A CHARACTERIZATION OF SEQUENTIAL EQUILIBRIUM IN GAMES OF SIMPLE INFORMATION TYPE★

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Abstract. We classify all sequential games into two categories based on their information structure: games of a simple and a complex information type. We study sequential games that can be solved using a generalized backward induction method and show that this method can be used *if and only if* the game is of the simple information type. We also show that if the game is of the simple information type then the generalized backward induction method yields the *entire* set of sequential equilibria of the game. The method consists of two parts: the roll-back procedure and the consistency check, the later being performed after the entire sequentially rational strategy profile has been constructed. We propose a method that allows to test for simple information type. The The majority of sequential games that arise in applications are of simple information type.

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1. Introduction

Kreps and Wilson [10] in introducing the concept of sequential equilibrium argue that solutions of sequential games should have some elements of sequential rationality and, in particular, one should be able to use some form of backward induction or the "rollback procedure" to find those solutions. In fact they state: "The criterion of sequential rationality is familiar in the analysis of single-person decision problems. It justifies the standard "roll-back" procedure for constructing a sequential optimal strategy in a problem described by a decision tree. Extensive games are multi-person decision problems. Though complicated by the interaction among players, they are not different in substance[1]." We show here that such a roll-back procedure can only be applied to a class of sequential games which are of the so-called *Simple Information Type*.

Although the idea of sequential rationality, which we rely upon, is well known and has been extensively used to find sequential equilibria in applications, our contribution is in the generality of the result. We demonstrate that for a certain class of games the backward induction method does not generalize in a useful and practical way. The main contribution is in providing the insight in terms of better understanding of the backward induction method. We present the necessary and sufficient condition for the method to fail, clarify the reasons for the failure, and validate the conjecture by Kreps and Wilson [10] regarding a generalization of the backward induction method for an important class of sequential games. For the latter, we show that indeed the backward induction method allows us to find the entire set of sequential equilibria for an arbitrary game of the simple information type.

It is well-known that the backward induction procedure works extremely well for sequential games with perfect information as was first demonstrated by [11] (also see Aliprantis [3] for a precise mathematical description of the backward induction). For sequential games with imperfect information, the issue is more subtle as one needs to account for players' beliefs. The optimal choice at a given information set depends on the beliefs at the information set, and these beliefs in turn depend on the strategies used by the players whose choices precede this information set. In contrast, in sequential games with perfect information all players know *exactly which node* they are at when making choices. What we show here is that in many cases it is still possible to use a more general version of the backward induction method to find all sequential equilibria of games with incomplete or imperfect information.

[1]See the first paragraph of the concluding remarks on page 885 of [10].

The method consists of two parts: the roll-back procedure and the verification of consistency. To perform the roll-back procedure, one can look at the best responses of each player for each possible belief at a given information set, and then roll back to find the optimal choices of the players at the information sets at the preceding stage of the game. The consistency check is performed after the entire sequentially rational strategy profile has been constructed rather than stage-by-stage along the game tree.

We show that by using the generalized backward induction procedure one can find *the entire set* of sequential equilibria for an important class of finite sequential games with perfect recall that we refer to as games of a *simple information type*. Roughly speaking this means that the nodes in the information sets do not interlock, i.e., have a clear chronological order. We show that for such games the backward induction process yields very good results. The simple information property is quite broad and easily verifiable. Indeed, the majority of the sequential games studied in economics, management, and computer science are of the simple information type. These include various models of sequential bargaining (see for instance [2, 13]), Entry Deterrence Game (see for example [13]), various signaling games (for instance, Joint Venture Entry Game [13], limit-pricing games [15], as well as various models of cheap talk games. One may also refer to [16] and [6] for other examples. We establish that the condition that the extensive form has a simple information structure is the necessary and sufficient condition for solvability by the generalized backward induction method. We examine an example of a game that is not of the simple information type and find that the generalized backward induction process does not work, and one then has to use the agent normal form to find sequential equilibria of the game.

Our paper is organized as follows. In Section 2 we describe the notation and give all the definitions regarding sequential games and sequential equilibrium. In the next section we classify all sequential games into two categories – those of simple and complex information type. In Section 4 we show that simple information type is the necessary and sufficient condition for the backward induction solvability. For games of simple information type we generalize the backward induction method and prove that the method identifies all sequential equilibria. In Section 5 we conclude.

2. Definitions and Methodology

As usual, a reflexive, antisymmetric, and transitive binary relation on a set X is called a *partial order*.

Definition 2.1. *A* ***pograph*** *is a pair* $(X, \succeq)$*, where* X *is a finite set of nodes, and* $\succeq$ *is a partial order on* X*. An arbitrary node of* X *will be denoted by* t*.*

Intuitively, the binary relation $\succeq$ designates precedence, and the notation $t_1 \succeq t_2$ informs us that t_2 is among the successors of t_1. The strict relation $\succ$ is defined as usual.

Definition 2.2. *If* $y \succeq x$*, then* y *is a* ***predecessor*** *of* x*, and* x *is a* ***successor*** *of* y*.*

Definition 2.3. *A node* $t \neq x$ *is the* ***immediate predecessor*** *(or the* ***parent****) of* x *if* $t \succeq x$ *and there is no other node* s *such that* $t \succ s \succ x$*. In this case* x *is called an* ***immediate successor****, or a* ***child*** *of* t*.*

A node with nonempty set of children would be referred to as a **decision node**, while a node having no children would be called a **terminal node**. A node with no parent would be called a **root**. Denote the set of all decision nodes by Y, the set of all terminal nodes by Z, and the set of roots by W. Clearly, $W \subseteq Y$ and $X = Y \cup Z$.

Definition 2.4. *If* $x \succ y$*, then the* ***path*** *from* x *to* y *is the chain*

$$x = x_1 \succeq x_2 \succeq \ldots \succeq x_n = y$$

such that x_{i-1} *is the immediate predecessor of* x_i *for each* $i = 2, \ldots, n$*.*

Definition 2.5. *A* ***frame*** T *is a pograph such that for every non-root node* x *there exists a unique root* w *such that:*

1. *there exists a unique path from* w *to* x*, and*
2. *for any* $y \in W$ *different from* w*, there is no path from* y *to* x*.*

Definition 2.6. *The length of a frame is the maximum number* L *such that* $w \succ x_1 \succ \cdots \succ x_L$*, where* $w \in W$ *and* $x_L \in Z$*.*

Note that this definition implies that every non-root node has exactly one parent. A frame is called **finite** if the set of nodes X is finite. In this paper we only consider finite frames (for a more general treatment of extensive games refer to Alós-Ferrer and Ritzberger [4] and [5]).

Definition 2.7. *An* ***n-player extensive form (or sequential) game*** G *is a tuple* $(T, \mathcal{P}, H, C, U)$*, whose elements are interpreted as follows.*

*T is a finite **frame** as described in Definition 2.5. We work with frames rather than trees to be more general and account for the possibility of multiple roots.*

*$\mathcal{P}$ is a **player partition**, where $\mathcal{P} = \{\mathcal{P}_0, \mathcal{P}_1, \dots \mathcal{P}_n\}$. Each $\mathcal{P}_i$ is called player i's set and is a set of all nodes at which player i is decisive. Player 0 incorporates a random mechanism that may determine the game path, and may be inactive.*

*H is an **information partition**, which is a refinement of the player partition $\mathcal{P}$. It partitions each set $\mathcal{P}_i$ into information sets. Denote the set of all information sets of player i by H_i. Each information set $\mathcal{I} \in H_i$ has the following properties:*

(1) *if $a, b \in \mathcal{I}$, then $a \not\succ b$, and*
(2) *for every node $a \in \mathcal{I}$, the set of choices available at a is the same.*

Given a decision node $x \in Y$, we will denote the information set containing x by $\mathcal{I}(x)$.

