Building Trust in Cooperative Relationships.

George Loginov*
University of Wisconsin-Madison

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1. Introduction

This paper contributes to the literature on the emergence of cooperation in repeated interactions which can be unilaterally ended by any participating party. In the model there is a continuum of agents who are matched in pairs to play a variant of repeated prisoners dilemma with a continuous action set. After each stage of the game the partners in a match may choose to end it and start a new one with a different agent.

Kranton (1996) shows that one way to sustain cooperation in this situation is to gradually increase its level in the initial phase of an interaction in order to make the existing partnership more valuable than a new one and thus prevent the agents from abandoning their current match. Focusing only on symmetric pure strategies, she demonstrates that the optimal cooperative behavior can be supported by strategies involving no more than two levels of cooperation, with the lower level employed at most once in the interaction.

Main results in Fujiwara-Greve and Okuno-Fujiwara (2009):

1. Under no information transmission trust-building strategies are played in equilibria. A trust-building strategy consists of T periods of defection followed by cooperation. The partnership is preserved as long as the play in the previous round is symmetric: (D, D) in the trust-building phase, (C, C) in the cooperation phase.
2. There exist monomorphic equilibria, in which all agents play the same trust-building strategy and no break ups occur on the equilibrium path.
3. There exist polymorphic equilibria, in which several different trust-building strategies coexist. Break ups occur on the equilibrium path when agents in a match play different strategies.
4. Polymorphic equilibria can be more efficient than monomorphic ones.

Allowing for some information transmission can also increase the efficiency of a monomorphic equilibrium. Proposed mechanism: reference letters. After each round of PD, players in a match simultaneously decide whether to continue the match and to issue (at a small cost) a reference letter (valid only for one period) to the opponent. In equilibrium, reference letters are issued during the cooperation stage only, and the average efficiency is increased since the average length of the trust-building phase is reduced. If both agents in a match have a letter, they can skip the trust-building phase and start cooperating immediately. The literal meaning of the reference letter: This agent cooperated in the previous period.

*loginov@wisc.edu

The main advantage of the reference letter system is that it is decentralized, and it is in the interest of each agent both to issue a letter to an opponent in a match and to seek one from the opponent. The drawbacks of reference letters: limited use in polymorphic equilibria because of the on-the-path break ups and efficiency loss due to the fact that newcomers don't have a letter and thus have to go through a trust-building phase.

2. Model

2.1 Stage game

There is a continuum of agents who are matched to play voluntary separable repeated prisoner's dilemma with variable stakes. Time is discrete, each agent survives to the next period with probability $\delta \in (0, 1)$. This probability also serves as the discount rate.
The timing of the game is as follows:

1. In the beginning of each period unmatched agents are put in pairs.
2. The agents in a match simultaneously announce their stakes $a_i, a_j \in [0, \infty)$.
3. The agents play a round of prisoner's dilemma with payoffs determined by $a = \min\{a_i, a_j\}$ (see Figure 1).
4. After observing the outcome of the game, agents simultaneously decide whether they want to keep their current partnership. The partnership is split if at least one player wants to end it.
5. Agents die with probability $(1 - \delta)$. Unmatched survivors return to the matching pool. Newborn agents replacing the dead are also added to the matching pool, so that the population mass is constant.

		2: C	2: D
1	C	a, a	$-L(a), T(a)$
	D	$T(a), -L(a)$	$0, 0$

Figure 1: Prisoner's dilemma with stake a.

The payoffs in the prisoner's dilemma in Figure 1 are defined as functions of the stake a: $T(a)$ is the temptation to defect and $L(a)$ is the loss from the opponent's defection. We impose the following assumptions on the functions:

$$T(0) = 0, \frac{dT}{da} > 1, \frac{d^2T}{da^2} > 0, L(0) = 0, \text{ and } L(a) > 0 \text{ for } a > 0.$$

These assumptions guarantee that the game in Figure 1 is in prisoner's dilemma class for every $a > 0$. In addition we assume that the temptation function is convex.

Strategies

If no information is transmitted between the agents, the game strategies can be contingent on the histories of interactions within a given partnership. Let $h_t \in H_t$ be the history at the beginning of period t of the partnership, and $H_t = [\mathbb{R}_+^2 \times \{C, D\}^2 \times \{k, e\}^2]^{(t-1)}$ be the set of all such histories.

A pure strategy is a triple $(a_t, x_t, z_t)_{t=1}^{\infty}$, in which
$a_t : H_t \to \mathbb{R}_+$ specifies the stake in period t,
$x_t : H_t \times \mathbb{R}_+^2 \to \{C, D\}$ specifies the action in the prisoner's dilemma, and
$z_t : H_t \times \mathbb{R}_+^2 \times \{C, D\}^2 \to \{k, e\}$ is the decision to *keep* or *exit* the current partnership.

