Prebidding vs. Postbidding in First-Price Auctions with and without Head-starts

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Abstract

We study the effect of prebidding and postbidding in first-price auctions with a single prize under incomplete information. All the bidders' values are private information except bidder 1's value which is commonly known. Bidder 1 places his bid either before (prebidding auction) or after (postbidding auction) all the other bidders. We show that for relatively small (high) values of bidder 1 the prebidding auction yields a lower (higher) expected highest bid than the postbidding auction. However, by giving head-starts, for relatively small (high) values of bidder 1, the prebidding auction yields a higher (lower) expected bid than the postbidding auction. In other words, head-starts may completely change the comparative benefit of the seller in prebidding and postbidding first-price auctions.

Keywords: First-price auctions, prebidding, postbidding, head-starts.

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1 Introduction

One of the most fundamental results in auction theory is the revenue equivalence theorem established by Myerson (1981) according to which the expected revenue of the seller in equilibrium is independent of the auction mechanism under quite general conditions.[1] One of these conditions is that the players are ex-ante symmetric. When players submit their bids sequentially the symmetry of the players is broken and the revenue equivalence of auction mechanisms does not hold.[2] Players submit their bids sequentially in auctions particularly if the seller wishes to favor specific bidders (Laffont and Tirole 1991). Favoritism of one (or several) of the bidders can occur when the favorite bidder obtains the option to observe the bids of all the other bidders and only then submits his bid (Arozamena and Weinschelbaum 2009), or, alternatively, when the seller offers the option for one of the bidders to change his bid for a bribe (Menezes and Monterio 2006). Also favoritism can occur by letting a bidder submit his bid before all the others and then giving him a head-start.

To illustrate such a sequential auction, consider a monopoly that won the exclusive license to operate during some period of time. When this period of time is over, the operating licence is resold by a first-price auction. The candidates to buy the licence through the auction include the monopoly that won it in the previous period and several new competitors who covet the license. The difference between them is that the information about the monopoly is commonly known, while the information about the new competitors is private information. Given this asymmetric information, the designer (seller) can choose between different forms of auctions. For example, the monopoly can be given the right to bid with or without a head-start before the other bidders who have observed the monopoly's bid before they place

[1] See Lorentziadis (2016) for more information about the optimal bidding in auctions from a game theory prespective.

[2] For revenue equivalence of asymmetric auctions, see, for example, Fibich and Gavious (2003, 2010) and Fibich et al. (2004).

their own bids. Alternatively, the monopoly can observe the bids of all the other bidders and only then place a bid. These sequential first-price auctions will be referred to as prebidding and postbidding (first-price) auctions.

Formally, we study sequential first-price auctions under incomplete information where bidder 1's value for the prize is commonly known while the other bidders' values are private information. In the prebidding auction, bidder 1's places a bid in the first stage. In the second stage, all the other bidders observe bidder 1's bid and simultaneously submit their own bids. In contrast, in the postbidding auction, the bidders whose values are private information simultaneously submit their bids in the first stage. Then, in the second stage, bidder 1 observes the other bids and then submits his bid. Our goal is to analyze each of these forms of auctions and, particularly, to understand which of them and under which conditions yields a higher expected highest bid.

The analysis of the postbidding auction is quite straightforward since bidder 1's bid in the second stage plays the role of a reserve price for all the other bidders in the first stage. Thus, if the value of bidder 1 is not higher than the optimal reserve prize of a simultaneous auction, he has a positive effect on the expected highest bid. However, if his value is sufficiently high then he might have a negative effect on the expected highest bid and it would be better to exclude him.

The analysis of the prebidding auction, on the other hand, is not straightforward. In that auction, although bidder 1's bid plays the role of a reserve prize for all the other bidders in the second stage, regardless of his value for the prize, he has a positive effect on the expected highest bid. Furthermore, the expected highest bid increases in bidder 1's value for the prize. The intuition for these results is that, in contrast to the postbidding auction or the standard simultaneous auction, the expected highest bid in the prebidding auction is larger than or equal to the reserve price that is bidder 1's bid in the first stage.