*An information partition has the property that there is an information set $\mathcal{I}_W$, called the **root information set**, which contains all nodes that have no predecessors, i.e., it is the set $\{w \in W\}$. A probability distribution ρ on $\mathcal{I}_W$ is specified and is interpreted as the **distribution of the initial "states of nature"**.*

*C is a **choice partition** that partitions the (finite) set of all choices available throughout the game, M into the subsets C_x, $x \in X$, where C_x represents the set of all choices (actions) available at decision node x. It satisfies the following condition: for every $\mathcal{I} \in H$ and $x, y \in \mathcal{I}$, we have $C_x = C_y$. Thus we can introduce another partition C', which partitions the set of all choices for the game into the subsets $C_{\mathcal{I}}$, each of those containing all choices available at the information set $\mathcal{I}$. To each choice c available at a decision node x there corresponds a unique edge originating from x, and vice versa.*

*U is a **payoff function**, which is a vector-valued function that assigns to every terminal node $z \in Z$ a vector $U(z) = (U_1(z), U_2(z), \cdots, U_n(z))$, whose components are the payoffs of players $1, \cdots, n$ at the terminal node z.*

Definition 2.8. *An information set $\mathcal{I}_i \in H_i$ of player i has **perfect recall** if the following condition is satisfied. Whenever a node $x \in \mathcal{I}'_i$ is a predecessor of a node $y \in \mathcal{I}_i$ and the unique path from x to y starts with a choice c, then every node $w \in \mathcal{I}_i$ has a predecessor $v \in \mathcal{I}'_i$ such that the unique path from v to w starts with c.*

Definition 2.9. *An extensive game G is said to have* ***perfect recall*** *if every information set has perfect recall.*

We only consider extensive games with perfect recall because the existence of sequential equilibria is guaranteed only for games with perfect recall.

Definition 2.10. *Given an information set $\mathcal{I} \in H_i$ of player i, define a* ***local strategy*** *$b_{i\mathcal{I}}$ to be a probability distribution over $C_{\mathcal{I}}$. Denote the set of all local strategies of player i at $\mathcal{I}$ by $B_{i\mathcal{I}} = \Delta_{d_{\mathcal{I}}}$, where $d_{\mathcal{I}}$ is the cardinality of $C_{\mathcal{I}}$ (the number of choices available at $\mathcal{I}$), and $\Delta_{d_{\mathcal{I}}}$ is the unit $(d(\mathcal{I}) - 1)$-simplex.*

A local strategy $b_{i\mathcal{I}}$ is called *completely mixed* if $b_{i\mathcal{I}} \in B^{\circ}_{i\mathcal{I}}$,[2] i.e., if every choice at $C_{\mathcal{I}}$ is played with some positive probability.

Definition 2.11. *A* ***behavior strategy*** *b_i of player i is a tuple $(b_{i\mathcal{I}})_{\mathcal{I} \in H_i}$, i.e., an assignment of some local strategy $b_{i\mathcal{I}}$ to every $\mathcal{I} \in H_i$. The set of all behavior strategies of player i is denoted by B_i, $B = \prod_{\mathcal{I} \in H_i} B_{i\mathcal{I}}$. A* ***behavior strategy combination*** *$b = (b_1, \cdots, b_n)$ is an n-tuple whose i^{th} component is a behavior strategy of player i.*

We will call a behavior strategy b_i *completely mixed* if for each $\mathcal{I} \in H_i$, $b_{i\mathcal{I}}$ is completely mixed. A behavior strategy combination b is *completely mixed* if each b_i is completely mixed.

Fix a behavior strategy combination $b \in B$. It induces a probability distribution P on Z as follows. Fix a terminal node $z \in Z$, without loss of generality let $w \in W$ be the unique root predecessor of z, and $w \succ x_1 \succ \cdots \succ x_{r-1} \succ x_r = z$ be the path from w to z. Given a non-root node x, denote by $b(x)$ the probability assigned by b to the edge connecting x with its parent (recall that in a frame every non-root node has exactly one parent). Then the realization probability of z given $b \in B$ is:

$$P(z|b) = \rho(w) \cdot \prod_{j=1}^{r} b(x_j).$$

Now we can define the expected payoff of player i given a behavior strategy combination b. Assume without loss of generality that the set of terminal nodes is $Z = \{z_1, \ldots, z_m\}$. Then the expected payoff of player i from a behavior strategy b can be calculated as follows.

$$E_i(b) = \sum_{j=1}^{m} U_i(z_j) P(z_j|b).$$

[2] As usual, $B^{\circ}_{i\mathcal{I}}$ denotes the interior of $B_{i\mathcal{I}}$.

Definition 2.12. *A* ***system of beliefs*** μ *is a function that prescribes to every information set* $\mathcal{I} \in H$ *a probability distribution over the nodes in* $\mathcal{I}$*.*

Given a sequential game G with perfect recall, the set of all beliefs systems will be denoted by M.

Definition 2.13. *An* ***assessment*** *is a pair* (μ, b)*, where* $\mu \in M$ *is a system of beliefs and* $b \in B$ *is a behavior strategy combination.*

Fix a completely mixed behavior strategy profile $b \in B^\circ$. Then, every node of the extensive form is reached with some positive probability. As Kreps and Wilson [10] argue, given $b \in B^\circ$, reasonable beliefs are computed from b via Bayes' rule. That is, given a non-root decision node x,

$$\mu(x) = \frac{P(x|b)}{P(\mathcal{I}(x)|b)} = \frac{P(x|b)}{\sum_{y \in \mathcal{I}(x)} P(y|b)}. \tag{2.1}$$

Denote by $\Psi^\circ \subseteq B^\circ \times M$ the set of all assessments (μ, b) such that b is a completely mixed behavior strategy and μ is computed from b via the above formula. Let Ψ be the closure of Ψ° in $B \times M$.

Definition 2.14. *An assessment* (μ, b) *is* ***consistent*** *if* $(\mu, b) \in \Psi$*, i.e.,* (μ, b) *is the limit point of some sequence* $(\mu_k, b_k) \subseteq \Psi^0$*.*

Definition 2.15. *The* ***belief correspondence*** $\phi : B \twoheadrightarrow M$ *is defined for each* $b \in B$ *as* $\phi(b) = \{\mu \in M : (\mu, b) \in \Psi\}$*.*

Clearly, ϕ is nonempty-valued and closed (i.e., has a closed graph). Note that it is single-valued on B°, because for any completely mixed behavior strategy, every information set is reached with some positive probability, so that the ratio in Equation 2.1 is well-defined for each non-root decision node.

For each information set $\mathcal{I} \in H$, denote by $Z(\mathcal{I}) \subseteq Z$ the set of all terminal successors of $\mathcal{I}$. A node $z \in Z$ belongs to $Z(\mathcal{I})$ if and only if some node $x \in \mathcal{I}$ is among the predecessors of z.

Given an assessment (μ, b), for every terminal node z and every information set $\mathcal{I} \in H$ we can calculate the conditional probability of reaching z given that the information set $\mathcal{I}$ is reached, as follows:

$$P_{\mathcal{I}}(z|\mu, b) = \begin{cases} \mu(p_m(z)) \cdot \prod_{l=0}^{m-1} b(p_l(z)) & \text{if} \quad z \in Z(\mathcal{I}) \\ 0 & \text{otherwise} \end{cases},$$

where $p_m(z) \in \mathcal{I}$ for some $m \in \mathbb{N}$ and p_k is the k^{th} predecessor of z.

