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ASYMMETRIC ALL-PAY AUCTIONS

FEI LI AND SÉRGIO O. PARREIRAS

Abstract. In the independent private values setting, we provide sufficient conditions for the continuity and uniqueness of the equilibrium of all-pay auctions and, an algorithm that computes the equilibrium.

1. Introduction

In numerous economic settings, agents compete by making costly and irreversible investment before the outcome is known, such as labor promotion contest (Lazear and Rosen, 1981), lobbying and rent-seeking activities (Baye et al., 1993; Ellingsen, 1991) and R&D races (Dasgupta, 1986). When some characteristics of agents, such as cost of investment, are privately known by themselves, it is natural to model the economic problem as an all-pay auction with incomplete information. To find a Bayesian-Nash equilibrium of in these all-pay auctions, simplifying assumptions such as – bid strategies are monotone and differentiable they are called upon such limitations: "The results are interesting but how do we know this is the only equilibrium? Even if the differentiable equilibrium is unique, how do we know there are not other equilibria in discontinuous strategies?". The purpose of this paper is two-fold. First, to provide sufficient conditions for equilibrium continuity, differentiability and uniqueness. Second, in doing so, to validate the first-order approach.

Our contribution is mostly to all-pay auctions with more than two asymmetric bidders. For two bidders, under independent private values, Amann and Leininger (1996)'s algorithm finds the equilibrium, which exists by Athey (2001) and, it is unique and differentiable by

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Lizzeri and Persico (1998).[1] For symmetric bidders, Krishna and Morgan (1997) allowing for affiliation characterize the symmetric equilibrium; under independent private values, Parreiras and Rubinchik (2010) prove it is unique.

We restrict attention to risk-neutral bidders with independent, private values for two reasons. First, risk aversion may create discontinuities (Parreiras and Rubinchik, 2010). Secondly, correlation or interdependent values may lead to non existence of monotonic equilibria (Lu and Parreiras, 2015a).

It should be self-evident that our model is not nested with models with correlation and/or interdependent values (Krishna and Morgan, 1997; Lizzeri and Persico, 1998; Lu and Parreiras, 2015b).

The more than two asymmetric bidders case, is relevant to many real-life applications. Moreover, the behavior implied by its equilibrium might be qualitatively different from the other cases (Kirkegaard, 2010; Parreiras and Rubinchik, 2010). It appears, the lack results establishing uniqueness and the validity of the first-order approach has been the major impediment to applied work. Notable exceptions are Kirkegaard (2010, 2013) and Parreiras and Rubinchik (2015).

1.1. **Outline and Preview of Results.** In section 3, we show that any equilibrium under independent, private values and risk-neutral bidders must be monotone. The result follows from the well-known fact that the allocation rule of any incentive compatible mechanism must be monotone.

In section 4, we obtain sufficient conditions for the continuity of equilibrium bid strategies. We adapt an argument of Bernard LeBrun's[2], which was originally used in the context of the first-price auction. In the all-pay auction, unlike the first-price auction, restrictions on the distribution of valuations may be need to ensure continuity.

In section 5, we use the continuity of bid strategies to prove the winning probabilities are differentiable. We show how to recover bid distributions from winning probabilities by solving a linear system of equations, which allow us to establish the cumulative distribution of bids are also differentiable.

Differentiability is not enough to justify the first-order approach because first-order conditions at bid level b only hold for those bidders who are active at b. In other auction formats, this is not a problem as typically, all bidders are always active. In the all-pay auction, however,

[1] Lizzeri and Persico (1998) also allow for affiliation.

[2] See Lebrun (1997). Beware that many results are not included in the published version, Lebrun (1999).

even when bidding is continuous, some bidders might not be active at some bid levels. To bypass such hurdle, in section 6, for $i = 1, \dots, N$, we determine the set of bidders who are active in the internal $[0, \bar{b}_i]$ where, $\bar{b}_i$ is the highest bid placed by bidder i. More importantly, we can determine those bidders who are active in $[0, \bar{b}_i]$ without the need to specify the value of the bid level $\bar{b}_i$. Recently and independently, Hubbard and Kirkegaard (2015) do the same for first-price auctions.[3]

In section 7, we present an algorithm that computes the unique equilibrium of the all pay-auction. The key idea is the since the first-order conditions are independent of the level of bids, the information on the set of active bidders in $[0, \bar{b}_i]$ that we gathered in section 7 is sufficient for us to write down the first-order of active bidders as a system of differential equations and solve it.

2. Model and Notation

There are $N \geq 2$ risk-neutral bidders who compete for a single, indivisible object. Bidder $i = 1, \dots, N$ privately learns the realization of her valuation, v_i, which is drawn from an absolutely continuous distribution, $V_i \sim F_i$, with support on $[\underline{v}_i, \overline{v}_i]$. Valuations are independently distributed. After learning valuations, bidders simultaneously post their bids thus a bidder's strategy is a map from the valuations to bids, $b_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}_+$. The object is allocated to the highest bid. If a tie happens, any random tie-breaking rule can be used.

Definition 1. Given a strategy, $b_i(\cdot)$, the corresponding cumulative probability distribution of bids is $G_i(b) \stackrel{\text{def}}{=} \Pr[b_i(V_i) \leq b]$.

We denote densities (whenever they are well-defined) by the corresponding lower cases; valuations and bids densities are $f_i(v)$ and $g_i(b)$.

Definition 2. For $H : \mathbb{R} \to \mathbb{R}$ and all a, b in $\mathbb{R}$, we define

$$\Delta H(a, b) \stackrel{\text{def}}{=} H(b) - H(a).$$

Definition 3 (Extreme Bids)**.** Given a strategy, $b_i(\cdot)$, bidder i's top and bottom bids are $\bar{b}_i \stackrel{\text{def}}{=} b_i(\overline{v}_i)$ and $\underline{b}_i \stackrel{\text{def}}{=} b_i(\underline{v}_i)$.

[3]To address the econometrics of auctions literature, they restrict attention to auctions with two classes of bidders where valuations are independently and, within each class, identically distributed. Section 6 suggests their results should more generally.

3. Any equilibrium is monotone

Athey (2001) proved the existence of an pure strategy, monotone equilibrium. At the end of this section, we shall see that under independent private values, any equilibrium pure strategy equilibrium is monotone.