The above argument suggests that the expected highest bid in the prebidding auction is higher than in the postbidding auction. Indeed, for relatively high values of bidder 1, this result holds since bidder 1 in the prebidding auction has a positive effect on the expected highest bid, while in the postbidding auction he has a negative effect. However, we show that for sufficiently low values of bidder 1, the expected highest bid in the postbidding auction might be higher than in the prebidding auction. The reasoning is that since bidder 1's value is always larger than his bid, the reserve price that the other bidders face in the postbidding auction is larger than in the prebidding auction. Therefore, since the expected highest bid increases in the reserve price for relatively small values, we obtain that the expected highest bid in the postbidding auction is larger than in the prebidding auction.

As a last step, in both the prebidding or postbidding auctions, we let the seller give head-starts to the bidders in the first stage. For this purpose, we assume that each of the bidders in the prebidding auction wins if his bid is larger than or equal to kb_1 where b_1 is bidder 1's bid in the first stage and k is a constant larger than one. Similarly, bidder 1 in the postbidding auction wins the contest if his bid is larger than or equal to $kb_{\max}$ where $b_{\max}$ is the highest bid of the bidders in the first stage. We show that when we compare between the prebidding and postbidding auctions the results are completely different with and without head-starts. In particular, with optimal head-starts, for relatively low values of bidder 1, the prebidding auction yields a higher expected bid than the postbidding auction, but for relatively high values of bidder 1 and if the number of players is sufficiently high, the postbidding auction yields a higher expected bid. The reason is that given the high value of bidder 1 and the head-start, the bidders in the second stage of the prebidding auction do not have any incentive to participate, and then the seller's revenue is limited by bidder 1's bid. On the other hand, in the postbidding auction, regardless of the value of bidder 1, the seller's revenue increases in the number of bidders and is limited only by the highest possible value of these bidders.

Several papers in the literature study head-starts in contests. Corns and Schotter (1999) demonstrated by theoretical and empirical arguments that a head-start in the form of a price preference policy that is given to a subset of firms might not only benefit that subset but can actually lower the purchasing cost of the government. Kirkegaard (2012) studied asymmetric all-pay auctions with head-starts under incomplete information where players simultaneously choose their efforts, and showed that the total effort increases if the weak contestant is favored with a head-start. Segev and Sela's (2014) revealed that in a sequential all-pay auction by giving a head-start to the bidder who places the first bid a designer can increase the players' expected highest effort.

Sequential auctions have also received some attention. For example, Pitchik and Schoter (1988) analyzed sequential first and second price auctions with a budget constraint and two different prizes. Benoit and Krishna (2001) analyzed sequential first and second price auctions with synergy between the stages and a budget constraint, and Pitchik (2009) analyzed a sequential auction with a budget constraint under incomplete information.[3] These papers deal with sequential auctions in which a prize is awarded in each stage of the auction when the link between the stages is made by the bidders' budget constraints. On the other hand, in our paper, like in Arozamena and Weinschelbaum (2009), only one prize is awarded in the last stage which links between the stages of the auction.

The rest of the paper is organized as follows: In Section 2 we analyze our postbidding and prebidding auctions and compare between them. In section 3 we compare between these two auctions by allowing for head-starts. In section 4 we conclude.

[3] See also Brusco and Lopomo (2008, 2009) who considered sequential auctions with a budget constraint and with and without synergy between the values of the prizes.

