# A 'marginalist' model of network formation*

By Norma Olaizola† and Federico Valenciano‡

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**Abstract**

We provide a model of network-formation where the "quality" of a link, i.e. the fidelity-level of its transmission, depends on the amount invested in it and is determined by a link-formation "technology", an increasing strictly concave function which is the only exogenous ingredient in the model. The revenue from the investment in links is the information that the players receive through the network. Two approaches are considered. First, assuming that the investments in links are made by a planner, the basic question is that of efficiency. Second, assuming that links are the result of investments of the players involved, according to such a technology, whose reward from forming them is the information they receive through the network that results. Then, there is the question of stability in the underlying network-formation game in the sense of Nash equilibrium if coordination is not feasible, or pairwise Nash if pairwise coordination is feasible.

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†*BRiDGE* group (http://www.bridgebilbao.es), Departamento de Fundamentos del Análisis Económico I, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; norma.olaizola@ehu.es.

‡*BRiDGE* group (http://www.bridgebilbao.es), Departamento de Economía Aplicada IV, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; federico.valenciano@ehu.es.

# 1 Introduction

This work is a contribution to the literature on economic models of network formation. In this line of work an increasing flow of research has been contributed by game-theorists and economists in general after Myerson (1977) and Aumann and Myerson (1988). After these pioneer works in the field, the two basic and most influential models of network formation are those of Jackson Wolinsky (1996) and Bala and Goyal (2000a), where networks are the result of creating links between pairs of individuals, be it by bilateral agreements in the first model or unilateral decisions in the second. In both models, the cost of a link and its quality are exogenously given, giving rise to two-parameter models. The simplicity of these basic models embodies some rigidity: either necessarily bilateral formation and compulsory equal share of the fixed cost of each link, or unilateral formation requiring full-covering of that fixed cost by the creator; and in both cases a fixed level of quality for the resulting link. The point of this work is to provide a more flexible model in both aspects: link-formation and link-performance.[1].

We provide a model of network formation where links are the result of investments. The "quality" of a link, i.e. the fidelity-level of its transmission, is never perfect and depends on the amount invested in it. A link-formation "technology" determines the quality of the resulting link and is the only exogenous ingredient in the model. Formally, a technology is assumed to be an increasing and strictly concave function whose range is $[0, 1)$, i.e. it is assumed that, whatever the investment in a link, there is always some friction. The revenue from the investment in links is the information that the nodes receive through the network that results. This model poses the following questions that are addressed.

A first approach assumes that the investments in links are made by a planner. In this scenario there are two basic questions. First, given an "infrastructure" specified by an underlying graph of feasible links, there is the question of existence and determination of an optimal investment (i.e. maximizing the aggregate payoff of the nodes connected by that infrastructure) in these links using them all. A second question is that of efficiency or, equivalently, about which infrastructures are efficient in the sense of maximizing in absolute terms the aggregate revenue for a given technology for an optimal investment in their links. Necessary conditions for the existence of an optimal investment in an infrastructure are established. This result is used to address the second and fully characterize the efficient structures. It is proved the only structures that can be efficient are the empty one, the complete network, the all-encompassing star and an interesting type of hybrid star-complete network, and conditions for each of them to be efficient for a given technology are established. In fact, both complete and all-encompassing stars are extreme cases of these hybrid structures. A remarkable

[1]In Olaizola and Valenciano's (2015a) model, costs and flow levels for links singly-supported and doubly-supported differ and three exogenous parameters specify the model. Other extensions of the seminal models are commented and compared with the one presented here after giving and outline of it.

aspect of these hybrid star-complete networks is that they can be described as core-periphery structures.

A second approach assumes that links are the result of investments of the node-players involved, whose reward from forming them is the information they receive through the network that results. Links are formed according to a technology available to all players. In this game-theoretic scenario, there is the question of stability in the underlying network-formation game. Stability in the sense of Nash equilibrium if coordination is not feasible, or pairwise Nash if pairwise coordination is feasible. Necessary conditions for stability of the type of structures proved to be efficient are established, yielding the conclusion that stable complete and stable star networks are not efficient, and reciprocally. Note that the question of efficiency in this context is covered by the second issue addressed in the first approach.

In addition to these basic issues, this model admits several variations and extensions that are briefly commented in the last section.

Bloch and Dutta (2009) is possibly the closest model to the one introduced here in spite of the obvious differences. In their model as in ours the strength of a link depends on the investment of the two players involved. Nevertheless, the coincidences do not go beyond this. They assume that players have a fixed endowment, and the link strength is an *additively separable convex* function of individual investments. Not surprisingly the results are completely different. With respect to their assumption about non-decreasing returns to investments they claim that "*While this assumption may seem at odds with the classical literature on productive investments, we strongly believe that convexity is the right assumption to make when one discusses investments in communication links.*" (Bloch and Dutta, 2009, p. 42). In this respect, we still believe that concavity is a reasonable assumption about link-formation technology. [2] In this point there is a similarity with Hojman and Szeidl (2008), again outweighed by the differences. In their model players choose their links, and a player's payoff is a strictly increasing and concave function of a weighted sum of the number players at different distances, with weights decreasing with distance. Thus the link-formation technology is the discrete one of the seminal models, w.r.t. which the crucial difference is in the payoff function, which embodies the decreasing returns assumption. The difference with our model is apparent, in it the non-discrete link-formation technology is the only exogenous ingredient, while the logic of the rest is as in the seminal models: *payoff = information - cost*. A comparison with Galeotti and Goyal (2010), who assume that

[2] Our model seems a more natural extension of the basic model of Jackson and Wolinsky (1996) and Bala and Goyal (2000a). Bloch and Dutta (2009) claim that the literature on discrete link formation assumes an extreme form of convexity. But an echelon-function of the form

$$\phi(x) = \begin{cases} s, & \text{if } x \geq c, \\ 0, & \text{if } x < c, \end{cases}$$

is an equaly extreme (if at all) form of concavity. In fact, such function can be seen as a limit case of one of the possible extensions of this model briefly commented in the last section.

returns from information (which can be acquired personally or through connections) are increasing and concave, while the costs of personally acquired information are linear, draws to a similar conclusion.

## 2 Notation and terminology

A *undirected weighted graph* consists of a set of *nodes* $N$ and a set of *links* specified by a map $g : N_2 \to \mathbb{R}_+$, where $N_2$ denotes the set of all subsets of $N$ with cardinality 2, and $\mathbb{R}_+$ denotes the set of non-negative real numbers. In the sequel $\underline{ij}$ stands for $\{i, j\}$ and $g_{\underline{ij}}$ for $g(\{i, j\})$ for any $\{i, j\} \in N_2$.[3] When $g_{\underline{ij}}$ only takes the values 0 or 1, $g$ is said to be *non-weighted*. When $g_{\underline{ij}} > 0$ it is said that a *link of weight* $g_{\underline{ij}}$ connects $i$ and $j$. $N^d(i, g) := \{j \in N : g_{\underline{ij}} > 0\}$, and $N(i, g)$ denotes the set of nodes connected with $i$ by a *path*, i.e. a sequence of distinct nodes s.t. every two consecutive nodes are connected by a link. If $g_{\underline{ij}} > 0$, $g - \underline{ij}$ denotes the graph that results from eliminating link $\underline{ij}$ i.e. $g - \underline{ij} = g'$ s.t. $g'_{\underline{ij}} = 0$ and $g'_{\underline{kl}} = g_{\underline{kl}}$ for all $\{k, l\} \neq \{i, j\}$. A graph is *connected* if any two nodes are connected by a path. A *component* of a graph is a maximal connected subgraph. A link is a *bridge* in a graph if its elimination causes the two nodes it connects to be in different components in the resulting graph. A graph has a *cycle* if there are two nodes connected by a link and also by a path of length 2 or more (the *length* of a path is the number of links it contains, i.e. that of nodes minus 1).

Undirected graphs underlie a variety of situations were actual links mean some sort of reciprocal connection or relationship. Such structures are commonly referred to as *networks*. As behind a network there always lies a graph as a most salient feature, we transfer the notions introduced so far for graphs to networks, identifying them with the underlying graph and refer the new ones directly to networks.

The *empty network* is the one for which $g_{\underline{ij}} = 0$ for all $i, j$ $(i \neq j)$. A *complete network* is one where $g_{\underline{ij}} > 0$ for all $i, j$ $(i \neq j)$.[4] A *tree* is a connected network with no cycles. An *all-encompassing star* consists of a network with $n - 1$ links in which a node (the *centre*) is connected with each of the remaining nodes by a link. A node is *peripheral* in a network if it is involved in one link only.

## 3 The model

The main ingredient in the model that has been briefly sketched in the introduction is a link-formation technology.

[3]The convenience of the distinction between $ij$ and $\underline{ij}$, especially as subindices, will be apparent soon. With this convention $g_{\underline{ij}} = g_{\underline{ji}}$, while $g_{ij} \neq g_{ji}$ in general.

[4]Note that there exist infinite complete *weighted* networks, but only one in case of 0-1 networks.

**Definition 1** *A link-formation technology is a differentiable map* $\delta : \mathbb{R}_+ \to [0,1)$ *s.t.* $\delta(0) = 0$*, and satisfies the following conditions:*

$(C.1)$ $\delta'(c) > 0$*, for all* $c \geq 0$*, i.e. it is increasing.*

$(C.2)$ *It is strictly concave.*

The interpretation of this function and the meaning of the assumptions are the following. If $c$ is the amount invested in a link to connect two *nodes* or *players,* $\delta(c)$ is interpreted as the level of fidelity of the transmission of information through the link.[5] More precisely, $\delta(c)$ is the fraction of information flowing through the link that remains intact.[6] Flow occurs only through links invested in ($\delta(0) = 0$), but a perfect fidelity in transmission between different nodes is never reached ($0 \leq \delta(c) < 1$). The smoothness of $\delta$ makes the use of differential calculus possible, which allows for a relatively simple formal "marginal analysis" without getting involved in sophisticated technical issues. Condition C.1 is quite plausible: the quality of a link is increasing with the investment in it. C.2 is a reasonable condition, at least "in the long run" given that $\delta'(c) > 0$ and the range of $\delta$.[7]

Based on this basic ingredient we consider two different scenarios.

**Scenario 1**: A set $N = \{1, 2, ..., n\}$ of nodes can be connected by links according to a link-formation technology. Investments are made by, say, a central planner. A *link-investment vector* is an $n(n-1)/2$-vector, $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$, where $c_{\underline{ij}}$ denotes the investment in link $\underline{ij} \in N_2$ through which the fidelity-level is $\delta(c_{\underline{ij}})$. Then $\delta^{\overline{\mathbf{c}}} = (\delta^{\overline{\mathbf{c}}})_{\underline{ij} \in N_2} = (\delta(c_{\underline{ij}}))_{\underline{ij} \in N_2}$ denotes the resulting *weighted network*. For a given link-investment vector $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$, a node $i$ receives from another node's value $v$, the fraction that reaches $i$ through the best possible route in the weighted network $\delta^{\overline{\mathbf{c}}}$.[8] Let $\mathcal{P}_{ij}(\delta^{\overline{\mathbf{c}}})$ denote the set of paths in $\delta^{\overline{\mathbf{c}}}$ connecting $i$ and $j$. For $p \in \mathcal{P}_{ij}(\delta^{\overline{\mathbf{c}}})$, let $\delta^{\overline{\mathbf{c}}}(p)$ denote the resulting fidelity-level determined by the product of the fidelity-levels through each link in that path. Then, $i$ values information originating from $j$ that arrives via $p$ by $v\delta^{\overline{\mathbf{c}}}(p)$. If information is routed via the best possible route from $j$ to $i$, then $i$'s valuation of the information originating from $j \neq i$ is

$$I_{ij}(\delta^{\overline{\mathbf{c}}}) = \max_{p \in \mathcal{P}_{ij}(\delta^{\overline{\mathbf{c}}})} v\delta^{\overline{\mathbf{c}}}(p) = v \max_{p \in \mathcal{P}_{ij}(\delta^{\overline{\mathbf{c}}})} \delta^{\overline{\mathbf{c}}}(p),$$

[5] We often prefer the term 'node' to avoid a biased language. Moreover, in the first of the two scenarios that we presently describe it is more appropriate to speak of nodes given their passive role.

