Decision Functions, Local Risk, and Local Risk Aversion

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Abstract

In general situations of decision making under risk there do not exist indices of risk and risk aversion that are relevant for all decision makers and for all risky assets. However, we show that for many decision-making problems that involve what we call *local risk*, such indices do exist. To formalize this idea we represent decision-making problems by *decision functions*. The relevance of indices to a decision function is formalized as a decision function's property, called *monotonicity with respect tor risk and risk aversion.* In this paper, local risks arise in situations that involve investments with infinitesimally small investment time horizons.

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1 Introduction

A typical situation of decision making under risk involves a risky asset and a decision maker ("agent"). Analysis of such situations depends mainly on two distinct considerations (Diamond and Stieglitz 1974):

1. The riskiness of the asset.

2. The attitude of the agent towards risk (risk aversion).

The central role of these concepts in the analysis of decision making has led to many different methods of measuring risk and risk aversion, and hence to many different orderings of risky assets and decision makers. However, for most interesting decision-making problems there do not exist orders of riskiness and risk aversion that are relevant for all decision makers and all risky assets. Consider for instance the "risk premium" problem first presented by Pratt (1964).[1] If we consider a standard set of decision makers and a standard set of lotteries (risky assets), then, for any order of risk aversion defined on the set of agents, the fact that agent i is more averse to risk than agent j does not imply that the risk premium of a lottery l for i is greater than the risk premium of the same lottery l for j, for any lottery l. Similarly, for any order of riskiness defined on the set of lotteries, the fact that lottery l_1 is riskier than lottery l_2 does not imply that the risk premium of l_1 is greater than that of l_2, for all decision makers. Since such indices do not exist we say that the risk premium (as well as many other decision-making problems) is not monotonic with respect to risk and risk aversion.

This observation is true also with regard to the well-known Arrow–Pratt coefficients of absolute and relative risk aversion: in most situations of decision making under risk, the coefficients of risk aversion do not represent the attitude of all (risk-averse) decision makers towards risk. However, Pratt (1964) notes that these indices measure what he calls *local risk aversion* that relates to risks "in the small." In other words, in the limit, when the variation of returns goes to zero, the indices do represent the attitude of all (risk-averse) decision makers towards risk and are useful for analyzing a large variety of decision-making situations. Since risk and risk aversion are very close concepts, it is only natural to study also the properties of risk in such situations that involve infinitesimal uncertainty, or, as we call it, "local risk."

[1] We consider here the risk premium as a decision making problem. For a formal definition of risk premium and of decision-making problem, see Sections 3 and 2, respectively.

In this paper we propose a formal and general framework for analyzing decision-making problems involving either "regular" or local risk. In our framework, a decision-making problem is represented by a *decision function*, which is a real-valued function defined on a cross-set of decision makers and risky assets. Decision makers are represented by a pair consisting of a von Neumann–Morgenstern utility function and a wealth level, and risky assets are characterized by a random variable or a random process. To keep the discussion general, we define an index of risk aversion simply as a real-valued function of decision makers, and we define an index of riskiness as a real-valued function of risky assets.[2] We say that indices of riskiness and risk aversion are relevant or *suitable* for a decision-making problem if the decision function representing the decision problem is monotonic with respect to the values of the indices. As noted above, for most decision functions, unless we limit the domain sets in a very restrictive way, there do not exist suitable indices of risk and risk aversion.[3] However, if we focus on decision functions that involve only local risks, suitable indices do exist.

Given a decision-making problem represented by a decision function, the property of monotonicity can be used as an axiomatic characterization of the most appropriate indices of risk and risk aversion. We apply this idea to several common decision-making problems when only local risk is involved and derive the suitable indices of risk and risk aversion. These problems include the asset allocation problem, the risk premium, and the certainty equivalent. Not surprisingly, the only suitable indices of risk aversion that this method produces are the Arrow–Pratt coefficients of risk aversion, both the absolute and the relative ones. However, our analysis derives several indices of risk, including the (instantaneous) variation, the (instantaneous) Sharpe ratio, the (instantaneous) variance-mean ratio, and several combinations of these indices with the initial price of the security.

Focusing on local risk is not new in the literature of decision making. In fact, several papers use different methods to generate local risk, although they do not call it by this name.[4] For example, Pratt (1964) showed that if the distribution of the returns is sufficiently concentrated, i.e., the third absolute central moment is sufficiently small compared with the variance, then for

[2]The word *riskiness* emphasizes the idea that risk is a property of an asset or an investment. We will use the terms "risk" and "riskiness" interchangeably.

[3]Appendix A analyzes several decision-making problems with restricted domains.

[4]Although Pratt (1964) defines his index as an index of local risk aversion, it seems that the word "local" refers to the aversion rather than to the risk.

any decision maker, the magnitude of the so-called *risk premium* is correlated with the level of the decision maker's risk aversion. Another similar interpretation of risk-aversion measures was developed independently by Arrow (1965). Pratt (1964) calls this type of risk "risk in the small". In addition, Samuelson (1970) showed that the classic mean-variance analysis, initiated by Markowitz (1959), applies approximately to all utility functions, in situations that involve what he calls "compact" distribution. Finally, Shorrer (2011) shows that the Arrow-Pratt risk aversion of an agent is correlated with her willingness to accept or reject small gambles. In his setup, small gambles are characterized by random variables that take only small values. All these papers refer to the risk in the limit, where decision makers' wealth can be changed only infinitesimally.

In the present paper we use the framework of stochastic calculus for generating local risks. More specifically, the returns of risky assets in our setting follow continuous-time random processes. Decision-making problems with regard to such securities may depend on the investment time horizon. Local risk arises when the investment time horizon is infinitesimally small.

The paper is organized as follows. In Section 2 we present our framework and define formally the concepts of decision functions, indices of risk and risk aversion, and monotonicity with respect to risk and risk aversion. In Section 3 we use monotonicity to derive suitable indices for several decision-making problems under local risk. In all the examples in this section, risky assets are characterized by continuous-time stochastic processes. Section 4 concludes. In addition, Appendix A derives indices of risk and risk aversion for several decision-making problems involving "regular" risks with restricted domains. Proofs are relegated to Appendix B.

2 Framework

In this section we propose a general and simple framework for analyzing decision-making situations that involve risk. The framework contains formal definitions of the concepts *decision function*, *index of riskiness* (or *index of risk*), *index of risk aversion*, and a property of decision functions called *monotonicity* that connects the first three concepts to each other. The definitions are general and do not refer specifically to local risks.

2.1 Decision Functions

We represent decision-making problems by *decision functions* whose domain is a cross-set of risky assets and decision makers. A *risky asset*, such as a gamble or a security, is characterized by a random variable or a random process whose values can be interpreted in several ways such as absolute returns, relative returns, prices, or values. A *decision maker* is characterized by two elements: a utility function and an initial level of wealth. Utilities are assumed to be von Neumann–Morgenstern utilities.

Let X_A be a set of risky assets and let X_{DM} be a set of decision makers. A *decision function* d is a real-valued function

$$d : X_A \times X_{DM} \to \mathbb{R}, \tag{1}$$

where $\mathbb{R}$ is the set of real numbers. Of course, not every such decision function has an interesting economical interpretation, but, as we show later, many well-known problems of decision making under risk can be presented in this way. This includes the asset allocation problem, acceptance or rejection of a gamble, certainty equivalence, risk premium, and many more.

