Mechanism Design with Financially Constrained Agents and Costly Verification*

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April 14, 2016

Latest Version

Abstract

A principal wishes to distribute an indivisible good to a population of budget constrained agents. Both valuation and budget are an agent's private information, but the principal may inspect an agent's budget through a costly verification process and impose a penalty. I characterize the (direct) efficiency-maximizing mechanism. I also show an implementation via a two-stage mechanism which features discriminatory cash subsidies and sales taxes.

Keywords: Mechanism Design, Budget Constraints, Efficiency, Costly Verification

JEL Classification: D45, D61, D82, H42

*Preliminary and Incomplete. Any comments and suggestions are welcomed. I would like to express my gratitude to Rakesh Vohra for his encouraging support since the very beginning of the project. I am also grateful to Aislinn Bohren, Hanming Fang, Daniel Hauser, Ilwoo Hwang, Nick Janetos, Vijay Krishna, George Mailath, Steven Matthews, Mariann Ollar, Mallesh Pai, Andrew Postlewaite, Debraj Ray, Nishant Ravi and the participants in seminars at University of Pennsylvania for valuable discussions. All remaining errors are my sole responsibility.

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1 Introduction

Consider a principal who has a limited supply of indivisible goods and wishes to distribute them to agents with quasi-linear utilities so as to maximize utilitarian efficiency. This can be achieved via a competitive market if agents are not financially constrained. Financial constraints mean that some agents with high valuations will not obtain the good while agents with low valuations but access to cash will. This is precisely the problem faced by governments in the allocation of goods such as housing and health care.

In Singapore, for example, 80% of the population's housing needs are met by the Housing Development Board (HDB). Founded in 1960, its aim is to provide affordable housing.[1] In the United States, Medicaid was created in 1965 to provide health care to individuals and families with low income and limited resources. Medicaid accounts for 16.1% of the state general funds[2] and provided health coverage to 80 million low-income people in 2014.[3] Similar public housing and social health care programs can be found in many other countries.[4][5]

In these instances, the principal relies on agents' reports of their ability to pay, and can and does verify this information. For example, applicants for HDB flats in Singapore and Medicaid in the United States are subject to a set of eligibility conditions such as age, family nucleus, monthly income etc. In practice, the verification process is costly. First, in some developing countries, there dose not exist verifiable records on a household's income and/or wealth and governments are lack of the administrative capacity to process this information. In this case, other verification methods such as a visit to the household to verify the visible living standard are not uncommon and often costly (see Coady et al. (2004)). Second, certain types of income such as tips, side-jobs and cash receipts are costly to verify. Similarly, governments have few ways to verify the income reports by individuals who are self-employed or run small business without performing a costly investigation.

[1] http://www.hdb.gov.sg/fi10/fi10320p.nsf/w/AboutUsPublicHousing?OpenDocument

[2] http://ccf.georgetown.edu/wp-content/uploads/2012/03/Medicaid-state-budgets-2005.pdf

[3] http://www.cbpp.org/research/health/policy-basics-introduction-to-medicaid?fa=view&id=2223

[4] https://en.wikipedia.org/wiki/Public_housing

[5] https://en.wikipedia.org/wiki/Universal_health_coverage_by_country

Last, but not the least, agents may be financially constrained due to limit access to the financial market or high expenditures such as medical expenses and/or education costs. This information is often costly to verify.

The natural question arises as to what is the efficiency-maximization mechanism in these circumstances. Specifically, I consider a mechanism design problem in which there is a unit mass of continuum of agents and a limited supply of indivisible goods. Each agent has two-dimensional private information — his valuation of the good $v \in [0, \overline{v}]$ and his exogenous budget constraint b. The budget constraint is a hard one in the sense that no one can be compelled to pay more than his budget. For simplicity, I assume there are only two possible types of budgets $b_2 > b_1$. The principal can inspect an agent's budget at a cost and impose a penalty. Inspection perfectly reveals an agent's budget. The principal is also subject to a budget balance constraint which requires that the revenue raised from selling the good must exceed the inspection cost. This constraint rules out this possibility that the principal can inject money and relieve all budget constraints. Invoking the revelation principle, I can focus on direct mechanism in which each agent reports his private information directly. Given the report, the mechanism specifies for each agent his probability of getting the good, his payment, his probability of being inspected and the penalty imposed on him.

In this paper, I first characterize the optimal direct mechanism that maximizes efficiency among all mechanisms that are incentive compatible and individually rational, and satisfy the resource constraint, agents' budget constraints and the principal's budget balance constraint.

If budgets are common knowledge, then the principal can treat agents with different budges separately. Only low-budget agents receive cash subsidies, i.e., $u(0, b_1) > u(0, b_2)$. Furthermore, there exist two cutoffs $v_2^{**} > v_1^*$. High-budget agents receive the good for sure and pay v_2^{**} if their valuations are above v_2^{**}. Low-budget agents receive the good with probability $a^* \leq 1$ and pay $a^* v_1^* - u(0, b_1)$ if their valuations above v_1^*. This creates incentives for high-budget agents whose valuations are below v_1^* to misreport as a low-budget type to receive cash subsidies as well as incentives for high-budget agents whose valuations are slightly above v_1^* to misreport as a low-budget type to receive the good with a lower probability but at a much lower price. As a result, if budgets are agents' private information and the principal cannot inspect them, agents receive the same amount of cash subsidies $u(0, b_1) = u(0, b_2)$ regardless of their budgets and high-budget agents whose valuations lie in $[v_1^*, v_2^{**}]$ are pooled with low-budget agents whose valuations are above v_1^*.[6]