[6] Nevertheless, other interpretations are possible. For instance, as a degree of reliability, as in Bala and Goyal (2000b), or the "strength of a tie" (Granovetter, 1973), i.e. a measure of the quality/intensity/value of a relationship as e.g. in personal relationships, where quality/strength of a "link" is a function of the "investments" of each of the two people involved. A link can also be a means for the flow of other goods, but we give preference here to the interpretation in terms of information.

[7] It would also be reasonable to assume $\delta$ to be convex up to a certain value of $c$, and concave beyond an inflection point. This and other variations of the model are considered in the concluding section.

[8] We assume homogeneity in values. A more general model would not assume this value to be the same for all $i$ and $j$.

and $i$'s overall revenue from $\delta^{\overline{\mathbf{c}}}$ is

$$I_i(\delta^{\overline{\mathbf{c}}}) = \sum_{j \in N(i;\delta^{\overline{\mathbf{c}}})} I_{ij}(\delta^{\overline{\mathbf{c}}}).$$

The *value* of the network resulting from a link-investment vector $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ is the aggregate payoff, i.e. the total value of the information received by the nodes minus the cost of the network:

$$v(\delta^{\overline{\mathbf{c}}}) := \sum_{i \in N} I_i(\delta^{\overline{\mathbf{c}}}) - \sum_{\underline{ij} \in N_2} c_{\underline{ij}}. \tag{1}$$

In this setting two questions are addressed. First, the problem of determining the link-investment vector that maximizes the aggregate value for a given "infrastructure" specified by an underlying graph of feasible links. Second, the characterization of efficient networks/link-investment vectors that maximize the aggregate value in absolute terms.

**Scenario 2**: Let $N$, $\delta$ and $v$ be as in Scenario 1, but now the nodes/players form the links by investing in them. The quality of a link depends on the total amount invested in it by the two players it connects, and it is assumed that a link-formation technology $\delta$ is available to all agents and determines the quality of a link as a function of the investment in it. An *investment profile* is specified by a matrix $\mathbf{c} = (c_{ij})_{i,j \in N}$, where $c_{ij} \geq 0$ (with $c_{ii} = 0$) is the investment of player $i$ in the link connecting $i$ and $j$, and determines a link-investment vector $\overline{\mathbf{c}}$

$$\mathbf{c} \to \overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2} \text{ s.t. } c_{\underline{ij}} := c_{ij} + c_{ji}.$$

The available link-formation technology, $\delta$, yields a *weighted network* for each investment profile $\mathbf{c}$. Namely, $\delta^{\mathbf{c}} := \delta^{\overline{\mathbf{c}}}$, where

$$\delta^{\mathbf{c}}_{\underline{ij}} = \delta^{\overline{\mathbf{c}}}_{\underline{ij}} = \delta(c_{\underline{ij}}) = \delta(c_{ij} + c_{ji})$$

whenever $i \neq j$. Thus, $i$'s payoff is the value of the information received by $i$ minus $i$'s investment:

$$\Pi_i^{\delta}(\mathbf{c}) = I_i(\delta^{\mathbf{c}}) - C_i(\mathbf{c}) = I_i(\delta^{\overline{\mathbf{c}}}) - \sum_{j \neq i} c_{ij}. \tag{2}$$

Note that a game in strategic form, where a *strategy* of a player is an $(n-1)$-vector of investments and the *payoff function* is given by (2), is implicitly defined. Therefore, the notion of Nash equilibrium can be applied: an investment profile is *Nash stable* if no player has an incentive to change his/her investments' vector. In fact, this strictly non-cooperative notion is not the only stability notion that makes sense in this context. A refinement of this notion consists of requiring also stability w.r.t. pairwise coordinated moves, i.e. two players agreeing on a joint investment in a link as introduced by Jackson and Wolinsky (1996).[9]

[9] This strong version of pairwise stability was suggested by Jackson and Wolinsky (1996) and applied by Goyal and Joshi (2003) and Belleflamme and Bloch (2004) among others. See also Bloch and Jackson (2006) for a discussion of different notions of equilibrium in network formation and references therein.

**Definition 2** *An investment profile* $\mathbf{c}$ *is pairwise Nash stable if it is Nash stable and there are no two players* $i$ *and* $j$ *that can improve their payoffs by investing coordinately in a link connecting them, that is, there is no* $\mathbf{c}'$ *s.t.* $c'_{ik} = c_{ik}$ *for all* $k \neq j$*, and* $c'_{jk} = c_{jk}$ *for all* $k \neq i$ *and*

$$\Pi_i^\delta(\mathbf{c}') > \Pi_i^\delta(\mathbf{c}) \quad \text{and} \quad \Pi_j^\delta(\mathbf{c}') > \Pi_j^\delta(\mathbf{c}).$$

We first consider Scenario 1 and address the question of efficiency in relative and absolute terms. Later, the question of stability is addressed in Scenario 2. All results assume a link-formation technology as specified in Definition 1. Thus we deal with a model with three "parameters": the number of nodes/players $n$, the value $v$ of the information at each node, and the link-formation technology represented by function $\delta$.[10]

# 4 Scenario 1: Efficiency

Let $\overline{\mathbf{c}}$ and $\overline{\mathbf{c}}'$ be two link-investment vectors and $v(\delta^{\overline{\mathbf{c}}})$ and $v(\delta^{\overline{\mathbf{c}}'})$ their values as defined by (1). If $v(\delta^{\overline{\mathbf{c}}}) \geq v(\delta^{\overline{\mathbf{c}}'})$ we say that $\delta^{\overline{\mathbf{c}}}$ *dominates* $\delta^{\overline{\mathbf{c}}'}$ (or that $\overline{\mathbf{c}}$ dominates $\overline{\mathbf{c}}'$). Network $\delta^{\overline{\mathbf{c}}}$ (or link-investment vector $\overline{\mathbf{c}}$) is said to be *efficient* if it dominates any other.[11] Thus efficiency can be seen as the goal of a planner investing in links with the objective of maximizing the aggregate benefit, i.e. the aggregate value received by the nodes minus the total cost of the network. We use the following notation: for all $i, j \in N, i \neq j$, $\overline{p}_{ij}$ denotes an optimal path connecting them (note it may be not unique), i.e. one for which the resulting fidelity-level is maximal:

$$\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) = \max_{p \in \mathcal{P}_{ij}(\delta^{\overline{\mathbf{c}}})} \delta^{\overline{\mathbf{c}}}(p),$$

where $\delta^{\overline{\mathbf{c}}}(p)$ is the product of the fidelity-levels of the links forming path $p$ for the link-investment vector $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$, and the aggregate payoff for this link-investment vector is

$$v(\delta^{\overline{\mathbf{c}}}) = \sum_{i \in N} I_i(\delta^{\overline{\mathbf{c}}}) - \sum_{\underline{ij} \in N_2} c_{\underline{ij}} = 2v \sum_{\underline{ij} \in N_2} \delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) - \sum_{\underline{ij} \in N_2} c_{\underline{ij}}. \tag{3}$$

Note that in principle the last expression in (3) may not be unique. This occurs if for some pair of nodes the optimal path connecting them is not unique.

We make use of the following notation: if $\overline{p}_{kl}$ is an optimal path connecting nodes $k$ and $l$ for a link-investment vector $\overline{\mathbf{c}}$ that contains link $\underline{ij}$, $\delta(\overline{p}_{kl}^{ij})$ denotes the product

[10] It can be assumed w.l.o.g. $v = 1$, which slightly simplifes the presentation, and this is assumed in the examples. Nevertheless, it is preferable not to do so and keep explicit this otherwise hidden parameter. In Scenario 1 this value can be interpreted as a subjective evaluation of the planner w.r.t. which the efficiency objective is specified. Nevertheless, the reader may choose to ignore all occurrences of $v$ assuming $v = 1$.

[11] This the strong efficiency notion introduced by Jackson and Wolinsky (1996).

of fidelity-levels for the path that would result from replacing in $\overline{p}_{kl}$ the link $c_{\underline{ij}}$ by a perfect one, i.e. by $c'_{\underline{ij}} = 1$. In other terms:

$$\delta(\overline{p}_{kl}^{ij}) = \frac{\delta^{\overline{\mathbf{c}}}(\overline{p}_{kl})}{\delta(c_{\underline{ij}})}.$$

Before addressing the question of efficiency in absolute terms, we address the problem of maximizing the aggregate payoff for a given "infrastructure" specified by an underlying graph of feasible links.

**Definition 3** *An infrastructure is a subset $S \subseteq N_2$ which specifies the set of links which must be invested in. And a link-investment vector $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ is optimal for an infrastructure $S$ if for all $\underline{ij} \in N_2$, $c_{\underline{ij}} > 0$ if and only if $\underline{ij} \in S$, and maximizes the aggregate payoff given this constraint.*

Then the following result establishes necessary conditions for a link-investment vector to be optimal for an infrastructure.

**Lemma 1** *For a link-investment vector $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ to be optimal for a given infrastructure $S \subseteq N_2$, the following are necessary conditions: $(i)$ For any two nodes connected in $S$ there is a unique optimal path connecting them, and $(ii)$ For all $\underline{ij} \in S$,*

$$\delta'(c_{\underline{ij}}) = \frac{1}{2v \sum_{\underline{kl} \in N_2 (\underline{ij} \in \overline{p}_{kl})} \delta(\overline{p}_{kl}^{ij})}. \tag{4}$$

**Proof.** Let $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ be a link-investment vector s.t. $c_{\underline{ij}} > 0$ if and only if $\underline{ij} \in S$. We prove first part $(ii)$. Assume $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ to be optimal for $S$, and $\underline{ij} \in S$, i.e. $c_{\underline{ij}} > 0$. Then link $\underline{ij}$ is part of at least one optimal path in $\delta^{\overline{\mathbf{c}}}$ (the one connecting $i$ and $j$, otherwise $\overline{\mathbf{c}}$ would not be optimal).[12] Then, in any of the possibly different but equivalent expressions of the right-hand side of (3), $\delta(c_{\underline{ij}})$ would appear at least once (i.e. $\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) = \delta(c_{\underline{ij}})$ if $\overline{\mathbf{c}}$ is optimal), and possibly also in the product yielding $\delta^{\overline{\mathbf{c}}}(\overline{p}_{kl})$ for other pairs of nodes $k, l$. Fix any choice of these (possibly multiple) optimal paths for every two connected nodes and let the aggregate payoff be given by the right-hand side of (3). The right-hand side is an up to $n(n-1)/2$-variable function with partial derivatives. (It is not claimed that these partial derivatives are those of the aggregate payoff nor this is needed.) Even if a slight modification of some $c_{\underline{ij}} > 0$ might cause a path not to be optimal, a non-null partial derivative w.r.t. $c_{\underline{ij}}$ means that by slightly increasing (if it were $> 0$) or decreasing (if it were $< 0$) the investment in link $\underline{ij}$ the aggregate payoff (through those the same paths, optimal or not but

[12] Note that $\delta(c_{\overline{ij}})$ actually appears in (3) only if $c_{\overline{ij}} > 0$.

still available) would surely increase, which contradicts $\overline{\mathbf{c}}$'s optimality for $S$. Then the partial derivative of the right-hand side of (3) w.r.t. $c_{\underline{ij}}$ must be 0, i.e.

$$\frac{\partial}{\partial c_{\underline{ij}}}(2v\sum_{\underline{kl}\in N_2}\delta^{\overline{\mathbf{c}}}(\overline{p}_{kl})-\sum_{\underline{kl}\in N_2}c_{\underline{kl}})=2v\delta'(c_{\underline{ij}})\sum_{\underline{kl}\in N_2(\underline{ij}\in\overline{p}_{kl})}\delta(\overline{p}_{kl}^{ij})-1=0,$$

which yields (4).