2 First-price auctions

2.1 The postbidding auction

We first consider a first-price auction with n risk neutral bidders and an indivisible prize where bidder 1 has a commonly known value for the prize v_1, while bidder $i, i = 2, 3, ..., n$ has a private value for the prize v_i which is drawn independently from a continuously differentiable distribution function $F(v)$ over the support $[0, 1]$ with a density f. In the first stage, $n - 1$ bidders who know the value of bidder 1 simultaneously submit their bids $b_i(v_i), i = 2, 3, ..., n$. In the second stage, bidder 1 observes the bids of the other $n - 1$ bidders and then submits his bid $b_1(v_1)$. The bidder with the highest bid wins the prize and pays his bid. In the case of a tie in which bidder 1 also submits the highest bid, he wins the prize. We term the above form of a sequential first-price auction a postbidding (first-price) auction.

In order to analyze the perfect Bayesian equilibrium of this auction we begin by considering the equilibrium effort function of bidder 1 in the second period. He submits a bid that is equal to the highest bid in the first stage as long as his value v_1 is larger than or equal to the highest bid; otherwise he stays out of the auction. Formally, the equilibrium bid of bidder 1 is given by

$$b_1(v_1) = \begin{cases} 0 & \text{if} \quad 0 \leq v_1 < b_{\max}(v_{2,...,}v_n) \\ b_{\max}(v_{2,...,}v_n) & \text{if} \quad b_{\max}(v_{2,...,}v_n) \leq v_1 \leq 1 \end{cases}$$

where $b_{\max}(v_{2,...,}v_n)$ is the highest bid of the $n - 1$ bidders in the first stage. Assume that in the first stage there is a monotonic and differentiable equilibrium bid function $x_i = b_i(v_i), i = 2, ..., n$. The inverse bid function in this case will be defined as $y_i(x_i)$. Then, the maximization problem of bidder i is

$$\max_x U_i(x) = F^{n-2}(y_i(x))(v_i - x)$$
$$s.t. \quad x \geq v_1$$

The solution of the above maximization problem gives us the equilibrium strategies of the standard (simultaneous) first-price auction with $n-1$ symmetric bidders who face a reserve price v_1. That is, the equilibrium of bidder $i, i = 2, .., n$ is[4]

$$b_i(v_i) = \begin{cases} 0 & \text{if} \quad 0 \leq v_i < v_1 \\ v_i - \frac{1}{F^{n-2}(v_i)} \int_{v_1}^{v_i} F^{n-2}(s)ds & \text{if} \quad v_1 \leq v_i \leq 1 \end{cases}$$

Thus, the seller's expected revenue in the postbidding auction is given by (see Krishna 2009)

$$R_{post} = 1 - v_1 F^{n-1}(v_1) - \int_{v_1}^{1} (n-1)F^{n-2}(v) - (n-2)F^{n-1}(v)dv \tag{1}$$

Since bidder 1's value in the postbidding auction acts as a reserve price for the $n-1$ bidders in the first stage, bidder 1's contribution to the seller's revenue is quite straightforward.

Proposition 1 *In the postbidding first-price auction there is a critical value* $0 < v^*_{post} < 1$ *such that for all values* $v_1 \leq v^*_{post}$ *bidder 1 has a positive effect on the expected highest bid while for all values* $v_1 > v^*_{post}$ *he has a negative effect and then it is better to exclude him from the auction.*[5]

In the following, we compare the seller's revenue in the postbidding first-price auction with that of the prebidding first-price auction.

[4] It is assumed that if bidder $i, i = 2, .., n$ has no chance to win he submits a bid of zero.

[5] Arozamena and Weinschelbaum (2009) showed that if the value of bidder 1 is not commonly known, as in our case, and instead is private information, then bidder 1 always has a positive effect on the seller's revenue in the postbidding first-price auction.

2.2 The prebidding auction

We again consider the same auction, but now in the first stage bidder 1 submits a bid $b_1(v_1)$, and in the second stage, the other $n-1$ bidders observe bidder 1's bid and then each of them simultaneously submits a bid $b_i(v_i), i = 2, 3, ..., n$. The bidder with the highest bid wins the prize and pays his bid. In the case of a tie, one of the $n-1$ bidders who placed the highest bid in the second stage wins. We term the above form of a sequential first-price auction a prebidding (first-price) auction.