The problem becomes much more complex when the principal can inspect agents' budgets at a cost. Even under monotone hazard rate assumption and decreasing density assumption, it is unclear that there exists a unique cutoff v_1^* such that all low-budget agents above this cutoff are pooled. This is because though a gradually increasing allocation rule for low-budget agents leads to lower realized values and lower revenue, it also reduces inspection cost. In the paper, I show it remains to be true despite of introducing costly verification. Given this result, it is intuitive that there exists another cutoff $v_2^* \in [v_1^*, v_2^{**}]$ such that only high-budget agents whose valuations lie in $[v_2^*, v_2^{**}]$ are pooled with low-budget agents whose valuations are above v_1^*. Similarly, it is intuitive that high-budget agents receive less cash subsidies $u(0, b_2) \leq u(0, b_1)$ in the presence of costly inspection. However, the effect of an increase in v_2^* and/or a decrease in $u(0, b_2)$ is ambiguous. Since the change increases realized values and revenue as well as inspection cost, the resulting mechanism may be infeasible or generate lower welfare.

Second, I provide a simple and intuitive implementation of the optimal direct mechanism using a two-stage mechanism. Unlike in the case without inspection, in which all agents are subsidized and regulated equally regardless of their budgets, the two-stage mechanism provides more subsidies to the poor in their initial purchases (the first stage) and imposes more restrictions on them in the resale market (the second stage). Specifically, all agents are asked to report their budgets in the first stage. The principal provides discriminatory cash subsidies to agents and assigns the goods randomly (with uniform probability) at discriminatory prices. Agents who report a low budget receive more cash subsidies and a lower price. In the second stage, a resale market opens, but it is regulated, with discriminatory sales taxes. Agents who report a low budget are subject to a higher per-unit sales tax. Only agents who report low budget are inspected.

The implementation resembles some features of the public housing program in Singapore. In

[6] The values of the two cutoffs are different from those in the common knowledge budgets case.

Singapore, buyers of resale HDB flats can apply for additional housing grants. They are required to physically stay in their flats for at least 5 years before they are eligible to resell or sublet their flats if their flats are purchased with housing grants. In contrast, flats purchased without housing grants are subject to no or shorter requirement.

Technically, this paper develops a novel method that can potentially be used in solving other mechanism design problems with multidimensional types. Note that if each agent has only one-dimensional private information — valuation, then the binding incentive compatibility (IC) constraints are the adjacent (IC) constraints. If each agent has two-dimensional private information but the principal cannot inspect the budget, then it is sufficient to consider two one-dimensional deviations. Pai and Vohra (2014) solves a similar problem in the discrete setting in which there is a finite number of agents. Unfortunately, this is no longer the case if each agent has two dimensional private information and the principal can inspect the budget. In this case, the binding (IC) constraints endogenously depend on the allocation rule and we must consider deviations in which an agent can misreport both dimensions of his private information. Thus, the standard arguments cannot be used here.

To overcome this difficulty, I approximate the allocation rule of each budget type using step functions. When restricting attention to step allocation functions, the binding (IC) constraints are those of high-budget types whose values are the jump discontinuity points of the high-budget's allocation rule. This allows one to write the optimal inspection rule as a function of the possible values and jump discontinuity points of the allocation rule. Then I solve a modification of the principal's problem in which the allocation rule of low-budget types are restricted to take at most M distinct values. Since for M sufficiently large the step-function can approximate the optimal allocation rule arbitrarily well, I can obtain characterization of the optimal mechanism in the limit.

The rest of the paper is organized as follows. Section 1.1 discusses related work. Section 2 presents the model. Section 3 characterizes the direct optimal mechanism when all agents' budget constraints are common knowledge. Section 4 characterizes the direct optimal mechanism when an agent's budget is his private information. Section 5 provides a simple and intuitive implementation.

Section 6 concludes. All the proofs are relegated to appendix.

1.1 Related Literature

This paper is related to two branches of literature. First, it contributes to the literature studying mechanism design problems when agents are financially constrained by incorporating costly inspection. Prior work analyzes the revenue or efficiency of a given auction form or the design of an optimal mechanism when agents have hard budget constraints. See Che and Gale (1998, 2006, 2000), Laffont and Robert (1996), Maskin (2000), Benoit and Krishna (2001), Brusco and Lopomo (2008), Malakhov and Vohra (2008) and Pai and Vohra (2014).

The closest papers to this are Che et al. (2012) and Richter (2013). In Che et al. (2012) and Richter (2013), like this paper, there is a unit mass of a continuum of agents and a limited supply of goods. In Richter (2013) agents have linear preferences for an unlimited supply of the goods. He finds that both the revenue-maximizing mechanism and efficiency-maximizing mechanism feature a linear price for the good. In addition, the efficiency-maximizing mechanism has a uniform cash subsidy. In both Che et al. (2012) and this paper, each agent has a unit demand of an indivisible good and the efficiency-maximizing mechanism can be implemented via a random assignment with regulated resale and cash subsidy scheme. However, Che et al. (2012) does not consider the possibility that the principal can inspect an agent's budget at a cost. This feature also distinguishes the current paper from all the other papers on mechanism design with financially constrained agents. As a result, unlike Che et al. (2012), in which all agents are subsidized and regulated equally regardless of their budgets in an optimal mechanism, I show that an optimal mechanism provides more subsidies to the poor in their initial purchases and imposes more restrictions on them in the resale market.

