Mechanism Design with Interdependent Valuations and Ambiguity*

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We consider a mechanism design setting with multidimensional signals and interdependent valuations. When agents' signals are statistically independent, Jehiel and Moldovanu [22] show that efficient and Bayesian incentive compatible mechanisms generally do not exist. In this paper, we extend the standard model to accommodate ambiguity averse agents. We obtain a characterization theorem for incentive compatible mechanisms. In a single object allocation setting, we exhibit necessary as well as sufficient conditions under which the efficient allocation can be implemented. In particular, we derive a condition that quantifies the amount of ambiguity necessary for efficient implementation. We further show that under some natural assumptions on the preferences, this necessary amount of ambiguity becomes sufficient for efficient implementation. Finally, we provide a definition of informational size such that given any nontrivial amount of ambiguity, the efficient allocation can be implemented if agents are sufficiently informationally small.

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1 Introduction

We consider a mechanism design setting with multidimensional signals and interdependent valuations. If agents are Bayesian and signals are statistically inde-

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pendent, Jehiel and Moldovanu [22] show that, except in some special, nongeneric cases, ex post efficient and interim incentive compatible mechanisms do not exist.[1] This paper investigates how the mechanism design problem is affected by the presence of ambiguity aversion. Specifically, agents are modeled using the maxmin expected utility model of Gilboa and Schmeidler [20]. Our main results provide necessary as well as sufficient conditions under which the ex post efficient allocation in a single object auction can be implemented.

Our first step is to derive a condition, Minimal Ambiguity, which specifies a nontrivial amount of ambiguity that is necessary for inducing truthful revelation of the agents' privately held information. That some ambiguity is necessary is consistent with the impossibility result obtained by Jehiel and Moldovanu [22] in the sense that without ambiguity the requirements of efficiency and incentive compatibility become incompatible.

Minimal Ambiguity quantifies the minimal amount of ambiguity that is sufficient for local incentive compatibility. That is, if Minimal Ambiguity holds, the mechanism designer can construct transfer schemes that satisfy local incentive compatibility constraints. To understand why ambiguity facilitates truthful revelation of private information, we focus on *full insurance transfer schemes*, which were first introduced by Bose et al. [7]. A full insurance transfer scheme ensures that so long as everyone reports truthfully, an agent's ex post utility is a constant function of the other agents' reports. Intuitively, the mechanism designer can use full insurance transfer schemes to induce truth telling: if all agents report truthfully, an agent is fully insured against ambiguity; if an agent misreports his signal, the expected return from misreporting is evaluated according to a worst-case belief—the incentive to lie is thus diminished.

Our next step is to identify conditions under which local incentive compatibility is sufficient for global incentive compatibility in the presence of ambiguity aversion. The question of when local incentive compatibility is sufficient in

[1] Jehiel and Moldovanu [22] generalize earlier results by Maskin [27] and Dasgupta and Maskin [15].

a Bayesian setting has been studied by Carroll [9] and Archer and Kleinberg [2]. The basic idea is to impose a suitable monotonicity condition under which global incentive compatibility constraints are obtained from adding up a sequence of local incentive compatibility constraints. There are two issues we need to address here. The first is to identify the monotonicity condition in our setting. The other is the nonadditivity of the maxmin representation: the belief used in each constraint is endogenously determined and, hence, the sum of these local constraints can differ from the global one. Regarding the first issue, the desired monotonicity condition turns out to be a multidimensional extension of the familiar single crossing condition from one-dimensional settings.[2] Regarding the second issue, if each agent's valuation function is linear in his own signal, such nonadditivity does not arise. Otherwise, the linearity condition on valuation functions can be replaced by two other restrictions on preferences: agents' valuation functions satisfy a familiar increasing differences condition and agents' preferences satisfy the comonotonic independence axiom of Schmeidler [39].

Another contribution of the paper is to identify conditions under which the amount of ambiguity sufficient for efficient implementation can be arbitrarily small. Specifically, we link the required size of ambiguity perceived by an agent to his informational size, a notion studied by McLean and Postlewaite [29, 30, 31, 32]. Intuitively, an agent is informationally small if his private information has a small marginal effect on other agents' valuations. We show that given any nontrivial amount of ambiguity, the efficient allocation can be implemented if agents are sufficiently informationally small. One instance in which informational smallness arises naturally is when each agent's valuation is affected only by the average of the information possessed by the other agents. If the number of agents is large, then each agent's informational size is small. Thus, the efficient allocation can be implemented even when each agent perceives only a small amount of ambiguity.

The paper is organized as follows. In Section 2, we describe the general social choice setting. In Section 3, we obtain a full characterization of when a social choice

[2]See Dasgupta and Maskin [15], Jehiel and Moldovanu [22] and Bergemann and Välimäki [3].

rule is implementable. In Section 4, we provide necessary and sufficient conditions for implementing the efficient social choice rule in a single object allocation setting. In Section 5, we present the results on informational smallness. We conclude with discussion in Section 6 and discuss the related literature in the final section.

2 The Model

Information structure. There are N agents, indexed by $i \in \mathcal{I} := \{1, ..., N\}$. The agents have to make a collective choice k from a set $\mathcal{K} := \{1, ..., K\}$ of possible alternatives. Agent i observes a signal (or type) s^i drawn from a space $S^i \subseteq \mathbb{R}^{K\times N}$. Assume that agent i's signal space S^i is compact and convex. Let $S := \times_{i=1}^N S^i$ with s as generic element and let $S^{-i} := \times_{j\neq i} S^j$ with s^{-i} as generic element.

Ex post payoffs. Assume that agent j's information affecting agent i's utility in alternative k is captured by a one-dimensional signal s^j_{ki}. Thus, agent i's ***valuation for alternative*** k is $v^i_k(s^1_{ki}, ..., s^N_{ki})$. This payoff structure is used by many models that appear in the literature on mechanism design with multidimensional signals, including Jehiel and Moldovanu [22] and Jehiel et al. [23]. Assume that signals are private information and the valuation functions are common knowledge.

Let $s^{-i}_{kj} := (s^1_{kj}, ..., s^{i-1}_{kj}, s^{i+1}_{kj}, ..., s^N_{kj})$ for every $i, j \in \mathcal{I}$, and $k \in \mathcal{K}$. For every $i \in \mathcal{I}$ and $k \in \mathcal{K}$, assume that the family of functions $\{v^i_k(\cdot, s^{-i}_{ki})\}_{s^{-i}_{ki}}$ is *equidifferentiable* at every s^i_{ki}: $\frac{v^i_k(t^i_{ki}, s^{-i}_{ki}) - v^i_k(s^i_{ki}, s^{-i}_{ki})}{t^i_{ki} - s^i_{ki}}$ converges uniformly as $t^i_{ki} \to s^i_{ki}$. Also, assume that $\frac{\partial v^i_k(s^i_{ki}, s^{-i}_{ki})}{\partial s^i_{ki}}$ is nonnegative and continuous in s^i_{ki} for every $s \in S, i \in \mathcal{I}$, and $k \in \mathcal{K}$, and that

$$\sup_{s\in S, i\in\mathcal{I}, k\in\mathcal{K}} \frac{\partial v^i_k(s^i_{ki}, s^{-i}_{ki})}{\partial s^i_{ki}} < \infty. \tag{1}$$

Agents have quasilinear preferences, so that $v^i_k(s^1_{ki}, ..., s^N_{ki}) + x^i$ represents agent i's utility when alternative k is selected and agent i receives a transfer $x^i \in \mathbb{R}$, conditional on all the signals $(s^1, ..., s^N)$.

Mechanisms. A ***social choice rule*** (SCR) is a function $p : S \to \mathbb{R}^K$ such that for every $s \in S$ and $k \in \mathcal{K}$, $0 \le p^k(s) \le 1$ and for every $s \in S$, $\sum_{k=1}^K p^k(s) = 1$. A SCR

p is ***efficient*** if

$$p^k(s) > 0 \Rightarrow k \in \operatorname*{argmax}_{\hat{k}} \sum_{i=1}^{N} v^i_{\hat{k}}(s) \quad \forall s \in S.$$

A ***transfer scheme*** is a function $x : S \to \mathbb{R}^N$. A ***direct revelation mechanism*** is a pair (p, x) where p is a SCR and x is a transfer scheme. For reported signals s, the term $p^k(s)$ is the probability that alternative k is selected and $x^i(s)$ is the transfer to agent i. A direct revelation mechanism is ***efficient*** if the associated SCR is efficient.

Interim payoffs. Let Σ^{-i} be the Borel algebra on S^{-i}. Let $\mathcal{F}^i$ be a set of probability measures on (S^{-i}, Σ^{-i}) that are absolutely continuous with respect to Lebesgue measure. The set $\mathcal{F}^i$, which we assume to be weak* compact, represents agent i's beliefs about the other agents' signals. A key assumption here is that agent i's set of beliefs $\mathcal{F}^i$ is independent of the realization of his signal. Assume that the sets $\mathcal{F}^i$ are common knowledge.

Assume that agent i believes that everyone else reports truthfully and assume that agent i receives signal s^i. Given a direct revelation mechanism (p, x), agent i's ***interim utility*** when he reports signal t^i is

$$u^i(t^i, s^i) := \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} \Big(\sum_{k=1}^{K} p^k(t^i, s^{-i}) v^i_k(s^i_{ki}, s^{-i}_{ki}) + x^i(t^i, s^{-i}) \Big) dF^i.$$

The function $\mu^i : S^i \to \mathbb{R}$ defined by $\mu^i(s^i) := u^i(s^i, s^i)$ for every $s^i \in S^i$, is called agent i's ***indirect utility function*** associated with (p, x).

3 Interim Incentive Compatible Mechanisms

We begin with a characterization theorem for interim incentive compatible mechanisms. In particular, it provides sufficient and necessary conditions for the implementation of the efficient SCR in a general social choice setting. Nonetheless, these conditions might be difficult to verify due to the presence of ambiguity aversion. In Section 4, we restrict our attention to a single object setting and provide simple and easily verifiable conditions on preferences under which the efficient SCR is implementable.

3.1 A Characterization Theorem

By the revelation principle, it is without loss of generality to restrict attention to incentive compatible direct revelation mechanisms. A direct revelation mechanism (p, x) is ***interim incentive compatible*** if

$$u^i(s^i, s^i) \geq u^i(t^i, s^i) \quad \forall s^i, t^i \in S^i, \forall i \in \mathcal{I}.^3$$

A SCR p is ***implementable*** if there exists a transfer scheme x such that the direct revelation mechanism (p, x) is interim incentive compatible.

One difficulty in analyzing mechanism design problems with ambiguity averse agents is that even though the valuation functions are assumed to be differentiable in signals, the nondifferentiability of the resulting interim utility functions arises from the presence of ambiguity aversion.[4] To deal with this problem, we adopt the techniques developed by Carbajal and Ely [8], who study mechanism design problems in which valuation functions are not differentiable.

For every $i \in \mathcal{I}$, $s^i, t^i \in S^i$, and $\alpha \in (0,1)$, define $\zeta^i(s^i, t^i, \alpha) := s^i + \alpha(t^i - s^i)$. Whenever the end points s^i and t^i are clear from the context, we write $\zeta^i(\alpha)$ in place of $\zeta^i(s^i, t^i, \alpha)$ to simplify notation. For every $i \in \mathcal{I}$, $s^i, t^i \in S^i$ and $\alpha \in (0,1)$, let

$$Y^i(s^i, t^i, \alpha, F^i) := \int_{S_{-i}} \sum_k p^k(\zeta^i(\alpha), s^{-i}) \frac{dv^i_k(\zeta^i_{ki}(\alpha), s^{-i}_{ki})}{d\alpha} dF^i.$$

For every $i \in \mathcal{I}$ and $s^i, t^i \in S^i$, define the correspondence $C^i(s^i, t^i, \cdot) : (0,1) \rightrightarrows \mathbb{R}$ as follows:

$$C^i(s^i, t^i, \alpha) := \{z \in \mathbb{R} | \min_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i) \leq z \leq \max_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i)\} \quad \forall \alpha \in (0,1).$$

Theorem 3.1. *The SCR p is implementable with associated indirect utility functions $\{\mu^i\}_{i=1}^N$ if and only if there exist integrable selections $\{c^i(s^i, t^i, \cdot) \in C^i(s^i, t^i, \cdot)\}_{s^i, t^i \in S^i, i \in \mathcal{I}}$ such that*[5]

[3]Observe that the definition of interim incentive compatibility only invokes pure strategies. This is without loss of generality under the assumption that agents cannot commit to randomize. See Wolitzky [42] for a more detailed discussion about this assumption.

[4]It is well known that maxmin preferences have "kinks at certainty". See, for example, Dow and Werlang [18].

[5]For any correspondence $C : (0,1) \rightrightarrows \mathbb{R}$, a ***selection*** c of the correspondence C is a function $\alpha \mapsto c(\alpha)$ such that $c(\alpha) \in C(\alpha)$ a.e. in $(0,1)$.

(i) for every $i \in \mathcal{I}$ and $s^i, t^i \in S^i$,

$$\max_{F^i \in \mathcal{F}^i} \int_{S_{-i}} \sum_k p^k(t^i, s^{-i}) \big(v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i}) \big) dF^i \geq \int_0^1 c^i(s^i, t^i, \alpha) d\alpha$$
$$\geq \min_{F^i \in \mathcal{F}^i} \int_{S_{-i}} \sum_k p^k(s^i, s^{-i}) \big(v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i}) \big) dF^i; \tag{2}$$

(ii) for every $i \in \mathcal{I}$ and $s^i, t^i \in S^i$,

$$\mu^i(t^i) - \mu^i(s^i) = \int_0^1 c^i(s^i, t^i, \alpha) d\alpha;$$

(iii) for every $i \in \mathcal{I}$ and $r^i, s^i, t^i \in S^i$,

$$\int_0^1 c^i(r^i, s^i, \alpha) d\alpha + \int_0^1 c^i(s^i, t^i, \alpha) d\alpha + \int_0^1 c^i(t^i, r^i, \alpha) d\alpha = 0.$$ [6]

The theorem above reduces to Theorem 3.1 in Jehiel and Moldovanu [22] when each agent has a single prior. Our result indicates that implementation with ambiguity averse agents can be easier than Bayesian implementation. More concretely, expanding $\mathcal{F}^i$ from a singleton to a nontrivial set will raise the first term and reduce the last term in (2) simultaneously and, hence, create a wedge so that the inequalities in (2) can be satisfied. Thus, expanding the sets of possible beliefs enlarges the set of implementable social choice rules.

3.2 Full Insurance Mechanisms

In this section, we define a class of transfer schemes, whose main feature is that an agent is fully insured against ambiguity so long as all agents report truthfully. The role of this class of transfer schemes is discussed after Theorem 4.1.