$(i)$ Assume that two nodes $g$ and $h$ are connected by two different optimal paths in $\delta^{\overline{\mathbf{c}}}$. Then there is at least one link, say $\underline{ij}$, that is part of one of these paths but not of the other. Then the right-hand side of (4) admits at least *two* different expressions: one where the optimal path between any pair of nodes $k,l$ is $\overline{p}_{kl}$, and another one where it is $\overline{q}_{kl}$, and such that for any pair $k,l$ different from pair $g,h$, $\overline{p}_{kl}=\overline{q}_{kl}$, while for $g$ and $h$ the optimal path is different, i.e. $\overline{p}_{gh}\neq\overline{q}_{gh}$, and only the first one contains $\underline{ij}$. In that case,

$$\frac{1}{2v\sum_{\underline{kl}\in N_2(\underline{ij}\in\overline{p}_{kl})}\delta^{\overline{\mathbf{c}}}(\overline{p}_{kl}^{ij})}\neq\frac{1}{2v\sum_{\underline{kl}\in N_2(\underline{ij}\in\overline{p}_{kl})}\delta^{\overline{\mathbf{c}}'}(\overline{q}_{kl}^{ij})}$$

because

$$\sum_{\underline{kl}\in N_2(\underline{ij}\in\overline{p}_{kl})}\delta^{\overline{\mathbf{c}}}(\overline{p}_{kl}^{ij})-\sum_{\underline{kl}\in N_2(\underline{ij}\in\overline{p}_{kl})}\delta^{\overline{\mathbf{c}}'}(\overline{q}_{kl}^{ij})=\delta^{\overline{\mathbf{c}}}(\overline{p}_{gh})>0,$$

which leads to a contradiction because (4) yields two different values for $\delta'(c_{\underline{ij}})$. ∎

We postpone a comment on the clear interpretation of condition (4) to the formulation of a result establishing necessary conditions for a link-investment vector to be efficient in absolute terms.

**Proposition 1** *For a link-investment vector $\overline{\mathbf{c}}=(c_{\underline{ij}})_{\underline{ij}\in N_2}$ to be efficient the following conditions are necessary: $(i)$ For any two connected nodes there exists a unique optimal path connecting them.*
*$(ii)$ For each $\underline{ij}\in N_2$ s.t. $c_{\underline{ij}}>0$ :*

$$\delta'(c_{\underline{ij}})=\frac{1}{2v\sum_{\underline{kl}\in N_2(\underline{ij}\in\overline{p}_{kl})}\delta(\overline{p}_{kl}^{ij})}. \tag{5}$$

*$(iii)$ For each $\underline{ij}\in N_2$ s.t. $c_{\underline{ij}}=0$, if $\delta'(0)>1/2v$,*

$$2v\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij})\geq 2v\delta(c^{\sharp})-c^{\sharp}, \tag{6}$$

*where $c^{\sharp}=\arg\max_{c>0}(2v\delta(c)-c)$.*

**Proof.** Assume $\overline{\mathbf{c}}=(c_{\underline{ij}})_{\underline{ij}\in N_2}$ to be efficient, then $\overline{\mathbf{c}}$ must be optimal for the infrastructure $S=\{\underline{ij}\in N_2:c_{\underline{ij}}>0\}$. Then $(i)$ and $(ii)$ follow immediately form Lemma 1.

$(iii)$ Assume $c_{\underline{ij}}=0$. Then no investment in link $\underline{ij}$ can increase the aggregate payoff, that is, for all $c>0$, $2v\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij})\geq 2v\delta(c)-c$, otherwise investing $c$ in link

$\underline{ij}$ would surely increase the aggregate payoff. This yields condition (6). Note that condition (6) has a bite only if $\delta'(0) > 1/2v$, otherwise, by the assumptions about function $\delta$, $2v\delta(c) - c < 0$ for all $c > 0$. ■

**Comment:** Part ($i$) establishes that in a network that results from an efficient link-investment vector any two nodes "see" each other through a unique best path connecting them. Therefore each node sees the others through a unique tree rooted at it, and all the other nodes see it through the same tree and consequently such trees for two different nodes have in common at least the optimal path connecting them. This is a consequence of part ($ii$) which establishes a set of equations on the non-zero investments in links for a link-investment vector which have a very intuitive interpretation. First note that the denominator of the right-hand side of (5) is $2v$ times the sum of the fidelity-levels through all optimal paths of which link $\underline{ij}$ is part of (discounting link $\underline{ij}$, i.e. divided by $\delta(c_{\underline{ij}})$). In other words, the actual flow of information that link $\underline{ij}$ supports. Thus this sum can be interpreted as a *measure of the importance or centrality of link* $\underline{ij}$ *in network* $\delta^{\overline{\mathbf{c}}}$. The interpretation of (5) is then clear: the greater this measure, the smaller $\delta'(c_{\underline{ij}})$ and consequently the greater $c_{\underline{ij}}$, that is, the greater the investment that efficiency imposes in that link. Note also that the less important link in this sense imposes a condition for the marginal fidelity-level w.r.t. investment at 0. Namely, if $\underline{ij}$ is the link for which the right-hand side of (5) is maximal (and consequently $c_{\underline{ij}} > 0$ minimal), for (5) to be feasible $\delta'(0)$ must be greater than this right-hand side, otherwise no such a $c_{\underline{ij}}$ exists. Condition ($iii$) is a necessary condition for the existence of efficient link-investment vectors where some link receives no investment: no investment in that link must be profitable for the two nodes because it would increase the aggregate payoff.

In different contexts the complete network and the star emerge as efficient structures[13]. In order to prove that this is also true in the current model we first establish necessary conditions for the efficiency of each of these structures based on Proposition 1.

**Proposition 2** *For a complete network to be efficient the following conditions are necessary:*
($i$) *The marginal fidelity-level w.r.t. investment at* 0 *is greater than* $1/2v$*, i.e.*

$$\delta'(0) > 1/2v. \tag{7}$$

($ii$) *All links are invested in the same amount, namely,* $c^\sharp$ *s.t.*

$$\delta'(c^\sharp) = 1/2v. \tag{8}$$

($iii$) *The following relation holds:*

$$c^\sharp \leq 2v(\delta(c^\sharp) - \delta(c^\sharp)^2). \tag{9}$$

[13]E.g. in Jackson and Wolinsky (1996), Bala and Goyal (2000), Olaizola nad Valenciano (2015c).

**Proof.** Assume $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ to be efficient and complete, i.e. $c_{\underline{ij}} > 0$ for all $\underline{ij} \in N_2$. Note first that every node sees every other node through the link connecting them better than through any other path. Otherwise the link would be superfluous. Therefore, for all $\underline{ij} \in N_2$, $\delta(\overline{p}_{ij}) = \delta(c_{\underline{ij}})$ and $\delta(\overline{p}^{ij}_{kl}) = 1$, and (5) becomes

$$\delta'(c_{\underline{ij}}) = \frac{1}{2v\delta(\overline{p}^{ij}_{ij})} = \frac{1}{2v}.$$

Thus all links must receive the same investment $c^\sharp$ s.t. $\delta'(c^\sharp) = 1/2v$. Note that condition $\delta'(c^\sharp) = 1/2v$, necessary for the complete network to dominate any other complete network for a given technology, is feasible only if $\delta'(0) > 1/2v$. As is easy to check, if $\delta'(0) \leq 1/2v$, then the empty network dominates *all* complete networks. Finally, condition (6) becomes (9) for this complete network. ■

**Comment:** Condition (7) imposes a lower bound on $\delta'(0)$, i.e. on the marginal fidelity-level at the origin, for a technology to admit a complete network as efficient. Condition (5) from Proposition 1 becomes (8) and is then feasible, characterizing the optimal investment for the complete infrastructure $S = N_2$. It establishes that all links must receive the same investment, precisely that for which the marginal fidelity-level is that lower bound $1/2v$. Note that condition (8), derived form (5), has a clear meaning. In an efficient complete network every node sees any other node directly through the link that connects them. Therefore, in order to be efficient the investment $c^\sharp$ must maximize the contribution to the aggregate payoff, i.e., $c^\sharp$ must be s.t.

$$2v\delta(c^\sharp) - c^\sharp = \max(2v\delta(c) - c), \tag{10}$$

or, equivalently:

$$c^\sharp = \arg\max(2v\delta(c) - c). \tag{11}$$

Expressions which, by the assumptions on the technology, are equivalent to (8). Additionally, condition (6) from Proposition 1 becomes (9) and is also necessary for it to be efficient in absolute terms: the fidelity-level through a two $c^\sharp$-links path should not be good enough to make superfluous a direct $c^\sharp$-link. It is important to note that none of these conditions involve the number of nodes.

We now establish necessary conditions for a star network to be efficient.

**Proposition 3** *For an all-encompassing star to be efficient the following conditions are necessary:*
*(i) All links are invested in the same amount $c^\times_n$ s.t.*

$$\delta'(c^\times_n) = \frac{1}{2v(1 + (n-2)\delta(c^\times_n))}. \tag{12}$$

*(ii) Additionally, if $\delta'(0) > 1/2v$,*

$$2v\delta(c^\times_n)^2 \geq 2v\delta(c^\sharp) - c^\sharp, \tag{13}$$

*with $c^\sharp$ given by* (8).

**Proof.** $(i)$ Consider an all-encompassing star whose center is node $n$ and where its link with each node $i$ is invested in $c_{\underline{in}} > 0$. Then, (5) becomes

$$\delta'(c_{\underline{in}}) = \frac{1}{2v(1 + \sum_{j \in N \setminus \{i,n\}} \delta(c_{\underline{jn}}))}.$$

But the investments in all links must be equal. Otherwise, assume $c_{\underline{in}} < c_{\underline{jn}}$, which implies $\delta(c_{\underline{in}}) < \delta(c_{\underline{jn}})$, and, by C.1 and C.2, implies also that $\delta'(c_{\underline{in}}) > \delta'(c_{\underline{jn}})$, which by the last equality relating $\delta'(c_{\underline{in}})$ and the remaining $\delta(c_{\underline{jn}})$'s is equivalent to

$$\frac{1}{2v(1 + \sum_{k \in N \setminus \{i,n\}} \delta(c_{\underline{kn}}))} > \frac{1}{2v(1 + \sum_{k \in N \setminus \{j,n\}} \delta(c_{\underline{kn}}))},$$

which contradicts that $\delta(c_{\underline{in}}) < \delta(c_{\underline{jn}})$. Therefore, all links are invested in the same amount, say $c_n^{\times}$. Then, the equality above becomes

$$\delta'(c_n^{\times}) = \frac{1}{2v(1 + (n-2)\delta(c_n^{\times}))}.$$

$(ii)$ In order to be efficient no link between peripheral nodes should improve the aggregate payoff, for which it is sufficient that it increases their payoffs. In other words, condition (6) becomes:

$$2v\delta(c_n^{\times})^2 \geq \max_{c>0}(2v\delta(c) - c),$$

which by (10) is equivalent to $2v\delta(c_n^{\times})^2 \geq 2v\delta(c^{\sharp}) - c^{\sharp}$ if $\delta'(0) > 1/2v$, and holds trivially if $\delta'(0) \leq 1/2v$. ■

**Comment:** Condition (5) from Proposition 1 becomes (12) and characterizes the optimal investment for the star infrastructure imposing the same investment in every link. Note that, unlike $c^{\sharp}$, $c_n^{\times}$ depends on the number of nodes, hence the subindex. Now condition (6) from Proposition 1 becomes (13) and prescribes that investing in a link to connect two peripheral in the star cannot increase the aggregate payoff. Nevertheless, the question of the feasibility of condition (12) arises. The following lemma shows that the existence of $c_n^{\times}$ such that (12) holds is guaranteed if $\delta'(0) > 1/2v$ whatever the number of nodes, and for $n$ big enough if $\delta'(0) \leq 1/2v$.

**Lemma 2** *Whatever the number of nodes, if $\delta'(0) > 1/2v$, then it is sure to exist $c_n^{\times}$ such that* (12) *holds, and also when $\delta'(0) \leq 1/2v$ for $n$ sufficiently large. On the contrary, for a fixed $n$, no such $c_n^{\times}$ exists if $\delta'(0) \leq \frac{1}{2v(1+(n-2)\delta(\infty))}$, where $\delta(\infty)$ denotes $\lim_{c\to\infty} \delta(c)$.*

**Proof.** Assume $\delta'(0) > 1/2v$, and let $\varphi$ be the function $\varphi(c) := \frac{1}{2v(1+(n-2)\delta(c))}$. We prove that $\varphi(c) = \delta'(c)$ holds necessarily for some $c > 0$. Note that, as $\delta(0) = 0$, $\varphi(0) = 1/2v < \delta'(0)$. On the other hand, $\varphi(c) = \frac{1}{2v(1+(n-2)\delta(c))} > \frac{1}{2v(1+(n-2)\delta(\infty))}$, for all $c > 0$.

Thus, $\varphi(c)$ is a decreasing function whose value is always greater than $\frac{1}{2v(1+(n-2)\delta(\infty))}$ and it is $1/2v$ at 0, while $\delta'(c)$ is a decreasing function s.t. $\delta'(0) > 1/2v = \varphi(0)$. Moreover, by the assumptions on function $\delta$, $\lim_{c\to\infty}\delta'(c) = 0$. Therefore at some point necessarily $\varphi(c) = \delta'(c)$, i.e. (12) holds.