Second, this paper is related to the costly state verification literature. The first contribution in the series is Townsend (1979) who studies a model of a principal and a single agent. In Townsend (1979) verification is deterministic. Border and Sobel (1987) and Mookherjee and Png (1989) generalize it by allowing random inspection. Gale and Hellwig (1985) consider the effects of costly verification in the context of credit markets. Recently, Ben-Porath et al. (2014) study the allocation problem in the costly state verification framework when there are multiple agents and monetary transfer is not possible. These models differ from what I consider here in that in their models each agent only has one-dimensional private information.

This paper is also somewhat related to the literature on costless or ex-post verification. Glazer and Rubinstein (2004) can be interpreted as a model of a principal and one agent with limited but costless verification and no monetary transfers. Mylovanov and Zapechelnyuk (2014) study a model of multiple agents with costless verification but limited punishments. This paper differs from these earlier studies in that each agent has two-dimensional private information, verification is costly and there are monetary transfers.

2 Model

There is a unit mass of continuum of agents. There is a mass $S \in (0, 1)$ of indivisible goods. Each agent has a private valuation of the good $v \in V := [0, \overline{v}]$, and a privately known budget $b \in B := \{b_1, b_2\}$. I assume that $b_2 > \overline{v}$. Thus, a high-budget agent is never budget constrained in an individually rational mechanism. The type of an agent is a pair consisting of his valuation and his budget: $t := (v, b)$; and the type space is $T := V \times B$.

I assume v and b are independent. Each agent has a high budget with probability π and a low budget with the rest of the probability, i.e., $\mathbb{P}(b_2) = \pi$ and $\mathbb{P}(b_1) = 1 - \pi$. The valuation v is distributed with cumulative distribution function F and strictly positive density f.

The principal can inspect an agent's budget at a cost $k > 0$, and can impose a monetary penalty up to $c \geq 0$. The inspection perfectly reveals an agent's budget. The cost to agents of providing information is zero. For the main body of the paper, I assume that the penalty is not transferable.

The usual version of the Revelation Principle (see, e.g., Myerson (1979) and Harris and Townsend (1981)) does not apply to models with verification. However, it is not hard to extend the argument to this type of environment.[7] Specifically, I show in Appendix B that it is without

[7] See Townsend (1988) for a discussion and an extension of the Revelation Principle to a class of verification models

loss of generality to restrict attention to direct mechanisms. A direct mechanism is a quadruple (a, p, q, θ), where $a : T \to [0, 1]$ denotes the probability an agent obtains the good, $p : T \to \mathbb{R}$ denotes the payment an agent must make, $q : T \to [0, 1]$ denotes the probability of inspecting and $\theta : T \times \{b_1, b_2, n\} \to [0, c]$ denotes the penalty imposed on an agent. In particular, $\theta(\hat{t}, n) \in [0, c]$ denotes the penalty imposed on an agent who reports $\hat{t}$ and is not inspected, and $\theta(\hat{t}, b) \in [0, c]$ denotes the penalty imposed on an agent who reports $\hat{t}$ and is inspected and whose budget is revealed to be b.

The utility of an agent who has type $t := (v, b)$ and reports $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b) & \text{if } p(\hat{t}) + \theta(\hat{t}, b) \leq b \text{ and } p(\hat{t}) + \theta(\hat{t}, n) \leq b, \\ -\infty & \text{otherwise.} \end{cases}$$

An agent has a standard quasilinear utility up to his budget constraint, and cannot pay more than his budget.

The welfare criterion we use is utilitarian efficiency. For why utilitarian efficiency is a reasonable welfare criterion, see Vickrey (1945). Given quasi-linear preferences, the total value realized minus total inspection cost and penalty is an equivalent criterion. The principal's problem is

$$\max_{a,p,q,\theta} \mathbb{E}_t \left[a(t)v - q(t)k \right], \tag{$\mathcal{P}$}$$

subject to

$$u(t) \equiv u(t, t) \geq 0, \ \forall t \in T, \tag{IR}$$

$$u(t) \geq u(\hat{t}, t), \ \forall t \in T, \hat{t} \in \left\{ \hat{t} \in T \ \middle| p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\} \leq b \right\}, \tag{IC}$$

$$p(t) + \max\{\theta(t, n), \theta(t, b)\} \leq b, \ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t \left[p(t) + (1 - q(t))\theta(t, n) + q(t)\theta(t, b) - q(t)k \right] \geq 0, \ \forall t \in T, \tag{BB}$$

$$\mathbb{E}_t \left[a(t) \right] \leq S. \tag{S}$$

which does not include the environment considered in this paper.

(IR) is the individual rationality constraint, which requires that each agent gets a non-negative expected payoff from participating in the mechanism. (BC) is the budget constraint which states that an agent cannot be ask to make a payment larger than his budget b. (IC) requires that it is weakly better for an agent to report his true type than any other type whose transfers he can afford. (BB) is the principal's budget balance constraint, which requires that the revenue raised from selling the good must exceed the inspection cost. (BB) rules out the possibility that the principal can inject money and relieve all budget constraints. (S) is the limited supply constraint, which requires that the amount of good assigned cannot exceed the supply. We say a mechanism (a, p, q, θ) is *feasible* if it satisfies constraints (IR), (IC), (BC), (BB) and (S).

First, I show that it is without loss of generality to focus on mechanisms in which an agent is penalized if and only if he is found to have lied about his budget, and whenever he is found to have lied he is imposed the maximum possible monetary penalty c.