Then, we can define the expected payoff of player i starting from the information set $\mathcal{I} \in H$, given the assessment (μ, b) as follows.

$$E_i(\mathcal{I}, b, \mu) = \sum_{z \in Z(\mathcal{I})} U_i(z) P_{\mathcal{I}}(z|\mu, b).$$

Definition 2.16. *An assessment (μ, b) is* ***sequentially rational*** *if for each player i and each $\mathcal{I} \in H_i$,*

$$E_i(\mathcal{I}, b, \mu) \geq E_i(\mathcal{I}, (b'_i, b_{-i}), \mu) \text{ for every } b'_i \in B_i.$$

Definition 2.17. *An assessment (μ, b) is called a* ***sequential equilibrium*** *if it is both consistent and sequentially rational.*

3. Games of the Simple and the Complex Information Types

In this section we extend the binary relation of predecession $\succeq$ defined in the previous section from the set of nodes to the set of information sets.

Definition 3.1. *Let $\mathcal{I}$ and $\mathcal{I}'$ be two information sets of an extensive game G. We say that $\mathcal{I}$* ***immediately precedes*** *$\mathcal{I}'$ if there are nodes $x \in \mathcal{I}$ and $y \in \mathcal{I}'$ such that x is an immediate predecessor of y, and write $\mathcal{I} \succeq_\star \mathcal{I}'$.*

Definition 3.2. *Given two distinct information sets $\mathcal{I}$ and $\mathcal{I}'$ of an extensive game G, $\mathcal{I}$ is said to* ***precede*** *$\mathcal{I}'$ if there is a finite (possibly empty) collection of information sets $\mathcal{I}_1, \cdots, \mathcal{I}_m$ such that $\mathcal{I} \succeq_\star \mathcal{I}_1 \succeq_\star \cdots \succeq_\star \mathcal{I}_m \succeq_\star \mathcal{I}'$, and write $\mathcal{I} \succeq_p \mathcal{I}'$. By convention, $\mathcal{I} \succeq_p \mathcal{I}$ for any $\mathcal{I} \in H$.*

As can be seen, the relation $\succeq_p$ is reflexive and transitive, but is not antisymmetric. The latter implies that in general the relation $\succeq_p$ is not a partial order. For each player i, denote by $\succeq_i$ the restriction of $\succeq_p$ to the information partition H_i. Since we are concerned with games of perfect recall only, [18, Theorem 2] implies that $\succeq_i$ is a partial order on H_i. However, in general the partial orders $\succeq_i$ cannot be extended to a common partial order on the set of all information sets. Games for which such extension is possible will be called games of simple information type.

Definition 3.3. *An extensive form game G with perfect recall is called a game of* ***simple information type*** *if the partial orders $\succeq_i$ on H_i can be extended to a common partial order $\succeq_p$ on H, i.e., there exists a partial order $\succeq_p$ on H such that for any $\mathcal{I}, \mathcal{I}' \in H_i$, $\mathcal{I} \succeq_i \mathcal{I}'$ if and only if $\mathcal{I} \succeq_p \mathcal{I}'$. Otherwise it is a game of* ***complex information type****.*

We shall see that for games of the simple information type a generalization of the backward induction process can be used to find all sequential equilibria. It is straightforward to show that G is of the simple information type if and only if $\succeq$ is antisymmetric.

Theorem 3.4. *An extensive form game G with perfect recall is of the simple information type if and only if the precedence relation $\succeq_p$ is antisymmetric (and thus a partial order).*

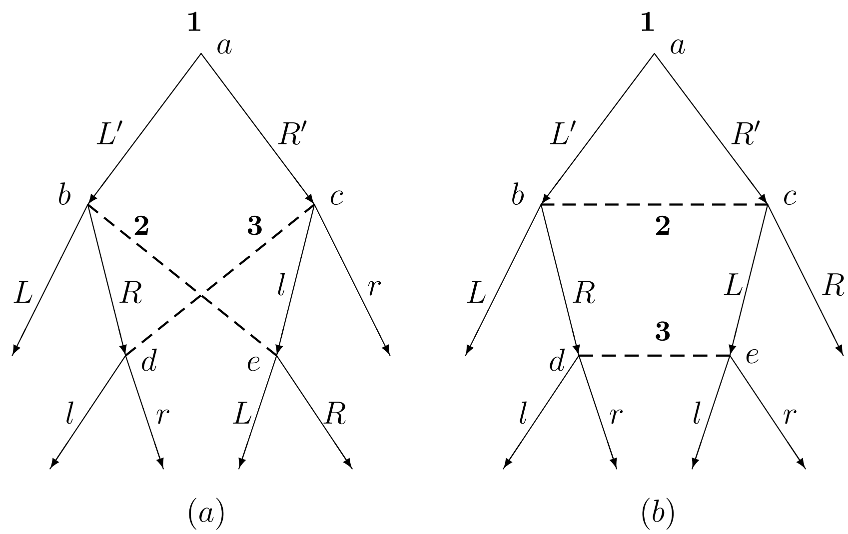

FIGURE 1

As an example, consider two sequential games depicted in Figure 1. The game of Figure 1(a) is a game of the complex information type: Here $\mathcal{I}_1 = \{a\}$, $\mathcal{I}_2 = \{b, e\}$, and $\mathcal{I}_3 = \{c, d\}$ are the information sets of players 1, 2 and 3, respectively. Notice that $\mathcal{I}_2 \succeq_p \mathcal{I}_3$ and $\mathcal{I}_3 \succeq_p \mathcal{I}_2$, however $\mathcal{I}_2 \neq \mathcal{I}_3$. Therefore this game is of the complex information type.

On the other hand, the game of Figure 1(b) is a game of the simple information type, as $\mathcal{I}_1 \succeq_p \mathcal{I}_2 \succeq_p \mathcal{I}_3$ and the relation $\succeq_p$ is a partial order.

We now describe a simple method that allows to test whether a given sequential game is of simple or complex information type. Note that since a game is finite, the number of terminal nodes is finite. Likewise, the number of information sets is finite. Enumerate all nontrivial information sets (i.e., those that contain at least two nodes) by natural numbers, 1 through M. The order of enumeration could be arbitrary. By properties of game trees, for each terminal node $z \in Z$ there exists a unique path from some root node $w \in W$ to z (note that w is determined uniquely given z), $w \succ x_1 \succ \cdots \succ x_m$. Form a tuple t^z as follows:

(1) If x_1 belongs to a nontrivial information set labeled ℓ_1, then assign to the first entry of t^z, t^z_1, a value of ℓ_1, otherwise if x_2 belongs to a nontrivial information set labeled ℓ_2, set $t^z_1 = \ell_2$, and so forth. If none of the nodes $x_1, \cdots, x_m$ belong to a nontrivial information set, let t^z be a singleton equal to 0.
(2) If $t^z \neq 0$, proceed the same way to determine t^z_2, excluding the node corresponding to t^z_2 at the start. If no nontrivial information sets are left, let t^z be the singleton t^z_1.
(3) Repeat the same steps until all nontrivial information sets no the path from w to z are exhausted.

In other words, the entries of t_z are the labels of the nontrivial information sets whose nodes appear on the path from w to z.

Definition 3.5. *Tuples t^z and $t^{z'}$ are called mutually incompatible if $t^z_i = t^{z'}_{i'} = \ell$, $t^z_j = t^{z'}_{j'} = \ell'$, and $(i - j')(j - i') > 0$.*

Intuitively, two tuples are mutually incompatible if the order of appearance of two labels ℓ and ℓ' is reversed. It is now straightforward to obtain the following result.

