STRATEGIC BUDGETS IN SEQUENTIAL ELIMINATION CONTESTS

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Abstract. We model endogenous budgets in a sequential elimination contest where contestants (e.g. campaigns) spend resources that are provided by strategic players called backers (e.g. donors). In the unique symmetric equilibrium, backers initially provide small budgets, increasing their contributions only if their contestant wins the preliminary round. If backers are only allowed to provide budgets at the start of the game as opposed to before each round, spending is higher. When unspent resources are refunded to the backer, total spending is higher than when all resources are sunk costs. Where there is an incumbent who is unopposed in the primary stage, we provide new insights into a documented phenomenon known as the incumbency advantage.

1. Introduction

In competitions for political office, donors provide resources to a campaign, and the campaign spends the resources in whatever way will best promote the candidate. In most models of political competition it is assumed that the budgets candidates receive are exogenous and as such the actions of donors are not explicitly modeled. Such a formulation, while adequate if our interest lies only in the spending stage of the game, is a partial analysis. To have a fuller understanding of the origin of budget constraints and the strategic role they can play in political contests, we need a model which generates the budgets candidates receive endogenously. Additionally, from a policy standpoint, a model of political competition with endogenously generated budgets can help a researcher investigate the effect of campaign finance reform on both budgets and expenditures in these contests. To wit, we develop a novel way of endogenizing budgets in a model of political competition by formally including donors in the framework, who act strategically.

Specifically, we focus on a sequential form of competition called an elimination contest. In an elimination contest, two groups of *contestants* compete in two preliminary rounds. The winners of each preliminary round compete for a prize in a final round. In most political competitions, from council elections at local municipalities to presidential elections, candidates have to go through preliminary stages to enter a final race. This contest structure is also common in sports and workplace promotions. Contestants in the elimination contest depend on a set of strategic players, called *backers*, to provide the resources that will be spent in the contest. Each contestant's objective is to spend resources to maximize the probability of winning the final contest, taking into account the other players' strategies. Contestants' spending is constrained by the resources they receive.

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Backers would like to increase the chance of their contestant winning, but the probability of winning is weighed against the cost of providing resources. This formulation highlights the different incentives to spend of the two sets of players. We derive the unique symmetric equilibrium of such a model under a variety of spending rules.

1.1. **Empirical Support.** If backers are allowed to give budgets and replenish them anytime during a contest, we find that, in equilibrium, they will give small budgets initially, increasing their contributions in later rounds if the the contestant is successful. In figure 1 we provide some empirical observations in support of this implication using campaign finance data from presidential elections in the US. The figure illustrates campaign contributions three presidential candidates received in 2008 and 2012.

In 2008 Barack Obama and John Mccain were the Democratic and Republican party nominees respectively.[1] They achieved the nominations by first competing in their party's respective primary elections. Similarly, in 2012, Mitt Romney was the republican presidential nominee.[2] In the figure 1 we have also illustrated some time markers, such as how many states/delegates a nominee had won by certain dates and when the closest competitor dropped out. As can be clearly seen, the contributions to all the eventual nominees were initially low and increased once they were successful in the primaries. This observation suggests that backers are indeed strategic in reality.[3]

1.2. **Policy Implications.** To illustrate the importance of modeling strategic backers, and consequently endogenous budgets, we discuss the effects of two policies. The first is a rule governing how unspent resources are used. Unspent budgets can occur in the elimination contest when a contestant loses the preliminary contest and has some budget remaining. In political contests, the way unspent campaign funds are used after the campaign is often regulated. Although donations can be returned to donors, more often, they are spent in a way that is an imperfect substitute for a refund. For example, the funds may be donated to a charity, another campaign, or a political party. We show that in an elimination contest in which resources are only provided at the beginning of the game, spending increases with the fraction of unspent resources that are returned to backers. If the backers are allowed to provide resources at each stage, we show there will be no unspent resources. We also show that when backers are able to provide resources throughout the contest, they initially give low budgets to the competitors, increasing it to a higher level only if they reach the final stage.

The second policy is whether funds may be replenished between the primary and final contest. We analyze two cases, one in which backers provide resources to contestants only at the beginning of the game and another in which backers are permitted to provide resources at each stage of the

[1]Since George W. Bush was in his second term he was not eligible for reelection

[2]President Obama was in his first term, hence was eligible to compete for a second term

[3]There are other additional reasons why budgets candidates receive increase substantially after they win the nomination. We are suggesting that such behavior is consistent with backers being strategic in their timing of contributing.

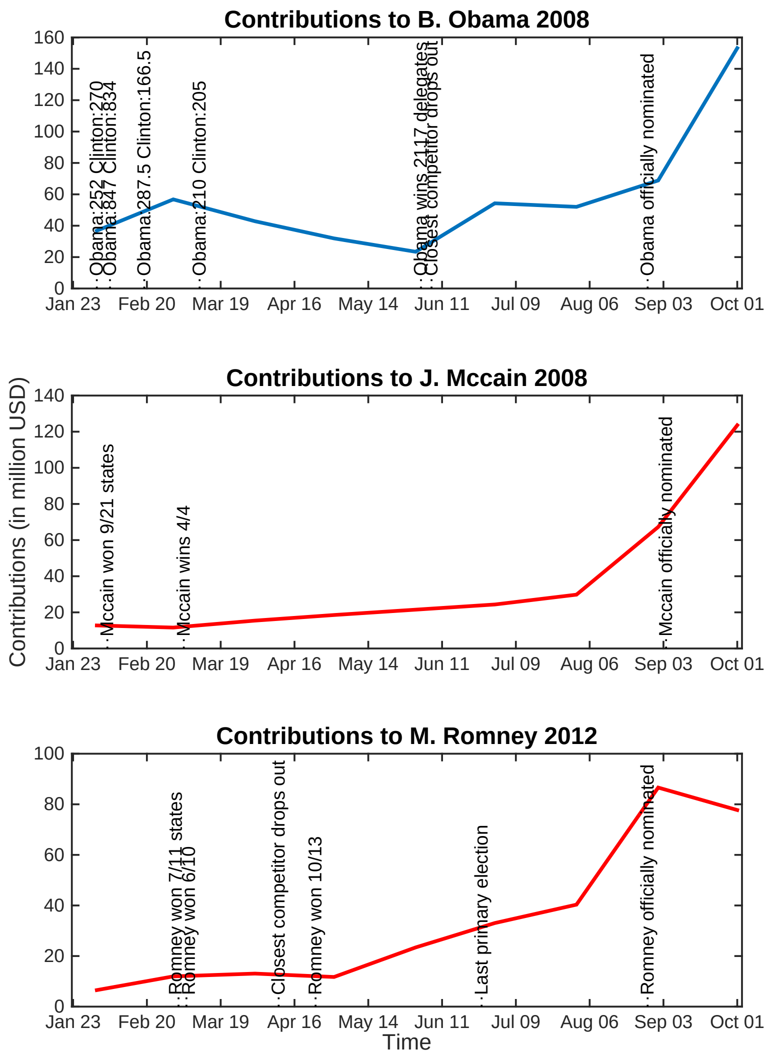

FIGURE 1. Contributions to Presidential Candidates

Source: Federal Elections Commission http://docquery.fec.gov/pres/

contest. We find the unique symmetric equilibrium of each case and show that expenditure is lowest when backers can provide resources throughout the contest. This result suggests that spending will be lower if a sufficient amount of time is provided between stages for contestants to raise more donations.

1.3. **Asymmetric teams and Incumbency Advantage.** We also analyze an asymmetric variation of the game described above. In this game there are two types of contestants. *Challengers* must compete in a primary. The second type of contestant is called an *incumbent* and does not need to participate in the primaries. The backers for the challengers have the same value of winning but the incumbent backer may value winning differently than the challengers. We find the unique equilibrium, and provide a novel explanation of the incumbency advantage.

The incumbency advantage is a widely observed empirical observation that incumbents are more likely to win in the general election than challengers [8, 5, 1, 13]. Stone, Maisel, and Maestas[17] cite several frequently studied sources of the incumbency advantage.[4] These include deterrence of potential challengers, incumbents being higher quality candidates than challengers on average, access to government resources only available to office holders, and district partisanship. An incumbency advantage exists in our model because of another strategic advantage that is not discussed in this literature.[5]

In our model, we study a case where the challenger backers can only give budgets before the start of the primary round but incumbent backers give budgets at its conclusion. In this case, when there is an incumbency advantage, it is due to the fact that challengers have inadequate time to raise and spend funds between the primary and general election and incumbents can observe the actions of the challengers, which is quiet realistic. We find that in equilibrium, strategic donors are reluctant to give a large amount of money to a challenger's campaign before the primary because there is some chance that the donor's chosen challenger will lose in the primary. This allows even moderately weak incumbents who are not challenged in the primary to win the general election more frequently than the challenger who wins the primary. Our model of endogenously determined budgets makes it possible to understand this type of incumbency advantage.[6]

In 2012, the Republican primaries took an unusually long time to select a clear winner. It has been suggested, that this was the reason there was limited time to raise funds for the general election and as such is considered as one of the factors contributing to the struggles of Mitt Romney[7]. As evidence of the importance of having time to receive donations after the primary, the convention that decides the Republican nominee has been moved up by more than five weeks relative to 2012 to insure that the 2016 primary will be formally decided much sooner, and by doing so, the winner will have more time to replenish funds. In a press release [4] that announced the July 18th date for the 2016 Republican National Convention, the Republican National Committee Chairman Reince Priebus stated the following.

> A convention in July is a historic success for our party and future nominee. The convention will be held significantly earlier than previous election cycles, allowing

[4]See the references in Stone, Maisel, and Maestas[17] for more references.

[5]Siegel[15] models the incumbency advantage as a head start. In our model an incumbency advantage exists even without a head start.

[6]Being an incumbent is not always an advantage. We establish conditions under which a weak incumbent faces a disadvantage in equilibrium. Under some conditions, the incumbent is completely deterred from competing

[7]At least amongst Republicans

access to crucial general election funds earlier than ever before to give our nominee a strong advantage heading into Election Day.

1.4. **Previous Literature.** The crucial difference between our model and existing models of dynamic contests with budgets is that the initial budget is endogenously determined by a set of strategic players. Harbaugh and Klumpp[9] and Stein and Rapoport [16] also study elimination contests with budgets but the initial budget is exogenous. In addition, Sela and Erez[14] study a dynamic contest with a different structure, again the initial budget is exogenous and the evolution of the budget is determined by the spending of contestants. Because we model backers who provide the budgets at the beginning and subsequently, we focus on a different set of questions.

A closely related model of a static contest with multiple component contests is Friedman's[7] advertizing contest. Advertizing departments compete against one another in multiple markets simultaneously. The resources that they use to compete are provided strategically by the management of the respective firms. Although our model of budgets is similar, the dynamic interaction in our the sequential elimination contest is central to the questions we address.

More generally the elimination contest is an example of a contest in which the outcome is determined by the outcome of a number of component contests. Borel's formulation of the Colonel Blotto game is another example of such a contest[3]. In the Colonel Blotto game two players simultaneously allocate resources from a exogenous budget to a number of battle fields in order to try to win as many as possible. Borel's static game has been extended by [12] to the case of costly effort and budgets. Our paper fits into this growing literature on contests with multiple components in both static and dynamic frameworks.[8]

2. Elimination Contest Model

Four contestants, $i \in \{1, 2, 3, 4\}$, compete for a prize in a two stage elimination contest. Contestant 1 and 2, and contestant 3 and 4 compete with each other in the preliminary contests (first round). The winners of the preliminary contests compete in a final contest to determine the prize winner. Contestants compete by expending resources in each round. b_i and B_i are contestant i's expenditure (bid), in the preliminary and final round, respectively. Expenditures are constrained to be non negative. The probability of a contestant winning a round, $P_i(\cdot, \cdot)$, is increasing in that contestant's expenditure and decreasing in the other, as modeled by Tullock[18]. This probability is often referred to as the contest success function.

$$P_i(x_i, x_j) = \begin{cases} \dfrac{x_i}{x_i + x_j} \text{ if } x_i + x_j > 0 \\ \frac{1}{2} \text{ otherwise} \end{cases}, \text{ where } (x_i, x_j) \in \{(b_i, b_j), (B_i, B_j)\} \tag{1}$$

[8]See Roberson and Kovenock[11] for survey of the growing literature on static contests with multiple component contests. For a survey of dynamic contests see Konrad[10].