Lemma 1 *It is without loss of generality to focus on mechanisms in which an agent is penalized if and only if he is found to have lied about his budget, and $\theta(\hat{t}, b) = c$ if $\hat{b} \neq b$.*

Using Lemma 1, one can rewrite the principal's problem as follows:

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v - q(t)k \right], \tag{$\mathcal{P}$}$$

subject to

$$u(t) \equiv u(t,t) \geq 0, \ \forall t \in T, \tag{IR}$$

$$u(t) \geq u(\hat{t}, t), \ \forall t \in T, \hat{t} \in \left\{ \hat{t} \in T \,\middle|\, p(\hat{t}) \leq b \right\}, \tag{IC}$$

$$p(t) \leq b, \ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t \left[p(t) - q(t)k \right] \geq 0, \ \forall t \in T, \tag{BB}$$

$$\mathbb{E}_t \left[a(t) \right] \leq S. \tag{S}$$

Hence, in the rest of the paper, I will use the triple (a, p, q) to denote a mechanism.

Throughout the paper, I assume that $S < 1 - F(b_1)$ since otherwise the first-best can be achieved via a competitive market. I also impose the following two assumptions throughout the paper.

Assumption 1 $\frac{1-F}{f}$ *is non-increasing.*

Assumption 2 f *is non-increasing.*

Assumption 1 is the standard monotone hazard rate condition, which is often adopted in the mechanism design literature. This assumption ensures that allocating more good to higher value agents from lower value agents generates higher revenues for the principal. Assumption 2 says that higher valuations are less likely than the lower valuations. These two assumptions are also imposed in Richter (2013) and Pai and Vohra (2014). These two assumptions are satisfied by the uniform distribution, exponential distribution and the left truncation of a normal distribution.

3 Common Knowledge Budgets

In this section , we analyze the case in which all agents' budget constraints are common knowledge. This can be viewed as the situation in which the principal can inspect an agent' budget for free.

Since budgets are common knowledge, the (IC) constraints hold as long as for each $b \in B$, no agent has incentive to misreport his value:

$$va(v,b) - p(v,b) \leq va(\hat{v},b) - p(\hat{v},b), \ \forall v, \hat{v}. \tag{IC-v}$$

The principal's problem becomes

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v \right],$$

subject to (IR), (IC-v), (BC), (S) and

$$\mathbb{E}_t[p(t)] \geq 0, \forall t \in T. \tag{BB}$$

The following theorem characterizes the optimal mechanism.

Theorem 1 *Suppose Assumption 2 hold, and budgets are common knowledge. There exists* $v_1^* \leq v_2^* < \overline{v}$ *and* $0 < u(0, b_1) \leq v_1^* - b_1$ *such that the optimal mechanism is given by*

$$a(v, b_1) = \chi_{\{v \geq v_1^*\}}(v)\frac{u(0, b_1) + b_1}{v_1^*}, \quad p(v, b_1) = \chi_{\{v \geq v_1^*\}}(v)(u(0, b_1) + b_1) - u(0, b_1),$$

$$a(v, b_2) = \chi_{\{v \geq v_2^*\}}(v), \qquad p(v, b_2) = \chi_{\{v \geq v_2^*\}}(v)v_2^*.$$

There are two cutoffs for pooling at the top: $v_1^* \leq v_2^*$. All high-budget agents whose valuations are above v_2^* are pooled, and all low-budget agents whose valuations are above v_1^* are pooled.

4 Privately Known Budgets

In this section, we analyze the case in which an agent's budget is his private information. The (IC) constraints can be separated into two categories:

$$\text{Misreport value: } va(v, b) - p(v, b) \geq va(\hat{v}, b) - p(\hat{v}, b), \ \forall v, \hat{v}, b, \tag{IC-v}$$

$$\text{Misreport both: } va(v, b) - p(v, b) \geq \chi_{\{p(\hat{v}, \hat{b}) \leq b\}} \left(a(\hat{v}, \hat{b})v - q(\hat{v}, \hat{b})c - p(\hat{v}, \hat{b})\right), \ \forall v, \hat{v}, b, \hat{b}. \tag{1}$$

By the standard argument, (IC-v) holds if and only if for all $b \in B$, $a(v, b)$ is non-decreasing in v and $p(v, b) = va(v, b) - \int_0^v a(\nu, b)\mathrm{d}\nu - u(0, b)$. The difficulty arises from (1). In the following analysis, we first consider a relaxed problem and replace (1) by the following constraint:

$$va(v, b_2) - p(v, b_2) \geq a(\hat{v}, b_1)v - q(\hat{v}, b_1)c - p(\hat{v}, b_1), \ \forall v, \hat{v}. \tag{IC-b}$$

Note that (IC-b) only requires that a high-budget agent has no incentive to misreport as a low-budget agent. Later, given the optimal mechanism of the relaxed problem, we verify that a low-budget agent has no incentive to misreport as a high-budget agent, i.e., (1) is satisfied.

The principal's relaxed problem is

$$\max_{a,p,a} \mathbb{E}_t[a(t)v - q(t)k], \qquad (\mathcal{P}')$$

subject to (IR), (IC-v), (IC-b), (BC), (BB) and (S).

4.1 No Inspection

In this section, we consider the case in which the principal does not inspect agents, i.e., $q \equiv 0$. As we will see in Section 4.2, this is the case if inspection cost, k, is sufficiently high relative to the punishment, c.