In order to analyze the perfect Bayesian equilibrium of this auction we begin with the second stage and go backwards to the previous one. Assume that in the second stage there is a monotonic and differentiable equilibrium bid function $x_i = b_i(v_i), i = 2, ..., n$. The inverse bid function in this case will be defined as $y_i(x_i)$. Then, the maximization problem of bidder i is

$$\begin{aligned} \max_x U_i(x) &= F^{n-2}(y_i(x))(v_i - x) \\ s.t. \qquad x &\geq b_1 \end{aligned}$$

where $b_1 = b_1(v_1)$ is bidder 1's bid in the first stage. The solution of the above maximization problem gives us the equilibrium strategies of the standard (simultaneous) first-price auction with $n-1$ symmetric bidders who face a reserve price of $b_1 = b_1(v_1)$. That is, for every $i = 2, .., n$,

$$b_i(v) = \begin{cases} 0 & \text{if} \quad 0 \leq v_i < b_1 \\ v_i - \frac{1}{F^{n-2}(v_i)} \int\limits_{b_1}^{v_i} F^{n-2}(s)ds & \text{if} \quad b_1 \leq v_i \leq 1 \end{cases} \tag{2}$$

Then, the maximization problem of bidder 1 in the first stage is

$$\max_x U_1(x) = F^{n-1}(x)(v_1 - x)$$

The F.O.C. is

$$\frac{\partial U_1(x)}{\partial x} = (n-1)F^{n-2}(x)f(x)(v_1 - x) - F^{n-1}(x) = 0$$

By substituting $y(x) = v$, $x = b_1(v)$ and rearranging, we obtain that the equilibrium strategy of bidder 1 is given by

$$\frac{F(b_1(v))}{(n-1)f(b_1(v))} = v - b_1(v) \tag{3}$$

It is quite obvious that the equilibrium strategy of bidder 1 in the prebidding auction in which his bid is revealed to all the other $n-1$ bidders before they submit their bids will be different than his equilibrium strategy in the simultaneous first-price auction in which his bid is not revealed to all the other bidders. Interestingly, however, there is a class of distribution functions of the bidders' values for which bidder 1's bid is the same whether or not his bid is revealed to all the other $n-1$ bidders. To see that, let $F(x) = x^m$, $m > 0$. Substituting $F(x) = x^m$ in (3) yields

$$b_1(v) = v - \frac{b_1(v)}{m(n-1)}.$$

Rearrangement implies that

$$b_1(v) = \frac{m(n-1)}{m(n-1)+1}v = b_{\mathrm{Si}}(v)$$

where $b_{\mathrm{Si}}(v)$ is the unique symmetric equilibrium in the simultaneous first-price auction with n bidders. Thus, the equilibrium strategy of bidder 1 in the prebidding first-price auction is identical to the equilibrium strategy of a bidder in the simultaneous first-price auction with n bidders.

Similarly to the postbidding auction, bidder 1's bid in the first stage of the prebidding auction acts as a reserve price for the other $n-1$ bidders in the second stage. While in the postbidding auction bidder 1's effect on the seller's revenue is ambiguous, in the prebidding auction his effect is always positive.

Proposition 2 *In the prebidding first-price auction, regardless of his value* v_1*, bidder 1 always has a positive effect on the expected highest bid. Therefore, it is never profitable to exclude bidder 1 from the auction.*

Proof. The seller's expected revenue in the prebidding auction is

$$R_{pre} = b_1 F^{n-1}(b_1) + \int_{b_1}^{1} b(v) dF^{n-1}(v)$$

where $b_1 = b_1(v_1)$ is bidder 1's bid given by (3), and $b(v)$ is the bid of a bidder with type v in the second stage given by (2). Integrating by parts we get

$$R_{pre} = b_1 F^{n-1}(b_1) + 1 - \int_{b_1}^{1} F^{n-2}(s)ds \; - b(b_1)F^{n-1}(b_1) - \int_{b_1}^{1} b'(v)F^{n-1}(v)dv$$

