

Informational Advantage in US Treasury Auctions

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Abstract: We extend Wilson (1979) share auction framework to model the uniform-price US Treasury auction as a two-stage multiple leader-follower game. We then explicitly represent the primary dealer's (follower) strategic choice of bids as a function of its customer's (leader) bids and show that an increase in a customer's bid leads to two types of its dealer's reaction at the dealer bid-points that are near, in terms of price, to the customer's bid-point – the *quantity effect* by which the primary dealer increases its quantity, and the *price effect* by which the primary dealer decreases its bid shading. We explain how these two effects are translated into the primary dealer bidding behavior in handling the risk of being short-squeezed or face the winner curse in the post-auction market. We also find that comparing to the direct bidding system, where all bidders submit their bids directly to Treasury, the primary dealer bidding system increases the competition, which leads to an increase in both Treasury revenue and revenue's volatility. Relatively to existing studies, this paper first extends the left continuous step demand schedule in Kastl (2011) to explain how primary dealers move their bid-points around customers'. Second, it complements Hortacsu and Kastl (2012) by explicitly representing the primary dealer's strategic choice of bids as a function of its customer's bids, and explaining how the primary dealer's informational advantage impact its bidding behavior in handling the risk of being short-squeezed or face the winner curse in the post-auction market. Third, it provides valuable insights into the detailed bid-level data study of US Department of the Treasury (2012) and explains why one of its findings differs from a result in Hortacsu et al. (2015).

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1. Introduction

To finance its public debt, the US Government currently auctions several Treasury securities including bills, notes, bonds, Treasury Inflation Protected Securities, and Floating Rate Notes. The Treasury auction market is considered one of the biggest and most active financial markets in the world. In 2012 alone, Treasury issued \$7,658 billion through 264 public auctions. The US Treasury auctions these securities on a regular and predictable basis. The primary dealers are the largest group of buyers at these auctions. However, there are several accounts of primary dealers using the information extract from their customers' bids to formulate their own bids. The following are two such accounts.

- (i) First, Harper and Kruger (2013) report the following from Steve Rodosky, who runs the Treasury and derivatives trading at PIMCO:¹ "PIMCO likes to bid directly instead of bidding indirectly, because of the anonymity the direct bidding offers and because it has reduced the price swings that used to occur before auctions when it bids indirectly as primary dealers react to the bids they received."
- (ii) Second, Flitter (2012) reports that China² can now bypass primary dealers when buying US Treasury securities and go straight to the auction. Then he draws the following conclusion: "Since primary dealers are not allowed to charge customers money to bid on their behalf at Treasury auctions, China is not saving money by cutting out commission fees; instead, China is preserving the value of specific information about its bidding habits. By bidding directly, China prevents primary dealers from trying to exploit its huge presence in a given auction by driving up the price."

Primary dealer bidding behavior after observing their customers' bids in US Treasury auctions is an important subject for both academics and policy makers. For academics, it would be important to understand how primary dealers formulate their in-house bids after observing their customers' bids, by identifying how each type of information extracted from customers' bids affect the price or the quantity in the primary dealers' bids. For policy makers, it would be important to analyze the impact of dealers' informational advantage on the auction revenue and revenue volatility under the primary dealer bidding system in comparison to other bidding systems such as the direct bidding system in which all bidders submit their bids directly to Treasury without intermediaries.

Several studies have tried to address the above subject. The first one is the empirical study conducted by US Department of the Treasury (2012) that analyzes detailed bidding data from auctions of Treasury notes and bonds conducted between June 2009 and September 2012. For each auction, it partitions the bid levels into three zones: (i) the *aggressive zone*, i.e. bids that reflect the bidders' real demand of Treasury securities, and that bidders expect to be fully

¹ PIMCO is the world's largest active bond Fund.

² China is the second US Treasury holder after the Fed.

awarded; (ii) the *value zone*, i.e. bids that are placed by price-sensitive bidders and are not expected to be awarded completely unless the auction clearing yield tails above this zone; and (iii) the *throw-away zone*, i.e. bids that are not expected to be awarded unless the auction clearing yield tails dramatically above the value zone. For each primary dealer, it calls each bid in the throw-away zone as bid-to-miss, i.e. bids placed only to fulfill the pro-rate bidding requirement. It draws the two following conclusions. First, the primary dealers construct their value bids based on their clients' value bids. Second, after taking out throw-away bids, primary dealers submit more aggressive bids than both indirect bidders and direct bidders.

Hortacsu et al. (2015) also analyze detailed bidding data from auctions of US Treasury bills and notes conducted between July 2009 and October 2013, and find that primary dealers consistently bid higher yields (i.e. lower prices) in auctions compared to direct and indirect bidders. Their result contradicts the one of US Department of the Treasury (2012). This contradiction is due to the fact that Hortacsu et al. (2015) do not take into account the fact that some primary dealers are not very active in all maturity segments of the Treasury security market. Therefore, when participating in an auction for a Treasury security on which it is not very active, the primary dealer would want to receive a very small award. Thus, it submits a large part of its quantity-bids with the purpose of missing the award. It submits these bids at relatively higher yields (or lower prices) just to fulfill its pro-rata quantity bidding requirement.

The following two works study the information interaction between primary dealers and indirect bidders in more advanced modeling frameworks. They also compare the outcomes of the primary dealer bidding system to the one of other alternative bidding systems. Boyarchenko et al. (2016) develop a theory of intermediaries where the primary dealers aggregate the information across customers for advising them, and show that the primary dealer system increases both the expected auction revenue and variance, and hence, contradict the prevailing thinking that the primary dealer system lowers auction revenue, but also revenue risk. These studies explain how primary dealers use the information obtained from their customers' bids to formulate their own bids. Kazumori and Tchoundjo (2016) develop a model in which the primary dealer system impacts the bidding behavior through two channels: The *information pooling channel* where each primary dealer pools its information with that of its indirect bidders; and the *competition channel* where the auction system maintains its competitiveness by keeping bids by indirect bidders rather than primary dealers initializing them. They find that the primary dealer system provides the lowest price volatilities while maintaining the equal level of auction prices in comparison with the alternative direct bidding system.

Hortacsu and Kastl (2012) study, which is the closest to our, uses detailed data for the Canadian Treasury auction to propose a structural model in which dealers observe their customer's bids while preparing their own bids. In their setting, dealers can use information on customer bids to learn about *competition*, i.e. the distribution of competing bids in the auction, and *fundamentals*, i.e. the ex-post value of the security being auctioned. They find that the information about competition contained in customer bids accounts for 13-27% of dealers' expected profits, and do not find any evidence that dealers are learning about fundamentals. This finding about competition was possible due to the fact that Canada Treasury auction data allows these authors to observe the following three types of bids: (i) dealers' initial-in-house bids, i.e. bids that dealers initially submit to Treasury before they receive their customers' bids; (ii) customers' bids, i.e.

bids that dealers receive from their customers and submit to Treasury;³ and (iii) dealers' final-in-house bids, i.e. bids that dealers submit (to update their previous bid submissions) to Treasury after they have routed their customers' bids.

To quantify the excess award that dealers receive due to their informational advantage Hortacsu and Kastl (2012) compute the difference between the following two ex-post equilibria: on the one hand from dealers' final-in-house bids and customers' bids; and on the other hand from dealers' initial-in-house bids and customers' bids. Their finding is the first documented evidence that primary dealers use their information advantage to front-run their customers. In this study we extend their work by modeling how primary dealers formulate their bids after observing their customers' bids.

In the context of US Treasury auctions, we attempt to analyze how primary dealers formulate their in-house bids after observing their customers' bids, and to compare the outcomes of the primary dealer system versus the ones of the direct bidding system. Our novel modeling approach is to explicitly derive closed-form representations of primary dealers' demand schedules as functions of customers' demand schedules.⁴ The two main research questions of this study are the following. (i) Given the perfect knowledge of their customers' demand schedules, how do US Treasury dealers formulate their in-house demand schedules? (ii) Comparing to the alternative direct bidding system, how does the primary dealer informational advantage in the primary dealer system impacts US Treasury revenue? According to the best of our knowledge the first question has never been investigated in the literature. The answer to this question will complement Hortacsu and Kastl (2012) study. Boyarchenko et al. (2016), and Kazumori and Tchuindjo (2016) and have recently investigated the second question but under different modeling frameworks. The answer to this question will strengthen their results.

To answer above mention two questions, we start with Wilson (1979) model of share auction in which both price and quantity are continuous variables, and bidders have independent private values. These assumptions are similar to the ones in the closest related study of Hortacsu and Kastl (2012). Then we extend Wilson (1979) model in the following three aspects. First, from a symmetric game to an asymmetric game with two different types of bidders: the customers (or indirect bidders) and the primary dealers. Second, we extend this model from a static game to a two-stage multiple-leaders-followers game, similar to a Stackelberg (1934) game. As the third extension, we explicitly model the front-running behavior in our auction game.

In order to achieve our goal, we model the Treasury auction as a two-stage game having $N \in \mathbb{N}_+$ leaders (that are indirect bidders) and N followers (that are primary dealers). We pair each primary dealer to an indirect bidder, in a way that we have exactly N customer-dealer pairs. Then we construct the game such that in the first stage each indirect bidder submits its demand

³ Dealers transfer customers' bid to Treasury as soon as they receive these bids.

⁴ It is worth noticed that one cannot directly apply Hortacsu and Kastl (2012) research methodology for Canadian Treasury auctions to US Treasury auctions. This is because in the US Treasury auction dealers update their in-house bids internally as they are receiving (and routing) their customers' bids, and submit their in-house bids once to Treasury just before the auction deadline. For this reason, US treasury auction data allows for the observation of only two types of bids – the bids customers submitted to dealers for routing; and dealers' final bids.

schedule to be routed to its primary dealer, who in the second stage of the game, first observes its indirect bidder's demand schedule, then formulates its own demand schedule, and finally submits both demand schedules to Treasury with a dual objective of maximizing its profit and front-run its indirect bidder (consistent with Flitter (2012), and Harper and Kruger (2013) accounts). Even though there is a perfect Bayesian equilibrium for our auction game, it is worth noticed that we are not looking to a full equilibrium. We assume the customer's demand schedule to be exogenous to the model, then we use the backward induction technique – an approach developed by Selten (1965) to solve dynamic games of incomplete information of this kind – to derive the primary dealer's demand schedule as the best response functional to the indirect bidder's demand schedule. Thereafter, under the assumptions of constant marginal values and additive separable demand schedules, we obtain closed-form solutions for the demand schedule of the risk-neutral primary dealer.