Given the SCR p, define a transfer scheme x_{full} as follows:

$$x_{full}^i(s^i, s^{-i}) := T^i(s^i) - \sum_k p^k(s^i, s^{-i}) v_k^i(s_{ki}^i, s_{ki}^{-i}), \quad \forall s \in S, \forall i \in \mathcal{I}, \tag{3}$$

where $T^i : S^i \to \mathbb{R}$ is the ***reward function*** for agent i. Observe that the transfer scheme x_{full} is constructed so that if everyone reports truthfully, an agent's ex

[6] The characterization theorem implies that the usual payoff equivalence principle may not hold. This point has already been noted by Bodoh-Creed [4] and Wolitzky [42]. Carbajal and Ely [8] state a similar result in a more general mechanism design setting.

post utility is a constant function of the other agents' reports and equal to the reward. Following Bose et al. [7], x_{full} is called a ***full insurance transfer scheme*** and (p, x_{full}) is a ***full insurance mechanism***.

The next result is an immediate consequence of Theorem 3.1. It indicates that a SCR is implementable if and only if it is implementable by a full insurance transfer scheme.

Corollary 3.1. *The SCR p is implementable with associated indirect utility functions $\{\mu^i\}_{i=1}^N$ if and only if p is implementable by a full insurance transfer scheme x_{full} generating the same indirect utility function for each agent.*

4 Implementation of the Efficient Social Choice Rule

In this section, we focus on the implementation of the efficient SCR when there is a single object for sale.

4.1 A Single Object Auction

Consider an auction for a single object. In this auction, each agent reports his signal and the object is awarded to the agent with the highest valuation conditional on all the reported signals. The set of allocations is given by $\mathcal{K} = \{1, ..., N\}$, where $k = i$ denotes the decision to assign the object to agent i. When an agent is not awarded the object, his utility is normalized to 0. As a result, we can simplify notation and consider an N-dimensional signal space for each agent. With a slight abuse of notation, we continue to denote each agent j's signal space by $S^j \subseteq \mathbb{R}^N$. Given a signal $s^j = (s_1^j, ..., s_N^j) \in S^j$, the coordinate s_i^j represents the part of agent j's information affecting agent i's valuation for the object. When agent i is awarded the object, his valuation is $v^i(s_i^1, ..., s_i^N)$. Assume that all the conditions on the signal spaces, beliefs and valuation functions imposed in Section 2 are satisfied. Further, assume that each function v^i is continuously differentiable and increasing in each of its arguments:

$$\frac{\partial v^i(s_i^j, s_i^{-j})}{\partial s_i^j} > 0 \quad \forall s \in S, \forall j \in \mathcal{I}. \tag{4}$$

From now on, we focus exclusively on the implementation of the efficient SCR, which we denote by p_*.

4.2 A Necessary Condition

This section presents a necessary condition for implementing the efficient SCR. We first impose one additional assumption on the signal space. For every $i, j \in \mathcal{I}$, let $S_j^i := \{s_j^i | s^i \in S^i\}$ and $S_j^{-i} := \times_{l \neq i} S_j^l$. For every $i \in \mathcal{I}$ and every $s_i^i \in S_i^i$, let $e(s_i^i) := \{\hat{s}^i \in S^i | \hat{s}_i^i = s_i^i\}$. Assume that each $e(s_i^i)$ is a sublattice in $\mathbb{R}^N$ according to the usual product order.[7]

For every $i \in \mathcal{I}$ and $j \neq i$, define the functions $\underline{s}_j^i : S_i^i \to S_j^i$ and $\overline{s}_j^i : S_i^i \to S_j^i$ as follows:

$$\underline{s}_j^i(s_i^i) := \min_{\hat{s}^i \in e(s_i^i)} \hat{s}_j^i \quad \text{and} \quad \overline{s}_j^i(s_i^i) := \max_{\hat{s}^i \in e(s_i^i)} \hat{s}_j^i.$$

Let $\underline{s}^i(s_i^i) := \big(\underline{s}_1^i(s_i^i), ..., s_i^i, ..., \underline{s}_N^i(s_i^i)\big)$ and $\overline{s}^i(s_i^i) := \big(\overline{s}_1^i(s_i^i), ..., s_i^i, ..., \overline{s}_N^i(s_i^i)\big)$. Since $e(s_i^i)$ is a sublattice, we have $\underline{s}^i(s_i^i), \overline{s}^i(s_i^i) \in e(s_i^i)$ for every $s_i^i \in S_i^i$ and $i \in \mathcal{I}$. By the assumption that $v^j(s_j^i, s_j^{-i})$ increases in s_j^i for every $j \neq i$, we can deduce that for every s^{-i}, the signals $\underline{s}^i(s_i^i)$ and $\overline{s}^i(s_i^i)$ respectively maximize and minimize agent i's probabilities of obtaining the object within each $e(s_i^i)$. That is,

$$p_*^i\big(\overline{s}^i(s_i^i), s^{-i}\big) \leq p_*^i\big(\hat{s}^i, s^{-i}\big) \leq p_*^i\big(\underline{s}^i(s_i^i), s^{-i}\big) \quad \forall \hat{s}^i \in e(s_i^i), \forall s^{-i} \in S^{-i}, \forall s_i^i \in S_i^i.$$

Thus, we can think of $\overline{s}^i(s_i^i)$ as the least favorable signal in $e(s_i^i)$ in terms of obtaining the object, while $\underline{s}^i(s_i^i)$ as the most favorable signal.

Now we are ready to present the key condition of the paper.

[7] Combined with (4), the assumption that $e(s_i^i)$ is a sublattice is not innocuous. For example, it rules out the case in which there exist $j, l \neq i$ such that $s_j^i + s_l^i \leq g(s_i^i)$ for every $s_i^i \in S_i^i$, where $g : S_i^i \to \mathbb{R}$.

Assumption 1 (**Minimal Ambiguity**)**.** For every $i \in \mathcal{I}$ and $s_i^i \in S_i^i$,

$$\max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p_*^i\left(\bar{s}^i(s_i^i), s^{-i}\right) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i \geq \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p_*^i\left(\underline{s}^i(s_i^i), s^{-i}\right) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.$$

To understand the assumption, consider the case when $\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i}$ is independent of s^{-i}. Then, the inequality above becomes

$$\max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p_*^i\left(\bar{s}^i(s_i^i), s^{-i}\right) dF^i \geq \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p_*^i\left(\underline{s}^i(s_i^i), s^{-i}\right) dF^i.$$

The latter inequality permits a simple interpretation: within each $e(s_i^i)$, agent i's best-case belief about obtaining the object when he reports his least favorable signal, should be no less than his worst-case belief when he reports his most favorable signal.

The next lemma shows that Minimal Ambiguity is necessary for implementing the efficient SCR. This result follows from the observation that Minimal Ambiguity can be derived from condition (i) in Theorem 3.1.

Lemma 4.1. *A necessary condition for implementing the efficient SCR is Minimal Ambiguity.*

4.3 Sufficient Conditions With Linear Valuation Functions

To obtain the existence of efficient and interim incentive compatible mechanisms, we adopt the first-order approach that is familiar from Bayesian settings (Mirrlees [35] and Myerson [36]): the first step is to construct mechanisms that satisfy local incentive compatibility constraints; the second step is to look for conditions under which local incentive compatibility is sufficient for global incentive compatibility. Following this approach, we first construct transfer schemes such that the associated efficient mechanisms satisfy local incentive compatibility constraints. Such transfer schemes exist if Minimal Ambiguity holds. We then demonstrate that when the valuation functions are linear, a familiar monotonicity condition guarantees the sufficiency of local incentive compatibility. When the valuation functions are not linear, we pursue another route and provide the details in the next section.

We start with the definition of linearity.

Assumption 2. (**Linearity**) For every $i \in \mathcal{I}$, there exist differentiable functions $g^i : S_i^i \to \mathbb{R}_+$ and $f^i, h^i : S_i^{-i} \to \mathbb{R}_+$ such that

$$v^i(s_i^1, ..., s_i^N) := g^i(s_i^i) f^i(s_i^{-i}) + h^i(s_i^{-i}) \quad \forall s \in S.$$

It should be noted that the above notion of "linearity", which is also used by Carroll [9] and Archer and Kleinberg [2], is very permissive. For example, additively or multiplicatively separable valuation functions are linear in the sense of Assumption 2.[8]

Next, we formalize the notion of monotonicity. Let $S^i(s^{-i})$ be the subset of S^i for which it is efficient to award the object to agent i given the reports s^{-i}:

$$S^i(s^{-i}) := \{s^i \in S^i | v^i(s_i^i, s_i^{-i}) \geq \max_{j \neq i} v^j(s_j^i, s_j^{-i})\}.$$

Assumption 3 (**Monotonicity**)**.** For every $i \in \mathcal{I}$, $s^{-i} \in S^{-i}$, and $s_i^i, t_i^i \in S_i^i$ such that $s_i^i < t_i^i$,

$$\underline{s}^i(s_i^i) \in S^i(s^{-i}) \Rightarrow \underline{s}^i(t_i^i) \in S^i(s^{-i}) \quad \text{and} \quad \bar{s}^i(s_i^i) \in S^i(s^{-i}) \Rightarrow \bar{s}^i(t_i^i) \in S^i(s^{-i}).$$

Equivalently, Monotonicity says that $p_*^i(\underline{s}^i(s_i^i), s^{-i})$ and $p_*^i(\bar{s}^i(s_i^i), s^{-i})$ are non-decreasing in s_i^i. It extends the familiar single crossing condition to settings with multidimensional signals.

Theorem 4.1. *Assume Linearity and Monotonicity. The efficient SCR is implementable if and only if Minimal Ambiguity is satisfied.*

To gain some intuition about the role of ambiguity in obtaining the possibility result, it is helpful to consider full insurance mechanisms. In the context of this section, a full insurance transfer scheme takes the following form:

$$x_{full}^i(s^i, s^{-i}) = T^i(s^i) - p_*^i(s^i, s^{-i}) v^i(s^i, s^{-i}) \quad \forall s \in S, \forall i \in \mathcal{I},$$

[8] Agent i's valuation function is ***additively separable*** if the function f^i in Assumption 2 is constant; similarly, agent i's valuation function is ***multiplicatively separable*** if the function h^i is constant.

where $T^i : S^i \to \mathbb{R}$ is the reward function. Under this transfer scheme, the agent who is awarded the object pays his valuation conditional on all the reports and *every* agent receives a reward which is solely a function of his report. By construction, agent i's indirect utility is

$$\mu^i(s^i) := \min_{F^i \in \mathcal{F}^i} \int_{S_{-i}} \left(p^i_*(s^i, s^{-i}) v^i(s^i, s^{-i}) + x^i_{full}(s^i, s^{-i}) \right) dF^i = T^i(s^i) \quad \forall s^i \in S^i. \tag{5}$$

Thus, if all agents report truthfully, agent i's interim utility is equal to the reward in the full insurance transfer scheme regardless of his beliefs. If agent i misreports, there are two cases to consider. The first case is when agent i's report results in a lower valuation. Then he pays a price lower than his true valuation and, hence, obtains a surplus conditioning on obtaining the object. Consequently, ambiguity aversion drives agent i to assign the *lowest* probability of obtaining the object. Likewise, when agent i reports a signal that induces a higher valuation, he pays a price higher than his true valuation. Hence, he suffers a loss conditioning on obtaining the object. Due to ambiguity aversion, he assigns the *highest* probability of obtaining the object. More generally, agent i is fully insured against ambiguity when he reports truthfully; however, when he misreports, he evaluates the expected return according to a worst-case belief. This observation leads to the conclusion that the presence of ambiguity aversion weakens the interim incentive compatibility constraints when full insurance transfer schemes are used. In fact, Minimal Ambiguity specifies a minimal amount of ambiguity such that there exist transfer schemes that satisfy local incentive compatibility constraints.

The role of Monotonicity and Linearity is to guarantee that local incentive compatibility is sufficient for global incentive compatibility and consequently, the efficient SCR is implementable with a minimal amount of ambiguity. We should point out that Monotonicity is *not* necessary for implementing the efficient SCR in a setting with ambiguity averse agents. For example, in the extreme case of complete ambiguity,[9] the efficient SCR is always implementable. In contrast, Dasgupta and Maskin [15] and Bergemann and Välimäki [3] show that Monotonicity is necessary

[9]Complete ambiguity means each agent's set of beliefs contains *all* probability measures over the other agents' signals.

in a Bayesian mechanism design problem with one-dimensional signals. A further discussion of the role played by Linearity is presented in Section 4.4.

Another observation from Theorem 4.1 is that complete ambiguity is not required for implementing the efficient SCR. It is important to emphasize that this result depends on agents' utilities being quasilinear in transfers. Only then, the mechanism designer can use transfers as instruments to fully insure agents against ambiguity under honest reporting but induce uncertainty otherwise. In contrast, de Castro and Yannelis [16] find that in a nonquasilinear environment, the efficient SCR is implementable if and only if agents have complete ambiguity.

4.4 Sufficient Conditions With Nonlinear Valuation Functions

When the valuation functions are not linear, Monotonicity alone cannot guarantee that local incentive compatibility implies global incentive compatibility. This section provides conditions on beliefs and valuation functions that restore the sufficiency of local incentive compatibility.

We begin with an example showing how ambiguity aversion, while necessary for implementing the efficient SCR, may cause complications in establishing the sufficiency of local incentive compatibility constraints. Suppose that Monotonicity holds and signal spaces are one-dimensional. Take $r^i, s^i, t^i \in S^i$ such that $r^i < s^i < t^i$. Suppose that when agent i receives signal t^i, he does not want to report s^i; when he receives signal s^i, he does not want to report r^i. These can be interpreted as "local" incentive compatibility constraints. However, these two incentive constraints do not necessarily imply that reporting r^i when he receives signal t^i is not profitable, which can be viewed as a "global" incentive compatibility constraint. To see why local incentive compatibility constraints fail to be sufficient, observe that each misreport maps the reports of the other agents s^{-i} to a return. Thus, each misreport can be interpreted as an asset, which is a mapping from S^{-i} to $\mathbb{R}$.[10] Heuristically, the asset that agent i reports r^i when his signal is

[10]Suppose that the mechanism designer uses a full insurance transfer scheme with reward functions $\{T^i\}_{i=1}^N$. If agent i reports s^i after receiving t^i, this misreport or asset is $T^i(s^i) + p^i_*(s^i, \cdot)\big(v^i(t^i_i, \cdot) -$

t^i is a combination of the two local misreports, or assets, previously defined. If combining the two assets leads to a reduction in ambiguity, then it is possible that the combination yields an expected positive return even when the expected return from each asset is negative.[11] It is the possibility of hedging that poses problems for deriving global incentive compatibility from local incentive compatibility.

There are three circumstances in which such hedging does not arise. One is when the agent has a single prior, that is, in a subjective expected utility framework: regardless of the valuation functions, the standard independence axiom precludes the possibility of hedging. The second is when valuation functions are linear in the sense of Assumption 2. Intuitively, Linearity ensures that the assets from distinct local misreports are affine transformation of each other. Within maxmin models, no matter how the set of beliefs is specified, perfectly correlated assets cannot hedge one another. Therefore, no restriction other than Minimal Ambiguity is imposed on the beliefs to obtain the possibility result in Section 4.3. The two circumstances discussed above impose strong restrictions either on the beliefs, or on the valuation functions. When neither of the two restrictions are satisfied, that is, agents perceive a nontrivial amount of ambiguity and their valuation functions are not linear, we pursue another route. Namely, we impose weaker joint restrictions on beliefs and on valuation functions: each agent's preference satisfies comonotonic independence axiom of Schmeidler [39] and each agent's valuation function satisfies a suitably defined increasing differences condition. Comonotonic independence is a standard assumption in the decision theoretic literature;[12] the increasing differences condition is also a familiar restriction on valuation functions in the mechanism design literature. We show that when both restrictions are imposed, there is no possibility of hedging when combining a sequence of local misreports.