Lemma 1 (No Atoms At Positive Bids)**.** *In equilibrium, the distribution of bids G_i is continuous.*

Proof. A discontinuity of G_i corresponds to bidder i choosing $b > 0$ with positive probability. At this $b > 0$, the winning probability of any other bidder distinct from i must be discontinuous at b. That is, bidders distinct from i do not place bids in the some left neighborhood of the bid, $(b-\varepsilon, b)$ where $\varepsilon > 0$. It follows that i's winning probability must be constant on $(b-\varepsilon, b)$ but then bidding $b-\varepsilon/2$ is strictly better rather than bidding b for some type $v_i > 0$ that is bidding b. ■

By Lemma 1, in equilibrium ties never happen at positive bids, so we can define:

Definition 4 (Winning Probability)**.** For $b > 0$, define bidder i's winning probability, $W_i(b) \stackrel{\text{def}}{=} \Pi_{j\neq i} G_i(b)$.

Lemma 2 (Bottom Bids are Zero)**.** *In equilibrium, $\underline{b}_i = 0$.*

Proof. Let $\underline{b} = \max_i \underline{b}_i$. On one hand, if $\underline{b} = 0$ then the proof is done. On the other hand, if $\underline{b} > 0$ then no bidder chooses $\underline{b}$ with positive probability (Lemma 1). Since bids at or below $\underline{b}$ never win and $\underline{b} > 0$ is costly, $\underline{b}$ can not be a best-response. ■

Lemma 3 (No Atoms At Zero For At Least One Bidder)**.** *In equilibrium, $\Pi_{i=1}^n G_i(0) = 0$.*

Proof. Assume that $\Pi_{i\neq j}^n G_i(0) > 0$. Bidding zero leads to ties with positive probability and so W_j must be discontinuous at zero. Bidding zero is optimal only for players with zero valuation. But $v_j = 0$ implies $G_j(0) = 0$ contradicting the assumption. ■

The next lemma is well-known; we omit its proof.

Lemma 4 (Winning Probabilities Are Strictly Increasing.)**.** *In equilibrium, W_i is strictly increasing on $[0, W_i^{-1}(1)]$.*

The next lemma is extremely useful.

Lemma 5 (Ranking Winning Probabilities)**.** *For any $0 \leq a < b$, $\Delta G_i(a, b) \geq \Delta G_j(a, b)$ if and only if $\Delta W_i(a, b) \leq \Delta W_j(a, b)$.*

Proof. It follows immediately from the identity,

$$\Delta W_i(a,b)\Delta G_j(a,b) \equiv \Delta W_j(a,b)\Delta G_i(a,b).$$

■

In words, $\Delta W_i(a,b)$ is the probability that players other than player i bid in the interval (a,b) and $\Delta G_i(a,b)$ is the probability that player i bids in (a,b). The product $\Delta W_i(a,b)\Delta G_i(a,b)$ is the probability that all players bid in (a,b) and so it must be invariant with respect to i.

The *interim* payoff of bidder i with valuation v who bids b is,

$$U_i(b|v) = W_i(b)\cdot v - b.$$

Lemma 6 (Bid ranking)**.** *Player i with valuation v weakly prefers to bid b rather than bid a, $U_i(b|v) - U_i(a|v) \geq 0$, if and only if:*

$$\Delta W_i(a,b) \geq \frac{b-a}{v}.$$

Lemma 7 (Incentive Compatibility)**.** *For $\underline{v}_i \leq t < v \leq \overline{v}_i$ and $b_i(\cdot) \in \operatorname{argmax}_b U_i(b|\cdot)$, we have:*

$$\Delta W_i(b_i(t), b_i(v))\cdot t \leq b_i(v) - b_i(t) \leq \Delta W_i(b_i(t), b_i(v))\cdot v.$$

Lemma 8 (Monotone)**.** *Any pure strategy equilibrium, $\mathbf{b} = (b_1,\ldots,b_N)$, is non-decreasing, $v \geq t$ implies $b_i(v) \geq b_i(t)$.*

Proof. The proof follows immediately from incentive compatibility (lemma 7) and the fact the winning prob. are strictly increasing (lemma 4). ■

Corollary 1. *In an equilibrium, for all i exists $\hat{v}_i \geq \underline{v}_i$ such that $b_i(v) = 0$ for all $v \leq \hat{v}_i$ and b_i is strictly increasing in $(\hat{v}_i, \overline{v}_i)$.*

Proof. Since valuations are absolutely continuous valuations, if the bid function were constant in a region, it would posses an atom. Lemma 1 rules out atoms at positive bids but not at zero. ■

Hereafter, for brevity, we write 'equilibrium' instead of 'pure strategy, monotone equilibrium'.

4. Sufficient Conditions For Continuity

We need some additional definitions. Given a non-decreasing function H, its *left and right limits* at v, which always exist, are denoted by $H^-(v) \overset{\text{def}}{=} \sup\{H(x) : x < v\}$ and $H^+(v) \overset{\text{def}}{=} \inf\{H(x) : x > v\}$. Moreover, a non-decreasing H is said to be *discontinuous* if and only if

$H^-(v) < H^+(v)$ for some v. In this instance, we say v is a *discontinuity point* of H and $(H^-(v), H^+(v))$ is a *gap* (in the image) of H.

As any equilibrium is monotone, any discontinuity induces a gap and vice-versa. When (c, d) is a gap of b_i, by taking limits at bid ranking inequality (lemma 6), we see that player i must be indifferent between c and d.