The main contribution of this work is about the primary dealers bidding behavior with respect to its informational advantage. First, this study explicitly represents the dealer's strategic choice of bids as a function of its customer's bids, and explain how the primary dealer's informational advantage impact its bidding behavior in handling the risk of being short-squeezed or face the winner curse in the post-auction market. This complements Hortacsu and Kastl (2012) who focus on the empirical estimation of the benefits the dealer obtains from using its knowledge of its customers' bids to formulate its own bids. Second, this study compares the expected revenue and revenue volatility of the primary dealer bidding system to the ones of the alternative direct bidding system.

The second contribution of this study is the fact that it is the first to model the bidder's set of bids as mollified step functions.⁵ A mollified step demand schedule has the double advantage to look closely like its corresponding discrete step demand schedule and to be analytically tractable like a strictly decreasing continuous demand schedule. For analytic tractability reasons many authors model the bidders' demand schedules in the Treasury auctions as strictly downward sloping continuous functions. But because each Treasury auction bidder submits a limited number of bid-points, Kastl (2011) explains it is unrealistic to represent the bidder demand schedule by a strictly downward sloping continuous function, and introduces the more realistic left continuous step demand schedule. But because of their lack of analytic tractability left continuous step demand schedules, when used in our model, does not allow to correctly explaining the primary dealer's reaction with respect to changes at bid-points of its customer's demand schedule. To overcome this limitation, we mollify the customer discrete step demand schedule to transform it to a continuously differentiable function. We also illustrate how one can use the normal distribution function to perform such transformation or mollification.

The two main results of this study are the following. First, the primary dealer's response to its customer's increase in quantity at a bid-point is to increase the quantity at its own bid-points that the prices are in the neighborhood of the customer customer's bid-point price. More clearly, the primary dealer increases the quantity in its bid-point that is near (in terms of price) to the

⁵ The purpose of function mollification, introduced in differential calculus by Friedrichs (1944), is to approximate a non-smooth function by a smooth function that is graphically very close to the original non-smooth function. This is done through the convolution of the non-smooth function with a smooth function, called mollifier, which satisfies certain conditions.

customer's bid-point. Meanwhile, the primary dealer decreases the quantity in its bid-points that are not near (in terms of price) to that customer's bid-point. We call this the *quantity effect*. As we explain later, comparing to the alternative direct bidding system,⁶ this effect leads to an increase in the auction clearing price, and hence, increases the Treasury revenue.

Second, the primary dealer's response to its customer's increase in quantity at a bid-point, which price is less than the primary dealer's value, is to decrease its bid shadings at its bid-points that the prices are in the neighborhood of its customer's bid-point price and to increase its bid shadings elsewhere. Alternatively, the primary dealer's response to its customer's increase in quantity at a bid-point, which price is higher than the primary dealer's value, is to increase its bid shadings at its bid-points that the prices are in the neighborhood of its customer's bid-point. We call this the *price effect*. As we explain later, comparing to the alternative direct bidding system, this effect increases Treasury revenue volatility.

The price effect result brings more clarification to the empirical study of Hortacsu et al. (2015) who find that primary dealers shade their bids more than direct and indirect bidders. These authors do not clearly explain how the primary dealer shades its bids along its entire demand schedule. In particular, they do not account for the fact that primary dealers submit certain bids simply to fulfill their pro-rata quantity bidding requirement, as US Department of the Treasury (2012) explains in its empirical study. Our results show more clearly that the primary dealer pushes its bid-points, which prices are in the neighborhood of its customers' price-bids, to the right (i.e. increase their prices). Meanwhile it pushes other bid-points to the left (i.e. decreases their prices). Therefore, the results of this paper why Hortacsu et al. (2015) conclusion differs from the one in US Department of the Treasury (2012).

Our results confirm the ones in two recent studies. First, the combination of two effects described above shows that primary dealers submit very aggressive bids at the neighborhoods of their customers' bids. Meanwhile, they also submit less aggressive bids, certainly to fulfill their pro-rata bidding requirement. These results are then consistent with the empirical study of US Department of the Treasury (2012) who finds that after taking out bids⁷ placed to fulfill their pro-rata bidding requirement, primary dealers submit more aggressive bids than both indirect bidders and direct bidders. Furthermore, although indirect bidders submit less quantity-bid than primary dealers, the indirect bidder bids are very valuable. US Department of the Treasury (2012) shows in the illustration of the 10-year note auction in June 2012 that in relative size the indirect bidders submit the highest proportion of their bids in aggressive zone. From the combination of both the quantity effect and the pricing effect explained above we can infer the following, which provides an intuitive explanation to US Department of the Treasury (2012) results. As a primary dealer observes an increase in its indirect bidders' quantity bids it reduces its quantity demand in its throw-away zone and increases its quantity demand in its aggressive zone, i.e. the primary dealer flattens its demand schedule.

⁶ Direct bidding system refers to an auction system in which every competitive bidder submits its bids directly to the seller without the service of an intermediary.

⁷ That it calls bid-to-miss or throw-away bids.

Second, our results show that comparing to the direct bidding system, the primary dealer bidding system increase both Treasury revenue and the revenue volatility. These results are then consistent with the ones of Boyarchenko et al. (2016) who develop a theory of intermediaries where the primary dealers aggregate the information across customers for advising them, and show that the primary dealer system increases both the expected auction revenue and variance. Our results are not inconsistent with the ones of Kazumori and Tchuindjo (2016) that the primary dealer system provides the lowest price volatilities while maintaining the equal level of auction prices in comparison with the alternative direct bidding system. In fact, these authors show that, in a full equilibrium where the customers' demands are endogenously determined, the customers will reduce their bids further in the primary dealer system in comparison with the direct bidding system since customers in the primary dealer system need to anticipate that their primary dealer will observe their bids. This effect will lead to the reduction of the volatility of auction prices in the primary dealer system in their model.

This paper contributes to the literature of informational advantage that has always been an important topic in auction. The following are some earlier works on this topic. Wilson (1967) analyzed the problem of competitive bidding, via sealed tenders, under uncertainty when one of the parties knows the value of the prize with certainty. He derived and characterized equilibrium strategies for computational purposes in terms of the solution to a related differential equation. Hughart (1975) developed a game-theory model of bidding behavior in offshore oil-lease sales for the case in which one bidder has superior information concerning the value of the tracts being leased. He found the system of sale to the highest bidder in sealed bidding at a price equal to the amount bid to be non-optimal. Milgrom and Weber (1982) explored bidders' incentive to gather information in auction, when there is one bidder with only public information and another with some private information. They found that the bidder with only public information makes no profit at equilibrium, while the bidder with private information generally makes positive profits. Engelbrech-Wiggangs (1983) considered the sale of an object by sealed-bid auction, when only one bidder has private information as in Milgrom and Weber (1982) but there are many others bidders who have access only to public information. They found that at equilibrium the informed bidder's distribution of bids is the same as the distribution of the maximum of the other bidders' bids, and the expected profit of the informed bidder is generally positive, while the other bidders have zero expected profits.

The rest of this study is structured as follows. Section 2 firstly describes the US Treasury auction process, then presents the participants, and finally explains why a leader-follower framework is suitable to model the US Treasury auction. In Section 3, we first present the indirect bidder demand schedule as a left continuous step function. Then, we show how it can be smoothed to a differentiable function through the mollification technique. Finally, we provide a numerical example. Section 4 proposes a strategic bidding model for the primary dealer. Section 5 solves the model for the dealer's demand schedule and proposes closed-form solutions. Section 6 shows how the primary dealer's informational advantage impacts its bidding behavior for the auctioned security. Section 7 analyses the impacts of primary dealer's informational on Treasury's revenue and revenue volatility. The last section provides comments on bidders' behavior, and proposes policy recommendations to US Treasury.

2. The US Treasury Auction

In this section, we first briefly describe the US Treasury auction process. Second, we present the different types of bidders who participate to the auction. Then, we explain why this auction can be modeled as a leader-follower game. Finally, for each type of bidder we provide a brief description of its demand schedule.

2.1. The auction mechanism

Two different classes of bidders participate to US Treasury auctions – *noncompetitive bidders* and *competitive bidders*. A noncompetitive bidder is allowed to submit a single bid, up to thirty minutes before the auction closes. This single bid specifies the quantity of securities the noncompetitive bidder is willing to be awarded at whatever equilibrium yield results from the competitive bidding. After the auction closes, the Treasury subtracts all noncompetitive bid amounts from the total offering amount. The remaining amount is the competitive amount to be awarded in a uniform-price format to competitive bidders who submit their bids in single or multiple yield-quantity pairs. Treasury accepts competitive bids, in ascending order of the yield (or discount rate for bills), until it has exhausted the competitive amount.⁸ The highest accepted yield is called the stop-out yield. All bids with yields less than the stop-out yield are filled in full, all bids with yield equal to the stop-out yield are filled on a pro rata basis, and all bids with yields higher than the stop-out yield are rejected. All noncompetitive bidders are awarded the quantity of security they requested at the stop-out yield. It is worth noticing that so far, the total amount of noncompetitive bids has always been relatively small compared to the total offering amount – in 2013 noncompetitive awards represented only around 2.05%, 0.21%, and 0.29% of the total offering amounts for bills, TIPS, and nominal coupon securities (notes and bonds) respectively.⁹

2.2. The auction participants

Competitive bidders can be grouped into the following classes.

The Dealers: They can be classified in two subgroups: (i) *Primary Dealers*, which are institutional investors,¹⁰ mostly large investment banks that have the obligation to participate in every Treasury auction. They are allowed to bid on behalf of their customers and for their own accounts. The total quantity bid by each primary dealer must be at least equal to a minimum proportion of the total offering amount set by Treasury.¹¹ According to New York Fed rules, primary dealers' bids must be "reasonable" compared with the range of prices of the to-be-auctioned security in the when-issued market. Primary dealers also provide market commentary and analysis that are helpful in conducting monetary policy and act as counterparties when the

⁸ Garbade and Ingber (2005) provide more details on the US Treasury auction process.