Now assume $\delta'(0) \leq 1/2v$. We prove that for $n$ sufficiently big there exists $c > 0$ s.t. $\varphi(c) < \delta'(c)$, for which it is sufficient to prove that for $n$ sufficiently big $\varphi(1) \leq \delta'(1)$. But it is easy to check that this is equivalent to

$$n - 2 \geq \frac{1 - 2v\delta'(1)}{2v\delta(1)\delta'(1)},$$

which is sure to hold for $n$ big enough.

Finally, if for a fixed $n$ inequality $\delta'(0) \leq \frac{1}{2v(1+(n-2)\delta(\infty))}$ holds, then the graphs of $\varphi(c)$ and $\delta'(c)$ do not intersect, because

$$\varphi(c) = \frac{1}{2v(1+(n-2)\delta(c))} > \frac{1}{2v(1+(n-2)\delta(\infty))} \geq \delta'(0) > \delta'(c)$$

for all $c$. Note that there is no contradiction with the preceding result: whatever the value of $\delta'(0)$, for $n$ big enough $\delta'(0) > \frac{1}{2v(1+(n-2)\delta(\infty))}$. ■

Conditions (9) and (13), necessary for the optimal complete network and the star to be efficient are:

$$2v\delta(c^\sharp)^2 \leq 2v\delta(c^\sharp) - c^\sharp \leq 2v\delta(c^\times)^2.$$

The first inequality ensures that by deleting a link in an optimal complete network does not increase the aggregate payoff, while the second guarantees that connecting by a $c^\sharp$-link two nodes connected by a two $c^\times$-links path does not increase the aggregate payoff. Observe that these conditions determine an interval for $c^\sharp$

$$2v(\delta(c^\sharp) - \delta(c^\times)^2) \leq c^\sharp \leq 2v(\delta(c^\sharp) - \delta(c^\sharp)^2),$$

where *both* conditions hold. As it turns out, outside this interval the only non-empty efficient structures are either the optimal complete or the optimal star, while inside it both *plus* a sort of hybrid structure can be efficient. We first describe this hybrid structure.

**Definition 4** *A hybrid star-complete structure consists of a set $N$ of $n$ nodes of which: those in $L \subseteq N$, with $l = \#L$, $1 \leq l \leq n$, are directly connected among themselves and with any of the other nodes in $N$, and those in $N \setminus L$ are arranged as spokes of a star of center $p^* \in L$ of which at most one of them, $m^*$, is connected with a subset $M$ (with $m = \#M$, $0 \leq m \leq n - l - 2$) of the other spoke nodes of this star.*

Note that this includes as extreme cases complete networks ($L = N$) and all-encompassing stars ($L = \{p^*\}$). In other words, the properly hybrid star-complete structure arises for $\{p^*\} \subsetneq L \subsetneq N$. Figure 1 illustrates two hybrid star-complete structures. The following lemma will be crucial for the characterization of the efficient networks, establishing necessary conditions for a hybrid star-complete network to be efficient, with $N, L, M$, $m^*$ and $p^*$ as in Definition 4, and $p = \#P$ with $P := N \setminus (L \cup M \cup \{m^*\})$.

(a) Hybrid star-complete (b) Optimal hybrid star-complete

Figure 1

As with the complete network and the star, Proposition 1 imposes some constraints for a hybrid star-complete network to be efficient.

**Proposition 4** *Let $\delta^{\overline{\mathbf{c}}}$ be a hybrid star-complete network with $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$. The following conditions are necessary for $\delta^{\overline{\mathbf{c}}}$ to be efficient:*
*$(i)$ All links except those forming the star, i.e. except those connecting $p^*$ with the nodes in $N \setminus L$, are invested in the same amount $c^\sharp$ s.t. $\delta'(c^\sharp) = 1/2v$.*
*$(ii)$ The links connecting $p^*$ with those in $N \setminus L$ can receive up to three different levels of investment. Namely, the link connecting $p^*$ and $m^*$, the links connecting $p^*$ with nodes in $M$, and those connecting $p^*$ and nodes in $P$. If $i_m$ and $i_p$ denote generic nodes in $M$ and in $P$ respectively:*

$$\delta'(c_{\underline{i_p p^*}}) = \frac{1}{2v(1 + (p-1)\delta(c_{\underline{i_p p^*}}) + m\delta(c_{\underline{i_m p^*}}) + \delta(c_{\underline{m^* p^*}}))}, \tag{14}$$

$$\delta'(c_{\underline{i_m p^*}}) = \frac{1}{2v(1 + (m-1)\delta(c_{\underline{i_m p^*}}) + p\delta(c_{\underline{i_p p^*}}))}, \tag{15}$$

$$\delta'(c_{\underline{m^* p^*}}) = \frac{1}{2v(1 + p\delta(c_{\underline{i_p p^*}}))}. \tag{16}$$

*$(iii)$ If $l \geq 2$,*

$$c^\sharp \leq 2v(\delta(c^\sharp) - \delta(c^\sharp)\delta(c_{\underline{m^* p^*}})). \tag{17}$$

*$(iv)$ If $M \neq \varnothing$,*

$$c^\sharp \geq 2v(\delta(c^\sharp) - \delta(c_{\underline{m^* p^*}})\delta(c_{\underline{i_p p^*}})). \tag{18}$$

*$(v)$ If $M = \varnothing$,*

$$c^\sharp \geq 2v(\delta(c^\sharp) - \delta(c_{\underline{i_p p^*}})^2) \tag{19}$$

**Proof.** $(i)$ Let $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ be s.t. $\delta^{\overline{\mathbf{c}}}$ is a hybrid star-complete network, with $N, L, M$, $m^*$ and $p^*$ as in Definition 4, and $p = \#P$, where $P := N \setminus (L \cup M \cup \{m^*\})$. Assume $\overline{\mathbf{c}}$ is efficient. In $\delta^{\overline{\mathbf{c}}}$ all pairs of nodes are directly linked with the sole exception of some pairs of spoke-nodes of the star, of which those in $M$ are linked with $m^*$. These spoke nodes of the star not directly connected are connected one another by two-link paths through the star. Efficiency implies that all links are used by optimal paths. This entails that all pairs of nodes but those connected through two-link paths in the star see each other through the link connecting them. Moreover, we show that all links but those in the star are *only* used by the two nodes involved. Let $i, j \in N$ be any pair of nodes. First, note that if $i$ or $j$ belongs to $L \setminus \{p^*\}$, they cannot be extremes of an optimal path connecting them unless this is the 1-link path consisting of link $\underline{ij}$. Remains to be seen that also the links connecting $m^*$ and nodes in $M$ are only used by the two nodes involved. But this must be so, otherwise some link of the star would be superfluous. Then, for all these links, as $\delta(\overline{p}_{ij}) = \delta(c_{\underline{ij}})$, $\delta(\overline{p}_{ij}^{\underline{ij}}) = 1$, and (4) yields

$$\delta'(c_{\underline{ij}}) = \frac{1}{2v}.$$

Note that by the assumptions on $\delta$, the existence of $c^\sharp$ s.t. $\delta'(c^\sharp) = 1/2v$ is guaranteed if and only if $\delta'(0) > 1/2v$, and in that case it would be unique.

(*ii*) Now consider the links which form the star. Condition (4) applies to all of them. From the definition of hybrid star-complete network $\delta^{\overline{\mathbf{c}}}$, it follows that there are three different possible types of links corresponding to three different situations: the link connecting $p^*$ and $m^*$, the links connecting $p^*$ and nodes in $M$, and those connecting $p^*$ and nodes in $P$. Only nodes in $N \setminus L$ use these links to see each other. Then, applying (4) to each of the three cases we have the following.

- The nodes in $P$ see all other spoke nodes in $N \setminus L$ through the star. Thus each link $\underline{i_p p^*}$ is in the optimal path connecting $p^*$ with $i_p$ and with the other nodes in $P$ (i.e. $P \setminus \{i_p\}$), with all nodes in $M$ and with $m^*$, therefore, by applying (5):

$$\delta'(c_{\underline{i_p p^*}}) = \frac{1}{2v(1 + \sum_{i \in P\setminus\{i_p\}} \delta(c_{\underline{ip^*}}) + \sum_{j \in M} \delta(c_{\underline{jp^*}}) + \delta(c_{\underline{m^*p^*}}))} \tag{20}$$

But all these links must receive the same support: Let $i_p, j_p \in P$ be two nodes in $P$, and assume $c_{\underline{i_p p^*}} < c_{\underline{j_p p^*}}$. Then $\delta(c_{\underline{i_p p^*}}) < \delta(c_{\underline{j_p p^*}})$, which by the assumptions on technology $\delta$, implies that $\delta'(c_{\underline{i_p p^*}}) > \delta'(c_{\underline{j_p p^*}})$, which by the last equation relating $\delta'(c_{\underline{i_p p^*}})$ and the investments in the remaining links in the star is equivalent to state that

$$\sum_{i \in P\setminus\{i_p\}} \delta(c_{\underline{ip^*}}) + \sum_{j \in M} \delta(c_{\underline{jp^*}}) + \delta(c_{\underline{m^*p^*}}) < \sum_{i \in P\setminus\{j_p\}} \delta(c_{\underline{ip^*}}) + \sum_{j \in M} \delta(c_{\underline{jp^*}}) + \delta(c_{\underline{m^*p^*}}),$$

i.e. $\delta(c_{\underline{j_p p^*}}) < \delta(c_{\underline{i_p p^*}})$, which is a contradiction. Therefore $c_{\underline{i_p p^*}} = c_{\underline{j_p p^*}}$ for all $i_p, j_p \in P$.

- All nodes in $M$ see directly $p^*$, and those in $M \setminus \{i_m\}$ and all nodes in $P$ through the star. Thus each link $\underline{i_m p^*}$ is in the optimal paths for these connections, therefore:

$$\delta'(c_{\underline{i_m p^*}}) = \frac{1}{2v(1 + \sum_{j \in M\setminus\{i_m\}} \delta(c_{\underline{jp^*}}) + \sum_{i \in P} \delta(c_{\underline{ip^*}}))}, \tag{21}$$

and an entirely similar reasoning as the one used to show it for the links connecting nodes in $P$ with $p^*$ leads to the conclusion that all these links must receive the same investments. Then equations (20) and (21) become (14) and (15).

- Finally, spoke node $m^*$ in the star sees $p^*$ and the nodes in $M$ directly, and the other nodes in $P$ via the star. Again by applying (5), if $i_p$ denote a generic node in $P$, we conclude that link $\underline{p^*m^*}$ is only used to see $m^*$ and in the optimal paths connecting $p^*$ with nodes in $P$, therefore:

$$\delta'(c_{\underline{p^*m^*}}) = \frac{1}{2v(1+\sum_{i\in P}\delta(c_{\underline{ip^*}}))} = \frac{1}{2v(1+p\delta(c_{\underline{i_pp^*}}))}.$$

$(iii)$ If $\#L \geq 2$, at least one node is directly connected with $p^*$ and $m^*$ by $c^\sharp$-links. Then, if $2v\delta(c^\sharp)\delta(c_{\underline{m^*p^*}}) > 2v\delta(c^\sharp) - c^\sharp$ the aggregate payoff would increase by eliminating one these $c^\sharp$-links.

$(iv)$ If $M \neq \varnothing$, the weakest connection between spoke nodes are between node $m^*$ and the nodes in $P$. So if $2v\delta(c_{\underline{m^*p^*}})\delta(c_{\underline{i_pp^*}})) < 2v\delta(c^\sharp) - c^\sharp$, then a $c^\sharp$-link between them would increase the aggregate payoff.

$(v)$ If $M = \varnothing$, and $2v\delta(c_{\underline{i_pp^*}})^2 < 2v\delta(c^\sharp) - c^\sharp$, then a $c^\sharp$-link between any two spoke nodes in $P$ would increase the aggregate payoff. ■

Then we have the following characterizing result.