By (2) we have $b'(v) = \frac{(n-2)f(v)}{F^{n-1}(v)} \int_{b_1}^{v} F^{n-2}(s)ds$. Thus, since $b_1 = b(b_1)$ we have

$$R_{pre} = 1 - \int_{b_1}^{1} F^{n-2}(s)ds - \int_{b_1}^{1} ((n-2)f(v) \int_{b_1}^{v} F^{n-2}(s)ds)dv$$

Again, integrating by parts we obtain

$$R_{pre} = 1 - \int_{b_1}^{1} (n-1)F^{n-2}(v) - (n-2)F^{n-1}(v)dv \tag{4}$$

If we exclude bidder 1 from the auction we get the expected highest bid in the simultaneous first-price auction with $n-1$ bidders. Then the seller's expected revenue is given by (see Krishna 2009)

$$R_{\text{Si}}^{n-1} = 1 - \int_{0}^{1} (n-1)F^{n-2}(v) - (n-2)F^{n-1}(v)dv \tag{5}$$

The difference between (4) and (5) is

$$R_{pre} - R_{\text{Si}}^{n-1} = \int_{0}^{b_1} (n-1)F^{n-2}(v) - (n-2)F^{n-1}(v)dv \geq 0$$

■

It is important to note that the main difference between the prebidding and the postbidding auctions is the fact that in the prebidding auction the prize is always given to one of the bidders and the lowest expected price is equal to bidder 1's bid in the first stage. This is the reason that bidder 1's bid positively contributes to the seller's expected revenue in the prebidding auction. Moreover, by (4) we have

$$\begin{aligned}\frac{dR_{pre}}{db_1} &= (n-1)F^{n-2}(b_1) - (n-2)F^{n-1}(b_1) \\ &= F^{n-2}(r)((n-1) - (n-2)F(r)) \geq 0\end{aligned}$$

Thus, the seller's revenue in the prebidding auction increases in bidder 1's value for the prize v_1. Therefore, the seller benefits from a higher bid of bidder 1 in the first stage although it acts as a reserve price for the $n-1$ bidders in the second stage and as such may decrease their bids.

2.3 A comparison of the postbidding and prebidding auctions

We have just shown that in the prebidding auction, independent of his value v_1, bidder 1 has a positive effect on the expected highest bid, while in the postbidding auction bidder 1 has a positive effect of the expected highest bid only for relatively small values. Thus, we obtain that If the value of bidder 1, v_1, is relatively high, then the prebidding first price auction yields a higher expected bid than the postbidding first-price auction. On the other hand, when the value of bidder 1, v_1, is relatively low the results of the comparison of the prebidding and postbidding auctions is not clear. On the one hand, the reserve price in the postbidding auction (v_1) is higher than the reserve price in the prebidding auction ($b_1(v_1)$) and since the expected highest bid increases in the value of the reserve price (given that it is smaller than the optimal value) the expected highest bid in the postbidding auction might be higher than in the prebidding auction. On the other hand, in the prebidding auction the expected highest bid is higher than or equal to the reserve price which is not the case in the postbidding auction, and therefore the expected highest bid in the prebidding auction might be higher than in the postbidding auction. In the following result, however, we provide sufficient conditions according to which for relatively small values of v_1 the postbidding auction yields a higher expected bid than the prebidding auction.