⁹ Source: Aggregates from TreasuryDirect – Announcement, Data & Results, http://www.treasurydirect.gov/instit/annceresult/annceresult_query.htm

¹⁰ The current system of primary dealers was set up in 1960 with 18 dealers. The number of primary dealers grew to 46 in 1988, declined to 21 by 2007 and stands at 23 in April 2016.

¹¹ Which is currently the total offering amount divided by the number of primary dealers.

Federal Reserve buys or sells Treasury securities; (ii) *Other Dealers and Brokers*, which are mostly financial institutions such as brokerage houses, and commercial banks. Like primary dealers, they are allowed to bid on behalf of their customers and for their own accounts. However, they do not have the obligation to participate in every auction, or to bid for a minimum quantity of the total offering amount. It is worth noticed that both primary dealers and other dealers and brokers are not allowed to charge customers money for bidding on their behalf at Treasury auctions.

Indirect Bidders: This group represents the customers who submit their bids through primary dealers or through other dealers and brokers. Among them are sovereign central banks, institutional investors, corporations, and individuals.

Direct Bidders: This group represents bidders who submit their bids directly to the auction, without intermediation. Among them are sovereign central banks and institutional investors that submit large bids when they appear in auctions. Direct bidders do not route customers' bids and do not have the obligation to participate in every auction, or to bid for a minimum quantity of the total offering amount.

Primary dealers and indirect bidders are the major players in the Treasury auction market. According to Fleming (2007), in all 576 Treasury auctions conducted between May 5, 2003 and December 28, 2005 primary dealers, direct bidders, and indirect bidders purchased 70.9%, 2.4%, and 21.6% respectively. The remaining 5.1% went to noncompetitive bidders. Thus during this period primary dealers and indirect bidders together received 97.5% of the competitive amount, and therefore, without loss of generality, in what follows we model the US Treasury auction with only these two groups of bidders.

2.3. Treasury auctions as Stackelberg games

Considering only primary dealers and indirect bidders, the US Treasury auction can be view as a game with multiple leaders and multiple followers, which are indirect bidders (or customers) and primary dealers respectively. First, for simplification, we make the following assumptions. First, the number of customers in the auction is exactly the same as the number of primary dealers. Second, and each customer is paired with one and only one primary dealer (conversely, each primary dealer is paired with one and only one customer).

We represent the Treasury auction process as a two-stage game, such that in the first stage of the game, each customer submits its demand schedule to be routed to its primary dealer, who in the second stage of the game, first observes its customer's demand schedule, then formulates its own demand schedule, and finally submits both demand schedules to Treasury. An interesting question is whether the US treasury auction can be represented as a Stackelberg (1934) game. It is worth noting that a two-player Stackelberg game is a two-stage game of perfect information in which the following three conditions are satisfied:¹²

¹² We consider the Treasury auction to be a repeated game, and we implicitly assume that the dealer has no means of committing to a future non-Stackelberg second-mover action and the customer knows this. Indeed, if the dealer could commit to a Stackelberg first-mover action and the customer knew this, the customer's best response would be to want to play a Stackelberg second-mover action, i.e. to route the dealer's bid.

Condition 1: The two players must move (choose their quantities or prices) sequentially, and they both must know that they are moving sequentially.

Condition 2: The second mover must observe the action resulting from the first mover's move, and must act based on this observation.

Condition 3: The first mover must know that the second mover will observe the action resulting from its (first) move, and that the second mover's action will be based on its observation.

The following three points strengthen our motivation to model the Treasury auction as a Stackelberg game:

- (i) By design of the US Treasury auction, Condition 1 is satisfied as the customer (the first mover) submits its bids, to be routed, through the primary dealer (the second mover).
- (ii) Although there are not enough evidences to prove the existence of Condition 2 in US Treasury auctions, the empirical study of US Department of the Treasury (2012) reports that primary dealers construct their value bids based on their clients' value bids. Furthermore, in Canadian Treasury auctions Hortacsu and Kastl (2012) empirically show the existence of this condition, i.e. they show that primary dealers use their knowledge of their customers' bids to form their own bids.
- (iii) Harper and Kruger (2013) report from PIMCO, presented above, is an illustration of Condition 3 in US Treasury auctions. We can also note that Hortacsu and Saren (2005) have proved the existence of this condition in Canadian Treasury auctions by documenting patterns of customers' bidding behavior consistent with a strategic response to dealers' use of the information contained in customers' bids.

2.4. The bidders' demand schedules

Both type of bidders (primary dealers and their customers) submit their bids in form of bid-points, which are yield-quantity pairs. Without loss of generality, in what follows we will represent bid-points as price-quantity pairs.¹³ Furthermore, we should note that each type of bidder has different need for the auctioned security.

The primary dealers are the most active players in the Treasury security market. Their demands for the auctioned security are the results of various purposes, such as portfolio allocation, collateral requirements, risk management, regulatory requirements, repurchase agreements, resale on the secondary market, and coverage of the short positions created on the when-issued market prior to the auction. Moreover, each primary dealer must participate to the auction and must submit bids for at least a pre-defined quantity. This is why generally each primary dealer spread its bids on a large range of prices, i.e. it submits multiple bid-points. As US Department

¹³ Treasury changed its rule in September 1974 from receiving bids in price-quantity pairs to yield-quantity pairs (see e.g. Garbade, 2004).

of the Treasury (2012) explains, some of these bid-points are throw-away bids, i.e. bids that the primary dealer submits just to fulfill its pro-rate bidding requirement. Because of this large number of bid-points the primary dealer demand schedule can be expected to look like a downward sloping continuous function. But, now, we do not assume any functional form for the primary dealer's demand schedule. Later we will derive its demand schedule as its best response to its customer's bids.

Comparing to primary dealers, each customer (or indirect bidder) has a specific purpose when participating to the Treasury auction. Furthermore, it is not required to participate to the auction. This reason can explain why each indirect bidder generally submits a single or just a few bid-points. Therefore, it would be unrealistic to represent its demand schedule as a strictly downward sloping continuous function. But the more realistic left continuous step function with a down-jump at the right of each bid-point as in Kastl (2011) is not analytically tractable at all, as we will see later in our model, this representation cannot correctly explain the primary dealer's reaction with respect to the change at the bid-points of its customer's demand schedule. As the customer's demand schedule is exogenous to our model, we mollify this demand schedule before including it to the model. The resulting mollified step demand schedule looks like its original discrete step demand schedule and it is differentiable everywhere, in the sense that it has a high (but bounded) first derivative at the neighborhood of each bid-point.

3. Demand Schedule Mollification

In this section, first, we describe the function mollification methodology. Second, from the bid-points of an indirect bidder we show how to construct its left continuous discrete step demand schedule. Then, we use a normal mollifier to smooth this left continuous discrete step demand schedule. Finally, we provide a numerical example.

3.1. Mollification of non-smooth functions

The purpose of function mollification, introduced in differential calculus by Friedrichs (1944), is to approximate a non-smooth function by a smooth function that is very close to the original non-smooth function. This is done through the convolution of the non-smooth function with a smooth function, called mollifier, which satisfies certain conditions. The following theorem explains the function mollification process.

Theorem 1: *Let f be a non-smooth function. If φ is a real-valued smooth function that satisfies the following three conditions*

- (i) φ is compactly supported
- (ii) $\int_{\mathbb{R}} \varphi(t) dt = 1$
- (iii) $\frac{1}{\varepsilon} \varphi_{\varepsilon}(t)$ converges to $\delta(t)$ as ε goes to 0,
where $\varphi_{\varepsilon}(t) \equiv \varphi\left(\frac{t}{\varepsilon}\right)$ and $\delta(t)$ is the Dirac's delta function

then the function f_ε such that $f_\varepsilon(x) = (\varphi_\varepsilon * f)(x) = \int_{\mathbb{R}} \varphi_\varepsilon(x-t)f(t)dt$ is a smooth function that converges (at least point-wise) to f as ε goes to 0.

In Theorem 1, the resulting function f_ε is a smooth function, as the convolution is a smoothing operation. This is due to the fact that if the function φ_ε is compactly supported and n times continuously differentiable, and f is locally integrable, then $\varphi_\varepsilon * f$ is also n times continuously differentiable with $\frac{d^n}{dp^n}(\varphi_\varepsilon * f) = \left(\frac{d^n}{dp^n} \varphi_\varepsilon\right) * f$.

3.2. Indirect bidder demand schedule and mollification

Let us assume an indirect bidder submits a nonempty, finite and countable set of price-quantity pairs, $\{(p_k, q_k)\}_{k \in \{1, 2, \dots, K\}}$, where $p_k > 0$ and $q_k > 0$ represent the price and the quantity in the k -th bid-point respectively. Also, assume that the price-quantity pairs are ordered such that $p_i < p_j$ for $i < j$. We can now define this indirect bidder's demand schedule on the real line as follows.

$$Q(p) = \begin{cases} Q_k : \text{if } p \in (p_{k-1}, p_k] & \text{where } p_0 = 0 \\ Q_1 : \text{if } p = 0 \\ 0 : \text{if } p \notin [0, p_K] \end{cases} \quad (1)$$

where

$$Q_k = \sum_{j=k}^K q_j \text{ for each } k \in \{1, 2, \dots, K\}.$$

This demand schedule is a step function, which is flat between bid-points, left continuous at each bid-point, and right continuous at zero. It can be mollified by any mollifier that satisfies the three conditions in Theorem 1. In what follows we show how the normal density function can be used to mollify this demand schedule.

Let $\{\phi_\sigma(\cdot)\}_{\sigma \in \mathbb{R}_+}$ be a family of normal probability density functions with zero-mean and standard deviation $\sigma > 0$. For each $t \in \mathbb{R}$ we can express $\phi_\sigma(t)$ as $\frac{1}{\sigma} \phi\left(\frac{t}{\sigma}\right)$, where ϕ is the standard normal density function. Note that the integral of ϕ over the entire real line equals 1, and the sequence of function ϕ_σ converges to the Dirac delta function as σ approaches 0. But ϕ is not compactly supported, and hence it is not a suitable mollifier. We propose to truncate ϕ and re-scale it such that the resulting function satisfies conditions (i) – (iii) of Theorem 1.