Each contestant's expenditure is constrained by a budget in each round. Contestant i's budget in the preliminary and final contest is given by w_i and W_i, respectively. In each stage, contestants may choose any expenditure between zero and the budget.

The novel contribution of this paper is to introduce a new set of players called *backers* who provide the contestants with *contributions*. There are four backers, one for each contestant.[9] The set of players containing contestant i and backer i is called *team* i. Contestants can only spend from the contributions they recieve. The contributions that backer i provides before the preliminary round and after the preliminary round but before the final round are given by e_i and E_i, respectively. Backers are assumed to not be financially constrained and thus can provide any non negative amount of contributions.

The relationship between contributions, bids, and budgets is the following. The budget available in the preliminary contest is the backer's initial contribution, $w_i = e_i$. The budget in the final contest is the budget in the preliminary contest less the contestant's preliminary bid plus any addition contribution after the preliminary contest, $W_i = e_i - b_i + E_i$.

Team i's probability of winning the prize is given by Q_i and depends on the bids placed by contestants. Suppose contestants i and j compete and contestants k and l compete in the preliminary contests. Then,

$$Q_i = P_i(b_i, b_j)\left(P_k(b_k, b_l)P_i(B_i, B_k) + P_l(b_l, b_k)P_i(B_i, B_l)\right). \tag{2}$$

Within a team, contestants and backers differ in their payoff functions. Contestants are only concerned with maximizing their probability of winning. Hence, Q_i can be viewed as their payoff function. Unlike contestants, backers are concerned with the *expected value* of winning, which depends on both the probability of winning and the costs associated with trying to win the contest. An important consideration in determining the backer's cost is how contributions that are not spent in the elimination contest are used. Specifically, consider the expected unspent budget, i.e. part of the budget that is not used, for competitor i, L_i.

$$L_i = (1 - P_i(b_i, b_j))(e_i - b_i) + P_i(b_i, b_j)(W_i - B_i). \tag{3}$$

We assume that a fraction $r \in [0, 1]$ of the unspent budget is returned to the backer. If r is equal to zero then the unspent budget is entirely a sunk cost, and if r is equal to one then the backer is able to recuperate all the unspent budget. Using Q, L and r, we can write down a backers expected cost function, C.

$$C_i = e_i + P_i(b_i, b_j)E_i - rL_i \tag{4}$$

The ex-post value of winning the prize for a backer i is given by v_i. Then the backer's payoff function is given by U_i.

$$U_i = v_iQ_i - C_i \tag{5}$$

[9] In reality donors give contributions to multiple, competing players. In order to keep things tractable, we focus on the case of each backer having allegiance with exactly one player and leave the general case for future research

The elimination contest has the following structure. In the preliminary round backers make contributions simultaneously. Contestants then observe only their own budgets and choose their bids. The winners of the preliminary round are determined. Each of the backers in the final contest make further contributions, if allowed, after observing all of the bids and contributions in the previous round. The contestants then submit final bids.

In order to isolate the forces at play when backers make their contributions, we will assume that each backer has the same value of winning v. In section 3 we study an extension of the basic model in which we allow asymmetric valuations.

2.1. **Fixed Budgets.** This section restricts attention to a situation in which backers can only contribute at the beginning of the game. That is, E_i is constrained to be zero. In such a game contestants need to decide how to split the budget between the two rounds of the elimination contest. Due to the form of the function P there is no equilibrium in which both contestants bid zero in a round since there would always be a gain to shifting some resources to that round to insure a win there without losing too much probability of winning the other round. Also, there is no equilibrium in which one contestant bids zero in a round since there is always a profitable deviation involving bidding something strictly positive in that round. Finally, in the final contest, contestants always bid their entire remaining budgets. This follows since contestants maximize the probability of winning, and spending more in the last period will always increase the probability of winning. With these observations in place, the following proposition states the main result in this section.

Proposition 2.1. *There is a unique symmetric equilibrium in which backers provide the same contribution e and contestants split budgets evenly between the two stages with*

$$e = \frac{2v}{8-r}; \text{ and } b_i = B_i = \frac{e}{2} \text{ for all } i$$

Proof. First, let us focus on the strategies of the contestants. Suppose all but one contestants were given identical budgets by their backers and these contestants choose to split that budget evenly across the two rounds. The next lemma shows that in such a case, it is optimal for the remaining contestant to also split his budget evenly between the two rounds.

Lemma 2.2. *Suppose $b_2 = b_3 = b_4 = B_2 = B_3 = B_4 = b$. Then, $b_1 = B_1 = \frac{w_1}{2}$ maximizes Q_1 for any $w_1 \geq 0$.*

Proof. Contestant 1's optimization problem is to maximize Q_1 subject to his budget constraint.

$$\max_{b_1 \in [0,w_1]} Q_1 = \max_{b_1 \in [0,w_1]} \left(\frac{b_1}{b_1+b}\right)\left(\frac{w_1-b_1}{w_1-b_1+b}\right) \tag{6}$$

Any solution, b_1^*, to the above will be interior since bidding zero in either round ensures a loss. This observation along with the differentiability of the objective function implies the first order condition for an optimal bid is in fact necessary. This condition is given by

$$\left(\frac{b}{(b_1^*+b)^2}\right)\left(\frac{w_1-b_1^*}{w_1-b_1^*+b}\right)-\left(\frac{b_1^*}{b_1^*+b}\right)\left(\frac{b}{(w_1-b_1^*+b)^2}\right)=0 \tag{7}$$

The unique solution is $b_1^* = \frac{w_1}{2}$. □

Lemma 2.2 shows that if it is an equilibrium strategy for the backers to provide their teammate contestants the same budgets then the contestants will split these budgets evenly across the two rounds. Furthermore, lemma 2.2 implies that if one backer should unilaterally deviate, then her teammate contestant would still split her budget evenly, as long as he thinks that other bidders are doing so. This follows since only the contestant whose backer deviates observes the deviation when preliminary bids are made, and in the final, all contestants bid their remaining budget. To complete the proof of proposition 2.1 we need to show that the backers want to provide identical budgets to their teammates.

The remaining proof is divided into two parts. First, we prove that the prescribed strategies are indeed an equilibrium (existence). Second, we show that the prescribed strategies constitute the unique symmetric equilibrium.

Existence

Suppose all but backer 1 use the same strategy. Using equation (5), lemma 2.2 and $e_2 = e_3 = e_4 = e$, the backer's optimization problem is

$$\max_{e_1\geq 0} U_1 = \max_{e_1\geq 0} v\left(\frac{e_1}{e_1+e}\right)^2 - e_1 + r\left(\frac{e}{e_1+e}\right)\frac{e_1}{2}$$

We know $E_1 = 0$ in the fixed budget case. Also, given the discussion at the beginning of this section, $W_1 - B_1 = 0$.

Since the other budgets are finite, backer 1 will choose a strictly positive contribution amount. Therefore we can describe the optimal contribution level, e_1^* using first order conditions.

$$2\frac{ve_1^*e}{(e_1^*+e)^3} - 1 + \frac{re}{2(e_1^*+e)} - \frac{ree_1^*}{2(e_1^*+e)^2} = 0. \tag{8}$$

Suppose, all but backer 1 provide a budget equal to $\frac{2v}{8-r}$ to their teammate contestants. Without loss of generality, suppose backer 1 provides a budget equal to $e_1^* = \frac{\alpha v}{8-r}$, where $0 \leqslant \alpha \leqslant 8-r$, the latter inequality due to individual rationality. We will show that α equal to 2 is the unique best response for backer 1. Substituting the values for e_i's in equation (8), the first order condition, as a function of α is,

$$\phi(\alpha) = 20\alpha + 4r - 2\alpha r - 8 - \alpha^3 - 6\alpha^2 = 0 \tag{9}$$

It is straightforward to check that $\phi(2) = 0$. In order to show that $\alpha = 2$ is the unique best response for backer 1, let us assume that is this is not the case. Suppose, there exists α', α'', not equal to each other and strictly positive, such that $\phi(\alpha') = \phi(\alpha'') = 0$. Without loss of generality, suppose, $\alpha' < \alpha''$.

If α is a best response for backer 1, then it must satisfy the second order conditions for a maxima, as well as the first. The second order conditions are found by taking the derivative of $\phi(\cdot)$, with respect to α.

$$\phi'(\alpha) = 20 - 2r - 3\alpha^2 - 12\alpha \tag{10}$$

Since α' is a best response, $\phi'(\alpha') \leqslant 0$. This would imply that $\phi'(\bar{\alpha}) < 0$ for all $\bar{\alpha} > \alpha'$.[10] This would imply, $\phi(\bar{\alpha}) < 0$, since $\phi(\alpha') = 0$. However, this would also imply that $\phi(\alpha'') < 0$, a contradiction. Finally, note that $\phi'(2) < 0$. Therefore, $\alpha = 2$ is the unique best response, which implies that the prescribed strategies form an equilibrium.

Uniqueness

We now prove uniqueness of the symmetric equilibrium. A symmetric equilibrium implies $e_1 = e_2 = e$, assuming the other backers are also providing a budget of e_2. Substituting in equation (8) implies

$$\frac{v}{4} - e + \frac{re}{4} - \frac{re}{8} = 0$$

which is solved uniquely by $e = \frac{2v}{8-r}$ □

Proposition 2.1 also shows that the contribution levels, and therefore the spending in each round of the contest, are increasing in the reimbursement rate r. The reason for this is that the higher the fraction of a dollar the backer can expect to recuperate, in the event of a first round loss of their teammate, lower is the marginal cost associated with contributions given to the contestants. Therefore higher r's will be associated with greater contributions.

Given that contributions are increasing in r, a natural question to ask is whether the expected cost of the contest (for the backers) is increasing or decreasing in the reimbursement rate. As r goes up, the contribution amounts rise, however it also leads to large portions of unspent resources being returned to the backer, thereby reducing cost. Therefore, the overall effect of reimbursement on expenditure is not obvious. To charecterize this effect, we can calculate expected cost to a backer in equilibrium using equation (4).

$$C_f = \frac{2v}{8-r} - \frac{r}{2}\left(\frac{v}{8-r}\right) = \frac{v(4-r)}{2(8-r)} \tag{11}$$

It is straightforward to check that C_f declines with r. Therefore, in the symmetric equilibrium expected cost is decreasing in the reimbursement rate, r.

2.2. **Varying Budgets.**

In the fixed-budget elimination contest, studied in the previous section, backers were not able to do anything in the game after the first round. We now turn to a contest where backers have to act in each period. They have to decide the amount of resources they provide their teammates with in each round.