Theorem 3.6. *A sequential game with perfect recall is of simple information type if and only if there exists no pair of mutually incompatible tuples.*

As an example, consider the game in Figure 1(a). There are six terminal nodes, A through F, and two nontrivial information sets: $\{b, e\}$, labeled by 1, and $\{b, e\}$, labeled by 2. The corresponding tuples are $t_A = 1$, $t_B = t_C = (1, 2)$, $t_D = t_E = (2, 1)$, and $t_F = 2$. Tuples t_B and t_D are incompatible, therefore the game is of complex information type.

4. Sequential Equilibria and the Backward Induction

We start by showing that the backward induction method could be generalized to an arbitrary game of the simple information type, which shows that the condition of the simple information type is a sufficient condition for solvability by the backward induction method. The generalized backward induction method consists of two parts: (1) Constructing a sequentially rational behavior strategy profile given a system of beliefs; (2) Verifying consistency. The second part is performed after the *entire* sequentially rational strategy profile is constructed rather than stage-by-stage along the game tree. These two parts reflect the dual nature of sequential equilibrium, which integrates sequential rationality and consistency, and are presented in the Subsections 4.1 and 4.2.

In Subsection 4.3 we demonstrate that a naive application of the method to games of the complex information type is problematic and establish that the simple information type of the extensive form is also a necessary condition for solvability by the backward induction. We prove that the set of consistent backward induction assessments so defined coincides with the set of sequential equilibria, and provide an alternative proof of the existence of sequential equilibrium.

4.1. **Sequential rationality.** Let G be an extensive form game of the simple information type. Let $\mathcal{F}_0$ be the set of all minimal elements of H with respect to the partial order $\succeq_p$, that is, the set of all information sets $\mathcal{I}$ such that there is no information set $\mathcal{I}'$ with the property $\mathcal{I} \succ_p \mathcal{I}'$.

Lemma 4.1. *The set $\mathcal{F}_0$ of the minimal elements is nonempty.*

Proof. Notice that since G is a finite game, the set H is finite and partially ordered by $\succeq_p$. Also H has finitely many chains, call them $\mathcal{C}_1, \cdots, \mathcal{C}_q$, and every chain has finitely many elements. Hence every chain $\mathcal{C}_i$ has a (unique) minimal element $\mathcal{I}_i$. Therefore $\mathcal{F}_0 = \{\mathcal{I}_1, \cdots, \mathcal{I}_q\} \neq \emptyset$, ■

Define the set of children of $\mathcal{F}_0$ as the set of all children of the information sets in $\mathcal{F}_0$, and denote it by $\mathcal{N}_0$. The following result is crucial for extending the backward induction to games of the simple information type. This result may not hold for games of the complex information type.

Lemma 4.2. *Let G be a sequential game of a simple information type. Then all nodes in $\mathcal{N}_0$ are terminal nodes.*

Proof. Assume G is a sequential game of the simple information type. By Lemma 4.1 we have $\mathcal{F}_0 \neq \emptyset$. Suppose by contradiction there exists an information set $\mathcal{I} \in \mathcal{F}_0$, node $x \in \mathcal{I}$, and a child y of x such that y is a non-terminal node (this implies y is a decision node). Without loss of generality, let $\mathcal{I}'$ be the information set containing y. Notice that $\mathcal{I}$ is different from $\mathcal{I}'$ by the definition of an information set (since we have $x \succ y$).

Also, since $x \succ y$, we have $\mathcal{I} \succeq_p \mathcal{I}'$. We claim that $\mathcal{I} \not\sim_p \mathcal{I}'$. Indeed, if $\mathcal{I} \sim_p \mathcal{I}'$, then the order $\succeq_p$ is not antisymmetric, and hence not a partial order, which is a contradiction. Then we have both $\mathcal{I} \succeq_p \mathcal{I}'$ and

$\mathcal{I} \not\succ_p \mathcal{I}'$, which implies $\mathcal{I} \succ_p \mathcal{I}'$. But then $\mathcal{I}$ is not a minimal element with respect to $\succeq_p$, contradiction. This completes the proof. ■

For example, for the game in Figure 1(b) the set $\mathcal{F}_0$ consists of the information set $\{d, e\}$. Note, however, that for the game in part (a) there is no information set whose all children are terminal nodes.

We are now in a position to formally outline the first part of the *generalized backward induction method*, namely, constructing a sequentially rational strategy profile $b^\star$ for a given system of beliefs $\mu^\star$. The second part, in which we check whether $b^\star$, constructed earlier, is consistent with $\mu^\star$, will be addressed in Subsection 4.2. The *first part* is decomposed into the following steps.

Step 1. Fix a system of beliefs $\mu^\star \in M$.

Step 2. Denote by $\mu^\star_{\mathcal{F}_0}$ the restriction of $\mu^\star$ to the set $\mathcal{F}_0$. Given $\mu^\star_{\mathcal{F}_0}$, pick a best response for each player who is decisive at $\mathcal{F}_0$, and denote this best response profile $b^*_{\mathcal{F}_0}$. Note that the set of such best responses, or sequentially rational mixed strategies at the information sets in $\mathcal{F}_0$ (denote it $\Lambda_{\mathcal{F}_0}(\mu)$) is a nonempty, convex, and compact subset of $B_{\mathcal{F}_0}$.

Step 3. Fix $b^*_{\mathcal{F}_0} \in \Lambda_{\mathcal{F}_0}(\mu)$. Truncate the game tree Γ of the sequential game G by deleting all nodes in $\mathcal{N}_0$ and assigning to each node in $\mathcal{F}_0$ the expected payoff vector generated by $b^*_{\mathcal{F}_0}$. [3] This step generates a new sequential game, call it G_1, with the corresponding extensive form Γ_1. It can be easily verified that $\succeq_p$ restricted to Γ_1 is a partial order.

Step 4. Denote by $\mathcal{F}_1$ the set of minimal elements among the information sets of G_1 with respect to the partial order $\succeq_p$ restricted to Γ_1. By the earlier argument, the set $\mathcal{F}_1$ is nonempty.

Steps 5, 6 and 7. Repeat Steps 2, 3 and 4, applied to the game G_1 in place of G.

Since $\succeq_p$ is a partial order, the roll-back procedure will stop at the root information set $\mathcal{I}_W$. This is due to the fact that $\mathcal{I}_W$ is a unique maximal element of H with respect to $\succeq_p$ for the game G, which is proven in the Appendix (Lemma 5.1). Thus, for a given system of beliefs $\mu^\star$, we can recursively construct a finite sequence of games $G_1, \cdots, G_r$ and a strategy profile b^* using this generalized backward induction process.

Note that although this process should be repeated for all possible belief systems, the roll-back procedure outlined above needs to be performed only *finitely* many times after one appropriately subdivides the belief space into finitely many components, exploiting the fact that

[3]We are using here Lemma 4.2, which guarantees that all nodes in $\mathcal{N}_0$ are terminal nodes.

beliefs are piecewise-constant (see the discussion on p. 18 following Example 4.12 for more details).

Definition 4.3. *Let* Γ *be an extensive form game of the simple information type. Given a system of beliefs* $\mu^\star$*, a strategy profile* b^* *generated by the generalized backward induction process is called a* ***backward induction strategy profile****.*

4.2. **Consistency.** In this subsection we outline the *second part* of the generalized backward induction method, where we check $\mu^\star$ and $b^\star$, constructed in Subsection 4.1, for consistency. The consistency check is performed after the *entire* sequentially rational behavior strategy is constructed, i.e., at the end of Part 1 of the backward induction method, rather than stage-by-stage along the game tree. If the assessment passes the check, then it is called a consistent backward induction assessment, otherwise it gets discarded.

We use the following result due to Kohlberg and Reny [9, Proposition 6.1, p. 307].