2.2 Indices of Riskiness and Risk Aversion

An *index of riskiness* is a real-valued function defined on a set of risky assets. Given an index of riskiness Q, we say that asset a_1 is Q-riskier than asset a_2 if $Q(a_1) > Q(a_2)$. Similarly, an *index of risk aversion* is a real-valued function defined on a set of decision makers. Given an index of risk aversion K, we say that decision maker dm_1 is K-more averse to risk than decision maker dm_2 if $K(dm_1) > K(dm_2)$. Obviously, these definitions are very general and it is not necessary that a measure of risk or risk aversion is good for anything at all.

2.3 Monotonicity

The existence of indices of risk and risk aversion that are relevant for a situation of decision making depends on the specific situation. In our setup, it is a property of decision functions called *monotonicity with respect to risk and risk aversion.*

Let X_A be a set of risky assets, X_{DM} a set of decision makers, and let d be a decision function whose domain is the cross-set $X_A \times X_{DM}$.

Definition 2.1.

1. *d is monotonically increasing with respect to an index of risk Q if for all* $a_i, a_j \in X_A$ *and for all* $dm \in X_{DM}$*:*

$$Q(a_i) > Q(a_j) \Leftrightarrow d(a_j, dm) > d(a_i, dm).$$

2. *d is monotonically increasing with respect to an index of risk aversion K if for all* $dm_i, dm_j \in X_{DM}$ *and for all* $a \in X_A$*:*

$$K(dm_i) > K(dm_j) \Leftrightarrow d(a, dm_j) > d(a, dm_j).$$

If a decision function is monotonically increasing with an index, its opposite function—also, by definition, a decision function—is *monotonically decreasing* with the index. Obviously, there is no general rule that says whether a decision function should increase or decrease with an index (of riskiness or risk aversion). This depends on the economical interpretation of the function and the index. Hence, we will use the following definition of monotonicity that refers to both increasing and decreasing functions.

Definition 2.2.

1. *Decision problem d is monotonic with respect to an index of risk Q, and Q is a suitable index for d, if d is either monotonically increasing or monotonically decreasing with Q.*

2. *Decision problem d is monotonic with respect to an index of risk aversion K, and K is a suitable index for d, if d is either monotonically increasing or monotonically decreasing with K.*

We say that a decision function d is *monotonic with respect to risk* if there exists an index of riskiness that is suitable for d. Similarly, d is *monotonic with respect to risk aversion* if there exists an index of risk aversion that is suitable for d. By definition, if an index of riskiness is suitable for a decision function, its opposite index, i.e., the index that takes exactly the opposite values, is also suitable for the same decision function. Since one of the two indices ranks the set of risky assets based on their riskiness, then the other index, which ranks them in the opposite direction, ranks it based on their "safeness." Equivalently, any index of risk aversion that is suitable for a decision function may rank decision makers based on their aversion to risk or the opposite—attraction to risk. There is no technical rule that says which of any two opposite indices of riskiness (risk aversion) relates to riskiness (risk aversion) and which to safeness (attraction to risk). This of course depends on the economical interpretation of the decision function and the index.

Note that being suitable to some decision function is not a special or a unique property of indices of riskiness and indices of risk aversion. In fact, any pair of indices of risk and risk aversion are suitable for many decision functions, such as the function defined as the multiplication of the indices by each other. Therefore, the starting point of our analysis is a decision-making problem. Indices of riskiness and risk aversion are interesting only if they are suitable for the specific decision functions that we are analyzing.

The question whether a decision function is monotonic with respect to risk and risk aversion obviously depends on the function's domain. It is always possible to limit a function's domain in a way that makes it monotonic with respect to risk and risk aversion. For instance, any decision function is monotonic (with respect to risk and risk aversion) if either the set of decision makers or the set of risky assets contains only one element. Moreover, one decision function might have different suitable indices for different domains. Hence, given a decision function, the challenge is to find the largest domain set for which the function is still monotonic with respect to risk and risk aversion.

3 Monotonic Decision Functions

In this section we examine several decision functions with quite general domains and show that when only local risk is involved, the functions become monotonic with respect to risk and risk aversion. We start by defining the domain sets of those functions, i.e., the set of decision makers and the set of risky assets.

Decision Makers.
A decision maker is characterized by a pair consisting of a von Neumann-Morgenstern utility function and an initial wealth level. Utility functions are assumed to be twice differentiable, with a positive first derivative and a

negative second derivative (implying risk aversion). We denote by DM the set of decision makers whose utilities are defined on the set of real numbers, like exponential utilities, and we denote by DM_+ the set of all decision makers with utilities defined only on positive real numbers, like power utilities. Accordingly, the initial wealth of decision makers in DM can take any real number, while the initial wealth of decision makers in DM_+ take only positive numbers.

Risky Assets.
Risky assets are securities whose values follow continuous-time random processes. Let s be such a security and let s_t denote the value of s at time t. We assume that s_t is the unique strong solution of a stochastic differential equation (SDE) of the form

$$ds_t = \mu_t dt + \sigma_t^T dW_t, \tag{2}$$

where W is a vector of K independent standard Wiener processes, and the superscript T means transpose. The drift $\mu_t = \mu(s_t, t)$ and the vector of diffusion $\sigma_t = \sigma(s_t, t)$ are both continuous functions of s_t and t. In addition, we assume also that $\mu_t > 0$ and $\sigma_t^T \sigma_t \neq 0$ for all t.[5] We denote by S the collection of all such securities, and we denote by S_+ the set of all such securities that take only positive values (with probability 1). A more rigorous description of the continuous-time framework is relegated to Appendix B.

All our examples here involve investments in a security, i.e., buying a number of shares of a security at one instant and selling it after a while. The interval of time between buying and selling is called the *investment time-horizon*. We assume that all securities purchases are done at time zero in which s_0 is already known, and all sales are done before time T, where T is a positive number. For simplicity we assume that the (net) risk-free interest rate is zero.

To clarify our notations further, if a decision maker with utility u and initial wealth w buys one unit of security s with investment time horizon t, his utility becomes random and is equal to $u(w - s_0 + s_t)$. Alternatively, if she invests all her initial wealth in that security, her utility will be $u(ws_t/s_0)$. We call $s_t - s_0$ the absolute return of s and s_t/s_0 the relative return of s. Naturally, when situations involve absolute risk we will refer to the sets DM

[5]The reason for these assumptions is that if $\mu_t \leq 0$, then we consider the riskiness of s at t as infinite, and if $\sigma_t^T \sigma_t = 0$ then s should not be considered as risky at t.

and S. By contrast, if relative returns are involved, we will refer to the sets DM_+ and S_+.

3.1 Time Horizon-dependent Decision Problems

We consider here several decision functions whose domain is the cross-set $DM \times S$ or $DM_+ \times S_+$. The functions are parametrized by time horizon t. For any $t > 0$ the functions are not monotonic with respect to risk and risk aversion. However, if t is infinitesimally small, i.e., if situations involve only "local risk," all these functions become monotonic with respect to risk and risk aversion.

1. Interest Risk Premium.
Let $i \in DM$ be a decision maker with utility function u and initial wealth level w, and let $s \in S$ be a security. The compound interest rate $r_t = r_t(i, s)$ is defined implicitly by the equation

$$E\Big[u\Big(w - s_0 + s_t\Big)\Big] = u\Big(w - s_0 + \mathrm{E}(s_t) - (e^{r_t t} - 1)\Big). \tag{3}$$

The expression on the left hand-side of (3) is interpreted as the expected utility of buying one unit of s at time zero and selling it at t. The right-hand side is the utility of having the expected value of this investment minus $(e^{r_t t} - 1)$, called the *risk premium.* Thus, r_t can be interpreted as the compound interest rate that makes i indifferent between the following two options: (1) buying one unit of s and (2) having the expectation of this investment but paying the (net) compound interest rate of r_t on a \$1 loan. Since decision makers are risk averse, the risk premium $(e^{r_t t} - 1)$ should be positive, which implies that r_t is positive as well.