Lemma 9 (Bidding in i's Gap, Reveals A Weakly Higher Valuation)**.** *Given $c < d$, if $b_i^-(v_i) = c$, $G_i(c, d) = 0$ and $b_j(v_j) \in (c, d]$. then:*

$$v_j \geq v_i.$$

Proof. Type v_i of player i weakly prefers c to any b in (c, d) so:

$$\frac{b - c}{v_i} \geq \Delta W_i(c, b) \quad \forall b \in (c, d). \tag{4.1}$$

Type v_j of player j prefers $b = b_j(v_j) \in (c, d]$ to c, so by lemma 6:

$$\Delta W_j(c, b) \geq \frac{b - c}{v_j} \tag{4.2}$$

Moreover, as $\Delta G_j(c, b) \geq \Delta G_i(c, b) = 0$ by lemma 5:

$$\Delta W_i(c, b) \geq \Delta W_j(c, b). \tag{4.3}$$

Combining inequalities 4.1, 4.2 and 4.3 we obtain: $v_j \geq v_i$. ∎

We shall use the following extensions of lemma 9:

Corollary 2. *If $\Delta G_j(c, \hat{b}) > 0$, 4.3's inequality is strict and so $v_j > v_i$.*

Corollary 3 (No Gaps at Zero)**.** *If $\underline{v}_i = \underline{v}$ for all bidders then in any gap (c, d), $c > 0$.*

Proof. Bidders with gaps $(0, d_i)$ and discontinuity points $v_i > \underline{v}$ must bid zero with positive probability. Any bidder j who bid continuously on $(0, \varepsilon)$, where $\varepsilon > \min d_i > 0$, has $v_j > v_i$. That is, any such bidder j also bids zero with positive probability. So, all must bid zero with positive probability contradicting Lemma 3. ∎

Lemma 10 (Gaps Do Not Overlap)**.** *Given any two gaps, (c, d) and (e, f), either their intersection is empty or one gap contains the other*: $(c, d) \cap (e, f) = \emptyset$ *or* $(c, d) \subset (e, f)$ *or* $(c, d) \supset (e, f)$.

Proof. Contrary to the lemma, suppose that there is an overlap of (e, f) and (c, d), that is either $c < e < d < f$ or $e < c < f < d$. Let v_i and v_j be the corresponding discontinuity points of b_i and b_j that are associated to the gaps. Obviously, we must have $i \neq j$. Using twice the observation, bidding in the gap of another player reveals a weakly higher valuation (lemma 9), yields $v_i = v_j$.

Without loss of generality, let's assume $c < e < d < f$ and so:

$$\frac{e-c}{v_i} \geq \Delta W_i(c,e) \geq \Delta W_j(c,e) \geq \frac{e-c}{v_j}. \tag{4.4}$$

The first and third inequalities follows from incentive compatibility (lemma 7). The second inequality follows by the ranking winning probabilities result (lemma 5).

Equation 4.4 together with $v_i = v_j$ implies $\Delta W_i(c,e) = \Delta W_j(c,e)$, which in turn, means that both bidders do not bid in the interval (c,e), that is $\Delta G_j(c,e) = \Delta G_i(c,e) = 0$.

By definition, $e = b_j^-(v_j)$ and so there is a sequence $v_j^n \nearrow v_j$ such that $b_j(v_j^n) \nearrow e$. Pick n_0 be such that for $n \geq n_0$, $b_j(v_j^n) \in (c,e)$. Since $0 < \Delta F_j(v_j^m, v_j) \geq \Delta G_j(c,e) = 0$, we obtain a contradiction.

■

Lemma 11 (Minimal Gap)**.** *In a discontinuous equilibrium there is an interval (c,d) such that:*
(1) *The interval (c,d) is a gap for some bidder i.*
(2) *Any bidder j either does not bid the interval, $\Delta G_j(c,d) = 0$, or bids continuously on it, $\Delta G_j(x,y) > 0$, $\forall c \leq x < y \leq d$.*

Proof. Since gaps do not overlap (lemma 10), gaps are either disjoint or ordered by inclusion. Since there is a finite number of bidders, any chain of gaps has a minimal gap which clearly satisfies (1) and (2). ■

Remark 1. Although, gaps do not overlap, gaps may coincide, be nested or disjoint.

Our next step is to characterize bidders' behavior at the gap. To analyze bidders' marginal incentives, we need the additional notation.

Definition 5. The set of active bidders at b is:

$$J(b) \stackrel{\text{def}}{=} \{j : \forall \varepsilon > 0, \Delta G_j(b-\varepsilon, b+\varepsilon) > 0\}.$$

Definition 6. The set of bids where bid densities are well defined is:

$$B \stackrel{\text{def}}{=} \{b : \forall i, G_i'^{+}(b) = G_i'^{-}(b)\}.$$

Since G_i is non-decreasing, the set B has full Lesbegue measure.

For a probability cumulative distribution H with density h, the *reversed hazard rate* or growth rate of H is h/H.

Lemma 12 (Marginal Winning Probabilities)**.** *For $b \in B$, the reverse hazard rate of the bid distribution of an active bidders is:*

$$\frac{g_i(b)}{G_i(b)} = \frac{\left(\sum_{j\in J(b)} \frac{G_j(b)}{F_j^{-1}(G_j(b))}\right) - \frac{(\#J(b)-1)G_i(b)}{F_i^{-1}(G_i(b))}}{(\#J(b)-1)\Pi_{j=1}^n G_j(b)}, \quad \forall i \in J(b), \tag{rhr}$$

and marginal winning probabilities are:

$$W_i'(b) = \begin{cases} \frac{1}{F_i^{-1}(G_i(b))} & \text{if } i \in J(b), \\ \frac{\sum_{j\in J(b)} \frac{G_j(b)}{F_j^{-1}(G_j(b))}}{(\#J(b)-1)G_i(b)} & \text{otherwise.} \end{cases} \tag{MW}$$

Proof of rhr. It follows directly from the first order-conditions, see Parreiras and Rubinchik (2010, lemma 7, p. 711). ■