We first observe that in this case the (IC-b) constraint holds if and only if (IC-v) holds and

$$va(v, b_2) - p(v, b_2) \geq a(v, b_1)v - p(v, b_1), \ \forall v. \qquad (2)$$

To see this, note that if (2) holds, then

$$\begin{aligned} va(v, b_2) - p(v, b_2) &\geq a(v, b_1)v - p(v, b_1) \\ &\geq a(\hat{v}, b_1)v - p(\hat{v}, b_1), \end{aligned}$$

where the second inequality follows from the (IC-v) constraint. Thus, if the principal does not inspect agents, it is sufficient to consider the two one-dimensional deviations: only misreport value and only misreport budget. Using the envelope condition, (2) can be rewritten as

$$u(0, b_2) + \int_0^v a(\nu, b_2)\mathrm{d}\nu \geq u(0, b_1) + \int_0^v a(\nu, b_1)\mathrm{d}\nu, \ \forall v. \qquad (3)$$

We first show that in an optimal mechanism, the lowest value agents with different budgets must get the same payoff.

Lemma 2 *Suppose Assumption 2 holds. In an optimal mechanism of $\mathcal{P}'$, $u(0, b_1) = u(0, b_2)$.*

Second we show that given any v, an optimal mechanism on average allocates weakly more resources to high-budget agents whose valuations are below v than to low-budget agents whose valuations are below v.

Lemma 3 *Suppose Assumptions 1 and 2 hold. In an optimal mechanism of $\mathcal{P}'$, the allocation rule satisfies*

$$\int_0^v a(\nu, b_2) f(\nu) \mathrm{d}\nu \geq \int_0^v a(\nu, b_1) f(\nu) \mathrm{d}\nu, \ \forall v. \tag{4}$$

The following theorem characterizes the optimal direct mechanism.

Theorem 2 *Suppose Assumptions 1 and 2 hold. There exists $b_1 < v^* \leq v_2^{**} \leq \overline{v}$ and $u^* > 0$ such that the optimal mechanism of $\mathcal{P}$ with no inspection satisfies*

$$\begin{aligned}
a(v, b_1) &= \chi_{\{v \geq v^*\}} \frac{u^* + b_1}{v^*}, \ p(v, b_1) = \chi_{\{v \geq v^*\}}(u^* + b_1) - u^*, \\
a(v, b_2) &= \chi_{\{v \geq v^*\}} \frac{u^* + b_1}{v^*} + \chi_{\{v \geq v_2^{**}\}} \left(1 - \frac{u^* + b_1}{v^*}\right), \\
p(v, b_2) &= \chi_{\{v \geq v^*\}}(u^* + b_1) + \chi_{\{v \geq v_2^{**}\}} \left(1 - \frac{u^* + b_1}{v^*}\right) v_2^{**} - u^*.
\end{aligned}$$

Not surprisingly the optimal allocation rule obtained here shares similar features with the one found in Pai and Vohra (2014). There are two cutoffs for pooling at the top: $v^* < v_2^{**}$. All high-budget agents whose valuations are above v_2^{**} are pooled, and all low-budget agents whose valuations are above v^* are pooled. In addition, high-budget agents whose valuations are in $[v^*, v_2^{**}]$ are pooled with low-budget agents whose valuations are at least v^*.

4.2 The General Case

In this section, we solve the general principal's problem. Using the envelope condition, (IC-b) becomes: For all v and $\hat{v}$,

$$u(0, b_2) + \int_0^v a(\nu, b_2) \mathrm{d}\nu \geq u(0, b_1) + a(\hat{v}, b_1)(v - \hat{v}) - q(\hat{v}, b_1)c + \int_0^{\hat{v}} a(\nu, b_1) \mathrm{d}\nu. \tag{IC-b}$$

First, for each $\hat{v}$, we identify the type of high-budget agents whose gain from falsely claiming to be a type $(\hat{v}, b_1)$ agent are the largest. (IC-b) holds if only if for each $\hat{v} \in V$, $q(\hat{v}, b_1)c \geq \sup_v \Delta(v, \hat{v})$, where

$$\Delta(v, \hat{v}) \equiv u(0, b_1) - u(0, b_2) - \int_0^v a(\nu, b_2)\mathrm{d}\nu + a(\hat{v}, b_1)(v - \hat{v}) + \int_0^{\hat{v}} a(\nu, b_1)\mathrm{d}\nu.$$

Since $\partial\Delta(v, \hat{v})/\partial v = -a(v, b_2) + a(\hat{v}, b_1)$ is non-increasing in v, $\Delta(v, \hat{v})$ is concave in v and achieves its maximum at $v = v^d(\hat{v})$, where

$$v^d(\hat{v}) \equiv \inf \left\{ v | a(v, b_2) \geq a(\hat{v}, b_1) \right\}. \tag{5}$$

Suppose the allocation rules for both budget types are continuous, then the high-budget agents who benefit most from falsely claiming to be $(\hat{v}, b_1)$ are those who get the good with the same probability as type $(\hat{v}, b_1)$ agents do. Since the principal's objective function is strictly decreasing in q, the optimal inspection rule satisfies

$$q(\hat{v}, b_1) = \frac{1}{c} \max \left\{ 0, \Delta(v^d(\hat{v})) \right\}. \tag{6}$$

Note that $v^d(\cdot)$ is defined using the allocation rule, i.e., which (IC-b) constraint binds endogenously depends on the allocation rule chosen. Consider, for example, the standard mechanism design problem with one-dimensional information – value. With only monetary transfer, it is always the nearby types who benefit most from misreporting as a particular type. With only costly verification, it is always the lowest type who benefit most from misreporting as other types. However, these are no longer true in the current setting, making the above two approaches invalid.