A similar problem can be defined in relative terms, where now $i \in DM_+$ and $s \in S_+$. We define $\widehat{r}_t = \widehat{r}_t(i, s)$ implicitly by the equation,

$$E\Big[u\Big(w\frac{s_t}{s_0}\Big)\Big] = u\Big(w\big(\frac{\mathrm{E}(s_t)}{s_0} - (e^{\widehat{r}_t t} - 1)\big)\Big). \tag{4}$$

The left-hand side of (4) is the expected utility of investing w in s with investment time horizon t (where w is the initial wealth), and the right-hand side is the utility of investing w in a risk-free asset whose return is equal to the expected return of s minus the relative risk premium $(e^{\widehat{r}_t t} - 1)$. Thus, $\widehat{r}_t$

can be interpreted as the compound interest rate that makes one indifferent between the following two options: (1) investing w in s and (2) having the expectation of this investment but paying the (net) compound interest rate of r_t on the sum of investment (w). Here again, for any risk averse decision maker the risk premium ($e^{r_t t} - 1$) will be positive, which implies that $r_t > 0$.

2. Portfolio Allocation.
We denote by $\alpha_t(i, s)$ the optimal number of shares of $s \in S$ that investor $i \in DM$ buys in order to maximize her utility, where the rest of her wealth has zero return (cash), and the investment time horizon is t. Formally,

$$\alpha_t(i, s) = \arg\max_{\alpha} E\left[u\left(w - \alpha s_0 + \alpha s_t\right)\right]. \tag{5}$$

Since the expected absolute (rather than relative) return of s is positive, α_t is positive as well.

The problem can be phrased in relative terms as follows. Let $\widehat{\alpha}_t$ be the optimal fraction of wealth to be invested in a security $s \in S_+$ by a decision maker $i \in DM_+$, where the investment time horizon is t. Formally,

$$\widehat{\alpha}_t(i, s) = \arg\max_{\alpha} E\left[u\left((1 - \alpha)w + \alpha w s_t / s_0\right)\right]. \tag{6}$$

As before, $\widehat{\alpha}_t$ is positive.

3. The Certainty Equivalent of an Optimal Allocation.
We denote by $z_t = z_t(i, s)$ the compound risk-free interest rate for decision maker $i \in DM$ and security $s \in S$ that makes i indifferent between accepting this return of z_t on \$1 and investing optimally in s. Formally, z_t is defined implicitly by

$$E\left[u\left(w - \alpha_t s_0 + \alpha_t s_1\right)\right] = u(w + e^{z_t t} - 1), \tag{7}$$

where $\alpha_t \equiv \alpha_t(dm, s)$. The expression $e^{z_t t} - 1$ is the certainty equivalent of investing optimally in s. Since α_t is positive $e^{z_t t} - 1$ is also positive, implying that z_t is positive as well.

Similarly, the relative certainty equivalent of an optimal allocation $\widehat{z}_t = \widehat{z}_t(i, s)$, where $i \in DM_+$ and $s \in S_+$, is defined as the compounded risk-free interest rate $\widehat{z}_t$ that makes the decision maker indifferent between investing all his wealth w with a return of $\widehat{z}_t$ and investing optimally in s. Formally, $\widehat{z}_t$ is defined implicitly by the equation

$$E\left[u\left((1-\widehat{\alpha}_t)w+\widehat{\alpha}_t w s_t/s_0\right)\right]=u(w(e^{\widehat{z}_t t})). \tag{8}$$

To sum up this section, we presented six decision functions: three refer to absolute returns and the other three refer to relative returns. The functions are parametrized by the investment time horizon t.

3.2 Local Risks

When the investment time horizon t is a positive number, i.e., $0 < t < T$, then all the previous decision functions involve what we call regular risks. However, "local risks" are involved if we relate to an infinitesimally small investment time horizon. Formally, based on the previous functions, we define six new decision functions that involve only local risk, as follows:

(1) $r(i,s) = \lim_{t\to 0} r_t(i,s)$.

(2) $\alpha(i,s) = \lim_{t\to 0} \alpha_t(i,s)$.

(3) $z(i,s) = \lim_{t\to 0} z_t(i,s)$.

(4) $\widehat{r}(i,s) = \lim_{t\to 0} \widehat{r}_t(i,s)$.

(5) $\widehat{\alpha}(i,s) = \lim_{t\to 0} \widehat{\alpha}_t(i,s)$.

(6) $\widehat{z}(i,s) = \lim_{t\to 0} \widehat{z}_t(i,s)$.

The following theorem claims that these functions are not only well defined but also monotonic with respect to risk and risk aversion.

Theorem 3.1. *Decision functions* $(1)-(6)$ *are monotonic with respect to risk and risk aversion. Moreover, the suitable measures are those that appear in Table 1.*

Table 1 summarizes the six decision functions and their suitable indices of risk and risk aversion. It is shown that the property of monotonicity derives two different indices of risk aversion that are the well-known coefficients of risk aversion of Pratt and Arrow, and five different indices of riskiness. Interestingly, one can notice the equivalence between indices of relative risk and the index of relative risk aversion. Like the index of relative risk aversion that depends on w, the indices of relative riskiness depend on the value of the security s_0. It seems that w and s_0 play a similar role in both cases as a reference point to amounts of money. Although decision function (6) relates also to relative returns, its suitable index of riskiness does not depend on s_0. A slight modification of this decision function can derive and index of risk that does depend on s_0, such as $\sigma^T\sigma/(s_0^2\mu^2)$.[6]

The indices of riskiness that appear in Table 1 are *instantaneous*; i.e., they depend on the parameters of securities at only one point in time. If we limit the parameter K to be 1 (only one Wiener process in the environment), the dimension of σ will be 1 as well. In this case the indices will be the instantaneous variation σ^2, the instantaneous var-mean ratio σ^2/μ, and the square of the inverse instantaneous Sharpe ratio $(\sigma/\mu)^2$. The other two indices of riskiness are simply combinations of these indices with the security's price.

3.3 Other Decision Functions

We have discussed here several examples of decision-making problems and used the property of monotonicity to derive the suitable indices of risk and risk aversion. Obviously, one can think of other decision problems that might be monotonic with respect to risk and risk aversion.

Note that not all decision functions involving only local risks are monotonic with respect to risk and risk aversion. For instance, Schreiber (2013) studies the decision problem of accepting or rejecting investments in the continuous-time framework. The decision function representing this problem can take only two values: 1 for an acceptance and 0 for a rejection, and therefore by definition it is not strictly monotonic with respect to risk and risk aversion. However, as Schreiber (2013) shows, the value of the function depends exclusively on a pair of indices, one of risk and the other of risk aversion, and it can be considered as weakly monotonic with respect to risk

[6] This would be the case if we exchanged $\widehat{\alpha}_t$ by α_t in (8).