**Proposition 5** *The only efficient structures are the optimal complete network (described in Proposition 2), the optimal all-encompassing star (described in Proposition 3), the hybrid star-complete structures (described in Proposition 4) and the empty network.*
*(i) If $\delta'(0) > 1/2v$ and $c^\sharp < 2v(\delta(c^\sharp) - \delta(c_n^\times)^2)$ the only efficient structure is the optimal complete network.*
*(ii) If $\delta'(0) > 1/2v$ and $c^\sharp > 2v(\delta(c^\sharp) - \delta(c^\sharp)^2)$ the only efficient structure is the optimal all-encompassing star.*
*(iii) If $\delta'(0) > 1/2v$ and $2v(\delta(c^\sharp) - \delta(c_n^\times)^2) \leq c^\sharp \leq 2v(\delta(c^\sharp) - \delta(c^\sharp)^2)$ the only efficient structures are hybrid star-complete networks.*
*(iv) If $\delta'(0) \leq \frac{1}{2v(1+(n-2)\delta(\infty))}$ the only efficient structure is the empty network, while if $\frac{1}{2v(1+(n-2)\delta(\infty))} < \delta'(0) \leq 1/2v$ the only efficient structure for $n$ big enough is the optimal all-encompassing star.*

**Proof.** First note that any complete network is dominated either by the *optimal* complete network described in Proposition 2 if $\delta'(0) > 1/2v$, or by the empty network if $\delta'(0) \leq 1/2v$. Let $\bar{\mathbf{c}} = (c_{\underline{ij}})_{ij\in N_2}$ be a link-investment vector such that $\delta^{\bar{\mathbf{c}}}$ is connected, with a number of links $m$ s.t. $(n-1) \leq m \leq n(n-1)/2$ and positive aggregate payoff (if it were negative it would be dominated by the empty network). At the end of the proof the result is extended to any network, connected or not. We sketch roughly the idea of the proof before formalizing it precisely. First, prove that any connected network is dominated by one formed in the following way. Start by forming an all-encompassing star with the best $n-1$ links in $\overline{\mathbf{c}}$, and if no links remain stop (this only occurs if $m = n-1$, i.e. the starting network is a tree). Otherwise, take the weakest of the remaining available links and connect the two worst connected peripheral nodes in this star with it if this improves their connection, otherwise dispose of the link. Now repeat the procedure with a new of the weakest of the available links and a new pair of worst connected nodes in the network under construction up to some link improves it or no link remains. In the first case, improve the connection using the link and repeat the procedure for the pair of nodes worst connected and the new weakest available link, but with a difference: instead of disposing of the link if it does not improve the connection, replace the last one used to connect spokes of the star with it. At the end of this process, a hybrid star-complete structure which yields a greater or equal aggregate value as that of the initial network at less or equal cost arises. The second part of the proof consists of imposing optimality conditions in order to refine the set of dominant structures, showing that outside the interval specified in $(iii)$ the only efficient hybrid star-complete structure is one of the two extreme cases, the optimal complete network, or the optimal all-encompassing star network, while within this interval optimal hybrid star-complete networks are dominant structures.

We now formalize the procedure outlined. Starting from $\overline{\mathbf{c}}$, we describe an algorithm to construct a new link-investment vector that yields a dominant hybrid star-complete network, $\overline{\mathbf{c}}' = (c'_{\underline{ij}})_{\underline{ij} \in N_2}$, as the final outcome of a sequence of investment vectors $\overline{\mathbf{c}}'_1, \overline{\mathbf{c}}'_2, ...$, each of them resulting from the preceding one by adding at most one link.

Step 1: Let $\overline{\mathbf{c}}'_1$ be the star that results from connecting node 1 with the other $n-1$ investing in each link exactly the same amount invested in each of the best $n-1$ links in $\overline{\mathbf{c}}$ and s.t. $c'_{\underline{12}} \leq c'_{\underline{13}} \leq ... \leq c'_{\underline{1n-1}} \leq c'_{\underline{1n}}$. And let $\overline{\mathbf{c}}_1$ be the result of eliminating in $\overline{\mathbf{c}}$ the $n-1$ best links.

Step 2: From now on proceed as follows with the current $\overline{\mathbf{c}}'_t$ to form $\overline{\mathbf{c}}'_{t+1}$: choose two of the nodes worst connected for $\overline{\mathbf{c}}_t$ in the following way: nodes 2 and 3 in the first iteration, and in general, if $c'_{\underline{ij}} > 0$ with $i < j$ is the last added link, then:

- if $j < n$ check whether $i$ and $j+1$ connection via node 1 in $\overline{\mathbf{c}}'_t$ can or cannot be improved with the worst available link, say $c_{\underline{kl}}$, in $\overline{\mathbf{c}}_t$, i.e. check if $2v\delta(c'_{\underline{1j}})\delta(c'_{\underline{1j+1}}) < 2v\delta(c_{\underline{kl}}) - c_{\underline{kl}}$ and in case affirmative, make $c'_{\underline{1j+1}} := c_{\underline{kl}}$;

- if $j = n$ ($i$ has already been connected directly with all peripheral nodes), check whether $j+1$ and $j+2$ connection via node 1 in $\overline{\mathbf{c}}'_t$ can or cannot be improved with the worst available link in $\overline{\mathbf{c}}_t$, say $c_{\underline{kl}}$, i.e. if $2v\delta(c'_{\underline{1j+1}})\delta(c'_{\underline{1j+2}}) < 2v\delta(c_{\underline{kl}}) - c_{kl}$, and in case affirmative, make $c'_{\underline{j+1j+2}} := c_{\underline{kl}}$.

In both cases, make $\overline{\mathbf{c}}_{t+1}$ by eliminating $c_{\underline{kl}}$ in $\overline{\mathbf{c}}_t$, and $\overline{\mathbf{c}}'_{t+1}$ by adding the new link to $\overline{\mathbf{c}}'_t$, and go back to Step 2. Otherwise, i.e. if the worst available link in $\overline{\mathbf{c}}_t$ does not improve the connection, replace with it the last connection made in the preceding step, dispose of the replaced one, and go back to Step 2 with $\overline{\mathbf{c}}_{t+1}$ resulting by eliminating from $\overline{\mathbf{c}}_t$ the link disposed of, and $\overline{\mathbf{c}}'_{t+1} = \overline{\mathbf{c}}'_t$. Figure 2 shows the dynamic of the process to generate a dominant hybrid star-complete structure $\overline{\mathbf{c}}'$, and the aspect of the final outcome.

2 or 3 steps & outcome

Figure 2

Obviously the process ends in $1+m-(n-1)=m-n+2$ iterations, when $\overline{\mathbf{c}}_t=\varnothing$ and no links remain. Then, we show that if $\overline{\mathbf{c}}'=\overline{\mathbf{c}}'_{m-n+2}$, we have that $v(\delta^{\overline{\mathbf{c}}'})\geq v(\delta^{\overline{\mathbf{c}}})$. As both $\delta^{\overline{\mathbf{c}}}$ and $\delta^{\overline{\mathbf{c}}'}$ are connected, $v(\delta^{\overline{\mathbf{c}}})$ and $v(\delta^{\overline{\mathbf{c}}'})$ are the sum of $\frac{n(n-1)}{2}$ terms, one for each pair of nodes. Each of these terms corresponds to one pair of nodes and gives the contribution to the aggregate payoff of the value that they receive from each other (minus the cost of the link if they are directly connected). In the first step, by organizing the the $n-1$ best links as a star, they connect all nodes: $n-1$ pairs directly and all other pairs by the maximal number $(\frac{(n-1)(n-2)}{2})$ of two-link connections using these best links. From then on, by the way in which $\overline{\mathbf{c}}'$ has been formed, when an "available link" in $\overline{\mathbf{c}}$ (i.e. in $\overline{\mathbf{c}}_t$ at stage $t$) of cost $c$ is discarded this is because its direct contribution (i.e. term $2v\delta(c)-c$ in the sum that yields $v(\delta^{\overline{\mathbf{c}}})$) is *smaller or equal than* any term in $v(\delta^{\overline{\mathbf{c}}'_t})$. As after Step 1, once the initial star has been formed, links added to form $\overline{\mathbf{c}}'$ are increasingly strong, one discarded link will never be missed. As a result, the aggregate value of the resulting network cannot be smaller than that of the initial one, and by construction the outcome is a hybrid star-complete network. Thus any network is dominated by such an structure, which must satisfy the conditions established in Proposition 4 to be efficient. That is, all links but those forming the star are $c^\sharp$-links, and those forming the star must be s.t. (14), (15) and (16).

Now, assuming $\delta'(0)>1/2v$, there are the following possibilities:

$(i)$ If $c^\sharp<2v\delta(c^\sharp)-2v\delta(c_n^\times)^2$, i.e. $2v\delta(c_n^\times)^2<2v\delta(c^\sharp)-c^\sharp$, with $c_n^\times$ given by (12), then by connecting any two peripheral nodes in an all-encompassing star star satisfying optimality condition (12) with a $c^\sharp$-link the aggregate payoff would increase. But then the same must occur for any two spoke nodes in the star of a hybrid star-complete network satisfying conditions (14), (15) and (16), because

$$\delta(c_n^\times)>\delta(c_{\underline{i_p p^*}})>\delta(c_{\underline{i_m p^*}})>\delta(c_{\underline{m^* p^*}}),$$

given that

$$\delta'(c_n^\times)<\delta'(c_{\underline{i_p p^*}})<\delta'(c_{\underline{i_m p^*}})<\delta'(c_{\underline{m^* p^*}}).$$

Then it must be $L=N$. In other words, the only efficient structure is the optimal complete network .

$(ii)$ By Proposition 4-$(iii)$, if $l\geq 2$ then $c^\sharp\leq 2v(\delta(c^\sharp)-\delta(c^\sharp)\delta(c_{\underline{m^*p^*}}))\leq 2v(\delta(c^\sharp)-\delta(c^\sharp)^2)$, which contradicts $c^\sharp>2v(\delta(c^\sharp)-\delta(c^\sharp)^2)$. Therefore it must be $l<2$, i.e. $L=\{p^*\}$, which means that the hybrid star-complete is actually an all-encompassing star.

$(iii)$ If $2v(\delta(c^\sharp)-\delta(c_n^\times)^2)\leq c^\sharp\leq 2v(\delta(c^\sharp)-\delta(c^\sharp)^2)$, none of the two preceding conclusions applies, but some hybrid star-complete structure is sure to be efficient.

Nevertheless it is not possible to conclude in general terms which hybrid star-complete network becomes efficient.

*(iv)* Now assume $\delta'(0) \leq 1/2v$. By Lemma 2, if $\delta'(0) \leq \frac{1}{2v(1+(n-2)\delta(\infty))}$ the only efficient structure is the empty network, while if $\frac{1}{2v(1+(n-2)\delta(\infty))} < \delta'(0) \leq 1/2v$ the only efficient structure for $n$ big enough is the optimal all-encompassing star.

Now assume that the initial $\delta^{\overline{\mathbf{c}}}$ is not connected, but a component of a network with a greater number of nodes. It is clear that the conclusion applies to it, i.e. a component is dominated by one of these structures with the same number of nodes. Then, it is easy to check that if $\overline{\mathbf{c}}_n^{\sharp}$ and $\overline{\mathbf{c}}_n^{\times}$ denote the link-investment vectors for the optimal complete network and the optimal star of $n$ nodes, then $v(\delta^{\overline{\mathbf{c}}_n^{\sharp}}) < v(\delta^{\overline{\mathbf{c}}_{n+1}^{\sharp}})$, and $v(\delta^{\overline{\mathbf{c}}_n^{\times}}) < v(\delta^{\overline{\mathbf{c}}_{n+1}})$, where $\delta^{\overline{\mathbf{c}}_{n+1}}$ is the result of adding to $\delta^{\overline{\mathbf{c}}_n^{\times}}$ one node connected with the center by a link in which the investment is also $c_n^{\sharp}$. Therefore $v(\delta^{\overline{\mathbf{c}}_n^{\times}}) < v(\delta^{\overline{\mathbf{c}}_{n+1}}) < v(\delta^{\overline{\mathbf{c}}_{n+1}^{\times}})$. Similarly, the aggregate value of a properly hybrid star-complete $n$-network increases by adding one node connected with the center of the star by a link as good as the best one in it. Therefore efficiency implies connectedness. In this way, the proof is complete. ■

**Comments:** (i) The idea of the proof is interesting by its novelty, imposed by the novelty of the framework, and is the following. First, prove constructively that any link-investment vector/network that yields a positive aggregate payoff is dominated by a hybrid-star complete structure, a sort of suboptimal mixture of complete and star networks. The construction of the dominant structure from any given one is based on the idea of reorganizing or rearranging the "available links" in the network given in a most efficient way. Then this type of dominant hybrid structure is further refined imposing optimality conditions. This yields optimal hybrid star-complete structures (characterized in Proposition 4), which include as particular extreme cases the optimal complete and the optimal star networks. Finally, it is shown that among them only the two extremes, the optimal complete and the optimal star networks, can be efficient outside an interval relating $c^{\sharp}, c_n^{\times}, \delta(c^{\sharp})$ and $\delta(c_n^{\times})$, while within this interval is not possible to conclude in general terms which hybrid structure is optimal.