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[10] since $\phi'(\cdot)$ is decreasing in α

This problem is related to a game in which backers directly control bidding so that the contestants are superfluous. Such a game is equivalent to the game studied in [16] where contestants do not have binding budgets and the cost of expenditure to the contestants is linear. The solution to the problem is found through backward induction. The value of winning the final round is the value of the prize. It is well known that with the contest success function the equilibrium in the symmetric case is to submit a bid equal to a fourth of the value of winning, i.e. $\frac{v}{4}$. If both sides use the same strategy, due to symmetry, each wins with probability half. Thus, the value of competing in the final contest is $\frac{1}{2}v - \frac{v}{4}$, which is equal to $\frac{v}{4}$. This means that the value of winning the preliminary contest is $\frac{v}{4}$. So in the preliminary stage all teams bid a fourth of $\frac{v}{4}$ which is $\frac{v}{16}$.

The game described above is inherently different from one we study, where backers can not directly control the contestants. However, as we will show in proposition 2.3, when backers are symmetric in their valuations they can induce the same bids as above even if they do not directly control the bids placed in each round. This will be the case because in equilibrium contestants will spend everything they receive in each round.

Proposition 2.3. *In the unique symmetric equilibrium of the elimination contest,* $e_i = b_i = \frac{v}{16}$ *and* $E_i = B_i = \frac{v}{4}$ *for all* i.

Any symmetric equilibrium of the game has to take one of two forms. Either contestants will spend everything they receive in each round or they will save from their first round budgets. Our result states that the equilibrium must always be the former. To gain some intuition consider the following. Suppose the backers believe that the contestants will spend their entire budgets in each round. In this case, they will supply budgets of $\frac{v}{16}$ and $\frac{v}{4}$ in the preliminary and final round respectively. In order to show that such beliefs are indeed correct, we need to show that given the backers' actions the contestants want to spend their entire budgets. This step shows that the prescribed strategies are an equilibrium. To rule out the other type of equilibrium, and thereby establishing uniqueness, we prove that backers do not have any incentive to give enough contributions in the preliminary round so that the contestants save. We prove the proposition through a series of lemmas, using backward induction.

First, we show that the actions described in the proposition are part of an equilibrium. This is done by deriving the unique equilibrium of the final stage sub-games beginning with sub-games in which all of the preliminary round contributions have been spent (lemmas 2.4). Using backward induction, the actions in the first round are also shown to be optimal (lemma 2.5 and 2.6), conditional on contestants spending their entire budgets each round. The other sub-games, where contestants may not spend their entire budget first round, are analyzed in lemma 2.7. Lastly we show that this is the only symmetric equilibrium. Particularly, there is no equilibrium in which budgets are so large that contestants spend less than their entire budget in the preliminary round (lemma 2.8).

Lemma 2.4. *If contestants spend their entire first round budgets in the first round, then* $E_i = B_i = \frac{v}{4}$.

Proof. The contestants will always bid the entire budget in the final round. Conditional on the contestants spending their entire budgets in the first round, the unique equilibrium of the final sub-game is for both backers to contribute $\frac{v}{4}$. To see this, note that the expected payoff to a backer i in the final round is

$$\frac{E_i}{E_i + E_j} v - E_i$$

The first order conditions for optimal contribution imply

$$E_i^* = \max\left\{0, \sqrt{vE_j} - E_j\right\}$$

$$E_j^* = \max\left\{0, \sqrt{vE_i} - E_i\right\}$$

The above equations describe the best responses of each backer, conditional n the other backer's behavior. E_i^* equal to $\frac{v}{4}$ and E_j^* equal to $\frac{v}{4}$ uniquely solve the above equations. □

The final round sub-games are completely described by the amount of resources each contestant who wins the first round carries over to the final. Without loss of generality, let players 1 and 2 win the initial round and make it to the final round. Then, the sub-games will be represented by the pair $(w_1 - b_1, w_2 - b_2)$.

Lemma 2.5. *If contestants spend their entire first round budgets in the first round, then backers will provide a contribution of* $e_i = \frac{v}{16}$.

Proof. If the contestants arrive in the final round with no budgets then the sub-game in the final round is of the form $(0, 0)$. In this case, we know from lemma 2.4 that the unique Nash equilibrium in the final round is for backers to supply $\frac{v}{4}$ each and for contestants to bid everything. Therefore, using equation (5) backer's payoff is

$$U_1 = \left(\frac{e_1}{e_1 + e_2}\right)\left(\frac{e_3}{e_3 + e_4}\frac{1}{2} + \frac{e_4}{e_3 + e_4}\frac{1}{2}\right) v - \left(e_1 + \frac{e_1}{e_1 + e_2}\frac{v}{4}\right)$$

$$= \left(\frac{e_1}{e_1 + e_2}\right)\frac{v}{4} - e_1 \tag{12}$$

In this sub-game there is no left over budget since the contestants always spend the entire budget they receive each round. It is straightforward to verify that the backer 1's optimal response to e_2 is $\max\{0, \sqrt{\frac{v}{4}e_2} - e_2\}$. Similarly the unique best response of backer 2 is to submit a budget equal to $\max\{0, \sqrt{\frac{v}{4}e_1} - e_1\}$. These two best response functions give rise to a unique optimal strategy in the first period where $e_1 = \frac{v}{16}$ and $e_2 = \frac{v}{16}$ conditional on contestant behavior. □

Lemma 2.6. *If* $e_i = \frac{v}{16}$, *then it is a Nash equilibrium for the contestants to spend their entire first round budgets in the first round.*

Proof. Suppose contestant 1 deviates and spends less than e_1 in the first round. That is he arrives in round 2 with $w_1 - b_1 = e_1 - b_1 > 0$. This implies sub-games of the type $(e_1 - b_1, 0)$ in the second round. Note that the contestants will always bid their entire budget in the final round.

Backer 2's payoff in the final round as a function of her contribution and the budget of the rival's is given by U_2^2, where the super-script represents the final round.

$$U_2^2 = \frac{E_2}{e_1 - b_1 + E_1 + E_2} v - E_2$$

Therefore backer 2's best response, by way of contribution in the final round, is

$$E_2^* = \max\left\{0, \sqrt{v(e_1 - b_1 + E_1)} - (e_1 - b_1 + E_1)\right\} \tag{13}$$

Similarly, backer 1's payoff when deciding final round contributions is given by

$$U_1^2 = \frac{e_1 - b_1 + E_1}{e_1 - b_1 + E_1 + E_2} v - E_1$$

In this payoff function we are using the fact that first round contributions are, in essence, sunk costs, if the contestant wins the first round. Backer 1's optimal contribution in the final round is therefore,

$$E_1^* = \max\left\{0, \sqrt{vE_2} - E_2 - (e_1 - b_1)\right\} \tag{14}$$

From equations (13) and (14) notice that if $e_1 - b_1 \leqslant \frac{v}{4}$, then $E_1^* = \frac{v}{4} - (e_1 - b_1)$ and $E_2^* = \frac{v}{4}$ will be the unique equilibrium strategies. This follows from the earlier discussion where we proved that there are no unilateral deviations in the final stage of the game and the unique equilibrium strategies are for backers to provide a budget of $\frac{v}{4}$. This implies that contestant 1 would never want to save less than $\frac{v}{4}$ in the first round, since doing so only reduces her probability of winning the first round, but leaves the probability of winning the final round untouched. But if $e_1 = \frac{v}{16}$, the contestant can not save the requisite amount. □

Lemma 2.7. *The following strategies constitute a Nash equilibrium,* $e_i = b_i = \frac{v}{16}$ *and* $E_i = B_i = \frac{v}{4}$ *for all* i*.*

Proof. We have already established that if the contestants arrive in the final round with no resources, then strategies in the statement of this lemma are in fact optimal. The only remaining case to consider is where one contestant arrives in the final round with some resources. We are only interested in unilateral deviations so without loss of generality, suppose backer 1 deviates and other backers and contestants use the prescribed strategies and thus the contestants arrive at the final stage with no resources. This implies sub-games of the type $(e_1 - b_1, 0)$ in the second round, with $e_1 - b_1 > 0$. Note that the contestants will always bid their entire budget in the final round.

Also, in the proof of lemma 2.6 we established that if the contestants save, it must be the case that $e_1 - b_1 > \frac{v}{4}$. Therefore, from equation (14) we can conclude, $E_1^* = 0$.[11] Using equation, (13),

[11]The maximum value the function $\sqrt{vE_2} - E_2$ can take is $\frac{v}{4}$

$E_2^* = \max\left\{0, \sqrt{v(e_1-b_1)} - (e_1-b_1)\right\}$. The function $\sqrt{v(e_1-b_1)} - (e_1-b_1)$ is strictly positive for $v \geqslant e_1 - b_1 > \frac{v}{4}$. Therefore $E_2^* = \sqrt{v(e_1-b_1)} - (e_1-b_1)$.[12]

Given the values of E_1^* and E_2^* and the fact that all backers other than backer 1 are providing a budget of $\frac{v}{16}$ in round one, contestant 1's problem is to maximize the following probability of winning with respect to his bid in the first round, b_1

$$Q_1 = \left(\frac{b_1}{b_1+\frac{v}{16}}\right)\left(\frac{e_1-b_1}{E_2^*+e_1-b_1}\right) = \left(\frac{b_1}{b_1+\frac{v}{16}}\right)\sqrt{\frac{e_1-b_1}{v}}$$

Clearly, $b_1 > 0$, otherwise the contestant would surely lose in round one. Since, $e_1 > \frac{v}{4}$, the bid, b_1^*, that maximizes the above probability must be an interior solution. Therefore the first order condition, given below, for an optimal first round bid is necessary.

$$\left.\frac{dQ_1}{db_1}\right|_{b_1=b_1^*} = \frac{v\sqrt{e_1-b_1^*}}{16(b_1^*+\frac{v}{16})^2} - \frac{b_1^*}{2(b_1^*+\frac{v}{16})\sqrt{e_1-b_1^*}} = 0$$

$$\Longrightarrow 16(b_1^*)^2 + 3vb_1^* - 2ve_1 = 0$$

Introducing a convenient transformation, $e_1 = \alpha v$, where $1 \geqslant \alpha > \frac{1}{4}$, the unique positive root of the above equation, and hence the unique pure strategy, conditional on not bidding his entire budget in the first round, played by contestant 1 in round one following a deviation by backer 1 in round one is given by

$$b_1^*(\alpha) = v\left(\frac{\sqrt{9+128\alpha}-3}{32}\right) \tag{15}$$

If contestant 1 uses the above bid in the first round, he will bid $\alpha v - b_1^*(\alpha)$ in the final round. From a previous discussion we know that if this happens, the contestant's own backer will not supply any more funds in the final round (as long as $\alpha v - b_1^*(\alpha) > \frac{v}{4}$, which must be true in the case we are analyzing). The other finalist backer will best respond to this strategy by providing her teammate contestant with a budget equal to $E_2^* = \sqrt{v(\alpha v - b_1^*(\alpha))} - (\alpha v - b_1^*(\alpha))$. In this case, the probability of winning the final round for contestant 1 is

$$\frac{\alpha v - b_1^*(\alpha)}{\alpha v - b_1^*(\alpha) + E_2^*} = \sqrt{\frac{\alpha v - b_1^*(\alpha)}{v}}$$

$b_1^*(\alpha)$ is indeed a best response to the backer providing him with the budget αv if the contestant does not wish to spend his entire budget in the first round and then receive a budget equal to $\frac{v}{4}$ in the final round. To show this let α^* be the value such that the contestant is indifferent between bidding $b_1^*(\alpha^*)$ and the entire budget in round one. That is,

$$\left(\frac{\alpha^* v}{\alpha^* v + \frac{v}{16}}\right)\frac{1}{2} = \frac{b_1(\alpha^*)}{b_1^*(\alpha^*)+\frac{v}{16}}\sqrt{\frac{\alpha^* v - b_1^*(\alpha^*)}{v}} \tag{16}$$