We first provide the representation of preferences that satisfy comonotonic in-

$v^i(s^i_i, \cdot)) - T^i(t^i) : S^{-i} \to \mathbb{R}$.

[11] The possibility of hedging from maxmin preferences is captured by the axiom of uncertainty aversion in Gilboa and Schmeidler [20].

[12] Preferences that satisfy comonotonic independence axiom are the intersection of maxmin and Choquet expected utility model.

dependence axiom. Let (Ω, Σ) be a measurable space and let $\Delta(\Omega)$ be the set of all probability measures on (Ω, Σ). We use Prokhorov metric to measure the distance between two probability measures.[13] A ***capacity*** is a function $\nu : \Sigma \to [0, 1]$ such that (i) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$; (ii) $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. A capacity is ***convex*** if it also satisfies

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B) \quad \forall A, B \in \Sigma.$$

The ***core of a capacity*** ν is

$$core(\nu) := \{\pi \in \Delta(\Omega) | \pi(A) \geq \nu(A), \forall A \in \Sigma\}.$$

Preferences that satisfy comonotonic independence axiom can be represented by maxmin expected utility with the agent's set of priors being the core of a convex capacity.[14] Thus, the desired assumption can be stated as follows.

Assumption 4 (Comonotonic Independence). For every $i \in \mathcal{I}$, agent i's set of beliefs $\mathcal{F}^i$ is the core of a convex capacity.

The ε-contamination model provides a natural class of preferences where an agent's set of beliefs is a core.[15] Given $F^i \in \Delta(S^{-i})$ and $\varepsilon \in (0, 1]$, agent i's set of beliefs $\mathcal{F}^i$ is

$$C_\varepsilon(F^i) := \{(1 - \varepsilon)F^i + \varepsilon H^i | H^i \in \Delta(S^{-i})\}.$$

Intuitively, agent i puts a weight of $1 - \varepsilon$ on the other agents' signals being drawn from the distribution F^i, but puts ε weight that the signals could be drawn from any other distribution. Thus, $1 - \varepsilon$ can be interpreted as the agent's confidence in his belief and ε captures the amount of ambiguity that the agent perceives. It can

[13] For any two probability measures $F^i, G^i \in \Delta(S^{-i})$, the Prokhorov metric is

$$d(F^i, G^i) := \inf\{\varepsilon > 0 | F^i(A) \leq G^i(A^\varepsilon) + \varepsilon, \forall A \in \Sigma^{-i}\},$$

where $A^\varepsilon := \{s^{-i} \in S^{-i} | \inf_{\hat{s}^{-i} \in A} \parallel s^{-i} - \hat{s}^{-i} \parallel \leq \varepsilon\}$.

[14] Shapley [40] shows that the core of a convex capacity is not empty.

[15] The axiomatic foundation for the ε-contamination model is provided by Kopylov [24]. Bose et al. [7] and Bose and Daripa [5] adopt this formulation to study the problem of optimal auction design.

be easily verified that $C_\varepsilon(F^i)$ is the core of the convex capacity

$$\nu^{F^i}(A) := (1-\varepsilon)F^i(A) \quad \forall A \in \Sigma \setminus \{\Omega\} \quad \text{and} \quad \nu^{F^i}(\Omega) := 1.$$

Next, we introduce the assumption on valuation functions. The valuation function $v^i : S \to \mathbb{R}$ has *increasing differences* if for all $s_i^{-i}, \hat{s}_i^{-i} \in S_i^{-i}$ and all $r_i^i, s_i^i, t_i^i \in S_i^i$ such that $r_i^i < s_i^i < t_i^i$, we have $v^i(t_i^i, s_i^{-i}) - v^i(t_i^i, \hat{s}_i^{-i}) \geq v^i(s_i^i, s_i^{-i}) - v^i(s_i^i, \hat{s}_i^{-i})$ implies that $v^i(s_i^i, s_i^{-i}) - v^i(s_i^i, \hat{s}_i^{-i}) \geq v^i(r_i^i, s_i^{-i}) - v^i(r_i^i, \hat{s}_i^{-i})$.

Assumption 5 (Increasing Differences). For every $i \in \mathcal{I}$, agent i's valuation function v^i has increasing differences.

Theorem 4.2. *Assume Monotonicity, Comonotonic Independence and Increasing Differences. The efficient SCR is implementable if and only if Minimal Ambiguity is satisfied.*

We give a heuristic argument to illustrate the role played by Comonotonic Independence and Increasing Differences using the example at the beginning of this section. We first introduce comonotonic functions. Two functions $g, h : \Omega \to \mathbb{R}$ are ***comonotonic*** if

$$\big(g(\omega) - g(\omega')\big)\big(h(\omega) - h(\omega')\big) \geq 0 \quad \forall \omega, \omega' \in \Omega.$$

Increasing Differences implies that the two assets—reporting r^i when agent i's signal is s^i and reporting s^i when agent i's signal is t^i—are comonotonic. Schmeidler [39] shows that if an agent's preference satisfies comonotonic independence axiom, combining two comonotonic functions does not reduce ambiguity. Thus, if neither asset generates an expected positive return, then a combination of them—reporting r^i when agent i's signal is t^i—does not generate an expected positive return either.

5 Informational Size

A natural question to ask is under what conditions the efficient SCR is implementable with an arbitrarily small amount of ambiguity. This section addresses this question by linking the required amount of ambiguity perceived by an agent to his informational size.

Our definition of informational size is a counterpart of the notion introduced in McLean and Postlewaite [30, 31, 32]: it measures the degree to which one agent's signal can affect the valuations of other agents. Formally, define the ***informational size*** of agent i as $\max_{j\neq i, s\in S} \frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j}$.[16] Observe that in a model with private values, the informational size of each agent is 0.

One final assumption on the spaces of signals is imposed before stating the main result of this section. The role of this assumption is discussed after the theorem.

Assumption 6 (**Lipschitz Continuity**). For every $i \in \mathcal{I}$ and $j \neq i$, the functions $\underline{s}^i_j$ and $\overline{s}^i_j$ are Lipschitz continuous.

Theorem 5.1. *Assume Linearity and Lipschitz Continuity. For every $\varepsilon \in (0, 1]$, if each agent i's set of beliefs $\mathcal{F}^i$ contains an ε-ball , then there exists a $\delta > 0$ such that whenever $\frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} < \delta$ for every $i \in \mathcal{I}$, $j \neq i$, and $s \in S$, the efficient SCR is implementable. The same conclusion holds if Linearity is replaced by Comonotonic Independence and Increasing Differences.*

While the technical details of the proof are provided in the Appendix, here we give a heuristic argument for the special case in which valuation functions are additively separable. A preliminary lemma shows that by Lipschitz Continuity, Monotonicity is always satisfied when each agent's informational size is sufficiently small. Thus, by Theorem 4.1, we only need to show that Minimal Ambiguity is also satisfied when each agent's informational size is sufficiently small. Observe that in this setting Minimal Ambiguity is equivalent to

$$\max_{F^i\in\mathcal{F}^i} F^i(\{s^{-i}\in S^{-i} | v^i(s^i_i, s^{-i}_i) \geq \max_{j\neq i} v^j(\overline{s}^i_j(s^i_i), s^{-i}_j)\})$$
$$\geq \min_{F^i\in\mathcal{F}^i} F^i(\{s^{-i}\in S^{-i} | v^i(s^i_i, s^{-i}_i) \geq \max_{j\neq i} v^j(\underline{s}^i_j(s^i_i), s^{-i}_j)\}) \quad \forall s^i_i \in S^i_i, \forall i \in \mathcal{I}.$$

[16] Equivalently, we can define the informational size of agent i as $\max_{j\neq i, s\in S}\left(v^j(\overline{s}^i_j(s^i_i), s^{-i}_j) - v^j(\underline{s}^i_j(s^i_i), s^{-i}_j)\right)$.

By the definitions of $\underline{s}^i$ and $\bar{s}^i$, we have $v^j(\bar{s}^i_j(s^i_i), s^{-i}_j) \geq v^j(\underline{s}^i_j(s^i_i), s^{-i}_j)$ for all $s^i_i \in S^i_i, s^{-i}_j \in S^{-i}_j, i \in \mathcal{I}$, and $j \neq i$. Thus, the required amount of ambiguity for Minimal Ambiguity and, hence, implementation of the efficient SCR, converges to zero as the number

$$\max_{i\in\mathcal{I}, j\neq i, s^i_i\in S^i_i, s^{-i}_j\in S^{-i}_j} v^j(\bar{s}^i_j(s^i_i), s^{-i}_j) - v^j(\underline{s}^i_j(s^i_i), s^{-i}_j), \tag{6}$$

decreases. Observe that the number in (6) is bounded above by

$$\max_{i\in\mathcal{I}, j\neq i, s^i_i\in S^i_i, \hat{s}^i_j\in S^i_j, s^{-i}_j\in S^{-i}_j} \frac{\partial v^j(\hat{s}^i_j, s^{-i}_j)}{\partial \hat{s}^i_j}\left(\bar{s}^i_j(s^i_i) - \underline{s}^i_j(s^i_i)\right). \tag{7}$$

Thus, the required amount of ambiguity for efficient implementation converges to zero as each agent's informational size converges to zero.

Intuitively, we can measure the conflict between Bayesian incentive compatibility and efficiency by the number in (7). Then, it is readily seen that there are two cases in which this conflict does not arise. The first is when $\frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} = 0$ for every $s \in S, i \in \mathcal{I}$ and $j \neq i$, which amounts to assuming private values. The other is when $\underline{s}^i_j(s^i_i) - \bar{s}^i_j(s^i_i) = 0$ for every $s^i_i \in S^i_i, i \in \mathcal{I}$ and $j \neq i$, which conforms to the model with one-dimensional signals. These observations are consistent with the possibility results in the mechanism design literature.[17]

One instance in which informational smallness arises naturally is when the valuation of each agent is determined only by his private information and the average of the information possessed by all agents. Then, as the number of agents increases, the informational size of each agent converges to zero. By Theorem 5.1, the required amount of ambiguity that induces truth telling also converges to zero. Next, we present this result through an example in which each agent's signal consists of a private element and a common element.[18] Let agent i's signal

[17]For example, see Vickrey [41], Clarke [11], Groves [21], Dasgupta and Maskin [15], Jehiel and Moldovanu [22] and Bergemann and Välimäki [3].

[18]Similar examples have been discussed by Maskin [27], Dasgupta and Maskin [15], Jehiel and Moldovanu [22] and Compte and Jehiel [12].

be $(\theta^i, c^i) \in [0,1] \times [0,1]$ and let his valuation for the object be

$$v^i(\theta^i, c^i, c^{-i}) := \theta^i + \frac{1}{N}\sum_j c^j \quad \forall \theta^i \in [0,1], \forall (c^i, c^{-i}) \in [0,1]^N. \tag{8}$$

The element θ^i is the private part of agent i's signal, as it is of interest to him only, while c^i is the common part as it is relevant to all agents.[19] Obviously, the valuation functions satisfy Linearity. Also, we can verify that Lipschitz Continuity holds. Since each agent's informational size is $\frac{1}{N}$, Theorem 5.1 implies that the efficient SCR is implementable when the number of agents is large enough.

Corollary 5.1. *Suppose that each agent i's valuation function is given by (8). For every $\varepsilon \in (0,1]$, if each agent i's set of beliefs $\mathcal{F}^i$ contains an ε-ball, then there exists $\overline{N} > 0$ such that whenever $N > \overline{N}$, the efficient SCR is implementable.*

6 Discussion

6.1 Common Knowledge Assumption

In the paper, we assume that each agent's set of priors is common knowledge. In fact, this assumption can be replaced by a weaker one: it is common knowledge that each agent's set of priors contains a certain set.[20] Formally, we assume it is common knowledge that the set of probability measures $\mathcal{F}^i_* \subseteq \Delta(S^{-i})$ is contained in agent i's set of priors for all $i \in \mathcal{I}$.[21] If we replace $\mathcal{F}^i$ with $\mathcal{F}^i_*$ in the definition of Minimal Ambiguity, the conclusions of Theorems 4.1 and 4.2 continue to hold. To see this, consider a full insurance transfer scheme that implements the efficient SCR when each agent i's set of priors is $\mathcal{F}^i_*$. Under this full insurance mechanism, agent i's interim utility is independent of his set of priors as long as all agents

[19]The notation in this example is in minor conflict with the general notation presented in Section 4 to take advantage of the information structure of the example: (i) s^i in this section simply represents θ^i and c^i, but more precisely it should be $s^i_i = \theta^i + \frac{1}{N}c^i$, $s^i_j = c^i$ for every $i \in \mathcal{I}$ and $j \neq i$; (ii) agent i's valuation function $v^i(\theta^i, c^i, c^{-i}) := v^i(s^i_i, s^{-i}_i) = s^i_i + \frac{1}{N}\sum_{j\neq i} s^j_i$ for every $i \in \mathcal{I}$ and $j \neq i$.

[20]I thank Pietro Ortoleva for suggesting this weakening of the common knowledge assumption.

[21]Let $\text{clh}(\mathcal{F}^i)$ denote the closed convex hull of $\mathcal{F}^i$. Since $\text{clh}(\mathcal{F}^i)$ and $\mathcal{F}^i$ generate the identical preference, an equivalent assumption is that it is common knowledge that $\mathcal{F}^i_*$ is contained in the closed convex hull of agent i's set of priors for all $i \in \mathcal{I}$.

report truthfully. However, when he misreports, his worst-case utility over $\mathcal{F}^i$ is at most as large as his worst-case utility over $\mathcal{F}^i_*$ due to ambiguity aversion. This means that the expected return from misreporting is lower when agent i is more ambiguity averse and, hence, he has less incentive to misreport.[22] Since truthful revelation is optimal when agent i's set of priors is $\mathcal{F}^i_*$, it remains optimal when agent i is actually more ambiguity averse.

Under this weaker common knowledge assumption, Theorem 5.1 can be stated as follows: Assume that Linearity (or Comonotonic Independence and Increasing Differences) and Lipschitz Continuity hold. If it is common knowledge that for every $i \in \mathcal{I}$, there exists $F^i \in \Delta(S^{-i})$ and $\varepsilon \in (0,1]$ such that $B_\varepsilon(F^i) \subseteq \mathcal{F}^i$, then the efficient SCR is implementable whenever agents are sufficiently informationally small.

