Robust Selling Mechanisms*

Vinicius Carrasco, Vitor Farinha Luz, Paulo Monteiro and Humberto Moreira†

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Abstract

We consider the problem of a seller who faces a privately informed buyer and only knows one arbitrary moment of the distribution from which valuations are drawn. In face of this uncertainty, the seller maximizes his worst-case expected profits. Insurance against uncertainty takes a simple form. Conditional on sales, the seller's ex-post profits are an affine transformation of the known moment. We use this restriction imposed by robustness on the seller's payoffs to derive the optimal mechanism. It entails distortions at the intensive margin, e.g., except for the highest buyer's valuation, sales will take place with probability strictly smaller than one. The seller can implement such allocation by committing to post prices drawn from a non-degenerate distribution. We extend the model to deal with the case in which multiple goods are sold and the buyer's private information is multidimensional. Selling the goods in a fully separable way is always optimal in the multidimensional screening problem. For the special case in which the buyer's expected values for each of the M goods are the same in the multidimensional problem, selling all goods in fixed proportions in a bundle is also optimal.

Keywords: Robust Mechanism Design, Pricing under Uncertainty, Price Discrimination, Multidimensional Screening.

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†Carrasco: Department of Economics, PUC-Rio, vncarrasco@gmail.com. Luz: The University of British Columbia, vitor.farinhaluz@ubc.ca. Monteiro: FGV/EPGE, pklm@fgv.br Moreira: FGV/EPGE, humberto@fgv.br.

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Contents

1 Introduction

This paper considers the problem faced by a monopolist or seller (he), who – except for knowing one moment of the distribution – is unaware of the distribution from which the buyer's (she) valuation is drawn. In face of such ambiguity, he seeks to design a mechanism that is robust, in the sense of maximizing his worst-case expected profits over all distributions satisfying the moment condition. The analysis encompasses the cases in which $M > 1$ goods are sold (therefore, valuation can be multidimensional).

Two sets of results pervade the analysis. First, the seller's hedge against uncertainty takes a very simple form. Suppose the buyer's valuation v for a good lies in the interval $[0, 1]$ and the only information about the cumulative distribution F held by the seller is the moment $\int \kappa(v)\, dF(v)$, for some non-decreasing function $\kappa(\cdot)$. The optimal robust mechanism leads to ex-post profits which are, conditional on sales, an affine transformation of function $\kappa(\cdot)$. If, for example, the seller knows the $n-th$ moment of the buyer's valuation, so that $\kappa(v) = v^n$, the optimal mechanism will induce ex-post profits which are linear in v^n for the states in which sales take place. Second, a strong form of separability ensues for the multidimensional screening analysis. Selling the goods in a fully separable way is always optimal in the multidimensional screening problem. For the special case in which the buyer's expected values for each of the M goods are the same in the multidimensional problem, bundling all goods in fixed proportions is also optimal. Therefore, robustness calls for mechanisms that do not rely on fine details of the problem to establish for which types of buyer's goods should be sold separately or bundled.

For most of the exposition, we consider the case in which the first moment of the type distribution is known. The assumption that the seller knows at least the first moment of the distribution of valuations makes the problem interesting. Under full ignorance of the distribution, the optimal mechanism would be trivial: set a price equal to the lowest valuation the buyer might attribute to the good. Moreover, it might well be the case that the seller, proceeding as an econometrician as proposed by Segal (2003), has access to enough data to estimate (say, through an hedonic regression) the mean of the distribution of valuations, but his sample falls short of data for a non-parametric (consistent) estimation of the whole distribution. Alternatively, it might be common knowledge that buyer expects her valuation to be k but, before purchase and after the seller offers her a mechanism, might acquire relevant information about her valuation. In such interpretation, due to Carroll (2013), if the seller is uncertain about the agent's information acquisition technology, all he will know is that valuations are drawn from a distribution with mean k.

To fix ideas, let us come back to the simple one-period case with single-dimensional private information. Since the seminal paper by Myerson (1981), much is known about such case when the buyer has a unit demand, and the seller knows the cumulative distribution F from which this valuation is drawn. Bulow and Roberts (1989) demonstrate that the optimal selling mechanism in the regular case resembles the solution of a standard monopolist's problem taught in introductory microeconomics. In fact, for a given price p, if one defines "quantities" sold as the probability of sales under F – namely, $1 - F(p)$ – total revenues as can be written as $p \cdot (1 - F(p))$. A monopolist would then compute marginal revenues, $p - \frac{1-F(p)}{f(p)}$, and sell if, and only if,they are larger than costs. The rule coincides with the one derived by Myerson (1981). Moreover, the seller can implement such optimal mechanism by posting a single price.

Although quite simple, the solution requires full knowledge of "demand" – the cumulative distribution F – and cannot be pursued if the seller only knows the expected valuation, k, of the buyer's valuation. Solving for the mechanism that maximizes expected profits under the worst-case distribution involves a couple of steps. First, we fix an arbitrary profit function induced by an incentive compatible mechanism, and consider the problem of finding the distribution that minimizes its expected value. This minimization problem is constrained by the fact that the distribution must integrate to one and yield expected value of k. Once one incorporates those constraints in a Lagrangian form, nature minimizes the expected value of the seller's profits subtracted by the constraints weighted by their shadow costs (i.e., Lagrangian multipliers).

Our first result shows that any solution of the minimization problem will only place mass on values $v \in [0, 1]$ for which the seller's profits equal to the linear function of the valuation $\xi v - \lambda$, where ξ is the shadow cost of the constraint that imposes that the average of the distribution must be k, and λ is the shadow cost of the constraint that the distribution must integrate to one. Hence, when only the mean is known, on the support of the worst-case distribution, ex-post profits are an affine function of the valuation. One interpretation for this linear format is that, knowing only the first moment of the distribution of valuations, the seller can only explore linearly higher buyer's valuations; otherwise, nature could move likelihood weights in a way that still preserves the shadow costs imposed by the restrictions on the distributions and reduce the seller's expected profits. Alternatively, under the information acquisition interpretation of the model, the stage at which the buyer obtains new information adds volatility to the seller's (degenerate) prior. A profit function with a call option format is then optimal.

When an arbitrary moment condition is known, we follow Bergemann and Schlag (2008, 2011) and recast the robust design problem as a zero-sum game played by the seller and an adversarial nature, who seeks to minimize his expected profits by choosing distributions satisfying the moment condition. A Nash equilibrium of such game will then correspond to the robust mechanism and the worst-case distribution. We prove the existence of a Nash equilibrium of the zero-sum game and characterize it. At the robust mechanism the seller's profits are, conditional on sales, an affine transformation of the known moment function $\kappa(v)$, i.e., robustness imposes this natural restriction on the seller's payoff. Once this restriction is established, finding the robust allocation is simple. In the case of a differentiable $\kappa(\cdot)$, by imposing that the derivative of the profit function with respect to valuations must vary linearly with the derivative of $\kappa(v)$, we derive an ordinary differential equation (ODE) whose solution yields the robust sale mechanism. In the particular case in which $\kappa(v) = v$ (i.e., the mean is known), the solution corresponds to the unique robust allocation.

As opposed to what prevails in standard Bayesian problems, in the robust mechanism, sales take place with probability smaller than one for buyers with valuation below the highest one.[1] Put differently, there are distortions at the intensive margin. The interpretation of why it is optimal to distort at the intensive margin is simple. Consider a posted price p, which only features distortions in the extensive margin. The connected ex-post profit function has a discontinuity at valuation $v = p$, i.e., profits are very sensitive to small variations of valuation around the price. This profit sensitivity is bad for an ambiguity averse seller: the worst-case distribution "exploits" this discontinuity and places a positive probability mass at values to the left of the price (at $v = p_-$). In order to hedge against such distributions, the seller offers all interior trade probabilities so that small variations in values lead to small variations in revenue.[2]

There are two different ways to indirectly implement such robust allocation. Since the robust allocation displays distortions at the intensive margin, standard non-linear prices can be used to implement it. The second and more interesting way to implement the robust allocation explicitly uses the lack of curvature in the buyer's payoff. In fact, the seller can implement the robust allocation by committing to pick a price from a well designed, non-degenerate, probability distribution of prices.[3] Therefore, randomizing over posted prices is an implementation of the optimal robust mechanism.

We extend the simple model to consider the case in which the seller sells M goods and all he knows is that the buyer has unit demand for each of the goods, and that her (multidimensional) valuation lies in $[0,1]^M$ and has expected value of $\boldsymbol{k} \in [0,1]^M$. We assume that the seller faces no technological constraints and can produce up to one unit of each good. In Bayesian settings, not much is known about the solution of such a problem and even standard features of single-good sales mechanisms, such as monotonicity of profits in buyer's valuations and the optimality

[1]In fact, this happens whenever $\kappa(\cdot)$ is continuous.

[2]If the seller is a standard expected utility maximizer, he will only find it optimal to discriminate if either the buyer's payoff or his cost function have some curvature.

[3]Although this second form of implementing the robust selling mechanism is appealing, it requires full commitment of the "adversary" nature and the seller to commit to their strategies in the simultaneous zero-sum game between them, which leads to some difficult to use this scheme in practice. On the other hand, committing to a schedule of non-linear pricing makes the seller immune to the hypothesis of simultaneous move. In other words, committing to a full schedule instead of to randomization of over posted prices does not allow the nature deviate in the interim stage of realization of the randomizing and the choice by the buyer.

of deterministic mechanisms, do not extend in general to multidimensional profit maximizing mechanisms (see, for example, Manelli and Vincent (2007) and Hart and Reny (forthcoming)).

In contrast, a general characterization of the robust multidimensional mechanism is not only feasible, but follows from similar arguments used in the single-dimensional problem. In the zero-sum game representation of the robust design problem, we consider the problem faced by nature in minimizing the seller's expected profits and construct a distribution for which the seller's best response attains at most the valuation he would obtain by selling the goods in an entirely separable way. As a mechanism that entails selling each good separately is always feasible to the seller, this shows that separate sales is a robust mechanism. For the case in which the buyer's expected valuations for the goods are the same, selling them in fixed proportions in a bundle is also a robust mechanism. This is established by simply verifying that such mechanism yields the same worst-case expected profits for the seller as selling separately the goods.