2.2 An equilibrium with constant stakes.

We first derive the condition under which the whole population chooses one and the same strategy and bids the same amount in every period. Consider a strategy c_T in which $a_t = a$ in all periods, $x_t = D$ for $t \leq T$ and $z_t = k$ if and only if the action profile in the previous round of the prisoner's dilemma is symmetric. That is, players in a match build trust for T periods by bidding the equilibrium amount a and defecting, and cooperate thereafter. Let p be the population state in which all agents play strategy c_T. The expected payoff to c_T in state p can be derived from the equation

$$V(c_T; p) = \delta^{2T} \frac{a}{1 - \delta^2} + \frac{\delta}{1 + \delta} V(c_T; p)$$

in which the first term is the expected payoff in a partnership and the second is the continuation value the agent receives if he outlives his partner. Thus

$$V(c_T; p) = \frac{a\delta^{2T}}{1 - \delta}$$

and c_T is an equilibrium if, once the cooperation phase of the partnership is reached, there is no profitable deviation:

$$T(a) + \delta V(c_T; p) \leq \frac{a}{1 - \delta^2} + \frac{\delta}{1 + \delta} V(c_T; p)$$

This condition determines the range of stakes at which strategy c_T can be sustained in an equilibrium for a given discount rate δ:

$$T(a) \leq \frac{1 - \delta^{2(T+1)}}{1 - \delta^2} a \qquad (1)$$

The right-hand side of inequality (1) is linear in a, while the temptation function is convex in a, therefore the set of stakes at which the inequality holds is bounded. It can be empty if marginal temptation is always greater than $\frac{1-\delta^{2(T+1)}}{1-\delta^2}$. The size of the set changes monotonically with respect to the discount rate and the number of periods in the trust-building phase.

Next we are concerned with the following question: is there a way to reduce the length of the trust-building phase by bidding less in the beginning stages of the interaction? And if so, can it increase the agents' expected payoffs? To answer these questions, we first consider the strategy c_1 which prescribes cooperation from period 2, and assume it is an equilibrium given the parameters a and δ. Consider the set of strategies $s_1(b)$ which only differ from c_1 in the first period. While s_1 suggests $c_1 = a$ and $x_1 = D$, under $s_1(b)$ we let $a_1 = b$ and $x_1 = C$ if both players bid b. Thus we can determine, if one bids a in every period and starts cooperating in period 2, can one bid b (presumably, less that a) and start cooperating in the first period, raising the bid to a thereafter?

Consider the population state in which all agents choose the same strategy $s_1(b)$. The expected payoff $V(s_1)$ in such state can be derived from the equation

$$V(s_1) = b + \frac{a\delta^2}{1-\delta^2} + \frac{\delta}{1+\delta} V(s_1)$$

so that

$$V(s_1) = (1+\delta)(b + \frac{a\delta^2}{1-\delta^2})$$

For s_1 to be an equilibrium, the temptation to betray must be low enough at both stakes a and b:

$$\begin{aligned} T(b) + \delta V(s_1) &\leq b + \frac{a\delta^2}{1-\delta^2} + \frac{\delta}{1+\delta} V(s_1) \\ T(a) + \delta V(s_1) &\leq \frac{a}{1-\delta^2} + \frac{\delta}{1+\delta} V(s_1) \end{aligned}$$

Simplifying the expressions we arrive at the following conditions:

$$\begin{aligned} T(b) - b &\leq \delta^2(a-b) \\ T(a) - a &\leq \delta^2(a-b) \end{aligned}$$

The properties of the function $t(a) = T(a) - a$ can be derived from those of $T(a)$. Since we assumed that $T(0) = 0$ and $T'(a) > 1$, we can conclude that $t(0) = 0$ and $t'(a) > 0$, so $T(a) - a$ is nonnegative and increasing. Non-negativity of t implies that $\delta^2(a-b) \geq 0$, so $a \geq b$, and our intuition that b must not exceed a is confirmed. Moreover, since $b \leq a$, it must be that $T(b) - b \leq T(a) - a$, so if the second inequality is satisfied, the first must also be satisfied. Intuitively, if one can suppress the temptation at the high stake one wouldn't defect at the lower stake when promised the high stake starting from the next period.

We are now able to give a positive answer to the first question we have stated: it is possible to reduce the length of the trust-building phase by bidding less in the beginning stages of the interaction.