6.2 Efficiency

There are three notions of efficiency for settings with incomplete information: ex ante efficiency, interim efficiency, and ex post efficiency. In a Bayesian setting with quasilinear utilities, if all the agents and the mechanism designer have the same ex ante belief, the three notions coincide.[23] However, they generally differ in a setting with ambiguity averse agents. The notion we use in the paper is ex post efficiency, whose definition is not affected by the presence of ambiguity aversion. In Appendix E , we show that under some assumptions on the mechanism designer's preferences, the full insurance mechanisms associated with the ex post efficient SCR are interim efficient. Thus, our results remain valid if we use the notion of interim efficiency. However, there exist situations in which ex ante efficiency cannot be achieved.

[22]Following Ghirardato and Marinacci [19], we say that an agent is more ambiguity averse if his set of beliefs is larger.

[23]See Laffont [25].

6.3 Surplus Extraction

A question commonly encountered in the literature is if there exists an efficient mechanism that leaves the agents no information rents. In a Bayesian setting, when agents' signals are independently distributed, it is well known that agents receive information rents due to the privacy of their information.[24] In our setting with ambiguity averse agents, we obtain a similar result: full surplus extraction is impossible except in the extreme case of complete ambiguity.[25] Appendix F provides the details.

6.4 Indirect Mechanisms

We show that in direct revelation mechanisms, the presence of ambiguity aversion facilitates truthful revelation of multidimensional information. However, the same conclusion does not extend to commonly used auctions—first price auctions, second price auctions, ascending auctions and descending auctions. To see this, consider a first price auction. Similar arguments apply to the other standard auctions. Take two types $s^i, t^i \in S^i$ such that $s^i_i < t^i_i$, and $p^i_*(s^i, s^{-i}) \geq p^i_*(t^i, s^{-i})$ for all $s^{-i} \in S^{-i}$ and $p^i_*(s^i, s^{-i}) > p^i_*(t^i, s^{-i})$ for some s^{-i}.[26] Suppose that q is the optimal bid of type s^i in an equilibrium. Since $v^i(s^i, s^{-i}) < v^i(t^i, s^{-i})$ for all $s^{-i} \in S^{-i}$, type t^i must also prefer the bid q to any lower bid. Thus, the optimal bid of type t^i is at least as large as q. On the other hand, efficiency requires that type s^i obtain the object with a higher probability than type t^i. Hence, type s^i should bid higher than type t^i, which is a contradiction. Intuitively, efficiency fails because when type s^i computes the optimal bid, only the one-dimensional signal s^i_i matters. Standard auctions therefore fail to elicit multidimensional information, no matter whether

[24] When agents' signals are statistically dependent and signal spaces are finite, Cremer and McLean [13, 14] show that the mechanism designer can extract all the surplus from the agents. McAfee and Reny [28] and Miller et al. [34] extend their results to allow for infinite signal spaces.

[25] In an independent private value setting, Bose and Daripa [5] show that a descending auction can exploit the ambiguity aversion of the agents to extract full surplus. However, their result does not extend to settings with interdependent valuations and multidimensional signals.

[26] Such a pair of types exists. In the example from Section 5, take two types (θ^i, c^i) and $(\hat{\theta}^i, \hat{c}^i)$ such that $\theta^i + \frac{1}{N}c^i < \hat{\theta}^i + \frac{1}{N}\hat{c}^i$ and $\theta^i > \hat{\theta}^i$.

the agents are ambiguity averse or not. One subject for future work is to design an indirect mechanism that can elicit multidimensional information by exploiting the ambiguity aversion of the agents.

7 Related Literature

Mechanism design with maxmin preferences. There is a growing literature studying mechanism design problems when agents are ambiguity averse. Bose et al. [7] study a single object auction in which the seller is only concerned with maximizing revenue. If the buyers face more ambiguity than the seller, they show that a full insurance mechanism is always optimal. Bodoh-Creed [4] obtains a payoff equivalence theorem in a setting with ambiguity averse agents and uses that theorem to characterize the revenue maximizing auction. Carroll [10] studies a multidimensional screening problem in which the seller is uncertain about the joint distribution of the buyer's types. He shows that the optimal selling mechanism for the seller is simply to screen along each component. There are two main differences between those papers and ours. First, those papers adopt a setting with private valuations, whereas we focus on a setting with interdependent valuations. Second, those papers address the issue of maximizing revenue, whereas we are mainly concerned with implementing the efficient SCR.

Wolitzky [42] studies the bilateral trading problem of Myerson and Satterthwaite [37] and characterizes when efficient and weakly budget balanced mechanisms exist. In contrast, we study a single object auction with interdependent valuations and multidimensional signals. Moreover, we focus on efficient mechanisms without imposing balanced budget constraint on the mechanism designer.

Di Tillio et al. [17] and Bose and Renou [6] both study the effects of introducing ambiguity in mechanisms. Di Tillio et al. [17] consider a screening model and show that a revenue maximizing seller can benefit from using ambiguous allocation rules. Rather than introducing ambiguity in the allocation rule, Bose and Renou [6] focus on situations in which there is no priori ambiguity, but the mechanism designer can create ambiguity deliberately through an ambiguous communication device. Consequently, the social choice rules that are not implementable with respect to the priors become implementable. Our paper complements their paper in the sense that we show precisely how much ambiguity is sufficient for implementing the efficient SCR and their result suggests that this amount of ambiguity can be generated through an ambiguous communication device. We provide an example in Appendix D to illustrate how to generate the required amount of ambiguity for efficient implementation through an ambiguous communication device.

Mechanism design with correlated information. In the literature on efficient mechanism design with interdependent valuations, a correlated information condition proposed by Cremer and McLean [13, 14] is used extensively to bypass the impossibility result.[27] The Cremer-McLean mechanism is sometimes criticized for the reason that the transfers can be increasingly large as beliefs of different signals get close. To overcome this criticism, McLean and Postlewaite [30, 31] point out that if signals are statistically dependent, the efficient allocations can be implemented with small payments when agents are sufficiently informationally small. Analogously, we adopt a similar notion of informational size and show that the efficient allocations can be implemented with a small amount of ambiguity when agents are sufficiently informationally small.

Approximate implementation. Another point of interest is whether efficient allocations can be implemented with a weaker notion of incentive compatibility. Theorem 1 in McLean and Postlewaite [31] addresses this question: efficient allocations can be implemented in an approximate ex post incentive compatible equilibrium if each agent's informational size is sufficiently small. Roughly speaking, a mechanism is approximately ex post incentive compatible if an agent would not misreport when there is only a small utility gain. The literature does not provide an explicit reason why an agent would forgo a small utility gain. Alternatively,

[27] Mezzetti [33] proposes a two-stage Groves mechanism, which can implement the efficient social choice rule if agents can observe their own realized utilities and transfers can be made based on the reported utilities.

we show that the realization of weaker interim incentive compatibility constraints arises endogenously as a result of ambiguity aversion and under natural assumptions on preferences, efficient allocations can be implemented.[28]

Appendix

A Appendix for Section 3.1

Lemma A 1. *If the SCR p is implementable, then for every $i \in \mathcal{I}$ and $s^i, t^i \in S^i$,*

$$\begin{aligned}\max_{F^i \in \mathcal{F}^i} \int \sum_k p^k(t^i, s^{-i})\big(v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i})\big)dF^i &\geq \mu^i(t^i) - \mu^i(s^i) \\ &\geq \min_{F^i \in \mathcal{F}^i} \int \sum_k p^k(s^i, s^{-i})\big(v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i})\big)dF^i.\end{aligned} \tag{9}$$

Proof. By definition, for every $i \in \mathcal{I}$ and $s^i, t^i \in S^i$,

$$\begin{aligned}u^i(t^i, s^i) &= \min_{F^i \in \mathcal{F}^i} \int \Big(\sum_k p^k(t^i, s^{-i}) v_k^i(s_{ki}^i, s_{ki}^{-i}) + x^i(t^i, s^{-i})\Big)dF^i \\ &\geq \mu^i(t^i) + \min_{F^i \in \mathcal{F}^i} \int \sum_k p^k(t^i, s^{-i})\big(v_k^i(s_{ki}^i, s_{ki}^{-i}) - v_k^i(t_{ki}^i, s_{ki}^{-i})\big)dF^i.\end{aligned}$$

Thus, the interim incentive compatibility constraint $\mu^i(s^i) \geq u^i(t^i, s^i)$ implies

$$\mu^i(s^i) \geq \mu^i(t^i) + \min_{F^i \in \mathcal{F}^i} \int \sum_k p^k(t^i, s^{-i})\big(v_k^i(s_{ki}^i, s_{ki}^{-i}) - v_k^i(t_{ki}^i, s_{ki}^{-i})\big)dF^i. \tag{10}$$

Reversing the roles of s^i and t^i, we obtain

$$\mu^i(t^i) \geq \mu^i(s^i) + \min_{F^i \in \mathcal{F}^i} \int \sum_k p^k(s^i, s^{-i})\big(v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i})\big)dF^i. \tag{11}$$

The desired inequalities in (9) follow by combining (10) and (11). □

Lemma A 2. *For every $i \in \mathcal{I}$ and $k \in \mathcal{K}$, there exists $M > 0$ such that*

$$|v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i})| \leq M|t_{ki}^i - s_{ki}^i| \quad \forall s^i, t^i \in S^i, \forall s^{-i} \in S^{-i}.$$

Proof. Fix $i \in \mathcal{I}$ and $k \in \mathcal{K}$. For every $s^i, t^i \in S^i$ and $s^{-i} \in S^{-i}$, the Mean Value

[28] We should point out that ambiguity aversion can only be used to weaken the *interim* incentive compatibility constraints, instead of the ex post constraints used by previous studies.

Theorem allows us to write $|v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i})| = |\frac{\partial v_k^i(\zeta_{ki}^i, s_{ki}^{-i})}{\partial \zeta_{ki}^i}(t_{ki}^i - s_{ki}^i)|$ for some ζ_{ki}^i between s_{ki}^i and t_{ki}^i. By (1), there exists $M > 0$ such that $|\frac{\partial v_k^i(\zeta_{ki}^i, s_{ki}^{-i})}{\partial \zeta_{ki}^i}| < M$ for all ζ^i and s^{-i}. This completes the proof. □

Lemma A 3. *If the SCR p is implementable, the associated indirect utility function μ^i is Lipschitz continuous on S^i for every $i \in \mathcal{I}$.*

Proof. Fix $i \in \mathcal{I}$. By Lemma A 1 and Lemma A 2, there exists $M > 0$ such that for every $s^i, t^i \in S^i$ with $\mu^i(s^i) - \mu^i(t^i) \geq 0$,

$$\begin{aligned} \mu^i(s^i) - \mu^i(t^i) &\leq \max_{F^i \in \mathcal{F}^i} \int \sum_k p^k(s^i, s^{-i})\big(v_k^i(s_{ki}^i, s_{ki}^{-i}) - v_k^i(t_{ki}^i, s_{ki}^{-i})\big) dF^i \\ &\leq \max_{k \in \mathcal{K}} M|s_{ki}^i - t_{ki}^i| \leq M \parallel s^i - t^i \parallel . \end{aligned} \tag{12}$$

Similarly, Lemma A 1 and Lemma A 2 imply that there exists $M' > 0$ such that for every $s^i, t^i \in S^i$ with $\mu^i(s^i) - \mu^i(t^i) \leq 0$,

$$\begin{aligned} \mu^i(t^i) - \mu^i(s^i) &\leq \max_{F^i \in \mathcal{F}^i} \int \sum_k p^k(t^i, s^{-i})\big(v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i})\big) dF^i \\ &\leq \max_{k \in \mathcal{K}} M'|s_{ki}^i - t_{ki}^i| \leq M' \parallel s^i - t^i \parallel . \end{aligned} \tag{13}$$

The combination of (12) and (13) implies that $\mu^i(s^i)$ is Lipschitz continuous. □

Take any $s^i \in S^i$ and let $l \in \mathbb{R}^{K \times N}$ be a directional vector for which $s^i + \rho l \in S^i$ for a sufficiently small scalar ρ. The two-sided directional derivative of $\mu^i(s^i)$ evaluated at $s^i \in S^i$ in the direction of l is

$$D\mu^i(s^i; l) := \lim_{\rho \to 0} \frac{\mu^i(s^i + \rho l) - \mu^i(s^i)}{\rho}.$$

For every $i \in \mathcal{I}$, the vector difference between signals $s^i, t^i \in S^i$ is denoted by $l(s^i, t^i) := t^i - s^i$ and the open line segment connecting s^i to t^i is given by $L(s^i, t^i) := \{s^i + \alpha(t^i - s^i) | \alpha \in (0,1)\}$. Since S^i is convex, $L(s^i, t^i) \subseteq S^i$ for all $s^i, t^i \in S^i$.

Lemma A 4. *If the SCR p is implementable, the associated indirect utility function μ^i admits two-sided directional derivatives in the direction $l(s^i, t^i)$ a.e. in $L(s^i, t^i)$ for every $s^i, t^i \in S^i$ and $i \in \mathcal{I}$.*

Proof. Fix $i \in \mathcal{I}$ and $s^i, t^i \in S^i$. Define the function U^i on $(0,1)$ by $U^i(\alpha) := \mu^i\big(\zeta^i(s^i, t^i, \alpha)\big) = \mu^i\big(\zeta^i(\alpha)\big)$. For any $\alpha, \alpha' \in (0,1)$, Lemma A 3 implies that there exists $M > 0$ such that

$$\begin{aligned} |U^i(\alpha) - U^i(\alpha')| = |\mu^i\big(\zeta^i(\alpha)\big) - \mu^i\big(\zeta^i(\alpha')\big)| &\leq M \parallel \zeta^i(\alpha) - \zeta^i(\alpha') \parallel \\ &= M \parallel s^i - t^i \parallel |\alpha - \alpha'|. \end{aligned}$$

Thus, we conclude that U^i is Lipschitz on $(0,1)$ and therefore, differentiable a.e. in $(0,1)$. In particular, if U^i is differentiable at $\alpha \in (0,1)$, we obtain

$$\frac{dU^i(\alpha)}{d\alpha} = \lim_{\rho \to 0} \frac{\mu^i\big(\zeta^i(\alpha) + \rho l(s^i, t^i)\big) - \mu^i\big(\zeta^i(\alpha)\big)}{\rho} = D\mu^i\big(\zeta^i(\alpha); l(s^i, t^i)\big).$$

□

For every $i \in \mathcal{I}$, $s^i, t^i \in S^i$, $\alpha \in (0,1)$, $F^i \in \mathcal{F}^i$, and $\rho \in (0, 1-\alpha)$, let

$$y^i(s^i, t^i, \alpha, F^i, \rho) := \int \sum_k p^k(\zeta^i(\alpha), s^{-i}) \frac{v^i_k(\zeta^i_{ki}(\alpha + \rho), s^{-i}_{ki}) - v^i_k(\zeta^i_{ki}(\alpha), s^{-i}_{ki})}{\rho} dF^i.$$

Lemma A 5. *For every $i \in \mathcal{I}$, $s^i, t^i \in S^i$, $\alpha \in (0,1)$, and $F^i \in \mathcal{F}^i$,*

$$\lim_{\rho \to 0} y^i(s^i, t^i, \alpha, F^i, \rho) = Y^i(s^i, t^i, \alpha, F^i).$$

