# Good Lies

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**Abstract**

Decision makers often face uncertainty both about the ability and the objectives of their advisors. If an expert is sufficiently concerned about establishing a reputation for being skilled and unbiased, she may truthfully report her private information about the decision-relevant state. However, we show that truthful revelation may not necessarily maximise the expected payoff of the decision maker. There is indeed a trade-off between the amount of information revealed about the decision-relevant state and what the decision maker learns about the advisor's type. While in a truth-telling equilibrium the decision maker learns only about the ability of the expert, in an equilibrium with some misreporting the decision maker also learns something about the expert's preferences. Therefore, although truthful revelation allows for more informed current decisions, it may lead to worst sorting. Thus, if a decision maker places enough weight on future choices relative to present ones, some lying may be preferred to truth-telling.

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# 1 Introduction

There are many settings in which decision makers ($DM$s) consult experts before making a decision. The assumption is that experts have better information than $DM$s about a decision-relevant state of the world. Ultimately, the value of the expert's information depends both on the ability of the expert and on the preferences of the expert over the ideal choice of action. While ability determines the accuracy of the information gathered by the expert, a misalignment between expert's and $DM$'s preferences may lead the expert not to reveal his accurate information truthfully. It is often the case that a $DM$ faces some uncertainty both about the ability and the preferences of the expert he consults.

This uncertainty becomes particularly problematic in contexts characterized by a high degree of contract incompleteness whereby the $DM$ cannot directly reward the expert for the accuracy of her advice or commit to a decision rule. Implicit incentives such as reputational concerns are understood to significantly shape these relationships. This is the case of ongoing relationships such as those between patients and doctors, firms and consultants, investors and financial advisors or politicians and policy advisors.

In these contexts, two questions arise. The first question is about how much information about the decision-relevant state can be credibly transmitted when both the ability and the preferences of the expert are unknown to the $DM$. The second question is about the scope of the $DM$'s learning about the type of expert with whom he interacts. The key result of the present paper is to highlight that these two issues are related one to the other. In particular, we show that there is a trade-off between the amount of information on the decision-relevant state that can be credibly transmitted and the amount of learning on the expert's type. In a truthtelling equilibrium, all the information of the expert about the decision-relevant state is passed on to the $DM$. However, since all experts use the same reporting strategy (i.e. they all report their information truthfully), the $DM$ does not learn anything about the preferences of the expert. From the DM's viewpoint, this outcome is shown to be inferior to the outcome of an equilibrium where the information about the state is only partially revealed. In a partially revealing equilibrium, the reporting strategies of biased experts differ from those of unbiased experts. Thus, the report of the expert reveals some information also about the preference type of the expert.

The existing literature on reputational cheap talk has explored the issue of how much information about the decision-relevant state can be credibly transmitted by focusing on models where uncertainty is either about the ability of the expert (Ottaviani-Sorensen, 2006) or the preferences of the expert (Morris, 2001). A contribution of the present paper is to propose a model where both sources of uncertainty coexist.

Our paper is also related to a more recent literature that addresses settings in which reputational concerns are non-linear. In particular, Li (2007) analyzes how convexity in reputational payoffs, induces experts to change their mind when sending sequential reports. Prat (2005) assumes that the decision maker's utility is convex in reputation as a proxy for learning, in order to analyze welfare.

The remainder of the paper is organized as follows. In Section 2, we introduce the general setup of the model. In Section 3 we characterize the informative equilibria in which the unbiased expert truthtells and in Section 4 we analyze welfare for these equilibria in order to illustrate our main point. In Section 5 we discuss other informative equilibria and in Section 6 we provide a complete mapping of all the equilibria providing general welfare results. Section 7 concludes.

# 2 Model

There are two periods indexed with $t = 1, 2$. In the first period, a Decision Maker ($DM$) has to choose an action $a_1 \in (0, 1)$. We can think of the $DM$'s choice as the choice of whether to invest ($a_1 = 1$) or not to invest ($a_1 = 0$) in a project. The return of the investment depends on the action chosen by the DM as well as on the state of the world $x_1 = 0, 1$, as follows:

$$R_1(a_1, x_1) = \begin{cases} r & \text{if } a_1 = 1, x_1 = 1 \\ -r & \text{if } a_1 = 1, x_1 = 0 \\ 0 & \text{if } a_1 = 0 \end{cases}$$

The $DM$ does not observe $x_1$ and it is common knowledge that $\Pr(x_1 = 1) = 1/2$.[1] Note that in the absence of further information, $E(R_1 \mid a_1 = 1) = 0$ , and the $DM$ is indifferent between investing and not investing. Upon paying a fixed fee $w_1$, the $DM$ can consult an

[1] The assumption of a fair prior for the state of the world is not without loss of generality. However the results that we present hold whenever the prior on the state is not too extreme. Note that a prior that is not too extreme represents the situation in which the uncertainty about the true state is highest and thus it is more likely that the DM seeks the advice of an expert.

expert.[2] The expert observes a signal $s_1 \in (0,1)$ about state $x_1$ and provides her advice in the form of a message $m_1 \in (0,1)$. The $DM$ observes $m_1$ and then chooses $a_1$.[3]

We assume that there is a finite pool of ex-ante identical experts and the DM can consult only one expert per period. Experts differ in their ability and preferences over the action chosen by the DM.

**Expert's ability.** An expert can be either smart ($S$) or dumb ($D$) ability. The ability of the expert affects the precision of his signal, as follows:

$$\Pr(s_1 = x_1 \mid x_1, S) = p > \Pr(s_1 = x_1 \mid x_1, D) = 1/2$$

Thus, a smart expert receives an informative signal while a dumb expert receives an uninformative signal about $x_1$. We let $\alpha$ denote the common prior about an expert being smart. As it is customary in models of career concerns, we assume that neither the DM nor the expert know the ability type of the expert. We denote the expected precision of the expert's signal with $q \equiv \alpha p + (1-\alpha)\frac{1}{2}$.

**Expert's preferences.** An expert can be either unbiased ($U$) or biased ($B$). An unbiased expert does not have any explicit preference in favor of a specific action chosen by the DM. Without loss of generality, we assume that a biased expert has a bias in favour of action $a_t = 1$. We assume that an expert knows his own preferences and let $\gamma$ denote the common prior about an expert being unbiased. Ability and preference types are independent.

At the end of the first period, state $x_1$ is revealed. The DM can then use the pair $(m_1, x_1)$ to update his prior belief about the incumbent being smart, $\widehat{\alpha}(m_1, x_1) \equiv \Pr(S \mid m_1, x_1)$, as well as his prior belief about the incumbent being unbiased, $\widehat{\gamma}(m_1, x_1) \equiv \Pr(U \mid m_1, x_1)$. We interpret them as the reputations that the incumbent expert has established at the end of the first period for being smart and for being unbiased respectively. We also denote the corresponding update on the expected precision of the incumbent expert's signal with

[2]We discuss fees more in the detail at the end of the present section. As it will become clear, the fee $w_1$ does not play any role in the present model. In principle, it could be set equal to zero. Hence, the participation constraint of the DM is trivially satisfied whenever some information of the expert is transmitted in equilibrium. Note also that we are implicitly assuming that the outside option of the expert is equal to zero.

[3]Since the ex-ante expected return of the investment in the absence of additional information is zero, the DM always finds it optimal to hire an expert in any equilibrium where the expert reveals some of his private information about the state.

$\widehat{q}(m_1, x_1) \equiv \widehat{\alpha}(m_1, x_1)p + (1 - \widehat{\alpha}(m_1, x_1))\frac{1}{2}$. This concludes the first period.

The second period is identical to the first period except for two notable features. First, at the beginning of the second period the $DM$ uses information $(m_1, x_1)$ to decide whether to retain the incumbent or replace him with a new expert. In this latter case, the new expert is randomly extracted from the original pool of experts. Second, while $w_2$ is still fixed, it is not set exogenously. It is instead set equal to the expected surplus generated by the expert's information in the second period.[4] Except for these two features, the game unfolds as in the first period with either the incumbent expert or a new expert, with state $x_2$, signal $s_2$, message $m_2$, action $a_2$ and return $R_2(a_2, x_2)$. States $x_1$ and $x_2$ as well as signals $s_1$ and $s_2$ are assumed to be *iid*.

It is worth discussing our modelling choice of the fees. Both $w_1$ and $w_2$ are fixed fees to reflect a world of highly incomplete contracts where both the true state of the world and the report of the expert are observable but not verifiable, and thus experts cannot be paid conditional on the accuracy of their reports, nor can contracts be written on reports. While $w_1$ is set exogenously at the beginning of the first period, $w_2$ is set equal to the expected surplus generated by the expert's information in the second period. This latter assumption serves two purposes. First, it allows us to create reputational concerns on the part of the incumbent expert since the expected value of his information depends on the values of reputations $\widehat{\alpha}(m_1, x_1)$ and $\widehat{\gamma}(m_1, x_1)$ that he has acquired in the first period. Thus, the incumbent's choice of $m_1$ takes into account the impact on $w_2$ via reputations $\widehat{\alpha}(m_1, x_1)$ and $\widehat{\gamma}(m_1, x_1)$. Second, it guarantees that the expected payoff of the incumbent is continuous in $\widehat{\alpha}(m_1, x_1)$ and $\widehat{\gamma}(m_1, x_1)$. This makes the set of informative equilibria richer than what we would have if $w_2$ were set equal to a fixed exogenous value.

The fee $w_2$ is not the only channel through which reputational concerns work. An extra reputational channel is represented by the ability of the $DM$ to fire the incumbent at the end of $t = 1$. In fact, we introduce the hiring and firing mechanism not so much to induce reputational concerns but to allow the $DM$ to fully exploit what he learns at the end of $t = 1$ about the ability and the preferences of the incumbent. Put differently, in the present model, hiring and firing have more a flavour of a sorting rather than a reputation mechanism.

We conclude this section by stating the payoffs of the players. In each period, the $DM$

[4]This is consistent with a standard setting in which the expert has monopoly power and is paid a wage equal to the expected value of the information provided.

receives a payoff equal to $R_t(a_t, x_t)$. Thus, the $DM$'s stage-payoff depends on the correctness of the decision gross of the monetary fee paid to the expert. This is equivalent to assuming that experts are single agents that seek to maximize their monetary payoff, while the DM's is simply concerned about taking the best state contingent action in each period.

In each period, a biased expert gets a payoff equal to $w_t + a_t$. We assume that both components $w_t$ and $a_t$ of the payoff are relation-specific. In particular, a biased expert benefits from DM's choice of action $a_t = 1$ if and only if the expert has been hired by DM.[5] In other words, a biased expert faces a relation specific conflict of interest. The stage payoff of an unbiased expert is simply equal to $w_t$. Put differently, an unbiased expert is a no-conflict type. Note that since the second period fee depends on the expert's reputation, an unbiased expert is only motivated by career concerns.

We assume that the $DM$ and the expert may assign different weights to period 1 and period 2 payoffs. We let $\delta_{DM} \in (0,1)$ and $\delta_E \in (0,1)$ measure the importance that the $DM$ and the expert respectively give to their future payoff relative to their current one. Accordingly, total utilities read:

$$\begin{aligned} U_{DM} &= (1-\delta_{DM})R_1(a_1, x_1) + \delta_{DM} R_2(a_2, x_2) \\ U_U &= (1-\delta_E)w_1 + \delta_E w_2 \\ U_B &= (1-\delta_E)(w_1 + a_1) + \delta_E(w_2 + a_2) \end{aligned}$$

# 3 Equilibrium Analysis - Equilibria in which the Good Expert Truthtells

We use the concept of Perfect Bayesian Equilibrium and restrict our attention to informative equilibria where the $DM$ changes his beliefs about the state of the world after some message on the equilibrium path.[6] For the sake of exposition we will refer to these equilibria simply as informative equilibria.[7]

[5]For example, a financial analyst can obtain some benefits from inducing an investor to make an investment (no matter the ex-post return of the investment).

[6]Without loss of generality, we restrict attention to informative equilibria in which DM interprets message 1 to be (weakly) correlated with signal 1 and hence state 1.

[7]In the present game, an equilibrium could be considered informative even when DM learns something about the type of the expert and nothing about $x_t$. It is straightforward to show that this can never be the

We are interested in addressing the following questions. In the equilibria in which the $DM$ learns something about the state, how much does the $DM$ learn about the ability and the preferences of the expert? How are learning about the state and learning about the expert's type related?

In this section we focus on equilibria in which the unbiased type ($U$) truthtells. This suffices to illustrate our main point. In sections 5 and 6, we discuss other informative equilibria, while in the appendix we provide a thorough analysis of the equilibria.

The game can be solved by backward induction. Notice that since $R_t(1,1) = -R_t(1,0)$ and $\Pr(x_t = 1) = \frac{1}{2}$, it is straightforward to verify that in any informative in which the $DM$ changes her beliefs about the state of the world after some message on the equilibrium path, she will also follow the expert's advice. Thus, in all the equilibria we focus on, $a_t(m_t) = m_t$.[8]

**Second Period**

In the last period, the expert will not be concerned about her reputation. Thus the bad expert will always claim to have observed signal 1 in order to induce DM to choose action 1. For a good expert with no explicit preferences in favour of a particular action, any strategy is a continuation equilibrium. In line with the rest of the literature on career concerns, we focus on the continuation equilibrium in which the good expert acts in the interest of the DM and thus truthfully reveals his signal.

At the beginning of the second period, DM chooses whether to retain the incumbent or hire a new expert. Let $V(m_1, x_1)$ denote the expected surplus generated by the incumbent's information. Similarly, let $V$ denote the expected surplus generated by the information of a new expert randomly chosen from the pool of experts. Since DM's expected return on the investment in the absence of further information is equal to zero, the expected surplus generated by the information of an expert coincides with the investment return that the DM expects to obtain conditional on consulting an expert. Given the second-period reporting strategies of a bad and good expert, it is straightforward to show that:

$$V(m_1, x_1) \equiv \frac{r}{2}\widehat{\gamma}(m_1, x_1)\left[2\widehat{q}(m_1, x_1) - 1\right] \qquad (1)$$

case.

[8] Put differently, in this model if an equilibrium is informative, it is also persuasive. With discrete actions and a prior that is not fair, an informative equilibrium may not be persuasive. For example, if either the prior on the state is extreme or the return in one state is extreme, a message by $E$ may induce $DM$ to revise his beliefs about the state. However, this revision may not be sufficient to induce the $DM$ to choose the action reccomended by $E$.

$$V \equiv \frac{r}{2}\gamma\left(2q-1\right) \tag{2}$$

Note that the expected value of the incumbent's information, $V(m_1, x_1)$ is strictly increasing in the incumbent's reputations $\widehat{\gamma}(m_1, x_1)$ and $\widehat{\alpha}(m_1, x_1)$. The values of these reputations - and hence the value of $V(m_1, x_1)$ itself- endogenously depend on the equilibrium played in the first period. The expected value of the information on a new expert, $V$ simply depends on the prior values of reputations $\gamma$ and $\alpha$. Furthermore, since both $q$ and $\widehat{q}(m_1, x_1)$ are greater than $\frac{1}{2}$ (i.e., in expectation the expert always has better information than the $DM$), both $V(m_1, x_1)$ and $V$ are strictly positive. Thus, the $DM$ always finds it optimal to consult an expert in period 2. In particular, the $DM$ will retain the incumbent whenever $V(m_1, x_1) \geq V$ and fire him otherwise. Formally, let $\imath(m_1, x_1)$ denote $DM$'s strategy where:

$$\imath(m_1, x_1) = \begin{cases} 1 \text{ if } V(m_1, x_1) \geq V \\ 0 \text{ otherwise} \end{cases} \tag{3}$$

**First Period**

We assume that the continuation equilibrium described in the previous subsection is played, and proceed to characterize those equilibria in which the good expert truthtells in the first period.