Related literature

Our paper is part of a growing literature on mechanism design with principals with maximin preferences.[4] Frankel (2014) and Carrasco and Moreira (2013) consider decision-making problems with non-transferable utility in which a maximin principal is unaware of the agent's bias (in Frankel (2014)) or the distribution of states (Carrasco and Moreira (2013)). In an otherwise standard procurement setting à la Laffont and Tirole (1986), Garrett (2014) considers the case of a principal who does not know the producer's disutility of effort, and shows that a simple fixed-price-cost-reimbursement (FPCR) menu minimizes the principal's maximum expected payment to the agent. In Carroll (2015a), the principal only partially knows the set of actions available to the agent; he shows that if the principal maximizes expected profits under worst-case set of actions, the optimal contract is linear in output. Our work differs from those listed above by considering a seller's pricing decision.

There are, nevertheless, a set of papers that focus on pricing with unknown distribution of values and posit that the seller has preferences for robustness.[5] Bergemann and Schlag (2008, 2011) are perhaps the first to do so. In their first paper, they consider the case in which the seller designs a mechanism to minimize the maximum regret, whereas in the second they also consider a maximin procedure. In both cases, they work within a static and single-dimensional case. Similar to what we find, they show that randomizing over prices is a way to insure against un-

[4] There is also a growing literature with maximin agents. Bose et al. (2006) and Wolitsky (2014) are examples of this kind of analysis in, respectively, optimal auction design and bilateral trade with maximin agents.

[5] Segal (2003) also considered the case in which a seller did not know the distribution from which valuations were drawn. Rather than positing that the seller has preferences for robustness, he considered a seller who proceeds as an econometrician and estimates, from the mechanisms offered to subset of buyers, the distribution of values.

certainty in the minimax problem. Pointing out that, without further restrictions, the maximin problem entails the trivial solution of charging the lowest possible valuation with probability one, they consider "local robustness", that is, maximin pricing over neighborhoods around a given distribution. They show that the optimal mechanism is to post a single price and establish that, starting at the certainty case, the charged price decreases as uncertainty (measured by the size of the neighborhood) increases. Instead of considering a minimax criterion or working with neighborhoods around a given distribution, to avoid a trivial solution for the maximin problem, we assume that a given moment of the distribution of values is known. This complements their minimax analysis, on the one hand, and allows for an analysis of the multidimensional, on the other.

In independent works, Carroll (2013) and Kos and Messner (2015) consider selling problems where the seller only knows the mean. In Carroll (2013), at the stage the seller offers her a contract, the buyer knows her expected valuation, but she can acquire information about it before purchasing the good. The seller, who only knows the prior from which the expected valuation is drawn, designs a mechanism to maximize the worst-case expected profits over information acquisition technologies, which amounts to choosing among mean preserving spreads of the prior.[6] In Kos and Messner (2015), the seller also maxminimizes worst-case payoffs, but on top of the case in which the seller knowns the mean of the distribution, they also consider an extension in which the first and second moments are known. Our single-good case corresponds to a special of Carroll (2013)'s when the seller's prior is degenerate, and is similar to Kos and Messner (2015)'s when only the mean is know. When we deal with the case of a single good, however, our analysis encompasses the case of knowledge of an arbitrary moment/statistic by the seller. We also extend the analysis to $M > 1$ goods (and multidimensional valuations).

Not much is known in general for Bayesian multidimensional design (see, for instance, Hart and Reny (forthcoming) and references therein). By looking at worst-case selling procedures, we are able to fully derive optimal mechanisms and show that they involve full separation (and full bundling for the symmetric case), in a stark contrast to the mixed-bundling solution of McAfee et al. (1989) and the literature that followed. In Carroll (2015b)[7] an additive separable multidimensional mechanism design problem with general (quasi-linear) preferences is considered for the case in which the designer only knows the marginal distributions and maxminimizes expected payoffs. Our setting is much more restrictive in terms of the payoff structure, but allows for a larger degree of ignorance by the designer (since he only knows a single moment of the joint distribution of the agent's private information, rather than the marginals). Both papers

[6]In the introduction, we have borrowed Carroll's story to justify why the seller might know the mean of the distribution from which valuations are drawn.

[7]After the first version of this paper was written, Carroll (2015b) was brought to our attention. We thank Lucas Maestri for calling our attention to this recent paper.

derive the full separability result. For what we call the symmetric case – in which the known moment is the same in each dimension –, our proof relies on the explicit derivation of a Nash Equilibrium of the zero-sum game played by nature and the seller. For such particular case, we also establish that full bundling is equally optimal. Our proof of separability for the asymmetric case is a variant, for the case in which only the expected value of the distribution of values is known, of what Carroll (2015b) describes as the "maximal positive correlation" case (see Section 3.2 of his paper). In fact, we use the worst-case distributions derived for the single-good case to construct what Carroll (2015b) calls a comonotonic joint distribution that induces expected profits by the seller which are bounded above by what he obtains under full separability.

Organization

Section 2 lays down the general model. In Section 3, we derive the robust mechanism for the setting in which the seller sells one good. We tackle the robust design problem using two different approaches. In the first one, we use standard Lagrangian technique, whereas, in the second, we recast the robust design problem in terms of a zero-sum game played by the seller and an adversary nature who chooses distribution to minimize his expected profits. This latter approach proves useful to derive the robust mechanism for an arbitrary moment condition that might be known by the seller. In Section 4 we consider the robust design for the case in which the seller sells M goods. We draw our concluding remarks in Section 5. The Appendix contains the proofs not presented in the text.

2 General model

A monopolist (or seller) can produce $M \geq 1$ indivisible and non-storable goods at zero cost. The seller faces a buyer who has valuation for the good denoted as $\boldsymbol{v} = \left(v^1, \ldots, v^M\right) \in [0,1]^M$. If quantity $\mathbf{q} \in [0,1]^M$ (we use bold to represent the vector of quantities) and transfers p are made, the utility obtained by the buyer is given by

$$\boldsymbol{v} \cdot \mathbf{q} - p,$$

and the seller's profits are pThe set of direct mechanisms is defined as

$$\mathcal{M} \equiv \left\{ m = (\mathbf{q}, p) : [0,1]^M \to [0,1] \times \mathbb{R} \text{ is } v\text{-measurable} \right\}.$$

In a direct mechanism, the realized payoff of an agent of type $\mathbf{v}$ who reports $\widehat{\boldsymbol{v}}$ is

$$\mathcal{U}_m\left(\widehat{\boldsymbol{v}} \mid \boldsymbol{v}\right) \equiv \left[\boldsymbol{v} \cdot \mathbf{q}\left(\widehat{\boldsymbol{v}}\right) - p\left(\widehat{\boldsymbol{v}}\right)\right],$$

and the realized firm profits are given by

$$\Pi_m\left(\widehat{\boldsymbol{v}} \mid \boldsymbol{v}\right) \equiv p\left(\widehat{\boldsymbol{v}}\right).$$

We will also use $\mathcal{U}_m\left(\boldsymbol{v}\right)$ and $\Pi_m\left(\boldsymbol{v}\right)$ to denote realized payoffs for the buyer (the rent function) and the seller (the profit function) under truth-telling.

We assume that the type distribution $F \in \Delta\left([0,1]^M\right)$ is not known neither by the buyers nor by the seller. The seller only knows that the set of possible distributions is $\mathcal{F} \subseteq \Delta\left([0,1]^M\right)$ but does not have a probability distribution over this set. Instead, the seller is ambiguity averse. The seller's problem amounts to designing a mechanism to maximize his worst-case expected profits over all distributions in $\mathcal{F}$. In this paper the defining property of this set is assumed to be the moment conditions, which are the only market information available to the seller. The definition of this set is presented for each case of interest studied. Over the next sections, to build the ideas that will allow us to find the robust mechanism for the general model we just laid out, we will consider the seller's problem for some special cases of interest.

3 One good

We first consider the case where the seller holds one indivisible good and only has information about the average of the type distribution. Formally, the set of possible distributions is

$$\mathcal{F} \equiv \left\{F \in \Delta\left([0,1]\right); \int v dF\left(v\right) = k\right\}, \tag{1}$$

for some $k \in (0,1)$.

The seller wants to maximize the profit guarantee given by this moment condition alone. This is done by considering the worst-case expected profits from the set of possible distributions in $\mathcal{F}$. We restrict attention to incentive compatible direct mechanisms.[8] The seller's problem is

[8]Since the buyer has complete information at the announcement stage, the set of optimal reporting strategies is independent of the actual type distribution $F \in \mathcal{F}$. Any optimal report-independent strategy implements an allocation that can also be implemented via (optimal) truthful reporting in a mechanism where the buyer only announces his valuation.

given by:

$$\max_{m \in \mathcal{M}} \min_{F \in \mathcal{F}} \int_0^1 p(v)\, dF(v), \tag{2}$$

subject to participation constraint

$$\mathcal{U}_m(v) := vq(v) - p(v) \geq 0, \tag{3}$$

and incentive compatibility constraints

$$\mathcal{U}_m(v) \geq vq(\widehat{v}) - p(\widehat{v}), \tag{4}$$

for all $v, \hat{v} \in [0,1]$. Any mechanism $m \in \mathcal{M}$ that satisfies conditions (3) and (4) is called a feasible mechanism.

Definition 3.1. We will call the mechanism that solves problem (2), whenever it exists, a (optimal) robust mechanism and the allocation that implements it the (optimal) robust allocation.

As usual under the single-crossing condition, incentive compatibility is equivalent to the monotonicity condition: $q(\cdot)$ is non-decreasing; and the envelope condition:

$$\mathcal{U}_m(v) = \mathcal{U}_m(0) + \int_0^v q(\tau)\, d\tau. \tag{5}$$

Substituting equation (5) in the objective function of (2) and noticing that, regardless of the worst-case distribution, the seller will always pick a mechanism with $\mathcal{U}_m(0) = 0$, which is equivalent to the participation constraint at the robust mechanism. His problem can then be equivalently rewritten as:

$$\max_{(q,p) \in \mathcal{M}} \min_{F \in \mathcal{F}} \int_0^1 \Pi_m(v) dF(v) \tag{6}$$

subject to $q(\cdot)$ non-decreasing and

$$\Pi_m(v) := p(v) = vq(v) - \int_0^v q(\tau)\, d\tau \tag{7}$$

is the profit function associated to mechanism $m = (q, p)$.

Given any non-decreasing allocation function $q : [0,1] \to [0,1]$, we can define the associated profit function Π by (7), which is non-decreasing and satisfies $\Pi(0) = 0$, where we are omitting the subindex m for convenience. In what follows it will be convenient to represent the mechanism through its associated profit function.