Proof. Fix $i \in \mathcal{I}$, $s^i, t^i \in S^i$, $\alpha \in (0,1)$ and $F^i \in \mathcal{F}^i$. Equidifferentiability implies

$$\begin{aligned} &|\lim_{\rho \to 0} y^i(s^i, t^i, \alpha, F^i, \rho) - Y^i(s^i, t^i, \alpha, F^i)| \\ &\leq \lim_{\rho \to 0} \int \sum_k p^k(\zeta^i(\alpha), s^{-i}) \Big| \frac{v^i_k(\zeta^i_{ki}(\alpha + \rho), s^{-i}_{ki}) - v^i_k(\zeta^i_{ki}(\alpha), s^{-i}_{ki})}{\rho} - \frac{dv^i_k(\zeta^i_{ki}(\alpha), s^{-i}_{ki})}{d\alpha} \Big| dF^i \\ &\leq \lim_{\rho \to 0} \sup_{k, s^{-i}} \Big| \frac{v^i_k(\zeta^i_{ki}(\alpha + \rho), s^{-i}_{ki}) - v^i_k(\zeta^i_{ki}(\alpha), s^{-i}_{ki})}{\rho} - \frac{dv^i_k(\zeta^i_{ki}(\alpha), s^{-i}_{ki})}{d\alpha} \Big| = 0. \end{aligned}$$

□

Lemma A 6. *For every $i \in \mathcal{I}$, $s^i, t^i \in S^i$, and $\alpha \in (0,1)$,*

$$\lim_{\rho \to 0} \min_{F^i \in \mathcal{F}^i} y^i(s^i, t^i, \alpha, F^i, \rho) = \min_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i).$$

Proof. [29]Fix $i \in \mathcal{I}$, $s^i, t^i \in S^i$ and $\alpha \in (0,1)$. It follows from Lemma A 5 that

$$\lim_{\rho \to 0} \min_{F^i \in \mathcal{F}^i} y^i(s^i, t^i, \alpha, F^i, \rho) \le \lim_{\rho \to 0} y^i(s^i, t^i, \alpha, \tilde{F}^i, \rho) = Y^i(s^i, t^i, \alpha, \tilde{F}^i) \quad \forall \tilde{F}^i \in \mathcal{F}^i.$$

Thus,

$$\lim_{\rho \to 0} \min_{F^i \in \mathcal{F}^i} y^i(s^i, t^i, \alpha, F^i, \rho) \le \min_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i). \tag{14}$$

On the other hand, for every $\rho \in (0, 1-\alpha)$, let $\underline{F}^i_\rho \in \mathcal{F}^i$ be such that

$$y^i(s^i, t^i, \alpha, \underline{F}^i_\rho, \rho) = \min_{F^i \in \mathcal{F}^i} y^i(s^i, t^i, \alpha, F^i, \rho). \tag{15}$$

Such $\underline{F}^i_\rho$ exists as $\mathcal{F}^i$ is compact. Furthermore, by passing to a subsequence, $\underline{F}^i_\rho$ converges to $\underline{F}^i \in \mathcal{F}^i$ as $\rho \to 0$. By (1), there exists $M > 0$ such that

$$\begin{aligned}
&|y^i(s^i, t^i, \alpha, \underline{F}^i_\rho, \rho) - Y^i(s^i, t^i, \alpha, \underline{F}^i)| \\
\le & |y^i(s^i, t^i, \alpha, \underline{F}^i_\rho, \rho) - y^i(s^i, t^i, \alpha, \underline{F}^i, \rho)| + |y^i(s^i, t^i, \alpha, \underline{F}^i, \rho) - Y^i(s^i, t^i, \alpha, \underline{F}^i)| \\
\le & M \left| \int (d\underline{F}^i_\rho - d\underline{F}^i) \right| + |y^i(s^i, t^i, \alpha, \underline{F}^i, \rho) - Y^i(s^i, t^i, \alpha, \underline{F}^i)|.
\end{aligned}$$

As $\rho \to 0$, the first term approaches 0 since $\underline{F}^i_\rho \to \underline{F}^i$; from Lemma A 5, the second term approaches 0 as well. Hence, by taking limits in equation (15), we obtain $Y^i(s^i, t^i, \alpha, \underline{F}^i) = \lim_{\rho \to 0} \min_{F^i \in \mathcal{F}^i} y^i(s^i, t^i, \alpha, F^i, \rho)$, which implies that

$$\min_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i) \le \lim_{\rho \to 0} \min_{F^i \in \mathcal{F}^i} y^i(s^i, t^i, \alpha, F^i, \rho). \tag{16}$$

Combining (14) with (16) completes the proof. □

Lemma A 7. *If the SCR p is implementable, for every $i \in \mathcal{I}$, $s^i, t^i \in S^i$, and a.e. $\alpha \in (0,1)$,*

$$\min_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i) \le D\mu^i(\zeta^i(\alpha); l(s^i, t^i)) \le \max_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i).$$

Proof. Fix $i \in \mathcal{I}$, $s^i, t^i \in S^i$ and $\alpha \in (0,1)$. For every $\rho \in (0, 1-\alpha)$, Lemma A 1

[29]This proof follows similar arguments as Proposition 2 in Bose et al. [7].

implies

$$
\begin{aligned}
&\mu^i(\zeta^i(\alpha)+\rho l(s^i,t^i))-\mu^i(\zeta^i(\alpha))\\
&\geq \min_{F^i\in\mathcal{F}^i}\int\sum_k p^k(\zeta^i(\alpha),s^{-i})\Big(v^i_k(\zeta^i_{ki}(\alpha+\rho),s^{-i}_{ki})-v^i_k(\zeta^i_{ki}(\alpha),s^{-i}_{ki})\Big)dF^i.
\end{aligned}
$$

If $\rho>0$, it follows from the above expression that

$$
\frac{\mu^i(\zeta^i(\alpha)+\rho l(t^i,s^i))-\mu^i(\zeta^i(\alpha))}{\rho}\geq \min_{F^i\in\mathcal{F}^i} y^i(s^i,t^i,\alpha,F^i,\rho). \tag{17}
$$

If $\rho<0$, then

$$
\frac{\mu^i(\zeta^i(\alpha)+\rho l(t^i,s^i))-\mu^i(\zeta^i(\alpha))}{\rho}\leq \max_{F^i\in\mathcal{F}^i} y^i(s^i,t^i,\alpha,F^i,\rho). \tag{18}
$$

By Lemma A 4, $\mu^i(\zeta^i(\alpha))$ admits two-sided directional derivatives for almost all $\alpha\in(0,1)$. Thus, taking the lower limit as $\rho\downarrow 0$ in (17) and the upper limit as $\rho\uparrow 0$ in (18), and applying Lemma A 6 yield the desirable result. □

The next two lemmas establish the existence of integrable selections from the correspondence $C^i(s^i,t^i,\cdot)$.

Lemma A 8. *Given the SCR $p: S\to\mathbb{R}^K$, the functions $\alpha\mapsto\max_{F^i\in\mathcal{F}^i}Y^i(s^i,t^i,\alpha,F^i)$ and $\alpha\mapsto\min_{F^i\in\mathcal{F}^i}Y^i(s^i,t^i,\alpha,F^i)$ are $\mathcal{B}(0,1)$-measurable for every $i\in\mathcal{I}$ and $s^i,t^i\in S^i$.*

Proof. Fix $i\in\mathcal{I}$ and $s^i,t^i\in S^i$. We are going to show that $\max_{F^i\in\mathcal{F}^i}Y^i(s^i,t^i,\alpha,F^i)$ is $\mathcal{B}(0,1)$-measurable. The other case can be similarly handled. Take $(\alpha,F^i)\in(0,1)\times\mathcal{F}^i$ and let (α_n,F^i_n) be a sequence from $(0,1)\times\mathcal{F}^i$ that converges to (α,F^i) with respect to the product topology. By (1), there exists $M>0$ such that

$$
\begin{aligned}
&|Y^i(s^i,t^i,\alpha_n,F^i_n)-Y^i(s^i,t^i,\alpha,F^i)|\\
\leq&|Y^i(s^i,t^i,\alpha_n,F^i_n)-Y^i(s^i,t^i,\alpha_n,F^i)|+|Y^i(s^i,t^i,\alpha_n,F^i)-Y^i(s^i,t^i,\alpha,F^i)|\\
\leq&M|\int(dF^i_n-dF^i)|+|Y^i(s^i,t^i,\alpha_n,F^i)-Y^i(s^i,t^i,\alpha,F^i)|.
\end{aligned}
$$

The first term converges to 0 by construction; the second term above converges to 0 as $n\to\infty$, because F^i is absolutely continuous with respect to Lebesgue measure.

Therefore, $Y^i(s^i, t^i, \alpha, F^i)$ is jointly continuous. By Measurable Maximum Theorem,[30] $\max_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i)$ is $\mathcal{B}(0,1)$-measurable. □

Lemma A 9. *For every $i \in \mathcal{I}$ and $s^i, t^i \in S^i$, the correspondence $C^i(s^i, t^i, \cdot)$ is measurable and integrably bounded.*[31]

Proof. Fix $i \in \mathcal{I}$. Given Lemma A 8, the measurability of $C^i(s^i, t^i, \cdot)$ follows from the proof of Lemma 2 in Carbajal and Ely [8]. Next we show that $C^i(s^i, t^i, \cdot)$ is integrably bounded. By (1), there exists $M > 0$ such that $\min_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i) \leq \max_{F^i \in \mathcal{F}^i} Y^i(s^i, t^i, \alpha, F^i) \leq M$ for every $\alpha \in (0,1)$. □

Proof of Theorem 3.1. "Only if": Given the previous lemmas, the argument follows the same lines as the proof of Theorem 1 in Carbajal and Ely [8].

"If": Let $\mathcal{C}^i = \{c^i(s^i, t^i, \cdot) \in C^i(s^i, t^i, \cdot)\}_{s^i, t^i \in S^i}$ be a family of integrable selections for which conditions (i) (ii) (iii) are satisfied for every $i \in \mathcal{I}$. Fix an arbitrary signal $\tau^i \in S^i$ for every $i \in \mathcal{I}$. Define $x : S \to \mathbb{R}^N$ by

$$x^i(s^i, s^{-i}) = \int_0^1 c^i(\tau^i, s^i, \alpha) d\alpha - \sum_k p^k(s^i, s^{-i}) v_k^i(s_{ki}^i, s_{ki}^{-i}) \quad \forall s \in S,$$

where $c^i(\tau^i, s^i, \cdot) \in \mathcal{C}^i$ for every s^i. We are going to show that (p, x) is an interim incentive compatible mechanism. Fix $i \in \mathcal{I}$. Take any two signals $s^i, t^i \in S^i \in S^i$. By definition, the indirect utility $\mu^i(t^i)$ is

$$\begin{aligned} \mu^i(t^i) &= \min_{F^i \in \mathcal{F}^i} \int \Big(\sum_k p^k(t^i, s^{-i}) v_k^i(t_{ki}^i, s_{ki}^{-i}) + x^i(t^i, s^{-i}) \Big) dF^i = \int_0^1 c^i(\tau^i, t^i, \alpha) d\alpha \\ &= \int_0^1 c^i(s^i, t^i, \alpha) d\alpha + \int_0^1 c^i(\tau^i, s^i, \alpha) d\alpha. \end{aligned}$$

The last equality follows from condition (iii). Also, agent i's interim utility when his true signal is t^i but he reports s^i is

$$u^i(s^i, t^i) = \min_{F^i \in \mathcal{F}^i} \int \sum_k p^k(s^i, s^{-i}) \big(v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i}) \big) dF^i + \int_0^1 c^i(\tau^i, s^i, \alpha) d\alpha.$$

[30] See Theorem 18.19 in Aliprantis and Border [1].

[31] A correspondence C is said to be a measurable correspondence if for every open set O in $\mathbb{R}$, the inverse image $C^{-1} = \{\alpha \in (0,1) | C(\alpha) \cap O \neq \varnothing\}$ belongs to $\mathcal{B}(0,1)$. The correspondence C is said to be integrably bounded if there exists a nonnegative Lebesgue integrable function g defined on $(0,1)$ such that $C(\alpha) \subseteq [-g(\alpha), g(\alpha)]$ for almost all $\alpha \in (0,1)$.

Interim incentive compatibility requires $\mu^i(t^i) \geq u^i(s^i, t^i)$, that is,

$$\int_0^1 c^i(s^i, t^i, \alpha)d\alpha \geq \min_{F^i \in \mathcal{F}^i} \int \sum_k p^k(s^i, s^{-i})\big(v_k^i(t_{ki}^i, s_{ki}^{-i}) - v_k^i(s_{ki}^i, s_{ki}^{-i})\big)dF^i,$$

which is implied by condition (i). Since s^i and t^i were arbitrarily chosen, this shows that (p, x) is interim incentive compatible, as desired. □

Proof of Corollary 3.1. Suppose that the SCR p is implementable with associated indirect utility functions $\{\mu^i\}_{i=1}^N$. Fix $\tau^i \in S^i$ and let $\mu_0^i := \mu^i(\tau^i)$ for every $i \in \mathcal{I}$. By condition (ii) in Theorem 3.1, we infer that for every $i \in \mathcal{I}$ and $s^i \in S^i$,

$$\mu^i(s^i) = \int_0^1 c^i(\tau^i, s^i, \alpha)d\alpha + \mu_0^i, \tag{19}$$

where $c^i(\tau^i, s^i, \cdot)$ is a selection of $C^i(\tau^i, s^i, \cdot)$ satisfying conditions (i) and (iii) in Theorem 3.1. Construct a full insurance transfer scheme x_{full} as in (3) with $T^i(s^i) := \int_0^1 c^i(\tau^i, s^i, \alpha)d\alpha + \mu_0^i$ for every $i \in \mathcal{I}$ and $s^i \in S^i$. Applying the "if" part of the proof of Theorem 3.1, we can conclude that x_{full} implements the SCR p.

Next, we show that (p, x_{full}) generates indirect utility function μ^i for every $i \in \mathcal{I}$. By the construction of x_{full}, if agent i receives signal s^i, his indirect utility is

$$\begin{aligned}&\min_{F^i \in \mathcal{F}^i} \int \big(\sum_k p^k(s^i, s^{-i})v_k^i(s_{ki}^i, s_{ki}^{-i}) + x_{full}^i(s^i, s^{-i})\big)dF^i \\ &= \int_0^1 c^i(\tau^i, s^i, \alpha)d\alpha + \mu_0^i = \mu^i(s^i).\end{aligned}$$

The last equality follows from (19). This completes the proof. □

B Appendix for Section 4

B .1 Preliminary Lemmas

Lemma B 10. *Suppose that the efficient SCR p_* is implementable with associated indirect utility functions $\{\mu^i\}_{i=1}^N$. For every $i \in \mathcal{I}$ and $s^i, \hat{s}^i \in S_i^i$ such that $s_i^i = \hat{s}_i^i$, we have $\mu^i(s^i) = \mu^i(\hat{s}^i)$.*

Proof. Fix $i \in \mathcal{I}$. Lemma A 1 implies that for every $s^i, \hat{s}^i \in S^i$,

$$\begin{aligned} \max_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i})\big(v^i(\hat{s}_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})\big)dF^i &\geq \mu^i(\hat{s}^i) - \mu^i(s^i) \\ &\geq \min_{F^i \in \mathcal{F}^i} \int p_*^i(s^i, s^{-i})\big(v^i(\hat{s}_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})\big)dF^i. \end{aligned} \tag{20}$$

If $s_i^i = \hat{s}_i^i$, then (4) implies $v^i(\hat{s}_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i}) = 0$ for every $s^{-i} \in S^{-i}$. Hence, inequalities in (20) imply that $\mu^i(s^i) = \mu^i(\hat{s}^i)$. □

Instead of proving Lemma 4.1, we show a stronger result, which can be used to examine the possibility of full surplus extraction.