In order to keep track of the binding (IC-b) constraints, we solve the principal's problem by approximating the allocation rule using step functions. Fix $M \geq 2$. Let $0 = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$ and $0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1$. Suppose the allocation rule for type b_1 agent takes M distinct values: $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \ldots, M$. The next lemma show that the optimal allocation rule for type b_2 agent can take at most $M+2$ distinct values: $a^0, a^1, \dots, a^{M+1}$.

Lemma 4 *Suppose Assumptions 1 and 2 hold. Suppose $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$. Then there exists $0 \le v_2^0 \le v_2^1 \le \dots \le v_2^M \le \bar{v}$ such that the optimal allocation rule for b_2 satisfies $a(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 1, \dots, M$, $a(v, b_2) = 0$ if $v < v_2^0$ and $a(v, b_2) = 1$ if $v > v_2^M$.*

We say an allocation rule a is a *M-step allocation rule* if there exists $0 = v_1^0 < v_1^1 < \dots < v_1^M = \bar{v}$, $0 \le v_2^0 \le v_2^1 \le \dots \le v_2^M \le \bar{v}$ and $0 = a^0 \le a^1 < a^2 < \dots < a^M \le a^{M+1} = 1$ for some $M \ge 2$ such that $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$ and $a(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 0, 1, \dots, M+1$. Lemma 4 shows that it is without loss of generality to focus on M-step-allocation rules among all the step allocation rules.

Consider a mechanism using a M-step allocation rule, then it is easy to see that for $v \in (v_1^{m-1}, v_1^m)$, the type b_2 agents who benefit most from falsely claiming to be a type (v, b_1) agent have valuations $v^d(v) = v_1^{m-1}$. Hence, we can keep track of the binding (IC-b) constraints by keeping track of the jump points of the allocation rule. In this case, the optimal inspection rule satisfies $q(v, b_1) = q^m$ if $v \in (v_1^{m-1}, v_1^m)$ and

$$q^m = \frac{1}{c} \max \left\{ 0, u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \right\} \tag{7}$$

for $m = 1, \dots, M$.

4.2.1 Optimal Mechanism with Step Allocation Rule

In this section, we consider the principal's problem $(\mathcal{P}')$ with two modifications:

$$\max_{a,p,a} \mathbb{E}_t[a(t)v - q(t)k], \tag{$\mathcal{P}'(M, D)$}$$

subject to (IR), (IC-v), (IC-b), (BC), (S),

$$a \text{ is a } M'\text{-step allocation rule for some } M' \le M,$$

$$\mathbb{E}[p(t) - q(t)k] \ge -D. \tag{BB-D}$$

The second modification is to relax the government's budget balance constraint by $D > 0$. As it will become clear later, any feasible mechanism of $\mathcal{P}'$ can be approximated arbitrarily well by a feasible mechanism of $\mathcal{P}'(M, D)$ for M sufficiently large and D sufficiently small.

Next we show that in an optimal mechanism either no one is inspected, or all low-budget agents are inspected with positive probabilities.

Lemma 5 *Suppose Assumptions 1 and 2 hold. An optimal mechanism of $\mathcal{P}'(M, D)$ satisfies one of the following two conditions:*

(C1) For all $m = 1, \dots, M$,

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \ge 0. \tag{8}$$

(C2) For all $m = 1, \dots, M$,

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \le 0. \tag{9}$$

If (C2) holds, then (IC-b) holds even if there is no inspection and the optimal mechanism is characterized in Section 4.1. In the optimal mechanism, the allocation rule is a 2-step allocation rule and satisfies (C1) with equality. Thus, we can conclude that an optimal mechanism of $\mathcal{P}'(M, D)$ satisfies (C1).

Corollary 1 *Suppose Assumptions 1 and 2 hold. An optimal mechanism of $\mathcal{P}'(M, D)$ satisfies (C1).*

Then the optimal inspection rule satisfies $q(v, b_1) = q^m$ if $v \in (v_1^{m-1}, v_1^m)$, where

$$q^m = \frac{1}{c}\left[u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})\right] \tag{10}$$

for $m = 1, \ldots, M$. Now the principal's problem $\mathcal{P}'(M, D)$ can be written as:

$$\max_{\substack{u(0,b_1),u(0,b_2),\\ \{a^m\}_{m=1}^M,\{v_1^m\}_{m=1}^{M-1},\{v_2^m\}_{m=0}^M}} \pi\sum_{m=1}^{M+1}\int_{v_2^{m-1}}^{v_2^m} a^m v f(v)\mathrm{d}v + (1-\pi)\sum_{m=1}^{M}\int_{v_1^{m-1}}^{v_1^m} a^m v f(v)\mathrm{d}v$$
$$-(1-\pi)\frac{k}{c}\sum_{m=1}^{M}\int_{v_1^{m-1}}^{v_1^m}\left[u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})\right] f(v)\mathrm{d}v$$

subject to

$$\pi\sum_{m=1}^{M+1} a^m[F(v_2^m) - F(v_2^{m-1})] + (1-\pi)\sum_{m=1}^{M} a^m[F(v_1^m) - F(v_1^{m-1})] \le S, \tag{β}$$

$$a^M v_1^{M-1} - \sum_{j=1}^{M-1} a^j(v_1^j - v_1^{j-1}) - u(0, b_1) \le b_1, \tag{η}$$