An expert that is randomly chosen to provide advice in the first period will be concerned about the impact that his message $m_1$ has on $V(m_1, x_1)$ since this determines both her chances of being retained as well as her expected fee.

For a bad expert that observes signal $s_1 = s_1^{'}$, the *expected* continuation payoff of choosing message $m_1 = m_1^{'}$ reads:

$$\sum\nolimits_{x_1} \Pr(x_1 \mid s_1^{'}) \left[V(m_1^{'}, x_1) + 1\right] \imath(m_1^{'}, x_1)$$

The previous expression can be explained as follows. For a given pair $(m_1^{'}, x_1)$, the continuation payoff of a bad expert reads $[V(m_1, x_1) + 1]\, \imath_{m_1, x_1}$. Given message $m_1^{'}$, the probability that pair $(m_1^{'}, x_1)$ realizes is equal to $\Pr(x_1 \mid s_1^{'})$. Therefore, a bad expert with signal $s_1 = s_1^{'}$ will chose $m_1$ to maximize:

$$(1-\delta_B)\left[w_1 + a(m_1)\right] + \delta_B \sum\nolimits_{x_1} \Pr(x_1 \mid= s_1^{'}) \left[V(m_1, x_1) + 1\right] \imath(m_1, x_1) \tag{4}$$

So, a bad expert is concerned about the impact that $m_1$ has both on his current payoff (since $m_1$ affects DM's current action) and his future payoff.

For an unbiased expert that observes $s_1 = s_1^{'}$, the expected continuation payoff of choosing message $m_1 = m_1^{'}$ reads:

$$\sum\nolimits_{x_1} \Pr(x_1 \mid s_1^{'}) \left[V(m_1^{'}, x_1)\right] \imath(m_1^{'}, x_1)$$

Thus, an unbiased expert with signal $s_1 = s_1^{'}$ will chose $m_1$ to maximize:

$$(1 - \delta_{DM})\, w_1 + \delta_{DM} \sum\nolimits_{x_1} \Pr(x_1 \mid= s_1^{'}) \left[V(m_1, x_1)\right] \imath(m_1, x_1) \tag{5}$$

So, an unbiased expert is concerned only about the reputational effects of $m_1$ (since $m_1$ does not affect the current payoff of the $U$ expert).

Given the incentives faced by the biased and the unbiased expert, we now analyze what informative equilibria exist where $U$ truthfully reveals her private information. The following proposition provides a characterization of these equilibria:

**Proposition 1** *Any informative equilibrium in which the $U$ expert truthtells is characterized by the $B$ expert truthfully revealing $s_1 = 0$ and truthfully revealing signal $s_1 = 0$ with probability $\lambda_B \in (0, 1]$ and by the $DM$ hiring the expert if and only if $m_1 = x_1$.*

To understand the forces at work, it is a good idea to start with the analysis of a putative truthtelling ($TT$) equilibrium, i.e., an equilibrium in which *both* $U$ and $B$ truthfully reveal their private signals in the first period.

A first key thing to notice is that in a $TT$ equilibrium, $U$ and $B$ perfectly separate with respect to the signal, but pool with respect to their preference type. In other words, both expert types use the same reporting strategy of truthfully revealing their signal. Thus, message $m_1$ does not convey any information about the preferences of the incumbent expert. On the other hand, since the message contains all the information of the expert about the state of the world, observing the message and the state of the world allows the $DM$ to make the sharpest inference about the ability type of the expert.[9] Formally, in a $TT$ equilibrium

[9] The same occurs in babbling equilibria where there is also no learning about ability.

we have that reputations are updated as follows:[10]

$$\gamma^{TT}(m_1, x_1) = \gamma \text{ for all } (m_1, x_1)$$

$$\underline{\alpha} \equiv \alpha^{TT}(1,0) = \alpha^{TT}(0,1) < \alpha < \alpha^{TT}(1,1) = \alpha^{TT}(0,0) \equiv \overline{\alpha}$$

Thus, learning takes place in a way that if the expert makes a correct (incorrect) call, his reputation for being smart increases (decreases), while there is no impact on his reputation for being unbiased. Based on these values of reputations, it is immediate to verify that:

$$V^{TT}(1,0) = V^{TT}(0,1) < V < V^{TT}(1,1) = V^{TT}(0,0)$$

This implies that in a putative $TT$ equilibrium, the $DM$ will retain the expert if she makes a correct call and fire her otherwise. This in turn provides the incumbent expert with the incentives to truthfully reveal her information. Indeed, since the signal is informative, reporting it truthfully maximizes the probability of being ex-post correct, i.e., that $m_1 = x_1$, and thus maximizes the expected continuation payoff of the expert. Based on (5), $U$ cares only about the impact of $m_1$ on her expected continuation payoff and thus will always truthtell. Based on (4), $B$ cares about the impact of $m_1$ both on her expected continuation payoff and on her current payoff. Clearly, if $B$ receives $s_1 = 1$, she has both a current incentive and a future incentive to report $m_1 = 1$. If $B$ receives $s_1 = 0$, then she faces a trade off between truthtelling and lying. Indeed, following the equilibrium strategy of truthtelling maximizes the expected continuation payoff but minimizes the current payoff (since $a_1(0) = 0 < a_1(1) = 1$), while the converse is true if $B$ deviates and lies with $m_1 = 1$. It is straightforward to show that there always exists a scalar $\delta_E^{TT} \in (0,1)$ such that if $\delta_E > \delta_E^{TT}$, then truthtelling by the bad expert can be supported in equilibrium. Thus, $TT$ is an equilibrium only if a bad experts is sufficiently concerned about her career.

Note that $TT$ could never be supported in equilibrium if there were only a reputational concern for being perceived as unbiased (as in Morris, 2001). It is the presence of a second dimension of reputation - that for being perceived as smart - that re-establishes the right reputational incentives to fully reveal information. From this perspective (i.e., the perspec-

[10]Since reputation values are determined endogenously in equilibrium, we use the superscript $TT$ to denote reputation values in a $TT$ equilibrium.

tive of disciplining the behavior of an expert), the two reputational channels can somehow be considered substitutes for one another: In an equilibrium with no lying, the preference channel is completely muted, but the ability channel is activated thus providing the incentives for truthful revelation.

What other informative equilibria exist other than $TT$? Note that if a bad expert does not care enough about her future payoff, she will refrain from truthfully revealing her information in the first period. In particular, the bad expert will have an incentive to over-report message 1 which is the message that maximizes her current payoff. In the appendix we show that there always exists a scalar $\delta_E^{PPG} \in \left(0, \delta_E^{TT}\right)$ such that if $\delta_E^{PPG} < \delta_E < \delta_E^{TT}$, then there exists an informative equilibrium in which $U$ always truthfully reports, while $B$ uses the following mixed strategy: she truthfully reports signal 1 and mixes on signal 0. We call this equilibrium $PP_G$ (Partial Pooling Good).

How does learning about the expert takes place in a $PP_G$ equilibrium? In a $PP_G$ equilibrium, $U$ and $B$ are using different strategies. In particular, $B$ is reporting message 1 more often than $U$. As a consequence, message 1 (0) conveys some information that the expert is likely to be $B(U)$. Thus, in a $PP_G$ equilibrium messages not only provide information about the state, but also function as signals about the preferences of the expert. Note that since messages contain some information about the state of the world, the observation of the pair $(m_1, x_1)$ still allows $DM$ to make some inference about the ability of the expert. However, due to the fact that $B$ lies, this inference is - at least in some states - less sharp than in $TT$. Formally, we have that:(0, 0)

$$\gamma^{PPG}(1,0) < \gamma^{PPG}(1,1) < \gamma < \gamma^{PPG}(0,1) = \gamma^{PPG}$$

$$\underline{\alpha} = \alpha^{PPG}(0,1) < \alpha^{PPG}(1,0) < \alpha < \alpha^{PPG}(1,1) < \alpha^{PPG}(0,0) = \overline{\alpha}$$

Figure 1 shows how the values of reputations change when $B$'s probability of lying $(1-\lambda_B)$ changes. Ceteris paribus, the larger the probability with which $B$ lies, the sharper the inference about preferences and the less sharp the inference about ability. In particular, as the probability with which $B$ lies increases towards 1, both $\gamma^{PPG}(0,1)$ and $\gamma^{PPG}(0,0)$ grow towards 1 too, while both $\gamma^{PPG}(1,0)$ and $\gamma^{PPG}(1,1)$ fall below the prior (i.e., message 0 (1) becomes increasingly effective in signalling that the expert sending $m_2 = 0$ $(m_2 = 1)$ is good (bad)). At the same time, the values of reputations for ability remain bounded away from 1 and relatively clustered around the prior $\alpha$.

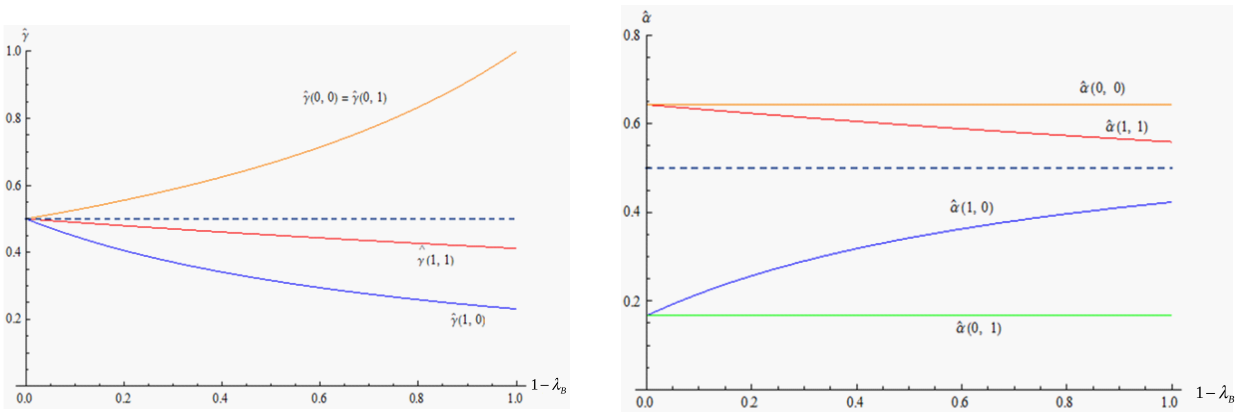


Figure 1: Reputation values as a function of $B'$s probability of misreporting $(1 - \lambda_B)$.

All this implies that as $B$'s probability of lying increases, $V^{PPG}(0,1)$ increases and $V^{PPG}(1,1)$ falls. As a consequence, both $U$'s incentives to truthfully report signal 1 and $DM$'s incentives to retain an incumbent that makes a correct call $(1,1)$ decrease. At some point, the reputational reward of reporting message 0 becomes so large that $V^{PPG}(0,1) > V^{PPG}(1,1)$ and either $U$ deviates from her $TT$ strategy or $DM$ chooses to fire an incumbent that made a correct call. In all cases, $PP_G$ is destroyed. Thus, in a $PP_G$ equilibrium, $B$'s probability of lying must be sufficiently small.

# 4 Welfare Analysis

To carry out our welfare analysis, we identify two distinct effects that emerge in equilibrium, namely a discipline effect and a sorting effect. The discipline effect results from the reputational incentives of the expert. The incentives of the expert to establish a good reputation induce the expert to reveal his private information in the first period, and this positively affects the expected return of the investment in the first period. The more information the expert reveals about $x_1$, the stronger is the discipline effect and the larger is the expected return of the investment in the first period. The strength of the discipline effect clearly depends on the equilibrium being played, with $TT$ being the equilibrium with the strongest discipline effect since all the expert's information is transmitted to the $DM$ in $TT$.

The sorting effect arises at the end of the first period when the $DM$ can observe the pair $m_1, x_1$, interpret in light of the equilibrium being played, and eventually learn about the incumbent's type. The sorting effect has an impact on the expected return of the investment in the second period. The more the $DM$ learns about the incumbent, the larger the expected return of the investment in the second period, and the stronger is the sorting effect.

Formally, let $\sigma$ denote an equilibrium in the first period and $E_0^\sigma [R_t(a_t, x_t)]$ denote the *ex-ante* expected return of period-$t$ investment conditional on equilibrium $\sigma$ being played. We say that equilibrium $\sigma'$ displays a stronger discipline effect than equilibrium $\sigma''$ if and only if $E_0^{\sigma'} [R_1(a_1, x_1)] > E_0^{\sigma''} [R_1(a_1, x_1)]$, and that equilibrium $\sigma'$ displays a stronger sorting effect with respect to $\sigma''$ if and only if $E_0^{\sigma'} [R_2(a_2, x_2)] > E_0^{\sigma''} [R_2(a_2, x_2)]$.

The discussion in the previous section about how reputation values are affected by the reporting strategy of the bad expert highlights that there may exist a trade off between, the amount of information that is revealed on state $x_1$, and learning about the type of expert. In particular, we know that the more information the expert reveals on the sate, the more the $DM$ learns about the ability type of the expert, but the less she learns about the preferences of the expert. The following proposition highlights how a $TT$ equilibrium fares with respect to a $PP_G$ equilibrium in terms of discipline and sorting.

**Proposition 2** *(i) $PP_G$ always reduces discipline with respect to $TT$; (ii) there always exists a scalar $\alpha^* \in (0, 1)$ such that $PP_G$ always improves sorting with respect to $TT$ whenever $\alpha > \alpha^*$.{Proof in the Appendix}*

$TT$ naturally involves the expert revealing the most possible information on the state in period 1, therefore dominating any partially revealing equilibrium in terms of discipline as stated in point (i). In terms of sorting however, $TT$ allows the expert to learn a lot about the expert's ability but does not reveal any information on the preferences of the expert, since both $U$ an $B$ advisors pool on their preference type by honestly reporting their signal. On the other hand , since $PP_G$ involves $U$ and $B$ experts adopting different strategies, this allows the $DM$ to learn something about the expert's preferences, but a bit less on ability with respect to $TT$. Point (ii) of proposition 2 highlights that when the prior on ability is sufficiently high, $PP_G$ can actually improve sorting with respect to $TT$. Indeed, when the prior on ability is sufficiently high, learning about the ability of the expert becomes relatively less relevant than learning about the preferences of the expert for making informed future decisions.