Proposition 3 *If the value of bidder 1, v_1, is relatively high, then the prebidding first price auction yields a higher expected bid than the postbidding first-price auction. However, if $f(x) > 0$ for all x, then for sufficiently small values of v_1, the expected highest bid in the postbidding first-price auction is higher than in the prebidding first-price auction.*

Proof. As we already mentioned the comparison of the prebidding and postbidding when v_1 is relatively high is quite clear. Therefore, we assume that v_1 is relatively low. We first show that if $f(x) > 0$ for all x, then $b_1'(0) < 1$. By (3), in the prebidding auction, bidder 1's bid is given by

$$\frac{F(b_1(v))}{(n-1)f(b_1(v))} = v - b_1(v)$$

Thus,

$$1 - b_1'(v) = \frac{(n-1)f(b_1(v))^2 - (n-1)F(b_1(v))f'(b_1(v))}{((n-1)f(b_1(v)))^2} b_1'(v)$$

Rearranging implies that

$$b_1'(v) = \frac{(n-1)f(b_1(v))^2}{nf(b_1(v))^2 - F(b_1(v))f'(b_1(v))}.$$

Let $f(x) > 0$ for all x, then

$$\lim_{v \to 0} b_1'(v) = \frac{n-1}{n} < 1$$

Now we can compare the seller's revenue in the postbidding auction given by (1) and the seller's revenue in the prebidding auction given by (4). The difference between these revenues is

$$R_{pre} - R_{post} = v_1 F^{n-1}(v_1) - \int_{b_1}^{v_1} (n-1)F^{n-2}(v) - (n-2)F^{n-1}(v) dv \tag{6}$$

and

$$\frac{d}{dv_1}R_{pre} - R_{post} = F^{n-1}(v_1) + v_1(n-1)F^{n-2}(v_1)f(v_1) - ((n-1)F^{n-2}(v_1) - (n-2)F^{n-1}(v_1)) + b'(v_1)((n-1)F^{n-2}(b(v_1)) - (n-2)F^{n-1}(b(v_1)))$$

Note that $\lim_{v_1 \to 0} \frac{d^k}{d(v_1)^k}(R_{pre}(v_1) - R_{post}(v_1)) = 0$ for $k = 1, ..., n-3$ and

$$\lim_{v_1 \to 0} \frac{d^{n-2}}{d(v_1)^{n-2}}(R_{pre}(v_1) - R_{post}(v_1)) = -(n-1)!(1 - b_1'(0))(f(0))^{n-2}$$

Thus, if $f(x) > 0$ for all x, then $b_1'(0) = \frac{(n-1)}{n}$ and we obtain that $\lim_{v_1 \to 0} \frac{d^{n-2}}{d(v_1)^{n-2}}(R_{pre}(v_1) - R_{post}(v_1)) < 0$. Hence, $\lim_{v_1 \to 0}(R_{pre}(v_1) - R_{post}(v_1)) < 0$. ■

We have seen that for sufficiently high values of bidder 1 the prebidding auction yields a higher expected bid than the postbidding auction, and if $f(x) > 0$ for all x, for sufficiently low values of bidder 1, the opposite is true. In the next section we will show that by giving head-starts the results of their comparison could be completely different.

3 First-price auctions with head-starts

3.1 The postbidding auction with head-starts

We assume now that in the postbidding auction bidder 1 wins the contest if his bid b_1 is larger than or equal to kb_h where $b_h = \max\{b_i\}_{i=2}^n$ is the highest bid of the $n-1$ bidders in the first stage, and k is a constant larger than one. The equilibrium bid of bidder 1 in the second stage is given by

$$b_1(v_1) = \begin{cases} 0 & \text{if} \quad 0 \leq v_1 < kb_h \\ kb_h & \text{if} \quad kb_h \leq v_1 \leq 1 \end{cases}$$

Assume that in the first stage there is a monotonic and differentiable equilibrium bid function $x_i = b_i(v_i), i = 2, ..., n$. The inverse bid function in this case will be defined as

$y_i(x_i)$. Then, the maximization problem of bidder i is

$$\begin{aligned}\max_x U_i(x) &= F^{n-2}(y_i(x))(v_i - x) \\ s.t. \quad x &\geq \frac{v_1}{k}\end{aligned}$$