Case	Decision Problem	Domain Set		Indices	
		Individuals	Assets	Risk	R Aversion
1	Risk Premium (A)	DM	S	$\sigma^T\sigma$	$-\frac{u''(w)}{u'(w)}$
2	Asset Allocation (A)			$\sigma^T\sigma/\mu$	
3	CE of Optimal (A)			$\sigma^T\sigma/\mu^2$	
4	Risk Premium (R)	DM_+	S_+	$\sigma^T\sigma/s_0^2$	$-w\frac{u''(w)}{u'(w)}$
5	Asset Allocation (R)			$\sigma^T\sigma/(\mu s_0)$	
6	CE of Optimal (R)			$\sigma^T\sigma/\mu^2$	

Table 1: The table presents suitable indices of risk and risk aversion for six decision functions that involve only local risk. Recall that a decision maker is characterized by a pair consisting of a utility function u and an initial wealth level w and that a risky asset s is characterized by an SDE of the form $ds_t = \mu_t dt + \sigma_t^T dW_t$. The table uses the notations $\sigma \equiv \sigma_0$ and $\mu \equiv \mu_0$. The first and second columns of the table are used to identify the specific decision function, as defined above. The third and fourth columns are the domain sets of these functions. The last two columns are the indices of risk and risk aversion. It can be seen that indices of riskiness are functions of the parameters of the risky assets at time zero and that indices of risk aversion are functions of the parameters characterizing decision makers with initial wealth w.

and risk aversion.

While this example of accepting or rejecting an investment is not monotonic with respect to risk and risk aversion for technical reasons only, it seems that the following decision function is not monotonic for a deeper reason. Let $ie_t(i, s)$ denote the *interest-rate equivalent* decision function, defined implicitly by the equation

$$E\left[u\Big(w - s_0 + s_t\Big)\right] = u\Big(w + (e^{ie_t t} - 1)\Big). \tag{9}$$

The expression $(e^{ie_t t} - 1)$ is known as the *cash equivalent* (of the absolute return $s_t - s_0$). For any security s, the cash equivalent ie_t is well defined for at least small values of t. Now, although the decision function $ie = \lim_{t\to 0} ie_t$ involves only local risk, it can be shown that it is not monotonic with respect to risk (but it is monotonic with respect to risk aversion).

4 Conclusion

The main argument of this paper is that many decision functions that involve only local risk are monotonic with respect to risk and risk aversion. This can be explained intuitively as follows: in the limit, when the investment time horizon goes to zero, only the first two elements of the Taylor series of the utility matter, which implies that the only relevant parameters are the initial wealth level, the first and second derivatives of utilities, and the first two moments of distribution. This of course simplifies the analysis of decision functions and brings them closer to each other. However, as we have shown in the paper, not every decision function that involves only local risk is monotonic with respect to risk and risk aversion. A rule that characterizes monotonic decision functions (involving only local risk) is a challenge that needs to be addressed in future research.

Our formal analysis supports and reflects several ideas with regard to the nature of risk and risk aversion. First, in general decision-making situations, i.e., those that involve regular risks, risk and risk aversion should be considered as subjective concepts in the sense that there do not exist indices of risk and risk aversion that are relevant for all decision makers and all risky assets. In this sense, risk, as an attribute of a risky asset, is analogous to beauty as a an attribute of a work of art as both cannot be measured objectively (this stands in contrast with the perception that risk is analogous to, say, body temperature which can be measured objectively; see Aumann and Serrano 2008). Second, the concepts of risk and risk aversion should be studied in the context of a specific situation of decision making under risk, in which the challenge is to find indices of risk and risk aversion that are the most appropriate to this situation. Such indices might be relevant only for some of the decision makers and definitely might be irrelevant in other decision-making situations. Third, risk is what risk averters dislike in the sense that if suitable indices for a decision problem exist, they should affect decisions in the same way. As we have shown, in many situations of decision making under local risk, all risk averters in the sense of Arrow and Pratt dislike the same types of local risks. In such situations, risk aversion and risk affect the decision function in a similar way. Finally, by our approach, the concepts of risk and risk aversion are situation-dependent and therefore risk is definitely not the opposite of attractiveness. By contrast, risk is a property of random distribution that is interpreted differently in every situation.

A final noteworthy remark about our indices of local risk is that since the analysis of infinitesimal risks concerns the first and the second elements of a Taylor series of utilities, the indices that we derive can be viewed as an approximation of indices in situations that involve regular risk. Schreiber (2014) shows that the indices of riskiness of Aumann and Serrano (2008) and of Foster and Hart (2009) coincide with one of the indices that are derived here when only local risk is involved. Similarly, we would expect any index of riskiness that is seemingly connected to utilities to coincide with one of our indices in situations that involve only local risk.

Appendix

A Standard Decision Functions

In this section we use our framework to analyze several examples of what we call "regular" decision-making problems. What makes them regular is the fact that they refer to risky assets characterized by random variables rather than random processes. Our goal is to use the property of monotonicity to find the most appropriate indices of risk and risk aversion. To make these functions monotonic with respect to risk and risk aversion we should restrict the domain sets in different ways. All the example in this section are summarized in Table 2. The rest of this section is an explanation of that table.

A.1 Decision Makers

The third column of Table 2, labeled DM, contains different sets of decision makers, characterized by their utilities and the range of values that w can take. The sets of decision makers are as follows:

1. *Quadratic*—the set of decision makers whose utilities have the form $u(w) = w - bw^2$, where b is a positive-valued parameter and $w < 1/2b$.

2. *CARA*—the set of decision makers whose utilities have a constant absolute risk aversion. There is essentially a unique CARA utility with parameter α, given by $u(w) = -e^{-\alpha w}$. α is assumed to be positive and w can take any real number.

3. *CRRA*—the set of decision makers whose utilities have a constant relative risk aversion. There is essentially a unique CRRA utility with parameter γ, given by

$$u_\gamma(x) = \begin{cases} \frac{(x^{1-\gamma}-1)}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(x) & \text{if } \gamma = 1 \end{cases} ,$$

Case	Decision Problem	Domain Sets		Suitable Measures	
		DM	**Assets**	**Risk**	**R Aversion**
1	Accept or Reject	Quadratic	G	μ_2/μ	ARA
2			R	μ_2/μ	RRA
3		CARA	G	R^{AS}	ARA
4			R	S^{AS}	RRA
5		CRRA	R	S	RRA
6	Risk Premium	CARA	G_N	σ	ARA
7			R_N	σ	RRA
8	A Asset Allocation	Quadratic	R^*	μ_2/μ	ARA
9		CARA	R_N	σ^2/μ	ARA
10	R Asset Allocation	Quadratic	R^*	μ_2/μ	RRA
11		CARA	R_N	σ^2/μ	RRA
12	CE A of Optimal	CARA	R_N	σ/μ	ARA
13	CE R of Optimal	CARA	R_N	σ/μ	RRA

Table 2: **Monotonic Decision Problems**
This table presents the suitable measures of risk and risk aversion for six different decision-making problems. The third and fourth columns are the domain sets of the problems. The third column, DM, is the column of decision makers, who can belong to one of three groups: quadratic utilities, constant absolute risk-aversion utilities (CARA), and constant relative risk-aversion utilities (CRRA). Assets (fourth column) belong to one of six sets: G for additive lotteries with a finite set of values, R for multiplicative lotteries with a finite set of values, G_N for normally distributed additive lotteries, R_N for normally distributed multiplicative lotteries, R_{LN} for log-normally distributed multiplicative lotteries, and R^* for general multiplicative lotteries. The fifth and sixth columns are the suitable measures of risk and risk aversion. The sixth column is the measures of riskiness defined as follows: if g is a gamble (additive lottery), $\mu = \mathrm{E}(g)$, $\mu_2 = \mathrm{E}(g^2)$, and $\sigma = \mathrm{E}(g - \mathrm{E}(g))$. For a multiplicative lottery r, $\mu = \mathrm{E}(r-1)$, $\mu_2 = \mathrm{E}((r-1)^2)$, and $\sigma = \mathrm{E}(r - \mathrm{E}(r))$. Three other measures of riskiness are: R^{AS}: the Aumann–Serrano index of riskiness of additive lotteries, S^{AS}: the Aumann–Serrano index of riskiness of multiplicative lotteries, and S: the Schreiber index of riskiness of multiplicative lotteries. The last column is the measure of risk aversion which can be either the absolute risk aversion (ARA) or the relative risk aversion (RRA).

where γ and w are assumed to be positive.