$$-(1-\pi)u(0, b_1) + (1-\pi)\sum_{m=1}^{M}\int_{v_1^{m-1}}^{v_1^m} a^m\left[v - \frac{1-F(v)}{f(v)}\right] f(v)\mathrm{d}v$$
$$-(1-\pi)\frac{k}{c}\sum_{m=1}^{M}\int_{v_1^{m-1}}^{v_1^m}\left[u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})\right] f(v)\mathrm{d}v \ge -D,$$
$$-\pi u(0, b_2) + \pi\sum_{m=1}^{M+1}\int_{v_2^{m-1}}^{v_2^m} a^m\left[v - \frac{1-F(v)}{f(v)}\right] f(v)\mathrm{d}v \tag{λ}$$

$$u(0, b_1) \ge 0, u(0, b_2) \ge 0, \tag{ξ_1, ξ_2}$$

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \ge 0,\ m = 1, \ldots, M, \tag{μ^m}$$

$$0 = a^0 \le a^1 \le a^2 \le \cdots \le a^M \le a^{M+1} = 1, \tag{$\alpha^1, \ldots, \alpha^{M+1}$}$$

$$0 = v_1^0 \le v_1^1 \le \cdots \le v_1^M = \overline{v}, \tag{$\gamma_1^1, \ldots, \gamma_1^M$}$$

$$0 \le v_2^0 \le v_2^1 \le \cdots \le v_2^M \le \overline{v}. \tag{$\gamma_2^0, \ldots, \gamma_2^{M+1}$}$$

The greek letters in parentheses denote the corresponding Lagrangian multipliers. Lemma 6 shows that in an optimal mechanism of $\mathcal{P}'(M, D)$ the allocation rule is a 2-step allocation rule. The proof of Lemma 6 is collected in Section D.3. Let $V(M, D)$ denote the value of $\mathcal{P}'(M, D)$, then we have:

Lemma 6 *Suppose Assumptions 1 and 2 hold. Then $V(M, D) = V(2, D)$ for all $M \geq 2$ and $D \geq 0$.*

The main result of this section is Theorem 3 which shows that in an optimal mechanism of the relaxed problem $\mathcal{P}'$ the allocation rule must be a 2-step allocation rule. Let V denote the value of $\mathcal{P}'$. We prove Theorem 3 by first showing that for any $D > 0$ there exists $\overline{M}(D) > 0$ such that for all $M > \overline{M}(D)$

$$V - V(M, D) \leq (1 - \pi)\left(1 + \frac{k}{c}\right)\frac{\mathbb{E}[v]}{M}.$$

By Lemma 6, $V - V(2, D) \leq (1 - \pi)(1 + k/c)\mathbb{E}[v]/M$ for all $D > 0$ and $M > \overline{M}(D)$. Fixing $D > 0$ and taking M to infinity yields $V(2, 0) \leq V \leq V(2, D)$ for all $D > 0$. Hence, $V = V(2, 0)$ by the continuity of $V(2, D)$.

Theorem 3 *Suppose Assumptions 1 and 2 hold. Then $V = V(2, 0)$.*

Corollary 2 *Suppose Assumptions 1 and 2 hold. There exists $0 \leq v_1^* \leq v_2^* \leq v_2^{**} \leq \overline{v}$, $u_1^* \geq u_2^*$ and $0 \leq a^* \leq 1$ such that the optimal mechanism of $\mathcal{P}$ is given by*

$$\begin{aligned}
a(v, b_1) &= \chi_{\{v \geq v_1^*\}} a^*, \;\; p(v, b_1) = \chi_{\{v \geq v_1^*\}} a^* v_1^* - u_1^*, \\
q(v, b_1) &= \frac{1}{c}\left[\chi_{\{v \geq v_1^*\}} a^*(v_2^* - v_1^*) + u_1^* - u_2^*\right], \\
a(v, b_2) &= \chi_{\{v \geq v_2^*\}} a^* + \chi_{\{v \geq v_2^{**}\}}(1 - a^*), \\
p(v, b_2) &= \chi_{\{v \geq v_2^*\}} a^* v_2^* + \chi_{\{v \geq v_2^{**}\}}(1 - a^*) v_2^{**} - u_2^*, \\
q(v, b_2) &= 0.
\end{aligned}$$

4.3 Properties of the Optimal Mechanism

In this section, I discuss the properties of the optimal mechanism.

Proposition 1 *Suppose Assumptions 1 and 2 hold. In the optimal mechanism, (S) holds with equality.*

This result seems to be intuitive but it is not obvious since the principal are required to be budget balanced. Mathematically, virtual value is associated with a Lagrange multiplier and enters the objective function. In other words, the principal is maximizing the weighted sum of total realized values and revenue minus inspection cost. It is well know that in revenue-maximizing mechanisms, the principal often limits allocation. Thus, it is possible for the principal to limit allocation here due to the revenue concern.

Intuitively, consider the extreme case in which low-budget agents have no money, i.e., $b_1 = 0$. In this case, the principal needs to raise all money from selling to high-budget agents. As she increases the amount of goods sold to high-budget agents, the revenue will start declining after some point. On the other hand, increasing the amount of goods allocated to low-budget agents raises inspection cost. Thus, it is not obvious that in the optimal mechanism all the goods are distributed to agents.

In the proof of Proposition 1 I show that if not all the goods are distributed to agents yet then the principal can increase the amounts of goods allocated to high-budget and low-budget agents simultaneously. For appropriately chosen allocation rule, the resulting mechanism is feasible and strictly improves welfare.