Proposition 2.1. *If the strategy c_1 (always bid a, defect in the first period, cooperate thereafter) can be supported in an equilibrium, then so can be the strategy $s_1(b)$ (bid b in the first period, bid a thereafter, always cooperate) for $b \leq a - \frac{T(a)-a}{\delta^2}$.*

In order to answer the second question (can $s_1(b)$ yield a higher payoff that c_1?) we need to introduce some notation. Recall that the following two conditions are crucial for the existence of equilibria:

$$\begin{aligned} T(a) &\leq (1+\delta^2)a \\ T(a) - a &\leq \delta^2(a-b) \end{aligned}$$

The first inequality guarantees that strategy c_1 with stake a is an equilibrium. Let $\bar{a}$ be the positive solution to $T(a) = (1+\delta^2)a$ (recall that T is nonnegative and convex, so the solution is unique). Then c_1 can be supported in an equilibrium on $[0, \bar{a}]$. The second inequality determines the range of stakes b for each $a \in [0, \bar{a}]$. Let $\bar{b}(a) = a - \frac{T(a)-a}{\delta^2}$, then for every $a \in [0, \bar{a}]$ strategy $s_1(b)$ can be supported in an equilibrium with $b \in [0, \bar{b}(a)]$.

Next recall that the expected lifetime payoff of the strategy c_1 with stake a is

$$V(c_1) = \frac{a\delta^2}{1-\delta},$$

while the payoff of the strategy s_1 with stakes b and a is

$$V(s_1) = (1+\delta)(b + \frac{a\delta^2}{1-\delta^2}).$$

Thus the payoff gain from strategy s_1 is $V(s_1) - V(c_1) = (1+\delta)b$, so if we fix $a \in [0, \bar{a}]$, the highest payoff gain is

$$(1+\delta)\bar{b}(a) = (1+\delta)[a - \frac{T(a)-a}{\delta^2}]$$

[Note: this value is maximized at a stake a that solves $T'(a) = 1 + \delta^2$, which is unique due to the properties of T.]

Figure 2 illustrates the relationship between $V(c_1)$ and $V(s_1)$: for every $a \in (0, \bar{a})$ strategy s_1 with stake $\bar{b}(a)$ yields higher payoff than the corresponding strategy c_1. However strategy c_1 at $\bar{a}$ may still yield the overall highest payoff. The payoff to the strategy c_1 is linear in the stake a and is

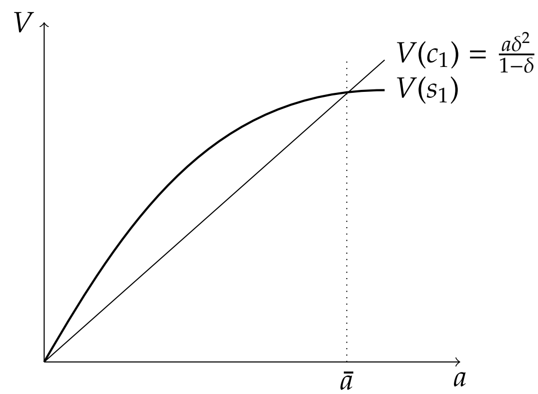


Figure 2: Payoffs to strategies c_1 and s_1 with $b = \bar{b}(a)$.

maximized at $a = \bar{a}$. The payoff to the strategy s_1 with $b = \bar{b}(a)$ is concave in a and thus, in principle, can be maximized at a stake lower than $\bar{a}$. The following proposition provides the condition when it is the case.

Proposition 2.2. *The maximum payoff of strategy s_1 is higher than that of strategy c_1 if and only if $T'(\bar{a}) > \frac{1}{1-\delta^2}$.*

Proof. Let $\bar{V}(c_1)$ be the highest payoff to c_1:

$$\bar{V}(c_1) = \max_{a \in [0,\bar{a}]} V(s_1) = \frac{\bar{a}\delta^2}{1-\delta}$$

The payoff loss from playing the strategy c_1 at stake $a \in (0, \bar{a})$ rather than at $\bar{a}$ equals

$$V(c_1, \bar{a}) - V(c_1, a) = \frac{(\bar{a} - a)\delta^2}{1 - \delta}$$

On the other hand, the payoff advantage of the strategy $s_1(a, \bar{b}(a))$ compared to $c_1(a)$ is $(1 + \delta)\bar{b}(a)$. Therefore for s_1 to yield the overall highest payoff, there must exist $a^* \in (0, \bar{a})$ such that

$$(1 + \delta)\bar{b}(a^*) > \frac{(\bar{a} - a^*)\delta^2}{1 - \delta} \tag{2}$$