Recall that by our approach, a decision maker is characterized by a pair consisting of a utility function and an initial wealth level. The values that the wealth can take depend on the specific utility function and are indicated above.

A.2 Risky Assets

All risky assets in the fourth column of Table 2 (labeled "assets") are characterized by random variables. The values that they take are interpreted either as absolute returns (additive lotteries) or relative returns (multiplicative lotteries). To clarify the distinction between additive and multiplicative lotteries note that if the initial wealth is w, accepting additive lottery g causes the wealth to distribute as $w + g$. On the other hand, investing the whole wealth in a multiplicative lottery r causes the wealth to distribute as wr. Following Aumann and Serrano (2008) we assume that the expectation of an additive lottery is positive and that an additive lottery takes at least one negative value with positive probability. Similarly, the geometric mean of a multiplicative lottery is assumed to be greater than one and a multiplicative lottery takes values lower than one with positive probability.

The different sets of risky assets are:

- G is the set of additive lotteries with a finite set of values.
- R is the set of multiplicative lotteries with a finite set of values.
- R^* is the set of multiplicative lotteries.
- G_N is the set of additive lotteries whose returns are distributed normally.
- R_N is the set of multiplicative lotteries whose returns are distributed normally.

A.3 Indices of Risk and Indices of Risk Aversion

Given a decision maker $dm = (u, w)$, the Arrow–Pratt indices of absolute and relative risk aversion are defined as follows:

$$\text{ARA} = -\frac{u''(w)}{u'(w)}, \tag{10}$$

$$\text{RRA} = -w\frac{u''(w)}{u'(w)}, \tag{11}$$

where ARA stands for absolute risk aversion and RRA stands for relative risk aversion.

The principle of monotonicity derives seven different indices of riskiness for the decision functions discussed above. Four of these indices are basically functions of the first and second moments of the distributions of the assets. Given a random variable x, we denote its first moment (expectation) by $\mu(x)$, its second moment by $\mu_2(x) \equiv \mu(x^2)$, and its variance by $\sigma^2(x)$.[7] These notations explain most of the indices that appear in Table 2. Three additional indices of riskiness are as follows:

1. R^{AS} is the Aumann and Serrano (2008) index of riskiness of additive gambles. If $g \in G$ is a lottery, R^{AS} is defined implicitly by

$$\mathrm{E}\, e^{-g/R^{AS}(g)} = 1. \tag{12}$$

2. S^{AS} is the Aumann and Serrano (2008) index of riskiness of multiplicative gambles (securities). If r is a lottery, S^{AS} is defined by

$$S^{AS}(r) \equiv R^{AS}(r-1). \tag{13}$$

3. S is the Schreiber (2014) index of relative riskiness of multiplicative lotteries. If r is a multiplicative lottery, S is defined by

$$S(r) \equiv R^{AS}(\log r). \tag{14}$$

A.4 Decision Problems

A.4.1 Acceptance or Rejection: Cases 1–5

The first decision function in Table 2 is whether to accept or reject a risky asset. The function takes only one of two values: 1 for an acceptance and 0 for a rejection. We say that a decision maker $dm = (u, w)$ accepts an additive lottery g if she benefits from having it, i.e., if $u(w+g) \geq u(w)$. Otherwise she rejects it. Similarly, a decision maker accepts a multiplicative lottery

[7] For a multiplicative lottery r we take x to be $x = r - 1$.

(security) r if she benefits from investing all her wealth in multiplicative lottery r, i.e., if $u(wr) \geq u(w)$. Otherwise she rejects it.

Since the solution of this problem is only one of two possible values, i.e., $\{0, 1\}$, the definition of suitable measures is based on weak monotonicity rather than monotonicity.[8]

Cases 1 and 2. Recall that a quadratic utility function has the form: $u(w) = w - bw^2$. By definition, accepting a lottery g implies that

$$\begin{aligned} \mathrm{E}\left[w + g - b(w+g)^2\right] &\geq w - bw^2 \\ 1 &\geq \frac{b}{1-2bw}\frac{\mathrm{E}\,g^2}{\mathrm{E}\,g}. \end{aligned} \tag{15}$$

Since the right-hand side of the equation is monotonic with $b/(1-2bw)$, which is ARA of a quadratic utility, ARA is a suitable index of risk aversion. Similarly, since the right-hand side is monotonic with $\mathrm{E}\,g^2/\,\mathrm{E}\,g$, μ_2/μ is a suitable index of riskiness. The proof of Case 2 is quite similar.

Cases 3-5. Cases 3 and 4 are proved in Aumann and Serrano (2008). Case 5 is proved at Schreiber (2014).

A.4.2 Risk Premium: Cases 6–7

The risk premium can be measured in terms of either money or in terms of relative return, depending on the type of the risky assets. To define both types of risk premium, let $dm = (u, w)$ be a decision maker and let g be a lottery. The risk premium x of g for dm is defined implicitly by

$$\mathrm{E}\,u(w+g) = u(w + \mathrm{E}\,g - x). \tag{16}$$

Here, x can be interpreted as the sum of money that makes a decision maker indifferent between accepting an additive lottery and accepting the expected

[8] We say that a decision function d is weakly monotonically increasing with respect to an index of riskiness R if for any decision maker dm and any pair of risky assets a_1 and a_2, $R(a_1) > R(a_2) \Rightarrow d(dm, a_1) \geq d(dm, a_2)$. We say that the decision function is weakly monotonic with respect to R if it is weakly monotonically increasing with R or with $-R$. Weak monotonicity of a decision function with respect to risk aversion is defined in a similar way.

value of the lottery but paying x. If r is a multiplicative lottery, the risk premium x of r for dm is defined implicitly by

$$\mathrm{E}\,u(wr) = u(w(\mathrm{E}\,r - x)). \tag{17}$$

Here, x can be interpreted as the interest rate that makes a decision maker indifferent between investing all her wealth in a multiplicative lottery and investing all her wealth in a risk-free interest rate equal to the expected return of the multiplicative lottery minus x.

Case 6. In this case decision makers are CARA and assets are additive lotteries whose absolute returns are distributed normally. The risk premium x is defined implicitly by

$$\begin{aligned}
\mathrm{E}\,e^{-\alpha(w+g)} &= e^{-\alpha(w+\mathrm{E}\,g-x)} \\
e^{-\alpha(w+\mathrm{E}\,g)+0.5\alpha^2\sigma_g^2} &= e^{-\alpha(w+\mathrm{E}\,g-x)} \\
-\alpha(w+\mathrm{E}\,g)+0.5\alpha^2\sigma_g^2 &= -\alpha(w+\mathrm{E}\,g-x) \\
x &= 0.5\alpha\sigma_g^2. \qquad (18)
\end{aligned}$$

The transition between the first and second lines is based on the following lemma: if y is a random variable distributed normally, then

$$\mathrm{E}\,e^y = e^{\mathrm{E}\,y+0.5\sigma_y^2}. \tag{19}$$

Since x is monotonically increasing with α (ARA) and with σ_g^2 (variance), they are the suitable measures in this problem.