Proposition 4.1. *For any finite extensive-form game there is a finite system of polynomial equations in the beliefs,* μ*, and the behavior strategies,* b*, such that* $(\mu^\star, b^\star)$ *is consistent if and only if it satisfies these equations. Moreover, the finite algorithm constructed in the proof provides this system of equations.*

The proof of this proposition can be found in [9, Proposition 6.1, pp. 307–308] and is omitted for brevity. Below we outline the algorithm presented in the proof, which describes a finite and fairly efficient procedure that allows to check for consistency by plugging $\mu^\star$ and $b^\star$ into the system of polynomial equations. Essentially the system consists of Bayes' rule equations and certain products of powers of those equations with cancellation of the variables that appear on both sides. In the subsequent analysis we closely follow [9].

Recall that a finite system of equations $R_j(x_1, \cdots, x_\ell) = r_j$, $j = 1, \cdots, k$ in the variables $x_1, \cdots, x_\ell$ has a **positive approximate solution** if there exists a sequence $\{x^n\}$ of strictly positive vectors such that

$$\lim_{n\to\infty} R_j(x_1^n, \cdots, x_\ell^n) = r_j, \;\; j = 1, \cdots, k.$$

Note that an assessment (μ, b) is consistent if and only if the following system of equations in the variables $\{x_c\}_{c\in C}$ has a positive approximate solution:

$$\frac{\prod_{c\in I_k} \; x_c}{\prod_{c\in I_j} \; x_c} = \frac{\mu_k}{\mu_j}, \;\; x_c = b_c, \tag{4.1}$$

where I_k (I_m) is the set of choices c on the path to node k (j), respectively. The following lemma provides the necessary and sufficient condition for consistency and appears in [9, Corollary 4.3, p. 302].

Lemma 4.4. *The system in 4.1 has a positive approximate solution if and only if the following holds: whenever any number of the left-hand sides of 4.1 or their reciprocals are multiplied together so that all of the variables cancel, then the same operation on the corresponding right-hand sides of 4.1, if well-defined, equals 1.*

Since the game is finite, there are finitely many, say M, pairs of nodes $\{k, j\}$ that belong to the same information set. For each $m = 1, \cdots, M$, let $\alpha_m = \mu_k \prod_{c\in I_j} x_c$, $\beta_m = \mu_j \prod_{c\in I_k} x_c$, $I'_m = I_k$, $J'_m = I_j$, then the first part of (4.1) becomes

$$1 = \frac{\alpha_m}{\beta_m}. \tag{4.2}$$

We now describe how to construct products of powers of (4.2) to obtain the desired system of polynomial equations. Taking logs in (4.1) produces a linear system of equations $Ax = r$, where $r_m = \mu_k - \mu_j$ for each m. Given vector r, the set of y such that $yA = 0$ and $\sum_{y_m \neq 0} y_m r_m$ is well-defined is the union of two cones:

$$K = \{y : \ yA = 0 \text{ and } y_m \geq 0 \text{ if } r_m = \infty, p_m \leq 0 \text{ if } r_m = -\infty\}, \tag{4.3}$$

and $-K$. Since M is finite and A is fixed, there are finitely many such cones, each of which is generated by finitely many extreme vectors q. Denote the collection of all such extreme vectors by Q. Note that since the matrix A has integer coefficients (either 1 or -1), each q can be chosen to have integer coefficients. Observe that (4.2) has a positive approximate solution if and only if for every such $q \in Q$ we have

$$\prod_{p_m>0} \alpha_m^{p_m} \prod_{p_m<0} \beta_m^{|p_m|} = \prod_{p_m>0} \beta_m^{p_m} \prod_{p_m<0} \alpha_m^{|p_m|} \tag{4.4}$$

Note that the consistency check does not rely on the actual solving of the system. In order to construct the system one needs to compute extreme vectors of cones K and $-K$, which can be done using one of the known algorithms (see for instance [12]), so consistency is readily verifiable.

Definition 4.5. *Let Γ be an extensive form game of the simple information type and b^* be a strategy profile generated by the roll-back procedure outlined in Subsection 4.1 given a system of beliefs $\mu^\star$. If $\mu^\star$ is consistent with $b^\star$, we call the assessment (μ, b^*) a **consistent backward induction assessment**.*

4.3. **Characterization of sequential equilibrium and its existence.** Having described the generalized backward induction method, we now go on to show that this method will give us the set of sequential equilibria of a game if and only if it is a game of the simple information type.

It is evident that if we apply the generalized backward induction method to a game of the complex information type, we will find a truncated game G_k for some $k \leq L$, where L is the length of the tree, such that the set $\mathcal{F}_{k-1}$ of the minimal elements with respect to $\succeq_p$ is empty since the relation $\succeq_p$ is not a partial order. Therefore we have the following result, which we state without a proof.

Lemma 4.6. *Let G be an extensive form game of the complex information type. Then the generalized backward induction procedure described in Steps 1 through 5 will yield an empty set of nodes $\mathcal{F}_{k-1}$ for some $k \leq L$.*

Let us introduce the following definition.

Definition 4.7. *An extensive form game G is said to be **backward induction solvable** if the backward induction procedure yields a nonempty set of nodes $\mathcal{F}_{k-1}$ for all $k \leq L$.*

The following result, which is an immediate consequence of Theorems 4.6, 4.9, and Corollary 4.11, is of some significance as it shows that the simple information type condition exactly characterizes the class of games that can be solved using the backward induction method.

Theorem 4.8. *(***Simple Information Type is Both Necessary and Sufficient***) An extensive form game G is backward induction solvable if and only if G is of the simple information type.*

The next set of results show that the consistent backward induction assessments are precisely the sequential equilibria of a sequential game G if the game is of the simple information type, and that this set is nonempty. This shows that every sequential equilibrium of a game of the simple information type can be found by the generalized backward induction. The result (Theorem 4.9) can be established using the mathematical induction.

Theorem 4.9. *(***Characterization of Sequential Equilibrium***) Let G be an extensive form game of the simple information type. Then an assessment (μ^*, b^*) is a consistent backward induction assessment if and only if it is a sequential equilibrium of G.*

Finally, we need to show that the set of consistent backward induction assessments is nonempty. [4] Define the correspondence $\tau : M \times B \twoheadrightarrow M \times B$ as follows. Given $(\mu, b) \in M \times B$, let

$$\tau(\mu, b) = \tilde{M} \times \tilde{B},$$

where $\tilde{M}$ is the set of all belief systems consistent with b, and $\tilde{B} = \tilde{B}_{\mathcal{F}_r} \times \cdots \times \tilde{B}_{\mathcal{F}_0}$, such that for each $j = 0, \cdots, r$, $\tilde{B}_{\mathcal{F}_j}$ is the set of the best responses at $\mathcal{F}_j$ given ρ (the probability distribution on the root information set $\mathcal{I}_W$), $\mu_{\mathcal{F}_j}$, and the truncation according to $(b_{\mathcal{F}_{j-1}}, \cdots, b_{\mathcal{F}_0})$.

It follows immediately from the definition of a consistent backward induction assessment that $(\mu, b) \in M \times B$ is a consistent backward induction assessment if and only if (μ, b) is a fixed point of τ.

It remains to show that the correspondence τ has a fixed point. Note that τ may be ill-behaved on the boundary, ∂B. If $b \in \partial B$, then the set of beliefs consistent with b is not necessarily convex. However, τ is nicely behaved in the interior of $M \times B$. We use the generalization of Kakutani's fixed point theorem that was proven in [20, Theorem 3.2], which is presented below for completeness.