Now let us consider a finite real constant $L \gg p_K$, and a new function $\theta(\cdot|L)$ defined as follows

$$\theta(t|L) = \begin{cases} \phi(t)\Psi(L) & \text{if } t \in (-L, L) \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where $\Psi(L) = (2\Phi(L) - 1)^{-1}$, and Φ represents the standard normal distribution function, i.e. $\Phi(x) = \int_{-\infty}^x \phi(t)dt$ for $x \in \mathbb{R}$. Note that $\theta(\cdot|L)$ is compactly supported, and its integral over the entire real line equals 1. Furthermore, if we define the function sequence $\{\theta_\sigma(\cdot|L)\}_{\sigma \in \mathbb{R}_+}$ such that for each $t \in \mathbb{R}$ we have $\theta_\sigma(t|L) = \frac{1}{\sigma} \theta(\frac{t}{\sigma}|L)$, then $\{\theta_\sigma(\cdot|L)\}_{\sigma \in \mathbb{R}_+}$ will converges to the Dirac delta function as σ approaches 0. Hence, $\theta(\cdot|L)$ can be used to mollify the indirect bidder's discrete demand schedule of Equation (1) as a direct application of Theorem 1. Therefore, the indirect bidder's mollified demand schedule will be given as follows

$$\tilde{Q}_\sigma(p|L) = \Psi(L) \left(Q_1 \Phi\left(\frac{p}{\sigma}\right) - Q_K \Phi\left(\frac{p-p_K}{\sigma}\right) - \sum_{k=2}^K (Q_{k-1} - Q_k) \Phi\left(\frac{p-p_{k-1}}{\sigma}\right) \right). \quad (3)$$

The detail derivation of the above formula is given in Appendix A. The following four important points are to be noted about the resulting indirect bidder's mollified demand schedule.

- (i) It is differentiable, as being a linear combination of differentiable functions.
- (ii) At the neighborhood of each price-bid its first derivative with respect to the price is negative and has an extremely high, but finite, absolute value. This absolute value converges to infinity as σ goes to 0.
- (iii) Outside the neighborhood of the price-bids its first derivative with respect to the price equals 0.
- (iv) Its graph resembles to the one of its corresponding discrete step function.

3.3. A numerical example

Let us assume that an indirect bidder submits the following three bid-points: (96; \$5,000,000), (98; \$10,000,000), and (99.5; \$5,000,000). As $p_1, p_2,$ and p_3 are 96, 98, and 99.5 respectively, and $q_1, q_2,$ and q_3 are \$5,000,000, \$10,000,000, and \$5,000,000 respectively, the bidder discrete demand schedule is given by the following function

$$Q(p) = \begin{cases} Q_1 = \$20,000,000 : \text{if } p \in [0, 96] \\ Q_2 = \$15,000,000 : \text{if } p \in (96, 98] \\ Q_3 = \$5,000,000 : \text{if } p \in (98, 99.5] \\ 0 : \text{if } p \notin [0, 99.5] \end{cases} \quad (4)$$

Let us choose $L = 1,000$, which is higher than $p_3 = 99.5$. Then, using the closed form formula of Equation (3), and the fact that $\Psi(1,000) \approx 1$ we can obtain the following results. For each positive price p , the bidder mollified demand schedule is given by (in million \$)

$$\begin{aligned} \tilde{Q}_\sigma(p|1,000) &= \Psi(1,000) \left(20\Phi\left(\frac{p}{\sigma}\right) - 5\Phi\left(\frac{p-99.5}{\sigma}\right) - 5\Phi\left(\frac{p-96}{\sigma}\right) - 10\Phi\left(\frac{p-98}{\sigma}\right) \right) \\ &\approx 20\Phi\left(\frac{p}{\sigma}\right) - 5\Phi\left(\frac{p-99.5}{\sigma}\right) - 5\Phi\left(\frac{p-96}{\sigma}\right) - 10\Phi\left(\frac{p-98}{\sigma}\right). \end{aligned} \quad (5)$$

4. A Model for the Treasury Auction

In this section, first, we present the auction participants, i.e. the seller and the bidders. Second, we describe the timing of the auction game. Finally, we propose a primary dealer's multi-attribute utility function, which is used to formulate an optimization problem that the solution is its demand schedule.

4.1. The seller

There is only one seller, the US Treasury, who has a known $M \in \mathbb{R}_+$ infinitely divisible quantity of securities to be auctioned. This amount is the auction amount or the competitive amount. It is the amount of securities that Treasury awards to competitive bidders at the end of the auction. This amount equals the total offering amount minus the total noncompetitive bids. Treasury receives noncompetitive bids until thirty minutes before the auction deadline. Then announces the amount of noncompetitive bids received. In practice, primary dealers usually submit their bids just before the auction deadline. We assume here that these primary dealers know the competitive amount with certainty when they submit their bids.¹⁴

4.2. The bidders

We suppose that there are $N \in \mathbb{N}$ customers or indirect bidders identified by elements of the index set $I_C \equiv \{ci : i = 1, 2, \dots, N\}$. Each customer submits only one set of bid-points. The customer does not interact directly with Treasury. It submits its bids indirectly through only one

¹⁴ One can still assume that competitive bidders who submit their bids more than thirty minutes prior to the time the auction deadline know the auction amount with certainty. This is because the total amount of noncompetitive bids is usually very small relative to the total offering amount. For example, in 2012 only 1.85% of the total offering was awarded to noncompetitive bidders. Therefore, without loss of generality, the auction amount can be approximated by the total offering amount.

primary dealer. We also assume that there are N primary dealers¹⁵ identified by elements of the index set $I_D \equiv \{di : i = 1, 2, \dots, N\}$. Each primary dealer submits two set of bids to Treasury – one set representing its own demand schedule and the other one representing the demand schedule of only one customer. We are clearly assuming that each customer routes its bids through only one primary dealer, and each primary dealer routes bids from only one customer. We thus form a set of N customer-dealer pairs, $\{(ci, di) : i = 1, 2, \dots, N\}$, understood as dealer di routes bids from customer ci . We can represent all bidders in a large set $I \equiv I_C \cup I_D$.

4.3. The timing of the Treasury auction game

The timing of the auction game is understood as follows.

- (i) Firstly, each bidder $i \in I$ receives a positive signal S_i , drawn from the interval $[0, 1]$ and distributed according a commonly known probability distribution function F_S . We assume the signals S_i 's are independently distributed across bidders.
- (ii) Secondly, each bidder $i \in I$ assigns a value to any unit of the to-be-auctioned securities using a commonly known marginal valuation function, $v : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}_+$, which is continuous and weakly decreasing in its first argument (the unit of the auction securities), and bounded and strictly increasing in its second argument (the realization of the signal).
- (iii) Thirdly, each customer $ci \in I_C$ formulates its set of bid-points and submits it to its corresponding dealer $di \in I_D$ for the routing service.
- (iv) Fourthly, each primary dealer di extracts information from its customer ci 's bid-points, and then formulates its own set of bid-points.
- (v) Finally, each dealer di submits both its demand schedule and the one received from its customer ci to Treasury.

Let \mathcal{D} to be the set of all functions defined on the positive real line that are positive, weakly decreasing, bounded, and differentiable. This set contains mollified step demand schedules that we described in the previous section. In what follows, we assume that each customer $ci \in I_C$ submit a set of bid-points that can be represented as a discrete demand schedule, i.e. a function that specifies the amount of securities the customer would need given the price. It is clear that the mollified version of this discrete demand schedule belong to the set \mathcal{D} .

¹⁵ Hereafter, the terms “primary dealers” or “dealers” refer to all *Primary Dealers* and those of *Other Dealers and Brokers* who simultaneously bid for their own needs and route their customers' bids.

4.4. The primary dealer's utility function

When formulating its demand schedule, we assume the primary dealer wants to maximize utilities from two objective functions. The first utility function $U: \mathbb{R} \rightarrow \mathbb{R}_+$ is a strictly increasing and differentiable function of the primary dealer's profit from the auction. The existence of the profit as objective can be explained by the fact that the primary dealer is a for-profit institution that the main goal is revenue generation. Profit maximization is a standard assumption in Treasury auction research.¹⁶

As we explained above, in order to make sure it will be awarded enough Treasury securities as the outcome of the auction, the primary dealer uses the knowledge of its customers' bids to adjust its own bids.¹⁷ Therefore, the second utility function, $V: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is new in the financial literature is a strictly increasing and differentiable function of a proxy of the primary dealer use of its informational advantage to front-run its customers. We assume the function V satisfies the von Neumann and Morgenstern (1947) axioms of utility theory. We thus suggest to proxy the primary dealer ex-post informational advantage benefit by the ratio of the primary dealer's award over its customer's award from the auction. An increase of this ratio provides more utility to the primary dealer, as its risk of being short squeezed on the post auction market is reduced. We choose to model the ex-post informational advantage benefit by this ratio because it is simple and easy to understand. However, it is not the only possible measure of the ex-post informational advantage benefit.

Note that there is a trade-off between the two primary dealer's objectives. By front-running its customer, the dealer increases its demand for the security to be auctioned at higher price levels. This increases its potential award, but raises the expected auction clearing price, and hence decreases its expected profit. We assume the primary dealer's multi-attribute utility function to be a multiplicative separable function of its two single utility functions, i.e. this multi-attribute utility function is defined as $W: \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that $W(x, y) = U(x)V(y)$. We choose this simple form to avoid dealing with arbitrary scaling constants.

4.5. The primary dealer's demand schedule

According to the timing of the auction game described above the Treasury auction can be modeled as a two-stage game. In the first stage of the auction each customer submits its demand schedule to its corresponding primary dealer, knowing that this primary dealer will use this demand schedule in the second stage of the game as private information to formulate its own demand schedule before submitting both demand schedules to Treasury. Our proposed Treasury game can be characterized as a dynamic game of incomplete and perfect information. This game is finite as it has only two stages. The perfect information aspect of the game is due to the fact that in the second stage, each primary dealer knows the exact history of the game, i.e. the demand schedule its corresponding customer submitted in the first stage. Kuhn's theorem guarantees that for each primary dealer there exists a pure-strategy demand schedule, i.e. a

¹⁶ See e.g. Kremer and Nyborg (2004).