Case 7. In this case decision makers are CARA and assets are multiplicative lotteries whose relative returns are distributed normally. We get

$$\begin{aligned}
\mathrm{E}\,e^{-\alpha(wr)} &= e^{-\alpha(w(\mathrm{E}\,r-x)} \\
e^{-\alpha(w\,\mathrm{E}\,r)+0.5\alpha^2w^2\sigma_r^2} &= e^{-\alpha(w(\mathrm{E}\,r-x)} \\
-\alpha w\,\mathrm{E}\,r+0.5\alpha^2w^2\sigma_r^2 &= -\alpha w(\mathrm{E}\,r-x) \\
x &= 0.5\alpha w\sigma_r^2. \qquad (20)
\end{aligned}$$

Here again, the transition between the second and third lines follows from (19). Since x is monotonically increasing with αw (RRA) and with σ_r^2 (variance), they are the suitable measures in this problem.

A similar result cannot be established if we assume that decision makers are CRRA and assets are log-normally distributed multiplicative lotteries. To see this, let r be such a multiplicative lottery and let dm be a CRRA decision maker with parameter γ. Let $y = \log(r)$. By definition, y is distributed normally. The risk premium x is defined implicitly by

$$\begin{aligned}
\mathrm{E}(wr)^{1-\gamma} &= (w(\mathrm{E}\, r - x))^{1-\gamma} \\
w^{1-\gamma}\,\mathrm{E}\, e^{(1-\gamma)y} &= w^{1-\gamma}(\mathrm{E}\, e^{y} - x)^{1-\gamma} \\
\mathrm{E}\, e^{(1-\gamma)y} &= (\mathrm{E}\, e^{y} - x)^{1-\gamma} \\
e^{(1-\gamma)\,\mathrm{E}\, y + 0.5(1-\gamma)^2\sigma_y^2} &= (\mathrm{E}\, e^{y} - x)^{1-\gamma} \\
e^{\mathrm{E}\, y + 0.5\sigma_y^2} - e^{\mathrm{E}\, y + 0.5(1-\gamma)\sigma_y^2} &= x. \qquad (21)
\end{aligned}$$

It is easy to see that x is monotonic with γ. Hence, in this case, RRA is a suitable index of risk aversion. However, unless we restrict the set of multiplicative lotteries there will be no suitable index of riskiness to this decision function. If we assume that $\mathrm{E}\, r = e^{\mathrm{E}\, y + 0.5\sigma_y^2} = c$, where c is some constant, then we get

$$x = c(1 - e^{-\gamma\sigma_y^2}), \qquad (22)$$

and x is monotonic with RRA and σ_y^2. To sum up this example, the risk premium problem for CRRA decision makers has no suitable index of riskiness unless the set of log-normal multiplicative lotteries has the same expectation.

A.4.3 Asset Allocation: Cases 8–11

The asset allocation problem can be described in either absolute or relative terms. Formally, given a decision maker $dm = (u, w)$ and a multiplicative lottery r, the absolute asset allocation is the solution of the problem

$$\arg\max_x \mathrm{E}\, u\big(w + x(r-1)\big), \qquad (23)$$

where x is the optimal amount of money to invest in r. The relative asset allocation problem is the solution of the problem

$$\arg\max_x \mathrm{E}\, u\big(w + xw(r-1)\big), \qquad (24)$$

where x is interpreted as the optimal fraction of wealth to be invested in r.

Cases 8–10.
For a quadratic decision maker, the solution of the absolute asset allocation problem is

$$\arg\max_x \mathrm{E}\left[w + x(r-1) - b\big(w + x(r-1)\big)^2\right]. \tag{25}$$

The first order condition of (25) is

$$\mathrm{E}\left[(r-1) - 2b(w + x(r-1))(r-1)\right] = 0, \tag{26}$$

or,

$$x = \frac{\mathrm{E}(r-1)}{\mathrm{E}(r-1)^2}\frac{1-2bw}{2b}. \tag{27}$$

In this case x is monotonic with ARA $(2b/(1-2bw))$ and monotonic with μ_2/μ, and hence ARA and μ_2/μ are the suitable measures. The proof of the relative case is quite similar.

Cases 9–11.
If decision makers are CARA and returns of multiplicative lotteries are normal, the absolute asset allocation problem becomes

$$\arg\max_x \mathrm{E}\left[e^{-\rho\left(w+x(r-1)\right)}\right]. \tag{28}$$

It follows from (19) that (28) is equivalent to

$$\arg\max_x e^{-\rho w - \rho x\,\mathrm{E}(r-1) + 0.5\rho^2 x^2 \sigma_r}, \tag{29}$$

which is equivalent to

$$\arg\max_x -\rho x\,\mathrm{E}(r-1) + 0.5\rho^2 x^2 \sigma_r. \tag{30}$$

The first-order condition is

$$-\mathrm{E}(r-1) + \rho x \sigma_r = 0, \tag{31}$$

and we get

$$x = \frac{1}{\rho} \frac{\mathrm{E}(r-1)}{\sigma_r^2}. \tag{32}$$

Since x is monotonic with ARA (ρ) and monotonic with σ_r^2/μ_r, ARA and σ_r^2/μ_r are the suitable measures in this case. The proof of the relative case (Case 11) is quite similar.

A.4.4 The Certainty Equivalent of Optimal Allocation: Cases 12–13

Assume that x^* is the solution of the absolute asset-allocation problem for decision maker $dm = (u, w)$ and multiplicative lottery r. The absolute certainty equivalent of the optimal allocation, denoted by y, is defined implicitly by

$$\mathrm{E}\, u(w + x^*(r-1)) = u(w + y). \tag{33}$$

In this case, y can be interpreted as the amount of money that makes the decision maker indifferent between investing optimally in r and accepting y. Similarly, the relative certainty equivalent of the optimal allocation, denoted by y, is defined implicitly by

$$\mathrm{E}\, u(w + x^*(r-1)) = u(w(1 + y)). \tag{34}$$

Here, y can be interpreted as the net risk-free interest rate that makes the decision maker indifferent between investing optimally in r and accepting net return y on w.

In Case 13, y is defined implicitly by the equation

$$\mathrm{E}\, e^{-\rho\big(w + x(r-1)\big)} = e^{-\rho\big(w+y\big)}. \tag{35}$$

According to (32) we substitute $x = \frac{1}{\rho}\frac{\mathrm{E}(r-1)}{\sigma_r^2}$ and get

$$\begin{aligned} \mathrm{E}\, e^{-\rho\big(w + \frac{1}{\rho}\frac{\mathrm{E}(r-1)}{\sigma_r^2}(r-1)\big)} &= e^{-\rho\big(w+y\big)} \\ e^{-\rho w - \frac{\mu_r^2}{\sigma_r^2} + 0.5\frac{\mu_r^2}{\sigma_r^2}} &= e^{-\rho\big(w+y\big)} \\ -\rho w - \frac{\mu_r^2}{\sigma_r^2} + 0.5\frac{\mu_r^2}{\sigma_r^2} &= -\rho\big(w+y\big) \\ y &= \frac{1}{2\rho}\frac{\mu_r^2}{\sigma_r^2}. \end{aligned} \tag{36}$$

Similarly, the certainty equivalent of the relative optimal allocation, denoted by y and defined implicitly by

$$\mathrm{E}\, e^{-\rho\Big(w+w\alpha(r-1)\Big)} = e^{-\rho\Big(w(y+1)\Big)}, \tag{37}$$

is

$$y = \frac{1}{2w\rho}\frac{\mu_r^2}{\sigma_r^2}. \tag{38}$$

In both cases, the solution of the decision function is a multiplication of an index of risk aversion (ARA or RRA) by an index of risk, namely, the Sharpe ratio.