Theorem 4.10. ***(Generalization of Kakutani's Theorem)*** *Let X be a nonempty, convex and compact subset of $\mathbb{R}^n$, and $\varphi : X \twoheadrightarrow X$ be a closed correspondence such that the set $\varphi(x)$ is convex and nonempty for each $x \in ri(X)$.[5] Then φ has a fixed point in X.*

Lemmata 5.2 and 5.3 presented in the Appendix imply that τ satisfies the hypotheses of 4.10. Also, $M \times B$ satisfies the hypotheses of 4.10, since M and B are nonempty, convex, and compact subsets of some Euclidean space. Therefore, we get the following existence result.

Corollary 4.11. *The correspondence τ has a fixed point in $M \times B$ and, consequently, the set of consistent backward induction assessments for an extensive game of the simple information type is nonempty.*

We now illustrate the backward induction method with an example.

Example 4.12. As an example consider the game depicted in Figure 2. Since there is only one nontrivial information set containing two nodes, consistency requires $b(R)b(L')\mu(C) = b(L)\mu(D)$. Partition the belief space of player 3 at the information set $\{C, D\}$ into the following three components; on each of the components we will construct a

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[4]The reader would correctly observe that this follows from Theorem 4.9 and the existence result by Kreps and Wilson [9, Proposition 1], however we present an alternative proof for completeness.

[5]Recall that $ri(X)$, the algebraic relative interior of a convex subset X of $\mathbb{R}^k$, is the interior of X which results when X is regarded as a subset of its affine hull.

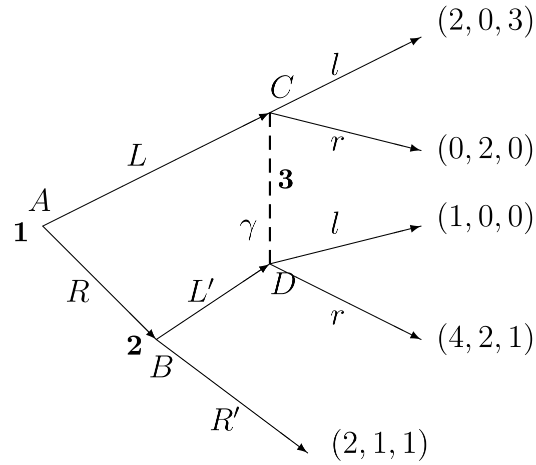

FIGURE 2

sequentially rational strategy profile, and then check it for consistency. Note that since there is only one nontrivial information set consisting of two nodes, a consistent assessment has to satisfy only one equation, $b_1(L)\mu(D) = b_1(R)b_1(L')\mu(C)$.

(1) $\mu(C) > \frac{1}{4}$, in which case the best response of player 3 is a singleton, $b_3^\star = \{l\}$. Truncate the game tree, assigning the vector $(2, 0, 3)$ to node C, and the vector $(1, 0, 0)$ to node D. Then, player 2's best response is $b_2^\star = \{R'\}$. After the final truncation, which assigns the vector $(2, 1, 1)$ to node B, it is evident that player 1 is indifferent between her pure strategies L and R. Thus $b_1^\star = \Delta(L, R)$, where $\Delta(L, R)$ denotes the set of all probability distributions on $\{L, R\}$. Note that the consistency of μ holds for any $0 \leq b_1^\star(L) \leq 1$.

(2) $\mu(C) = \frac{1}{4}$, which implies $b_3^\star = \Delta(l, r)$. Consider the following subcases.

a) $b_3^\star(r) > \frac{1}{2}$; it is easy to verify that the best responses of player 2 and player 1 are $b_2^\star = \{L'\}$ and $b_1^\star = \{R\}$, respectively. However, the constructed strategy profile $b^\star$ is inconsistent with $\mu(C) = \frac{1}{4}$, because $b^\star$ requires $\mu(D) = 1$.

b) $b_3^\star(r) = \frac{1}{2}$, which implies $b_2^\star = \Delta(L', R')$ and $b_1^\star = \{R\}$. However, only $b_2^\star = \{R'\}$ is consistent with the beliefs $\mu(C) > \frac{1}{4}$.

c) $b_3^\star(r) < \frac{1}{2}$, and similarly $b_2^\star = \{R'\}$ and $b_1^\star = \{R\}$.

(3) $\mu(C) < \frac{1}{4}$, which induces $b_3^\star = \{r\}$. It is easy to verify that $b_2^\star = \{L'\}$ and $b_1^\star = \{R\}$. Consistent beliefs of player 3 must satisfy $\mu(D) = 1$, which is consistent with $\mu(C) < \frac{1}{4}$.

Consistency check narrows down the set of sequentially rational strategy profiles, ruling out those which are not consistent with the corresponding belief system. Here, the assessments $\left\{(R, L', \Delta(l,r), \mu(C) = \frac{1}{4}\right\}$ and $\left\{(R, R', \frac{1}{2}\ell + \frac{1}{2}r, \frac{1}{4} < \mu(C) \leq 1)\right\}$ are sequentially rational but not consistent. To summarize, all consistent backward induction assessments (and therefore, sequential equilibria) for this game are as follows.

(1) $\{(\alpha L + (1-\alpha)R, R', l, \mu(C) = 1) : 0 \leq \alpha \leq 1\}$
(2) $\left\{(R, R', l, \mu(C)) : \frac{1}{4} \leq \mu(C) \leq 1\right\}$
(3) $\{(R, L', r, \mu(C) = 0)\}$
(4) $\left\{(R, R', \beta l + (1-\beta)r, \mu(C) = \frac{1}{4}) : \frac{1}{2} \leq \beta \leq 1\right\}$

□

For games of the simple information type the information sets play the same role as the decision nodes in the standard backward induction method that works for sequential games of perfect information. To solve the game of Figure 2 one starts by calculating the best responses of player 3 at her information set $\{C, D\}$, then work back to the best responses of player 2, and finally, find the best responses of player 1. This example suggests that for games of the simple information type the proposed method can be used to find not just one of several sequential equilibria, but the entire set of sequential equilibria (Theorem 4.9).

It is important to emphasize that for a given $\mu \in M$, the generalized backward induction process may yield infinitely, in fact uncountably many behavior strategy profiles. This is because the set of best responses at any step may be uncountable. Also, in order to find all consistent backward induction assessments, one needs to perform the generalized backward induction for every system of beliefs $\mu \in M$.

However, the generalization of the backward induction becomes tractable after we subdivide the belief space and the strategy space into finitely many components as follows. First, note that at each step there is a *finite* partition of the belief space, such that the best responses are constant for all beliefs in each element of the partition. For instance, in the game of Figure 2 the belief space at information set γ could be partitioned into three components: $\left\{\mu(C) > \frac{1}{4}\right\}$, $\left\{\mu(C) = \frac{1}{4}\right\}$, and $\left\{\mu(C) < \frac{1}{4}\right\}$. This is of course due to the fact that the game is finite. Thus, one needs to check only finitely many cases on each step. Second, if the best response at some information set is not a singleton and consists of a continuum of choices, one can break the corresponding behavior strategy space into finitely many components, each producing a unique best response at the preceding information set. In the game of Figure 2 such components for the case $\mu(C) = \frac{1}{4}$ are: 1) $\{b_3^\star(r) > \frac{1}{2}\}$,

2) $\{b_3^\star(r) = \frac{1}{2}\}$, and 3) $\{b_3^\star(r) < \frac{1}{2}\}$.[6] Thus, one needs to check only *finitely* many cases to find *all* sequential equilibria.

Now we introduce a string of definitions which would help illustrate that it may be impossible to exploit sequential structure of the game to find sequential equilibria and that the only way to achieve the former task is to use the agent-normal form.