[12] $e_1 > v$ is not individually rational, therefore, $v \geqslant e_1 - b_1$ and we know that $e_1 - b_1 > \frac{v}{4}$ from the previous paragraph

The first term, on both the left hand and the right hand sides, is the probability of winning the first round and the second term is the probability of winning the final round.[13] Equation (16) has a unique solution at $\alpha^* \approx 0.545$. For all $\alpha > \alpha^*$, the right hand side of equation (16) is larger than the left hand side, implying that the contestant prefers to bid $b_1^*(\alpha)$ in round one, thereby saving part of his budget and bidding that remaining amount in the final round. Therefore the contestant's best response to his backers strategy of giving him a budgets equal to αv is summarized by the function $\beta_1(\alpha)$, with $1 \geqslant \alpha \geqslant \frac{v}{4}$

$$\beta_1(\alpha) = \begin{cases} b_1^*(\alpha) \text{ for } \alpha > \alpha^* \\ \alpha v \text{ otherwise} \end{cases} \tag{17}$$

Now we turn to the decision problem of backer 1. Recall, that backer 1 is the player that deviates from the prescribed equilibrium path in the final round. Clearly, supplying a budget less than $\alpha^* v$ would imply that the backer would be back on the equilibrium path. This is true since for any budget level below $\alpha^* v$, the contestant would bid the entire budget in the first round. This would lead the backer to provide $\frac{v}{16}$ to the contestant in the first round, since it is the unique best response to the contestant's and other backers' strategies. Therefore, for a deviating backer $e_i \geqslant \alpha^* v$. We can now write down the deviating backer's payoff as a function of α

$$\begin{aligned} U_1 &= v\left(\frac{b_1^*(\alpha)}{b_1^*(\alpha)+\frac{v}{16}}\right)\frac{\alpha v - b_1^*(\alpha)}{\alpha v - b_1^*(\alpha) + E_2} - \alpha v + r(\alpha v - b_1^*(\alpha))\left(\frac{\frac{v}{16}}{b_1^*(\alpha)+\frac{v}{16}}\right) \\ &= v\left(\frac{b_1^*(\alpha)}{b_1^*(\alpha)+\frac{v}{16}}\right)\sqrt{\frac{\alpha v - b_1^*(\alpha)}{v}} - \alpha v + r(\alpha v - b_1^*(\alpha))\left(\frac{\frac{v}{16}}{b_1^*(\alpha)+\frac{v}{16}}\right) \end{aligned} \tag{18}$$

where the second equality follows from substituting $E_2 = E_2^*$. It is sufficient to focus on the case $r = 1$ since the backer's payoff from inducing the contestant to save is always highest when any remaining resources are returned in the event of a loss in the first round. Substituting $r = 1$ and then carrying out some manipulation (18) implies,

$$U_1 = \left(\frac{b_1^*(\alpha)}{b_1^*(\alpha)+\frac{v}{16}}\right)\left(v\sqrt{\frac{\alpha v - b_1^*(\alpha)}{v}} - \alpha v - \frac{v}{16}\right) \tag{19}$$

The backer's problem is to maximize U_1 with respect to α, where $\alpha \in [\alpha^*, 1]$. The term inside the first parentheses in equation (19) is positive for any value of α. Therefore the sign of the backer's

[13] In the equation, the probability of winning the final round accounts for the best response of the other finalist team, which depends on the resources that contestant 1 brings into the final round

payoff will be the same as sign of the term inside the second parentheses for any value of α.

$$\begin{aligned}\text{sign}\,(U_1) &= \text{sign}\left(v\sqrt{\frac{\alpha v - b_1(\alpha)}{v}} - \alpha v - \frac{v}{16}\right)\\ &= \text{sign}\left(\sqrt{\frac{\alpha v - b_1(\alpha)}{v}} - \alpha - \frac{1}{16}\right)\\ &= \text{sign}\left(\sqrt{\alpha - \frac{\sqrt{9+128\alpha}-3}{32}} - \alpha - \frac{1}{16}\right)\end{aligned}$$

The second equality follows from the fact that v is positive. The third equality is a results of canceling the common terms. Notice that the final term on the right hand side depends only on α. Since $\alpha \in [\alpha^*, 1]$, it is straightforward to check that this term is negative over the relevant range as can be seen in figure 2. Clearly, the deviating strategy gives the backer a negative payoff, violating individual rationality. Therefore we have established that the proposed strategies in the proposition in fact constitute an equilibrium.

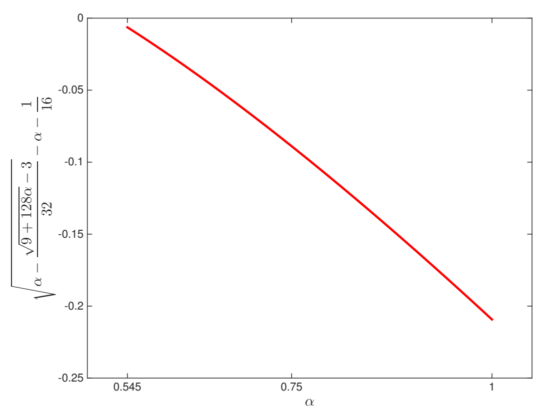

FIGURE 2. U_1 is negative if the backer deviates

□

Lemma 2.8. *The strategies* $e_i = b_i = \frac{v}{16}$ *and* $E_i = B_i = \frac{v}{4}$ *for all* i *constitute the unique symmetric Nash Equilibrium of the elimination game.*

Proof. To show uniqueness, we only need to show that all backers in the initial round never provide a budget different than $\frac{v}{16}$. To begin with, notice that if the backers provide any budget less than $\frac{v}{4}$ in the first round, then the contestants never save. If the contestants don't save then lemma 2.4 establishes the uniqueness of final round strategies. Therefore providing anything less than $\frac{v}{4}$ which is not equal to $\frac{v}{16}$ will clearly not constitute an equilibrium since the contestants will not save anything in this case and the payoff for the backer will be given by equation (12), which leads to the unique best response of $\frac{v}{16}$ for the backers.

Suppose all backers provide budgets greater than $\frac{v}{16}$ to their teammate contestant. The budgets must be high enough that the contestants are induced to save for the final round. That is $e_i > \frac{v}{4}$, for all i. However, we also know, from a previous discussion, that a contestant would never want to save less than $\frac{v}{4}$ for the final round (from her first round budget), because the backer would provide her with a total budget of at least $\frac{v}{4}$ in the final round. Again, from a previous discussions, if $e_i - b_i > \frac{v}{4}$, then from equation (14) we can conclude, $E_i = 0$. So in essence, all the backers provide a budget to their teammates in the first round and nothing thereafter. This makes the game identical to the fixed budget game studied in section 2.1. That game had a unique symmetric equilibrium as stated in proposition 2.1, where the contestants evenly split their budget of $e_i = \frac{2v}{8-r}$ between the two rounds. However, this would imply that $w_i - b_i = \frac{v}{8-r} < \frac{v}{4}$ leading to a contradiction. □

The varying budget game allows a backer to wait and see the result of the first round before giving her teammate with further contributions for the final. This allows the backer to provide a low budget initially and then provide a higher budget their teammate condition on winning the first round. To put this result in perspective of an application, consider a game of sequential elections, such as presidential races. Our result suggests that if donors wait for the outcome of early elections before providing their candidate with funds, it could be due to strategic reasons as opposed to jumping on a the band-wagon of a winning team. As mentioned in the introduction this is consistent with what we observe in the data.

2.3. **Costs and Expenditures.** The total expected cost to a backer is given by equation (4). Suppose, in the fixed budget game the expected cost to a backer is C_f. Since we know the equilibrium strategies of the backers and contestants in this game, it is straightforward to calculate C_f as was done previously in equation (11). We showed that, C_f is decreasing in r, which implies, that the minimum value C_f can take is $\frac{3v}{14}$.[14]

In the varying budget game the expected cost to the backer, given by C_v, is $\frac{3v}{16}$, which is found by substituting the equilibrium contribution and bid levels in equation (4).[15] Clearly $C_f > C_v$ for any value of r. So the varying budget game has significantly lower equilibrium expected costs than the fixed budget game. Allowing a backer to wait, which is what happens in the varying budget game, before committing to a budget for the final round is what allows her to lower her expected cost.

[14] C_f calculated at $r = 1$

[15] This is the same expected expenditure as in a game in which four players compete for the prize in a single contest.[18]

3. Incumbent Vs Challenger

Till now we have studied a game in which the teams were identical. That is, each contestant and backer that formed a team were no different from their counterparts in another team. While this environment is apt in many settings, in many other scenarios, the teams and players may not be identical to each other. Consider the case of presidential elections in the US. If there is an incumbent president residing in office who has not completed two terms, usually that politician will be their party's nominee in the next election. As such, they do not have to contest in the primaries. This is typically not true for the other party's nominee. They usually have to go through a primary first, defeating other challengers before being able to compete against the incumbent.

In order to accommodate scenarios like the one mentioned above consider the following modification to the original model. Each team k still consists of a contestant and a backer. However, now there are three teams instead of four. Teams $k = 1, 2$ are called the *challengers*. These teams still have to go through two rounds of competition to win the prize. In the first round they compete against each other. If they win the first round they go the final round and have to compete against team I, the *incumbent*. Since there are two types of teams, it makes sense to distinguish these teams by letting them value winning the contest differently. As such, let V_c and V_I be the value the challengers and incumbents place on winning, respectively. The notation from the previous section regarding contributions, budgets and bids is carried forward.

Given this formulation, the probability of winning the contest for a team k is given by

$$Q_k = \begin{cases} P_k(b_k, b_j)P_k(B_k, B_I) \text{ for } k = 1, 2; k \neq j \\ P_k(B_k, B_j) \text{ for } k = I; k \neq j \end{cases} \tag{20}$$

The leftover budget, the cost and payoff of the backer is given by equations (3), (4) and (5), respectively.

The elimination contest has a very similar structure to the original game with a few modifications. In the preliminary round challenger backers make contributions simultaneously. Contestants then observe only their own budgets and choose their bids after which the winner of the preliminary contest is determined. In the final round, the backer in the victorious challenger team and incumbent backer make contributions after observing all of the bids and contributions in the previous round. The contestants then submit final bids.

3.1. **Incumbent Behavior.** The incumbent team does not compete in the preliminary round. However, it is able to observe the actions of the backers and contestants competing in the first round. We assume that the incumbent backer provides its teammate contestant with a contribution after the conclusion of the preliminary round.[16] The incumbent contestant cares only about winning, just like the contestants of the original game.

[16] We believe that this is an apt way to model the game with an incumbent. Since the incumbent and his backers are not required to participate in the preliminary round, we delay their decision making till the final round

3.2. **Fixed Budgets.** We assume the challenger backers can only contribute at the beginning of the game. So, $E_k = 0$ for $k = 1, 2$. The main result of this section is stated in the following proposition that characterizes backers' contributions and the contestants' bids in a symmetric equilibrium. The symmetric nature of the equilibrium is with respect to the behavior of challenger backers and challenger contestants.