A.5 Other Decision-making Problems

Although a decision function is a function of decision makers and risky assets, a decision function does not necessarily depend on the decision maker's utility function. For instance, the decision problem described in Foster and Hart (2009), which deals with avoiding bankruptcy, has nothing to do with utilities. The suitable index of riskiness of their problem is the index of riskiness that is defined there. Interestingly, they say nothing about the suitable index of risk aversion that turns out to be the level of wealth of the decision maker. Another example of a decision-making problem that does not depend on utilities appears in Meilijson (2009). It is shown there that the Aumann-Serrano index of riskiness can be used as a relevant index in a decision making problem that also has to do with avoiding bankruptcy. Utility functions are irrelevant there too. Note that our definition of a decision function is even more general and includes probabilistic questions whose solutions do not depend on any parameter of decision makers. Decision-making problems of this kind are beyond the scope of the present paper.

A.6 Stochastic Dominance

As Aumann and Serrano (2008) write, the most uncontroversial, widely accepted notions of riskiness are provided by the concepts of stochastic dominance (Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970)). We say that a random variable x first-order dominates

(FOD) y if $x \geq y$ for sure and $x > y$ with positive probability; and x second-order dominates (SOD) y if x may be obtained from y by "mean-preserving spreads," i.e., by replacing some of x's values with random variables whose mean is that value. We say that x stochastically dominates y if there is a random distributed variable like x that dominates y.

It is interesting to note that most of the indices of riskiness in Table 2 (except for the R^{AS}, S^{AS}, and S) are not compatible with stochastic dominance in the sense that they may consider a certain asset as riskier than others even though it may stochastically dominate the others. The explanation is simple: if one risky asset stochastically dominates another risky asset, it means that all decision makers will prefer to have the first risky asset. It is not necessarily relevant to other decision-making problems.

B Proofs

B.1 The Securities Model

The uncertainty in this model is generated by K standard Wiener processes $W^1, \ldots, W^K$ defined on a filtered probability space (Ω, F_T, F, P) that satisfies the so-called usual conditions. The filtration $F = (F_t)_{t\in[0,T]}$ is the augmentation of the natural filtration F^W, generated by the vector $W = \{W(t) = W^1(t)...W^K(t), t \in [0,T]\}$ of standard independent Wiener processes; see Karatzas and Shreve (1998).

B.2 Proof of Theorem A

Throughout this section we use Ito's lemma several times. It is worthwhile to recall a simple version of this lemma. If s is a random process described by

$$ds = \mu dt + \sigma dW, \tag{39}$$

and $f(s,t)$ is a twice-differentiable function, then

$$df = [\mu_t f_s + 0.5\sigma\sigma f_{ss} + f_t]dt + f_s \sigma dW, \tag{40}$$

where f_s and f_{ss} are the first and second derivatives of f in relation to s, and f_t is the first derivative of f in relation to t.

The following two lemmas will be useful for our proofs.

Lemma B.1. *Let $F_t(y)$ be a set of real-valued, continuous, and monotonic functions, with $0 < t \leq T$ and $y \in \mathbb{R}$. Assume that there exists a continuous and monotonic function $F(y)$ such that*

1. *$\forall y, F(y) = \lim_{t \to 0} F_t(y)$.*
2. *$\exists y^*$, s.t. $F(y^*) = 0$.*

Then, there exists ϵ s.t.

$$\forall t < \epsilon \ \exists y_t \ s.t. \ F_t(y_t) = 0,$$

and

$$\lim_{t \to 0} y_t = y^*.$$

Proof. Given $\delta > 0$ we have to show that there exists ϵ s.t. $\forall t < \epsilon \ |y_t - y^*| < \delta$ and that y_t satisfies $F_t(y_t) = 0$. Since $F(y)$ is monotonic (either increasing or decreasing), there exists a positive number C such that $|F(y^* - \delta)| > C$ and $|F(y^* + \delta)| > C$. Condition 1 implies that there exists ϵ s.t. $\forall t < \epsilon$,

$$|F_t(y^* + \delta) - F(y^* + \delta)| < C,$$

and

$$|F_t(y^* - \delta) - F(y^* - \delta)| < C.$$

Hence, either $F_t(y^* - \delta) < 0$ and $F_t(y^* + \delta) > 0$ or $F_t(y^* - \delta) > 0$ and $F_t(y^* + \delta) < 0$. Since F_t is continuous, $\exists y_t \in (y^* - \delta, y^* + \delta)$ s.t. $F(y_t) = 0$. □

Lemma B.2. *Let $F_t(\alpha)$ be a set of twice-differentiable concave functions where $0 < t \leq T$ and $\alpha \in \mathbb{R}$, and let F be a twice-differentiable concave function such that*

1. *$\forall \alpha, \ F(\alpha) = \lim_{t \to 0} F_t(\alpha)$.*
2. *$\alpha^* = \arg\max_\alpha F(\alpha)$.*

Then, there exist $\epsilon > 0$ such that

$$\forall t < \epsilon, \exists \alpha_t \ s.t. \ \alpha_t = \arg\max_\alpha F_t(\alpha),$$

and

$$\lim_{t \to 0} \alpha_t = \alpha^*.$$

Proof. We have to show that given $\delta > 0$, there exists $\epsilon > 0$ such that $\forall t < \epsilon$, $\exists \alpha_t$, which maximizes $F_t(\alpha)$, and that $|\alpha_t - \alpha^*| < \delta$.

Let $\delta_1 = \min\{F(\alpha^*) - F(\alpha^* - \delta), F(\alpha^*) - F(\alpha^* + \delta)\}$. There exists ϵ s.t. $\forall t < \epsilon$,

$$|F_t(\alpha^*) - F(\alpha^*)| < \delta_1/3$$
$$|F_t(\alpha^* + \delta) - F(\alpha^* + \delta)| < \delta_1/3$$
$$|F_t(\alpha^* - \delta) - F(\alpha^* - \delta)| < \delta_1/3.$$

Hence, $\forall t < \epsilon$

$$F_t(\alpha^*) > F_t(\alpha^* - \delta) \text{ and } F_t(\alpha^*) > F_t(\alpha^* + \delta).$$

Since for all t, F_t is concave, there exists $\alpha_t \in (\alpha^* - \delta, \alpha^* - \delta)$, which is the argmax of F_t. □

Proofs of the Results of Table 2

Proof. **(Cases** 1 **and** 5**)**
The absolute risk premium r_t is defined implicitly by

$$\mathrm{E}\left[u\Big(w - s_0 + s_t\Big)\right] = u\Big(w + \mathrm{E}(s_t) - s_0 - (e^{r_t t} - 1)\Big). \tag{41}$$

We define a set of functions, F_t, for all $t > 0$, as follows.

$$F_t(r) = \Big(u\big(w - s_0 + \mathrm{E}(s_t) - (e^{rt} - 1)\big) - \mathrm{E}\left[u\big(w - s_0 + s_t\big)\right]\Big)\Big/t. \tag{42}$$

Obviously, for every $t > 0$, if $F_t(r) = 0$, then $r = r_t$.