(ii) Again, in a setting that, compared to the basic link-formation models of Jackson and Wolinsky's (1996) connections' model and Bala and Goyal's (2000) two-way flow model, leaves many degrees of freedom, the only efficient structures in those models continue to be efficient. Moreover, in a setting where there are infinite complete and star structures, only a particular type of each of such structures with the highest degree of symmetry, as specified by Propositions 2 and 3, emerges as efficient. Nevertheless, a most relevant result is the emergence of some other interesting hybrid structures among the efficient ones.

(iii) Strictly speaking, the proof leaves unanswered the question of whether properly hybrid structures, i.e. different from the optimal all-encompassing star and complete are actually efficient for some technologies.[14] The following example shows that for

[14]In fact, these structures were thought as a way of "cornering" the presumably only efficient ones, i.e. the star and the complete. It was to our surprise that they turned out actually efficient for some

certain technologies neither the complete nor the star are efficient, and consequently a properly speaking hybrid star-complete must be efficient.

**Example:** Assume $n = 12, v = 1$ and a technology $\delta$ such that $c^\sharp = 0.25$; $\delta(c^\sharp) = 0.75$, $\delta'(c^\sharp) = 0.5$; and $c_{12}^\times = 3.5$, $\delta(c_{12}^\times) = 0.8$, and $\delta'(c_{12}^\times) = \frac{1}{18}$. The reader may check that these figures are consistent with the assumptions on $\delta$ of the model.[15]

Then we are in case (*iii*) in Proposition 5 because $c^\sharp = 0.25$ and

$$2(\delta(c^\sharp) - \delta(c_{12}^\times)^2) = 0.22 < 0.375 = 2(\delta(c^\sharp) - \delta(c^\sharp)^2).$$

So there is room for hybrid structures different from the star and the complete to be efficient. In fact we have:

$$\begin{array}{l} v(\mathbf{c}_{12}^\sharp) = \frac{n(n-1)}{2}(2v\delta(c^\sharp) - c^\sharp) = 49.5, \\ v(\mathbf{c}_{12}^\times) = 2v(n-1)\delta(c_{12}^\times) + v(n-1)(n-2)\delta(c_{12}^\times)^2 - (n-1)c_{12}^\times = 49.5, \end{array}$$

So, there is a *tie,* therefore if $c_{12}^\times < 3.5$: $v(\mathbf{c}_{12}^\times) > v(\mathbf{c}_{12}^\sharp)$; but if $c_{12}^\times > 3.5$: $v(\mathbf{c}_{12}^\times) < v(\mathbf{c}_{12}^\sharp)$.

Let $\mathbf{c}_{12}^{hsc}$ be an optimal hybrid star-complete network s.t. 11 nodes form a 10-link optimal star and the remaining node is connected with all the others by $c^\sharp$-links. Then we have:

$$v(\mathbf{c}_{12}^{hsc}) = v(\mathbf{c}_{11}^\times) + 11 \times 2(\delta(c^\sharp) - c^\sharp) = 20\delta(c_{11}^\times) + 90 \times \delta(c_{11}^\times)^2 - 10c_{11}^\times + 16.5.$$

Now assume $c_{11}^\times = 0.3$; $\delta(c_{11}^\times) = 0.77778$ and $\delta'(c_{11}^\times) = \frac{1}{16}$.[16] Then we have:

$$v(\mathbf{c}_{12}^{hsc}) = 83.5.$$

That is, network $\mathbf{c}_{12}^{hsc}$ beats *both* the 12-star and the 12-complete! This does not mean that this particular hybrid star-complete network is efficient, but the counterexample proves that actually only properly hybrid star-complete networks are efficient for some technologies.

(iv) An interesting feature of the *optimal* hybrid star-complete networks is the different levels of information received by different nodes in these structures. As has been show in Proposition 4, the links that form the star may receive up to 3 different levels of investment. By comparing (14), (15) and (16) it is clear that $\delta'(c_{\underline{ip}p^*}) < \delta'(c_{\underline{im}p^*}) < \delta'(c_{\underline{m^*p^*}})$, that is

$$c_{\underline{ip}p^*} > c_{\underline{im}p^*} > c_{\underline{m^*p^*}}.$$

The weakest among these links is the one connecting $m^*$ and the center, the next level is for the links connecting those in $M$ and the center, and the highest level of

technologies.

[15] Note that $\delta'(c_{12}^\times) = \frac{1}{18} = \frac{1}{2(1+10\times 0.8)} = \frac{1}{2(1+(n-2)\delta(c_{12}^\times))}$.

[16] Note that $\frac{1}{2(1+9\times\delta(c_{11}^\times))} = \frac{1}{16}$ yields $\delta(c_{11}^\times) = 0.77778$, thus everything is consistent if $\delta'(c_{11}^\times) = 1/16$, because $1/18 < 1/16 < 1/2$.

investment is for the links connecting the remaining spoke nodes (i.e. those in $P$) with the center. In view of these investments, it is easy to conclude that $p^*$ receives the highest amount of information, followed by nodes in $P$, then those in $M$ followed by node $m^*$, and finally those in $L$. If $M = \varnothing$, there are only 3 levels because all spoke nodes in the star are in the same conditions, which become 2 levels if $L = \varnothing$, and only one level if $L = N$.

# 5 Scenario 2: Stability

We now consider the second scenario described in Section 3, where nodes are *players* who form the links by investing in them. An *investment profile*, specified by a matrix $\mathbf{c} = (c_{ij})_{i,j \in N}$, where $c_{ij} \geq 0$ (with $c_{ii} = 0$) is the investment of player $i$ in the link connecting $i$ and $j$, determines a link-investment vector $\overline{\mathbf{c}}$

$$\mathbf{c} \to \overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2} \text{ s.t. } c_{\underline{ij}} := c_{ij} + c_{ji}.$$

The available link-formation technology, $\delta$, yields a *weighted network* for each investment profile $\mathbf{c}$. Namely, $\delta^{\mathbf{c}} := \delta^{\overline{\mathbf{c}}}$, where

$$\delta^{\mathbf{c}}_{\underline{ij}} = \delta^{\overline{\mathbf{c}}}_{\underline{ij}} = \delta(c_{\underline{ij}}) = \delta(c_{ij} + c_{ji}),$$

and payoffs are given by (2), i.e.

$$\Pi_i^{\delta}(\mathbf{c}) = \sum_{j \in N(i;\delta^{\overline{\mathbf{c}}})} I_{ij}(\delta^{\overline{\mathbf{c}}}) - \sum_{j \in N^d(i,\delta^{\mathbf{c}})} c_{ij} = v \sum_{j \in N(i;\delta^{\overline{\mathbf{c}}})} \delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) - \sum_{j \in N^d(i,\delta^{\mathbf{c}})} c_{ij}. \qquad (22)$$

This scenario poses the question of stability. We address here the stability of the structures that have emerged as efficient: the empty network, the complete network, the star and the hybrid star-complete networks. Although properly speaking one should refer to *stability of investment profiles*, we often express our results in terms of the resulting networks. Thus a Nash or pairwise Nash "stable network" should be read as a *weighted network that results from a Nash or pairwise Nash stable investment profile*. The following result establishes a necessary and sufficient condition for the *empty network* (which results from no investments) to be stable.

**Proposition 6** *The empty network is*
*(i) A Nash network if and only if* $\delta'(0) \leq 1/v$.
*(ii) A pairwise Nash stable network if and only if* $\delta'(0) \leq 1/2v$.

**Proof.** *(i)* Let $\delta^{\mathbf{0}}$ be the empty network, which results when no link is invested in, i.e. $c_{ij} = 0$ for all $i, j \in N$. In these conditions a player has an incentive to invest $c > 0$ in a link with another (or any number of them) only if $v\delta(c) - c > 0$. But under conditions C.1 and C.2, if $\delta'(0) \leq 1/v$ and $c > 0$, then $\delta(c) < c\delta'(0) \leq c/v$. Assume now that $\delta'(0) > 1/v$. Then, there exists $c > 0$ s.t. $\delta(c) > c/v$, and it is advantageous to invest $c$ in a link with another player. Therefore $(i)$ is proved.

$(ii)$ If pairwise coordination is feasible two players may form a link by jointly investing $c$ by investing $c/2$ each. If $\delta'(0) \leq 1/2v$, then $\delta(c) < c\delta'(0) \leq c/2v$. On the contrary, if $\delta'(0) > 1/2v$, then there exists $c > 0$ s.t. $\delta(c) > c/2v$, and it is advantageous to any two players to invest $c/2$ each in a link connecting them. ■

**Comments:** (i) Thus, it all depends on the technology $\delta$ and $v$, namely on the marginal fidelity-level w.r.t. the first unit invested: the empty network is Nash (pairwise Nash) stable if and only if this marginal fidelity-level is equal or less than $1/v$ $(1/2v)$. In other words, the greater the value of the information at each player, the smaller this marginal fidelity-level must be for the empty network to be stable. When pairwise coordination is feasible stability is more demanding.

(ii) Thus the empty network is Nash stable being surely inefficient $(1/2v < \delta'(0) \leq 1/v)$ or possibly inefficient $(\delta'(0) \leq 1/2v)$, and is pairwise Nash stable only in the latter case.

In order to address the stability of complete and star networks we establish a result very similar to Lemma 1, establishing necessary conditions for an investment profile to be "stable for a given infrastructure" in the following sense.

**Definition 5** *A link-investment vector $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij}\in N_2}$ is Nash (pairwise Nash) stable for an infrastructure $S$ if for all $\underline{ij} \in N_2$, $c_{\underline{ij}} > 0$ if and only if $\underline{ij} \in S$, and it is Nash (pairwise Nash) stable in the strategic network formation game that results when the strategies of players are constrained to invest only in links in $S$.*

**Lemma 3** *For an investment profile $\mathbf{c} = (c_{ij})_{i,j\in N}$ to be stable for a given infrastructure $S \subseteq N_2$, the following are necessary conditions: $(i)$ If a player invests in two different links, the sets of players connected by optimal paths containing each of them are disjoint. $(ii)$ For all $\underline{ij} \in S$ s.t. $c_{ij} > 0$:*

$$\delta'(c_{\underline{ij}}) = \frac{1}{v\sum_{k\in N(i;\delta^{\overline{\mathbf{c}}})(\underline{ij}\in\overline{p}_{ik})}\delta(\overline{p}_{ik}^{ij})}. \tag{23}$$

*$(iii)$ For all $\underline{ij} \in S$ s.t. $c_{\underline{ij}} > 0$ and $c_{ij} = 0$:*

$$\delta'(c_{\underline{ij}}) \leq \frac{1}{v\sum_{k\in N(i;\delta^{\overline{\mathbf{c}}})(\underline{ij}\in\overline{p}_{ik})}\delta(\overline{p}_{ik}^{ij})}. \tag{24}$$