¹⁷ Hortacsu and Kastl (2012) have also evidenced that Canadian Treasury dealers use their knowledge of customers' bids to formulate their own bids.

demand schedule that belong to the set \mathcal{D} , given its customer's demand schedule. We assume the customer's demand schedule to be exogenous to the model, and we look for the primary dealer's demand schedule. In what follows we use the backward induction technique to find a solution for primary dealer di 's demand schedule, given customer ci 's demand schedule.

In the first stage of our Treasury auction game customer ci looks forward to the sub-game that will result in the second stage as the consequence of the choice of its first stage demand schedule. In this sub-game perfect equilibrium, customer ci anticipates that primary dealer di will use customer's ci demand schedule to formulate a demand schedule that maximizes the primary dealer expected multi-attribute utility function. By making this anticipation, customer ci assumes dealer di is rational, and hence, eliminates all non-credible threat¹⁸ from dealer di . Thus, customer ci anticipates that primary dealer di optimization problem can be formulated as follows

$$\max_{D_{di}(p|s_{di}, D_{ci}(p))} \mathbb{E} \left[U \left(\int_0^{D_{di}(p|s_{di}, D_{ci}(p))} v(u, s_{di}) du - p D_{di}(p|s_{di}, D_{ci}(p)) \right) V \left(\frac{D_{di}(p|s_{di}, D_{ci}(p))}{D_{ci}(p)} \right) \right]. \quad (6)$$

We are explicitly saying the following. Given the realization s_{di} of primary dealer di 's private information, and customer ci 's demand schedule, $D_{ci} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, primary dealer di 's resulting response is a demand schedule $D_{di} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that is the best response to all other bidders' demand schedules, i.e. the primary dealer's pure bidding strategy is a mapping $B_d : \mathbb{R}_+ \times [0, 1] \times \mathcal{D} \rightarrow \mathcal{D}$,

5. A Solution for the Model

From the first order condition for the optimization problem (6) we obtain dealer di 's best response functional to customer ci 's demand schedule and other bidders' demand schedules, given dealer di 's marginal valuation function. Then we propose restricted assumptions that allow to obtain closed-form solutions for dealer di 's demand schedule.¹⁹

5.1. The dealer's optimal demand schedule

Following Wilson (1979) framework, from primary dealer di 's perspective, we can define, over the realization of the stop-out price, a probability distribution function $H : \mathbb{R}_+ \times \mathcal{D} \rightarrow [0, 1]$ that is twice continuously differentiable with respect to each argument, as follows

¹⁸ A non-credible threat would be that after observing customer ci 's demand schedule, dealer di submits its own bids with very high prices such that it is sure to be award all quantity its requires. But this will not be in the best interest of dealer di , as it will lead for sure to a negative profit for dealer di .

¹⁹ Let us note that the indirect bidders might have a different utility maximization problem than the primary dealers. The question of how each indirect bidder uses it paired primary dealer's best response functional to formulate its own demand schedule is important, but it is not relevant in the context of this analysis, as we are concerned only about the strategic responses of the primary dealers to their customers' demand schedules.

$$H\left(p\left|D_{di}\left(p\left|s_{di}, D_{ci}(p)\right.\right)\right.\right) \equiv \Pr\left(\tilde{p} \leq p\left|D_{di}\left(p\left|s_{di}, D_{ci}(p)\right.\right)\right.\right). \quad (7)$$

In the preceding expression $H\left(p\left|D_{di}\left(p\left|s_{di}, D_{ci}(p)\right.\right)\right.\right)$ represents the distribution of the auction clearing price, \tilde{p} , as seen by the dealer di , given that it submits the demand schedule $D_{di}\left(p\left|s_{di}, D_{ci}(p)\right.\right)$. From now on, for simplicity of notations let us remove s_{di} from the arguments of dealer di 's marginal valuation function, and both s_{di} and $D_{ci}(p)$ from the arguments of its demand schedule. Thus, $D_{di}\left(p\left|s_{di}, D_{ci}(p)\right.\right)$ and $v(u, s_{di})$ will be rewritten as $D_{di}(p)$ and $v_{di}(u)$ respectively. Using (7), expression (6) can be rewritten as follows

$$\max_{D_{di}(p)} \int_0^1 \left(U \left(\int_0^{D_{di}(p)} v_{di}(u) du - p D_{di}(p) \right) V \left(\frac{D_{di}(p)}{D_{ci}(p)} \right) \right) dH(p|D_{di}(p)). \quad (8)$$

Assumption 1: *Given the realization of its private signal, and based on its bidder type, each bidder assigns a constant marginal value to each unit of the security to be auctioned.*

According to Assumption 1 there is a function $h_c : [0, 1] \rightarrow \mathbb{R}_+$ such that for each indirect $ci \in I_C$, its signal is transformed into its marginal value as $h_c(s_{ci}) \equiv v(x, |s_{ci}) = v_{ci}$, for all $x \in \mathbb{R}_+$. Also there is a function $h_d : [0, 1] \rightarrow \mathbb{R}_+$ such that for each primary dealer $di \in I_D$, its signal is transformed into its marginal value as $h_d(s_{di}) \equiv v(x, |s_{di}) = v_{di}$, for all $x \in \mathbb{R}_+$.

Assumption 2: *Each primary dealer is risk-neutral with respect of its utility functions.*

Under Assumptions 1 and 2, Proposition 1 provides an equation that dealer di 's demand schedule is a solution.

Proposition 1: *Primary dealer di 's demand schedule is a solution of the following equation*

$$D_{di}(p) = - \frac{2}{\frac{1}{(v_{di}-p)} + \frac{D_{ci}(p)}{D_{ci}(p)}} \frac{H_p(p|D_{di}(p))}{H_D(p|D_{di}(p))}$$

Proof: See Appendix B

In Proposition 1 the term $H_p(p|D_{di}(p))$ represents the density of the auction stop-out price when primary dealer di submits the demand schedule $D_{di}(p)$. As Tchindjo (2015) explains, $H_D(p|D_{di}(p))$ represents the shift in the probability distribution of the auction clearing price due to the change in primary dealer di 's demand schedule. This derivative captures primary

dealer di 's market power and it is always negative because an increase in demand raises the auction clearing price, increasing the probability that the auction clearing price is higher than a given price.

5.2. Closed-form solutions for dealer's demand schedule

For the sake of obtaining closed-form solutions for dealer di 's demand schedule we consider the following assumption.

Assumption 3: *The customer's demand schedule can be represented as two additively separable functions of the price and its private signal. The primary dealer's demand schedule can be represented as two additively separable functions. The first is a function of its private signal, and the second is a function of both the price and its customer's demand schedule.*

According to Assumption 3 customer ci 's demand schedule and dealer di 's demand schedule can be represented (respectively) as

$$D_{ci}(p, s_{ci}) \equiv \alpha_c(p) + \beta_c(s_{ci}), \quad (9)$$

$$D_{di}(p, s_{di}, D_{ci}(p, s_{ci})) \equiv \alpha_d(p, D_{ci}(p, s_{ci})) + \beta_d(s_{di}), \quad (10)$$

where $\alpha_c : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a decreasing and differentiable function, $\alpha_d : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is a differentiable function that decreases and increases with its first and second arguments respectively, $\beta_c : [0, 1] \rightarrow \mathbb{R}$ and $\beta_d : [0, 1] \rightarrow \mathbb{R}$ are increasing functions of the signal. Here we assume that between customers, the demand schedules differ only by their private signals, and between primary dealers the demand schedules differ by their private signals as well as by their customers' demand schedules they observe. Assumption 3 generalizes the one of Hortacsu (2002) who considers the bidder's demand schedule to be linear in both the price and its private signal.

Proposition 2: *Given Assumption 3, the dealer's demand schedule is a solution to the following ordinary differential equation*

$$D'_{di}(p) + \frac{1}{2N} \left(\frac{1}{v_{di} - p} + \frac{D'_{ci}(p)}{D_{ci}(p)} \right) D_{di}(p) = -D'_{ci}(p)$$

Proof: See Appendix C.

Proposition 2 shows that the primary dealer's demand schedule is a solution of a first order non homogeneous linear ordinary differential equation (ODE) with variable coefficients. The following proposition presents the unique solution of this ODE.

Proposition 3: *Given $D_{di}(0)$, primary dealer di 's demand at price zero, a closed-form solution for its demand schedule can be uniquely determined as*

$$D_{di}(p) = \left(\frac{v_{di} - p}{D_{ci}(p)} \right)^{\frac{1}{2N}} \left(D_{di}(0) \left(\frac{D_{ci}(0)}{v_{di}} \right)^{\frac{1}{2N}} - \int_0^p D_{ci}'(t) \left(\frac{D_{ci}(t)}{v_{di} - t} \right)^{\frac{1}{2N}} dt \right).$$

Proof: The proof is in two step, which are the existence and the uniqueness on the solution, and the computation of the solution.

Step1: Existence and uniqueness of a solution: Let us rewrite the ODE in Proposition 2 as follows. $D_{di}'(p) = F(p, D_{di}(p))$. Clearly the functional F has the following two features. (i) It is continuous in p as $p \in [0, v_{di}]$, (ii) it is Lipschitz continuous in $D_{di}(p)$ as the function D_{ci} belongs to \mathcal{D} . Furthermore, $D_{di}(p)$ is bounded as the function D_{di} belongs to \mathcal{D} . Thus, by Picard-Lindelöf theorem there exists a unique solution to the above ODE associated to the initial value $D_{di}(0)$.

Step2: Computation of the solution: The result can be obtained by the integrating factor technique, via the following integrating factor $I(t) \equiv \exp\left(\int \frac{1}{2N} \left(\frac{1}{v_{di} - t} + \frac{D_{ci}'(t)}{D_{ci}(t)} \right) dt\right) = \left(\frac{D_{ci}(t)}{v_{di} - t} \right)^{\frac{1}{2N}}$.