Proof of MW. Recall that the set B has full measure as the G_i are monotone. Pick $b \in B$. We have $W_i'(b) = \sum_{j\neq i} g_j(b)\Pi_{k\neq i,j}G_k(b)$ so $\frac{W_i'(b)}{\Pi_{j\neq i}G_j(b)} = \sum_{j\neq i}\frac{g_j(b)}{G_j(b)}$. Since $j \notin J(b)$ implies $g_j(b) = 0$ and $j \in J(b)$ implies $g_j(b)/G_j(b)$ is given by rhr, we obtain:

$$\frac{W_i'(b)}{\Pi_{j\neq i}G_j(b)} = \sum_{j\in J(b)\setminus\{i\}} \frac{\sum_{k\in J(b)} \frac{G_k(b)}{F_k^{-1}(G_k(b))} - (\#J(b)-1)\frac{G_j(b)}{F_j^{-1}(G_j(b))}}{(\#J(b)-1)\Pi_{k=1}^n G_k(b)} =$$

$$= \begin{cases} \frac{(\#J(b)-1)\cdot\left(\sum_{k\in J(b)} \frac{G_k(b)}{F_k^{-1}(G_k(b))}\right) - \sum_{j\in J(b)\setminus\{i\}} (\#J(b)-1)\frac{G_j(b)}{F_j^{-1}(G_j(b))}}{(\#J(b)-1)\Pi_{k=1}^n G_k(b)} & \text{if } i \in J(b) \\ \frac{\#J(b)\cdot\left(\sum_{k\in J(b)} \frac{G_k(b)}{F_k^{-1}(G_k(b))}\right) - \sum_{j\in J(b)} (\#J(b)-1)\frac{G_j(b)}{F_j^{-1}(G_j(b))}}{(\#J(b)-1)\Pi_{k=1}^n G_k(b)} & \text{otherwise.} \end{cases}$$

Simplifying the above expression, one gets MW. ■

Remark 2. If $J(b)$ is constant in a neighborhood of b and $i \in J(b)$ then by rhr, g_i is continuous in this neighborhood.

Lemma 13 (Local Incentives At The Gap)**.** *Let (c, d) be a minimal gap, as defined in Lemma 11, corresponding to a discontinuity of $b_i(\cdot)$ at v_i and consider the set J of bidders that bid continuously in (c, d). We have:*

$$\sum_{j\in J} \frac{G_j(c)}{F_j^{-1}(G_j(c))} \leq (\#J-1)\frac{F_i(v_i)}{v_i} \leq \sum_{j\in J} \frac{G_j(d)}{F_j^{-1}(G_j(d))}. \tag{4.5}$$

Proof. First, the sets $J(b)$ and J are identical for any $b \in B \cap (c, d)$. Secondly, the type v_i is indifferent between c and d and weakly prefers

c or d to any other bid b in (c, d). Third, for any sequences $c_n \searrow c$ and $d_n \nearrow d$ in B there exists n_0 such that for $n \geq n_0$:

$$W_i'(c_n) \leq \frac{1}{v_i} \leq W_i'(d_n) \tag{4.6}$$

By MW since i does not bid in (c_n, d_n) and $J(c_n) = J(d_n) = J$:

$$\frac{\sum_{j\in J} \frac{G_j(c_n)}{F_j^{-1}(G_j(c_n))}}{(\#J-1)G_i(c_n)} \leq \frac{1}{v_i} \leq \frac{\sum_{j\in J} \frac{G_j(d_n)}{F_j^{-1}(G_j(d_n))}}{(\#J-1)G_i(d_n)} \tag{4.7}$$

Since $G_i(c_n) = G_i(d_n) = F_i(v_i)$ as G_i is constant on (c, d) and moreover as F_j^{-1} and G_j are continuous for all j, taking the limit $n \nearrow +\infty$ in 4.7 yields

$$\sum_{j\in J} \frac{G_j(c)}{F_j^{-1}(G_j(c))} \leq (\#J-1)\frac{F_i(v_i)}{v_i} \leq \sum_{j\in J} \frac{G_j(d)}{F_j^{-1}(G_j(d))}.$$

■

Lemma 14. *Consider the set J of bidders the bid continuously on (c, d) and the set of K of bidders with (c, d) as a gap. If $c > 0$, the first inequality in Lemma 13 binds,*

$$\sum_{j\in J} \frac{G_j(c)}{F_j^{-1}(G_j(c))} = (\#J-1)\frac{F_i(v_i)}{v_i} \leq \sum_{j\in J} \frac{G_j(d)}{F_j^{-1}(G_j(d))}. \tag{4.8}$$

Proof. For notational simplicity, write

$$\alpha_j \stackrel{\text{def}}{=} \tfrac{G_j(c)}{F_j^{-1}(G_j(c))}\ \forall j \in J \quad \text{and} \quad \beta_k \stackrel{\text{def}}{=} \tfrac{G_k(c)}{F_k^{-1}(G_k(c))}\ \forall k \in K. \tag{4.9}$$

Pick $\varepsilon > 0$ so that the set of bidders who bid continuously on $(c-\varepsilon, c)$ is the union $J \cup K$. For any sequence $c_n \nearrow c$, $g_k(c_n) \geq 0$ and so by taking limits on rhr we get that for any k in K,

$$\sum_{j=1}^{\#J} \alpha_j + \sum_{k=1}^{\#K} \beta_k \geq (\#J + \#K - 1) \cdot \beta_k, \tag{4.10}$$

and by lemma 13,

$$\sum_{j=1}^{\#J} \alpha_j \leq (\#J-1) \cdot \beta_k. \tag{4.11}$$

Summing 4.10 and 4.11 over K yields $\#K \sum_{i=1}^{\#J} \alpha_j \geq (\#J-1) \sum_{k=1}^{\#K} \beta_k$ and $\#K \sum_{i=1}^{\#J} \alpha_j \leq (\#J-1) \sum_{k=1}^{\#K} \beta_k$. Both inequalities in both 4.10 and 4.11 are binding, which implies the β_k are identical. ■

Corollary 4. *If i and j have a common minimal gap (c, d) with $c > 0$, then $v_i = v_j$ and $F_i(v_i) = F_j(v_j)$, where v_i and v_j are their discontinuity points.*

Proof. First, by lemma 9, the discontinuity points $v_k = F_k^{-1}(G_k(c))$ are identical. Secondly, by the lemma's proof, the β_k are identical. Finally, remember that $\beta_k = F_k(v_k)/v_k$. ■

For each valuation (random variable) V_i consider an auxiliary random variable, $H_i \sim \mathrm{U}[0, \overline{v}_i]$. Shaked and Shanthikumar (2007) show that V_i is larger (smaller) than H_i in the reverse hazard rate order, if and only if, the ratio $F_i(v)/F_{H_i}(v) = F_i(v) \cdot \overline{v}_i/v$ is increasing (decreasing) in v.