To better understand how the sorting effect works, let us consider the expressions for the ex-ante expected return of the investment in period 2 in a $TT$ and in a $PP_G$ equilibrium respectively:

$$\begin{aligned} E_0^{TT}\left[R_2(a_2,x_2)\right] &= \qquad (6) \\ \Pr(m_1 &= 1, x_1 = 1|TT)V^{TT}(1,1) + \Pr(m_1 = 0, x_1 = 0|TT)V^{TT}(0,0) \\ +\Pr(m_1 &= 1, x_1 = 0|TT)V + \Pr(m_1 = 0, x_1 = 1|TT)V \end{aligned}$$

$$\begin{aligned} E_0^{PPG}\left[R_2(a_2,x_2)\right] &= \qquad (7) \\ \Pr(m_1 &= 1, x_1 = 1|PP_G)V^{PP_G}(1,1) + \Pr(m_1 = 0, x_1 = 0|PPG)V^{PP_G}(0,0) \\ +\Pr(m_1 &= 0, x_1 = 1|PP_G)V + \Pr(m_1 = 0, x_1 = 1|PP_G)V \end{aligned}$$

Now, let us compare the bites of (6) and (7) that refer to the events in which the expert makes a mistake and is fired. In all these cases, $DM$ hires a new manager. The ex-ante expected return of the investment using a new manager is $V$. The following expression represents the net expected benefit of replacing an expert in $PP_G$ with respect to $TT$,which we denote the *replacement component*:

$$\begin{aligned} & \Pr(m_1 = 0, x_1 = 1|PP_G)V + \Pr(m_1 = 0, x_1 = 1|PP_G)V \\ & -\left[\Pr(m_1 = 1, x_1 = 0|TT)V + \Pr(m_1 = 0, x_1 = 1|TT)V\right] \\ = \; & \left[\Pr(m_1 \neq x_1|PPG) - \Pr(m_1 \neq x_1|TT)\right]V \qquad (8) \end{aligned}$$

Note that expression (8) is always positive since the probability of making an incorrect evaluation is higher in $PP_G$ where there is misreporting. Therefore, replacing the incumbent expert with a new expert occurs more often in $PP_G$ than in $TT$. Since the value of a new expert is the same in both equilibria (i.e. $V$), the benefit of replacement is higher in $PP_G$ than in $TT$.

Now, let us compare the net expected benefit of continuing to rely on the advice of the incumbent in the two equilibria under consideration, which we denote the *continuation component*:

$$\begin{aligned} \Pr(m_1 &= 1, x_1 = 1|PP_G)V^{PP_G}(1,1) + \Pr(m_1 = 0, x_1 = 0|PP_G)V^{PP_G}(0,0) \\ &\quad - \left[\Pr(m_1 = 1, x_1 = 1|TT)V^{TT}(1,1) + \Pr(m_1 = 0, x_1 = 0|TT)V^{TT}(0,0)\right] \end{aligned}$$

Since $q^{PPG}(0,0) = q^{TT}(0,0)$, the expression above can be written as follows:

$$\begin{aligned} &\frac{r}{4}q\gamma\left(2q^{PP_G}(1,1) - 1\right) + \frac{r}{4}q\gamma(2q^{PP_G}(0,0) - 1) - \frac{r}{2}q\gamma(2q^{TT}(1,1) - 1) = \\ = &\ \frac{r}{2}q\gamma\left[q^{PP_G}(1,1) - q^{TT}(1,1)\right] \end{aligned} \tag{9}$$

As we stressed in the previous section, $TT$ is the equilibrium with the sharpest learning on ability. Thus, $q^{TT}(1,1) > q^{PP_G}(1,1)$ and expression (9) is always negative. Accordingly, the net benefit of retaining an incumbent that provides a correct evaluation is lower in $PP_G$ than in $TT$.

In order for $PP_G$ to improve sorting over $TT$, it must be that expression (8) be sufficiently larger than expression (9). This occurs when the prior on ability is sufficiently high. Intuitively, when $\alpha$ is sufficiently high, there is less scope for learning on the skill dimension. Thus, the distance between $q^{PP_G}(1,1)$ and $q^{TT}(1,1)$ becomes less pronounced, and the benefit of retaining an incumbent that provides a correct evaluation becomes essentially the same in both equilibria. Which equilibrium performs better in terms of sorting then depends on the benefit replacement which is always higher in $PP_G$ for any value of $\alpha$.[11]

Now let's consider the ex-ante expected utility of the $DM$ in equilibrium $\sigma$:

$$E_0^\sigma\left[U_{DM}\right] = (1 - \delta_{DM})\, E_0^\sigma\left[R_1(a_1, x_1)\right] + \delta_{DM} E_0^\sigma\left[R_2(a_2, x_2)\right]$$

Proposition 2 suggests that it may not always be the case that $TT$ is the welfare maximizing equilibrium. While $TT$ always allows for a higher expected utility of current decisions (discipline), $PPG$ may imply better expected decisions in the future (sorting). Whether one equilibrium dominates the other ultimately depends on the decision maker's preferences for the future versus the present as stated in the following proposition:

[11]Formally, when $\alpha \to 1$, expression (9) shrinks to 0 while expresion (8) remains strictly positive.

**Proposition 3** *For any partially revealing equilibrium that improves sorting with respect to* $TT$*, there always exists a* $\delta^*_{DM} \in (0,1)$ *such that the partially revealing equilibrium increases (decreases) the* $DM'$*s expected utility with respect to the* $TT$ *equilibrium, if the decision maker assigns a high (low) enough weight to future decisions relative to present decisions (*$\delta_{DM} > (<)\ \delta^*_{DM}$*)*

**Proof.** *It is straightforward to show that for any partially revealing equilibrium which we define as* $\sigma = P$*, it follows that* $E_0^P(R_1) < E_0^{TT}(R_1)$*. If* $P$ *improves sorting we have that* $E_0^P(R_2) > E_0^{TT}(R_2)$*. Since* $E_0^{\sigma}(U_{DM})$ is monotonic in $\delta_{DM}$, this completes the proof. ■

So far we have restricted our analysis to the class of equilibria in which the $U$ expert always truthfully reports her signal. However, in the next section we consider a wider array of informative equilibria, and Proposition 3 is therefore useful to establish that any partially revealing equilibrium that improves sorting has the potential to be welfare improving

# 5 Other Equilibria: Political Correctness

It is well known that signalling games such as the one we are analyzing are characterized by multiple equilibria. So far we have focused exclusively on the equilibria that involve the unbiased expert truthfully reporting her information, however there are also equilibria that have the "Political Correctness" feature mentioned by Morris (2001), in which the unbiased expert misreports when receiving $m_1 = 1$ in order to signal that she is not biased. A natural question that may arise is whether these equilibria may also have the potential to improve sorting.

In order to answer this question we first consider the possible equilibria of this type that may exist. The following proposition provides full characterization of these equilibria according to the equilibrium strategy of the $DM$, namely whether experts are hired with positive probability after both messages 0 and 1.

**Proposition 4** *Any informative equilibrium in which the* $U$ *misreports is characterized by one of the following profiles of* $DM$ *and expert strategies:*

*i)* $PP_B$ *(Partial Pooling Bad) - the* $DM$ *hires the expert with positive probability both after receiving* $m_1 = 1$ *and* $m_1 = 0$*, the* $B$ *expert truthfully reveals* $s_1 = 1$ *and truthfully* *reveals signal $s_1 = 0$ with probability $\lambda_B \in (0, 1]$, and the G expert truthfully reveals one of the two signals and truthfully reveals the other with probability $\lambda_U \in (0, 1)$.*

*ii) PG (Pooling Good) - the DM hires the expert only if $m_1 = 0$, the B either always truthtells or misreports on one of the two signals, and the G expert always sends message* 1.

The first thing to notice is that the first type of equilibria ($PP_B$) represent situations in which sufficient information is revealed by message 1 on both ability and integrity, so that the $DM$ is willing to continue to listen to the expert after both messages, while the second case ($PG$) represents the more extreme scenario in which the unbiased expert will never communicate the evaluation favored by the biased expert in order to signal his type. For the sake of completeness we evaluate the welfare effects of both classes of equilibria in order to highlight some interesting properties.

## 5.1 Political Correctness equilibria that can never be preferred to truthful revelation

Considering the $PP_B$ equilibrium the first thing to notice is that this may involve the $B$ expert telling the truth if her reputational concerns $\delta_E$ are above a certain threshold (or randomizing if $\delta_E$ is exactly equal to a threshold value) when receiving signal 0. As for the $PP_G$ equilibria analyzed in the previous sections, if reputational concerns are too low babbling is the only equilibrium.[12]

Notice also, that $PP_B$ is supported by two possible misreporting strategies of the $U$ expert. The first, which we denote $PP_B(1)$ shares the features of the political correctness effect described by Morris (2001), in that the $U$ expert may misreport when receiving the high signal in the attempt to signal to the $DM$ that she is unbiased. The second, which we denote $PP_B(0)$, is specific to our setting and involves the $U$ expert misreporting with positive probability when receiving $s_1 = 0$ to signal her type. This equilibrium arises principally from the fact that the Biased expert can benefit from inducing the $DM$ to take action 1 only if she is rehired, in other words the bias is relation specific. Indeed, an equilibrium in which $U$ attempts to signal her type by sending message 1 more often, must involve the $B$ expert

[12] Although we cannot establish whether the threshold value of $\delta_E$ below which a $PP_B$ equilibrium does not exist is greater or less than that of $PP_G$, in section 6 we show that for low enough reputational concerns only babbling equilibria exist.

sending the same message less often. In our setting this can occur, because when the $B$ expert has high enough concerns for the future (i.e., for high enough $\delta_E$) she may even be willing to truthtell in order to maximize her chances of being rehired, and continue to profit from inducing the $DM$ to credibly take incorrect actions in the future.

At a first glance, as occurs for the $PP_G$ equilibrium, it may seem that this partially revealing equilibrium may also improve sorting with respect to the truthtelling equilibrium. However, rather surprisingly this is never the case. Intuitively, this occurs because in the attempt to signal her type, the $U$ expert must misreport, therefore generating less learning on ability. This ultimately leads the overall the sorting effect of $PP_B$ to always be dominated by truthful revelation. This following proposition summarizes this result:

**Proposition 5** *There does not exist a $PP_B$ equilibrium in which the $B$ expert truthfully reports, that can improve welfare with respect to $TT$, since it never improves sorting.*

To gather further intuition for this result, it is useful to compare $PP_B$ with $PP_G$.[13] In particular, we can illustrate this by considering $PP_B(1)$ but a similar argument applies for $PP_B(0)$ First notice that, as for $PP_G$ there are two components of sorting that work in opposite directions, while misreporting leads to better sorting (with respect to $TT$) in terms of the net benefit of replacing the incumbent (replacement component), in terms of the net expected benefit of continuing to rely on the services of the incumbent advisor (continuation component). Unlike in $PP_G$ however, the negative component always outweighs the positive one, and this result is driven by the upside sorting component.

More specifically, notice that since in $PP_G$ there is some misreporting on message 1, providing a correct evaluation when sending message 0 generates a higher update on reputation for ability with respect to correctly providing message 1. The opposite holds for $PP_B$, where the higher ex-post level of reputation results from correctly sending message 1, precisely because the $U$ expert is misreporting on message 0 to signal her type. In order to focus on continuation sorting which drives the result, we equalize replacement sorting in both equilibria by considering a fair prior on $\gamma$ and the same degree of misreporting, so that $\lambda_U = \lambda_B = \lambda$. These assumptions imply that the updates on expected precision of the

[13] Notice that $PP_B$ equilibria in which the $B$ expert randomizes share some features of $PP_G$. To the extent that the probability of misreporting of the $U$ expert tends to 0, $PP_B$ converges to $PP_G$ and has the potential to improve sorting with respect to truthtelling.

incumbent in the two equilibria have the following properties:

$$\overline{q}_{low} \equiv q^{PP_B}(0,0) = q^{PP_G}(1,1) < q^{PP_B}(1,1) = q^{PP_G}(0,0) \equiv \overline{q}.$$

Now if we analyze the difference in the expected benefit of sorting between the two equilibria, it simplifies to the following expression:

$$E_0^{PP_G}(R_2) - E_0^{PP_B}(R_2) = \gamma(1-\lambda)[q(2\overline{q}-1) - (1-q)(2\overline{q}_{low}-1)] > 0 \tag{10}$$

Considering the terms in square brackets, the first (second) term is positive (negative), since the state that reveals more information on ability (i.e., (0,0) in $PP_G$ and (1,1) in $PP_B$) is more (less) likely to occur in $PP_G$ precisely because in $PP_B$, the $U$ expert misreports when receiving $s_1 = 1$. The positive term always outweighs the negative one, and (10) is greater than zero, because the state that reveals more on ability is also reached with higher probability in $PP_G$, since the $U$ expert is not forced to "lie" to signal her integrity.

## 5.2 Political Correctness equilibria that may be preferred to truthful revelation

If we consider the more extreme (or conservative) equilibria in which the $DM$ never rehires an expert that provides $m_1 = 1$, it turns out that these may improve sorting with respect to truthful revelation. We therefore obtain the following:

**Proposition 6** *There always exists a PG equilibrium that improves sorting with respect to $TT$ and therefore may improve welfare if the $DM$ is sufficiently concerned about the future (i.e. $\delta_{DM}$ is sufficiently high) (Proof in the Appendix)*

The proof of this proposition essentially relies on identifying an instance in which this may occur. To gather an intuition for this result, consider the $PG$ equilibrium in which the $DM$ hires only after $(m_1, x_1) = (0,0)$ and the $B$ expert always truthfully reports her information, that is supported by intermediate values of reputational concerns, $\delta_E$ In this case, little information is revealed about ability since the $U$ expert has strong incentives to signal her type, and therefore never truthfully reports her information.

First of all notice that this equilibrium exists only when the market for experts is characterized by great chances of encountering low skilled and biased experts, in other words for low values of the priors on $\alpha$ and $\gamma$. To see this notice that in order for the above mentioned strategy of the $DM$ to be consistent with equilibrium, a necessary condition is that $\alpha$ must be low otherwise learning on preferences is always valuable, and the expert would always be hired when sending message 0. However, this is not sufficient and $\gamma$ must also be sufficiently small so that observing an incorrect combination $(m_1, x_1) = (0, 1)$ leads to a sufficiently (negative) update on reputation for ability to induce the expert to be fired. This can be seen by observing that

$$\alpha^{PG}(0,1) = \frac{\alpha[\gamma + (1-\gamma)(1-p)]}{[\gamma + (1-\gamma)(1-q)]}$$

and that this is clearly increasing in $\gamma$. In other words, the update on ability is greater the greater is the probability of encountering a $B$ expert that truthfully reveals her information and provides an incorrect forecast.