The solution of the above maximization problem gives us the equilibrium strategies of the standard (simultaneous) first-price auction with $n-1$ symmetric bidders who face a reserve price $\frac{v_1}{k}$. That is, for $i = 2, .., n$,

$$b_i(v_i) = \begin{cases} 0 & \text{if} \quad 0 \leq v_i < \frac{v_1}{k} \\ v_i - \frac{1}{F^{n-2}(v_i)} \int\limits_{\frac{v_1}{k}}^{v_i} F^{n-2}(s)ds & \text{if} \quad \frac{v_1}{k} \leq v_i \leq 1 \end{cases}$$

Thus, the seller's expected revenue in such a case is given by

$$R_{post-k} = 1 - \frac{v_1}{k} F^{n-1}(\frac{v_1}{k}) - \int\limits_{\frac{v_1}{k}}^{1} (n-1)F^{n-2}(v) - (n-2)F^{n-1}(v)dv \tag{7}$$

3.2 The prebidding auction with head-starts

We assume here that each of the bidders in the second stage wins the contest if his bid b_i, $i = 2, .., n$ is larger than or equal to kb_1, where b_1 is bidder 1's bid in the first stage and k is a constant larger than one. The equilibrium bid of bidder $i, i = 2, .., n$ in the second stage is given by

$$b_i(v_i) = \begin{cases} 0 & \text{if} \quad 0 \leq v_i < kb_1(v_1) \\ v_i - \frac{1}{F^{n-2}(v_i)} \int\limits_{kb_1(v_1)}^{v_i} F^{n-2}(s)ds & \text{if} \quad kb_1(v_1) \leq v_i \leq 1 \end{cases} \tag{8}$$

The maximization problem for bidder 1 in the first stage is now given by

$$\max_{b_1} U_1(x) = F^{n-1}(kb_1)(v_1 - b_1)$$

When $k > 1$, the bidder 1's equilibrium strategy is then given by

$$b_1 = \begin{cases} b_1(v_1) & v_1 \le v^*_{pre} \\ \frac{1}{k} & v_1 > v^*_{pre} \end{cases}$$

where $b_1(v_1)$ is the solution of

$$\frac{F(kb_1(v))}{(n-1)kf(kb_1(v))} = v - b_1(v) \tag{9}$$

and v^*_{pre} satisfies

$$F^{n-1}(kb_1(v^*_{pre}))(v^* - b_1(v^*_{pre})) = v^*_{pre} - \frac{1}{k} \tag{10}$$

Thus, when $v_1 > v^*_{pre}$ bidder 1's bid in the first stage is $b_1 = \frac{1}{k}$ and then all the $n-1$ bidders in the second stage stay out of the auction since in order to win they have to place a bid that is higher than 1, namely, higher than their values. Assume now that $v_1 \le v^*_{pre}$. Then, the seller's revenue is given by

$$R_{pre-k} = \frac{kb_1(v_1)}{k} F^{n-1}(kb_1(v_1)) + \int_{kb_1(v_1)}^{1} b(v) dF^{n-1}(v)$$

Integrating by parts yields

$$R_{pre-k} = \frac{kb_1(v_1)}{k} F^{n-1}(kb_1(v_1)) + 1 - \int_{kb_1(v_1)}^{1} F^{n-2}(s)ds - b(kb_1(v_1))F^{n-1}(kb_1(v_1)) - \int_{kb_1(v_1)}^{1} b'(v)F^{n-1}(v)dv$$

By (8) we have

$$b'(v) = \frac{(n-2)f(v)}{F^{n-1}(v)} \int_{kb_1(v_1)}^{v} F^{n-2}(s)ds$$

Thus, since $kb_1(v_1) = b(kb_1(v_1))$ we have

$$R_{pre-k} = 1 - kb_1(v_1)F^{n-1}(kb_1(v_1))(1-\frac{1}{k}) - \int_{kb_1(v_1)}^{1} F^{n-2}(s)ds - \int_{kb_1(v_1)}^{1} ((n-2)f(v) \int_{kb_1(v_1)}^{v} F^{n-2}(s)ds)dv$$

Integrating by parts again yields

$$R_{pre-k} = 1 - kb_1(v_1)F^{n-1}(kb_1(v_1))(1 - \frac{1}{k}) - \int_{kb_1(v_1)}^{1} (n-1)F^{n-2}(v) - (n-2)F^{n-1}(v)dv \tag{11}$$

In the next subsection we compare the seller's revenue in the prebidding and postbidding auctions with head-starts.