3.1 Worst-case distributions

In this subsection we fix an arbitrary non-decreasing profit function $\Pi(\cdot)$ such that $\Pi(0) = 0$ to analyze the worst-case payoff problem, i.e., the minimization problem in (6). Because of the monotonicity of $\Pi(\cdot)$, we can assume without loss of generality that $\Pi(\cdot)$ is left-continuous.

To apply the Kuhn-Tucker technique to the minimization problem in (6) we relax the equality constraints that define the set $\mathcal{F}$ and which require they are cumulative distributions and that have mean k. Formally, let us define the problem of finding the worst-case expected profits:

$$\min_{F \in \mathcal{D}} \int \Pi(v)\, dF(v) \tag{8}$$

subject to aggregate mass being smaller than or equal to one:

$$\int dF(v) \leq 1, \tag{9}$$

and to the least mean condition:

$$k - \int v dF(v) \leq 0, \tag{10}$$

where $\mathcal{D} = \{F : [0,1] \to [0,1] \text{ is non-decreasing and right continuous}\} \subset \mathcal{F}$.

The next lemma guarantees the existence of solution of (8) and shows that at the robust mechanism the constraints must bind.

Lemma 3.1. *(Existence) There is a solution to the problem (8). At a robust mechanism, this solution belongs to $\mathcal{F}$.*

Let $F^* \in \mathcal{D}$ be a solution of the problem in (8). Standard arguments (see, for instance, Luenberger (1969)) imply that there exists a Lagrangian functional $L : \mathcal{D} \times \mathbb{R}^2_+ \to \mathbb{R}$ of problem (8) defined by

$$L(F, \lambda, \xi) = \int \Pi(v) dF(v) + \lambda \left(\int dF(v) - 1 \right) + \xi \left(k - \int v dF(v) \right)$$

so that the optimal distribution $\tilde{F} \in \mathcal{D}$ and multipliers $\tilde{\lambda} \geq 0$ and $\tilde{\xi} \geq 0$ must satisfy the saddle point condition:

$$L(\tilde{F}, \lambda, \xi) \leq L(\tilde{F}, \tilde{\lambda}, \tilde{\xi}) \leq L(F, \tilde{\lambda}, \tilde{\xi}),$$

for all $(F, \lambda, \xi) \in \mathcal{D} \times \mathbb{R}_+^2$. Hence, ignoring constant terms,

$$\tilde{F} \in \arg\min_{F \in \mathcal{D}} \int \left[\Pi(v) + \tilde{\lambda} - \tilde{\xi} v\right] dF(v).$$

Clearly, problem (8) is well defined only if $\Pi(v) + \tilde{\lambda} - \tilde{\xi} v \geq 0$, for all $v \in [0,1]$.[9] We can then define the set of valuations where the profit function coincides with the envelope affine function $\tilde{\xi} v - \tilde{\lambda}$, $I = \left\{v \in [0,1]; \Pi(v) = \tilde{\xi} v - \tilde{\lambda}\right\}$, and its complement $I^c = \left\{v \in [0,1]; \Pi(v) > \tilde{\xi} v - \tilde{\lambda}\right\}$. At an optimal solution $\tilde{F} \in \mathcal{D}$, the mass of valuations where the profit is above the affine envelope must be zero, i.e., $d\tilde{F}(I^c) = 0$. Otherwise, by moving mass from I^c to I, the expected profit could be reduced even further and the objective of the minimization problem could then be improved.

More interestingly, at the robust mechanism, the worst-case distribution must assign positive likelihood for all values above a certain cutoff valuation. The interpretation is that the adversarial "nature"[10] who minimizes the seller's shadow profits assigns positive likelihood for valuations in I for which shadow profits $\Pi(v) + \tilde{\lambda} - \tilde{\xi} v$ are zero, and assigns zero likelihood for valuations in I^c, for which shadow profits are positive. Hence, if an interval of the form $[\tilde{v}, 1]$ were to have positive shadow profits, the seller could reduce the amount sold to those types in a way that simultaneously brings *shadow* profits slightly below zero (and, therefore, induces the nature to assign positive likelihood to $[\tilde{v}, 1]$) and still yields *profits* that are larger than the ones obtained from valuations smaller than $\tilde{v}$ (since $\Pi(\cdot)$ is non-decreasing). This redistribution would ensure larger expected profits for the seller.

Since $\Pi(v) + \tilde{\lambda} - \tilde{\xi} v \geq 0$ and nature only places positive likelihood on $I = \left\{v \in [0,1]; \Pi(v) = \tilde{\xi} v - \tilde{\lambda}\right\}$, the following lemma shows that we can replace the profit function $\Pi(\cdot)$ by its lower piecewise linear envelope. More precisely, the lemma shows that the lower envelope is itself a profit function generated by an allocation with sales also realized in the interval $[0,1]$.

Lemma 3.2. *(Envelope profit) Let $q : [0,1] \to [0,1]$ be any non-decreasing allocation that satisfies*

$$\Pi(v) := vq(v) - \int_0^v q(\tau) d\tau \geq \tilde{\xi} v - \tilde{\lambda},$$

[9] In fact, if there exists $v \in [0,1)$ such that $\Pi(v) + \tilde{\lambda} - \tilde{\xi} v < 0$, one would have $L\left(N.H_v, \tilde{\lambda}, \tilde{\xi}\right) \to -\infty$ when $N \to \infty$, where H_v is the Heaviside function at v, i.e., $H_v(x) = 0$ for $x \in [0, v]$ and $H_v(x) = 1$ for $x \in (v, 1]$. Since $\Pi(v)$ is left-continuous and non-decreasing, then $\Pi(1) + \tilde{\lambda} - \tilde{\xi} \geq 0$.

[10] In Subsection 3.3 we will explore this interpretation to derive that the robust mechanism as a part of a Nash equilibrium between the seller and the adversary nature in a zero-sum game.

for all $v \in [0,1]$ *and for some* $\tilde{\xi} > 0$ *and* $\tilde{\lambda} > 0$*. Then, the allocation defined by*

$$\tilde{q}(v) := \frac{1}{\tilde{v}} \int_0^{\tilde{v}} q(\tau) d\tau + \tilde{\xi} \ ln \ \left(\frac{v}{\tilde{v}}\right)$$

is non-decreasing and below $q(\cdot)$ *and its associated profit* $\tilde{\Pi}(v) = v\tilde{q}(v) - \int_0^v \tilde{q}(\tau)d\tau = \tilde{\xi}v - \tilde{\lambda}$*, for all* $v \in [0,1]$*, is a linear function, where* $\tilde{\lambda} = \tilde{\xi}\tilde{v}$*.*

The next proposition formally establishes that, at the robust mechanism, the profit function is the piecewise linear.

Proposition 3.1. *(Piecewise linear robust profit) If* $\Pi^*(\cdot)$ *is the profit function associated to a robust mechanism, then* $\Pi^*(v) = \max\left\{\tilde{\xi}v - \tilde{\lambda}, 0\right\}$*, for some* $\tilde{\xi} > 0$ *and* $\tilde{\lambda} > 0$*, i.e., it is piecewise linear.*

Proposition 3.1 says that, if all the seller knows is the mean of the buyer's valuation, the best he can do to insure against ambiguity is to design a mechanism that induces ex-post profits which are, conditional on sales, linear in valuations. This mechanism guarantees him expected profits which are bounded below by a linear function on the piece of information he has. Alternatively, consider the interpretation that, ex-ante, the seller knows the buyer's expected willingness to pay, but the latter acquires additional information after the mechanism is offered. Then, the convexity (or call-option format) of the profit function is the seller's optimal response to the added volatility in the buyer's valuation stemming from the information acquisition stage.

3.2 Robust mechanism

Let $\Pi^*(\cdot)$ is the profit function associated to a robust mechanism.From Proposition 3.1, if $\tilde{F} \in \mathcal{F}$ is a worst-case distribution, the seller's expected profits at a robust mechanism is

$$\int \Pi^*(v)\, d\tilde{F}(v) = \int \max\left\{\tilde{\xi}v - \tilde{\lambda}, 0\right\} d\tilde{F}(v) = \tilde{\xi}\,[k - \tilde{v}]\,,$$

where $\tilde{v} = \tilde{\lambda}/\tilde{\xi}$ is the marginal type, i.e., the type such that the mechanism prescribes sales for all $v > \tilde{v}$. Moreover, at a robust allocation $q^*(\cdot)$,

$$\Pi^*(v) \equiv vq^*(v) - \int_0^v q^*(\tau)\, d\tau = \tilde{\xi}v - \tilde{\lambda},$$

for all $v \in [\tilde{v}, 1]$. Differentiating the above condition, we get

$$v\frac{dq^*}{dv}(v) = \tilde{\xi} \Rightarrow q^*(v) = \tilde{\xi} \ln\left(\frac{v}{\tilde{v}}\right),$$

for all $v \in [\tilde{v}, 1]$. The seller's problem can then be simplified to

$$\max_{\tilde{v}\in[0,1],\tilde{\xi}\geq 0} \tilde{\xi}\left[k-\tilde{v}\right] \tag{11}$$

subject to

$$\tilde{\xi}\ln\left(\frac{v}{\tilde{v}}\right) \leq 1, \text{ for all } v \in (0,1]. \tag{12}$$

For the next proposition define the (implicit) solution $v^* \in (0, k)$ of equation:[11]

$$v^*\left(1 - \ln v^*\right) = k. \tag{13}$$

Then, the solution to problem (11) yields:

Proposition 3.2. *(Robust mechanism) The robust mechanism is given by the pair allocation and transfer*

$$q^*(v) = \begin{cases} 0, & \text{if } v < v^* \\ 1 - \frac{\ln v}{\ln v^*}, & \text{if } v \geq v^* \end{cases} \tag{14}$$

and

$$p^*(v) = vq^*(v) - \int_0^v q^*(\tau)\, d\tau. \tag{15}$$

The expected robust profit is given by v^*.