This $PG$ equilibrium may improve sorting with respect to $TT$ when the $DM$ faces higher odds of encountering a biased expert. Intuitively, the conservative replacement strategy implied by $PG$ allows the DM to better discriminate biased versus unbiased experts, while also learning something about ability. In order to gather intuition for this result, we can breakup the net welfare gain of $PG$ with respect to $TT$ into the usual two components, namely the net expected benefit of firing an expert (replacement):

$$\begin{aligned} & [\Pr((m_1, x_1) \neq (0,0))|PG) - \Pr(m_1 \neq x_1|TT)]\, V \\ = \; & \frac{r}{4}\gamma\,(2q-1)\,[q(1-\gamma) + 2(1-q)(1-\gamma) + \gamma], \end{aligned}$$

and the net expected benefit of continuing to rely on the advice of an incumbent (continuation):

$$\begin{aligned} \Pr((m_1, x_1) \; & = \; (0,0))|PPG)V^{PG}(0,0) - \Pr(m_1 \neq x_1|TT)\overline{V} \\ & = \; \frac{r}{2}\gamma[2q\overline{q} - q_{00}^{PG} + (1/2 - q)]. \end{aligned}$$

Notice that for lower values of $\gamma$ both components increase. This implies that there always exists (i.e., for any value of $\alpha$) a lower bound on $\gamma$, such that for values of gamma below this value, $PG$ always improves sorting. In particular, the greater are the chances of encountering a biased expert the greater is the replacement component. This is because since biased experts truthfully send message 1 in equilibrium, they separate themselves from $U$ experts on this message. As the share of biased experts increases, the chances of observing message 1 rise, and therefore the probability of correctly firing a biased expert also increases. On the other hand, the continuation component is also increasing, implying that the DM will tend to retain more qualified incumbents in expectations. Precisely because biased experts are truthtelling, an increase in the share of biased experts leads $q^{PG}(0,0)$ to increase, since more truthful reporting allows for more learning on ability.

# 6 A Complete Mapping of Equilibria and Welfare Implications

So far we have established that $TT$ equilibria may sometimes be dominated by other equilibria that involve some degree of misreporting. In order to provide a more complete picture of our results and suggest some policy implications, it is useful to represent a mapping of all the equilibria based on the priors that represent the information environment. In particular we characterize the equilibria with respect to the career concerns of the experts represented by parameter $\delta_E$. This allows us to establish which classes of equilibria may exist for the different regions of $\delta_E$, in order to better comprehend in which cases truthtfull equilibria may or may not be welfare maximizing.

The first thing to notice is that informative equilibria exist whenever the expert assigns a high enough weight to future payoffs. Moreover, for all the values of $\delta_E$ for which informative equilibria exist there is always a potential for multiplicity. Another relevant feature is that truthtelling equilibria may never coexist with $PP_G$, as shown in section 2. The following proposition formally represents this situation:

**Proposition 7** *There exist a $\underline{\delta}_E, \overline{\delta}_E \in (0,1)$ with $\underline{\delta}_E < \overline{\delta}_E$ such that:*

*a) For $\delta_E < \underline{\delta}_E$ no informative equilibria exist;*

*b) For $\delta_E \in (\underline{\delta}_E, \overline{\delta}_E)$ there always exists a non-empty set of informative equilibria that includes at least one, and at most all of the following: $PP_G, PP_B(1)$ and $PG$;*

*c) For $\delta_E \in (\underline{\delta}_E, \overline{\delta}_E)$ there always exists a non-empty set of informative equilibria that includes at least one, and at most all of the following $TT, PP_B$, and $PG$.*

*(Proof in the Appendix)*

Although equilibrium multiplicity does not allows us to uniquely establish which equilibrium will be played, the welfare maximizing equilibrium represents the best possible outcome attainable for a given range of $\delta_E$. This complete mapping of the equilibria allows us to state that truthful revelation may not necessarily be welfare maximizing. In particular, the following general welfare results apply. First of all, when $\delta_E$ is sufficiently high, $TT$ is feasible and welfare maximizing as long as $PG$ does not exist. However, whenever the $PG$ equilibrium exists, it may dominate $TT$ as we observed in the previous section. Furthermore, when experts do not care enough about future payoffs, truthful revelation break downs, but there may exist other partially revealing equilibria that involve some degree of misreporting (either $PP_G$ or $PG$), that may even generate higher levels of welfare with respect to truthful revelation.

# 7 Conclusion

Decision makers often seek the advice of experts before making a decision. The presumption is that an expert has access to valuable information (not available to the decision maker) that is relevant to make the correct decision and that the expert will truthfully report such information to the decision maker. In fact, experts may differ in their abilities to retrieve accurate information and may well have objectives that are not necessarily in line with those of decision makers.

In the present paper we analyze a model of cheap talk where the credibility of the expert's advice hinges upon the decision maker's beliefs about how unbiased and competent the expert is. When the expert and the decision maker interact repeatedly, the expert can use present interaction to affect the beliefs of the decision maker and establish a reputation for being unbiased and competent thereby increasing the credibility of his future advice.

We show that these reputational concerns on the part of the expert may suffice to achieve truthtelling. However, we show that truthtelling may not necessarily be the outcome preferred by the decision maker. In particular, we highlight the existence of a trade off between the amount of information on the decision-relevant state and the information about preferences (i.e. the bias) of the expert. In particular, in a dynamic setting in which a decision maker has to make current and future decisions, this trade off may be such that the decision maker prefers to give up some information on the current state of the world and learn more about the preferences of his advisor since this will allow the decision maker to make better decisions in the future.

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# 8 Appendix

**Characterization of Equilibria**

We characterize the informative equilibria where the DM changes his beliefs about the state of the world after some message on the equilibrium path. We will refer to those equilibria simply as informative equilibria. We first establish a lemma that will make it easier to analyze the game by backward induction.

**Lemma 1** *In any informative equilibria, DM chooses* $a_t(m_t) = m_t$*.*

**Proof.** If the equilibrium is informative, $\Pr(x_t = 1 \mid m_2 = 0) < \Pr(x_t = 1) < \Pr(x_t = 1 \mid m_2 = 1)$. Since $R_t(1,1) = -R_t(1,0)$ and $\Pr(x_t = 1) = \frac{1}{2}$, $E(R_t(1,x_t) \mid m_t = 1) > E(R_t(0,x_t) \mid m_t = 1)$ and $E(R_t(0,x_t) \mid m_t = 0) > E(R_t(1,x_t) \mid m_t = 0)$. ■

We now proceed by backward induction.

**Second Period**

**Lemma 2** *In the second period: i) a bad expert sends* $m_2 = 1$ *irrespective of* $s_2$*; ii) a good expert truthtells.*

**Proof.** In the last period, the expert will not be concerned about his reputation. Thus the bad expert will always claim to have observed signal 1 in order to induce DM to choose action 1. For a good expert with no explicit preferences in favour of a particular action, any strategy is a continuation equilibrium. In line with the rest of the literature on career concerns, we focus on the continuation equilibrium in which the good expert acts in the interest of the DM and thus truthfully reveals his signal. ■

At the beginning of the second period, DM chooses whether to retain the incumbent or hire a new expert. Given the second-period reporting strategies of a bad and good expert, it is straightforward to show that:

$$\begin{aligned} V &\equiv \Pr(m_2 = 0)a_2(m_2 = 0)E(R_2 \mid m_2 = 0) + \Pr(m_2 = 1)a_2(m_2 = 1)E(R_2 \mid m_2 = 1) = \\ &= \frac{r}{2}\gamma\,(2q-1) \end{aligned} \tag{11}$$

$$\begin{aligned} V(m_1, x_1) &\equiv \Pr(m_2 = 0 \mid m_1, x_1) a_2(m_2 = 0) E(R_2 \mid m_2 = 0, m_1, x_1) + \\ + \Pr(m_2 &= 1 \mid m_1, x_1) a_2(m_2 = 1) E(R_2 \mid m_2 = 1, m_1, x_1) = \\ &= \frac{r}{2} \widehat{\gamma}(m_1, x_1) \left[ 2\widehat{q}(m_1, x_1) - 1 \right] \end{aligned} \tag{12}$$

**Lemma 3** *The DM retains the incumbent if* $V(m_1, x_1) \geq V$ *and hires a new expert if* $V(m_1, x_1) < V$*.*

**Proof.** Since both $q$ and $\widehat{q}(m_1, x_1)$ are greater than $\frac{1}{2}$ (i.e. in expectation the expert always has better information than the $DM$), both $V(m_1, x_1)$ and $V$ are strictly positive. Thus, the DM always finds it optimal to consult an expert in period 2. In particular, the DM will retain the incumbent whenever $V(m_1, x_1) \geq V$ and fire him otherwise.. ■

**First Period**

Let $\imath(m_1, x_1)$ denote DM's retaining strategy as describe in Lemma 3. That is:

$$\imath(m_1, x_1) = \begin{cases} 1 \text{ if } V(m_1, x_1) \geq V \\ 0 \text{ otherwise} \end{cases}$$

Assuming that experts and the DM behave as described by Lemmas 2 and 3, the continuation payoff of a bad expert at the end of the first period (i.e. when combination $(m_1, x_1)$ has been realized and observed) can be written as $\left[ V(m_1, x_1) + 1 \right] \imath_{m_1, x_1}$. Similarly, the continuation payoff of a good type can be written as $V(m_1, x_1)\imath_{m_1, x_1}$.

Now let's go back to the time when the expert observes signal $s_1$. For a bad expert who observes signal $s_1$, the expected continuation payoff of choosing message $m_1$ reads:

$$\pi_B(m_1, s_1) = \sum\nolimits_{x_1} \Pr(x_1 \mid s_1) \left[ V(m_1, x_1) + 1 \right] \imath_{m_1, x_1}$$

The previous expression can be explained as follows. Expression $\left[ V(m_1, x_1) + 1 \right] \imath_{m_1, x_1}$represents the continuation payoff that the bad expert receives if combination $(m_1, x_1)$ realizes. For a given message $m_1$ that is chosen by an expert with signal $s_1$, the probability of combination $(m_1, x_1)$ is equal to $\Pr(x_1 \mid s_1)$.

Similarly, for a good incumbent expert who observes $s_1$, the expected continuation payoff of choosing message $m_1$ reads:

$$\pi_G(m_1, s_1) = \sum\nolimits_{x_1} \Pr(x_1 \mid s_1) \left[ V(m_1, x_1) \right] \imath_{m_1, x_1}$$

It is key to note that the value of the expected continuation payoff depends on the equilibrium values of reputations $\widehat{\alpha}(m_1, x_1)$ and $\widehat{\gamma}(m_1, x_1)$ which determine both $V(m_1, x_1)$ and $\imath_{m_1,x_1}$. Thus the value of the expected continuation payoff arises endogenously in equilibrium.

In order to fully characterize the equilibria, we first write down the conditions for which each type of expert has a weak incentive to truthfully report each signal.

For the $B$ expert these are

$$\begin{aligned} \delta_E \pi_B(m_1 &= 0, s_1 = 0) - (1 - \delta_E) - \delta_E \pi_B(m_1 = 1, s_1 = 0) \geq 0 \text{ if } s_1 = 0 \quad (13) \\ (1 - \delta_E) + \delta_E \pi_B(m_1 &= 1, s_1 = 1) - \delta_E \pi_B(m_1 = 0, s_1 = 1) \geq 0 \text{ if } s_1 = 1 \quad (14) \end{aligned}$$

and for the $U$ expert they are:

$$\begin{aligned} \pi_U(m_1 &= 0, s_1 = 0) - \pi_U(m_1 = 1, s_1 = 0) \geq 0 \text{ if } s_1 = 0 \quad (15) \\ \pi_U(m_1 &= 1, s_1 = 1) - \pi_U(m_1 = 0, s_1 = 1) \geq 0 \text{ if } s_1 = 1 \quad (16) \end{aligned}$$

We therefore establish the following Lemma that states the properties that informative equilibria will *never* have:

**Lemma 4** *There cannot exist an informative equilibrium that satisfies one or more of the following properties:*

*i) The DM fires the incumbent whenever $m_1 = 0$ is observed and retains the incumbent whenever $m_1 = 1$ is observed.*

*ii) The $U$ expert always sends $m_1 = 1$.*

*iii) One or more expert types (i.e., $B$ or $U$) randomizes after receiving both signals*

*iv) The $B$ expert randomizes after receiving $s_1 = 0$ when the $U$ expert sends both messages with positive probability.*

**Proof.** i) By contradiction. Suppose such an equilibrium exists. Then $U$, whose only concern is to be retained, will always send $m_1 = 1$. Similarly, B will always send $m_1 = 1$

because $m_1 = 0$ generates neither a present nor a future benefit for $B$. But then, since both $B$ and $U$ always send $m_1 = 1$, the equilibrium cannot be informative.

ii) If this were true, in order for the equilibrium to be informative, the B would have to send message 0 with positive probability in equilibrium, but sending 0 would immediately allow the DM to identify the expert as $B$. Therefore, (13) and (14) are always respectively negative and positive, implying that the $B$ expert would always have an incentive to deviate to $m_1 = 1$, and so this cannot be an equilibrium.

iii) First notice that $\pi_i(m_1 = s_1) > \pi_i(m_1 \neq s_1)$ for $i \in \{U, B\}$, since $q > 1/2$. So when (14) is satisfied with equality it follows that (13) is always $> 0$, and viceversa. The same argument applies for (15) and (16).

iv) Notice that whenever $U$ sends both messages in equilibrium, it must be that (16) is satisfied. By the definition of $\pi_i(m_1, s_1)$, this also implies that $\pi_B(1, 1) - \pi_B(0, 1) > 0$ and so (14) is always satisfied with strict inequality. ■

As we will show, a set of informative equilibria exists that is consistent with the equilibrium play described in Lemmas 2 and 3. Lemma 4 implies that we can restrict our attention to those informative equilibria where:

A) If $m_1 = 1$ is sent, then there is at least an event $(m_1 = 1, x_t)$ after which DM retains the incumbent; If $m_1 = 0$ is sent, then there is at least an event $(m_1 = 0, x_t)$ after which DM retains the incumbent. That is, from the perspective of the incumbent, there is always a positive probability of being retained when sending either $m_1 = 1$ or $m_1 = 0$.

B) DM fires the incumbent whenever $m_1 = 1$ is observed and retains the incumbent whenever $m_1 = 0$ is observed.

We can now prove proposition 1.

