divide the prize differently according to each of their employees' responsibility, offering a higher share of the prize for the shop manager and a lower one for sellers. Our result suggests that the costly is the contest, the lower the difference in prize share allocated among employees. Another example can be found in some education systems. For instance, according to the new Italian law, every year the heads of the schools have a budget (bonus) to allocate to the teachers. This budget depends (in part) on the number of students enrolled in the school. Clearly the higher is the quality of teaching, the higher the number of students in the school and, in turn, the higher the next year bonus. Thus, the heads of the schools have to allocate the budget not only to award the best teachers but also to increase the overall effort exerted by the teaching staff.

Finally, we extended our analysis to the cooperative case. Our results are in line with the intuition that, without the free-riding, a cooperative group is more effective than a non-cooperative one. Moreover, it is always optimal to allocate a positive part of the prize to all players as long as they fully cooperate.

7 References

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