While, as in standard Bayesian selling mechanisms, there is no distortion at the top (i.e., $q^*(1) = 1$), the robust mechanism entails sales with probability smaller than one for all valuations $v < 1$. Hence, the mechanism distorts the allocation at the intensive margin. The reason for this distortion is to reconcile the need for hedging against uncertainty with the goal of charging higher prices from high valuation consumers. Considering the worst-case scenario, the seller will find it desirable to sell to buyers with low valuations. If he was, however, to sell with probability one to them, the amount he would be able to charge from infra marginal buyers would be small. By selling with probability smaller than one to low valuation buyers, the seller can charge more from infra marginal buyers. Price discrimination is the way by which the seller simultaneously insures against uncertainty and charge higher prices from inframarginal buyers,

Implementation

There are many ways to implement the allocation in Proposition 3.2. Perhaps the most immediate one is through a non-linear tariff. In fact, for the allocation $q^*(\cdot)$ and transfer $p^*(\cdot)$ defined

[11] From the proof of Proposition 3.2, it is clear that v^* exists and is unique.

by (14) and (15), we can then make use of the taxation principle to implement the robust direct mechanism through an indirect mechanism $\{(q, P^*(q)); P^*(q) = p^*(v) \text{ for } q = q^*(v)\}$.

A more interesting way – and that explicitly uses the fact that the buyer's payoff is linear – to implement the robust mechanism is, however, through a *distribution* of posted prices. Indeed, notice that the direct mechanism calls for a buyer of type v to be assigned the good with probability $q^*(v)$. At any given price p, the buyer will buy if and only if $p \leq v$. Now, assume that the seller commits to posting a price $p \in [v^*, 1]$ drawn from the cumulative distribution

$$q^*(p), \text{ for all } p \in [v^*, 1],$$

with $q^*(\cdot)$ from equation (14). It is easy to see that, if prices are drawn from $q^*(p)$, the probability that a buyer of valuation v buys is exactly $q^*(v)$. Hence, we have:

Proposition 3.3. *(Implementation) Committing to posting a price drawn from the distribution $q^*(p)$, for all $p \in [v^*, 1]$, implements the robust selling mechanism.*

However, as we did indicate in the introduction the implementation protocol proposed in Proposition 3.3 requires that the adversary nature also commits to choose the the critical distribution of Nash equilibrium characterized in the next subsection. This is especially sensitive to the commitment of the nature. On the other hand, schedule of prices are immune to this assumption.

3.3 Nash equilibrium

An alternative to the approach that makes use of Lagrangian techniques is to recast, as Bergemann and Schlag (2008) do in their minimax pricing problem, the robust design problem in terms of a zero-sum game between the seller and an adversarial nature.

In such game, the seller chooses feasible mechanisms in $\mathcal{M}$ and the nature selects distributions in $\mathcal{F}$. The seller has a von-Neumann-Morgenstern utility on profit function Π,whose Bernoulli expression at state v is $\Pi(v)$, whereas the nature has the negative reciprocal payoff $-\Pi(v)$. As argued in Bergemann and Schlag (2008), if (m^*, F^*) is a Nash equilibrium of such game, then m^* is a robust mechanism and F^* is a worst-case distribution.

To characterize the Nash equilibrium, we start by guessing that the worst-case distribution F^*. Let f^* be the absolutely continuous part of this distribution. Notice that, for any non-decreasing allocation $q(\cdot)$, integrating by parts the associated expected profit yields:

$$\int_0^1 \left[vq(v) - \int_0^v q(\tau)\, d\tau \right] dF^*(v) = \int_0^1 \left[v - \frac{1 - F^*(v)}{f^*(v)} \right] q(v) f^*(v)\, dv + \left(1 - F^*_-(1)\right) q(1),$$

where we are allowing for a mass point only possibly at $v = 1$. By making the virtual surplus identical to zero, i.e.,

$$v - \frac{1 - F^* (v)}{f^* (v)} = 0, \tag{16}$$

the nature guarantees that the seller will be indifferent among any feasible mechanism with $q(1) = 1$. In particular, the one in equation (14) is a best reply by the seller if equation (16) is satisfied. Solving (16) gives the distribution

$$F^* (v) \equiv \begin{cases} 0, & \text{if } v \in [0, v^*) \\ 1 - \frac{v^*}{v}, & \text{if } v \in [v^*, 1) \\ 1 & \text{if } v = 1, \end{cases} \tag{17}$$

where v^* is the lowest valuation with positive sales characterized by (13). This distribution, which has an absolutely continuous part and a singular characterized by the Dirac measure $v = 1$ with mass equals to v^*, is called the critical distribution.

The following proposition formally characterizes the robust mechanism and the critical distribution as the Nash equilibrium of the zero-sum game between the seller and the nature. We also show that this Nash equilibrium is unique.

Proposition 3.4. *(Nash equilibrium) Let $m^* = (q^*, p^*)$ be the mechanism characterized by (14) and (15) and the distribution F^* by (17). Then, (m^*, F^*) is the unique Nash equilibrium of the zero-sum game played by the nature and the seller.*

3.4 Arbitrary moment condition

In this subsection, we show that the above analysis extends to the case in which all the seller knows is an arbitrary moment condition. In particular, the restriction imposed by robustness on the seller's ex-post profit still holds. Such restriction, in turn, fully pins down the robust allocation.

Consider a continuously differentiable function $\kappa : \mathbb{R} \to \mathbb{R}$ satisfying[12] $\kappa' (\cdot) > 0$, and assume that all the seller knows that the distribution belongs to the set

$$\mathcal{F} = \left\{ F \in \Delta [0, 1] ; \int \kappa (v) \, dF (v) = k \right\}, \tag{18}$$

for some fixed $k \in (\kappa (0), \kappa (1))$.

[12]The argument can be extended to the more general case of bounded non-decreasing functions. So the analysis goes beyond the case of moments in the form v^n.

We follow the analysis of Section 3.3 and recast the robustness design problem as a zero-sum game played by the seller and nature. The robust mechanism is then derived by constructing a strategy profile for the seller and the nature and showing it is a Nash equilibrium of such game.

Distribution

Define for each $\underline{v} \in (0,1)$ the distribution $F^{\underline{v}}$ as

$$F^{\underline{v}}(v) \equiv \begin{cases} 0, & \text{if } v < \underline{v} \\ 1 - \frac{\underline{v}}{v}, & \text{if } v \in [\underline{v}, 1) \\ 1, & \text{if } v = 1. \end{cases}$$

A higher $\underline{v}$ leads to a first-order stochastic increase in the distribution $F^{\underline{v}}$. Hence, the function that takes $\underline{v}$ to $\int \kappa(\tau)\, dF^{\underline{v}}(\tau)$ is continuous and strictly increasing. Also, it converges to $\kappa(0)$ when $\underline{v} \to 0$ and to $\kappa(1)$ when $\underline{v} \to 1$. By the intermediate value theorem, there is a unique point $v^* \in (0,1)$ that satisfies

$$\int \kappa(v)\, dF^{v^*}(v) = k.$$

Let us define the critical distribution as $F^* = F^{v^*}$.

Mechanism

For each $\xi \geq 0$, define $q^\xi : [0,1] \to \mathbb{R}$ that solves the differential equation $vq'(v) = \xi \kappa'(v)$ on $[v^*, 1]$, with final condition $q(1) = 1$, and identical to zero on $[0, v^*)$:

$$q^\xi(v) \equiv \begin{cases} 0, & \text{if } v < v^* \\ 1 - \xi \int_v^1 \frac{\kappa'(\tau)}{\tau} d\tau, & \text{if } v \geq v^*. \end{cases}$$

Simple derivations show that the profit function of the seller implied by the allocation q^ξ is given by

$$\Pi^a(v) = \begin{cases} 0, & \text{if } v < v^* \\ \xi\left[\kappa(v) - \kappa(v^*)\right], & \text{if } v \geq v^*. \end{cases}$$

Let $\xi^* \equiv \left[\int_{v^*}^1 \frac{\kappa'(\tau)}{\tau} d\tau\right]^{-1}$, and define the allocation $q^* = q^{\xi^*}$ and transfer p^* by (15), which leads to the mechanism $m^* = (q^*, p^*)$. This mechanism also leads to the profit function $\Pi^* \equiv \Pi^{\xi^*}$ that satisfies

$$\Pi^*(v) = \xi^*\left[\kappa(v) - \kappa(v^*)\right].$$

Nash equilibrium

The next two lemmata provide a characterization of the Nash equilibrium of the zero-sum game between the seller and the nature. For the first lemma, notice that $\Pi^*(v) = \xi^*[\kappa(v) - \kappa(v^*)]$ if $v \geq v^*$ and $\Pi^*(v) > \xi^*[\kappa(v) - \kappa(v^*)]$ if $v < v^*$. Hence, it follows that

$$\int \Pi^*(v)\, dF(v) \geq \int \xi^*[\kappa(v) - \kappa(v^*)]\, dF(v) = \xi^*[k - \kappa(v^*)],$$

for any $F \in \mathcal{F}$, where the first inequality holds as an equality if and only if $supp\,(F) \subseteq [v^*, 1]$. Therefore, we have:

Lemma 3.3. *(Nature's problem) The critical distribution F^* solves*

$$\min_{F\in\mathcal{F}} \int \Pi^*(v) dF(v).$$

The next lemma shows that m^* solves the seller's problem. For this, notice that for any incentive compatible mechanism $m = (q, p) \in \mathcal{M}$,

$$p(v) \equiv vq(v) - \int_0^v q(\tau)\, d\tau,$$

which, using the definition of the critical distribution F^*, leads to

$$\int p(v)\, dF^*(v) = (1 - F^*(1))\, q(1).$$

Any feasible mechanism with $q(1) = 1$ maximizes profit. Therefore, we have:

Lemma 3.4. *(Seller's problem) The robust mechanism m^* solves*

$$\max_{m=(q,p)\in\mathcal{M}} \int \Pi_m(v)\, dF^*(v),$$

subject to $\Pi_m(v) := p(v) = vq(v) - \int_0^v q(\tau)\, d\tau$ and $q(\cdot)$ is non-decreasing.

Combining Lemma 3.3 and 3.4 leads immediately to the following result.

Proposition 3.5. *(Robust mechanism for arbitrary moment) The mechanism $m^* = (q^*, p^*)$ solves the robust mechanism problem under the moment condition given by (18).*

Once again, if all the seller knows is the moment condition $\int \kappa(v)\, dF(v) = k$, the best he can do to hedge against uncertainty is to design a mechanism that induces ex-post profits which are an affine transformation of $\kappa(v)$, so to guarantee expected profits bounded below by a linear function of the piece of information he has. Much as in the case of a known first moment, the robust allocation entails distortions at the intensive margin whenever $\kappa(\cdot)$ is a continuous function.