**Proof.** Let $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij}\in N_2}$ be the link-investment vector associated with investment profile $\mathbf{c}$, s.t. $c_{\underline{ij}} > 0$ if and only if $\underline{ij} \in S$. We prove first part $(ii)$. Assume $\mathbf{c}$ to be Nash stable for $S$, and $\underline{ij} \in S$. Then $i$ and/or $j$, at least one of them, say $i$, invests $c_{ij} > 0$. Then link $\underline{ij}$ is part of at least one optimal path in $\delta^{\overline{\mathbf{c}}}$ for $i$'s information (the one connecting $i$ and $j$, otherwise $i$ would withdraw support to it). Then, in any of the possibly different but equivalent expressions of the right-hand side of (22), $\delta(c_{\underline{ij}})$ would appear at least once ($\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) = \delta(c_{\underline{ij}})$ if $\mathbf{c}$ is stable), and possibly also in the product yielding $\delta^{\overline{\mathbf{c}}}(\overline{p}_{ik})$ for other of nodes $k$. Fix any choice of these (possibly multiple) optimal paths connecting $i$ with every other node with whom $i$ is connected and let $i$'s payoff be given by the right-hand side of (22). The right-hand side is an up to $n(n-1)$-variable function with partial derivatives. A non–null partial derivative w.r.t. $c_{ij}$ of this expression means that by slightly increasing (if it were $> 0$) or decreasing (if it were $< 0$) the investment of $i$ in link $\underline{ij}$ would increase $i$'s payoff (through the same available paths), which contradicts $\mathbf{c}$'s stability for $S$. Then the partial derivative of the right-hand side of (22) w.r.t. $c_{ij}$ must be 0, i.e., using the same notation as in Section 4,

$$\frac{\partial}{\partial c_{ij}}(v \sum_{k \in N(i;\delta^{\overline{\mathbf{c}}})} \delta^{\overline{\mathbf{c}}}(\overline{p}_{ik}) - \sum_{k \in N^d(i,\delta^{\mathbf{c}})} c_{ik}) = v\delta'(c_{\underline{ij}}) \sum_{k \in N(i;\delta^{\overline{\mathbf{c}}})(\underline{ij} \in \overline{p}_{ik})} \delta(\overline{p}_{ik}^{ij}) - 1 = 0,$$

which yields (23).[17]

($i$) Assume that player $i$ invests in links with two nodes $j$ and $k$, $c_{ij} > 0$ and $c_{ik} > 0$, and for some player $l$ there are two different optimal paths $\overline{p}_{il}$ and $\overline{p}'_{il}$ such that $\underline{ij} \in \overline{p}_{il}$ and $\underline{ik} \in \overline{p}'_{il}$. Then the right-hand side of (23) admits at least *two* different expressions where the optimal path connecting $i$ and any other player but $l$ is the same, where one uses $\overline{p}_{il}$ and another uses $\overline{p}'_{il}$. In that case, (23) yields two different values for $\delta'(c_{\underline{ij}})$, which is a contradiction.

($iii$) Assume now that $c_{\underline{ij}} > 0$ and $c_{ij} = 0$. A similar argument to the one used to prove part ($ii$) leads in this case to the conclusion that

$$v\delta'(c_{\underline{ij}}) \sum_{k \in N(i;\delta^{\overline{\mathbf{c}}})(\underline{ij} \in \overline{p}_{ik})} \delta(\overline{p}_{ik}^{ij}) - 1 \leq 0,$$

otherwise player $i$ would have an incentive to invest in link $\underline{ij}$, which yields (24). ■

As in the case efficiency, we have an immediate consequence for stability in general:

**Proposition 7** *For an investment profile $\mathbf{c} = (c_{ij})_{i,j \in N}$ to be Nash stable the following conditions are necessary: ($i$) If a player invests in two different links, the sets of players connected by optimal paths that contain each of them are disjoint.*
*($ii$) For each $\underline{ij} \in N_2$ s.t. $c_{ij} > 0$ :*

$$\delta'(c_{\underline{ij}}) = \frac{1}{v \sum_{k \in N(i;\delta^{\overline{\mathbf{c}}})(\underline{ij} \in \overline{p}_{ik})} \delta(\overline{p}_{ik}^{ij})}.$$

[17] Just note that by the chain rule

$$\frac{\partial}{\partial c_{ij}}(\delta(c_{ij} + c_{ji})) = \delta'(c_{ij} + c_{ji}) \cdot 1 = \delta'(c_{\underline{ij}}).$$

(*iii*) *For each* $\underline{ij} \in N_2$ *s.t.* $c_{\underline{ij}} > 0$ *and* $c_{ij} = 0$ *:*

$$\delta'(c_{\underline{ij}}) \leq \frac{1}{v \sum_{k \in N(i;\delta^{\overline{\mathbf{c}}})(\underline{ij} \in \overline{p}_{ik})} \delta(\overline{p}_{ik}^{ij})}.$$

(*iv*) *For each* $\underline{ij} \in N_2$ *s.t.* $c_{\underline{ij}} = 0$*, if* $\delta'(0) > 1/v$*,*

$$v\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) \geq v\delta(c^{\#}) - c^{\#}, \tag{25}$$

*where* $c^{\#} = \arg\max_{c>0}(v\delta(c) - c)$.
(*v*) *To be pairwise Nash stable condition* (25) *must be replaced by*

$$2v\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) \geq 2v\delta(c^{\sharp}) - c^{\sharp}, \tag{26}$$

*where* $c^{\sharp}$ *is given by* (8) *or* (11).

**Proof.** Assume $\mathbf{c} = (c_{ij})_{i,j \in N}$ to be Nash stable, then $\mathbf{c}$ must be Nash stable for the infrastructure $S = \{\underline{ij} \in N_2 : c_{\underline{ij}} > 0\}$. Then (*i*), (*ii*) and (*iii*) follow immediately from Lemma 3.

(*iv*) Assume $c_{\underline{ij}} = 0$. Then no investment in link $\underline{ij}$ from $i$ can increase $i$'s payoff, that is, for all $c > 0$, $v\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) \geq v\delta(c) - c$, otherwise investing $c$ in link $\underline{ij}$ would surely increase $i$'s payoff. This yields condition (25). Note that if $\delta'(0) \leq 1/v$, then $v\delta(c) - c < 0$ for all $c > 0$.

(*v*) If pairwise coordination is feasible and $2v\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) < 2v\delta(c^{\sharp}) - c^{\sharp}$, players $i$ and $j$ have an incentive to invest $c/2$ each on link $\underline{ij}$. ■

**Comment:** Part (*i*) establishes that what any player "sees" through different links in which he/she invests do not overlap: if $i$ sees $l$ through an optimal path that contains $\underline{ij}$, it cannot be the case that $i$ sees $l$ through another optimal path that contains $\underline{ik} \neq \underline{ij}$. Note the similarity and the *difference* with part (*i*) in Proposition 1. As it occurs in Proposition 1, (*i*) is a consequence of (*ii*), which also here has a clear interpretation. If player $i$ invests in a link with $j$, the denominator of the fraction in formula (23) that yields $\delta'(c_{\underline{ij}})$ is $v$ times the sum of the fidelity-levels through all optimal paths containing link $\underline{ij}$ (discounting that of link $ij$) through which player $i$ receives information. In other words, the actual amount of information that reaches $j$ on its optimal way to $i$. Thus this sum is a measure of the importance of link $\underline{ij}$ *to player* $i$: the greater this amount, the smaller $\delta'(c_{\underline{ij}})$, i.e. the greater $c_{ij}$ and $\delta(c_{\underline{ij}})$. Note here the similarity and difference with the meaning of similar expression (4) for an efficient link-investment vector: there it was the overall importance of the link for the flow of information, while here it is the importance for a player who invests in it. Parts (*iii*) and (*iv*) refer to links not invested in and impose necessary conditions for Nash and pairwise Nash stability, which amount to lack of incentives to invest in them when coordination is or not unfeasible.

The following corollary establishes that in a Nash network links are entirely supported by the player who benefits most from the link, or shared in any way if both benefit equally.

**Corollary 1** *If two players* $i$ *and* $j$ *are connected by a link in the network resulting from a Nash investment profile* $\mathbf{c}$*, then if both players benefit equally from the link any way of sharing the investment* $c_{\underline{ij}}$ *s.t. is consistent with equilibrium; otherwise, the investment is made entirely by the player who benefits the most from the existence of the link.*

**Proof.** Let $\mathbf{c}$ be a Nash investment profile and assume $c_{\underline{ij}} > 0$. Then if both invest in link $\underline{ij}$, condition (23) must hold for $i$ and $j$, which are compatible only if the denominator in the right-hand side of equation (23) are equal for $i$ and $j$. In other words, only if both players benefit equally from the existence of the link. If they benefit differently from it, both conditions are incompatible, and equilibrium is possible only if the player who benefits the most covers entirely the investment. In this way conditions (23) and (24) hold. ■

We now address the question of the stability of *complete networks*. The following proposition gives necessary and sufficient conditions for a complete network to be stable.

**Proposition 8** *Let* $\mathbf{c}$ *be an investment profile such that* $\delta^{\mathbf{c}}$ *is complete, then* $\delta^{\mathbf{c}}$ *is a Nash network* (*and pairwise Nash*) *if and only if the following conditions hold:*
(*i*) *The marginal fidelity-level w.r.t. investment at* 0 *must be greater than* $1/v$*, i.e.*

$$\delta'(0) > 1/v. \tag{27}$$

(*ii*) *All links receive the same joint investment, i.e.* $c_{ij} + c_{ij} = c^{\#} > 0$*, for all* $i \neq j$*, given by*

$$\delta'(c^{\#}) = 1/v. \tag{28}$$

(*iii*) *The only* $c^{\#}$ *that satisfies* (28) *is such that*

$$c^{\#} \leq 2v(\delta(c^{\#}) - \delta(c^{\#})^2). \tag{29}$$

(*iv*) *For all* $i, j$ $(i \neq j)$ :

$$c_{ij} \leq v(\delta(c^{\#}) - \delta(c^{\#})^2). \tag{30}$$

**Proof.** (*i*) Assume $\delta^{\mathbf{c}}$ is complete, i.e. $c_{ij} + c_{ji} > 0$ for all $i, j \in N$ $(i \neq j)$. In other words, for all $i, j$ at least one of the two players invests in the link, say $i$. Then, as all links work, a necessary condition for a player $i$ not to have an incentive to withdraw its investment $c_{ij} > 0$ in link $\underline{ij}$ is that $i$ sees $j$ strictly better through it than any other way. In other words, each link itself is the only optimal path it belongs to. Then, as $\delta(\overline{p}_{ij}) = \delta(c_{\underline{ij}})$, $\delta(\overline{p}_{ij}^{\,-ij}) = 1$, and (23) yields

$$\delta'(c_{\underline{ij}}) = \frac{1}{v}.$$

But, by C.1 and C.2, only if $\delta'(0) > 1/v$ there exists $c^{\#} > 0$ s.t. $\delta'\left(c^{\#}\right) = 1/v$, which in that case is *unique*. Note that pairwise Nash stability does not actually refine Nash stability for the complete network because all links exist and pairwise coordination does not actually provide new options.

($ii$) Therefore, in equilibrium all links must receive the same joint investment $c^{\#}$ s.t. $\delta'\left(c^{\#}\right) = 1/v$ and consequently support the same fidelity-level. Note that whatever the investments $c_{ij} > 0$ and $c_{ji} > 0$, as long as $c_{ij} + c_{ji} = c^{\#}$, none of the two players has an incentive to modify his/her investment in that link.

($iii$) & ($iv$) Finally, for $\delta^{\mathbf{c}}$ to be Nash stable no player must have an incentive to withdraw support to a link, i.e. for all $i, j$ $(i \neq j) : \delta(c^{\#})v - c_{ij} \geq \delta(c^{\#})^2 v$, which yields (30). Which is compatible with $c_{ij} + c_{ji} = c^{\#}$ if and only if (29). ■

**Comments:** (i) Condition ($i$) only concerns the technology $\delta$ and $v$, the number of players does not enter the conclusion. The marginal fidelity-level w.r.t. the first unit invested must be greater than $1/v$, which is a necessary condition for the existence of $c^{\#}$ such that (28) holds. The same applies to pairwise Nash stability as it does not actually refine Nash stability for the complete network given that all links exist and pairwise coordination does not actually provide new options. Note that (27) is the complementary of the necessary and sufficient condition for the empty network to be Nash stable according to Proposition 6. Condition ($ii$) establishes that, as a result of technological homogeneity, all links must receive the same joint support for a complete network to be Nash stable, and (28) is required for the joint investment $c^{\#}$ to be optimal. The existence of such a $c^{\#}$ is guaranteed by condition ($i$). Condition ($iii$) is a condition which involves $v$, the technology $\delta$ and the optimal investment $c^{\#}$, and it is necessary and sufficient for condition ($iv$) to be feasible. Condition ($iv$) is required because otherwise player $i$ would have an incentive to withdraw support from the link with $j$. Note that the first three conditions concern $\delta$, $\delta(c^{\#})$ and $v$, while only ($iv$) concerns directly the way in which the cost of the link is shared by setting an upper bound to the investment of each player. That is, if conditions ($i$)-($iii$) hold, complete networks supported by any investment profile satisfying ($iv$) are Nash stable. As condition ($i$) guarantees the existence of a unique $c^{\#}$ s.t. (28) holds, the stability of complete networks hinges upon condition (29).