Lemma B 11. *If the efficient SCR p_* is implementable, then for every $i \in \mathcal{I}$ and $s^i \in S^i$,*

$$\begin{aligned} \max_{F^i \in \mathcal{F}^i} \int p_*^i(\bar{s}^i(s_i^i), s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i &\geq \frac{\partial \mu^i(s^i)}{\partial s_i^i} \\ &\geq \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(s_i^i), s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i. \end{aligned}$$

Proof. Fix $i \in \mathcal{I}$ and $r^i, s^i \in S^i$ such that $s_i^i > r_i^i$. Since the efficient SCR p_* is implementable, Lemma A 1 implies that

$$\max_{F^i \in \mathcal{F}^i} \int p_*^i(s^i, s^{-i})\big(v^i(s_i^i, s_i^{-i}) - v^i(r_i^i, s_i^{-i})\big)dF^i \geq \mu^i(s^i) - \mu^i(r^i).$$

Dividing the expression above by $s_i^i - r_i^i$ yields

$$\max_{F^i \in \mathcal{F}^i} \int p_*^i(s^i, s^{-i}) \frac{v^i(s_i^i, s_i^{-i}) - v^i(r_i^i, s_i^{-i})}{s_i^i - r_i^i} dF^i \geq \frac{\mu^i(s^i) - \mu^i(r^i)}{s_i^i - r_i^i}.$$

Since the above inequality holds for all $r^i, s^i \in S^i$ with $s_i^i > r_i^i$, we have

$$\max_{F^i \in \mathcal{F}^i} \int p_*^i(\bar{s}^i(s_i^i), s^{-i}) \frac{v^i(s_i^i, s_i^{-i}) - v^i(r_i^i, s_i^{-i})}{s_i^i - r_i^i} dF^i \geq \frac{\hat{\mu}^i(s_i^i) - \hat{\mu}^i(r_i^i)}{s_i^i - r_i^i},$$

where $\hat{\mu}^i(s_i^i) := \mu^i(\hat{s}^i)$ for every $\hat{s}^i \in e(s_i^i)$ and $\hat{\mu}^i(r_i^i) := \mu^i(\hat{r}^i)$ for every $\hat{r}^i \in e(r_i^i)$. The notation is well defined due to Lemma B 10. From Lemma A 3, μ^i is Lipschitz continuous and therefore $\hat{\mu}^i$ is differentiable almost everywhere. Thus, taking the limit $r_i^i \uparrow s_i^i$ yields

$$\max_{F^i \in \mathcal{F}^i} \int p_*^i(\bar{s}^i(s_i^i), s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i \geq \frac{d\hat{\mu}^i(s_i^i)}{ds_i^i} = \frac{\partial \mu^i(s^i)}{\partial s_i^i}.$$

Note that the validity of interchanging the limits and integrals follows from Lemma A 6. The second inequality in the lemma can be established analogously. □

Lemma B 12. *For every $i \in \mathcal{I}$, $s^i \in S^i$, and differentiable function $h^i : S^{-i} \to \mathbb{R}_+$,*

$$\min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(s_i^i), s^{-i}) h^i(s^{-i}) dF^i \geq \min_{F^i \in \mathcal{F}^i} \int p_*^i(s^i, s^{-i}) h^i(s^{-i}) dF^i,$$
$$\max_{F^i \in \mathcal{F}^i} \int p_*^i(\bar{s}^i(s_i^i), s^{-i}) h^i(s^{-i}) dF^i \leq \max_{F^i \in \mathcal{F}^i} \int p_*^i(s^i, s^{-i}) h^i(s^{-i}) dF^i.$$

Proof. We prove the first inequality and the second follows from analogous arguments. Fix $i \in \mathcal{I}$ and $s^i \in S^i$. Let $\underline{F}^i \in \text{argmin}_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(s_i^i), s^{-i}) h^i(s^{-i}) dF^i$. Hence, by the definitions of $\underline{F}^i$ and $\underline{s}^i$, we can conclude that

$$\min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(s_i^i), s^{-i}) h^i(s^{-i}) dF^i = \int p_*^i(\underline{s}^i(s_i^i), s^{-i}) h^i(s^{-i}) d\underline{F}^i$$
$$\geq \int p_*^i(s^i, s^{-i}) h^i(s^{-i}) d\underline{F}^i \geq \min_{F^i \in \mathcal{F}^i} \int p_*^i(s^i, s^{-i}) h^i(s^{-i}) dF^i.$$

□

B .2 Proof of Theorem 4.1

The necessity of Minimal Ambiguity is given by Lemma B 11. We now prove the sufficiency of Minimal Ambiguity given Linearity and Monotonicity.

Let p_* be the efficient SCR satisfying Minimal Ambiguity. For every $i \in \mathcal{I}$, define a reward function $\underline{T}^i : S^i \to \mathbb{R}$ as follows:

$$\underline{T}^i(s^i) := \int_{\tau_i^i}^{s_i^i} \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p_*^i(\underline{s}^i(t_i^i), s^{-i}) \frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} dF^i dt_i^i,$$

where $\tau_i^i := \min_{s^i \in S^i} s_i^i$. Then the associated full insurance transfer scheme $\underline{x}_{full}$ is

$$\underline{x}_{full}^i(s) = \underline{T}^i(s^i) - p_*^i(s) v^i(s) \quad \forall s \in S, \forall i \in \mathcal{I}. \tag{21}$$

We are going to show that the full insurance transfer scheme $\underline{x}_{full}$ can implement

p_*. Fix $i \in \mathcal{I}$. By Linearity, $v^i(s_i^1, ..., s_i^N) := g^i(s_i^i)h^i(s_i^{-i}) + H^i(s_i^{-i})$ for every $s \in S$. By the assumption that v^i increases in s_i^i, the function g^i also increases in s_i^i.

Take $s^i, \hat{s}^i \in S^i$. By (5), we obtain

$$\mu^i(s^i) - \mu^i(\hat{s}^i) = \underline{T}^i(s^i) - \underline{T}^i(\hat{s}^i) = \int_{\hat{s}_i^i}^{s_i^i} \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(t_i^i), s^{-i})h^i(s_i^{-i})\frac{dg^i(t_i^i)}{dt_i^i}dF^i dt_i^i$$
$$= \int_{g(\hat{s}_i^i)}^{g(s_i^i)} \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(t_i^i), s^{-i})h^i(s_i^{-i})dF^i dg^i(t_i^i). \tag{22}$$

By Monotonicity, $\int p_*^i(\underline{s}^i(t_i^i), s^{-i})h^i(s_i^{-i})dF^i$ increases in t_i^i for every $F^i \in \mathcal{F}^i$. Hence, $\min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(t_i^i), s^{-i})h^i(s_i^{-i})dF^i$ also increases in t_i^i. If $\hat{s}_i^i \leq s_i^i$, then

$$\begin{aligned}\underline{T}^i(s^i) - \underline{T}(\hat{s}^i) &\geq \left(g^i(s_i^i) - g^i(\hat{s}_i^i)\right) \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(\hat{s}_i^i), s^{-i})h^i(s_i^{-i})dF^i \\ &\geq \left(g^i(s_i^i) - g^i(\hat{s}_i^i)\right) \min_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i})h^i(s_i^{-i})dF^i.\end{aligned} \tag{23}$$

The second equality follows from Lemma B 12. If $\hat{s}_i^i > s_i^i$, then

$$\begin{aligned}\underline{T}^i(s^i) - \underline{T}(\hat{s}^i) &= -\int_{g(s_i^i)}^{g(\hat{s}_i^i)} \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(t_i^i), s^{-i})h^i(s_i^{-i})dF^i dg^i(t_i^i) \\ &\geq -\int_{g(s_i^i)}^{g(\hat{s}_i^i)} \max_{F^i \in \mathcal{F}^i} \int p_*^i(\overline{s}^i(t_i^i), s^{-i})h^i(s_i^{-i})dF^i dg^i(t_i^i) \\ &\geq -\left(g^i(\hat{s}_i^i) - g^i(s_i^i)\right) \max_{F^i \in \mathcal{F}^i} \int p_*^i(\overline{s}^i(\hat{s}_i^i), s^{-i})h^i(s_i^{-i})dF^i \\ &\geq -\left(g^i(\hat{s}_i^i) - g^i(s_i^i)\right) \max_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i})h^i(s_i^{-i})dF^i.\end{aligned} \tag{24}$$

The equality follows from (22); the first inequality follows from Minimal Ambiguity; the second inequality follows from Monotonicity; the last inequality follows from Lemma B 12. Therefore, combining (23) and (24) yields

$$\mu^i(s^i) = \underline{T}^i(s^i) \geq \underline{T}(\hat{s}^i) + \min_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i})\left(g^i(s_i^i) - g^i(\hat{s}_i^i)\right)h^i(s_i^{-i})dF^i = u^i(\hat{s}^i, s^i).$$

Since s^i and $\hat{s}^i$ were arbitrarily chosen, this shows that the incentive compatibility constraints are satisfied.

B .3 Proof of Theorem 4.2

Lemma B 13. *The valuation function $v^i : S \to \mathbb{R}$ has increasing differences if and only if $\frac{\partial v^i(s_i^i,\cdot)}{\partial s_i^i}$ and $\frac{\partial v^i(t_i^i,\cdot)}{\partial t_i^i}$ are comonotonic for all $s_i^i, t_i^i \in S_i^i$.*

Proof. Suppose that v^i has increasing differences. The converse direction can be proved analogously. Take $s_i^i \in S_i^i$ and $s_i^{-i}, \hat{s}_i^{-i} \in S_i^{-i}$. Without loss of generality, assume $\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} > \frac{\partial v^i(s_i^i, \hat{s}_i^{-i})}{\partial s_i^i}$. We are going to show that $\frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} \geq \frac{\partial v^i(t_i^i, \hat{s}_i^{-i})}{\partial t_i^i}$ for all $t_i^i \in S_i^i$.

We claim that for every $t_i^i > \hat{s}_i^i > s_i^i$,

$$v^i(t_i^i, s_i^{-i}) - v^i(\hat{s}_i^i, s_i^{-i}) > v^i(t_i^i, \hat{s}_i^{-i}) - v^i(\hat{s}_i^i, \hat{s}_i^{-i}). \tag{25}$$

The proof is by contradiction. Suppose that there exists $s_i^i < \hat{s}_i^i < t_i^i$ such that

$$v^i(t_i^i, s_i^{-i}) - v^i(\hat{s}_i^i, s_i^{-i}) \leq v^i(t_i^i, \hat{s}_i^{-i}) - v^i(\hat{s}_i^i, \hat{s}_i^{-i}).$$

Rearrangement shows that

$$v^i(\hat{s}_i^i, \hat{s}_i^{-i}) - v^i(\hat{s}_i^i, s_i^{-i}) \leq v^i(t_i^i, \hat{s}_i^{-i}) - v^i(t_i^i, s_i^{-i}).$$

Since v^i has increasing differences, we obtain

$$v^i(s_i^i, \hat{s}_i^{-i}) - v^i(s_i^i, s_i^{-i}) \leq v^i(\hat{s}_i^i, \hat{s}_i^{-i}) - v^i(\hat{s}_i^i, s_i^{-i}).$$

Again, rearrangement yields

$$v^i(\hat{s}_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i}) \leq v^i(\hat{s}_i^i, \hat{s}_i^{-i}) - v^i(s_i^i, \hat{s}_i^{-i}).$$

Dividing both sides by $\hat{s}_i^i - s_i^i$ and taking the limit $\hat{s}_i^i \downarrow s_i^i$ yields

$$\lim_{\hat{s}_i^i \downarrow s_i^i} \frac{v^i(\hat{s}_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{\hat{s}_i^i - s_i^i} \leq \lim_{\hat{s}_i^i \downarrow s_i^i} \frac{v^i(\hat{s}_i^i, \hat{s}_i^{-i}) - v^i(s_i^i, \hat{s}_i^{-i})}{\hat{s}_i^i - s_i^i}.$$

By the assumption that v^i is differentiable, the above inequality is equivalent to $\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} \leq \frac{\partial v^i(s_i^i, \hat{s}_i^{-i})}{\partial s_i^i}$, a contradiction.