Proposition 3.1. *There is a unique symmetric equilibrium in which the challenger backers provide the same contribution e_c and the incumbent backer provides a contribution E_I, which are*

$$e_c = \begin{cases} \dfrac{25}{8V_I}\left(\dfrac{V_c}{5-r}\right)^2; & \text{if } V_I > \frac{5}{4(5-r)}V_c \\ 2V_I; & \text{if } \frac{5}{4(5-r)}V_c \geqslant V_I \geqslant \frac{V_c}{4+r} \\ \dfrac{V_c + V_I(4-r)}{4}; & \text{if } V_I < \frac{V_c}{4+r} \end{cases} \tag{21}$$

$$E_I = \begin{cases} \dfrac{5V_c(4V_I(5-r) - 5V_c)}{16V_I(5-r)^2}; & \text{if } V_I \geqslant \frac{5}{4(5-r)}V_c \\ 0; & \text{otherwise} \end{cases} \tag{22}$$

The bids of both the challenger contestants are given by,

$$b_c = \begin{cases} \dfrac{e_c}{2}; & \text{if } V_I > \frac{V_c}{4+r} \\ e_c - V_I; & \text{otherwise} \end{cases} \tag{23}$$

$$B_c = \begin{cases} \dfrac{e_c}{2}; & \text{if } V_I > \frac{V_c}{4+r} \\ V_I; & \text{otherwise} \end{cases} \tag{24}$$

The incumbent contestant bids his entire budget.

Proof. We divide the proof of this proposition into two parts. First, we derive the equilibrium, assuming one exists, and prove its uniqueness. Second, we establish, in the appendix, the existence of an equilibrium in this game by showing the prescribed strategies constitute an equilibrium. To begin with, we establish the optimal behavior of the players in different rounds of the game as a function of parameter values and as a response to each other's actions.

Incumbent Backers and Contestants: Final Round

In the final round $B_I = E_I = W_I$, since there is no benefit to contestants from having funds left over. The other player that moves in the final is the incumbent backer. She knows the remaining budget of the winning challenger at the time of deciding her contribution. With this in mind, the payoff to the incumbent backer is given by

$$U_I = V_I \frac{E_I}{E_I + w_k - b_k} - E_I$$

where k is the winning team from the preliminary round. Maximizing U_I with respect to E_I and then using first order conditions gives a familiar expression for the optimal contribution of the backer as function of her valuation and the opponents action.

$$E_I(w_k - b_k) = \max\left\{0, \sqrt{V_I(w_k - b_k)} - (w_k - b_k)\right\} \tag{25}$$

Challenger Contestants: Preliminary and Final Round

In the final round, the contestants bid their remaining budgets, that is, $B_k = W_k$ for $k = 1, 2$. Without loss of generality, consider the behavior of contestant 1 in the preliminary round. He will respond to e_1 and the opponent's bid b_2 taking into account the behavior in the final sub-game that will be induced if he wins the preliminary round. Knowing the actions of the incumbent backer and incumbent contestant in the final round, we can write down the probability of winning for challenger 1, using equation (20).

$$Q_1 = \begin{cases} \dfrac{b_1}{b_1 + b_2}\left(\dfrac{w_1 - b_1}{w_1 - b_1 + E_I}\right); \text{ if } b_1 \geqslant w_1 - V_I \\ \dfrac{b_1}{b_1 + b_2}; \text{ otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{b_1}{b_1 + b_2}\sqrt{\dfrac{e_1 - b_1}{V_I}}; \text{ if } b_1 \geqslant e_1 - V_I \\ \dfrac{b_1}{b_1 + b_2}; \text{ otherwise} \end{cases} \tag{26}$$

where the second equality follows from $w_1 = e_1$ and substituting the value of E_I in the first. From its functional form, we can observe that Q_1 is continuous. However, it is (only) differentiable almost everywhere. The only point of non-differentiability occurs when $b_1 = e_1 - V_I$. To see this, notice that $b_1 \leqslant e_1 - V_I$ implies the challenger contestant bids weakly more than V_I in the final round, which implies the incumbent provides zero contribution, which ensures a winning probability of one in the final round for the challenger. For $b_1 > e_1 - V_I$, the incumbent will bid a positive amount and therefore the challenger's probability of winning the final round will be $\sqrt{\frac{e_1 - b_1}{V_I}}$. Hence we distinguish between the right hand and left hand slopes of Q_1.[17]

$$\frac{dQ_1}{db_1}_+ = \begin{cases} \dfrac{2b_2(e_1 - b_1) - b_1(b_1 + b_2)}{2(b_1 + b_2)^2\sqrt{V_I(e_1 - b_1)}}; \text{ if } b_1 \geqslant e_1 - V_I \\ \dfrac{b_2}{(b_1 + b_2)^2}; \text{ otherwise} \end{cases} \tag{27}$$

$$\frac{dQ_1}{db_1}_- = \begin{cases} \dfrac{2b_2(e_1 - b_1) - b_1(b_1 + b_2)}{2(b_1 + b_2)^2\sqrt{V_I(e_1 - b_1)}}; \text{ if } b_1 > e_1 - V_I \\ \dfrac{b_2}{(b_1 + b_2)^2}; \text{ otherwise} \end{cases} \tag{28}$$

If $e_1 > V_I$ then probability of winning is increasing in b_1 as long as $b_1 \in [0, e_1 - V_I]$. If $b_1 \in [0, e_1 - V_I]$, then $B_1 \geqslant V_I$, which implies the incumbent is bidding zero and so the challenger always wins the final contest. Thus increases in b_1 within this interval increase the probability of winning

[17] where $\frac{dQ_1}{db_1}_+ = \lim_{\epsilon \to 0} \frac{Q_1(b_1+\epsilon) - Q_1(b_1)}{\epsilon}$ and $\frac{dQ_1}{db_1}_- = \lim_{\epsilon \to 0} \frac{Q_1(b_1) - Q_1(b_1-\epsilon)}{\epsilon}$

the preliminary round without decreasing the probability of winning the final. Therefore, either the optimal first round bid is in the interval $(\max\{0, e_1 - V_I\}, e_1)$ or is equal to $e_1 - V_I$.[18]

Suppose the first round bid that maximizes Q_1, b_1^*, is in $(\max\{0, e_1 - V_I\}, e_1)$. In this interval Q_1 is differentiable. Therefore, the first order conditions imply $\frac{dQ_1}{db_1}\Big|_{b_1=b_1^*} = 0$, which simplifies to

$$(b_1^*)^2 + 3b_1^* b_2 - 2b_2 e_1 = 0$$

This equation has only one positive root which is given in the following equation.

$$b_1^*(b_2, e_1) = \frac{-3b_2 + \sqrt{9(b_2)^2 + 8b_2 e_1}}{2} \tag{29}$$

The function $b_1^*(\cdot,\cdot)$ is contestant 1's best response to a bid by contestant 2, b_2, and also a best response to the backer giving him a budget of e_1. In order for $b_1^*(\cdot,\cdot)$ to be an optimal first round bid, it must also fall in the relevant range. Clearly, $b_1^*(\cdot,\cdot) \geqslant 0$. However, it must also be the case that $b_1^*(b_2, e_1) \geq e_1 - V_I$. That is

$$\frac{-3b_2 + \sqrt{9(b_2)^2 + 8b_2 e_1}}{2} \geq e_1 - V_I \tag{30}$$

Condition (30) is equivalent to $\frac{dQ_1}{db_{1+}}\Big|_{b_1=e_1-V_I} \geq 0$. This along with the fact that the first order condition has a unique positive solution implies that when condition (30) is satisfied, $b_1^*(b_2, e_1)$ is the unique maximizer. When condition (30) is not satisfied, $\frac{dQ_1}{db_{1+}}\Big|_{b_1=e_1-V_I} < 0$, which implies that $e_1 - V_I$ is the unique maximizer of Q_1, as we have already established that Q_1 is strictly increasing in $b_1 \in [0, e_1 - V_I]$

Let $\bar{e}$ be the value of e_1 such that (30) holds with equality. For all $e_1 \leqslant \bar{e}$, condition (30), holds. To see this, note that $\frac{\partial b_1^*(b_2,e_1)}{\partial e_1}$, which is the slope of the left hand side of (30), is less than 1, whereas slope of the right hand side of (30) is equal of 1. Hence the left hand side intersects the right from above.

Challenger Backers: Preliminary round

Without loss of generality, backer 1's payoff, U_1, as a function of the contribution can be defined piecewise over two regions. The first will be smaller contributions, $e_1 \leqslant \bar{e}$, which will be split by her teammate contestant into $b_1^*(b_2, e_1)$ and $e_1 - b_1^*(b_2, e_1)$ between the two rounds. The other region will be where the contestant bids $e_1 - V_I$ in the preliminary round and saves V_I for the final. In this case, the incumbent backer makes no contribution.

$$U_1 = \begin{cases} \underline{U}_1 \equiv \left(\dfrac{b_k^*}{b_k^* + b_l}\sqrt{\dfrac{e_k - b_k^*}{V_I}}\right) V_c - e_k + \left(\dfrac{b_l}{b_k^* + b_l}\right) r(e_k - b_k^*); \text{ for } e_1 \leq \bar{e} \\ \overline{U}_1 \equiv \left(\dfrac{e_k - V_I}{e_k - V_I + b_l} - e_k + \dfrac{b_l}{e_k - V_I + b_l} V_I\right); \text{ for } e_1 > \bar{e} \end{cases} \tag{31}$$

[18] $b_1 \neq 0, b_1 \neq e_1$ due to familiar arguments. Therefore, the bid in the first round that maximizes her probability of winning must be an interior solution

Equilibrium Derivation and Uniqueness

In what follows we derive the symmetric equilibrium and prove its uniqueness. The equilibrium can be characterized by the challenger backers' contribution, e_c. Using the threshold value, $\bar{e}$, there are three cases to be considered: (i)$e_c < \bar{e}$, (ii)$e_c > \bar{e}$ and (iii) $e_c = \bar{e}$. When $e_c = \bar{e}, U_k$ is not differentiable, and hence requires special treatment. In the appendix we prove the existence of the symmetric equilibrium by showing that the strategies stated in the proposition in fact constitute an equilibrium.

Case (i) $e_c < \bar{e}$. In this case we know condition (30) holds and the contestants will bid $b_k^*(\cdot,\cdot)$ as their first round bid. We establish the following lemma regarding contestant's bids.

Lemma 3.2. *In any symmetric equilibrium where $e_c < \bar{e}$, contestants split their budgets equally between the two rounds. That is, if $e_1 = e_2$ then $b_1 = b_2 = \frac{e_k}{2}$.*

Proof. Consider a symmetric equilibrium, $e_1 = e_2 < \bar{e}$. Without loss of generality, this implies $b_1 > e_1 - V_I$ in equilibrium. Therefore, from equation (27) we know that

$$\frac{dQ_1}{db_1} = \frac{2b_2(e_1-b_1) - b_1(b_1+b_2)}{2(b_1+b_2)^2\sqrt{V_I(e_1-b_1)}} = 0$$

$$\Longrightarrow \left(\frac{b_2}{b_1+b_2}\right)(e_1-b_1) - \frac{b_1}{2} = 0 \tag{32}$$

where the equality to zero is true if b_1 is optimal. Taking the above implication for the contestant 2 as well and dividing one equation by the other we get

$$\left(\frac{b_1}{b_2}\right)^2 = \frac{e_1-b_1}{e_2-b_2} \tag{33}$$

If $e_1 = e_2$, but, say, $b_1 > b_2$, the right hand side of the above equation will be less than one and left hand side greater than one. Therefore, $b_1 = b_2$. Substituting in equation (32) implies $b_1 = \frac{e_1}{2}$. □