**Proof of Proposition 1**

When $U$ truthtells, Lemma 4 point (iv) implies that the $B$ expert will always truthfully report after $s_1 = 1$. This implies that the only two candidate equilibrium expert strategies are the following:

a1) Both $U$ and $B$ truthfully report after $s_1 = 0, 1$ $(TT)$

a2) $U$ truthfully reports after $s_1 = 0, 1$; $B$ truthfully reports after $s_1 = 1$ and randomizes after $s_1 = 0$ $(PP_G)$

We show that each one is compatible with equilibrium and is consistent with case (A)

**a1) Both Experts truthfully report** $(TT)$

**(i) Proof of existence of truthfully revealing equilibrium ($TT$) that satisfies case (A)**

Let $\widehat{\alpha}^{TT}(m_1, x_1)$ and $\widehat{\gamma}^{TT}(m_1, x_1)$ denote the value of reputations in a (putative) truthtelling equilibrium. It is straightforward to verify that:

$$\begin{aligned} &\underline{\alpha} \equiv \widehat{\alpha}^{TT}(0,1) = \widehat{\alpha}^{TT}(1,0) < \widehat{\alpha}^{TT}(1,1) = \widehat{\alpha}^{TT}(0,0) \equiv \overline{\alpha} \\ &\widehat{\gamma}^{TT}(m_1, x_1) = \gamma \text{ for any } (m_1, x_1) \end{aligned}$$

Now let $V^{TT}(m_1, x_1)$ denote the value of $V(m_1, x_1)$ in a truthtelling equilibrium. Given the above values of reputations, it is straightforward to show that in a truthtelling equilibrium the following chain of inequalities holds:

$$\overline{V}^{TT} \equiv V^{TT}(0,0) = V^{TT}(1,1) > V > V^{TT}(0,1) = V^{TT}(1,0) \equiv \underline{V}^{TT} \tag{17}$$

**DM's firing strategy**. From (17), it follows that in a truthtelling equilibrium the DM will retain the incumbent whenever $m_1 = x_1$ and fire him otherwise, which is consistent with case (A) of Lemma 4. That is:

$$\imath_{m_1,x_1} = \begin{cases} 1 \text{ if } m_1 = x_1 \\ 0 \text{ if } m_1 \neq x_1 \end{cases} \tag{18}$$

**Biased incumbent's strategy**. By Lemma 4 (iv) we can focus on the case in which a bad expert receives $s_1 = 0$ given by (13), which can be written as:

$$\begin{aligned} &(1-\delta_B)\, a(0) + \delta_E \sum\nolimits_{x_1} \Pr(x_1 \quad | \quad s_1 = 0)\,[V(0,x_1)+1]\,\imath_{0,x_1} \\ &-(1-\delta_E)\, a(1) + \delta_E \sum\nolimits_{x_1} \Pr(x_1 \quad | \quad s_1 = 0)\,[V(1,x_1)+1]\,\imath_{1,x_1} \geq 0 \end{aligned} \tag{19}$$

By using (17) and (18), condition (19) can be simplified as follows:

$$\delta_B\,[\Pr(x_1 = 0|s_1 = 0) - \Pr(x_1 = 1|s_1 = 0)]\left(\overline{V}^{TT} + 1\right) \geq (1-\delta_B) \tag{20}$$

The LHS of (20) represents the weighted net expected future payoff of truthtelling. The RHS of (20) represents the weighted net current payoff of deviating from $TT$. Since the expert's signals are informative in expectations, $\Pr(x_1 = 0|s_1 = 0) - \Pr(x_1 = 1|s_1 = 0) > 0$. It is then straightforward to notice that there always exists a scalar $\overline{\delta}_E$ such that condition (20) is satisfied when $\delta_E > \underline{\delta}_E^{TT}$. Thus, $TT$ is an equilibrium whenever condition (20) is satisfied.

Since $\Pr(x_1 = 0|s_1 = 0) = q$, condition (20) can eventually be written as:

$$\delta_E \geq \frac{1}{(2q-1)\overline{V}^{TT} + 2q} \equiv \underline{\delta}_E^{TT} \tag{21}$$

As we should expect, if a bad incumbent expert cares enough about future payoffs, his reputational concerns lead him to truthell. As one can note by inspecting (21), $TT$ exists as long as the expert attributes enough weight to future payoffs relative to current ones.

**Unbiased incumbent type's strategy**. By Lemma 4 (iii) we can focus on the case in which a good expert receives $s_1 = 0$, given by (15), which can written as:

$$\sum\nolimits_{x_1} \Pr(x_1 \mid s_1 = 0) V^{TT}(0, x_1)\imath_{0,x_1} - \sum\nolimits_{x_1} \Pr(x_1 \mid s_1 = 0) V^{TT}(1, x_1)\imath_{1,x_1} \geq 0 \tag{22}$$

By using (17) and (18), condition (22) simplifies to:

$$[\Pr(x_1 = 0|s_1 = 0) - \Pr(x_1 = 1|s_1 = 0)]\,\overline{V}^{TT} \geq 0 \tag{23}$$

Note that (23) is always verified because the signal is informative.

**a2)** $U$ **truthfully reports after** $s_1 = 0, 1$**;** $B$ **truthfully reports after** $s_1 = 1$ **and randomizes after** $s_1 = 0$ **(**$PP_G$**)**

We denote with $\lambda_B \in (0, 1)$ the probability that the bad expert truthfully reveals his signal when receiving $s_1 = 0$.

Let $\widehat{\alpha}^{PP}(m_1, x_1)$ and $\widehat{\gamma}^{PP}(m_1, x_1)$ denote the value of reputations in a (putative) $PP_G$ equilibrium. It is straightforward to verify that:

$$\widehat{\alpha}^{PP_G}(0,0) > \widehat{\alpha}^{PP_G}(1,1) > \alpha > \widehat{\alpha}^{PP_G}(0,1) > \widehat{\alpha}^{PP_G}(1,0)$$
$$\widehat{\gamma}^{PP_G}(0,0) = \widehat{\gamma}^{PP_G}(0,1) > \gamma > \widehat{\gamma}^{PP_G}(1,1) > \widehat{\gamma}^{PP_G}(1,0)$$

In words, since in this equilibrium a bad expert sends message 1 more often than a good expert, the observation of message 0 (1) leads the $DM$ to revise the reputation for being unbiased upwards (downwards). With regard to the reputation for ability, making a correct (incorrect) forecast rewards (penalizes) reputation as in a truthtelling equilibrium.

Now let $V^{PP_G}(m_1, x_1)$ denote the value of $V(m_1, x_1)$ in this partial pooling equilibrium. Given the above values of reputations, it is straightforward to show the following result.

**Lemma 5:** $V^{PP_G}(0,0) > V(\varnothing) > V^{PP_G}(1,0)$ for any value of $\lambda_B$.

**Proof:** Straightforward calculation.

Therefore, in this partial pooling equilibrium the DM will always fire the incumbent after he sends $m_1 = 1$ incorrectly (i.e. after event $(1,0)$), and will always retain the incumbent after he sends $m_1 = 0$ correctly (i.e. after event $(0,0)$). Based on Lemma 2, we know that the existence of the equilibrium under consideration necessarily requires that the incumbent be retained after event $(1,1)$. However, the ordering of $V(\varnothing)$, $V^{PP}(0,1)$ and $V^{PP}(1,1)$ is ambiguous. That is because sending $m_1 = 0$ and making a mistake (i.e. event $(0,1)$) lowers the reputation for ability but still increases that for being unbiased, while the opposite occurs when $m_1 = 1$ is correct ex-post (i.e. event $(0,1)$). However, it is possible to prove the folllwing Lemma:

**Lemma 6:** For any $\alpha \in (0,1)$, $\gamma \in (0,1)$ and $p \in \left(\frac{1}{2}, 1\right)$, there always exists a scalar $\widehat{\lambda}_B \in [0,1)$ such that $V^{PP_G}(1,1) > V(\varnothing)$ if and only if $\lambda_B \in \left[\widehat{\lambda}_B, 1\right]$, and for these values of $\lambda_B$ it always holds that $V(\varnothing) > V^{PP_G}(0,1)$.

We know that in any equilibrium in which the unbiased expert sends message 1 with positive probability, she must be hired after sending message 1. Since by Lemma 5 for $(m_1, x_1) = (0,0)$ and $(m_1, x_1) = (1,0)$ the expert will always be respectively hired and fired, it follows that she therefore must be hired after $(1,1)$.

It remains to show that when the expert is hired in $(1,1)$ she is always fired in $(0,1)$

It is simple to show that there exists a $\widehat{\lambda}_B$ such that for $\lambda_B > \widehat{\lambda}_B$ the expert is hired with $(1,1)$ and the following condition is satisfied:

$$\gamma^{PP_G}(1,1)(2q^{PP_G}(1,1) - 1) > \gamma(2q - 1) \tag{24}$$

and a $\widetilde{\lambda}_B$ such that for $\lambda_B > \widetilde{\lambda}_B$ the expert is fired with $(0,1)$ and the following condition is satisfied:

$$\gamma^{PP_G}(0,1)(2q^{PP_G}(0,1) - 1) < \gamma(2q - 1) \tag{25}$$

In order to complete the proof it is sufficient to prove that $\widehat{\lambda}_B > \widetilde{\lambda}_B$ because if this is the case:

- when $\lambda_B > \widehat{\lambda}_B > \widetilde{\lambda}_B$ both (24) and (25) are satisfied
- when $\lambda_B \in (\widetilde{\lambda}_B, \widehat{\lambda}_B)$ (24) is not satisfied and $PP_G$ is not an equilibrium
- when $\widehat{\lambda}_B > \widetilde{\lambda}_B > \lambda_B$ (24) is not satisfied and $PP_G$ is not an equilibrium

These imply that (25) is satisfied if and only if (24) is also satisfied

In order to prove that $\widehat{\lambda}_B > \widetilde{\lambda}_B$ we proceed in two steps: 1) find a closed form expression for $\widetilde{\lambda}_B$; 2) show that for this value (24) is never satisfied

1) $\widetilde{\lambda}_B$ is the value of $\lambda_B$ that satisfies (25) with equality, substituting the corresponding reputations $\gamma(0,1)$ and $q(0,1)$ and $\widehat{q} = \widehat{\alpha}(p - 1/2) + 1/2$ we obtain:

$$\frac{\gamma}{\gamma + \widetilde{\lambda}_B(1-\gamma)} \left(2\left(\frac{\alpha(1-p)}{(1-q)}(p-1/2) + 1/2\right) - 1\right) = \gamma(2\left(\alpha(p-1/2) + 1/2\right) - 1)$$

which simplifies to:

$$\widetilde{\lambda}_B = \frac{1}{(1-\gamma)}\left[\frac{(1-p)}{(1-q)} - \gamma\right]$$

2) Using $\gamma(1,1)$ and $q(1,1)$ (29) can be written in the following way:

$$\begin{aligned} & \frac{\gamma q}{q + (1-q)(1-\lambda_B)(1-\gamma)} \left(2\left(\left(\frac{\alpha[p + (1-p)(1-\lambda_B)(1-\gamma)]}{q + (1-q)(1-\lambda_B)(1-\gamma)}\right)(p-1/2) + 1/2\right) - 1\right) \\ = \; & \gamma(2\left(\alpha(p-1/2) + 1/2\right) - 1) \end{aligned}$$

which simplifies to:

$$\frac{q[p + (1-p)(1-\lambda_B)(1-\gamma)]}{[q + (1-q)(1-\lambda_B)(1-\gamma)]^2} > 1$$

substituting the closed form solution for $\lambda_B^{'}$ obtained in step 1, this simplifies to:

$$\frac{2pq - q^2 - p^2}{pq(1-q)} > 0$$

substituting $q = \alpha(p - 1/2) + 1/2$ we obtain:

$$(p - p^2 - 1/4)(1 - \alpha(1-\alpha)) > 0$$

Since $(p - p^2 - 1/4) < 0$ for $p > 1/2$ and $(1 - \alpha(1 - \alpha)) > 0$, this implies that $(p - p^2 - 1/4)(1 - \alpha(1 - \alpha)) < 0$. It follows that (24) is never satisfied for $\lambda_B = \widetilde{\lambda}_B$ which completes the proof.

**DM's firing strategy**.Lemmas 5 and 6 imply that if the probability with which the bad incumbent truthtells is sufficiently high, the DM follows the same firing strategy he follows in a $TT$ equilibrium, and that this is the only hiring and firing strategy that is consistent with the $PP_G$ equilibrium.

**Unbiased incumbent type's strategy**. In this putative equilibrium, in order to verify that truthful reporting by the $U$ expert is consistent with equilibrium. By Lemma 4 (iii), it is sufficient to consider when (16) is strictly positive.[14] Given the firing strategy of the $DM$ in the equilibrium under consideration, condition (16) boils down to:

$$\Pr(x_1 = 1 \mid s_1 = 1)V^{PP}(1,1) \geq \Pr(x_1 = 0 \mid s_1 = 1)V^{PP}(0,0) \tag{26}$$

Now note that $V^{PP}(1,1) < V^{PP}(0,0)$ while $\Pr(x_1 = 1 \mid s_1 = 1) \geq \Pr(x_1 = 0 \mid s_1 = 1)$. Since $V^{PP_G}(1,1)$ and $V^{PP_G}(0,0)$ are respectively decreasing and increasing in $\lambda_B$, and $V^{PP_G}(1,1) = V^{PP_G}(0,0)$ when $\lambda_B = 1$ and $V^{PP_G}(1,1) = 0$ and $V^{PP_G}(0,0) > 0$ when $\lambda_B = 0$, it follows that there always exists a scalar $\lambda''_B \in [0,1)$ such that for $\lambda_B \in [\lambda''_B, 1]$, (26) is satisfied.

Based on the analysis above we define $\lambda^*_B = \max[\widehat{\lambda}_B, \lambda''_B]$.

**Biased incumbent type's strategy**. Let us now analyze the putative strategy of the bad expert of truthtelling when he receives $s_1 = 1$ and randomizing when he receives $s_1 = 0$. By Lemma 4 (iv), this boils down to verifying if and when condition (13) is satisfied with equality:

$$\Pr(x_1 = 0 \mid s_1 = 0)\delta_E\left[V^{PP_G}(0,0) + 1\right] - (1-\delta_E) + \delta_E \Pr(x_1 = 1 \mid s_1 = 0)\left[V^{PP_G}(1,1) + 1\right] = 0$$

We can rewrite the equality above as:

$$\begin{aligned} & (1 - \delta_E) + \delta_E \Pr(x_1 = 1 \mid s_1 = 0) - \delta_E \Pr(x_1 = 0 \mid s_1 = 0) = \\ = \; & \delta_E\left[\Pr(x_1 = 0 \mid s_1 = 0)V^{PP_G}(0,0) - \Pr(x_1 = 1 \mid s_1 = 0)V^{PP_G}(1,1)\right] \end{aligned}$$

[14] It is easy to show that if the condition for truthtelling when $s_1 = 1$ is satisfied, then also the condition when $s_1 = 0$ is given the

It is straightforward to show that for every $\lambda_B \in (0, 1)$ there always exists a $\delta_E \in (0, 1)$ for which it is satisfied.

Using the fact that $\Pr(x_1 = 1 \mid s_1 = 0) = 1 - q$ and $\Pr(x_1 = 0 \mid s_1 = 0) = q$, after rearranging terms we can write the previous condition as follows:

$$\delta_E = \frac{1}{[qV^{PP_G}(0,0)] - (1-q)V^{PP_G}(1,1)] + 2q} \equiv \underline{\delta}_E^{PP_G}(\lambda_B) \tag{27}$$

Note that:

a) $\underline{\delta}_E^{PP_G}(\lambda_B)$ is strictly increasing in $\lambda_B$.

b) When $\lambda_B \to 1$, $V^{PP_G}(0,0) \to V^{TT}(0,0)$ and $V^{PP_G}(1,1) \to V^{TT}(1,1)$, and therefore $\underline{\delta}_E^{PP_G}(1) \to \underline{\delta}_E^{TT}$, i.e. the $RHS$ of (27) tends to coincide with the $RHS$ of the truthtelling condition (21).

c) Since $\underline{\delta}_E^{PP_G}(\lambda_B)$ is strictly increasing in $\lambda_B$, and $\lambda_B^* < 1, \underline{\delta}_E^{PP_G}(\lambda_B^*) < \underline{\delta}_E^{TT}$.