Using the fact that $T(\bar{a}) = (1 + \delta^2)\bar{a}$, inequality (2) yields

$$\frac{T(\bar{a}) - T(a^*)}{\bar{a} - a^*} > \frac{1}{1 - \delta^2} \tag{3}$$

Finally, convexity of T implies concavity of $V(s_1)$, so if inequality (3) holds for some $a^* < \bar{a}$, it must hold for all $a \in (a^*, \bar{a})$ and therefore by letting $a \to \bar{a}$ we show the "only if" part of the statement:

$$T'(\bar{a}) > \frac{1}{1 - \delta^2}$$

Reversing the argument would prove the "if" part of the statement. ■

Thus we have shown that depending on the properties of the temptation function the strategy employing the highest possible level of cooperation may yield inefficient payoffs.

2.3 Eliminating the trust building phase

In the previous section we showed that introducing intermediate stakes may reduce the length of the trust-building phase and increase the expected payoff of the partnership. In this section we generalize this approach by constructing an equilibrium s_T strategy from a c_T strategy for arbitrary T. Recall that strategy c_T prescribes bidding $a_t = a$ in all periods, defecting in the first T periods and cooperating thereafter, as long as the opponent follows suit. A corresponding s_T strategy prescribes bidding b in the first T periods, bidding a thereafter and cooperating as long as the opponent cooperates.

Recall that the expected payoff of the strategy c_T is $V(c_T) = \frac{a\delta^{2T}}{1-\delta}$, whereas the payoff to s_T equals

$$V(s_T) = (1 + \delta)[b(1 + \delta^2 + \ldots + \delta^{2(T-1)}) + \frac{a\delta^{2T}}{1 - \delta^2}] = \frac{b(1 - \delta^{2T}) + a\delta^{2T}}{1 - \delta}$$

The one shot deviation principle requires that in order for s_T to be an equilibrium, it should be not profitable to deviate both in the first period of the partnership and when a higher cooperation level of a is reached, thus we have two inequalities:

$$T(b) + \delta V(s_T) \leq \frac{b(1 - \delta^{2T}) + a\delta^{2T}}{1 - \delta^2} + \frac{\delta}{1 + \delta} V(s_T)$$
$$T(a) + \delta V(s_T) \leq \frac{a}{1 - \delta^2} + \frac{\delta}{1 + \delta} V(s_T)$$

which result in two constraints

$$T(b) \leq b + \delta^{2T}(a - b) \tag{4}$$

(5) $$T(a) \leq a + \delta^2(\frac{1 - \delta^{2(T-1)}}{1 - \delta^2})(a - b) + \delta^{2T}(a - b)$$

If $b = 0$ then inequality (5) becomes inequality (1), therefore s_T is an equilibrium only if c_T is.

Proposition 2.3. *Inequalities (4) and (5) are binding at the point that maximizes the payoff of strategy* s_T*.*

Proof. Consider the following constrained optimization problem

$$\begin{aligned} \max\ & b(1 - \delta^{2T}) + a\delta^{2T} \\ \text{s.t. } & T(b) - b(1 - \delta^{2T}) - a\delta^{2T} \leq 0 \\ & T(a) - a(\frac{1 - \delta^{2(T+1)}}{1 - \delta^2}) - b\delta^2(\frac{1 - \delta^{2T}}{1 - \delta^2}) \leq 0 \end{aligned}$$

The Lagrangian for this problem is

$$L(a, b, \lambda, \mu) = b(1-\delta^{2T})+a\delta^{2T}-\lambda(T(b)-b(1-\delta^{2T})-a\delta^{2T})-\mu(T(a)-a(\frac{1 - \delta^{2(T+1)}}{1 - \delta^2})-b\delta^2(\frac{1 - \delta^{2T}}{1 - \delta^2}))$$

The first-order conditions with respect to a and b are

$$\begin{aligned} \frac{dL}{db} =& (1 - \delta^{2T}) - \lambda(T'(b) - 1 + \delta^{2T}) + \mu\delta^2(\frac{1 - \delta^{2T}}{1 - \delta^2}) = 0 \\ \frac{dL}{da} =& \delta^{2T} + \lambda\delta^{2T} - \mu(T'(a) - \frac{1 - \delta^{2(T+1)}}{1 - \delta^2}) = 0 \end{aligned}$$

If $\mu = 0$ then $fracdLda = 0$ implies $\lambda = -1 < 0$, therefore μ must be positive.

If $\lambda = 0$ then $\frac{dL}{db} = 0$ implies $\mu = -\frac{1-\delta^2}{\delta^2} < 0$, therefore λ must be positive.