(ii) A comparison of conditions (27), (28) and (29) for stability (Proposition 8) with conditions (7), (8) and (9) for efficiency (Proposition 2) is pertinent here. In the first case ($\delta'(0) > 1/v$ vs. $\delta'(0) > 1/2v$), the condition on the technology for stability of the complete network is more demanding. But even when both conditions hold, from $\delta'(c^{\#}) = 1/v$ vs. $\delta'(c^{\sharp}) = 1/2v$, the conclusion is clear: the optimal complete network is *not* stable. The reason is clear: as $\delta'(c^{\sharp}) = \delta'(c^{\#})/2$, then $c^{\sharp} > c^{\#}$, i.e. efficiency entails a joint investment in each link greater than what stability requires, and consequently in a complete efficient network players have an incentive to diminish their support to links and take advantage of the externalities (inexistent in the optimal complete) that free-riding generates. Finally, in the third case, the conditions on $c^{\#}$ and $c^{\sharp}$ respectively are entirely similar.

We now address the question of stability of *stars*. The following proposition gives necessary and sufficient conditions for a star to be Nash stable.

**Proposition 9** *Let* $\mathbf{c}$ *be an investment profile such that* $\delta^{\mathbf{c}}$ *is an all-encompassing star, then* $\delta^{\mathbf{c}}$ *is a Nash network if and only if (i)* $\delta^{\mathbf{c}}$ *is a periphery-sponsored star where all peripheral players invest the same amount* $c_n^*$ *in the only link in which each of them is involved given by:*

$$\delta'(c_n^*) = \frac{1}{v(1 + (n-2)\delta(c_n^*))}. \tag{31}$$

*(ii) Additionally, if* $\delta'(0) > 1/v$,

$$v\delta(c_n^*)^2 \geq v\delta(c^{\#}) - c^{\#}, \tag{32}$$

*with* $c^{\#}$ *given by* (28).
*(iii)* $\delta^{\mathbf{c}}$ *is a pairwise Nash network if and only if* (31) *and*

$$2v\delta(c_n^*)^2 \geq \max_{c>0}(2v\delta(c) - c) = 2v\delta(c^{\sharp}) - c^{\sharp}. \tag{33}$$

**Proof.** $(i)$ That a Nash stable all-encompassing star must be periphery-sponsored is directly concluded from Corollary 1: in a star peripheral players benefit more than the centre from their links. Now assume that two peripheral players, $i$ and $j$, support their links with different investments, say, $c_{ik} < c_{jk}$, where $k$ is the center, which entails $\delta_{ik}^{\mathbf{c}} < \delta_{jk}^{\mathbf{c}}$. By Lemma 3, it must hold $\delta'(c_{ik}) = \frac{1}{v+I_k(\delta^{\mathbf{c}}-\underline{ik})}$, and $\delta'(c_{jk}) = \frac{1}{v+I_k(\delta^{\mathbf{c}}-\underline{jk})}$. But note that

$$I_k(\delta^{\mathbf{c}} - \underline{ik}) - I_k(\delta^{\mathbf{c}} - \underline{jk}) = (\delta_{\underline{jk}}^{\mathbf{c}} - \delta_{\underline{ik}}^{\mathbf{c}})v > 0,$$

i.e. $I_k(\delta^{\mathbf{c}}-\underline{ik}) > I_k(\delta^{\mathbf{c}}-\underline{jk})$, but by this implies $\delta'(c_{ik}) < \delta'(c_{jk})$, which entails $c_{ik} > c_{jk}$, by the assumptions on function $\delta$, which contradicts the initial assumption $c_{ik} < c_{jk}$. Thus, all peripheral players must invest the same amount in their links. This amount $c_n^*$ is determined by (23):

$$\delta'(c_n^*) = \frac{1}{v\sum_{k\in N(i;\delta^{\overline{\mathbf{c}}})(\underline{ij}\in\overline{p}_{ik})}\delta(\overline{p}_{ik}^{ij})} = \frac{1}{v(1 + (n-2)\delta(c_n^*))}.$$

$(ii)$ For any two peripheral players $\delta^{\overline{\mathbf{c}}}(\overline{p}_{ij}) = \delta(c_n^*)^2$ and condition (25) becomes (32).

Remains only to be checked that it is worth for any peripheral player to invest $c^*$, i.e. that

$$c_n^* \leq \delta(c_n^*)(1 + (n-2)\delta(c_n^*))v.$$

But if (31) holds, this is equivalent to check that $c_n^* \leq \delta(c_n^*)/\delta'(c_n^*)$ or, equivalently that

$$\delta'(c_n^*) \leq \delta(c_n^*)/c_n^*$$

which follows from the smoothness and concavity of $\delta$.

$(iii)$ Assuming pairwise coordination feasible, condition (26) becomes (33). ■

**Comments:** (i) That only periphery-sponsored stars can be Nash stable is a direct consequence of Corollary 1. Symmetry, i.e. that all peripheral players must invest the same in equilibrium, follows from Lemma 3 and the assumptions about technology $\delta$. Condition (31) is then necessary for an $n$-player periphery-sponsored star where each peripheral player invests $c_n^*$ to be Nash stable, otherwise there would be an incentive to increase or decrease the investment. But this is not sufficient. If $c_n^*$ is the investment of each peripheral player in its link with the center s.t. (31) condition holds, condition (32) sets a lower bound for $c_n^*$ which ensures that no peripheral player has an incentive to invest in a link with any other peripheral player. Note that, like for the empty network and unlike for the complete one, this lower bound is strictly more demanding for pairwise Nash stability.

(ii) A comparison of conditions (31), (32) and (33) for stability with those for efficiency for the same structure (Proposition 3) yields the following conclusions. As it occurs with the complete network, the optimal star network is not stable: from the conditions that determine $\delta'(c_n^*)$ and $\delta'(c_n^\times)$, it follows that $\delta'(c_n^*) > \delta'(c_n^\times)$, which implies $c_n^* < c_n^\times$. In other words, the optimal star is not stable because efficient investments would invite to free-riding. As to the lower bounds for $\delta(c_n^*)^2$ for Nash and pairwise Nash stability, the first one is stronger than (13), but latter $(2v\delta(c_n^*)^2 \geq \max_{c>0}(2v\delta(c) - c))$ is identical.

Remains to be discussed the question of existence of $c_n^*$ satisfying conditions (31) and (32). The following lemma, entirely similar to Lemma 2 and whose proof is omitted for this reason, gives a sufficient condition for (31).

**Lemma 4** *Whatever the number of nodes, if $\delta'(0) > 1/v$, then it is sure to exist $c_n^*$ such that (31) holds, and also when $\delta'(0) \leq 1/v$ for $n$ sufficiently large. On the contrary, for a fixed $n$, no such $c_n^*$ exists if $\delta'(0) \leq \frac{1}{v(1+(n-2)\delta(\infty))}$, where $\delta(\infty)$ denotes $\lim_{c\to\infty}\delta(c)$.*

**Proposition 10** *The periphery-sponsored star is the only non-empty possibly Nash stable network without cycles.*

**Proof.** Assume that $\delta^{\mathbf{c}}$ is a non-empty Nash stable network without cycles in which there are two peripheral players, $i$ and $j$, at a distance greater than 2. By Corollary 1, both players must support their only link. It is then easy to see that if $\Pi_i^\delta(\mathbf{c}) \leq \Pi_j^\delta(\mathbf{c})$, then $i$'s payoff increases by switching $i$'s investment from his only link to a link with the same player with whom $j$ supports a link. ■

**Example 2** (an example here where both star an complete are stable)

Remains to be discussed the stability of *properly* hybrid star-complete networks. Let $N$, $L$, $M$, $P$, $m^*$ and $p^*$ be as in Definition 4 and Figure 1, but let us forget the optimal investments there and let us deduce them from (23) by imposing stability. First, note that all links connecting nodes in $L$ among themselves, nodes in $L \setminus \{p^*\}$ with nodes in $N \setminus L$, and also the links connecting $m^*$ with each node in $M$, generate no externalities because they are used for each pair of players involved only to see each

other. Therefore, in view of (23), all these links must receive an investment $c^{\#}$, s.t. $\delta'(c^{\#}) = 1/v$. As to the nodes in $N \setminus L$, as they see each other through the star they form, it is clear that stability imposes that they must support entirely the investment in their links with the center $p^*$. Again by applying (23), if $i_m$ and $i_p$ denote a generic node in $M$ and $P$ respectively, we conclude that:[18]
- $m^*$ sees through link $\underline{m^*p^*}$ only $p^*$ and nodes in $P$, therefore:

$$\delta'(c_{m^*p^*}) = \delta'(c_{\underline{m^*p^*}}) = \frac{1}{v(1 + p\delta(c_{\underline{i_p p^*}}))};$$

- each $i_m \in M$ sees $p^*$ through link $\underline{i_m p^*}$, the other nodes in $M$ (i.e. $M \setminus \{i_m\}$) and all nodes in $P$, therefore:

$$\delta'(c_{i_m p^*}) = \delta'(c_{\underline{i_m p^*}}) = \frac{1}{v(1 + (m-1)\delta(c_{\underline{i_m p^*}}) + p\delta(c_{\underline{i_p p^*}}))};$$

- each $i_p \in P$ sees through link $\underline{i_p p^*}$ the other nodes in $P$ (i.e. $P \setminus \{i_p\}$), all nodes in $M$ and $p^*$, therefore:

$$\delta'(c_{i_p p^*}) = \delta'(c_{\underline{i_p p^*}}) = \frac{1}{v(1 + (p-1)\delta(c_{\underline{i_p p^*}}) + m\delta(c_{\underline{i_m p^*}}) + \delta(c_{\underline{m^*p^*}}))}.$$

Additionally, there are further necessary conditions:
- for all $i, j \in L$ $(i \neq j) : c_{ij} \leq v(\delta(c^{\#}) - \delta(c^{\#})^2)$, which is compatible with $c_{ij} + c_{ji} = c^{\#}$ if and only if $c^{\#} \leq 2v(\delta(c^{\#}) - \delta(c^{\#})^2)$.
- for links connecting $m^*$ and any $i_m \in M$, $2v\delta(c_{m^*p^*})\delta(c_{i_m p^*}) \leq 2v\delta(c^{\#}) - c^{\#}$ (note that this condition implies the preceding one).

These conditions impose specific constraints for a hybrid star-complete network to be stable very similar to those imposed by efficiency.

# 6 Concluding remarks

We have introduced a simple "marginalist" model of network formation that is a natural extension of the seminal models of Jackson and Wolinsky (1996) and Bala and Goyal (2000a). A full characterization of efficient structures has been achieved in this setting. An interesting result is the emergence of hybrid star-complete structures among the efficient ones in addition to the extreme cases of such structures, complete networks and stars, and the empty network. Necessary conditions for Nash and pairwise Nash stability have been obtained, which permit to give necessary and sufficient conditions for complete networks and stars to be stable in either sense. These conditions show

[18] An entirely similar argument to the one used in Proposition 4 leading to the conclusion that all links of each of the three types must receive the same invetments for optimality, leads to the same conclusion for stability, and is omitted.

that hybrid structures cannot be stable, and that efficient complete and stars networks are not stable.

Apart from further exploring the model, there are a number of extensions worth investigating. These are some of them:

(i) Exploring the impact of assuming heterogeneity, technological and/or in individual values.

(ii) Exploring some variants of the technology function, as the following ones:

- Assuming $\delta : \mathbb{R}_+ \to [0, \bar{\delta})$ with $\bar{\delta} < 1$, i.e. setting an upper bound $\bar{\delta}$ lower than 1 for $\delta(c)$.

- Assuming $\delta(c) > 0$ only for $c > \bar{c}$, i.e. setting a "threshold" or minimal joint investment for a link to admit flow (translating the assumptions about $\delta$ to a map $\delta : [\bar{c}, \infty) \to [0, 1)$, with $\delta(c) = 0$ for $0 \leq c \leq \bar{c}$.)

- Assuming $\delta : \mathbb{R}_+ \to (-\alpha, 1)$, including the option to punish neighbors (this possibly makes other changes in the model necessary).

- Assuming $\delta$ first convex up to an inflection point, then concave: this is intuitively appealing and would yield as limiting cases Jackson and Wolinsky's (1996) connections model and Bala and Goyal's (2000a) two-way flow model.

(iii) Enriching the model, the basic one or any of its extensions, introducing some dynamics.

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