Based on *a*-*c*, a necessary and sufficient condition for the existence of $PP_G$ is that $\underline{\delta}_E^{PP_G}(\lambda_B^*) \leq \delta_E \leq \underline{\delta}_E^{TT}$. Otherwise stated, when $\delta_B \notin (\underline{\delta}_E^{PP_G}(\lambda_B^*), \underline{\delta}_E^{TT})$ it is not possible to find a value $\lambda_B \in (\lambda_B^*, 1)$ that satisfies (21)

**Proof of Proposition 2**

(i) Discipline is worst since

$$E_0^{TT}(R_1) = \frac{r}{2}(2q-1) > E_0^{PP_G}(R_1) = \frac{r}{2}(2q-1)[1-(1-\gamma)(1-\lambda_B)]$$

(ii) A sufficient condition for $PP_G$ to improve sorting with respect to $TT$ is:

$$\begin{aligned} E_0^{PP_G}(R_2) &> E_0^{TT}(R_2), \\ & [\Pr(0,1 \mid PP_G) + \Pr(1,0 \mid PP_G)]\, \frac{r}{2}\gamma(2q-1) + \frac{r}{4}q\gamma[(2\overline{q}-1) + (2q(1,1)-1)] \\ &> 2(1-q)\frac{r}{4}\gamma(2q-1) + 2q\frac{r}{4}\gamma(2\overline{q}-1) \end{aligned}$$

Bringing the first term on the RHS to the LHS and the second term on the LHS to the RHS of the equation, the above expression simplifies to:

$$(1-\gamma)(1-\lambda_B)(2q-1)^2 > 2q(\overline{q} - q(1,1)) \tag{28}$$

where the LHS represents the net benefit of receiving an incorrect message in $PP_G$ which is always positive, and the RHS represents the net loss of receiving a correct message in $PP_G$.

Now considering the RHS, $(\overline{q} - q(1,1))$ can be rewritten as:

$$(p - 1/2)\left(\frac{\alpha p}{q} - \alpha(1,1)\right),$$

and therefore the RHS becomes:

$$2(p-1/2)\left(\frac{\alpha(1-\lambda_B)(1-\gamma)(p-q)}{[q+(1-q)(1-\lambda_B)(1-\gamma)]}\right).$$

(28) can therefore be rewritten as:

$$(2q-1)^2 > 2(p-1/2)\left(\frac{\alpha(p-q)}{[q+(1-q)(1-\lambda_B)(1-\gamma)]}\right).$$

Now since $(2q-1)^2 = 2\alpha(p-1/2)^2$ and $\alpha(p-q) = (\alpha-\alpha^2)(p-1/2)$, the above expression becomes:

$$2\alpha > \left(\frac{(1-\alpha)}{[q+(1-q)(1-\lambda_B)(1-\gamma)]}\right).$$

We define $x = (1-\lambda_B)(1-\gamma)$, which represents the share of bad experts that are misreporting and substitute $q = \alpha(p-1/2)+1/2$ and rewrite the above expression in the following way:

$$f(\alpha, x) > 0$$

where $f(\alpha,x) = \alpha^2[(2p-1)(1-x)] + \alpha(2+x) - 1$. Since the second derivative of $f(\alpha,x)$ with respect to $\alpha$ is positive, it follows that the RHS is a convex function of $\alpha$, and the inequality has two solutions, $\frac{-(2+x)-[(2+x)^2+4(2p-1)(1-x)]^2}{2(2p-1)(1-x)} < 0$ and $\frac{-(2+x)+[(2+x)^2+4(2p-1)(1-x)]^2}{2(2p-1)(1-x)} > 0$. Since $\alpha \in (0,1)$, it follows that the only relevant solution is the positive one and that the above inequality is satisfied for $\alpha \in (\alpha^*(x), 1)$, where $\alpha^*(x) = \frac{-(2+x)+[(2+x)^2+4(2p-1)(1-x)]^2}{2(2p-1)(1-x)}$

Now, since

$$\frac{\partial f(\alpha,x)}{\partial x\mid_{\alpha>0}} > 0$$

this implies that :

1) $\partial\alpha^*(x)/\partial x < 0$. In other words $PP_G$ is more likely to improve sorting the greater is the share of Bad experts that misreport.

2) Using (1) and the result of Proposition 1 (existence of $PP_G$ equilibrium) we know that $PP_G$ exists for every $\alpha \in (0,1)$ and $\delta_B \in (\delta_B(\lambda_B^*), \delta_B(1))$ $PP_G$. It follows that all $PP_G$ equilibria improve sorting with respect to $TT$ if and only if $\alpha \in (\alpha^*(0), 1)$.

At this point the only thing that is left to show in order to prove that there always exists a sufficiently high value of $\alpha$ for which *every* $PP_G$ equilibrium improves sorting, is that $\alpha^*(0) \in (0,1)$.

We therefore compute $\alpha^*(0)$ :

$$\alpha^*(0) = \frac{(2p)^{1/2} - 1}{2p - 1} \in (0,1)$$

This follows immediately from the fact that high ability experts have informative but not completely informative signals, so that $2p > 1$ and $(2p)^{1/2} < 2p$

**Proof of Proposition 4**

When $U$ sends both messages in equilibrium, Lemma 4 implies that the $B$ expert will always truthfully report after $s_1 = 1$. This implies that the only candidate equilibrium expert strategies are the following:

a3) $B$ truthfully reports after $s_1 = 0, 1$; $G$ truthfully reports after $s_1 = 0$ and randomizes after $s_1 = 1$ ($PP_B(1)$)

a4) $B$ truthfully reports after $s_1 = 0, 1$; $G$ truthfully reports after $s_1 = 1$ and randomizes after $s_1 = 0$ ($PP_B(0)$)

a5) $B$ truthfully reports (randomizes) after $s_1 = 1$ ( $s_1 = 0$), and $U$ truthfully (randomizes) reports after $s_1 = 0$ ( $s_1 = 1$)

a6) $B$ truthfully reports (randomizes) after $s_1 = 1$ ( $s_1 = 0$), and $U$ truthfully (randomizes) reports after $s_1 = 1$ ( $s_1 = 0$)

Instead when $U$ sends always sends one message in equilibrium, by Lemma 4, this must be $m_1 = 0$. It follows that in this case, the only putative $B$ expert equilibrium strategies are the following:

b1) The $B$ truthfully reports ($PG$)

b2) The $B$ truthfully (randomizes) reports after $s_1 = 1$ ( $s_1 = 0$) ($PG(0)$)

b3) The $B$ truthfully (randomizes) reports after $s_1 = 0$ ( $s_1 = 1$) ($PG(1)$)

In order to complete the proof we show that all these putative equilibria exist, and show that (a3), (a4) (a5) and (a6) are consistent with case (A), while (b1), (b2), and (b3) are consistent with case (B)

**a3) $B$ truthfully reports after $s_1 = 0, 1$; $U$ truthfully reports after $s_1 = 0$ and randomizes after $s_1 = 1$ ($PP_B(1)$)**

We denote with $\lambda_U \in (0,1)$ the probability that the Unbiased expert truthfully reveals his signal when receiving $s_1 = 1$.

Let $\widehat{\alpha}^{PP_B}(m_1, x_1)$ and $\widehat{\gamma}^{PP_B}(m_1, x_1)$ denote the value of reputations in a (putative) $PP_G$ equilibrium. It is straightforward to verify that:

$$\widehat{\alpha}^{PP_B}(1,1) > \widehat{\alpha}^{PP_B}(0,0) > \alpha > \widehat{\alpha}^{PP_B}(0,0) > \widehat{\alpha}^{PP_B}(1,0)$$
$$\widehat{\gamma}^{PP_B}(0,0) > \widehat{\gamma}^{PP_B}(0,1) > \gamma > \widehat{\gamma}^{PP_B}(1,1) = \widehat{\gamma}^{PP_B}(1,0)$$

In words, since in this equilibrium a good expert sends message 0 more often than a good expert, the observation of message 0 (1) leads the $DM$ to revise the reputation for being unbiased upwards (downwards). With regard to the reputation for ability, making a correct (incorrect) forecast rewards (penalizes) reputation as in a truthtelling equilibrium but .

Now let $V^{PP_B}(m_1, x_1)$ denote the value of $V(m_1, x_1)$ in this partial pooling equilibrium. Given the above values of reputations, it is straightforward to show the following result.

**Lemma 7:** $V^{PP_B}(0,0) > V(\varnothing) > V^{PP_B}(1,0)$ for any value of $\lambda_U$.

In order to prove existence we proceed in the following way:

1) Prove that $(1,1)$ and $(0,0)$ are the only states in which the Expert will be hired in a putative $PP_B$ equilibrium and that this occurs whenever the Unbiased expert's probability of truthfully reporting $s_1 = 1$, which we call $\lambda_U$ is sufficiently high.

2) Show that for these values of $\lambda_U$ the DM's Hiring and Firing strategy is consistent with the strategies of the experts to play $PP_B$

1) **Proof that (1,1) and (0,0) are the only states in which the Expert will be hired in $PP_B$**

We first prove that in order for $PP_B$ to hold the expert must be hired in state $(1,1)$ which implies that he will always be fired in $(0,1)$

i) Since by Lemma 7 for $(m_1, x_1) = (0,0)$ and $(m_1, x_1) = (1,0)$ the expert will always be respectively hired and fired, and

ii) In any equilibrium in which the Good expert sends message 1 with positive probability, she must be hired after sending message 1, given (i),she therefore must be hired after $(1,1)$.

Finally we are left to show that whenever the expert is hired in $(1,1)$ she is always fired in $(0,1)$

It is simple to show that there exists a $\lambda''_U$ such that for $\lambda_U > \lambda''_U$ the expert is hired with $(1,1)$ and the following condition is satisfied:

$$\gamma^{PP_B}(1,1)(2q^{PP_B}(1,1)-1) > \gamma(2q-1) \tag{29}$$

and a $\lambda'_U$ such that for $\lambda_U > \lambda'_U$ the expert is fired with $(0,1)$ and the following condition is satisfied:

$$\gamma^{PP_B}(0,1)(2q^{PP_B}(0,1)-1) < \gamma(2q-1) \tag{30}$$

In order to complete the proof it is sufficient to prove that $\lambda^{''}_U > \lambda^{'}_U$ because if this is the case:

- when $\lambda_U > \lambda''_U > \lambda'_U$ both (29) and (30) are satisfied
- when $\lambda_U \in (\lambda'_U, \lambda^{''}_U)$ (a) is not satisfied and $PP_B$ is not an equilibrium
- when $\lambda''_U > \lambda'_U > \lambda_U$ (29) is not satisfied and $PP_B$ is not an equilibrium

These imply that (29) is satisfied if and only if (30) is also satisfied

In order to prove that $\lambda''_U > \lambda'_U$ we proceed in two steps: 1) find a closed form expression for $\lambda^{''}_U$; 2) show that for this value (30) is never satisfied

1) $\lambda''_U$ is the value of $\lambda_U$ that satisfies (29) with equality, substituting the corresponding reputations $\gamma(0,1)$ and $q(0,1)$ and $\widehat{q} = \widehat{\alpha}(p-1/2)+1/2$ we obtain:

which gives us:

$$\lambda^{''}_U = (1-\gamma)\left(\frac{q}{p-\gamma q}\right)$$

2) Using $\gamma(0,1)$ and $q(0,1)$ (30) can be written in the following way:

$$\begin{aligned} & \frac{\gamma[(1-q)+q(1-\lambda_U)]}{(1-q)+q\gamma(1-\lambda_U)}\left(2\left(\left(\frac{\alpha[(1-p)+p\gamma(1-\lambda_U)]}{(1-q)+q\gamma(1-\lambda_U)}\right)(p-1/2)+1/2\right)-1\right) \\ < \;& \gamma(2\left(\alpha(p-1/2)+1/2\right)-1) \end{aligned}$$

which simplifies to:

$$\frac{[(1-q)+q(1-\lambda_U)][(1-p)+p\gamma(1-\lambda_U)]}{[(1-q)+q\gamma(1-\lambda_U)]^2}<1$$

substituting the closed form solution for $\lambda_U''$ obtained in step 1, this simplifies to:

$$\frac{(p-\gamma q-\gamma q^2-q^2)(p-p^2-\gamma q-p^2\gamma)}{[p-pq-\gamma q+p\gamma q]^2}<1$$

multiplying both sides by the denominator, and after some computations we obtain:

$$q^3\gamma(1-\gamma)-p^3(1-\gamma)-q^2p-\gamma^2p^2q<p^2q(\gamma-2)+pq^2(-2\gamma^2+2)$$

which becomes:

$$(1-\gamma)[q^3\gamma-p^3]+p^2q(-\gamma^2-\gamma+2)+pq^2(2\gamma^2-\gamma+1)<0$$

$$(1-\gamma)[q^3\gamma-p^3+(\gamma+2)p^2q-2pq^2(\gamma+1/2)]<0$$

this becomes:

$$(2pq-q^2-p^2)(p-\gamma q)<0$$

and since $(2pq-q^2-p^2)<0$ and $(p-\gamma q)>0$ the above inequality is always satisfied.

To complete the proof we show that If the expert is hired in $(1,1)$ this necessarily implies that she is also hired in $(0,0)$ in any $PP_B$ equilibrium. To see this observe that in any $PP_B$ equilibrium in order for the $U$ to randomize it must be that (16) is satisfied with equality, which becomes:

$$qV^{PP_B}(1,1)=(1-q)V^{PP_B}(0,0),$$

so that $V^{PP_B}(1,1) = ((1-q)/q)\, V^{PP_B}(0,0)$. Now whenever the expert is hired in $(1,1)$ it must be that

$$V^{PP_B}(1,1) > \gamma(2q-1),$$

since $V^{PP_B}(0,0) > ((1-q)/q)\, V^{PP_B}(0,0)$ because $q > 1/2$, it follows that the condition for the expert to be hired in (0,0):

$$V^{PP_B}(0,0) > \gamma(2q-1).$$

is also always satisfied.