Now dividing both sides of the inequality in (25) by $t_i^i - \hat{s}_i^i$ and taking the limit

$\hat{s}^i_i \uparrow t^i_i$ yields

$$\lim_{\hat{s}^i_i \uparrow t^i_i} \frac{v^i(t^i_i, s^{-i}_i) - v^i(\hat{s}^i_i, s^{-i}_i)}{t^i_i - \hat{s}^i_i} \geq \lim_{\hat{s}^i_i \uparrow t^i_i} \frac{v^i(t^i_i, \hat{s}^{-i}_i) - v^i(\hat{s}^i_i, \hat{s}^{-i}_i)}{t^i_i - \hat{s}^i_i}.$$

Thus, $\frac{\partial v^i(t^i_i, s^{-i}_i)}{\partial t^i_i} \geq \frac{\partial v^i(t^i_i, \hat{s}^{-i}_i)}{\partial t^i_i}$ for every $t^i_i > s^i_i$. By applying analogous arguments, we can show that $\frac{\partial v^i(t^i_i, s^{-i}_i)}{\partial t^i_i} \geq \frac{\partial v^i(t^i_i, \hat{s}^{-i}_i)}{\partial t^i_i}$ for every $t^i_i < s^i_i$. □

Lemma B 14. *Assume Comonotonic Independence and Increasing Differences. For every $i \in \mathcal{I}$ and $s^i, \hat{s}^i$ such that $s^i_i > \hat{s}^i_i$, we have*

$$\int_{\hat{s}^i_i}^{s^i_i} \min_{F^i \in \mathcal{F}^i} \int p^i_*(\underline{s}^i(t^i_i), s^{-i}) \frac{\partial v^i(t^i_i, s^{-i}_i)}{\partial t^i_i} dF^i dt^i_i \geq \min_{F^i \in \mathcal{F}^i} \int p^i_*\left(\underline{s}^i(\hat{s}^i_i), s^{-i}\right) \left(v(s^i_i, s^{-i}_i) - v(\hat{s}^i_i, s^{-i}_i)\right) dF^i$$

and

$$\max_{F^i \in \mathcal{F}^i} \int p^i_*\left(\bar{s}^i(s^i_i), s^{-i}\right) \left(v(s^i_i, s^{-i}_i) - v(\hat{s}^i_i, s^{-i}_i)\right) dF^i \geq \int_{\hat{s}^i_i}^{s^i_i} \max_{F^i \in \mathcal{F}^i} \int p^i_*(\bar{s}^i(t^i_i), s^{-i}) \frac{\partial v^i(t^i_i, s^{-i}_i)}{\partial t^i_i} dF^i dt^i_i.$$

Proof. We prove the first inequality and the second one follows from analogous arguments. Fix $i \in \mathcal{I}$. Take any $s^i, \hat{s}^i \in S^i_i$ such that $s^i_i > \hat{s}^i_i$. By Monotonicity,

$$\int_{\hat{s}^i_i}^{s^i_i} \min_{F^i \in \mathcal{F}^i} \int p^i_*(\underline{s}^i(t^i_i), s^{-i}) \frac{\partial v^i(t^i_i, s^{-i}_i)}{\partial t^i_i} dF^i dt^i_i \geq \int_{\hat{s}^i_i}^{s^i_i} \min_{F^i \in \mathcal{F}^i} \int p^i_*(\underline{s}^i(\hat{s}^i_i), s^{-i}) \frac{\partial v^i(t^i_i, s^{-i}_i)}{\partial t^i_i} dF^i dt^i_i. \tag{26}$$

For any $t^i_i \in S^i_i$, let

$$\underline{F}^i(t^i_i) \in \operatorname*{argmin}_{F^i \in \mathcal{F}^i} \int p^i_*(\underline{s}^i(\hat{s}^i_i), s^{-i}) \frac{\partial v^i(t^i_i, s^{-i}_i)}{\partial t^i_i} dF^i.$$

By Lemma B 13, Increasing Differences implies that $\frac{\partial v^i(t^i_i, \cdot)}{\partial t^i_i}$ and $\frac{\partial v^i(\hat{t}^i_i, \cdot)}{\partial \hat{t}^i_i}$ are comonotonic functions for all $t^i_i, \hat{t}^i_i \in S^i_i$. Since $p^i_*(\underline{s}^i(\hat{s}^i_i), \cdot)$ is either 1 or 0 almost everywhere, the functions $p^i_*(\underline{s}^i(\hat{s}^i_i), \cdot) \frac{\partial v^i(t^i_i, \cdot)}{\partial t^i_i}$ and $p^i_*(\underline{s}^i(\hat{s}^i_i), \cdot) \frac{\partial v^i(\hat{t}^i_i, \cdot)}{\partial \hat{t}^i_i}$ are comonotonic as well. By Comonotonic Independence, Proposition 3 in Schmeidler [38] and Proposition 18 in Marinacci and Montrucchio [26] together imply that $\underline{F}^i(\cdot)$ is a constant function of t_i^i. That is, $\underline{F}^i(t_i^i) = \underline{F}^i(\hat{t}_i^i)$ for all $t_i^i, \hat{t}_i^i \in S_i^i$. With slight abuse of notation, we denote $\underline{F}^i := \underline{F}^i(t_i^i)$ for any $t_i^i \in S_i^i$. Thus, we obtain

$$\begin{aligned} &\int_{\hat{s}_i^i}^{s_i^i} \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p_*^i(\underline{s}^i(\hat{s}_i^i), s^{-i}) \frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} dF^i dt_i^i \\ &= \int_{\hat{s}_i^i}^{s_i^i} \int_{S^{-i}} p_*^i(\underline{s}^i(\hat{s}_i^i), s^{-i}) \frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} d\underline{F}^i dt_i^i. \end{aligned} \tag{27}$$

By changing the order of intergration, the right hand side of the equality above becomes

$$\begin{aligned} &\int_{S^{-i}} p_*^i(\underline{s}^i(\hat{s}_i^i), s^{-i}) \int_{\hat{s}_i^i}^{s_i^i} \frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} dt_i^i d\underline{F}^i \\ &= \int_{S^{-i}} p_*^i(\underline{s}^i(\hat{s}_i^i), s^{-i})(v(s_i^i, s_i^{-i}) - v(\hat{s}_i^i, s_i^{-i})) d\underline{F}^i \\ &\geq \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p_*^i(\underline{s}^i(\hat{s}_i^i), s^{-i})(v(s_i^i, s_i^{-i}) - v(\hat{s}_i^i, s_i^{-i})) dF^i. \end{aligned}$$

Combining the inequalities above, (26) and (27) implies the desired inequality. □

Proof of Theorem 4.2. We are going to show that $(p_*, \underline{x}_{full})$ is interim incentive compatible, where $\underline{x}_{full}$ is defined in (21). Fix $i \in \mathcal{I}$ and $s^i, \hat{s}^i \in S^i$. By the construction of $\underline{x}_{full}$, we have

$$\mu^i(s^i) - \mu^i(\hat{s}^i) = \underline{T}^i(s^i) - \underline{T}(\hat{s}^i) = \int_{\hat{s}_i^i}^{s_i^i} \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(t_i^i), s^{-i}) \frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} dF^i dt_i^i. \tag{28}$$

There are two cases to consider. Suppose first that $\hat{s}_i^i \leq s_i^i$. It follows from Lemmas B 14 and B 12 that

$$\begin{aligned} &\int_{\hat{s}_i^i}^{s_i^i} \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(t_i^i), s^{-i}) \frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} dF^i dt_i^i \\ &\geq \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(\hat{s}_i^i), s^{-i}) \big(v(s_i^i, s_i^{-i}) - v(\hat{s}_i^i, s_i^{-i})\big) dF^i \\ &\geq \min_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i}) \big(v(s_i^i, s_i^{-i}) - v(\hat{s}_i^i, s_i^{-i})\big) dF^i. \end{aligned} \tag{29}$$

Hence, the equalities in (28) and the inequalities in (29) together imply that

$$\mu^i(s^i) - \mu^i(\hat{s}^i) \geq \min_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i}) \big(v(s_i^i, s_i^{-i}) - v(\hat{s}_i^i, s_i^{-i})\big) dF^i. \tag{30}$$

Suppose $\hat{s}_i^i > s_i^i$. Using (28), Minimal Ambiguity and Lemma B 14, we obtain

$$\begin{aligned}
\mu^i(s^i) - \mu^i(\hat{s}^i) &= -\int_{s_i^i}^{\hat{s}_i^i} \min_{F^i \in \mathcal{F}^i} \int p_*^i(\underline{s}^i(t_i^i), s^{-i}) \frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} dF^i dt_i^i \\
&\geq -\int_{s_i^i}^{\hat{s}_i^i} \max_{F^i \in \mathcal{F}^i} \int p_*^i(\overline{s}^i(t_i^i), s^{-i}) \frac{\partial v^i(t_i^i, s_i^{-i})}{\partial t_i^i} dF^i dt_i^i \\
&\geq -\max_{F^i \in \mathcal{F}^i} \int p_*^i(\overline{s}^i(\hat{s}_i^i), s^{-i}) \big(v(\hat{s}_i^i, s_i^{-i}) - v(s_i^i, s_i^{-i})\big) dF^i.
\end{aligned}$$

Combining the last inequality above with Lemma B 12, we obtain

$$\begin{aligned}
\mu^i(s^i) - \mu^i(\hat{s}^i) &\geq -\max_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i}) \big(v(\hat{s}_i^i, s_i^{-i}) - v(s_i^i, s_i^{-i})\big) dF^i \\
&= \min_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i}) \big(v(s_i^i, s_i^{-i}) - v(\hat{s}_i^i, s_i^{-i})\big) dF^i.
\end{aligned} \tag{31}$$

Thus, the combination of (30) and (31) yields

$$\mu^i(s^i) \geq \mu^i(\hat{s}^i) + \min_{F^i \in \mathcal{F}^i} \int p_*^i(\hat{s}^i, s^{-i}) \big(v(s_i^i, s_i^{-i}) - v(\hat{s}_i^i, s_i^{-i})\big) dF^i = u^i(\hat{s}^i, s^i).$$

The equality follows from the construction of the transfer scheme $\underline{x}_{full}$. Since s^i and $\hat{s}^i$ were arbitrarily chosen, this shows that the incentive compatibility constraints are satisfied, as desired. □

C Appendix for Section 5

A sufficient condition for Monotonicity is that the valuation functions satisfy ***generalized single crossing***: for every $i \in \mathcal{I}$, $j \neq i$, $s_i^i \in S_i^i$ and $s^{-i} \in S^{-i}$,

$$\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} \geq \frac{\partial v^j(\underline{s}_j^i(s_i^i), s_j^{-i})}{\partial s_j^i} \frac{d\underline{s}_j^i(s_i^i)}{ds_i^i}, \quad \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} \geq \frac{\partial v^j(\overline{s}_j^i(s_i^i), s_j^{-i})}{\partial s_j^i} \frac{d\overline{s}_j^i(s_i^i)}{ds_i^i}.$$

Clearly, the generalized single crossing condition imposes restrictions on both the valuation functions and the signal spaces. A direct observation is that it reduces to the standard single crossing condition in one-dimensional settings. Moreover, in the example from Section 5, this condition is satisfied if and only if

$$\frac{\partial v^i(\theta^i, c^i, c^{-i})}{\partial c^i} \geq \frac{\partial v^j(\theta^j, c^i, c^{-i})}{\partial c^i} \quad \forall \theta^i, c^i, c^j \in [0,1], \forall i \in \mathcal{I}, j \neq i.$$

Lemma C 15. *Assume Lipschitz Continuity. There exists a* $\delta > 0$ *such that whenever* $\frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} < \delta$ *for every* $i \in \mathcal{I}$, $j \neq i$ *and* $s \in S$, *Monotonicity holds.*

Proof. Since for every $i \in \mathcal{I}$ and $j \neq i$, the functions $\underline{s}^i_j$ and $\overline{s}^i_j$ are Lipschitz continuous, there exists $M > 0$ such that $|\frac{d\underline{s}^i_j(s^i_i)}{ds^i_i}| < M$ and $|\frac{d\overline{s}^i_j(s^i_i)}{ds^i_i}| < M$ for every $s^i_i \in S^i_i$. Take $\delta := \min_{i \in \mathcal{I}, s \in S} \frac{1}{M} \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i}$. Clearly, if $\frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} < \delta$ for every $i \in \mathcal{I}$, $j \neq i$ and $s \in S$, the generalized single crossing condition and hence Monotonicity hold. □

Lemma C 16. *For every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that whenever* $\frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} < \delta$ *for every* $i \in \mathcal{I}$, $j \neq i$ *and* $s \in S$, *we have*

$$0 \leq \max_{j \neq i} v^j\left(\overline{s}^i_j(s^i_i), s^{-i}_j\right) - \max_{j \neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right) < \varepsilon \quad \forall i \in \mathcal{I}, s^i_i \in S^i_i, s^{-i} \in S^{-i}. \tag{32}$$

Proof. The first inequality in (32) follows from the definitions of $\underline{s}^i$ and $\overline{s}^i$. To prove the second inequality, take $\delta := \frac{\varepsilon}{\max_{i \in \mathcal{I}, j \neq i, s^i_i} \left(\overline{s}^i_j(s^i_i) - \underline{s}^i_j(s^i_i)\right)}$. By construction, for every $i \in \mathcal{I}, s^i_i \in S^i_i$ and $s^{-i} \in S^{-i}$, we obtain

$$\begin{aligned}
\max_{j \neq i} v^j\left(\overline{s}^i_j(s^i_i), s^{-i}_j\right) - \max_{j \neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right) &\leq \max_{j \neq i} \left(v^j\left(\overline{s}^i_j(s^i_i), s^{-i}_j\right) - v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right) \right) \\
&\leq \max_{j \neq i, \hat{s}^i_j \in S^i_j} \frac{\partial v^j(\hat{s}^i_j, s^{-i}_j)}{\partial \hat{s}^i_j} \left(\overline{s}^i_j(s^i_i) - \underline{s}^i_j(s^i_i)\right) < \varepsilon.
\end{aligned}$$

□

Fix $i \in \mathcal{I}$ and $s^i \in S^i$. Let $z(s^{-i}) := v^i(s^i_i, s^{-i}_i) - \max_{j \neq i} v^j\left(s^i_j, s^{-i}_j\right)$ for every $s^{-i} \in S^{-i}$. Define a semimetric on S^{-i} by

$$\tilde{d}(s^{-i}, \hat{s}^{-i}) := |z(s^{-i}) - z(\hat{s}^{-i})| \quad \forall s^{-i}, \hat{s}^{-i} \in S^{-i}.$$

Since $\tilde{d}(\cdot, \cdot)$ is a semimetric, there exist distinct signals $s^{-i}, \hat{s}^{-i} \in S^{-i}$ such that $\tilde{d}(s^{-i}, \hat{s}^{-i}) = 0$. Define

$$D^i(s^i) := \{s^{-i} \in S^{-i} | \max_{j \neq i} v^j\left(\overline{s}^i_j(s^i_i), s^{-i}_j\right) > v^i(s^i_i, s^{-i}_i) \geq \max_{j \neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right)\}$$

$$D^*(s^i) := \{s^{-i} \in S^{-i} | v^i(s^i_i, s^{-i}_i) = \max_{j \neq i} v^j\left(s^i_j, s^{-i}_j\right), \text{ where } \max_{j \neq i, s} \frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} = 0\}.$$

Lemma C 17. *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $\frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} < \delta$ for every $i \in \mathcal{I}$, $j \neq i$ and $s \in S$, we have $d_H\big(D^i(s^i), D^*(s^i)\big) < \varepsilon$ for every $i \in \mathcal{I}$ and $s^i \in S^i$, where d_H represents Hausdorff distance.*

Proof. Take any $\varepsilon > 0$ and let $\delta := \frac{\varepsilon}{\max_{i \in \mathcal{I}, j \neq i, s^i_i}\left(\bar{s}^i_j(s^i_i) - \underline{s}^i_j(s^i_i)\right)}$. Fix $i \in \mathcal{I}$ and $s^i \in S^i$. By the definition of $\underline{s}^i$, we have $z(s^{-i}) \leq v^i(s^i_i, s^{-i}_i) - \max_{j \neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right)$ for all $s^{-i} \in S^{-i}$. Also, for every $s^{-i} \in D^i(s^i)$, we have $v^i(s^i_i, s^{-i}_i) < \max_{j \neq i} v^j\left(\bar{s}^i_j(s^i_i), s^{-i}_j\right)$. The last two observations imply

$$z(s^{-i}) < \max_{j \neq i} v^j\left(\bar{s}^i_j(s^i_i), s^{-i}_j\right) - \max_{j \neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right) < \varepsilon \quad \forall s^{-i} \in D^i(s^i). \tag{33}$$

The second inequality above follows from Lemma C 16. Similarly, for every $s^{-i} \in D^i(s^i)$, we have

$$\begin{aligned} -\varepsilon &< \max_{j \neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right) - \max_{j \neq i} v^j\left(\bar{s}^i_j(s^i_i), s^{-i}_j\right) \\ &\leq v^i(s^i_i, s^{-i}_i) - \max_{j \neq i} v^j\left(\bar{s}^i_j(s^i_i), s^{-i}_j\right) \leq z(s^{-i}). \end{aligned} \tag{34}$$