Proposition 1 (Continuity Part I)**.** *Any equilibrium is continuous on condition that*:
(1) *$F_i(v)/v$ is non-decreasing in v $\forall i$;*
(2) *$\exists j$ such that $F_j(v)/v$ is increasing in v and*
(3) *$\underline{v}_i = \underline{v}$ $\forall i$.*

Proof. As the sum of the ratios is increasing and G_i is constant on (c, d), the marginal winning probability W_i' is increasing on (c, d) by MW. Noted that corollary 3 and the assumption all $\underline{v}_i$ are identical implies $c > 0$ by. Therefore,

$$\tfrac{d-c}{v_i} = \Delta W_i(c, d) \geq \int_c^d W_i'(b)\mathrm{d}b > (d - c) \cdot W_i'^{+}(c). \tag{4.12}$$

In 4.12, the equality holds true because i is indifferent between c and d, the weak inequality holds because W_i might not be absolutely continuous, the strict inequality because W_i' is increasing. From 4.12, we deduced that $\frac{1}{v_i} > W_i'^{+}(c)$, which contradicts the equality in 4.5. ■

When bidders have distinct valuation lower bounds, $\underline{v}_i \neq \underline{v}_j$:

Corollary 5. *Suppose that*:
(1) *$F_i(v)/v$ is non-increasing in v $\forall i$ and*
(2) *$\exists j$ such that $F_j(v)/v$ is decreasing in v, then*
the gap of any discontinuity has zero as lower bound.

Proof. Let (c, d) be a minimal gap associate to some discontinuity. By the proof of Proposition 1, $c = 0$. Now, consider some gap which is not minimal, (e, f). As gaps are ordered (lemma 10), there is some minimal gap $(0, d) \subset (e, f)$, which implies $e = 0$. ■

Next we extend Proposition 1 to cover other cases:

Proposition 2 (Continuity Part II)**.** *Any equilibrium is continuous provided that*:
(1) $F_i(v)/v$ *is non-increasing* $\forall i$;
(2) $\exists j$ *such that* $F_j(v)/v$ *is decreasing.*

Proof. In this case, $\sum_{j\in J} \frac{G_j(c)}{F_j^{-1}(G_j(c))}$ is decreasing in b so 4.5 is violated. ■

Note that $F_i(v)/v$ non-increasing implies $\underline{v}_i = 0$. An example for Proposition 2 is $F_i(v) = (v/\overline{v}_i)^{\alpha_i}$, where $0 < \alpha_i < 1$.

Proposition 3 (Continuity Part III)**.** *Any equilibrium is continuous provided that* $F_i(v)/v$ *is constant* $\forall i$*.*

Proof. In this case, valuations are uniformly distributed[4] with $V_i \sim \mathrm{U}[0, \overline{v}_i]$ and so $F_i(v)/v = 1/\overline{v}_i$.

Consider a minimal gap (c, d) and the respective sets J and K of bidders who bid continuously on (c, d) and of bidders who have (c, d) as a gap. As $F_i(v)/v = 1/\overline{v}_i$, Lemma 13 implies,

$$\overline{v}_k^{-1} = (\#J - 1) \sum_{j\in J} \overline{v}_j^{-1}, \quad \forall k \in K. \tag{4.13}$$

Pick $\delta > 0$ such that for $0 < \varepsilon < \delta$, the set of active bidders at $b \in (d - \varepsilon, d)$ is $J(b) = J \cup K$. But then, the equality 4.13 and rhr imply $g_k = 0$ on $(d - \varepsilon, d)$ which contradicts $k \in K$ being active. ■

5. Any Continuous Equilibrium is Differentiable

Proposition 4. *If the equilibrium is continuous then it is differentiable everywhere with possibly the exception of* $N + 1$ *points*: $0, \overline{b}_1, \ldots, \overline{b}_N$*.*

Before proceeding with the proof, we need the auxiliary result:

Lemma 15. *If* $b_i(\cdot)$ *is continuous then* W_i *differentiable on* $(0, \overline{b}_i)$*.*

Proof. Pick v_i such that $\overline{v}_i > v_i$, set $b = b_i(v) > 0$ and consider some sequence $b^n \nearrow b$. As $b_i(\cdot)$ is strictly increasing and continuous, there are v_i^n with $b_i(v_i^n) = b^n$. Incentive compatibility (lemma 7) implies,

$$\frac{1}{v^n} \leq \frac{\Delta W_i(b^n, b)}{b - b^n} \leq \frac{1}{v} \tag{5.1}$$

Taking the limit, we obtain $W_i^-(b) = \frac{1}{v}$. By considering a sequence $b^n \searrow b_i$, we similarly obtain $W_i^+(b) = \frac{1}{v}$. ■

[4] Parreiras and Rubinchik (2015) obtain closed-form expressions for the equilibrium of the uniform case with $\underline{v}_i = 0$.

Remark 3. If $b \notin [0, \overline{b}_i]$ then the bid distribution G_i is differentiable at b since G_i is constant outside this interval as $G_i(b) = 0$ for $b < \underline{b}_i$ and $G_i(b) = 1$ for $b > \overline{b}_i$.

Next we prove Proposition 4:

Proof of Proposition 4. First, we safely ignore the bids $b > \max \overline{b}_i$, since no one bids in this range. Second, we linearize the identity $W_i = \Pi_{j \neq i} G_i$ by taking logs, $\hat{G}_i = \ln(G_i)$ and $\hat{W}_i = \ln(W_i)$, so that:

$$M \cdot \begin{pmatrix} \hat{G}_1 \\ \vdots \\ \hat{G}_N \end{pmatrix} = \begin{pmatrix} \hat{W}_1 \\ \vdots \\ \hat{W}_N \end{pmatrix}$$

where M is an invertible $N \times N$-matrix with

$$M_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

Pick $b \notin \{0, \overline{b}_1, \ldots, \overline{b}_N\}$, re-ordering the bidders if necessary, we can assume there exists $m \geq 0$ such that $b < \overline{b}_i$ for $i \leq m$ and $b > \overline{b}_i$ for $i > m$. Decomposing M in blocks, with M_1 being $n \times n$,