We use a first order approach to find the equilibrium strategies. The derivative of U_1 with respect to e_1 in this region is given by

$$\frac{d\underline{U}_1}{de_1} = \left(\frac{db_1^*}{de_1}\left(\frac{b_2}{(b_1^*+b_2)^2}\right)\sqrt{\frac{e_1-b_1^*}{V_I}} + \frac{1}{2}\left(\frac{b_1^*}{b_1^*+b_2}\right)\left(\frac{1-\frac{db_1^*}{de_1}}{\sqrt{(e_1-b_1^*)V_I}}\right)\right)V_c - 1 \tag{34}$$

$$- \frac{db_1^*}{de_1}\left(\frac{b_2}{(b_1^*+b_2)^2}\right)r(e_1-b_1^*) + \left(\frac{b_1^*}{b_1^*+b_2}\right)r\left(1-\frac{db_1^*}{de_1}\right)$$

$$= \frac{db_1^*}{de_1}\left(\frac{b_2}{(b_1^*+b_2)^2}\right)\left(V_c\sqrt{\frac{e_1-b_1^*}{V_I}} - r(e_1-b_1^*)\right) - 1 \tag{35}$$

$$+ \left(\frac{b_1^*}{b_1^*+b_2}\right)\left(1-\frac{db_1^*}{de_1}\right)\left(\frac{V_c}{2\sqrt{(e_1-b_1^*)V_I}} + r\right)$$

Using equation (29), which charecterizes the optimal first round bid $b_1^*(\cdot,\cdot)$, the derivative of $b_1^*(b_2, e_1)$ with respect to e_1 is given by

$$\frac{\partial b_1^*}{\partial e_1} = \frac{8b_2}{4\sqrt{9(b_2)^2 + 8b_2e_1}}. \tag{36}$$

Using the fact that in a symmetric equilibrium, $e_1 = e_2$, and lemma 3.2 we get

$$\left.\frac{\partial b_1^*}{\partial e_1}\right|_{e_1=e_2} = \frac{2}{5}. \tag{37}$$

In a symmetric equilibrium $e_1 = e_2 = e$. Using lemma 3.2 and equation (37),(34) reduces to

$$\frac{dU_1}{de_1} = \left(\frac{2}{10\sqrt{2V_Ie}} + \frac{3}{10\sqrt{2V_Ie}}\right)V_c - 1 - \frac{r}{10} + \frac{3r}{10} \tag{38}$$

Setting equation (38) to zero gives a necessary condition for optimal contribution e. The unique solution to the first order condition is

$$e_c = \frac{25}{8V_I}\left(\frac{V_c}{5-r}\right)^2 \tag{39}$$

Using equation (30) with an equality sign and lemma 3.2, we can show that in a symmetric equilibrium $\bar{e} = 2V_I$. For the solution in equation (39), to be in the required region $e_c \leqslant \bar{e} = 2V_I$ which implies that $V_I \geqslant \frac{5V_c}{4(5-r)}$ as is required in the proposition. As stated previously, in the appendix we show that this is indeed an equilibrium strategy.

Case (ii) $e_c > \bar{e}$. In this case $U_1 = \overline{U}_1$. The derivative of U_1 with respect to e_1 is given by,

$$\frac{d\overline{U}_1}{de_1} = \left(\frac{b_2}{(e_1 - V_I + b_2)^2}\right)V_c - 1 - \left(\frac{b_2}{(e_1 - V_I + b_2)^2}\right)rV_I \tag{40}$$

In a symmetric equilibrium $e_1 = e_2 = e_c$. Substituting in the above equation, and equating it to zero, we get

$$e_c = \frac{V_c + V_I(4-r)}{4} \tag{41}$$

Again, noting that in a symmetric equilibrium $\bar{e} = 2V_I$, for the above solution to be in the required region $e_c > 2V_I$ which implies that $V_I < \frac{V_c}{4+r}$ as is required in the proposition.

Case (iii) $e_c = \bar{e}$. As stated previously, in any symmetric equilibrium, $\bar{e} = 2V_I$. Then, a necessary condition for $e_c = \bar{e}$ to be optimal is for the right-hand derivative and left hand derivative of U_k evaluated at $e_k = \bar{e}$ to be non positive. Note, if $e_1 = e_c = 2V_I$, then $b_1^* = V_I$. Then, $\left.\frac{\partial b_1^*}{de_1}\right|_{e_1=e_c} = 1$. Also, since we are considering a symmetric equilibrium, $b_2^* = b_1^* = V_I$. First, consider the right hand slope of U_1. Substituting various values in (34),

$$\left.\frac{dU_k}{de_k}_+\right|_{e_k=2V_I} = \left(\frac{V_I}{(2V_I)^2}\sqrt{\frac{V_I}{V_I}}\right)V_c - 1 - \left(\frac{V_I}{(2V_I)^2}\right)rV_I \tag{42}$$

For $e_1 = e_c$ to be a local maxima, $\frac{dU_1}{de_1}_+\Big|_{e_1=2V_I}$ must be weakly less than zero. This would imply, $V_I \geqslant \frac{V_c}{4+r}$.

Now, consider the left hand slope of U_1. Again, from equation (37) we know that left hand slope of b_1^* with respect to e_1 evaluated at e_c is equal to $\frac{2}{5}$. Carrying out an identical exercise as above we can evaluate the left hand slope of U_1.

$$\frac{dU_1}{de_1}_-\Big|_{e_1=2V_I} = \left(\frac{2}{5}\left(\frac{V_I}{(2V_I)^2}\right)\sqrt{1} + \frac{1}{4}\left(\frac{3}{5\sqrt{(V_I)^2}}\right)\right)V_c - 1 - \frac{2}{5}\left(\frac{V_I}{(2V_I)^2}\right)rV_I + \frac{3}{10}r$$

$$= \frac{V_c}{4V_I} - 1 + \frac{r}{5} \quad (43)$$

For $e_1 = e_c$ to be a local maxima, $-\frac{dU_1}{de_1}_-\Big|_{e_1=2V_I}$ must be weakly less than zero. This implies, $V_I \leqslant \frac{5}{4(5-r)}V_c$. These bounds are as in the statement of the proposition.

The challenger contestants actions are easily stated in terms of e_c using lemma 3.2. The incumbent's contribution level can be found by substituting the value of e_c in equation (25).

□

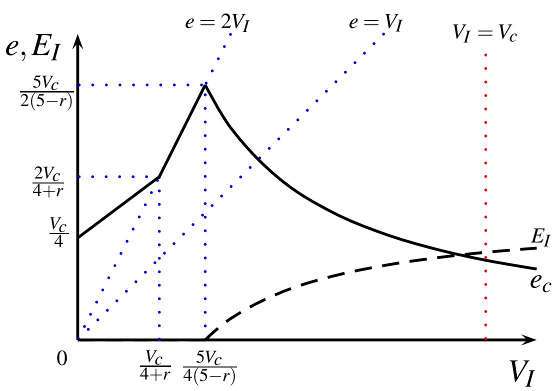

FIGURE 3. Challenger and Incumbent contributions

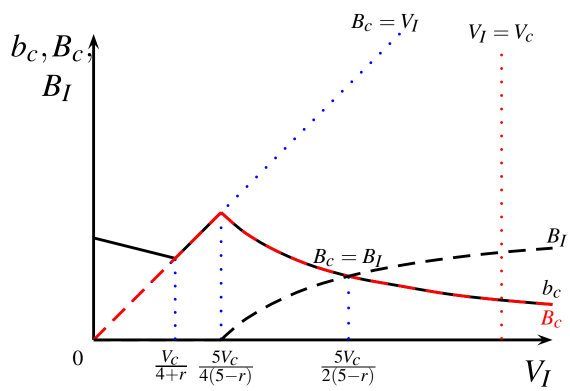

FIGURE 4. Challenger and Incumbent bids

Figures 3 and 4 illustrate the equilibrium budgets and bids as a function of the incumbent's value V_I. An interesting aspect of the challenger contributions e_c is that it is non-monotonic in V_I. Intuitively, if the incumbent's value is low enough, the challenger backers know that the incumbent backer will not invest heavily in her team-mate, the incumbent. Therefore, it is worthwhile for the challenger backer to provide enough resources to her team-mate which ensures a victory in the final round if they were to reach it. In the standard model without budgets, in equilibrium, both contestants always have some positive probability of winning, but with budgets there is a region in which the incumbent does not bid anything. Thus being an incumbent is a great disadvantage for a particularly weak backer/incumbent (low valuations). However, eventually, if V_I is high enough, this is no longer the case.

Also of interest is the probability of winning. As can be seen from figure 4, if $V_I \leqslant \frac{5V_c}{4(5-r)}$, the incumbent has zero probability of the winning the contest. However, this is a particularly weak incumbent. If $V_I \geqslant \frac{5V_c}{2(5-r)}$, the incumbent's probability of winning is greater than or equal to $\frac{1}{2}$, further increasing in V_I. The highest value $\frac{5V_c}{2(5-r)}$ can take is $\frac{5V_c}{8}$. This suggests that as long as the incumbent is not too weak, he will have a higher probability of retaining office. We can think of this as some sort of incumbency advantage. In our model, this advantage is born out of the strategic nature of the backers and the fact that the incumbent backer gets to observe the actions of the other backers and contestants before making her contribution.

3.3. **Varying Budgets.** In the fixed-budget game, we observed the incumbency advantage persists even when $V_c = V_I$. This was due to the advantage the incumbent team had by way of observing the actions of the challengers in the preliminary round. Suppose the challenger backers are now allowed to replenish budgets between rounds. Then, this advantage disappears. To see this, note that if $V_I = V_c$, then, in a sense, for the final round, we are back to the original model where the equilibrium of the game was stated in proposition 2.3. With these strategies, the probability of winning the contest, conditional on reaching the final round, is $\frac{1}{2}$ for both, challenger and incumbent, teams.[19]

3.4. **Costs and Expenditure.** For comparing costs, we assume $V_c = V_I = v$. Then, using proposition 3.1 we can calculate the contributions of the challengers and the incumbents in the fixed budget game. Given the equilibrium behavior of the contestants in the fixed budget game we can also calculate the expected cost for each backer using equation (4). These are stated in Table 1. Similarly, we can calculate the expected costs of the backers in the variable budgets game. For the challenger, this cost has been calculated previously in section 2.3. For the incumbent the expected cost is simply the contribution in the final round, given they are playing a variable budgets game, which is $\frac{v}{4}$.

The total expected cost is twice the challenger's expected cost, since there are two of them, plus the incumbents cost. It is straightforward to check that each of the expected costs in the fixed budget game are increasing in r. Substituting $r = 1$, we can see that the expected costs in the fixed budget game are always less than the expected costs in the variable budget game. These results are the reverse of what we found in the symmetric situation in which there is no incumbent.

The fixed budget game with an incumbent differs from the symmetric game because the incumbent is able to respond to the remaining budget that the challenger brings to the final stage. In the range of bids that occur in equilibrium, the incumbent's best response is increasing in the

[19] For $V_I \neq V_c$, it is straightforward to find the optimal strategies, assuming the challenger contestants spend their entire first round budgets in the preliminary round. These actions are an equilibrium for a wide range of parameter values. We focus on the symmetric case here and in the next subsection to make clear the effect of introducing an incumbent to the standard model.

Table 1. Expected Cost of Backers

	Expected Cost C	
	Fixed Budgets	Variable Budgets
Challenger (C_c)	$\frac{25v(4-r)}{32(5-r)^2}$	$\frac{3v}{16}$
Incumbent (C_I)	$\frac{5v(15-4r)}{16(5-r)^2}$	$\frac{v}{4}$
Total ($2C_c + C_I$)	$\frac{5v(35-9r)}{16(5-r)^2}$	$\frac{5v}{8}$

challenger's final bid (or his budget remaining after the first round). Thus the higher contributions by the challenger backers has an escalation effect on the contributions to the incumbent. The escalation effect implies that for the challengers' backers the marginal benefit of increasing the contribution in the fixed budget game is lower in the incumbent game than in the symmetric version.[20]

The escalation effect explains why the fixed budget games produces lower costs than the varying budget game in the incumbent version whereas the opposite is true in symmetric version. In the incumbent game, the preliminary contributions are lower for challengers than for backers in the symmetric game because of the escalation effect. There is no difference in behavior between varying budget games with or without an incumbent because final round contributions are given simultaneously in both cases.