2) **Show that for these values of $\lambda_U$ the DM's Hiring and Firing strategy is consistent with the strategies of the experts to play $PP_B$**

We begin by defining the binding conditions for the two types of experts, that must be satisfied in order for a $PP_B$ equilibrium to exist, that are consistent with the H/F conditions of the $DM$ defined in point (1) of the proof. The first refers to the condition that must be satisfied in order for the $U$ expert to randomize when receiving $s_1 = 1:$

$$qV^{PP_B}(1,1) = (1-q)V^{PP_B}(0,0), \tag{31}$$

and the second refers to the $B$ expert's condition for truthfully reporting her signal when receiving $s_1 = 0:$

$$\delta_B q[V^{PP_B}(0,0)+1] > (1-\delta_B) + \delta_B(1-q)[V^{PP_B}(1,1)+1]. \tag{32}$$

The relevant updates on reputation that are consistent with this putative equilibrium are

$$\gamma^{PP_B}(1,1) = \frac{\lambda_U\gamma}{\lambda_U\gamma+(1-\gamma)}; \gamma^{PP_B}(0,0) = \frac{\gamma[q+(1-q)(1-\lambda_U)]}{q+(1-q)(1-\lambda_U)} \tag{33}$$

and

$$\alpha^{PP_B}(1,1) = \frac{\alpha p}{q}; \alpha^{PP_B}(0,0) = \frac{\alpha[p+\gamma(1-p)(1-\lambda_U)]}{q+(1-q)(1-\lambda_U)} \tag{34}$$

The rest of the proof proceeds in the following steps,:

(i) Show that the LHS and RHS of (31) are respectively decreasing and increasing in $q$

(ii) Show that at $q = 1/2$ and $\lambda_U = \lambda_U''$ the LHS of ((31) is $\leq$ than the (RHS) for every $\gamma \in (0, 1)$

(iii) Observing that at $\lambda_U = 1$ the LHS of (31) is greater than the RHS for every $q \in (1/2, 1)$

By continuity (i), (ii) and (iii) imply that there exists $\overline{q} \in (0, 1)$, such that for $q < \overline{q}$ there always exists a $\lambda_U \in (\lambda_U'', 1)$ for which (31) is satisfied with equality.

(i) Substituting the equilibrium reputation values (33) and (34) we obtain expressions for $V^{PP_B}(1, 1)$ and $V^{PP_B}(0, 0)$, and dividing both sides of (31) by $(1 - q)$ we can write the $LHS$ and $RHS$ of in the following way:

$$
\begin{aligned}
LHS &= \frac{\lambda_U p}{(1-q)(\lambda_U + (1-\gamma))} \\
RHS &= \frac{(q + (1-q)(1-\lambda_U)(p + \gamma(1-p)(1-\lambda_U))}{(q + \gamma(1-q)(1-\lambda_U)}.
\end{aligned}
$$

It is straightforward to show that the $LHS(RHS)$ is increasing (decreasing) in $q$

(ii) Now fixing $q = 1/2$ and $\lambda_U = \lambda_U''$ the $LHS = 1$, while the $RHS$ simplifies to:

$$
RHS = \left(\frac{4p - 1 - \gamma}{p + \gamma(p-1)}\right)\left(\frac{p^2 + \gamma(p - p^2 - 1)}{p + \gamma(p-1)}\right)
$$

Now since the $RHS$ is decreasing in $\gamma$, in order to prove this point it is sufficient to show that the $LHS < RHS$ for $\gamma = 1$.Plugging in $\gamma = 1$, the $RHS$ simplifies to 1, which proves that the $LHS \leq RHS$ for every $\gamma \in (0, 1)$

(iii) Now when $\lambda_U = 1$ we have that $V^{PP_B}(1, 1) = V^{PP_B}(0, 0) = V^{TT}(m_1 = x_1)$ so that the $LHS > RHS$,which completes the proof.

**a4) $B$ truthfully reports after $s_1 = 0, 1$; $U$ truthfully reports after $s_1 = 1$ and randomizes after $s_1 = 0$ $(PP_B(0))$**

Let me call this equilibrium $PP_B(0)$ and let $\lambda_{U0}$ denote the probability with which $U$ sends $m_1 = 0$ when he has observed $s_1 = 0$. Let us first consider the conditions that must hold for $U$ and $B$'s strategies to be equilibrium strategies.

**Unbiased expert.**

If $s_1 = 1$:

$$\Pr(x_1 = 1 \mid s_1 = 1) V^{PP_B(0)}(1,1) > \Pr(x_1 = 0 \mid s_1 = 1) V^{PP_B(0)}(0,0) \tag{35}$$

If $s_1 = 0$:

$$\Pr(x_1 = 0 \mid s_1 = 0) V^{PP_B(0)}(0,0) = \Pr(x_1 = 1 \mid s_1 = 0) V^{PP_B(0)}(1,1) \tag{36}$$

**Biased expert.**

If $s_1 = 1$:

$$(1-\delta_E) + \Pr(x_1 = 1 \mid s_1 = 1)\delta_E \left[V^{PP_B(0)}(1,1) + 1\right] > \delta_E \Pr(x_1 = 0 \mid s_1 = 1) \left[V^{PP_B(0)}(0,0) + 1\right] \tag{37}$$

If $s_1 = 0$:

$$\Pr(x_1 = 0 \mid s_1 = 0)\delta_E \left[V^{PP_B(0)}(0,0) + 1\right] > (1-\delta_E) + \delta_E \Pr(x_1 = 1 \mid s_1 = 0) \left[V^{PP_B(0)}(1,1) + 1\right] \tag{38}$$

**Firing strategy.**

Now let us consider the firing strategy of the DM. Since both messages $m_1 = 0$ and $m_1 = 1$ are sent with positive probability on the equilibrium path, it must necessarily be that the DM retains the incumbent with positive probability both after $m_1 = 0$ and $m_1 = 1$. In this putative equilibrium, $V^{PP_B(0)}(1,1) > V > V^{PP_B(0)}(0,1)$ implying that the DM retains the incumbent after observing $(m_1 = 1, x_1 = 1)$ and fires the incumbent after observing $(m_1 = 0, x_1 = 1)$. But then, it must be that the DM retains the incumbent after $(m_1 = 0, x_1 = 0)$. This requires that:

$$V^{PP_B(0)}(0,0) > V \tag{39}$$

Since the signal structure is symmetric it is straightforward to show that as for $PP_B(1)$ also in $PP_B(0)$,the expert will be hired only in states (1,1) and (0,0)

**Analysis**

Condition (39) requires that $\lambda_{U0} \in (\lambda_{U0}^{*}, 1)$ where $\lambda_{U0}^{*} \in (0,1)$ solves $V^{PP_B(0)}(0,0) = V$.

Now consider condition (36). When $\lambda_{U0}=1$,$V^{PP_B(0)}(1,1) = V^{PP_B(0)}(0,0)$ and the LHS is larger than the RHS. When $\lambda_{U0}=0$,$V^{PP_B(0)}(0,0) = 0$ while $V^{PP_B(0)}(1,1) > 0$ and the LHS is smaller than the RHS. Thus there always exists a scalar $\lambda_{U0}^{**}$ such that (36) is satisfied. Consistency with the firing condition imposes that $\lambda_{U0}^{**}$ belong to the range $(\lambda_{U0}^{*}, 1)$. This occurs only if $\alpha$ is sufficiently small.

Now note that condition (38) is satisfied for $\delta_E$ sufficiently large whenever (36) is satisfied. To see this, rewrite condition (36) as

$$\Pr(x_1 = 0 \mid s_1 = 0)V^{PP_B(0)}(0,0) - \Pr(x_1 = 1 \mid s_1 = 0)V^{PP_B(0)1}(1,1) = 0$$

and conditions (38) as

$$\begin{aligned} &\delta_E \left[ \underbrace{\Pr(x_1 = 0 \mid s_1 = 0)V^{PP_B(0)}(0,0) - \Pr(x_1 = 1 \mid s_1 = 0)V^{PP_B(0)}(1,1)}_{0 \text{ (if (36) holds)}} \right] \\ &+\delta_E \left[ \Pr(x_1 = 0 \mid s_1 = 0) - \Pr(x_1 = 1 \mid s_1 = 0) \right] \\ > \; & 1 - \delta_E \end{aligned}$$

Note that $\Pr(x_1 = 0 \mid s_1 = 0) - \Pr(x_1 = 1 \mid s_1 = 0) > 0$ because the signal is informative. Thus, if (36) holds, there always exists a scalar $\delta_E^{*}$ such that conditions (38) is satisfied for $\delta_E^{*} > \delta_E$.

Finally, note that conditions (35) and (37) are always satisfied because $\Pr(x_1 = 1 \mid s_1 = 1) > \Pr(x_1 = 0 \mid s_1 = 1)$ and $V^{PP_B(0)}(1,1) > V^{PP_B(0)}(0,0)$.

**Conclusion:** $PP_B(0)$ exists only if $\alpha$ is sufficiently small and is characterized by a probability of lying by $U$ that is sufficiently small.

**a5) $B$ truthfully reports (randomizes) after $s_1 = 1$ ( $s_1 = 0$), and $U$ truthfully (randomizes) reports after $s_1 = 0$ ( $s_1 = 1$)**

We prove this in two steps

1) By Lemma 4 (iv) we know whenever the $U$ expert's strategy is as mentioned in (a5), this is never consistent with an equilibrium in which $B$ randomizes after $s_1 = 1$

2) We need to prove that there exists $(\lambda_B, \lambda_U) \in (0,1)^2$ such that:

$$qV(1,1) = (1-q)V(0,0), \tag{40}$$

$$\delta_E q[V(0,0)+1] > (1-\delta_E) + \delta_E(1-q)[V(1,1)+1], \tag{41}$$

$$V(0,1) < V < V(1,1). \tag{42}$$

We prove this by construction based on the existence of $PP_G$ and $PP_B$

**Unbiased Incumbent's Condition:**

When $\lambda_B = 1$ and $\lambda_U = \lambda_U^{PP_B} \in (0,1) : qV(1,1) = (1-q)V(0,0)$.

Since $V(1,1)$ is strictly increasing in $\lambda_B$ while $V(0,0)$ is strictly decreasing in $\lambda_B$, we have that:

When $\lambda_B = 1-\varepsilon$ and $\lambda_U = \lambda_U^{PP_B} \in (0,1) : qV(1,1) < (1-q)V(0,0)$.

By $PP_G$ we also know that:

When $\lambda_B = 1-\varepsilon$ and $\lambda_U = 1 : qV(1,1) > (1-q)V(0,0)$.

But then, keeping $\lambda_B = 1-\varepsilon$, there must exist a $\lambda_U \in \left(\lambda_U^{PP_B}, 1\right)$ such that $qV(1,1) = (1-q)V(0,0)$.

**Hiring/firing**

$V(0,0) < V < V(1,1)$.

When $\lambda_B = 1$ and $\lambda_U = \lambda_U^{PPB} \in (0,1) : V(0,1) < V < V(1,1)$.

When $\lambda_B = 1-\varepsilon$ and $\lambda_U = \lambda_U^{PPB} \in (0,1)$, the above inequality still holds (in fact, since $V(1,1)$ is strictly increasing in $\lambda_B$ and $V(0,1)$ strictly decreasing in $\lambda_B$, $\varepsilon > 0$ must be chosen small enough to ensure that this inequality is still valid; by continuity, such $\varepsilon$ exists).

When $\lambda_B = 1-\varepsilon$ and $\lambda_U \in \left(\lambda_U^{PPB}, 1\right)$, the inequality above holds a fortiori (because $V(1,1)$ is strictly increasing in $\lambda_U$ and $V(0,1)$ is strictly decreasing in $\lambda_U$).

**Biased Incumbent's Condition:**

$\delta_E q[R(0,0)+1] > (1-\delta_E) + \delta_E(1-q)[R(1,1)+1]$

We need to show that there exists a value of $\delta_E \in (0,1)$ that guarantees that (41) holds. From (41) we obtain:

$$\delta_B = \frac{1}{[qR(0,0) - (1-q)R(1,1) + 2q]}$$

If (40) is satisfied and since $q > 1/2$, $qR(0,0) - (1-q)R(1,1) > 0$ which implies that $\delta_B \in (0,1)$.

**a6) $B$ truthfully reports (randomizes) after $s_1 = 1$ ( $s_1 = 0$), and $U$ truthfully (randomizes) reports after $s_1 = 1$ ( $s_1 = 0$)**

It is straighforward to show that applied in (a5) can used to prove the existence of this equilibrium.

**b1) $U$ always sends $m_1 = 0$ and $B$ truthfully reports ($PG$).**

Let $\widehat{\alpha}^{PG}(m_1, x_1)$ and $\widehat{\gamma}^{PG}(m_1, x_1)$ denote the value of reputations in a (putative) in partially revealing $PG$ equilibrium in which the $U$ always sends $m_1 = 0$ and $B$ always truthfully reports. It is straightforward to verify that:

$$\underline{\alpha} \equiv \widehat{\alpha}^{PG}(1,0) < \widehat{\alpha}^{PG}(0,1) < \alpha < \widehat{\alpha}^{PG}(0,0) < \widehat{\alpha}^{PG}(1,1) = \overline{\alpha}$$
$$0 = \widehat{\gamma}^{PG}(1,0) = \widehat{\gamma}^{PG}(1,1) < \gamma < \widehat{\gamma}^{PG}(0,0) = \widehat{\gamma}^{PG}(0,1)$$

Now let $V^{PG}(m_1, x_1)$ denote the value of $V(m_1, x_1)$ in a $PG$. Given the above values of reputations, it is straightforward to show that in a truthtelling equilibrium the following chain of inequalities holds:

$$V^{PG}(0,0) > V \gtreqless V^{PG}(0,1) > V^{PG}(1,0) = V^{PG}(1,1) \equiv 0 \tag{43}$$

**DM's firing strategy**. From (43), it follows that in a truthtelling equilibrium the DM will always retain the incumbent whenever $m_1 = x_1 = 0$ and possibly also when $(m_1, x_1) = (0,1)$, and fire him otherwise.

**Unbiaed Incumbent's Strategy** Notice that (15) is always positive and (16) is always negative implying that the $G$ expert's strategy is consistent with equilibrium.

**Biased Incumbent's strategy**. Consider the case in which a bad expert receives $s_1 = 0$, as long as $\delta_E$ is sufficiently high (13) is always satisfied, while when $s_1 = 1$, as long as $\delta_E$ is not too high, (14) is always satisfied. It follows that for intermediate values of $\delta_E$ truthtelling is indeed consistent with equilibrium for the $B$ expert.

When $\delta_E$ is sufficiently low so that the $B$ expert always sends $m_1 = 1$ no informative equilibrium satisfying case (B) exists. To prove this notice given the putative equilibrium strategies of $U$, to always sends $m_1 = 0$, and the $DM$ to hire only after $m_1 = 0$, the $B$ expert would never be listened to or rehired after sending $m_1 = 1$, and therefore always has an incentive to deviate to $m_1 = 0$.

Finally notice that these equilibrium strategies of the experts are consistent with both of the possible hiring/firing strategies of $DM$. The $DM$ will hire when $(m_1, x_1) = (0, 1)$ depending on the priors on $\alpha$ and $\gamma$ as we show in the proof of Proposition 6.

**b2)** $U$ **always send** $m_1 = 0$**;** $B$ **truthfully reports (randomizes) after** $s_1 = 1$ **(** $s_1 = 0$**)** $(PG(0))$

It is immediate to show that since (13) is increasing in $\delta_E$ and strictly positive (negative) when $\delta_E = 1(\delta_E = 0)$ there always exists a $\delta_E^{'}$ such that (13) is satisfied with equality. Given this $B$ expert strategy the chain of inequalities given by (43) continues to hold which implies that this is consistent with the above mentioned strategy of the $U$ expert and the Hiring/Firing strategy of the DM.