Lemma B.3. *For all* $r \in \mathbb{R}$,

$$\lim_{t \to 0} F_t(r) = -u'(w)r - \frac{1}{2}\sigma_0^2 u''(w). \tag{43}$$

Proof.
It will be easier to calculate the limit of $F_t(r)$ as the difference between two functions:

$$g_t(r) \equiv \Big(u\big(w - s_0 + \mathrm{E}(s_t) - (e^{rt} - 1)\big)\Big)\Big/t - u(w)/t,$$

and

$$h_t \equiv \mathrm{E}\left[u\big(w - s_0 + s_t\big)\right] \Big/ t - u(w)/t.$$

Clearly,

$$F_t(r) = g_t(r) - h_t.$$

Since the three functions u, $E[s_t]$, and e^{rt} are differentiable with respect to t, the limit of $g_t(r)$ as t goes to zero can be calculated using the L'Hopital rule:

$$\lim_{t \to 0} g_t(r) = u'(w)(\mu_0 - r). \tag{44}$$

From Ito's lemma it follows that

$$h_t = \mathrm{E}\left[\int_0^t \big(u'_q \mu_q + \frac{1}{2} u''_q \sigma_q^2\big) dq\right] \Big/ t. \tag{45}$$

As t goes to zero, the limit of h_t is equal to the integrand at $t = 0$, i.e.,

$$\lim_{t \to 0} h_t = u'(w)\mu_0 + \frac{1}{2} u''(w)\sigma_0^2. \tag{46}$$

From (44) and (46) we get the desired result:

$$\begin{aligned} \lim_{t \to 0} F_t(r) &= \lim_{t \to 0} g_t(r) - \lim_{t \to 0} h_t(r) \\ &= -u'(w)r - \frac{1}{2}\sigma_0^2 u''(w). \end{aligned} \tag{47}$$

□

Now, let F(r) be defined as the limit of F_t's, i.e.,

$$F(r) \equiv \lim_{t \to 0} F_t(r) = -ru'(w) - \frac{1}{2}\sigma_0^2 u''(w). \tag{48}$$

In addition, let r^* be the real number s.t. $F(r^*) = 0$, i.e.,

$$r^* = -\frac{1}{2} \frac{u''(w)}{u'(w)} \sigma_0^2. \tag{49}$$

Since the two conditions of lemma B.1 are satisfied we conclude that

$$\lim_{t \to 0} r_t = r^* \tag{50}$$

Hence, r^* is the solution for the local absolute risk premium problem. It is easy to see that r^* is monotonic with respect to $-u''(w)/u'(w)$ and with respect to σ_0^2, which are an index of risk aversion and an index of riskiness, respectively. The proof of the relative case (Case 5) is quite similar. □

Proof. (**Cases 2 and 6**)
The (absolute) optimal allocation $\alpha_t(dm, s)$ is defined by

$$\alpha_t(dm, s) = \arg\max_{\alpha} E\left[u\left(w - \alpha s_0 + \alpha s_1\right)\right]. \tag{51}$$

We start by defining a set of functions as follows:

$$F_t(\alpha) = \left(E\left[u\left(\alpha w - \alpha s_0 + \alpha s_1\right)\right] - u(w)\right)\Big/ t. \tag{52}$$

Using Ito's lemma,

$$F_t(\alpha) = E_0\left[\int_0^t \alpha \mu_q u'_q + \frac{1}{2}\alpha^2 \sigma_q^2 u''_q \; dq\right]\Big/ t, \tag{53}$$

where $u'_q \equiv du(x_q)/d(x_q)$, $u''_q \equiv du^2(x_q)/d^2(x_q)$, and where $x_q = w - \alpha s_0 + \alpha s_q$. We define $F(\alpha)$ to be the limit of $F_t(\alpha)$ as t goes to zero:

$$\begin{aligned} F(\alpha) &\equiv \lim_{t \to 0} F_t(\alpha) \\ &= \alpha \mu_0 u'(w) + \frac{1}{2}\alpha^2 \sigma_0^2 u''(w). \end{aligned} \tag{54}$$

We denote by α^* the value of α that maximizes F:

$$\begin{aligned} \alpha^* &= \arg\max_{\alpha} F(\alpha) \\ &= -\frac{u'(w)}{u''(w)} \frac{\mu_0}{\sigma_0^2}. \end{aligned} \tag{55}$$

To show our result we use Lemma B.2. To see that for all t, F_t is a twice-differentiable and concave function, we rewrite F_t as the sum of two expressions:

$$F_t(\alpha) = \alpha \, \mathrm{E}_0 \left[\int_0^t \mu_q u'_q\right]\Big/ t + \alpha^2 \, \mathrm{E}_0 \left[\frac{1}{2}\sigma_q^2 u''_q \; dq\right]\Big/ t. \tag{56}$$

The assumption that the second derivative of the utility function is negative implies that the first derivative of $F_t(\alpha)$ decreases, which means that F_t is a concave function.

Since the two conditions of Lemma B.2 are satisfied, there exists $\epsilon > 0$ s.t. for all $t < \epsilon$, α_t does exist and:

$$\lim_{t\to 0} \alpha_t = \alpha^*. \tag{57}$$

It is easy to see that α^* is monotonic with $-u''(w)/u'(w)$ and with $\sigma_0^2/(\mu_0)$. That completes the proof of the absolute case. The proof of the relative case is quite similar. □

Proof. (**Cases 3 and 6**)
The certainty equivalent of the optimal allocation $z_t(dm, s)$ is defined implicitly by

$$u(w + e^{z_t t} - 1) = E\left[u\left(w - \alpha_t s_0 + \alpha_t s_t\right)\right]. \tag{58}$$

where $\alpha_t \equiv \alpha_t(dm, s)$, defined in (55). Let F_t be a set of functions, defined as follows:

$$F_t(z) = \left(u(w + e^{zt} - 1) - E\left[u\left(w - \alpha_t s_0 + \alpha_t s_1\right)\right]\right)\Big/ t \tag{59}$$

It is easy to see that if $F_t(z) = 0$ then $z = z_t$. To calculate the limit of z_t, it will be easier to look at F_t as the difference between two functions g_t and h_t, defined by

$$g_t(z) = \left(u(w + e^{zt} - 1) - u(w)\right)\Big/ t \tag{60}$$

and

$$h_t = \left(E\left[u\left(w - \alpha_t s_0 + \alpha_t s_1\right)\right] - u(w)\right)\Big/ t. \tag{61}$$

$$\tag{62}$$

Clearly,

$$F_t(z) = g_t(z) - h_t$$

for every value of z.

The limit of $g_t(z)$ as t goes to zero can be calculated by using the L'Hopital rule:

$$\lim_{t\to 0} g_t(z) = u'(w) \cdot z. \tag{63}$$

Recall that according to 55,

$$\alpha^* = -\frac{u'(w)}{u''(w)} \frac{\mu_0}{\sigma_0^2};$$

substituting α^* in h_t and taking the limit we get

$$\lim_{t\to 0} h_t = -\frac{u'^2 \mu_0^2}{u'' \sigma^2} + \frac{1}{2} \frac{u'^2 \mu_0^2}{u'' \sigma^2}. \tag{64}$$

Now, we define $F(z)$ to be the limit of $F_t(z)$, where t goes to zero:

$$\begin{aligned} F(z) &\equiv \lim_{t\to 0} F_t(z) \\ &= \lim_{t\to 0} g_t(z) - \lim_{t\to 0} h_t(z) \\ &= u_0' z + \frac{1}{2} \frac{(u_0')^2}{u_0''} \big(\frac{\mu_0}{\sigma_0}\big)^2 \end{aligned} \tag{65}$$

We define z^* to be the value that results in $F(z^*) = 0$:

$$z^* = -\frac{1}{2} \frac{u_0'}{u_0''} \big(\frac{\mu_0}{\sigma_0}\big)^2 \tag{66}$$

Since the conditions of Lemma B.1 are satisfied, we get:

$$\lim_{t\to 0} z_t = z^*. \tag{67}$$

It is easy to see that z^* is monotonic with the measures of risk and risk aversion, as appears in Table 2. The proof of the relative case is quite similar.
□

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