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

we obtain:

$$\begin{pmatrix} \hat{G}_1 \\ \vdots \\ \hat{G}_m \end{pmatrix} = M_1^{-1} \cdot \left(\begin{pmatrix} \hat{W}_1 \\ \vdots \\ \hat{W}_m \end{pmatrix} - M_2 \cdot \begin{pmatrix} \hat{G}_{m+1} \\ \vdots \\ \hat{G}_N \end{pmatrix} \right) \tag{5.2}$$

$$\begin{pmatrix} \hat{W}_{m+1} \\ \vdots \\ \hat{W}_N \end{pmatrix} = M_3 \cdot M_1^{-1} \cdot \begin{pmatrix} \hat{W}_1 \\ \vdots \\ \hat{W}_m \end{pmatrix} + \left(M_4 - M_3 \cdot M_1^{-1} M_2\right) \cdot \begin{pmatrix} \hat{G}_{m+1} \\ \vdots \\ \hat{G}_N \end{pmatrix}$$

By lemma 15 W_i is differentiable at b for $i \leq m$. By remark 3, G_i is differentiable at b for $i > m$. In sum, $(W_1, \ldots, W_m, G_{m+1}, \ldots, G_N)$ is differentiable. As a result, 5.2 implies $(G_1, \ldots, G_m, W_{m+1}, \ldots, W_N)$ is also differentiable. ∎

Next we prove W_i and G_i are differentiable everywhere with the possible exception of $\overline{b}_1$ and zero.

Lemma 16. *Suppose: 1) The function $f:[0,1]\to[0,1]$ is continuous, differentiable in $[0,1]\backslash\{a\}$ and non-decreasing; 2) There is a full Lebesgue-measure set X such that for any sequence x_n in X converging to a, the derivative $f'(x_n)$ converges to c. Then the function f is absolutely continuous on $[0,1]$, differentiable at a with $f'(a)=c$.*

We thank David C. Ullrich for Lemma 16's proof.[5]

Proof. There is a measure μ such that $\mu([0,x])=f(x)$ for all x and $\mu=\mu_{\text{ac}}+\mu_{\text{s}}$, where μ_{ac} is absolutely continuous and μ_{s} is singular. Since f is continuous at a, $0=\mu(\{a\})\geq\mu_{\text{s}}(\{a\})\geq 0$.

We also must have that $\mu_{\text{s}}([0,1]\backslash\{a\})=0$. Otherwise f would not be differentiable at some point x in $[0,1]\backslash\{a\}$ since $\mathrm{d}\mu_{\text{s}}/\mathrm{d}\lambda=+\infty$ almost everywhere with respect to μ_{s} (Rudin, 1987, Theorem 7.15, p. 143) and $f'=\mathrm{d}\mu_{\text{ac}}/\mathrm{d}\lambda+\mathrm{d}\mu_{\text{s}}/\mathrm{d}\lambda$ where λ is the Lebesgue measure.

We conclude that $\mu_{\text{s}}\equiv 0$, that is f is absolutely continuous, which in turn implies:

$$\frac{f(a)-f(x)}{a-x}=\frac{\int_x^a f'(z)dz}{a-x}. \tag{5.3}$$

Since for all $\varepsilon>0$, exists $\delta>0$ such that $|f'(z)-c|<\varepsilon$ for almost all z with $|a-z|<\delta$, equation 5.3 give us that $f'(a)=c$. ■

Corollary 6 (Bid Densities Are Continuous At The Top)**.** *If i's top bid is not the highest top bid, $\bar{b}_i<\max\bar{b}_j$, then G_i is differentiable and $g_i(\bar{b}_i)=0$.*

Proof. Remember that G_i is continuous. Moreover, Proposition 4 implies there is $\varepsilon>0$ such that G_i is differentiable everywhere in $(\bar{b}_i-\varepsilon,\bar{b}_i)$ and remark 3 implies G_i is differentiable in $(\bar{b}_i,+\infty)$. We are left to prove that $g_i(b_n)\overset{n\to\infty}{\longrightarrow}0$ as $b_n\overset{n\to\infty}{\longrightarrow}\bar{b}_i$. Once this is established, Lemma 16 delivers the result.

Consider the sets K and J of bidders for which $\bar{b}_i$ is the top bid and of bidders who bid in a neighborhood of $\bar{b}_i$. That is, $K=\{k:b_k(\bar{v}_k)=\bar{b}_i\}$ and $J=\{j:b_j(\underline{v}_j)<\bar{b}_i<b_j(\bar{v}_j)\}$. We have that:

$$\frac{(\#J-1)}{\bar{v}_k}\geq\sum_{j\in J}\frac{G_j(\bar{b}_i)}{F_j^{-1}(G_j(\bar{b}_i))}\quad\text{and} \tag{5.4}$$

$$\sum_{\ell\in J\cup K}\frac{G_\ell(\bar{b}_i)}{F_\ell^{-1}(G_\ell(\bar{b}_i))}\geq\frac{\#K+\#J-1}{\bar{v}_k}\quad\forall k\in K \tag{5.5}$$

Eq. 5.4 follows from $\lim_{b_n\searrow\bar{b}_i}\bar{v}_k\cdot W_k'(b_n)-1\leq 0$, MW and $G_k(\bar{b}_i)=1$. Eq. 5.5 follows from rhr and $\lim_{b_n\nearrow\bar{b}_i}g_k(b_n)\geq 0$. Adding these two

[5]See http://math.stackexchange.com/questions/1377034.

equations and simplifying, we obtain: $0 \geq 0$. Thus, the inequality in 5.5 must bind and so $\lim_{b_n \nearrow \bar{b}_i} g_k(b_n) = 0$. Clearly, we also have $\lim_{b_n \searrow \bar{b}_i} g_k(b_n) = 0$ as $g_k(b_n) = 0$ for $b_n > \overline{v}_i$. ■

A similar result also holds for the winning probability.

Corollary 7. *The winning probability W_i is differentiable at $\bar{b}_i$.*

Proof. The idea and notation (the definition of the sets K and J) is the same as in the proof of corollary 6.