6. Impacts of the Dealer Informational Advantage on his Bidding Behavior

From the closed-form solutions obtained in Proposition 3 this section presents comparative static predictions with respect to the primary dealer's informational advantage of observing its customer's demand schedule. We also explain how this informational advantage impact the primary dealer's bidding behavior.

6.1. Informational advantage and primary dealer's demand

Given the price level p , the change in primary dealer di 's demand with respect to the change in customer ci 's demand can be computed using the result in Proposition 3 as follows

$$\frac{\partial}{\partial D_{ci}(p)} D_{di}(p) = -\frac{1}{2N D_{ci}(p)} \left(\frac{v_{di} - p}{D_{ci}(t)} \right)^{\frac{1}{2N}} \left(K_0 - \int_0^p D_{ci}'(t) \left(\frac{D_{ci}(t)}{v_{di} - t} \right)^{\frac{1}{2N}} dt \right) - \frac{1}{2N} \frac{D_{ci}'(p)}{D_{ci}(p)}. \quad (11)$$

From Proposition 3 we can replace $K_0 - \int_0^p D_{ci}'(t) \left(\frac{D_{ci}(t)}{v_{di} - t} \right)^{\frac{1}{2N}} dt$ in Equation (11) by

$D_{di}(p) \left(\frac{D_{ci}(p)}{v_{di} - p} \right)^{\frac{1}{2N}}$, and hence Equation (11) becomes

$$\frac{\partial}{\partial D_{ci}(p)} D_{di}(p) = -\frac{1}{2N} \frac{D_{di}(p)}{D_{ci}(p)} \left(1 + \frac{D_{ci}'(p)}{D_{ci}(p)} \right). \quad (12)$$

The sign of the change in the primary dealer di 's demand with respect to the change in its customer's demand is determined by the sign of the expression in parenthesis on the right hand side of the preceding equation. As the customer's demand schedule is a mollified step function, we have the following two cases.

- (i) At the neighborhood of each customer's bid-price the first derivative of this customer's mollified step demand schedule with respect to the price is negative and has an extremely high, but finite, absolute value. Hence $|D'_{ci}(p)| \geq D_{di}(p)$ and thus,

$$\text{sgn}\left(1 + D'_{ci}(p)/D_{di}(p)\right) < 0. \text{ Therefore, } \frac{\partial}{\partial D_{ci}(p)} D_{di}(p) \geq 0.$$

- (ii) Outside the neighborhoods of the customer's bid-prices the first derivative of the customer's mollified step demand schedule with respect to the price is zero. Hence,

$$\frac{\partial}{\partial D_{ci}(p)} D_{di}(p) \leq 0.$$

Therefore, the primary dealer's response to its customer's increase in quantity at a bid-point is to reallocate its own quantity around that customer's bid-point. More clearly, the primary dealer increases the quantity in its bid-point that is near (in terms of price) to the customer's bid-point. Meanwhile, the primary dealer decreases the quantity in its bid-points that are not near (in terms of price) to that customer's bid-point. We call this the *quantity effect*.

6.2. Informational advantage and primary dealer's differential bid shading

Let us rewrite the expression of primary dealer di 's demand schedule in Proposition 3 as

$$p = v_{di} - \zeta_{di}(D_{ci}(p)), \quad (13)$$

where

$$\zeta_{di}(D_{ci}(p)) = D_{ci}(p)D_{di}(p)^{2N} \left(K_0 - \int_0^p D'_{ci}(t) \left(\frac{D_{ci}(t)}{v_{di}-t} \right)^{\frac{1}{2N}} dt \right)^{-2N} \quad (14)$$

represents primary dealer di 's differential bid shading. The partial derivative of primary dealer di 's differential bid shading with respect to its customer's demand is given by

$$\begin{aligned} \frac{\partial}{\partial D_{ci}(p)} \zeta_{di}(D_{ci}(p)) &= \frac{D_{di}(p)^{2N}}{\left(K_0 - \int_0^p D'_{ci}(t) \left(\frac{D_{ci}(t)}{v_{di}-t} \right)^{\frac{1}{2N}} dt \right)^{2N}} \\ &+ \frac{D'_{ci}(p)D_{di}(p)^{2N} \frac{D_{ci}(p)}{v_{di}-p} \left(\frac{D_{ci}(p)}{v_{di}-p} \right)^{\frac{1}{2N}-1}}{\left(K_0 - \int_0^p D'_{ci}(t) \left(\frac{D_{ci}(t)}{v_{di}-t} \right)^{\frac{1}{2N}} dt \right)^{2N+1}} \end{aligned} \quad (15)$$

Again, from Proposition 3 we can replace $K_0 - \int_0^p D_{ci}'(t) \left(\frac{D_{ci}(t)}{v_{di}-t} \right)^{\frac{1}{2N}} dt$ in Equation (15) by $D_{di}(p) \left(\frac{D_{ci}(p)}{v_{di}-p} \right)^{\frac{1}{2N}}$, and hence the Equation (15) becomes

$$\frac{\partial}{\partial D_{ci}(p)} \zeta_{di}(D_{ci}(p)) = \frac{v_{di}-p}{D_{ci}(p)} \left(1 + \frac{D_{ci}'(p)}{D_{ci}(p)} \right). \quad (16)$$

$p \leq v_{di}$. The sign of the change in primary dealer di 's differential bid shading (at a primary dealer bid-point) with respect to its customer's demand is determined by the sign of the expression in parenthesis on the right hand side of the preceding equation and as well as by whether the primary dealer's bid-price (at that bid-point) is less than or higher than²⁰ the primary dealer's value for the auctioned security. Again, as the customer's demand schedule is a mollified step function, we have the following two cases.

- (i) At the neighborhood of each customer's bid-price the first derivative of the customer's mollified step demand schedule with respect to the price is negative and has an extremely highly, but finite, absolute value. Hence $|D_{ci}'(p)| \geq D_{di}(p)$ and thus, $\text{sgn}(1 + D_{ci}'(p)/D_{di}(p)) < 0$. Therefore,

$$\frac{\partial}{\partial D_{ci}(p)} \zeta_{di}(D_{ci}(p)) \begin{cases} \leq 0 & \text{if } p \leq v_{di} \\ \geq 0 & \text{if } p \geq v_{di} \end{cases}.$$

- (ii) Outside the neighborhoods of the customer's bid-prices the first derivative of the customer's mollified step demand schedule with respect to the price is zero. Hence,

$$\frac{\partial}{\partial D_{ci}(p)} \zeta_{di}(D_{ci}(p)) \begin{cases} \geq 0 & \text{if } p \leq v_{di} \\ \leq 0 & \text{if } p \geq v_{di} \end{cases}.$$

Therefore, the primary dealer's response to its customer's increase in quantity at a bid-point, which price is less than the primary dealer's value, is to decrease its bid shading at its bid-point that is in the neighborhood (in terms of price) of the customer's bid-point and to increase its bid shading elsewhere (i.e. in other bid-points). More clearly, the dealer pushes its bid-point, which price is in the neighborhood of the customer price-bid, to the right. Meanwhile it pushes other bid-points to the left. Alternatively, the primary dealer's response to its customer's increase in quantity at a bid-point, which price is higher than the primary dealer's value, is to increase its bid shading at its bid-point that is in the neighborhood of the customer's bid-point and to decrease its bid shading elsewhere. More clearly, the dealer pushes its bid-point, which price is in the neighborhood of the customer price-bid, to the left. We call this the *price effect*.

²⁰ In the uniform-price Treasury auction, in order to increase its expected award, a bidder can submit strategic bids, which are bid-points that their prices are higher than the bidder's value for the auctioned security. However, the bidder expects the auction stop-out price to be less than these bid prices.

6.3. Informational advantage and primary dealer's bidding behavior

To make the analysis simple, let us assume that the customer submits a single bid-point to the primary dealer. After observing this bid-point, the primary dealer faces one of the following two alternative cases:

- (i) If the customer's bid-price is less than the primary dealer's value for the auctioned security, then the primary dealer will increase both the price and the quantity at its bid-point that the price is around the customer's bid-point price. This means that the primary dealer will become more aggressive. One the reasons of this aggressive behavior is to limit the risk of being shot-squeezed in the post-auction market.
- (ii) If the customer's bid-price is higher than the primary dealer's value for the auctioned security, then the primary dealer will decrease the price and increase the quantity at its bid-point that the price is around the customer's bid-point price. The reasons for the primary dealer behavior can be explained as follows. It increases the quantity not make sure it would be awarded enough securities (not to be short-squeezed). Also it decreases the price because of the risk of increasing the auction clearing price as the results of increasing the quantity in the zone where the price is higher than its value for the auction security. This limit the risk of a probable winner curse.

7. Impacts of the Dealer Informational on the Treasury Revenue

In this section, we illustrate how both the quantity effect and the price effect impact Treasury's revenue in the primary dealer bidding system, in comparison to the alternative direct bidding system in which all bidders submit their bids directly to Treasury without intermediary.

7.1. Preliminary assumptions

In what follows, without loss of generality, let us assume that each customer (indirect bidder) submits a single bid point while each primary dealer submits a complete demand schedule, which is a nonempty set of bid-points.

First, let us consider the primary dealer bidding system as described above, and let us assume that there are $N \in \mathbb{N}$ primary dealers paired to N customers. Now let:

- $(p_{PD}^{ci}, q_{PD}^{ci})$ be customer ci 's price-quantity bi-point;
- $D_{PD}^{di,I} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $D_{PD}^{di,F} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be primary dealer di 's initial and final demand schedules, formulated before and after observing customer ci 's bid-point respectively;
- \tilde{p}_{PD} be the ex-post auction clearing price.

Second, let us consider the alternative direct bidding system, in which all bidders (including primary dealer di and customer ci) submits their bids directly to Treasury. Now let:

- $(p_{DB}^{ci}, q_{DB}^{ci})$ be customer ci 's price-quantity bid-point;
- $D_{DB}^{di} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be primary dealer di 's demand schedule;
- \tilde{p}_{DB} be the ex-post auction clearing price.

Third, let us consider the following two assumptions:

H1: $(p_{PD}^{ci}, q_{PD}^{ci}) = (p_{DB}^{ci}, q_{DB}^{ci})$, i.e. customer ci would submits the same bid point in both bidding systems. This assumption will be released later.