Combining (33) and (34) yields $|z(s^{-i})| < \varepsilon$ for every $s^{-i} \in D^i(s^i)$. In addition, $z(\hat{s}^{-i}) = 0$ for every $\hat{s}^{-i} \in D^*(s^i)$. Thus, $\tilde{d}(s^{-i}, \hat{s}^{-i}) < \varepsilon$ for all $s^{-i} \in D^i(s^i)$ and $\hat{s}^{-i} \in D^*(s^i)$, as desired. □

Lemma C 18. *Suppose that the probability measure F^i is absolutely continuous with respect to Lebesgue measure for every $i \in \mathcal{I}$. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $\frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} < \delta$ for every $i \in \mathcal{I}$, $j \neq i$ and $s \in S$, we have*

$$F^i\big(D^i(s^i)\big) < \varepsilon \quad \forall i \in \mathcal{I}, \forall s^i \in S^i.$$

Proof. Recall that each valuation function v^i increases in s^l_i for all $l \in \mathcal{I}$. Thus, for every $i \in \mathcal{I}$, $s^i \in S^i$, and $s^{-i} \in S^{-i}$, the set $\{\hat{s}^{-i} \in S^{-i} | \tilde{d}(s^{-i}, \hat{s}^{-i}) = 0\}$ has Lebesgue measure zero. In particular, for every $i \in \mathcal{I}$ and $s^i \in S^i$, the set $D^*(s^i)$ has Lebesgue measure zero and $F^i\big(D^*(s^i)\big) = 0$ since F^i is absolutely continuous with respect to Lebesgue measure. Lemma C 17 shows that $D^i(s^i)$ converges uniformly to $D^*(s^i)$ as $\max_{i \in \mathcal{I}, j \neq i, s} \frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j}$ converges to 0. Therefore, $F^i\big(D^i(s^i)\big)$ converges uniformly to 0 as $\max_{i\in\mathcal{I},j\neq i,s} \frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j}$ converges to 0. □

Fix $i \in \mathcal{I}$. For every $\varepsilon \in (0,1]$ and every probability measure $F^i \in \Delta(S^{-i})$, let $B_\varepsilon(F^i) := \{G^i \in \Delta(S^{-i}) | d(F^i, G^i) \leq \varepsilon\}$, where d is the Prokhorov metric. Given any $s^{-i} \in S^{-i}$, the Dirac measure $\delta_{s^{-i}}$ on (S^{-i}, Σ^{-i}) is defined by $\delta_{s^{-i}}(A) = 0$ if $s^{-i} \notin A$ and $\delta_{s^{-i}}(A) = 1$ if $s^{-i} \in A$.

Lemma C 19. *For every $\varepsilon \in (0,1]$, if agent i's set of beliefs $\mathcal{F}^i$ contains an ε-ball, then there exist $\hat{\varepsilon} \in (0,1]$ and a probability measure $F^i \in \Delta(S^{-i})$ that is absolutely continuous with respect to Lebesgue measure such that $B_{\hat{\varepsilon}}(F^i) \subseteq \mathcal{F}^i$.*

Proof. The statement of the lemma is equivalent to the set of probability measures that are absolutely continuous with respect to Lebesgue measure being dense in $\Delta(S^{-i})$. Theorem 15.10 in Aliprantis and Border [1] says that the set of probability measures with finite support is dense in $\Delta(S^{-i})$. Also, any probability measure that has finite support can be written as a convex combination of Dirac measures corresponding to points in the support. Thus, we only need to show that any Dirac measure is the limit of a sequence of probability measures that are absolutely continuous with respect to Lebesgue measure.

Fix $s^{-i} \in S^{-i}$. Construct a sequence of probability measures as follows:

$$F_n(A) := n\lambda\big(A \cap B_{\frac{1}{n}}(s^{-i})\big) \quad \forall A \in \Sigma^{-i},$$

where λ denotes Lebesgue measure and $B_{\frac{1}{n}}(s^{-i})$ denotes a neighborhood of s^{-i} that has Lebesgue measure $\frac{1}{n}$. By construction, each probability measure F_n is absolutely continuous with respect to Lebesgue measure. It is straightforward to verify that for every open set $A \in \Sigma^{-i}$, we have $\liminf_n F_n(A) \geq \delta_{s^{-i}}(A)$. By Theorem 15.3 in Aliprantis and Border [1], $F_n \to \delta_{s^{-i}}$ in the weak* topology, which completes the proof. □

Proof of Theorem 5.1. Lemma C 15 shows that Monotonicity is satisfied when the informational size of each agent is sufficiently small. We only need to show that Minimal Ambiguity also holds when each agent's informational size is sufficiently small. Then if Linearity is satisfied, we apply Theorem 4.1 to complete the proof; if instead Comonotonic Independence and Increasing Differences are satisfied, we apply Theorem 4.2.

By Lemma C 19, there exist an $\hat{\varepsilon} \in (0,1]$ and a probability measure $F^i \in \Delta(S^{-i})$ that is absolutely continuous with respect to Lebesgue measure such that $B_{\hat{\varepsilon}}(F^i) \subseteq \mathcal{F}^i$ for every $i \in \mathcal{I}$. By Lemma C 18, there exists a $\hat{\delta} > 0$ such that whenever $\max_{i\in\mathcal{I},j\neq i,s} \frac{\partial v^j(s^i_j, s^{-i}_j)}{\partial s^i_j} < \hat{\delta}$, we have $F^i(D^i(s^i)) < \hat{\varepsilon}$ for every $i \in \mathcal{I}$ and $s^i \in S^i$. Let $\delta := \min\{\hat{\delta}, \min_{i\in\mathcal{I},s\in S} \frac{1}{M} \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i}\}$. By Lemma C 15, Monotonicity is satisfied. It remains to show that Minimal Ambiguity is satisfied as well.

Fix $i \in \mathcal{I}$ and $s^i \in S^i$. Since $F^i(D^i(s^i)) < \hat{\varepsilon}$, there exists a probability measure $\hat{G}^i \in B_{\hat{\varepsilon}}(F^i) \subseteq \mathcal{F}^i$ such that $\hat{G}^i\big(D^i(s^i)\big) = 0$. Therefore, we obtain

$$
\begin{aligned}
&\min_{G^i\in\mathcal{F}^i} \int p^i_*(\underline{s}^i(s^i_i), s^{-i}) \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} dG^i \\
&= \min_{G^i\in\mathcal{F}^i} \left(\int p^i_*(\bar{s}^i(s^i_i), s^{-i}) \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} dG^i + \int_{D^i(s^i)} \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} dG^i \right) \\
&\leq \int p^i_*(\bar{s}^i(s^i_i), s^{-i}) \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} d\hat{G}^i \leq \max_{G^i\in\mathcal{F}^i} \int p^i_*(\bar{s}^i(s^i_i), s^{-i}) \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} dG^i.
\end{aligned}
$$

The inequalities above imply that Minimal Ambiguity is satisfied, as desired. Applying Theorem 4.1 or Theorem 4.2 completes the proof. □

D Ambiguous Communication Devices

In this section, we explicitly construct an ambiguous communication device that generates the required amount of ambiguity for implementing the efficient SCR in the example from Section 5. Formally, an ambiguous communication device is a tuple $\big(\{\Theta^i\}_{i\in\mathcal{I}}, \{M^i\}_{i\in\mathcal{I}}, \Psi\big)$, where Θ^i is a set of messages that agent i can send to the mechanism designer, M^i is a set of messages that agent i can receive from the mechanism designer, and Ψ is a set of probability systems such that $\varphi : \times_i\Theta^i \to \Delta(\times_i M^i)$ for every $\varphi \in \Psi$. Before the allocation mechanism (p, x) is executed, each agent i sends a confidential message $\theta^i \in \Theta^i$ to the mechanism designer; then the mechanism designer sends a confidential message $m^i \in M^i$ to each agent i, where the messages are drawn according to a probability system $\varphi \in \Psi$. When Ψ is not a singleton, the agents are ambiguous about the probability system that has been used by the mechanism designer. If we assume that the agents adopt full Bayesian updating, then even though they are a priori Bayesian, they perceive ambiguity after one round of communication—the mechanism designer creates ambiguity through this ambiguous communication device.

Consider the example from Section 5 with $N = 2$. Suppose that each agent i's prior distribution of θ^j is uniform for every $i \in \mathcal{I}$ and $j \neq i$. We now construct an ambiguous communication device to induce the required amount of ambiguity for implementing the efficient SCR. Let $\Theta^i := [0,1]$ and $M^i = \{a, b\}$ for every $i \in \mathcal{I}$. Let $\Psi = \{\varphi, \hat{\varphi}\}$. Define $\varphi^i : [0,1] \to \Delta(M^i)$, where $\varphi^i(m^i|\theta^j)$ is the probability that the mechanism designer sends message $m^i \in M^i$ to agent i when he receives message $\theta^j \in [0,1]$ from agent $j \neq i$ and let $\varphi := \varphi^1 \times \varphi^2$. Similarly, the other probability system $\hat{\varphi} := \hat{\varphi}^1 \times \hat{\varphi}^2$. For every $i \in \mathcal{I}$ and $j \neq i$, let

$$\varphi^i(a|\theta^j) := \begin{cases} 1 & \text{if } \theta^j \in [0, \frac{1}{2}], \\ 0 & \text{otherwise;} \end{cases} \qquad \hat{\varphi}^i(a|\theta^j) := \begin{cases} 0 & \text{if } \theta^j \in [0, \frac{1}{2}], \\ 1 & \text{otherwise.} \end{cases}$$

We can verify that Minimal Ambiguity is satisfied with respect to the posteriors. Since Linearity and Monotonicity are also satisfied, Theorem 4.1 implies that the efficient SCR is implementable.

E Ex ante, Interim, and Ex post Efficient Mechanisms

This section formalizes different notions of efficiency and illustrate their relationships in our setting. For any signal $s \in S$, assume that the mechanism designer's ex post utility is $\sum_i x^i(s)$. The mechanism designer's ex ante and interim preferences are represented by maxmin expected utility and $\mathcal{G}_i^M$ denotes his set of ex ante beliefs about agent i's signals. Further, assume that the mechanism designer believes that all the signals are independently distributed and $\mathcal{G}^M := \{\times_i G_i^M | G_i^M \in \mathcal{G}_i^M\}$ denotes his set of ex ante beliefs about all agents' signals. Also, define $\mathcal{G}_{-i}^M :=$

$\{\times_{j\neq i} G_j^M | G_j^M \in \mathcal{G}_j^M\}$. In the ex ante stage, agents have not observed their signals. Each agent's ex ante preference is represented by maxmin expected utility with $\mathcal{G}^i$ being the set of ex ante beliefs of agent i. In the interim stage, each agent has observed his own signal, but not the signals of the others. Recall that $\mathcal{F}^i$ denotes the set of interim beliefs of agent i. Finally, in the ex post stage, all the signals are publicly revealed.

A mechanism (p, x) is ***ex ante efficient*** if there is no other mechanism $(\hat{p}, \hat{x})$ that yields a higher ex ante payoff to some agent or the mechanism designer, without lowering the ex ante payoffs of the others. Interim and ex post efficient mechanisms can be defined analogously. Let E_A, E_I, and E_P denote the sets of mechanisms that are respectively ex ante, interim, and ex post efficient.

In a Bayesian setting with quasilinear utilities, if all agents and the mechanism designer share the same ex ante belief, the three notions of efficiency coincide: $E_A = E_I = E_P = \{(p, x) | p \in P_*\}$, where $P_* := \{p | p^k(s) > 0 \Rightarrow k \in \text{argmax}_{\hat{k}} \sum_{i=1}^N v_{\hat{k}}^i(s), \forall s \in S\}$. In our setting, the three notions generally differ. Obviously, the set of ex post efficient mechanisms remains the same as in a Bayesian setting. Next, we examine how the sets of ex ante and interim efficient mechanisms are affected by the presence of ambiguity aversion. Define a transfer scheme x_C as follows: for every $s \in S$ and $i \in \mathcal{I}$, we have $x_C^i(s) := R^i - \sum_k p^k(s) v_k^i(s)$ for some $R^i \in \mathbb{R}$. Under a transfer scheme x_C, each agent is fully insured against ambiguity in the ex ante stage. Denote the set of all such transfer schemes by X_C and denote the set of all full insurance transfer schemes by X_{full}.

Proposition E 1. *If $\mathcal{G}^M \subseteq \mathcal{G}^i$ for all $i \in \mathcal{I}$, then $\{(p, x_C) | p \in P_*, x_C \in X_C\} \subseteq E_A$;If $\mathcal{G}_{-i}^M \subseteq \mathcal{F}^i$ for all $i \in \mathcal{I}$, then $\{(p, x_{full}) | p \in P_*, x_{full} \in X_{full}\} \subseteq E_I$.*

The proof of Proposition E 1 follows the same lines as the proof of Proposition 1 in Bose et al. [7]. The intuition behind this proposition is simple: when the mechanism designer faces less ambiguity than the agents, he can improve social welfare by fully insuring the agents.

An implication of Proposition E 1 is that the full insurance mechanisms we construct to implement the efficient SCR are interim efficient. Thus, if we use the notion of interim efficiency in the paper, our results will remain the same. However, ex ante efficient and interim incentive compatible mechanisms may not exist. For example, Bose et al. [7] establish conditions under which $\{(p, x_C) | p \in P_*, x_C \in X_C\} = E_A$. Then given any ex ante efficient mechanism, each agent's indirect utility is independent of his realized signal. It is readily seen that such a mechanism is not interim incentive compatible except in the case of complete ambiguity.

F Surplus Extraction

Assume that each agent receives zero utility by not participating in the mechanism. Given the realized signal $s^i \in S^i$, agent i's *surplus* or *information rent* is then defined by $\mu^i(s^i)$. An immediate consequence of Lemma B 11 is the following proposition.

Proposition F 2. *Suppose that the efficient SCR p_* is implementable with associated indirect utility functions $\{\mu^i\}_{i=1}^N$. For every $i \in \mathcal{I}$, if $\mu^i(\hat{\tau}^i) = 0$ for every $\hat{\tau}^i \in e(\tau_i^i)$, then $\mu^i(s^i) \geq \underline{T}^i(s^i)$ for every $s^i \in S^i$.*

Proposition F 2 provides a lower bound for each agent's information rents in any efficient mechanism that generates zero utility for the lowest type. In particular, it implies that generally all the types except the lowest receive information rents.

Another implication from Proposition F 2 is that each agent's lowest information rents decrease as he becomes more ambiguity averse. To see this, recall that the lower bounds $\underline{T}^i$ can be attained by the mechanism $(p_*, \underline{x}_{full})$. When agents are more ambiguity averse, $\underline{T}^i$ are lower and the mechanism designer can extract more surplus. In the case of complete ambiguity, the mechanism designer can extract the full surplus.

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