3.3 A comparison of postbidding and prebidding auctions with head-starts

We have shown that without head-starts, for sufficiently high values of bidder 1 the prebidding auction yields a higher expected bid than the postbidding auction and vice versa (under some conditions) for sufficiently low values of this bidder. When we allow the seller to give head-starts we obtain the opposite result as follows.

Proposition 4 *The expected highest bid in the prebidding first-price auction with head-starts is higher than in the postbidding first-price auction for sufficiently low values of bidder 1. However, if the number of bidders is sufficiently high, then the expected highest bid in the prebidding first-price auction with head-starts is lower than in the postbidding first-price auction for sufficiently high values of bidder 1.*

Proof. In the postbidding auction, the optimal head-start k^*_{post} is

$$r^* = \frac{v_1}{k^*_{post}}$$

where r^* is the optimal reserve price in the simultaneous first-price auction without head-starts and k^*_{post} is the head-start given to the $n-1$ bidders in the first stage of the postbidding auction.

Consider now the prebidding auction when $v_1 < v^*_{pre}$, and let the head-start k_{pre} be given by

$$b_1(v_1, k_{pre}) = k_{pre} b_1(v_1) = r^*$$

where $b_1(v_1)$ is given by (3). Thus, the difference between (11) and (7) is

$$R_{pre-k} - R_{post-k} = \frac{r^*}{k_{pre}} F^{n-1}(r^*) > 0.$$

However, if $v_1 \geq \frac{1}{k_{pre}}$ and the number of bidders is sufficiently high, we obtain by (10) that $\lim_{n\to\infty} v^*_{pre} = \frac{1}{k_{pre}}$ and therefore the bid of bidder 1 in the prebidding auction is $b_1(v) = \frac{1}{k_{pre}}$. Then, the other $n-1$ bidders do not have any incentive to participate in the second stage of the prebidding auction, and by (11) we obtain that for every $k_{pre} > 1$ the seller's revenue is

$$\lim_{n\to\infty} R_{pre-k} = \frac{1}{k_{pre}}$$

On the other hand, in the postbidding auction, by (7), for every k_{post} we have

$$\lim_{n\to\infty} R_{post-k} = 1.$$

Thus, if $v_1 \geq \frac{1}{k_{pre}}$ and the number of bidders is sufficiently high, the expected highest bid in the postbidding auction is higher than in the prebidding auction. ■

By Proposition 4, we can see that when bidder 1 has a relatively high value and if he has a head-start, the other $n-1$ bidders in the prebidding auction stay out of the auction. Then, the contribution of bidder 1 to the expected highest bid might be negative. Therefore, it is better for the seller to remove bidder 1 from the prebidding auction and let the other bidders compete simultaneously in a first-price auction. In the postbidding auction, however, bidder 1 never makes a negative contribution to the expected highest bid given the optimal head-start k^*_{post}.

4 Concluding remarks

In this paper we examined the effect of prebidding and postbidding in first-price auctions and compared between them. We showed that the results are ambiguous since each of the auction forms can yield a higher revenue for the seller. Furthermore, by allowing head-starts, the results are completely different than when there are no head-starts. The main consequence of our analysis is that if there is a bidder whose information is different from that of the other bidders, by allocating him either before or after the other bidders, the designer can ensure that the bidder with asymmetric information can positively contribute to the expected highest bid.

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