H2: $D_{PD}^{di, I}(p) = D_{DB}^{di}(p)$ for all $p \in \mathbb{R}_+$, i.e. primary dealer di 's initial demand schedule (prior to observing its customer bid) in the primary dealer bidding system is identical to the demand schedule it would have submitted in the direct bidding system.

7.2. Impact of quantity effect on Treasury revenue

As the quantity effect shows that in the primary dealer bidding system primary dealer di increases its quantity around the price p_{PD}^{ci} , after observing customer di 's bid-point, comparing to the direct bidding system one of the following two cases can happen.

- (i) $p_{PD}^{ci} \leq \tilde{p}_{DB}$, i.e. customer ci 's bid-price in the primary dealer bidding system is less than the price that would have cleared the direct bidding system. As primary dealer di , after observing its customer's bid, $(p_{PD}^{ci}, q_{PD}^{ci})$, will move some initial quantity bids, which prices are greater than p_{PD}^{ci} to increase the quantities in its bid-points that are around p_{PD}^{ci} , the primary dealer system clearing price \tilde{p}_{PD} will be pushed to the left. As the result, this will lead to $\tilde{p}_{PD} \leq \tilde{p}_{DB}$, i.e. the primary dealer bidding system will provide less revenue to Treasury than the direct bidding system.
- (ii) $p_{PD}^{ci} \geq \tilde{p}_{DB}$, i.e. customer ci 's bid-price in the primary dealer bidding system is higher than the price that would have cleared the direct bidding system. As primary dealer di , after observing its customer's bid $(p_{PD}^{ci}, q_{PD}^{ci})$, will move some initial quantity bids, which prices are less than p_{PD}^{ci} to increase the quantity in its bid-points that are around p_{PD}^{ci} , the primary dealer system clearing price will be pushed to the right. As the result, this will lead to $\tilde{p}_{PD} \geq \tilde{p}_{DB}$, i.e. the primary dealer bidding system will provide more revenue to Treasury than the direct bidding system.

Which one of the above cases is more likely to occur? To respond to this question let us recall that according to Condition 3 of the Stackelberg game, in the primary dealer bidding system customer ci knows that primary dealer di will observe its bid-point $(p_{PD}^{ci}, q_{PD}^{ci})$ and that primary dealer di will formulate its final demand schedule, $D_{PD}^{di,F} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, based on this observation. Therefore, not to be front-run by primary dealer di , customer ci will submit a more aggressive bid than the one it would have submitted in the alternative direct bidding system, i.e. its bid-point $(p_{PD}^{ci}, q_{PD}^{ci})$ will be such that $q_{PD}^{ci} = q_{DB}^{ci}$ and $p_{PD}^{ci} \geq p_{DB}^{ci}$. Hence, this increases the likelihood that $\tilde{p}_{PD} \geq \tilde{p}_{DB}$, which is case (ii) above. Thus, the primary bidding system provides higher expected revenue to Treasury than the alternative direct bidding system.

7.3. Impact of price effect on Treasury revenue

The following analysis is conducted assuming for primary dealer's bid-points that the prices are less than the primary dealer's value for the auctioned security.²¹ As the price effect shows that in the primary dealer bidding system primary dealer di decreases its bid shading around the price p_{PD}^{ci} , after observing customer di 's bid-point, comparing to the direct bidding system one of the following two cases can happen.

- (i) $p_{PD}^{ci} \leq \tilde{p}_{DB}$, i.e. customer ci 's bid-price in the primary dealer bidding system is less than the price that would have cleared the direct bidding system. As primary dealer di , after observing its customer's bid $(p_{PD}^{ci}, q_{PD}^{ci})$, will reduce the price in some bid-points, which prices are greater than p_{PD}^{ci} to increase the prices in its bid-points that are around p_{PD}^{ci} , the primary dealer system clearing price \tilde{p}_{PD} will be pushed to the left. As the result, this will lead to $\tilde{p}_{PD} \leq \tilde{p}_{DB}$, i.e. the primary dealer bidding system will provide less revenue to Treasury than the direct bidding system.
- (ii) $p_{PD}^{ci} \geq \tilde{p}_{DB}$, i.e. customer ci 's bid-price in the primary dealer bidding system is higher than the price that would have cleared the direct bidding system. As primary dealer di , after observing its customer's bid $(p_{PD}^{ci}, q_{PD}^{ci})$, will reduce the price in some bid-points, which prices are less than p_{PD}^{ci} to increase the prices in its bid-points that are around p_{PD}^{ci} . In this case, the clearing price will be pushed either right or left, depending whether p_{PD}^{ci} is close to or far from \tilde{p}_{DB} respectively. This can be explained as follows.
 - (a) If p_{PD}^{ci} is just greater than \tilde{p}_{DB} , i.e. $p_{PD}^{ci} = \tilde{p}_{DB} + \varepsilon$ (where ε is relatively small), primary dealer di will reduce the prices in bid-points that prices are less than \tilde{p}_{DB} . This will push the clearing price to the right. As the result, this will lead to

²¹ A similar analysis can easily be conducted for primary dealer's strategic bid-points, i.e. bid-points that the prices are higher than the primary dealer's value for the auctioned security.

$\tilde{p}_{PD} \geq \tilde{p}_{DB}$, i.e. the primary dealer bidding system will provide higher revenue to Treasury than the direct bidding system.

- (b) If p_{PD}^{ci} is far greater than \tilde{p}_{DB} , i.e. $p_{PD}^{ci} = \tilde{p}_{DB} + \delta$ (where δ is not too small), primary dealer primary dealer di will reduce the price in some bid-points that the prices are less than \tilde{p}_{DB} . This will push the clearing price to the right. But primary dealer di might also decrease the price in some bid-points that prices are between \tilde{p}_{DB} and p_{PD}^{ci} . This will push the clearing price to the left. As the result, we might have either $\tilde{p}_{PD} \geq \tilde{p}_{DB}$ or $\tilde{p}_{PD} \leq \tilde{p}_{DB}$, i.e. it is not clear either the primary dealer bidding system will provide higher or less revenue to Treasury than the direct bidding system. Intuitively, this sub-case can be explained by the fact that if customer ci bids extremely aggressively, the primary dealer will be discouraged to compete, and hence the primary dealer will submit very low bid-prices. Therefore, auction will clear at a lower price.

Which one of the above cases is more likely to occur? As we explained in the preceding subsection case (i) is less likely to occur. Case (ii) is more likely to occur, but comparing to the alternative direct bidding system it can lead to either higher or less revenue to treasury. Thus, the primary bidding system provides more volatile revenue to Treasury than the alternative direct bidding system.

8. Comments and policy recommendations

A reason that explains why in US Treasury auctions primary dealers use information contained in their customers' bids to formulate their own bids is the following. US Treasury dealers are the major players in the when-issued market, and generally they short-sell large amounts of the security to be auctioned prior to the auction. Therefore, in submitting its own set of bids to Treasury, each primary dealer wants to be awarded enough auctioned securities to cover the short positions it created in its book. Otherwise, it bears the risk of being short squeezed in the post-auction market. So in order to make sure it will be awarded enough Treasury securities as the outcome of the auction, the primary dealer might move its bids a little bit above its customers', but not too far above because its action can push the auction clearing price higher, and thus can make it have a loss when closing its pre-auction short positions or can reduce its benefit in the post-auction market.

Clearly, indirect bidders in US Treasury auction are aware about the use of information extracted from their bids by primary dealers to formulate their own bids. But why do these indirect bidders still submit their bids through primary dealers? It is important to note that indirect bidders might receive some benefits from submitting their bids through primary dealers. For example, Harper and Kruger (2013) report the following from Richard Prager, the global head of trading at BlackRock:²² "Even though BlackRock is aware of primary dealers using information containing

²² BlackRock is the world's largest asset manager firm.

in costumers' bids in their advantage, it prefers to bid indirectly because it wants to reward primary dealers for their research and other trading helps.”²³

Our results show that the primary dealers bid more aggressively as they observe quantity increases in their costumers' bids. In return these costumers, as they are aware of their primary dealers' behavior, bid more aggressively to avoid being front-run. This increases the competition, and hence leads to higher clearing prices. Thus in order to increase its auction revenue, Treasury should encourage direct bidders to bid indirectly, i.e. through primary dealers. As Mackenzie (2014) mention direct bidding makes it harder for primary dealers to gauge demand for Treasury debt sales, and reduces the allure of being a primary dealer. This reduces the competition and Treasury's revenue. A possible extension of this study would be to add direct bidders as the third player type in the Treasury auction game model.

Appendix A: Derivation of the Indirect Bidder's Mollified Demand Schedule: Eq. (3)

From Theorem1, we have the follows.

$$\tilde{Q}_\sigma(p|L) = (\theta_\sigma * Q)(p|L) = \int_{-\infty}^{+\infty} \theta_\sigma(p-t|L)Q(t)dt. \quad A1$$

Since $-L < p-t < L$ implies $p-L < t < p+L$, we have

$$\tilde{Q}_\sigma(p|L) = \Psi(L) \int_{p-L}^{p+L} \phi_\sigma(p-t)Q(t)dt \quad A2$$

By replacing $Q(t)$ with its expression found in Eq. (1), A1 becomes

$$\begin{aligned} \tilde{Q}_\sigma(p|L) &= \Psi(L) \sum_{k=1}^K \left(\int_{p_{k-1}}^{p_k} \phi_\sigma(p-t)Q(t)dt \right) \\ &= \Psi(L) \sum_{k=1}^K \left(\frac{Q_k}{\sigma} \int_{p_{k-1}}^{p_k} \phi\left(\frac{p-t}{\sigma}\right)dt \right) \\ &= -\Psi(L) \sum_{k=1}^K \left(Q_k \int_{\frac{p-p_{k-1}}{\sigma}}^{\frac{p-p_k}{\sigma}} \phi(u)du \right) \\ &= \Psi(L) \left(Q_1 \Phi\left(\frac{p}{\sigma}\right) - Q_K \Phi\left(\frac{p-p_K}{\sigma}\right) - \sum_{k=2}^K (Q_{k-1} - Q_k) \Phi\left(\frac{p-p_{k-1}}{\sigma}\right) \right). \quad A3 \end{aligned}$$

Appendix B: Proof of Proposition 1

²³ In futures markets, Fishman and Longstaff (1992) also find that costumers of dual-trading brokers do better than costumers of non-dual-trading brokers. A dual-trading broker is a broker that acts as an agent (buying and selling for its customer accounts) and simultaneously acting as a dealer (buying and selling for its in-house account).