4 Multiple goods

We now move to the case in which the seller can sell $M > 1$ goods. The seller has to design a mechanism $m = (\boldsymbol{q}, p) \in \mathcal{M}$, where $\boldsymbol{q} = (q^1, ..., q^M)$, to maximize his worst-case expected profits under all cumulative probability distributions on $[0,1]^M$ with the vector of mean $\boldsymbol{k} = (k^1, ..., k^M) \in [0,1]^M$, i.e., the set of possible distributions is given by

$$\mathcal{F} \equiv \left\{ F \in \Delta\left([0,1]^M\right); \int v^i dF(\boldsymbol{v}) = k^i, \text{ for } i = 1, \ldots, M \right\}. \tag{19}$$

The seller's problem reads

$$\max_{m\in\mathcal{M}} \min_{F\in\mathcal{F}} \int p(\boldsymbol{v})\, dF(\boldsymbol{v}) \tag{20}$$

subject to participation constraint

$$\mathcal{U}_m(\boldsymbol{v}) := \boldsymbol{v} \cdot \boldsymbol{q}(\boldsymbol{v}) - p(\boldsymbol{v}) \geq 0, \tag{21}$$

and incentive compatibility constraint

$$\mathcal{U}_m(\boldsymbol{v}) \geq \boldsymbol{v} \cdot \boldsymbol{q}(\widehat{\boldsymbol{v}}) - p(\widehat{\boldsymbol{v}}), \tag{22}$$

for all $\boldsymbol{v}, \widehat{\boldsymbol{v}} \in [0,1]^M$. We refer to a mechanism satisfying these as a feasible mechanism.

It is standard to show that a mechanism $m = (\boldsymbol{q}, p)$ is incentive compatible if and only if

$$\nabla\mathcal{U}_m(\boldsymbol{v}) = \boldsymbol{q}(\boldsymbol{v}) \ \text{ for a.e. } \boldsymbol{v} \in [0,1]^M \tag{23}$$

and

$$\mathcal{U}_m(\cdot) \ \text{ is convex,} \tag{24}$$

where ∇ represents the gradient operator.

Using equation (23), one can write the seller' profit function as the difference between total surplus and the buyer's utility:

$$\Pi_m(\boldsymbol{v}) := p(\boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{q}(\boldsymbol{v}) - \mathcal{U}_m(\boldsymbol{v}) = \boldsymbol{v} \cdot \nabla\mathcal{U}_m(\boldsymbol{v}) - \mathcal{U}_m(\boldsymbol{v}). \tag{25}$$

Therefore, each feasible mechanism $m = (\boldsymbol{q}, p)$ can be associated to a non-negative rent function $\mathcal{U}_m$ that satisfies (23) and (24). Reciprocally, given a non-negative convex function rent $\mathcal{U}_m$ that satisfies (23), then the mechanism $m = (\boldsymbol{q}, p)$ satisfying (23), (24) and (25) is feasible.

Plugging this is in the seller's objective we get

$$\max_{\{\mathcal{U}_m(\cdot) \text{ positive and convex}\}} \min_{F \in \mathcal{F}} \int \underbrace{[\boldsymbol{v} \cdot \nabla \mathcal{U}_m(\boldsymbol{v}) - \mathcal{U}_m(\boldsymbol{v})]}_{\Pi_m(\boldsymbol{v})} dF(\boldsymbol{v}). \tag{26}$$

It is convenient to derive a lower bound for the expected profits the seller can get when facing the adversarial nature. Toward that, let v^{i*} be the solution of the equation (13) when $k = k^i$, for all $i = 1, ..., M$, and define $v_A^* = \sum_{i=1}^{M} v^{i*}$. The next proposition shows that by selling the goods separately, the seller can guarantee payoff at least v_A^*.

Proposition 4.1. *By selling the goods separately, the seller can guarantee at least expected profits v_A^*. That is, v_A^* is a lower bound for the maximin value of the zero-sum game between the seller and the nature.*

Proposition 4.1 derives from the fact that the seller always has always the option to sell each good separately. When exercising this option, he is bounded to obtain as expected profits at least the sum of the single-good worst-case expected profits derived in Proposition 3.4. Put differently, nature's equilibrium payoff is bounded below by v_A^*. What we now show is that nature can construct distributions that attain such bound, so that v_A^* is the value of the zero-sum game it plays with the seller. We split the analysis in two cases. First, we consider the case where $\boldsymbol{k} = k(1, ..., 1)$. Then, using the insight of the symmetric case, we consider the general case $\boldsymbol{k} = (k^1, ..., k^M)$, where k^i's are not necessarily the same.

4.1 Nash equilibrium in the symmetric case

Consider the following distribution:

$$F^{M*}(v^1, ..., v^M) = F^*(\min_{1 \le i \le M} v^i),$$

where F^* is defined by (17).

This distribution leads to perfectly correlated values. Moreover, the marginal distribution of each coordinate is equal to the distribution derived in Proposition 3.4, for the one-good case. Also define the mechanism $m^* = (\mathbf{q}^*, p^*)$ such that

$$q^{i*}(\boldsymbol{v}) \equiv \begin{cases} 1 - \frac{\ln v^i}{\ln v^*}, & \text{if } v^i \ge v^* \\ 0, & \text{if } v^i < v^*, \end{cases} \tag{27}$$

for all $i = 1, ..., M$ and

$$p^* (\boldsymbol{v}) = \sum_{i=1}^{M} \left[v_i q^{i*} (\boldsymbol{v}) - \int_0^{v^i} q^{i*} \left([\boldsymbol{v}, s]^i \right) ds \right], \tag{28}$$

where v^* is defined by (13) and

$$[\boldsymbol{v}, s]^i \equiv \left(v^1, \ldots, v^{i-1}, s, 0, \ldots, 0 \right)$$

is a specific truncation of the vector $\boldsymbol{v}$.

In the following two lemmata we will show that $\left(m^*, F^{M*} \right)$ is a Nash equilibrium of the zero-sum game between the seller and the nature.

Lemma 4.1. *(Critical distribution) The critical distribution F^{M*} solves*

$$\min_{F \in \mathcal{F}} \int p^* (\boldsymbol{v}) \, dF (\boldsymbol{v}) \, .$$

Proof of Lemma 4.1. Using (27) and (28) we have that

$$p^* (\boldsymbol{v}) = \sum_{i=1}^{M} \frac{\left(v^i - v^* \right)^+}{- \ln v^*}.$$

Notice that, for any $\boldsymbol{v} \in [0, 1]^M$,

$$p^* (\boldsymbol{v}) \geq \sum_{i=1}^{M} \frac{v^i - v^*}{- \ln v^*},$$

and with strict inequality if $\boldsymbol{v} \notin \times_{i=1}^{M} [v^*, 1]$. Also, for any $F \in \mathcal{F}$, we have that

$$\int p^*(\boldsymbol{v}) dF (\boldsymbol{v}) \geq \int \sum_{i=1}^{M} \frac{v^i - v^*}{- \ln v^*} dF (\boldsymbol{v}) = \sum_{i=1}^{M} \frac{k - v^*}{- \ln v^*} = M v^*.$$

The first inequality holds as an equality if *supp* $(F) \subseteq \times_{i=1}^{M} [v^*, 1]$ and as a strict inequality otherwise. Finally, notice that $F^{M*} \in \mathcal{F}$ and *supp* $\left(F^{M*} \right) = \times_{i=1}^{M} [v^*, 1]$. □

We now show that, given distribution $F^{M*} \in \mathcal{F}$, the seller finds it optimal to choose mechanism m^*.

Lemma 4.2. *(Robust mechanism) The mechanism m^* solves the profit maximization problem defined by F^{M*}, i.e.,*

$$m^* \in \arg \max_{m = (p, \boldsymbol{q}) \in \mathcal{M}} \int p (\boldsymbol{v}) \, dF^{M*} (\boldsymbol{v})$$

subject to the existence of a non-negative function $\mathcal{U}_m$ that satisfies (23), (24) and (25).

Proof of Lemma 4.2. Notice that using incentive constraints relative to types in the diagonal vectors $\boldsymbol{v}^M := v\mathbf{1}$, $\boldsymbol{v}'^M := v'\mathbf{1} \in [0,1]^M$, where v and v' are (with some abuse of notation) scalars, we have that

$$\sum_{i=1}^{M} vq_i\left(\boldsymbol{v}^M\right) - p\left(\boldsymbol{v}^M\right) = -p\left(0^M\right) + \int_0^v q^a(s)\,ds,$$

where $q^a(s) \equiv \sum_{i=1}^{M} q_i\left(s^M\right)$ and $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^M$ is the vector formed by "1" in all entries. Hence, by standard arguments, we can rewrite the profit as

$$\int \left[vq^a(v) + p\left(0^M\right) - \int_0^v q^a(s)\,ds\right] dF^*(v) = \int q^a(v)\left[v - (1 - F^*(v))\right] dF^*(v) + p\left(0^M\right)$$

which is equal to

$$(1 - F^*(1))\, q^a(1) + p\left(0^M\right).$$

Taking into account the constraint $q^a(1) \leq M$ and $p\left(0^M\right) \leq 0$, we have that the mechanism is indeed optimal. □

It follows from the above lemmata that selling each of the goods separately is a robust mechanism. Rather than selling each good separately, the seller could also bundle the goods. Somewhat surprisingly, the seller can fare equally well by selling each of the goods in fixed proportions in a bundle. Indeed, given $\widetilde{\boldsymbol{v}}, \xi \in [0,1]^M$, define the allocation

$$\boldsymbol{q}^b(\boldsymbol{v}) = \begin{cases} 0, & \text{if } \xi \cdot (\boldsymbol{v} - \widetilde{\boldsymbol{v}}) < 0 \\ \ln\left(\frac{\xi \cdot \boldsymbol{v}}{\xi \cdot \widetilde{\boldsymbol{v}}}\right)\xi, & \text{if } \xi \cdot (\boldsymbol{v} - \widetilde{\boldsymbol{v}}) \geq 0 \end{cases} \tag{29}$$

coupled with price function p^b that satisfies[13] (25) defines a feasible mechanism $m^b = (\boldsymbol{q}^b, p^b)$. It can be readily seen that it induces $\Pi^b(\boldsymbol{v}) = \max\left\{\xi.\left(\boldsymbol{v} - \widetilde{\boldsymbol{v}}\right), 0\right\}$ as a profit function. By similar arguments to the ones presented in Section 3.1, when facing shadow costs $\tilde{\xi}$ of the constraints that impose $\int v^i dF(\boldsymbol{v}) = k$, for $i = 1, ..., M$, and shadow cost $\tilde{\lambda} = \xi \cdot \widetilde{\boldsymbol{v}}$ on the constraint $\int dF(\boldsymbol{v}) = 1$, the nature is willing to choose any distribution F with support on valuations v satisfying $\tilde{\xi} \cdot (\boldsymbol{v} - \widetilde{\boldsymbol{v}}) \geq 0$. Since F^{M*} is one of such distributions, the seller's worst-case profits under mechanism m^b is

$$\int \Pi^b(\boldsymbol{v})\, dF^{M*}(\boldsymbol{v}) = \int \tilde{\xi} \cdot (\boldsymbol{v} - \widetilde{\boldsymbol{v}})\, dF^{M*}(\boldsymbol{v}) = \tilde{\xi} \cdot [\boldsymbol{k} - \widetilde{\boldsymbol{v}}].$$

[13]In this case it is straightforward to define a non-negative and convex rent function $\mathcal{U}^b$ that satisfies (23) (24) and (25) for the mechanism $m^b = (\boldsymbol{q}^b, p^b)$.