Definition 4.13. *Two distinct information sets $\mathcal{I}_1$ and $\mathcal{I}_2$ are called **simultaneous** if both $\mathcal{I}_1 \succeq_\star \mathcal{I}_2$ and $\mathcal{I}_2 \succeq_\star \mathcal{I}_1$.*

Define the equivalence relation $\sim_\star$ on the set of information sets of the extensive form game as follows.

$$\mathcal{I}_1 \sim_\star \mathcal{I}_2 \text{ if } \mathcal{I}_1 \succeq_\star \mathcal{I}_2 \text{ and } \mathcal{I}_2 \succeq_\star \mathcal{I}_1.$$

Observe that the relation $\succeq_\star$ modulo $\sim_\star$ is a partial order on the set of information sets of a sequential game.

Definition 4.14. *An equivalence class of the relation $\sim_\star$ is called a **knot**.*

Let $\mathbb{K}$ denote the set of all knots for the extensive form game G. Notice that since G is a finite game, the set $\mathbb{K}$ is finite, without loss of generality $\mathbb{K} = (K_1, K_2, \cdots, K_m)$. Define the binary relation $\succeq_e$ on $\mathbb{K}$ as follows.

$$K_1 \succeq_e K_2 \text{ if } \mathcal{I} \succeq \mathcal{I}'$$

for some information sets $\mathcal{I} \in K_1$ and $\mathcal{I}' \in K_2$. Then, the notation $K_1 \succeq_e K_2$ tells us that knot K_1 **precedes** knot K_2. It is easy to show that $\succeq_e$ is a partial order on $\mathbb{K}$.

The backward induction method fails for games of the complex information type since some information sets interlock and it is not possible to define a clear chronological order on them. Note, however, that since the relation $\succeq_e$ is a partial order on the set of all knots $\mathbb{K}$, the knots are positioned in a clear chronological order. Let $\mathcal{J}_0$ be the set of all minimal elements of $\mathbb{K}$ with respect to the partial order $\succeq_e$. Follow the lines of Lemma 4.1 to conclude that the set $\mathcal{J}_0$ is nonempty. One might be tempted to apply the backward induction method to knots instead of information sets. However, if some knot K is nontrivial, there is no clear-cut procedure that produces the best responses at the information sets in K since those best responses are mutually related. The task of finding those best responses is reminiscent to the task of finding the set of Nash equilibria in the truncated subgame. At the same time, the former is not equivalent to the latter since K may include

[6] In general the subdivisions of the belief space have to be derived as the solution spaces of systems of multivariate polynomial inequalities.

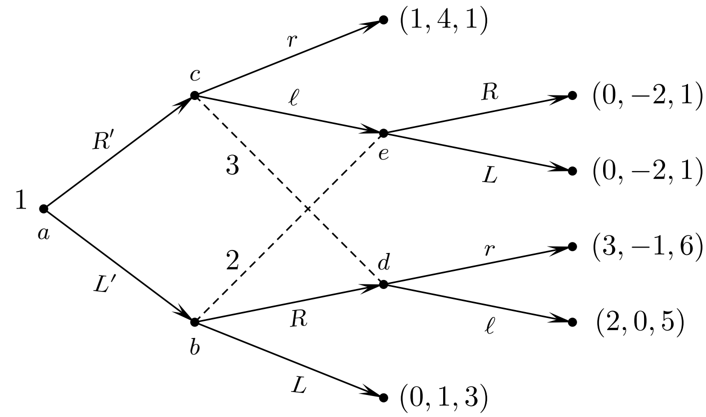

FIGURE 3

several information sets that belong to the same player (though such information sets are positioned in a clear chronological order among themselves due to perfect recall, see [18]).

Example 4.15. Consider a game of the complex information type depicted in Figure 3. It is easy to see that the set $\mathcal{F}_0$ is empty, therefore we cannot apply the backward induction method here. Note that since each player moves only once in this game, sequential equilibria could be found by solving for equilibrium in the agent normal form and then checking that consistency holds. Using this approach one can check that the following are sequential equilibria:

$$b(L') = 0, b(L) = 1, b(l) = 0, 0 \leq \mu(b) \leq 1, \mu(c) = 1,$$
$$b(L') = 1, b(L) = 1, \frac{1}{2} \leq b(l) \leq 1, \mu(b) = 1, 0 \leq \mu(c) \leq 1.$$

5. Conclusion

In this paper we show that one must exercise care while attempting to generalize the well-known backward induction method to games of imperfect information. Although such generalization may seem obvious in some cases, this is only possible if certain conditions are met. We show that the simple information structure condition is both necessary and sufficient for solvability by the backward induction. For games of the simple information type we formulate an appropriate generalization of the backward induction method. Though the generalized backward induction process involves checking uncountably many belief vectors, the problem becomes tractable if one exploits the fact that at each step there is a *finite* partition of the belief space, such that the best responses are the same for all beliefs in each element of the partition. Hence, one needs to check only finitely many cases on each step. Further, in case the best response at some information set is not a singleton and consists of a continuum of choices, one can break the set of best responses into finitely many components, each producing a unique best response at the preceding information set.

While we have presented the results for sequential equilibria, these results extend to the closely related concept of perfect Bayesian equilibrium, see [7] for a full discussion of the concept. As perfect Bayesian equilibria are explored in many applications, see for example [2], [6] and [13], it is important to have a well understood method of finding these equilibria. As is well-known, in a perfect Bayesian equilibrium beliefs off equilibrium path could be arbitrary, so the second part of the backward induction method would involve checking Bayesian update along the equilibrium path only, i.e., only at those information sets that are reached with a positive probability. However, consistency not only requires Bayes' rule to be satisfied along the equilibrium path, but imposes subtle restrictions on beliefs and strategies *off* equilibrium path. This easily establishes the well-known fact that every sequential equilibrium is perfect Bayesian. The results presented here are, therefore, useful in analyzing not only sequential equilibria, but also perfect Bayesian equilibria.

Our results indicate that there is a justification for the intuition that a “roll-back procedure” should provide an efficient way of solving sequential games. Note that the roll-back procedure alone does not guarantee a sequential equilibrium; it is also necessary to check for consistency. An especially interesting aspect of the results is showing that the fundamental intuition is valid if and only if the information structure of the game satisfies certain patterns. The results identify these patterns and thus fully describe the class of sequential games for which the backward induction method could be applied. The results also indicate that for the class of games for which the method could be applied, it works very well indeed, as it identifies the entire set of sequential equilibria. Our results may have implications beyond the class of finite games analyzed here, and it is possible that some of the results could be extended to a more general class of sequential games, for instance, games with infinite number of information sets and/or stages. We know that the backward induction process works well in general cases for most decision problems, and therefore one would expect that the method could provide insights for a more general class of sequential games.

Appendix

Lemma 5.1. *Let G be an extensive form game of the simple information type. Then $\mathcal{I}_W$, the root information set, is the unique maximal element of H with respect to the partial order $\succeq_p$.*

Proof. First, let us show that $\mathcal{I}_W$ is a maximal element. Suppose by contradiction there exists an information set $\mathcal{I} \in H$ such that $\mathcal{I} \succ_p \mathcal{I}_W$. Hence there exist nodes $x \in \mathcal{I}$ and $w \in \mathcal{I}_W$ such that $x \succ w$, that is, x is a predecessor of w. But this is impossible since $w \in \mathcal{I}_W$ is a root.

Next, let us show that there is no maximal element other than $\mathcal{I}_W$. Suppose by contradiction there exists an information set $\mathcal{I}'$ different from $\mathcal{I}_W$, such that $\mathcal{I}'$ is a maximal element of H with respect to $\succeq_p$. Fix a node $y \in \mathcal{I}'$. By assumption y is a non-root node, hence there exists a root node $w \in \mathcal{I}_W$ such that there is a unique path from w to y. But this implies $w \succ y$, hence $\mathcal{I}_W \succ_p \mathcal{I}'$. This contradicts our assumption that $\mathcal{I}'$ is a maximal element. Therefore $\mathcal{I}_W$ is the unique maximal element. ■

Proof of Theorem 4.9. ($\Rightarrow$) Assume (μ^*, b^*) is a consistent backward induction assessment. Then the system of beliefs μ^* is consistent with b^* by definition. It remains to show that (μ^*, b^*) is sequentially rational. This can be established using the method of mathematical induction. Without loss of generality, suppose the backward induction process stops at $\mathcal{F}_r$ for some $r \in \mathbb{N}$, i.e., the root information set I_W is in $\mathcal{F}_r$.