This proof is in two parts. In the first part, we derive the equation that the primary dealer's demand schedule is a solution for the general case. In the second part, we apply the constant marginal value assumption (Assumption 1) and the risk-neutrality assumption (Assumption 2).

Part 1 of the proof:

Let us rewrite the optimization problem defined in expression (8) as

$$\max_{D_{di}(p)} \int_0^\infty \Psi(D_{di}(p), D'_{di}(p), p) dp, \quad \text{B1}$$

where

$$\begin{aligned} \Psi(D_{di}(p), D'_{di}(p), p) \equiv & \left(U \left(\int_0^{D_{di}(p)} v_{di}(u) du - p D_{di}(p) \right) V \left(\frac{D_{di}(p)}{D_{ci}(p)} \right) \right) \\ & \times \left(H_p(p|D_{di}(p)) + D'_{di}(p) H_D(p|D_{di}(p)) \right), \end{aligned} \quad \text{B2}$$

and

$$\begin{aligned} H_p(p|D_{di}(p)) & \equiv \frac{\partial}{\partial p} H(p|D_{di}(p)), \\ H_D(p|D_{di}(p)) & \equiv \frac{\partial}{\partial D_{di}(p)} H(p|D_{di}(p)), \\ D'_{di}(p) & \equiv \frac{d}{dp} D_{di}(p). \end{aligned}$$

The first order condition of the optimization problem of expression B1 can be obtained through the Euler-Lagrange equation as follows.

$$\frac{\partial}{\partial D_{di}(p)} \Psi(D_{di}(p), D'_{di}(p), p) - \frac{d}{dp} \left(\frac{\partial}{\partial D'_{di}(p)} \Psi(D_{di}(p), D'_{di}(p), p) \right) = 0. \quad \text{B3}$$

From B3, on the one hand we have

$$\begin{aligned} \frac{\partial}{\partial D_{di}(p)} \Psi(D_{di}(p), D'_{di}(p), p) & = U(\cdot) V(\cdot) \left[H_{pD}(p|D_{di}(p)) + D'_{di}(p) H_{DD}(p|D_{di}(p)) \right] \\ & + \left((v_d(D_d(p)) - p) U'(\cdot) V(\cdot) + \frac{1}{D_c(p)} V(\cdot) U(\cdot) \right) \left[H_p(p|D_d(p)) + D'_{di}(p) H_D(p|D_d(p)) \right], \end{aligned} \quad \text{B4}$$

where

$$\begin{aligned} U(\cdot) & \equiv U \left(\int_0^{D_{di}(p)} v_{di}(u) du - p D_{di}(p) \right) \\ U'(\cdot) & \equiv U' \left(\int_0^{D_{di}(p)} v_{di}(u) du - p D_{di}(p) \right) \end{aligned}$$

$$\begin{aligned}
V(\cdot) &\equiv V\left(\frac{D_{di}(p)}{D_{ci}(p)}\right) \\
V'(\cdot) &\equiv V'\left(\frac{D_{di}(p)}{D_{ci}(p)}\right) \\
H_{DD}(p|D_{di}(p)) &\equiv \frac{\partial}{\partial D_{di}(p)} H_D(p|D_{di}(p)) \\
H_{pD}(p|D_{di}(p)) &\equiv \frac{\partial}{\partial p} H_D(p|D_{di}(p))
\end{aligned}$$

From B3, on the other hand we have

$$\begin{aligned}
\frac{d}{dp} \left(\frac{\partial}{\partial D_{di}(p)} \Psi(D_{di}(p), D_{di}'(p), p) \right) &= \frac{d}{dp} \left(U(\cdot) V(\cdot) H_D(p|D_{di}(p)) \right) \\
&= \left[\left(v_{di}(D_{di}(p)) D_{di}'(p) - D_{di}(p) - p D_{di}'(p) \right) U'(\cdot) V(\cdot) \right. \\
&\quad \left. + \frac{D_{di}'(p) D_{ci}(p) - D_{di}(p) D_{ci}'(p)}{D_{ci}^2(p)} V'(\cdot) U(\cdot) \right] H_D(p|D_{di}(p)) \\
&\quad + U(\cdot) V(\cdot) \left(H_{pD}(p|D_{di}(p)) + D_{di}'(p) H_{DD}(p|D_{di}(p)) \right), \tag{B5}
\end{aligned}$$

where

$$D_{ci}'(p) \equiv dD_{ci}(p)/dp$$

Substituting B4 and B5 into B3 leads to the following result.

$$D_{di}(p) = - \frac{\left(v_{di}(D_{di}(p)) - p \right) U'(\cdot) V(\cdot) + \frac{1}{D_{ci}(p)} V'(\cdot) U(\cdot) H_p(p|D_d(p))}{U'(\cdot) V(\cdot) + \frac{D_{ci}'(p)}{D_{ci}^2(p)} V'(\cdot) U(\cdot)} \frac{H_p(p|D_d(p))}{H_D(p|D_d(p))}, \tag{B6}$$

Part 2 of the proof:

According to Assumption 1 we have $v_{di}(D_{di}(p)) = v_{di}$, and according to Assumption 2, for all $(x, y) \in \mathbb{R}_+^2$ we have $U(x) = x$ and $V(y) = y$. Therefore, $U'(x) = 1$ and $V'(y) = 1$. Thus, Equation B6 becomes

$$D_{di}(p) = - \frac{2}{\frac{1}{(v_{di}-p)} + \frac{D_{ci}'(p)}{D_{ci}(p)}} \frac{H_p(p|D_{di}(p))}{H_D(p|D_{di}(p))} \tag{B7}$$

Appendix C: Proof of Proposition 2

When the auction clears, any price p above the auction clearing price \tilde{p} represents an excess supply, i.e. in the point of view of primary dealer di the event $\{\tilde{p} \leq p | D_{di}(p)\}$ can be

$$\text{represented as } \left\{ D_{di}(p) + \sum_{j=1, j \neq i}^N D_{dj}(p) + \sum_{k=1}^N D_{ck}(p) \leq M \right\}.$$

Let us consider the constant marginal value in Assumption. Now in Assumption 3, the demand schedules of indirect bidder ci and primary dealer di can be rewritten (respectively) as

$$D_{ci}(p, s_{ci}) = \alpha_c(p) + \beta_c(h_c^{-1}(v_{ci})), \quad \text{C1}$$

and

$$D_{di}(p, s_{di}, D_{ci}(p, s_{ci})) = \alpha_d(p, D_{ci}(p, s_{ci})) + \beta_d(h_d^{-1}(v_{di})). \quad \text{C2}$$

Hence

$$\{\tilde{p} \leq p | D_{di}(p)\} \equiv \left\{ \theta_{di} \leq M - D_{di}(p) - N\alpha_c(p) - \sum_{j=1, j \neq i}^N \alpha_d(p, D_{cj}(p)) \right\} \quad \text{C3}$$

where $\theta_{di} = \sum_{k=1}^N \beta_c(h_c^{-1}(v_{ci})) + \sum_{j=1, j \neq i}^N \beta_d(h_d^{-1}(v_{cj}))$ is a random variable specific to primary dealer di , and that is assumed to be independent to the price.

From the point of view of primary dealer di , the following expression, $M - D_{di}(p) - N\alpha_c(p) - \sum_{j=1, j \neq i}^N \alpha_d(p, D_{cj}(p))$, is a deterministic function of the price. Let $F_{\theta_{di}}(\cdot)$ and $f_{\theta_{di}}(\cdot)$ be the cumulative distribution function and probability distribution function of θ_{di} respectively. Therefore, we have

$$\begin{aligned} H_D(p | D_{di}(p)) &= \frac{\partial}{\partial D_{di}(p)} \Pr(\tilde{p} \leq p | D_{di}(p)) \\ &= \frac{\partial}{\partial D_{di}(p)} \Pr \left\{ \theta_{di} \leq M - D_{di}(p) - N\alpha_c(p) - \sum_{j=1, j \neq i}^N \alpha_d(p, D_{cj}(p)) \right\} \\ &= \frac{\partial}{\partial D_{di}(p)} F_{\theta_{di}} \left(M - D_{di}(p) - N\alpha_c(p) - \sum_{j=1, j \neq i}^N \alpha_d(p, D_{cj}(p)) \right) \\ &= -f_{\theta_{di}} \left(M - D_{di}(p) - N\alpha_c(p) - \sum_{j=1, j \neq i}^N \alpha_d(p, D_{cj}(p)) \right). \end{aligned} \quad \text{C4}$$

We also have

$$\begin{aligned}
H_p(p|D_d(p)) &= \frac{\partial}{\partial p} \Pr(\tilde{p} \leq p | D_{di}(p)) \\
&= \frac{\partial}{\partial p} F_{\theta_{di}} \left(M - D_{di}(p) - N\alpha_c(p) - \sum_{j=1, j \neq i}^N \alpha_d(p, D_{cj}(p)) \right) \\
&= - \left(\frac{\partial}{\partial p} D_{di}(p) + N \frac{d}{dp} \alpha_c(p) + \sum_{j=1, j \neq i}^N \frac{d}{dp} \alpha_d(p, D_{cj}(p)) \right) \\
&\quad \times f_{\theta_{di}} \left(M - D_{di}(p) - N\alpha_c(p) - \sum_{j=1, j \neq i}^N \alpha_d(p, D_{cj}(p)) \right)
\end{aligned} \tag{C5}$$

From Assumption 3, we have

$$\frac{d}{dp} \alpha_c(p) = \frac{\partial}{\partial p} D_{ci}(p), \tag{C6}$$

and

$$\sum_{j=1, j \neq i}^N \frac{d}{dp} \alpha_d(p, D_{cj}(p)) = (N-1) \frac{d}{dp} \alpha_d(p, D_{ci}(p)) = (N-1) \frac{\partial}{\partial p} D_{di}(p). \tag{C7}$$

Substituting C4, C5, C6 and C7 into the equation in Proposition 2.

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