Therefore, the best among all such mechanisms solves

$$\max_{\tilde{\xi},\tilde{\boldsymbol{v}}\geq 0} \tilde{\xi}\cdot[\boldsymbol{k}-\tilde{\boldsymbol{v}}] \tag{30}$$

subject to

$$\ln\left(\frac{\tilde{\xi}\cdot\boldsymbol{v}}{\tilde{\xi}\cdot\tilde{\boldsymbol{v}}}\right)\tilde{\xi}\leq\mathbf{1},\ \text{for all } \boldsymbol{v}.$$

The next proposition characterizes the optimal mechanism in such class.

Proposition 4.2. *(Full bundling) The solution to problem (30) entails full bundling in fixed proportions, i.e.,*

$$q^i(\boldsymbol{v}) = \begin{cases} 0, & \text{if } \mathbf{1}\cdot\boldsymbol{v} < Mv^* \\ 1 - \frac{\ln\frac{\mathbf{1}\cdot\boldsymbol{v}}{M}}{\ln v^*}\mathbf{1}, & \text{if } \mathbf{1}\cdot\boldsymbol{v} \geq Mv^*, \end{cases}$$

where v^* is the solution of (13). Moreover, the optimal profit is Mv^*.

As an immediate corollary of Proposition 4.2 we can show the following:

Proposition 4.3. *(Optimal full bundling) A mechanism that entails sales of all goods in the same proportion (full bundling) attains the same worst-case expected profits for the seller as a mechanism that sells the goods in a fully separable fashion. Both mechanisms are then optimal.*

Proof of Proposition 4.3. The proof follows from noticing that the expected profit generated by the mechanism in Proposition 4.2, Mv^*, is the same as the one he obtains by selling the goods separately. □

In the symmetric case, to insure against uncertainty, the seller can rely on two quite different mechanisms.[14] It can either ignore the fact that all goods are consumed by the same buyer and proceed as if he was dealing with M different buyers, and sell each good in a fully separable fashion. Or he can bundle the different goods in fixed proportions and, de facto, sell a single (bundled) good to the buyer. In this latter mechanism, the seller constructs and aggregate measure of willingness to pay, $\mathbf{1}\cdot\boldsymbol{v}$, and sells amounts of the bundled good that depend fully on $\mathbf{1}\cdot\boldsymbol{v}$. What is common between these two sales strategies is their simplicity: rather than bundling goods or selling them separately as a function of type announcements (as in Bayesian multidimensional mechanisms), sell them in a fully separable or fully bundled way.

Next we will show that selling goods separately is still a robust mechanism in the asymmetric case. However, the full bundling protocol proposed in Proposition 4.3 is no longer a robust selling mechanism when the average mean is not the same for all goods.

[14] In fact, it can combine the two mechanisms in whatever way he likes.

4.2 Nash equilibrium in the asymmetric case

We now move to the more general case where the means are not necessarily the same. Proposition 4.1 below shows that nature can limit the seller's expected profit to v_A^*. That is, the minimax value of the zero-sum game played between the seller and the nature is at most the payoff the seller can guarantee selling goods separately (see Proposition 4.1). As a consequence, the value of the zero-sum game is exactly v_A^*, that can be attained by a mechanism that sells the goods separately.

The proof of this result follows very closely the approach adopted by Carroll (2015b), in which a comonotonic distribution that bounds the seller's expected payoff is explicitly constructed. For the sake of completeness, we provide a detailed of proof of such construction in our framework.

Proposition 4.4. *The nature can guarantee at least v_A^* in the zero-sum game played against the seller. That is, v_A^* is an upper bound for the minimax value of the zero-sum game. Therefore, a mechanism that sells the goods separately is a robust mechanism.*

5 Conclusion

We have considered a seller's problem in designing a robust mechanism when facing a privately informed buyer and having knowledge of a single moment of the distribution from which buyer's multidimensional valuations are drawn. The results and their interpretations have been extensively discussed in the introduction and the main text, so we conclude with avenues for future research. Before doing so, however, we point out that most of the results we derived in this paper go through in a setting in which payoffs display curvature. This happens, for instance, in settings in which production can be larger than one and the buyer's marginal utility decays with consumption (or the seller's marginal cost of production is increasing). In a companion paper (Carrasco et al. (2015a)), we use the zero-sum game interpretation of the robust design problem to derive optimal robust mechanisms in quasi-linear settings with payoffs that are non-linear in the allocation.

Regarding other extensions, it would be nice, in the spirit of what Carroll (2013) does, to consider robust design for more general priors than the degenerate ones we consider here. Dealing with multiple buyers would also be a natural extension of what we have done in this paper. The main difficulties in tackling such extensions are technical. Regarding the former, extending the zero-sum game approach that we use to verify optimality for the case of M goods and T periods is not straightforward. Even for the single good case with general priors (the case considered by Carroll (2013)), it is not quite clear how to compute Nash equilibria of the zero-sum game played by the seller and nature. The case in which there are more than one buyer is even more challenging. In fact, if we were to use Myerson's trick of substituting the incentive compatible representation of the buyer's payoff into the seller's objective, and proceed as we did in this paper, we would end up with a system of Partial Differential Equations (PDE), whose solution is hard to obtain. We could, instead, assume symmetric buyers and combine Myerson's trick with Border (1991)'s conditions to solve for the optimal reduced form robust auction. The difficulty is then to solve a single buyer robust selling mechanism adding the constraints implied by Border (1991). Although challenging, we hope future research addresses all those questions.

Yet another interesting possibility is to consider the case in which the buyer and seller interact over time (see Carrasco et al. (2015b)), and he is uncertain regarding the stochastic process by which her valuation evolves over time. One might assume that the seller knows the distribution of the first period valuation but is unaware of its evolution or, in the spirit of what we do in this paper, the seller might be fully ignorant and only know an ex-ante moment condition. We are, in fact, currently working on these possibilities.

Appendix

Proof of Lemma 3.1. The set $\mathcal{D}$ is compact and the constraints of problem (8) are closed with respect to the weak topology. It is straightforward to see that the objective function is also lower semi-continuous with respect to the weak topology ($\Pi(v)$ is a left-continuous and non-decreasing function). The second constraint must bind. Otherwise, using the Lagrangian approach presented in text, we know that if $\tilde{\xi} = 0$, then the distribution that attains the minimum would be concentrated at $v = 0$ since $\Pi(v) + \tilde{\lambda}$ is a positive function for $v \in (0, 1]$. However, this violates the second constraint (unless $k = q = \tilde{\lambda} = 0$, in which case the Dirac measure concentrated at $v = 0$ is the optimal distribution). Hence, $\tilde{\xi} > 0$ and the second constraint must bind. In the proof of Proposition 3.1 we argue that if $\tilde{\xi} > 0$, then $\tilde{\lambda} > 0$ and, consequently, the first constraint should bind at the robust mechanism. $\square$

Proof of Lemma 3.2. Defining $\psi(v) = \int_0^v q(\tau)d\tau$, the hypothesis of the lemma is equivalent to $v\psi'(v) - \psi(v) \geq \tilde{\xi}(v - \tilde{v})$, for all $v \in [0, 1]$. Now, notice that

$$\begin{aligned} \tfrac{\psi(v)}{v} - \tfrac{\psi(\tilde{v})}{\tilde{v}} - \lambda^* \left(\tfrac{1}{v} - \tfrac{1}{\tilde{v}}\right) &= \int_{\tilde{v}}^v \left(\tfrac{\psi(\tau)}{\tau}\right)' d\tau + \tilde{\lambda} \int_{\tilde{v}}^v \tfrac{1}{\tau^2} d\tau \\ &= \int_{\tilde{v}}^v \tfrac{\tau\psi'(\tau) - \psi(\tau)}{\tau^2} d\tau + \tilde{\lambda} \int_{\tilde{v}}^v \tfrac{1}{\tau^2} d\tau \\ &\geq \int_{\tilde{v}}^v \tfrac{\tilde{\xi}(\tau - \tilde{v}) + \tilde{\lambda}}{\tau^2} d\tau = \tilde{\xi} \ln \left(\tfrac{v}{\tilde{v}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} q(v) &\geq \tfrac{\psi(v)}{v} + \tilde{\xi}\left(1 - \tfrac{\tilde{v}}{v}\right) \\ &\geq \tfrac{\psi(\tilde{v})}{\tilde{v}} + \tilde{\xi} \ln \left(\tfrac{v}{\tilde{v}}\right) + \tilde{\xi}\left(1 - \tfrac{\tilde{v}}{v}\right) + \tilde{\lambda}\left(\tfrac{1}{v} - \tfrac{1}{\tilde{v}}\right) \\ &= \tfrac{1}{\tilde{v}} \int_0^{\tilde{v}} q(\tau)d\tau + \tilde{\xi} \ln \left(\tfrac{v}{\tilde{v}}\right). \end{aligned}$$

It is straightforward to verify that $v\tilde{q}(v) - \int_0^v \tilde{q}(\tau)d\tau = \tilde{\xi}v - \tilde{\lambda}$, for all $v \in [0, 1]$. $\square$