If $\mu > 0$ and $\lambda > 0$, then both constraints are binding - need to check what happens in that case. ■

Consider an increasing sequence of stakes $\{a_n\}$ s.t. $\sum_{i=0}^{\infty} a_i \delta^{2i} < \infty$. If it is bounded, then so is the sequence $\{A_n\}$ of the continuation values of the partnership in period n, which is defined as $A_n = \sum_{i=n}^{\infty} a_i \delta^{2(i-n)}$. If $\lim_{n\to\infty} a_n = C$ for some $C \in \mathbb{R}$, then for all $n \in \mathbb{N}$ $a_n \leq C$ and hence $A_n \leq \frac{C}{1-\delta^2}$. Conversely, if $\lim_{n\to\infty} a_n = \infty$, then for any $C \in \mathbb{R}$ there exists $k \in \mathbb{N}$ s.t. $a_n > C$ for $n > k$, and thus $A_n > \frac{C}{1-\delta^2}$ so that $\lim_{n\to\infty} A_n = \infty$.

For $\{a_n\}$ to be supported in equilibrium, for every n we must have:

$$T(a_n) + \delta V \leq A_n + \frac{\delta}{1+\delta} V$$

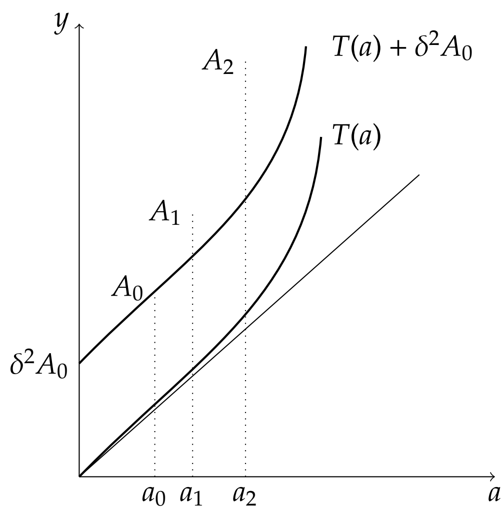
where V is the initial value of a partnership. The condition is equivalent to

$$T(a_n) \leq A_n - \frac{\delta^2}{1+\delta} V$$

where $V = (1+\delta)A_0$, so it can be rewritten as

$$A_n \geq T(a_n) + \delta^2 A_0$$

Possible source of contradiction: A_n has to grow as fast as $T(a)$, whereas $a_n \delta^2 \to 0$.



and the conditions for n and $n+1$ can be related via increments:

$$T(a_n) \leq A_n - \frac{\delta^2}{1+\delta} V$$

$$T(a_{n+1}) \leq A_{n+1} - \frac{\delta^2}{1+\delta} V$$

$$T(a_n) + [T(a_{n+1}) - T(a_n)] \leq A_n + [A_{n+1} - A_n] - \frac{\delta^2}{1+\delta}$$

The left hand side is increased by $[T(a_{n+1}) - T(a_n)]$, while the right hand side is increased by $[A_{n+1} - A_n] = (a_{n+1} - a_n) + \delta^2(a_{n+2} - a_{n+1}) + \ldots$. If $\{a_n\}$ converges, the right hand side can be expressed through its limit (?).

Where do we go from now? One possible question is how can we introduce intermediate stake levels to more complex strategies in order to increase their payoffs? Another question: can cooperation be sustained at levels higher that $\bar{a}$ for a given discount rate? Can we have a strategy that supports cooperation for an unbounded increasing sequence of stakes? The next proposition conjectures it is unlikely.

Proposition 2.4. *Let $p \in (1, \frac{1}{\delta^2})$ and consider a sequence $\{a_n\}$ s.t. $a_{n+1} = pa_n$. Then a strategy prescribing cooperation in each period with stakes from $\{a_n\}$ can not be supported in a monomorphic equilibrium.*

Proof. Let $a_0 = a$. To maintain cooperation at each level, the following inequality must hold for every $n \in \mathbb{N}$:

$$T(p^n a) + \delta V \leq \frac{ap^n}{1 - p\delta^2} + \frac{\delta}{1 + \delta} V$$

Here V is the payoff to the strategy. Rearranging we yield

$$T(p^n a) \leq \frac{ap^n}{1 - p\delta^2} - \frac{\delta^2}{1 + \delta} V \Rightarrow$$
$$T(p^n a) \leq \frac{ap^n}{1 - p\delta^2} \Rightarrow$$
$$\frac{T(p^n a)}{ap^n} \leq \frac{1}{1 - p\delta^2}$$

Since T is convex, the last inequality can only hold for a finite number of integers, therefore the strategy can not be sustained in a monomorphic equilibrium. ■