Proposition 4 implies $W_i'^{-}(\bar{b}_i) = 1/\overline{v}_i$ and MW implies $\lim_{b_n \searrow \bar{b}_i} W_i'(\bar{b}_n) = \sum_{\ell \in J \cup K} \frac{G_\ell(\bar{b}_i)}{F_\ell^{-1}(G_\ell(\bar{b}_i))} / (\#K + \#J - 1)$ and since 5.5 binds and $i \in K$, we have $\lim_{b_n \searrow \bar{b}_i} W_i'(\bar{b}_n) = W_i'^{-}(\bar{b}_i)$. We can now apply Lemma 16, which proves the lemma. ■

To recover the equilibrium by integrating the first-order conditions, we must prove it is absolutely continuous.

Proposition 5. *The winning probabilities W_i and the bid distributions G_i are absolutely continuous on $[0, \bar{b}_1]$.*

Proof. It follows directly from Lemma 16. ■

6. Determining Active Bidders

When there are more than two asymmetric bidders, even if bid strategies are continuous, we lack prior knowledge about which bidders are active at some bid. The set of active bidders at some bid is endogenously determined. Moreover, since first-order conditions only hold for active bidders, at first-glance, it might seem hopeless to apply the first-order approach. As it turns out, because bid levels do not show-up in the first-order conditions, it suffices for us to "rank the top bids" in order to compute the equilibrium.

For, first-price auctions, Hubbard and Kirkegaard (2015) obtain similar results.

Lemma 17 (Top Bids Are Ordered By Top Valuations)**.** *A strictly higher top bid imply a strictly higher valuation, $\bar{b}_j < \bar{b}_i \Longrightarrow \overline{v}_j < \overline{v}_i$.*

Proof. If $\bar{b}_j < \bar{b}_i$ then $\Delta G_j(\bar{b}_j, \bar{b}_i) = 0 < \Delta G_i(\bar{b}_i, \bar{b}_i)$, which by lemma 5 implies $\Delta W_j(\bar{b}_j, \bar{b}_i) > \Delta W_i(\bar{b}_i, \bar{b}_i)$. Given this, the result follows by incentive compatibility (lemma 7):

$$\Delta W_j(\bar{b}_j, \bar{b}_i)\, \overline{v}_j \leq \bar{b}_i - \bar{b}_j \leq \Delta W_i(\bar{b}_j, \bar{b}_i)\, \overline{v}_i.$$

■

Hereafter, without any loss of generality, we assume:

Notation. Bidders are ordered by top valuations: $\overline{v}_1 \geq \overline{v}_2 \geq \ldots \geq \overline{v}_n$.

Lemma 17 does not imply that if $\overline{b}_i = \overline{b}_j$ then $\overline{v}_i = \overline{v}_j$, which clearly is false as $\overline{b}_1 = \overline{b}_2$ in any equilibrium even if $\overline{v}_1 > \overline{v}_2$. However we can say that:

Corollary 8 (Ordering Bid's Supports)**.** *If $\overline{b}_i < \overline{b}_1$ and $\overline{b}_j = \overline{b}_i$ then $\overline{v}_i = \overline{v}_j$.*

Proof. It follows from $g_j(\overline{b}_j) = g_i(\overline{b}_i) = 0$ (corollary 6) and rhr. ■

Corollary 8 is silent regarding who places the highest top bid $\overline{b}_1$. The next two corollaries remedy this omission.

Corollary 9. *If the ratio $F_i(v)/v$ is decreasing in v for all i, the set of bidders that choose $\overline{b}_1$ is*

$$J(\overline{b}_1) = \{j : \sum_{i=1}^{j} \frac{1}{\overline{v}_i} \geq (j-1)\frac{1}{\overline{v}_j}\}.$$

Proof. Assume $\overline{b}_1 = \ldots = \overline{b}_i > \overline{b}_{i+1}$ in equilibrium.

Part I: Since $g_i(\overline{b}_1) \geq 0$,

$$\sum_{j=1}^{i} 1/\overline{v}_j \geq (i-1)/\overline{v}_i$$

and since $\overline{v}_j \geq \overline{v}_i$ for any $1 \leq j \leq i$, it follows that $\sum_{k=1}^{j} 1/\overline{v}_k \geq (j-1)/\overline{v}_j$ for all $1 \leq j \leq i$, which proves that $J(\overline{b}_1) \supset \{j : \sum_{i=1}^{j} \frac{1}{\overline{v}_i} \geq (j-1)\frac{1}{\overline{v}_j}\}$.

Part II: In equilibrium $g_{i+1}(\overline{b}_{i+1}) = 0$ (corollary 6) and so by rhr, $\sum_{j=1}^{i+1} F_j(v_j)/v_j = i/\overline{v}_{i+1}$ where $v_j = F_j^{-1}(G_j(\overline{b}_{i+1}))$, that is type v_j bids as type $\overline{v}_{i+1}$, $b_j(v_j) = \overline{b}_{i+1}$. As the ratios $F_j(v)/v$ are decreasing,

$$\sum_{j=1}^{i+1} 1/\overline{v}_j < \sum_{j=1}^{i+1} F_j(v_j)/v_j = i/\overline{v}_{i+1},$$

and as a result, $i+1 \notin \{j : \sum_{i=1}^{j} \frac{1}{\overline{v}_i} \geq (j-1)\frac{1}{\overline{v}_j}\}$.

Repeate the above argument for players $k \geq i+1$: $\sum_{j=1}^{k} 1/\overline{v}_j < \sum_{j=1}^{k} F_j(v_j)/v_j = (k-1)/\overline{v}_k$ where, $v_j = F_j^{-1}(G_j(\overline{b}_k))$; so $k \notin \{j : \sum_{i=1}^{j} \frac{1}{\overline{v}_i} \geq (j-1)\frac{1}{\overline{v}_j}\}$ and $J(\overline{b}_1) \subset \{j : \sum_{i=1}^{j} \frac{1}{\overline{v}_i} \geq (j-1)\frac{1}{\overline{v}_j}\}$.