Proof of Proposition 3.1. Let q be any non-decreasing allocation and $\Pi(v)$ the associated profit function. Without loss of generality, we can suppose that Π is not identical null. Using the Lagrangian approach presented in the text we know that there exist $\tilde{\xi} > 0$ and $\tilde{\lambda} \geq 0$ such that $\Pi(v) \geq \tilde{\xi}v - \tilde{\lambda}$, for all $v \in [0, 1]$. Notice that we can assume $\tilde{\lambda} > 0$ without loss of generality. By Lemma 3.2, $q(v) \geq \frac{1}{\tilde{v}} \int_0^{\tilde{v}} q(\tau)d\tau + \tilde{\xi} \ln \left(\frac{v}{\tilde{v}}\right)$, for all $v \in [0, 1]$, where $\tilde{v} = \tilde{\lambda}/\tilde{\xi}$. In particular, since $q(1) \leq 1$, we have that $\tilde{\xi} \ln \left(\frac{1}{\tilde{v}}\right) \leq 1$. Therefore, if $\tilde{q}(v) = \max \left\{\tilde{\xi} \ln \left(\frac{v}{\tilde{v}}\right), 0\right\}$, then $\tilde{q}(1) \leq 1$ and

$$\Pi(v) \geq \Pi^*(v) := v\tilde{q}(v) - \int_0^v \tilde{q}(\tau)d\tau = \max \left\{\tilde{\xi}(v - \tilde{v}), 0\right\},$$

i.e., $\tilde{q}(\cdot)$ is a non-decreasing allocation and attains the envelope profit function, which proves the result. Finally, notice that the robust mechanism, the constraint $\tilde{\xi}\ \ln\ \left(\frac{1}{\tilde{v}}\right) \leq 1$ should bind, which implies that $\tilde{v} > 0$, since $\tilde{\xi} > 0$. Hence, $\tilde{\lambda} > 0$. □

Proof of Proposition 3.2. Notice that, at the optimal solution, constraint (12) of problem (11) must be binding exactly at $v = 1$, i.e., $\tilde{\xi} = -1/\ \ln\ \tilde{v}$. The problem then simplifies to

$$\max_{\tilde{v}\in[0,1]} \frac{\tilde{v} - k}{\ln\ \tilde{v}}.$$

The first-order condition amounts to $\frac{1}{\ln\ \tilde{v}} + \frac{k-\tilde{v}}{\tilde{v}(\ \ln\ \tilde{v})^2} = 0$ which is equivalent

$$\tilde{v}(1 -\ \ln\ \tilde{v}) = k.$$

Since the objective function is strictly concave, this last equation has a unique solution, which we call v^*, and is the solution of our maximization problem. Now the expected profit is

$$\xi^*[k - v^*] = -\frac{1}{\ln\ v^*}[k - v^*] = v^*,$$

which concludes the proof. □

Proof of Proposition 3.4. 1) Characterization. Let f^* be the absolutely continuous part of distribution F^*. For any implementable allocation $q(\cdot)$, integration by parts yields

$$\int_0^1 \left[vq(v) - \int_0^v q(\tau)\, d\tau\right] f^*(v)\, dv = \int_0^1 \left[v - \frac{F^*_-(1) - F^*(v)}{f^*(v)}\right] q(v) f^*(v)\, dv + q(1) - \int_0^1 q(\tau)\, d\tau.$$

Therefore,

$$\int_0^1 \left[vq(v) - \int_0^v q(\tau)\, d\tau\right] f^*(v)\, dv = \int_0^1 \left[v - \frac{1-F^*(v)}{f^*(v)}\right] q(v) f^*(v)\, dv + \big(1 - F^*_-(1)\big)\, q(1)\,.$$

By making

$$v - \frac{1 - F^*(v)}{f^*(v)} = 0, \tag{31}$$

the nature guarantees that the seller will be indifferent among any incentive compatible mechanism such that $q(1) = 1$. In particular, the one in equation (14) is a best reply by the seller if equation (31) is satisfied. Solving (31) amounts to solving

$$\frac{d}{dv}\left[vF^*(v)\right]=1,$$

which implies that

$$F^*(v)=1-\frac{a}{v},$$

and $f^*(v)=\frac{a}{v^2}$, for all $v\in[a,1]$, where $a=1-F^*_-(1)$. Now, since $F^*\in\mathcal{F}$ we must have

$$k=\int_0^1 vdF^*(v)=\int_a^1 vf^*(v)\,dv+1-F^*_-(1),$$

which implies that $k=a\int_a^1\frac{dv}{v}+a$, or $k=a(1-\ln\ a)$, i.e., $a=v^*$.

The discussion that precedes Proposition 3.1 and the derivation of the mechanism in (14) Proposition 3.2 establishes that nature is indifferent among any distribution in $\mathcal{F}$ with support contained in $[v^*,1]$; if the seller chooses the mechanism in equations (14) and (15). Hence, F^* is a nature's best response.

2) Uniqueness. Given the Nash equilibrium $((q^*,p^*),F^*)$ characterized above,

$$v^*=\int\Pi^*(v)dF^*(v)\leq\int\Pi^*(v)dF(v),$$

for all $F\in\mathcal{F}$, and for all non-decreasing positive allocation $q(\cdot)$ such that $q(1)\leq 1$

$$\int\Pi(v)dF^*(v)=v^*\left(q(1)-\int_0^{v^*}q(\tau)\,d\tau\right)\leq v^*, \tag{32}$$

where $\Pi(v)=vq(v)-\int_0^v q(\tau)d\tau$.

Suppose that there exists another Nash equilibrium $((\bar{q},\bar{p}),\bar{F})$. The mechanism q^* assures at least an expected profit of v^* and therefore the mechanism $\bar{q}$ gives

$$\int\bar{\Pi}(v)d\bar{F}(v)\geq v^*,$$

where $\bar{\Pi}(v)=v\bar{q}(v)-\int_0^v\bar{q}(\tau)d\tau$. On the other hand, since the nature is minimizing the expected profit, it cannot attain a payoff lower than v^* when deviating to distribution F^*. Then, using (32) we must have

$$\int\bar{\Pi}(v)dF^*(v)=v^*\left(\bar{q}(1)-\int_0^{v^*}\bar{q}(\tau)\,d\tau\right)=v^*.$$

We then necessary have $\bar{q}(1) = 1$ and $\bar{q}\left(v_{-}^{*}\right) = 0$. Hence, by continuity $\bar{\Pi}(v^{*}) = 0$ and, by Proposition 3.1, $v^{*} \leq \bar{v}$, where $\bar{I} = [\bar{v}, 1]$ contains the support of $\bar{F}$. This implies that $\bar{F}_{-}(v^{*}) = 0$. If $x\left(1 - \bar{F}(x)\right) > v^{*}$, then the mechanism

$$q(v) = \begin{cases} 0, & \text{if } v \leq x \\ 1, & \text{if } x < v \leq 1 \end{cases}$$

gives expected profit $\int \left(vq(v) - \int_0^v q(\tau)\, d\tau\right) d\bar{F}(v) = \int_x^1 x d\bar{F}(x) = x\left(1 - \bar{F}(x)\right) > v^{*}$. Hence, $x\left(1 - \bar{F}(x)\right) \leq v^{*}$, for all x. Now

$$k = \int_0^1 \left(1 - \bar{F}(x)\right) dx = v^{*} + \int_{v^{*}}^1 \left(1 - \bar{F}(x)\right) dx \leq v^{*} + \int_{v^{*}}^1 \frac{v^{*}}{x} dx = v^{*} + v^{*} \ln \left(\frac{1}{v^{*}}\right) = k.$$

Therefore, $x\left(1 - \bar{F}(x)\right) = v^{*}$, for all $x > v^{*}$, and hence $\bar{F} = F^{*}$.

From Proposition 3.1, there exist $\bar{\xi} > 0$ and $\bar{\lambda} \geq 0$ such that $\bar{\Pi}(v) \geq \bar{\xi}(v - \bar{v})$, for all v, where $\bar{k} = \frac{\bar{\lambda}}{\bar{\xi}}$. By Lemma 3.2, we have

$$\bar{q}(v) \geq \frac{\int_0^{\bar{v}} \bar{q}(\tau)\, d\tau}{\bar{k}} + \bar{\xi} \ln \left(\frac{v}{\bar{v}}\right), \text{ for all } v.$$

In particular, $1 \geq \bar{\xi} \ln \left(\frac{1}{\bar{v}}\right)$. Since,

$$\bar{\xi}(k - \bar{v}) \leq \frac{k - \bar{v}}{\ln \left(\frac{1}{\bar{v}}\right)} < \frac{k - v^{*}}{\ln \left(\frac{1}{v^{*}}\right)} = v^{*},$$

if $\bar{v} \neq v^{*}$ we conclude that $\bar{v} = v^{*}$. Hence, $\bar{q}(v) \geq q^{*}(v)$, for all v. Let $G \in \mathcal{F}$ be a distribution such that $dG = a\delta_{v^{*}} + (1 - a)\delta_1$, where $a = \frac{1-k}{1-v^{*}}$ for $\int v dG(v) = k$ and δ_v is the Dirac measure at v. Then,

$$\begin{aligned} \int \bar{\Pi}(v)\, dG(v) &= (1 - a)\left(\bar{q}(1) - \int_0^1 \bar{q}(x)\, dx\right) + av^{*} = (1 - a)\left(1 - \int_{v^{*}}^1 \bar{q}(x)\, dx\right) + av^{*} \\ &\leq (1 - a)\left(1 - \int_{v^{*}}^1 q^{*}(x)\, dx\right) + av^{*} = \int \Pi^{*}(v)\, dH(v) = v^{*}. \end{aligned}$$

Therefore, $\bar{q}(x) = q^{*}(x)$ almost surely, and then $\bar{q} = q^{*}$. □

Proof of Proposition 4.1. Let $m^{*} = (\boldsymbol{q}^{*}, p)$ be the mechanism where q^{i*} is the robust allocation in one good case with mean is k^i:

$$q^{i*}(\boldsymbol{v}) = \begin{cases} 0, & \text{if } v^i < v^{i*} \\ 1 - \frac{\ln v^i}{\ln v^{i*}}, & \text{if } v^i \geq v^{i*}, \end{cases}$$

for $i = 1, ..., M$, and the price p is defined by (23) and (25). From (23), we get

$$\mathcal{U}_{m^*}(\boldsymbol{v}) = \sum_{i=1}^{M} U_i^*(v^i),$$

where $U_i^*(v^i) = \int_0^{v_i} q^{i*}(\tau) d\tau$. Therefore,

$$\Pi_{m^*}(\boldsymbol{v}) = \sum_{i=1}^{M} \Pi_i^*(v^i).$$

Given any distribution $F \in \mathcal{F}$, we have that

$$\int \Pi_{m^*}(\boldsymbol{v}) dF(\boldsymbol{v}) = \sum_{i=1}^{M} \int \Pi_i^*(v^i) dF_i(v^i),$$

where $\Pi_i^*(v^i) = v^i q^{i*}(v^i) - U_i^*(v^i)$ and $dF_i(v^i) = \int_{v^{-i}} dF(v^i, v^{-i})$ is the i-marginal distribution of F. Let F_i^* be the critical distribution of the one-good case with mean k^i characterized by (17) when $k = k^i$. Since the marginal distribution $F_i(v^i)$ has mean k^i, the robustness of the allocation in the one-good case implies that

$$\int \Pi_i^*(v^i) dF_i(v^i) \geq \int \Pi_i^*(v^i) dF_i^*(v^i) = v^{i*}$$

and, therefore,

$$\int \Pi_{m^*}(\boldsymbol{v}) dF(\boldsymbol{v}) \geq \sum_{i=1}^{M} v^{i*} = v_A^*.$$