The challenger backers contribution is increasing in r because a higher value of r lowers the marginal cost of providing resources. Since the contestants split the budget across the stages, this implies higher spending in final round by the challenger and by the incumbent because of the escalation effect. The fact that the challenger backers cost are increasing in r is also attributed to the fact that spending starts out lower than in the symmetric game. This means that the marginal benefit of giving more contributions (measured as additional probability of winning the first round due to higher contributions) is higher. Because of this r increases the contribution at a much higher rate than in the symmetric game.

4. Conclusion

By introducing a new set of strategic players, called backers, we have endogenized budgets for a dynamic contest. In the symmetric case we find sharp predictions about the expected expenditure in the two games. The expenditure is lower when backers can contribute throughout the game than when backers only contribute at the beginning. In the elimination contest we see that total spending increases as the fraction of unused resources returned increases. However, the expected cost to a

[20]The escalation effect is related to the Stackleberg contest games studied in Dixit[6] and Baik and Shogren[2].

backer decreases in the fraction returned. When there is an incumbent we show cases where it is advantageous or disadvantageous to be an incumbent. However, as long as the incumbent is not too weak, in terms of having a low valuation of winning, we find that the incumbency advantage, measured in terms of probability of winning the final round, is present. The strategic nature of the backers is what allows this advantage to persist.

Although our main application is political competition, the model of endogenous budgets is applicable to many other situations. Other applications include research departments competing with each other using funds provided by their institutions and military commanders that depend on their civilian governments for supplies. The central feature is that competitors depend on strategic backers for the resources that they use to compete, and a competitor's current spending affects the incentive for backers to provide resources in the future.

5. Appendix

Existence of Symmetric Equilibrium for Incumbent vs Challenger Contest

Our existence proof shows that the strategies stated in proposition 3.1 constitute an equilibrium. Given the behavior of the incumbent backer, stated in equation (25), we know that the challenger contestants will follow the strategy

$$b_k(b_l, e_k) = \begin{cases} b_k^*(b_l, e_k); & \text{if } e_k \leqslant \bar{e} \\ e_k - V_I; & \text{otherwise} \end{cases} \tag{44}$$

where, given the value of $b_l, \bar{e}$ solves the equation (30) with an equality. We show that the challenger backers do not want to deviate from the prescribed equilibrium strategies if the other backer is following the equilibrium strategy. Since the strategies depend on the parameter values, we will establish existence for one of the cases. The remaining cases follow an identical line of reasoning and similar techniques.

Case (i) $V_I > \frac{5}{4(5-r)} V_c$.

Suppose challenger backer 2 follows the strategies as stated in proposition 3.1. Since a contestant only observes their own backer's actions, we can assume that the challenger contestant also follows the prescribed equilibrium strategies. Now that we have fixed the strategies of team 2, we can find the value $\bar{e}$ for team 1. $\bar{e}$ solves the following equation

$$b_1^*(b_c, \bar{e}) = b_1^*\left(\frac{e_c}{2}, \bar{e}\right) = \frac{-3e_c + \sqrt{9e_c^2 + 16\bar{e}e_c}}{4} = \bar{e} - V_I \tag{45}$$

The value of $\bar{e}$ that solves the above equation is unique. To see this, consider the function $b_1^*(\frac{e_c}{2}, e_1) - e_1 + V_I$. The slope of this function with respect to e_1 is $\frac{2e_c}{\sqrt{9e_c^2+16e_1e_c}} - 1$ which is strictly negative since $e_1 \geqslant 0$.

Suppose challenger backer 1 follows a strategy of giving his team mate a budget of $e_1 = \alpha e_c$. If α is equal to one then the backer is also following the equilibrium strategy. However, if she chooses an $\alpha \neq 1$, then this backer is deviating from the prescribed behavior.

Let α^* be the value of α such that $\alpha^* e_c = \bar{e}$. Substituting in equation (45), we get an equation for α^*.

$$\frac{-3e_c + e_c\sqrt{9+16\alpha^*}}{4} = \alpha^* e_c - V_I \tag{46}$$

Suppose $\alpha^* < 1$. We know that $b_1^*(b_c, \alpha e_c) < \alpha e_c - V_I$, for $\alpha > \alpha^*$, since $b_1^*(b_c, e_1) - e_1 + V_I$ is decreasing in e_1 as shown previously. This would imply $b_1^*(b_c, e_c) < e_c - V_I$, that is $\frac{e_c}{2} < e_c - V_I$. Substituting the value of e_c, this implies, $V_I < \frac{5}{4(5-r)} V_c$, a contradiction to the case we are analyzing. Therefore $\alpha^* \geqslant 1$. For notational ease, define a function $x(\alpha) = \sqrt{9+16\alpha}$. We can re-write $b_1^*(b_c, e_1)$ as a function of α.

$$b_1^*\left(\frac{e_c}{2}, \alpha\right) = e_c\left(\frac{x(\alpha)-3}{4}\right) \tag{47}$$

Let $b_1(b_c, \alpha)$, be contestant 1's optimal strategy as a function of α, conditional on the other backer and contestant playing according to the prescribed equilibrium. Then

$$b_1\left(\frac{e_c}{2}, \alpha\right) = \begin{cases} e_c\left(\frac{x(\alpha)-3}{4}\right); & \text{if } \alpha \leqslant \alpha^* \\ \alpha e_c - V_I; & \text{otherwise} \end{cases} \tag{48}$$

Similarly, we can redefine backer 1's payoff function as a function of α.

$$U_1 = \begin{cases} \underline{U}_1; \text{ for } \alpha \leqslant \alpha^* \\ \overline{U}_1; \text{ otherwise} \end{cases} \tag{49}$$

Since α^* is unique, we can focus on two sub cases, depending on the value of α.

Subcase (i) $\alpha \leqslant \alpha^*$ In this case, from the previous discussion, we know that $b_1 = b_1^*$ and $U_1 = \underline{U}_1$. Using equation (47) we can rewrite $\underline{U}_1$ in equation (31) as

$$\underline{U}_1 = \left(\frac{e_c\left(\frac{x(\alpha)-3}{4}\right)}{e_c\left(\frac{x(\alpha)-3}{4}\right)+\frac{e_c}{2}}\sqrt{\frac{\alpha e_c - e_c\left(\frac{x(\alpha)-3}{4}\right)}{V_I}}\right)V_c - \alpha e_c + \left(\frac{\frac{e_c}{2}}{e_c\left(\frac{x(\alpha)-3}{4}\right)+\frac{e_c}{2}}\right) r\left(\alpha e_c - e_c\left(\frac{x(\alpha)-3}{4}\right)\right)$$

For notational parsimony, let $x = x(\alpha)$. Also, note that $\alpha = \frac{x^2-9}{16}$. Substituting in the above equation and simplifying,

$$\begin{aligned} \underline{U}_1 &= e_c\left(\frac{(x-3)^{\frac{3}{2}}V_c}{4(x-1)^{\frac{1}{2}}\sqrt{e_c V_I}} - \frac{x^2-9}{16} - \frac{r(x-3)}{8}\right) \\ &= e_c\left(\frac{(x-3)^{\frac{3}{2}}(5-r)}{5\sqrt{2}(x-1)^{\frac{1}{2}}} - \frac{x^2-9}{16} - \frac{r(x-3)}{8}\right) \end{aligned} \tag{50}$$

where the second equality follows from substituting the value of e_c inside the parenthesis.

If the backer is to deviate profitably from the prescribed equilibrium strategies, then she must choose an $\alpha \neq 1$, that is $x \neq 5$, that maximizes $\underline{\mathrm{U}}_1$. Since contributing zero can not give a higher payoff than playing the equilibrium strategy[21], if there is a profitable deviation, it must be one where $\alpha > 0$. That is one where $x > 3$. Since the deviating strategy is an interior solution, if one exists, we can find it using first order conditions. The first step in stating the condition is to calculate the derivative of $\underline{\mathrm{U}}_1$ with respect to the choice variable, α, or equivalently x.

$$\frac{d\underline{\mathrm{U}}_1}{dx} = e_c\left(\frac{5-r}{5\sqrt{2}}\left(\frac{3}{2}\left(\frac{x-3}{x-1}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{x-3}{x-1}\right)^{\frac{3}{2}}\right) - \frac{x}{8} - \frac{r}{8}\right)$$

$$= e_c\left(\frac{5-r}{10\sqrt{2}}\left(\frac{x-3}{x-1}\right)^{\frac{1}{2}}\left(\frac{2x}{x-1}\right) - \frac{x}{8} - \frac{r}{8}\right) \tag{51}$$

It is straightforward to check that for $\alpha = 1$, i.e. $x = 5$ the above equation equals zero. Hence $x = 5$ is a critical point. Now, notice that the part of the above equation inside the parenthesis only depends on x and r, and as such is independent of V_c and V_I. Therefore, since $e_c > 0$, we can find the critical x's by setting the equation inside the parenthesis equal to zero. For a fixed r, let this equation be given by the function $S_r(x)$.

$$S_r(x) = \frac{5-r}{10\sqrt{2}}\left(\frac{x-3}{x-1}\right)^{\frac{1}{2}}\left(\frac{2x}{x-1}\right) - \frac{x}{8} - \frac{r}{8} \tag{52}$$

We can graph the function in equation (52) in order isolate the values of x for which the first order conditions are satisfied. In figure 5 we illustrate the value the slope takes for three values of r and various values of α. From the picture it is clear, that for $r \leqslant 1$, $\alpha = 1$, i.e. $x = 5$, is the unique maximizer of $\underline{\mathrm{U}}_1$.

Furthermore, we can evaluate the second order condition and show, formally, that $\alpha = 1$ is a maxima. Deviating (51) with respect to x,

$$\frac{d^2\underline{\mathrm{U}}_1}{dx^2} = e_c\left(\frac{5-r}{10\sqrt{2}}\left(\frac{x-1}{x-3}\right)^{\frac{1}{2}}\left(\frac{6}{(x-1)^3}\right) - \frac{1}{8}\right) \tag{53}$$

Clearly, $\frac{d^2\underline{\mathrm{U}}_1}{dx^2}$ is decreasing in r. Evaluating $\frac{d^2\underline{\mathrm{U}}_1}{dx^2}$ with $r = 0$

$$\frac{d^2\underline{\mathrm{U}}_1}{dx^2} = e_c\left(\frac{3(x-1)^{\frac{1}{2}}}{\sqrt{2}(x-3)^{\frac{1}{2}}(x-1)^3} - \frac{1}{8}\right) \tag{54}$$

Substituing $x = 5$, we can see that $\frac{d^2\underline{\mathrm{U}}_1}{dx^2} < 0$. Therefore $x = 5$, i.e. $\alpha = 1$ is a local maxima.