**b3) The** $B$ **truthfully reports (randomizes) after** $s_1 = 0$ **(** $s_1 = 1$**)** $(PG(0))$

The same logic as for part b2) holds. Since (14) is decreasing in $\delta_E$ and strictly positive (negative) when $\delta_E = 0(\delta_E = 1)$ there always exists a $\delta_E^{''}$, (where $\delta_E^{''} < \delta_E^{'}$) such that (14) is satisfied. Also in this case the chain of inequalities given by (43) continues to hold which implies that this profile of strategies is consistent equilibrium.

**Proof of Proposition 5**

Since the signal structure is symmetric it straightforward to show that this proof holds for both $PP_B(1)$ and $PP_B(0)$. We therefore consider $PP_B(1)$ for the sake of exposition.

*Discipline is worst since* $E_0^{TT}(R_1) = \frac{r}{2}(2q-1) > E_0^{PP_B}(R_1) = \frac{r}{2}(2q-1)[1-\gamma(1-\lambda_U)]$

Therefore *a necessary condition for* $PP_B$ *to improve sorting with respect to* $TT$ *is:*

$$\begin{aligned} E_0^{PP_B}(R_2) &> E_0^{TT}(R_2), \\ & [\Pr(0,1 \mid PP_B) + \Pr(1,0 \mid PP_B)]\,\gamma(2q-1) \\ & +\gamma q \lambda_U[(2\overline{q}-1) + \gamma[q + (1-\lambda_U)(1-q)(2q(0,0)-1)] \\ &> 2(1-q)\gamma(2q-1) + 2q\gamma(2\overline{q}-1) \end{aligned}$$

*Bringing the first term on the RHS to the LHS and the second term on the LHS to the RHS of the equation, the above expression simplifies to:*

$$\gamma(1-\lambda_B)(2q-1)^2 > 2q(\overline{q} - q(0,0)) + (1-\lambda_U)[q(2\overline{q}-1) - (1-q)(2q(0,0)-1)] \quad (44)$$

*where the LHS represents the net benefit of receiving an incorrect message in* $PP_B$ with respect to $TT$ *which is always positive, and the RHS represents the net benefit of receiving a correct message in* $TT$ wrt $PP_B$ which is also positive.

*Now considering the RHS,* $(\overline{q} - q(0,0))$ *can be rewritten as:*

$$(p - 1/2)\left(\frac{\alpha p}{q} - \alpha(0,0)\right),$$

*and therefore the RHS becomes:*

$$2(p-1/2)\left(\frac{\alpha\gamma(1-\lambda_U)(p-q)}{[q+(1-q)\gamma(1-\lambda_U)]}\right) + x(1-\lambda_U)$$

where $x = [q(2\overline{q}-1) - (1-q)(2q(0,0)-1)]$*(28) can therefore be rewritten as:*

$$(2q-1)^2 > 2(p-1/2)\left(\frac{\alpha(p-q)}{[q+(1-q)\gamma(1-\lambda_U)]}\right) + x/\gamma.$$

*Now since* $(2q-1)^2 = 2\alpha(p-1/2)^2$ *and* $\alpha(p-q) = (\alpha-\alpha^2)(p-1/2)$*, the above expression becomes:*

$$2\alpha > \left(\frac{(1-\alpha)}{[q+(1-q)\gamma(1-\lambda_U)]}\right) + \frac{x}{2\alpha\gamma(p-1/2)^2}.$$

Now we can simplify $x/\left(2\alpha\gamma(p-1/2)^2\right)$ which becomes:

$$\frac{p(2q-1)+\gamma(1-\lambda_U)(1-q)(2p-1)}{\gamma(2p-1)[q+(1-q)\gamma(1-\lambda_U)]}$$

so that the initial expression can be rewritten in following way:

$$2\alpha > \left(\frac{2\alpha p + \gamma(1-\alpha) + \gamma(1-\lambda_U) - \gamma\alpha(2p-1)(1-\lambda_U)}{\gamma[q+(1-q)\gamma(1-\lambda_U)]}\right)$$

simplifying we obtain a quadratic function of $\alpha$ :

$$\alpha^2[(2p-1)\gamma(1-\gamma(1-\lambda_U)] + \alpha[2\gamma + \gamma(1-\lambda_U)(\gamma + (2p-1) - 2p] - (\gamma + \gamma(1-\lambda_U) > 0$$

To ease notation we define the above expression in the following way:

$$f(\alpha,(1-\lambda_U))>0$$

Since the second derivative of $f(\alpha,(1-\lambda_U))$ with respect to $\alpha$ is positive, it follows that the LHS is a convex function of $\alpha$, and the inequality has two solutions, one positive and one negative.

Since $\alpha\in(0,1)$, it follows that the only relevant solution is the positive one which we denote with $\alpha^*(1-\lambda_U)$, implying that $\alpha^*$ is a function of the probability of misreporting of the Unbiased expert. The above inequality is therefore satisfied for $\alpha\in(\alpha^*(1-\lambda_U),1)$

Now, since

$$\frac{\partial f(\alpha,(1-\lambda_U))}{\partial(1-\lambda_U)\mid_{\alpha>0}}<0$$

it follows that $\partial\alpha^*(1-\lambda_U)/\partial(1-\lambda_U)>0$. This implies that there exists *some* $PP_B$ equilibrium that improves sorting with respect to $TT$ if and only if $\alpha\in(\alpha^*(0),1)$.

At this point the only thing that is left to show in order to prove that there *never* exists a sufficiently high value of $\alpha$ for which every $PP_B$ equilibrium improves sorting, is that $\alpha^*(0)>1$.

We therefore compute $\alpha^*(0)$ :

$$\begin{aligned}\alpha^*(0) &= \frac{-(\gamma-p)+(\gamma^2-\gamma+p^2)^{1/2}}{\gamma\,(2p-1)}\gtreqless 1\\ \text{if } p &> 1/2 \text{ it follows that } \alpha^*(0)>1\end{aligned}$$

from which we obtain that if $p>1/2$ it follows that $\alpha^*(0)>1$, and since $p>1/2$ by construction, this completes the proof.

**Proof of Proposition 6**

In order to find an instance in which $PG$ may improve welfare with respect to $TT$, we consider the equilibrium in which the expert is hired only after (0,0)

Since it straightforward that discipline is always worst in $PG$,a necessary condition for

$PG$ to improve welfare with respect to $TT$ is that it must improve sorting:

$$\begin{aligned}
E_0[R_2 \quad | \quad & PG] > E_0[R_2 \mid TT], \\
& 1/2[q(1-\gamma)+(1-q)(1-\gamma)+\gamma+(1-\gamma)(1-q)]\gamma\,(2q-1) \\
& +1/2\gamma(2q_{00}^{PG}-1) \\
> \quad & q\gamma(2\overline{q}-1)+(1-q)(2q-1)
\end{aligned}$$

bringing the last two terms on the LHS to the RHS and last term on the RHS to the LHS this simplifies to:

$$1/2[1+(1-q)(1-\gamma-2)]\gamma\,(2q-1) > \gamma[2q\overline{q}-q_{00}^{PG}+(1/2-q)]$$

using the the fact that $2\widehat{q}-1=2\widehat{\alpha}(p-1/2)$ and $\alpha_{00}^{PG}=\frac{\alpha[p+\gamma(1-p)]}{[q+\gamma(1-q)]}$ and obtain:

$$\begin{aligned}
& 1/2[q(1-\gamma)-\gamma)]2\alpha(p-1/2) \\
> \quad & \left(2q\left[\frac{\alpha p}{q}(p-1/2)+1/2\right]-\left[\frac{\alpha[p+\gamma(1-p)]}{[q+\gamma(1-q)]}(p-1/2)+1/2\right]+(1/2-q)\right)
\end{aligned}$$

$$1/2[q(1-\gamma)-\gamma)]2\alpha(p-1/2) > 2\alpha p(p-1/2)+q-\frac{\alpha[p+\gamma(1-p)]}{[q+\gamma(1-q)]}(p-1/2))-q$$

$$[q(1-\gamma)-\gamma]\alpha(p-1/2) > \alpha(p-1/2)\left[2p-\frac{[p+\gamma(1-p)]}{[q+\gamma(1-q)]}\right]$$

$$[q(1-\gamma)-\gamma] > 2p-\frac{[p+\gamma(1-p)]}{[q+\gamma(1-q)]}$$

which we can rewrite in the following way:

$$[q(1-\gamma)-\gamma]-2p+\frac{[p+\gamma(1-p)]}{[q+\gamma(1-q)]} > 0 \tag{45}$$

Notice that (45) has the following properties: 1) it is strictly decreasing in $\alpha$ and strictly decreasing in $\gamma$; 2) Since when $\alpha=1$ there exists a $\overline{\gamma}$ such that for $\gamma<\overline{\gamma}$ (45) is satisfied, then for any $\alpha \in (0,1)$ there always exists a function $\overline{\gamma}(\alpha) \in (0,1)$ such that for for $\gamma < \overline{\gamma}(\alpha)$ (45) is satisfied. Moreover, it can be shown that $\overline{\gamma}(\alpha = 0) < 1$

Now we need to show that there is a non empty set of values of $\gamma$ and $\alpha$ for which the DM will hire the expert only in state $(0,0)$ Since it is straightforward to show that the hiring condition is always satisfied in state $(0,0)$, it is sufficient to analyze the condition for Firing the expert in state $(0,1)$, this will occur if the following condition is satisfied:

$$\gamma^{PG}(0,1)(2q^{PG}(0,1) - 1) > (<)\gamma(2q - 1)$$

where we can write $\gamma^{PG}(0,1) = \frac{\gamma}{\gamma+(1-\gamma)(1-q)}$ and $\alpha_{01}^{PG} = \frac{\alpha[\gamma+(1-\gamma)(1-p)]}{[\gamma+(1-\gamma)(1-q)]}$ then the above equation simplifies to:

$$\frac{[\gamma + (1-\gamma)(1-p)]}{[\gamma + (1-\gamma)(1-q)]^2} > (<)1$$

$$[\gamma + (1-\gamma)(1-p)] > (<)\gamma^2 + 2\gamma(1-\gamma)(1-q) + (1-\gamma)^2(1-q)^2$$

which simplifies

$$(1-\gamma)[\gamma + (1-p) - 2\gamma(1-q) - (1-\gamma)(1-q)^2] > (<)0$$

since $(1-\gamma)$ is always positive we can focus on when the term in square brackets is positive. This term simplifies to:

$$\gamma + (1-p) - (1-q)(\gamma(1+q) + (1-q)) > (<)0$$

which becomes:

$$\gamma > (<)\frac{(1-q)^2 - (1-p)}{q^2} \equiv \gamma'(\alpha) \tag{46}$$

Observation of (46) leads to the following conclusions regarding the RHS of (46)

i) It is decreasing in $\alpha$ so that for higher values of $\alpha$, the threshold value of $\gamma$ for which it is satisfied is decreasing and negative as $\alpha \to 1$, which implies that the expert will always be hired after $(0,1)$. We define $\alpha(PG)$ as the threshold value of $\alpha$ for which $\gamma'(\alpha) = 0$.

ii) This implies that for higher $\alpha$ it is more likely that the expert will be hired since there is less to learn about ability, and revealing information on integrity is relatively more valuable. Another way of putting it is that $\gamma$ has to be lower for the expert to be fired when sending 0, since with small $\gamma$ the update on integrity does not vary much.

Putting together point 1 and 2 we define $\gamma(PG) = \min[\gamma(\alpha), \gamma'(\alpha)]$, which allows us to state the following: for $\alpha < \alpha(PG)$ there always exists a $\gamma(PG) \in (0, 1)$ such that $PG$ improves sorting with respect to $TT$ for $\gamma < \gamma(PG)$.This completes the proof.

**Proof of Proposition 7**

First note that observing (15) and (16), $\delta_E$ does not affect the behavior of the $U$ expert, and we can concentrate on that of the $B$

By Lemma 4 point (iv), it follows that (13) is the only condition that is relevant for existence of $TT$ and $PP_B$. Since (13) is strictly increasing in $\delta_E$ and $> 0 (< 0)$ for $\delta_E = 1 (\delta_E = 0)$, it provides threshold value of $\delta_E$ above which each type of equilibria exists. We define these values for each type of equilibrium respectively as $\underline{\delta}_E^{TT}$ and $\underline{\delta}_E^{PP_B}$.

Regarding $PG$, since $\pi_B(m_1 = s_1 = 0) > \pi_B(m_1 = 0 \neq s_1)$ and $\pi_B(m_1 = 1, \cdot) = 0$ and since (13) and (14) are respectively increasing and decreasing in $\delta_E$, we have that there exists a $0 < \underline{\delta}_E^{PG} < \overline{\delta}_E^{PG} < 1$, such that $PG$ holds for $\delta_E \in (\underline{\delta}_E^{PG}, \overline{\delta}_E^{PG})$, where $\underline{\delta}_E^{PG}$ are $\overline{\delta}_E^{PG}$ respectively defined by the values the values of $\delta_E$ for which conditions (13) and (14) are satisfied with equality.

The following points allow us to complete the proof

1) Characterization of Proposition 1, in terms of (a1) and (a2) imply that $PP_G$ and $TT$ never coexist, so that $PP_G$ exists for $\delta_E \in (\underline{\delta}_E^{PP_G}, \overline{\delta}_E^{TT})$ where $\underline{\delta}_E^{PP_G} < \overline{\delta}_E^{TT}$

2) $\underline{\delta}_E^{PP_B(0)} > \overline{\delta}_E^{TT}$ and $\underline{\delta}_E^{PP_B(1)} < \overline{\delta}_E^{TT}$ : $PP_B$ exist both for when $TT$ exist and when $TT$ does not exist

3) $\overline{\delta}_E^{PG} \lesseqgtr \overline{\delta}_E^{TT}$ it straightforward to show that the sign of the expression may vary based on the values of $\gamma, \alpha$ and $p$

4) When $\delta_E < \min[\underline{\delta}_E^{PP_G}, \underline{\delta}_E^{PP_B(1)}, \underline{\delta}_E^{PP_G}]$ no informative equilibria exist. To prove this, notice that given any strategy of $U$, for these values of $\delta_E$ the $B$ expert will always send $m_1 = 1$. Given this strategy of $B$, any putative equilibrium in which $U$ is sending both signals, can never be an equilibrium since by the proofs of Propositions 1 and 4, the $DM$ will never hire after both messages, when the probability of misreporting of the $B$ expert is too high. By Lemma 4, the only other plausible equilibrium involves $U$ always sending $m_1 = 0$ and the $DM$ hiring only after $m_1 = 0$. By the Proof of Proposition 4, this can never

be an equilibrium because the $B$ expert would never be listened to or rehired after sending $m_1 = 1$, and therefore always has an incentive to deviate to $m_1 = 0$.