This implies that by selling the goods separately, the seller can guarantee payoff at least v_A^*. In other words, the maximin value of the game is at least v_A^*. □

Proof of Proposition 4.2. For completeness we characterize the solution of problem (30) for general vector $\boldsymbol{k}$. Defining $\eta = \mathbf{1} \cdot \xi$ and $\tilde{v} = \xi \cdot \tilde{\boldsymbol{v}}$, problem (30) is equivalent to

$$\begin{aligned} &\max_{\eta,\xi,\tilde{v}\geq 0} \xi \cdot \boldsymbol{k} - \tilde{v} \\ &s.t.\ \mathbf{1} - (\ln \eta - \ln \tilde{v})\, \xi \geq 0 \\ &\qquad \eta - \xi \cdot \mathbf{1} \geq 0. \end{aligned}$$

Let $a \in \mathbb{R}^M_+$ and $b \geq 0$ be the Lagrangian multipliers of the first and second constraints. The following first-order conditions

$$\begin{cases} -\frac{\xi \cdot a}{\eta} + b \leq 0 \\ -1 + \frac{\xi \cdot a}{\tilde{v}} \leq 0 \\ \boldsymbol{k} - (\ln \eta - \ln \tilde{v})\, a - b\mathbf{1} \leq 0 \end{cases}$$

are necessary for optimality[15] with the usual slackness conditions.

Guessing (and then verifying) that these first-order conditions and slackness conditions are binding, we must have that $\tilde{v} = \xi \cdot a = b\eta$,

$$a = \frac{1}{\ln \eta - \ln \tilde{v}} (\boldsymbol{k} - b\mathbf{1}) \text{ and } \xi = \frac{1}{\ln \eta - \ln \tilde{v}} \mathbf{1}.$$

Hence,

$$\eta = \mathbf{1} \cdot \xi = \frac{M}{\ln \eta - \ln \tilde{v}} \text{ and } \tilde{v} = \xi \cdot a = \frac{\mathbf{1} \cdot \boldsymbol{v} - bM}{(\ln \eta - \ln \tilde{v})^2} = b\eta = \frac{bM}{\ln \eta - \ln \tilde{v}},$$

which implies that

$$\frac{\mathbf{1} \cdot \boldsymbol{k}}{Mb} - 1 = \ln \eta - \ln \tilde{v} = \frac{M}{\eta}.$$

and, therefore, $\tilde{v} = \eta \exp\ [-M/\eta]$. However, $\tilde{v} = b\eta$ implies that $b = \exp\ [-M/\eta]$. Plugging this back into the previous expression, we get

$$b(1 - \ln b) = \frac{\mathbf{1} \cdot \boldsymbol{k}}{M},$$

[15]Notice that, for each $\eta \geq 0$, the Lagrangian is a concave functional of $(\xi, \tilde{v})$. Therefore, at the optimum choice of η, the first-order and slackness conditions for $(\xi, \tilde{v})$ are sufficient for optimality. We are deriving the necessary conditions of Lagrangian when the optimum η is interior. For sufficiency, we only have to discard corner solution for η. However, an easy inspection shows that the optimum η must be interior.

which gives a unique solution for b and pins down the values of $\tilde{v}$ and ξ:

$$\tilde{v} = -\frac{Mb}{\ln b} \text{ and } \xi = -\frac{1}{\ln b}\mathbf{1}.$$

Notice that substituting these values into (29) we get the optimal allocation and the optimal profit, given by

$$\xi \cdot \boldsymbol{k} - \tilde{v} = -\frac{\mathbf{1} \cdot \boldsymbol{k}}{\ln b} + \frac{Mb}{\ln b} = \frac{Mb - \mathbf{1} \cdot \boldsymbol{k}}{\ln b} = Mb.$$

□

Proof of Proposition 4.4. For each $i = 1, ..., M$, define the function $\varphi : [0,1] \to [0,1]^M$ by

$$\varphi^i(x) := \min\left\{\frac{v^{i*}}{1-x}, 1\right\}, \text{ for all } i = 1, ..., M.$$

Notice that φ^i is increasing, Lipschitz and differentiable except at $x = 1-v^{i*}$.[16] Also, $\varphi^i([0, 1 - v^{i*})) = [v^{i*}, 1)$ and $\varphi^i([0,1]) = [v^{i*}, 1]$.

Define the distribution $F \in \mathcal{F}$ as the push-forward of the Lesbegue measure with respect to the function φ:

$$F(\boldsymbol{v}) = \text{ Prob } \left(\left\{x \in [0,1]; \varphi^i(x) \le v^i, \text{ for all } i = 1, ..., M\right\}\right).$$

We have that

$$F(v^i, \mathbf{1}^{-i}) = \left(1 - \frac{v^{i*}}{v^i}\right)^+ + v^{i*} H_1\left(v^i\right),$$

for all $v^i \in [0,1]$, where H_1 is the Heaviside at 1 (whose "derivative" in the distributional sense is the Dirac measure concentrated at 1). Indeed, $F\left(v^i, \mathbf{1}^{-i}\right) = \Pr\left(\{x \in [0,1]; \varphi^i(x) \le v^i\}\right)$, since $\varphi^i \le 1$. It is clear that $F(\mathbf{1}) = 1$. Since $\varphi^i \ge v^{i*}$, we conclude that $F\left(v^i, \mathbf{1}^{-i}\right) = 0$ when $v^i < v^{i*}$. Now, if $v^{i*} \le v^i < 1$, then $\varphi^i(x) \le v^i$ if and only if $0 \le x \le 1 - \frac{v^{i*}}{v^i} = F(v^i, \mathbf{1}^{-i})$.

Fix an arbitrary incentive compatible mechanism, and let $\Pi(\boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{q}(\boldsymbol{v}) - \mathcal{U}(\boldsymbol{v})$ be the profit function it induces. Since the support of F is contained in $\times_{i=1}^M [v^{i*}, 1]$, it is without loss of generality to restrict attention to mechanisms such that $\mathcal{U}(\boldsymbol{v}^*) = 0$. Let $\bar{\boldsymbol{q}}(x) = \boldsymbol{q}(\varphi(x))$ and $U(x) = \mathcal{U}(\varphi(x))$ be the allocation and rent functions induced by the mechanism. From the change of variable's theorem, we get

$$\int \Pi(\boldsymbol{v})\, dF(\boldsymbol{v}) = \int_0^1 \left[\varphi(x) \cdot \bar{\boldsymbol{q}}(x) - U(x)\right] dx. \tag{33}$$

[16]Indeed, for all $x \in [0, 1 - v^{i*})$, $\varphi^i(x) = \frac{v^{i*}}{1-x}$ and then $\varphi^{i\prime}(x) = \frac{v^{i*}}{(1-x)^2} \le 1$. For $x > 1 - v^{i*}$, $\varphi^{i\prime}(x) = 0$.

First, suppose that $\mathcal{U}$ is differentiable. Then, the derivative of $U(x)$ is given by

$$U'(x) = \nabla\mathcal{U}(\varphi(x)) \cdot D\varphi(x) = \bar{\boldsymbol{q}}(x) \cdot D\varphi(x), \tag{34}$$

where $D\varphi$ is the differential of φ. Since $U(x) = \int_0^x \bar{\boldsymbol{q}}(\tau) \cdot D\varphi(\tau) d\tau + U(0)$ and $U(0) = \mathcal{U}(v^*) = 0$, we have that

$$\begin{aligned} \textstyle\int_0^1 U(x)\,dx &= \textstyle\int_0^1 \left(\int_0^x U'(\tau)\,d\tau\right) dx \\ &= \textstyle\int_0^1 (1-x)\,U'(x)\,dx \\ &= \textstyle\int_0^1 (1-x)\,\bar{\boldsymbol{q}}(x) \cdot D\varphi(x)dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \int \Pi(\boldsymbol{v})\,dF(\boldsymbol{v}) &= \sum_{i=1}^{M} \left(\int_0^1 \varphi^i(x)\,\bar{q}^i(x)\,dx - \int_0^1 (1-x)\,\bar{q}^i(x)\,\varphi^{i\prime}(x)\,dx \right) \\ &= \sum_{i=1}^{M} \int_0^1 \left(\varphi^i(x) - (1-x)\,\varphi^{i\prime}(x)\right) \bar{q}^i(x)\,dx \\ &= \sum_{i=1}^{M} \int_{1-\tilde{k}^i}^1 \bar{q}^i(x)\,dx \le v_A^*, \end{aligned}$$

where we have used the fact that $\varphi^i(x) - (1-x)\,\varphi^{i\prime}(x)$ in $[0, 1-v^{i*})$ and $\varphi^i(x) - (1-x)\,\varphi^{i\prime}(x) = 1$ in $[1-v^{i*}, 1]$.

For the general case, consider $x', x'' \in [0,1]$. Since $\mathcal{U}$ is a convex function

$$\begin{aligned} U(x') - U(x'') &\le (\varphi(x') - \varphi(x'')) \cdot \bar{\boldsymbol{q}}(x'') \text{ and} \\ U(x'') - U(x') &\le (\varphi(x'') - \varphi(x')) \cdot \bar{\boldsymbol{q}}(x'). \end{aligned}$$

Hence, $|U(x') - U(x'')| \le 2\,|x' - x''|$, because $\varphi^{i\prime}, \bar{q}^i \le 1$. Hence, U is differentiable and

$$U'(x) = \sum_{i=1}^{M} \varphi^{i\prime}(x)\,\bar{q}^i(x),$$

i.e., we get the same expression of (34) and the same result goes through. Therefore, minimax value of the zero-sum game is v_A^*. Combining this with Proposition 4.1, we obtain that selling the goods separately is a robust mechanism. □

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