■

Corollary 10. *If the ratio $F_i(v)/v$ is increasing in v for all i, the set of bidders who bid $\overline{b}_1$ is $J(\overline{b}_1) = \{j : \sum_{i=1}^{j} \frac{1}{\overline{v}_i} \geq (j-1)\frac{1}{\overline{v}_j}\}$.*

Proof. Assume $\overline{b}_1 = \ldots = \overline{b}_i > \overline{b}_{i+1}$ in equilibrium. The argument used in first part of the proof of corollary 9, give us that

$$\sum_{k=1}^{j} 1/\overline{v}_k \geq (j-1)/\overline{v}_j, \quad \forall 1 \leq j \leq i.$$

Moreover, by Lemma 17, $g_{i+1}(\overline{b}_{i+1}) = 0$ and so, by MW, $W'_{i+1}(\overline{b}_{i+1}) = \sum_{j=1}^{i} \frac{G_j(\overline{b}_{i+1})}{F_j^{-1}(G_j(\overline{b}_{i+1}))} \frac{1}{i-1} = 1/\overline{v}_{i+1}$. As the ratios $F_j(v)/v$ are increasing, so are the ratios $G_j(b)/F_j^{-1}(G_j(b))$ and so is the marginal winning probability, $W'_{i+1}(b)$. We have $\Delta W_{i+1}(\overline{b}_{i+1}, \overline{b}_1) = \int_{\overline{b}_{i+1}}^{\overline{b}_1} W'_{i+1}(b)\mathrm{d}b > (\overline{b}_1 - \overline{b}_{i+1}) \cdot W'_{i+1}(\overline{b}_{i+1}) = \frac{\overline{b}_1 - \overline{b}_{i+1}}{\overline{v}_{i+1}}$, so $\Delta W_{i+1}(\overline{b}_{i+1}, \overline{b}_1) > \frac{\overline{b}_1 - \overline{b}_{i+1}}{\overline{v}_{i+1}} \Leftrightarrow \Delta W_{i+1}(\overline{b}_1, \overline{b}_{i+1}) < \frac{\overline{b}_{i+1} - \overline{b}_1}{\overline{v}_{i+1}}$, which contradicts $\overline{b}_{i+1}$ is weakly preferred to $\overline{b}_1$ by type $\overline{b}_{i+1}$ (lemma 6). ■

7. Uniqueness

First consider the following algorithm.

Proposition 6. *Assume $F_i(v)/v$ increasing and $\underline{v}_i = \underline{v}$ for $i = 1, \ldots, N$. There is a unique equilibrium and Algorithm 1 computes it.*

Proof. Step (3) pins-down the set of bidders that bid at $\overline{b}_1$, see cor. 10. Step (11) uniquely identifies bidders that become active: Since $\sum_{i=1}^{j} G_i(b)/F_i^{-1}(G_i(b)) - (k-1)/\overline{v}_{k+1}$ is strictly increasing in b it must cross zero only once, exactly where bidder $k+1$ becomes active: by corollary 6 $g_{k+1}(\overline{b}_{k+1}) = 0$ implies $\sum_{i=1}^{k} G_i(\overline{b}_{k+1})/F_i^{-1}(G_i(\overline{b}_{k+1})) = (k-1)/\overline{v}_{k+1}$.

Since the set of active bidders is uniquely determined, any equilibrium must satisfy the first-order conditions or equivalently, rhr. As, for any $\varepsilon > 0$, the right hand side of rhr is Lipschitz continuous in the G_is provided $G_i > \varepsilon$. The solution of rhr exists and it is can be extended uniquely in the region where $G_i > \varepsilon$ (Agarwal and Lakshmikantham, 1993, p. 1–5). Finally, we take the limit $\varepsilon \searrow 0$ and extend (uniquely) G_i by the continuity.

Given the initial conditions, the solution is unique. However, if we do not impose initial conditions, as the system is autonomous (b does not appear directly in the right-hand side of rhr), any translation of the solution also solves rhr. That is, if $(G_i(b))_{i=1}^{N}$ is a solution then $(G_i(b+\tau))_{i=1}^{N}$ is also a solution for any value of τ. This allows us to

Algorithm 1

1: **procedure** SOLVE ALL-PAY($F_1, F_2, \ldots, F_N$)
2: Sort bidders by top valuations, $\overline{v}_1 \geq \overline{v}_2 \geq \ldots \geq \overline{v}_N$.
3: $J \leftarrow \{j : \sum_{i=1}^{j} \frac{1}{\overline{v}_i} \geq (j-1)\frac{1}{\overline{v}_j}\}$ ▷ Initial set of active bidders
4: $k \leftarrow \#J$ ▷ Initial number of active bidders
5: $b \leftarrow \overline{v}_1$ ▷ Initial top bid (before re-scaling)
6: $G_i(b) \leftarrow 1$ for all i ▷ Initial conditions for G_i
7: **while** $\Pi_{i=1}^{N} G_i(b) > 0$ **do**
8: $J(b) \leftarrow J$ ▷ Set of active bidders
9: Solve locally at b the ODE system given by rhr.
10: Decrease b continuoulsy
11: **while** $\sum_{i=1}^{k} \frac{G_i(b)}{F_i^{-1}(G_i(b))} = \frac{k-1}{\overline{v}_{k+1}}$ **do**
12: $J \leftarrow J \cup \{k+1\}$ ▷ Bidder $k+1$ becomes active
13: $k \leftarrow k+1$ ▷ Update number of active bidders
14: **end while**
15: **end while**
16: $\overline{b}_1 \leftarrow \overline{v}_1 - b$ ▷ Re-scale top bid
17: $G_i(z) \leftarrow G_i(z-b)$ for all i and all $z \in [0, \overline{b}_1]$ ▷ Re-scale bids
18: **end procedure**

use the following trick: We start solving the system with an arbitrary initial condition (step 5), however we know that for the right initial condition, the lowest bid is zero and so we re-scale the bids (by an appropriate translation) so the the last value of b corresponds to zero (steps 16 and 17). ■

Proposition 7. *Assume $F_i(v)/v$ is constant for $i = 1, \ldots, N$ There is a unique equilibrium. Moreover, the equilibrium can be obtained by the algorithm.*

Proof. The proof is similar to the above and so omitted. The main difference is the set of active bidders in this case is constant for all b. A bidder either is always active or always inactive: step (5di) of the algorithm is never reached. See Parreiras and Rubinchik (2015) for an explicit derivation of equilibrium strategies. ■

8. CONCLUSIONS

We provided sufficient conditions for the equilibrium of the all-pay auction be continuous and differentiable, which allow us to use the first-order approach.

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