[21] It is straight forward to check that playing the prescribed equilibrium strategies give a strictly positive payoff to a backer, if the other backers and contestants are playing the equilibrium strategies. Simply substitute $x = 5$ in equation (50) to see that $\underline{\mathrm{U}}_1 > 0$

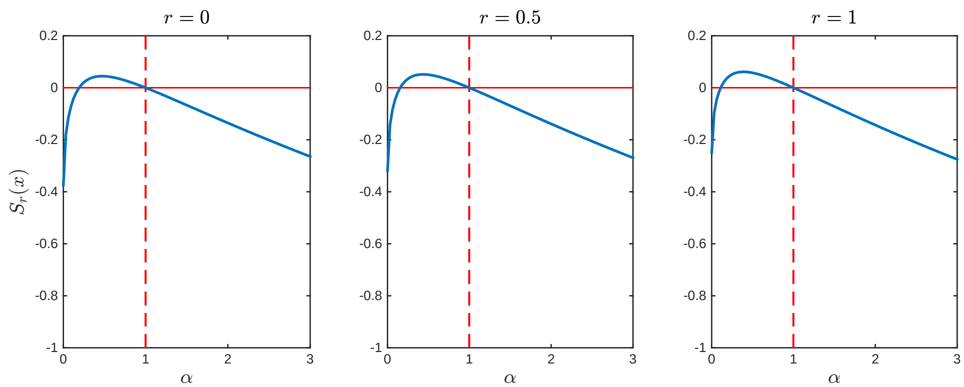

FIGURE 5. $\text{sign}\left(\frac{dU_1}{dx}\right) = \text{sign}(S_r(x))$

Subcase (ii) $\alpha > \alpha^*$ In this case backer 1 is deviating from the equilibrium strategy of $\alpha = 1$ by giving her teammate a much higher budget. We will show that there are no parameter values V_c, V_I and r such that this deviation will occur. We know that α^* uniquely solves equation (46) and $\alpha^* \geqslant 1$. Substituting $x^* = x(\alpha^*) = \sqrt{9 + 16\alpha^*}$ in equation (46) we get

$$16V_I = e_c(x^* - 1)(x^* - 3) \tag{55}$$

This equation gives us a unique value x^* for any V_I, V_c and r combination. Note that $x^* \geqslant 5$, since $\alpha^* \geqslant 1$. Also note that as long as $V_I > 0$ and $V_c > 0$, there will always exist a finite and real value of x^* that solves equation (55). Using this equation we can also evaluate how x^* changes as V_I changes. Taking natural logarithm on both sides does not affect the equality.

$$\ln 16 + \ln V_I = \ln e_c + \ln(x^* - 1) + \ln(x^* - 3)$$
$$\implies \frac{1}{V_I} = \frac{1}{e_c}\frac{de_c}{dV_I} + \frac{dx^*}{dV_I}\left(\frac{1}{x^* - 1} + \frac{1}{x^* - 3}\right)$$

where the implication follows from taking derivatives with respect to V_I. The value for $e_c = \frac{25V_c^2}{8V_I(5-r)^2}$. Then $\frac{de_c}{dV_I} = -\frac{25V_c^2}{8V_I^2(5-r)^2} = -\frac{e_c}{V_I}$. Subsituting in the above equation and simplifying,

$$\frac{dx^*}{dV_I} = \frac{(x^* - 1)(x^* - 3)}{V_I(x^* - 2)} \tag{56}$$

Suppose backer 1 finds it profitable to deviate to a strategy $\hat{\alpha} > \alpha^*$. Let us suppose that $\hat{\alpha}$ maximizes backer 1's payoff if the other backer and contestant are using the prescribed strategies, e_c and b_c respectively. Then, since $\hat{\alpha}$ is interior it must solve the first order condition for the case of $\alpha > \alpha^*$ given in equation (40).[22] We have previously referred to this case as $e_1 > \bar{e}$. Substituting

[22]This is the case since $\hat{\alpha} > \alpha^*$ implies the contestant will bid V_I in the final round

the value for $e_1 = \alpha e_c$ and $b_l = b_2 = b_c = \frac{e_c}{2}$ in equation (40)

$$\frac{d\overline{U}_1}{de_1} = \left(\frac{2e_c}{(2\alpha e_c - 2V_I + e_c)^2}\right)(V_c - rV_I) - 1 \tag{57}$$

Since $\hat{\alpha}$ maximizes $\overline{U}_1$, it must be the case that $\left.\frac{d\overline{U}_1}{de_1}\right|_{\alpha=\hat{\alpha}} = 0$. This implies that $\left.\frac{d\overline{U}_1}{de_1}\right|_{\alpha=\alpha^*} > 0$, since $\frac{d\overline{U}_1}{de_1}$ is decreasing in α. The final inequality implies

$$\begin{aligned} 2e_c(V_c - rV_I) >& (2\alpha^* e_c - 2V_I + e_c)^2 \qquad (58)\\ =& e_c^2\left(2\alpha^* - \frac{2V_I}{e_c} + 1\right)^2 \\ =& e_c^2\left(\frac{16\alpha^* - (x^*-3)(x^*-1)+8}{8}\right)^2 \\ =& e_c^2\left(\frac{x^{*2} - 9 - (x^*-3)(x^*-1)+8}{8}\right)^2 \\ =& e_c^2\frac{(x^*-1)^2}{4} \\ =& \frac{e_c^2}{4}\left(\frac{16V_I}{e_c} + 2(x^*-1)\right) \qquad (59) \end{aligned}$$

The third and sixth equality follow from equation (55). The fourth equality follows from the definition of $x(\cdot)$ and x^*. The final condition implies

$$2(V_c - V_I(r+2)) > e_c\frac{(x^*-1)}{2} \tag{60}$$

Let $f(V_I) = 2(V_c - V_I(r+2))$ and $g(V_I) = e_c\frac{(x^*-1)}{2}$, where the latter is a function of V_I since e_c and x^* are functions of V_I. Note that both $f(\cdot)$ and $g(\cdot)$ are continuous and differentiable.

Suppose there exists parameter values for V_c and V_I that satisfy equation (60). Then it must be the case that $V_I \in \left[\frac{5V_c}{4(5-r)}, \frac{V_c}{r+2}\right)$. The lower bound is established since this is the case we are studying. If $V_I \geqslant \frac{V_c}{r+2}$, then $f(V_I) < 0$, however $g(V_I) > 0$, which implies the inequality in equation (60) can not be true.

Suppose, $V_I = \underline{V}_I = \frac{5V_c}{4(5-r)}$. Then, $f(\underline{V}_I) = \frac{V_c(10-9r)}{2(5-r)}$ and $g(\underline{V}_I) = \frac{5V_c}{5-r}$. The latter is found by substituting the value of V_I in the equation for e_c, which gives $e_c = \frac{5V_c}{2(5-r)}$, and by calculating the value of x^* from equation (55), which gives $x^* = 5$. Clearly, $f\left(\underline{V}_I\right) < g\left(\underline{V}_I\right)$. Now, suppose $V_I = \overline{V}_I = \frac{V_c}{r+2}$. Then $f(\overline{V}_I) = 0$ and $g(\overline{V}_I) > 0$. So, again, $f(\overline{V}_I) < g(\overline{V}_I)$.

Therefore, if the inequality (60) has to be true for some parameter values, then there must exist $\hat{V}_I \in \left(\underline{V}_I, \overline{V}_I\right)$, such that $f(\hat{V}_I) = g(\hat{V}_I)$ and $\left|f'(\hat{V}_I)\right| < \left|g'(\hat{V}_I)\right|$. These conditions follow from the intermediate value theorem, the continuity and differentiability of $f(\cdot)$ and $g(\cdot)$ and the fact that

$f(\underline{V}_I) < g(\underline{V}_I)$. In other words, the function g has to eventually cross f from above in order for the inequality (60) to be true. $f(\hat{V}_I) = g(\hat{V}_I)$ implies

$$2(V_c - \hat{V}_I(r+2)) = \hat{e}_c \frac{(\hat{x}^* - 1)}{2} \tag{61}$$

where $\hat{e}_c = \frac{25}{8\hat{V}_I}\left(\frac{V_c}{5-r}\right)^2$ and $\hat{x}^*$ solves equation (55) with $V_I = \hat{V}_I$.

It is straightforward to show $f'(\hat{V}_I) = -2(2+r)$. We then calculate $g'(V_I)$.

$$\begin{aligned} g'(V_I) =& \frac{1}{2}\left(\frac{de_c}{dV_I}(x^* - 1) + e_c \frac{dx^*}{dV_I}\right) = \left(-\frac{e_c(x^*-1)}{V_I} + \frac{e_c(x^*-1)(x^*-3)}{V_I(x^*-2)}\right) \\ =& -\frac{e_c(x^*-1)}{2V_I(x^*-2)} = -\frac{g(V_I)}{V_I(x^*-2)} \end{aligned}$$

Substituting $V_I = \hat{V}_I$,

$$g'(\hat{V}_I) = -\frac{f(\hat{V}_I)}{V_I(\hat{x}^*-2)} = -\frac{2(V_c - \hat{V}_I(r+2))}{\hat{V}_I(\hat{x}^*-2)} \tag{62}$$

If $\left|f'(\hat{V}_I)\right| < \left|g'(\hat{V}_I)\right|$ is true then

$$\begin{aligned} 2 + r &< \frac{V_c - \hat{V}_I(r+2)}{\hat{V}_I(\hat{x}^*-2)} \\ \implies \hat{V}_I &< \frac{V_c}{(r+2)(\hat{x}^*-1)} \end{aligned} \tag{63}$$

We also know that $\hat{V}_I \geqslant \frac{5V_c}{4(5-r)}$. These inequalities imply $5(r+2)(\hat{x}^*-1) < 4(5-r)$. However, since $r \geqslant 0$ and $\hat{x}^* \geqslant 5$, the final inequality can not be satisfied.

References

[1] Stephen Ansolabehere and James M Snyder Jr. The incumbency advantage in us elections: An analysis of state and federal offices, 1942-2000. *Election Law Journal*, 1(3):315–338, 2002.

[2] Kyung H. Baik and Jason F. Shogren. Strategic behavior in contests: Comment. *The American Economic Review*, 82(1):pp. 359–362, 1992.

[3] E. Borel. The theory of play and integral equations with skew symmetric kernels. *Econometrica*, 21(1):pp. 97–100, 1953.

[4] Republican National Committee. *RNC Announces July 2016 Convention Dates*. 2015.

[5] Gary W Cox and Jonathan N Katz. Why did the incumbency advantage in us house elections grow? *American Journal of Political Science*, pages 478–497, 1996.

[6] Avinash Dixit. Strategic behavior in contests. *The American Economic Review*, 77(5):pp. 891–898, 1987.

[7] L. Friedman. Game-theory models in the allocation of advertising expenditures. *Operations Research*, pages 699–709, 1958.

[8] Andrew Gelman and Gary King. Estimating incumbency advantage without bias. *American Journal of Political Science*, pages 1142–1164, 1990.

[9] R. Harbaugh and T. Klumpp. Early round upsets and championship blowouts. *Economic Inquiry*, 43(2):316–329, 2005.

[10] K. Konrad. Dynamic contests. *WZB Markets and Politics Working Papers*, 2010.

[11] D. Kovenock and B. Roberson. Conflicts with multiple battle fields. *CESifo Working Paper Series*, 2010.

[12] Dmitriy Kvasov. Contests with limited resources. *Journal of Economic Theory*, 136(1):738 – 748, 2007.

[13] Steven D Levitt and Catherine D Wolfram. Decomposing the sources of incumbency advantage in the us house. *Legislative Studies Quarterly*, pages 45–60, 1997.

[14] A. Sela and E. Erez. Dynamic contests with resource constraints. *Working Paper*, 2011.

[15] Ron Siegel. Asymmetric contests with head starts and nonmonotonic costs. *American Economic Journal: Microeconomics*, 6(3):59–105, 2014.

[16] W.E. Stein and A. Rapoport. Symmetric two-stage contests with budget constraints. *Public Choice*, 124(3/4):pp. 309–328, 2005.

[17] Walter J Stone, L Sandy Maisel, and Cherie D Maestas. Quality counts: Extending the strategic politician model of incumbent deterrence. *American Journal of Political Science*, 48(3):479–495, 2004.

[18] G. Tullock. Efficient rent seeking. In James Buchanan, Robert Tollison, and Gordon Tullock, editors, *Toward a Theory of the Rent Seeking Society*, pages 269–282. Texas A&M Press, 1980.

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