Fix a natural number $0 \leq k < r$. By the induction hypothesis, assume (μ^*, b^*) is sequentially rational starting from any information set in $\mathcal{F}_j$, for all $0 \leq j \leq k$. We need to show that (μ^*, b^*) is sequentially rational starting from any information set in $\mathcal{F}_{k+1}$. Let $u \in \mathcal{F}_{k+1}$ be an information set owned by some player i.

If player i is not decisive at any node y such that $x \succeq y$ for some $x \in \mathcal{I}$, we are done, since $b^*_{i\mathcal{I}}$ is a best response of player i at $\mathcal{I}$ on the $(k+2)^{th}$ step of the generalized backward induction process.

Assume player i moves at some node y such that $x \succeq y$ for some $x \in \mathcal{I}$. Suppose by contradiction player i wants to deviate, starting from $\mathcal{I}$, at a nonempty collection V_i of information sets. Consider two possible cases:

(1) $V_i \cap \mathcal{F}_{k+1} = \emptyset$. Notice that the set V_i is finite and partially ordered by $\succeq_p$. Fix a chain $\mathcal{C}$ in V_i. Since it is finite and totally ordered, it has a unique maximal element, call it $\mathcal{I}'$. But this implies that player i wants to deviate at $\mathcal{I}'$, which contradicts our induction hypothesis, since $\mathcal{I}' \in \mathcal{F}_j$ for some $0 \leq j \leq k$.

(2) $V_i \cap \mathcal{F}_{k+1} \neq \emptyset$, which implies $V_i \cap \mathcal{F}_{k+1} = \mathcal{I}$. Then we have the following.

(a) If $V_i \backslash \mathcal{I} = \emptyset$, we arrive at a contradiction, since $b^*_{i\mathcal{I}}$ is a best response of player i at $\mathcal{I}$ on the $(k+2)^{th}$ step of the generalized backward induction process.

(b) Suppose $V_i \backslash \mathcal{I} \neq \emptyset$. Without loss of generality let $\tilde{b} \in B$ be a behavior strategy profile such that $\tilde{b}_m = b_j$ for all $m \neq i$, and $E_i(\mathcal{I}, \tilde{b}, \mu) > E_i(\mathcal{I}, b, \mu)$. The perfect recall assumption together with Lemma **??** implies that for every choice $s_j \in C_{\mathcal{I}}$ of player i at $\mathcal{I}$ we have

$$E_i(\mathcal{I}, (s_j, \tilde{b} \backslash s_j), \mu) > E_i(\mathcal{I}, (s_j, b \backslash s_j), \mu).$$

But then, deviating to $\tilde{b}_i$ from b_i by player i is equivalent to deviating at the information set $\mathcal{I}$ only. As before, this is a contradiction, since $b^*_{i\mathcal{I}}$ is a best response of player i at $\mathcal{I}$ on the $(k+2)^{th}$ step of the generalized backward induction process.

This shows (μ^*, b^*) is sequentially rational starting from any information set in $\mathcal{F}_{k+1}$. Hence by the mathematical induction (μ^*, b^*) is sequentially rational.

($\Leftarrow$) Let (μ^*, b^*) be a sequential equilibrium. We want to show that it is a consistent backward induction assessment. Since (μ^*, b^*) is a sequential equilibrium, $b^*_{\mathcal{F}_0}$ is optimal starting from any information set in $\mathcal{F}_0$ given $\mu^*_{\mathcal{F}_0}$. Therefore $b^*_{\mathcal{F}_0}$ is selected in the first step of the generalized backward induction process.

Assume by the induction hypothesis that $b^*_{\mathcal{F}_k}$ is selected on the $(k+1)^{th}$ step of the backward induction process, after truncating the extensive form according to $(b^*_{\mathcal{F}_l})_{l=0,\cdots,k-1}$. Suppose $b^*_{\mathcal{F}_{k+1}}$ is not optimal starting from some information set $\mathcal{I} \in \mathcal{F}_{k+1}$, after truncating the extensive form according to $(b^*_{\mathcal{F}_l})_{l=0,\cdots,k}$. But this implies the player who is decisive at u has an incentive to deviate from b^*, starting at $\mathcal{I}$, hence (μ^*, b^*) is not sequentially rational, which is a contradiction. This completes the proof. ■

Lemma 5.2. *The correspondence τ is convex-valued in the interior of $M \times B$.*

Proof. Fix $(\mu, b) \in M^\circ \times B^\circ$, and without loss of generality let $\tau(\mu, b) = \tilde{M} \times \tilde{B}$. Then $\tilde{M}$ is a singleton, hence it is a convex subset of M.

For each $j = 0, \cdots, r$, $\tilde{B}_{\mathcal{F}_j}$ is the set of best replies at $\mathcal{F}_j$ given ρ, $\mu_{\mathcal{F}_j}$, and the truncation according to $(b_{\mathcal{F}_{j-1}}, \cdots, b_{\mathcal{F}_0})$. Therefore

$\tilde{B}_{\mathcal{F}_j}$ is nonempty and convex for each $j = 0, \cdots, r$. This implies $\tilde{B} = \tilde{B}_{\mathcal{F}_r} \times \cdots \times \tilde{B}_{\mathcal{F}_0}$ is a convex subset of B. Consequently, $\tau(\mu, b) = \tilde{M} \times \tilde{B}$ is convex. ■

Lemma 5.3. *The correspondence τ is nonempty-valued and closed.*

Proof. (i) First let us show that τ is closed. Fix a sequence $(\mu^k, b^k) \subseteq M \times B$ such that $(\mu^k, b^k) \to (\mu^*, b^*) \in M \times B$ as $k \to \infty$. Let $(\tilde{\mu}^k, \tilde{b}^k) \in \tau(\mu^k, b^k)$ for each $k \in \mathbb{N}$ such that $(\tilde{\mu}^k, \tilde{b}^k) \to (\tilde{\mu}^*, \tilde{b}^*)$. We need to show that $(\tilde{\mu}^*, \tilde{b}^*) \in \tau(\mu^*, b^*)$.

Without loss of generality let $\tau(\mu^k, b^k) = M^k \times B^k$ for each $k \in \mathbb{N}$ and $\tau(\mu^*, b^*) = M^* \times B^*$. By the closedness of the belief correspondence ϕ, we have $\tilde{\mu}^* \in M^*$.

Fix $j \in \{0, \cdots, r\}$, and notice that the payoff of the player decisive at an information set $u \in \mathcal{F}_j$, given ρ, $\mu_{\mathcal{F}_j}$ and the truncation according to $(b_{\mathcal{F}_{j-1}}, \cdots, b_{\mathcal{F}_0})$, is jointly continuous in μ and b. Therefore $b^* \in B^*$. This shows that $(\tilde{\mu}^*, \tilde{b}^*) \in \tau(\mu^*, b^*)$, hence τ has a closed graph.

(ii) By the closedness of τ and compactness of $M \times B$ it suffices to show that τ is nonempty-valued in the interior of $M \times B$. Fix $(\mu, b) \in M^\circ \times B^\circ$, and follow the lines of the proof of Lemma 5.2 to establish that $\tau(\mu, b) \neq \emptyset$. ■

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