Proposition 2 *Suppose Assumptions 1 and 2 hold. The first-best allocation is achieved if and only if $S \geq \hat{S}(b_1)$, where $\hat{S}(b_1)$ is the solution to*

$$b_1 - v^* F(v^*) = 0$$

with $v^* = F^{-1}(1 - S)$. Furthermore, $\hat{S}(b_1)$ is strictly decreasing in b_1.

Proposition 3 *Suppose Assumptions 1 and 2 hold. Suppose also that* $S < \hat{S}(b_1)$*, i.e., the first-best cannot be achieved. In the optimal mechanism, (BB) and (BC) hold with equality.*

5 Implementation

In this section, I provide a simple and intuitive implementation of the direct optimal mechanism characterized in Section 4. This implementation resembles the features of some public programs.

Consider the following *random assignment with regulated resale and cash subsidy* scheme, which consists of two stages.

1. In the first stage, each agent reports his budget. Agents who report low budget are inspected with probability $(u_1^* - u_2^*)/c$. The principal offers cash subsidies u_1^* to low-budget agents and u_2^* to high-budget agents. The principal assigns the goods randomly (with uniform probability) at price $p_1^* := a^* v_1^*$ to low-budget agents and $p_2^* := a^* v_2^*$ to high-budget agents.

2. In the second stage, the resale market opens. The per-unit sales taxes are $\tau_1^* := v_2^{**} - v_1^*$ for low-budget agents and $\tau_2^* := v_2^{**} - v_2^*$ for high-budget agents. Agents who report low budget in the first stage and choose not to sell the good in the second stage are inspected with probability $(v_2^* - v_1^*)/c$.

It is straightforward to verify the following result.

Proposition 4 *Suppose Assumptions 1 and 2 hold. The optimal mechanism is implemented by RwRRC with* $0 \le v_1^* \le v_2^* \le v_2^{**} \le \overline{v}$*,* $u_1^* \ge u_2^*$ *and* $0 \le a^* \le 1$ *given by Corollary 2.*

If inspection is sufficiently costly or the principal cannot to inspect agents, then in the RwRRC scheme agents receive the same amount of cash subsidies $u_1^* = u_2^*$ and the same price $p_1^* = p_2^*$ in the first stage and face the same sales taxes $\tau_1^* = \tau_2^*$ in the second stage regardless of their budgets. This is consist with the findings in Che et al. (2012). If inspection is not too costly, the principal provides financial aids to low-budget agents ($u_1^* \le u_2^*$, $p_1^* \ge p_2^*$) in the first stage and

Table 1: Minimum Occupation Periods (MOP) of HDB flats

Types of HDB flats	Minimum Occupation Periods	
	Sell	Sublet
Resale flats w/ Grants	5-7 years	5-7 years
Resale flats w/o Grants	0-5 years	3 years

Sources. — Sell: http://www.hdb.gov.sg/fi10/fi10322p.nsf/w/SellFlatMinimumOccupationPerid; and Sublet: http://www.hdb.gov.sg/fi10/fi10323p.nsf/w/RentOpenMktRentOutWholeFlat?OpenDocument

discourages them from reselling by imposing a higher sales tax in the second stage. This resembles some features of the public housing in Singapore. In Singapore, buyers of resale HDB flats can apply for CPF housing grant, which is a housing subsidy to help eligible households. Buyers "must have physically occupied your flat for a period of time, referred to as the Minimum Occupation Period (MOP)"[8], before they are eligible to resell or sublet their flats. HDB flats purchased with CPF housing grant are subject to longer MOPs as illustrated by Table 1.

6 Conclusion

In this paper, I study the problem of a principal wishes to distribute an indivisible good to a population of budget constrained agents. Both valuation and budget are an agent's private information, but the principal may inspect an agent's budget through a costly verification process and impose an penalty. I characterize the (direct) efficiency-maximizing mechanism. I also show an implementation via a two-stage mechanism which features discriminatory cash subsidies and sales taxes.

[8] http://www.hdb.gov.sg/fi10/fi10322p.nsf/w/SellFlatMinimumOccupationPerid

A Appendix

Proof of Lemma 1. Consider types $t := (v, b)$ and $\hat{t}$ such that $p(\hat{t}) = p^*(\hat{t}) \leq b$. Then (IC) requires that

$$\begin{aligned}&a(t)v - p(t) - (1 - q(t))\theta(t, n) - q(t)\theta(t, b)\\ \geq &a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b)\end{aligned}$$

Consider an alternative mechanism $(a^*, p^*, q^*, \theta^*)$ with $a = a^*$ and $q = q^*$. Let $\theta^*(t, n) = \theta^*(t, b) = 0$ for all t and $\theta^*(\hat{t}, b) = c$ for all $\hat{t}$, and $\hat{b} \neq b$. Let $p^*(t) = p(t)+(1-q(t))\theta(t, n)+q(t)\theta(t, b)$. Since $p(t) + \max\{\theta(t, n), \theta(t, b)\} \leq b$, we have $p^*(t) \leq b$, i.e., (BC) holds. It is easy to see that the new mechanism also satisfies (IR), (BB) and (S) and does not affect the welfare.

Finally, I show that (IC) holds. Consider types $t := (v, b)$ and $\hat{t}$ such that $p^*(\hat{t}) + c \leq b$. If $\hat{b} = b$, then (BC) in the old mechanism implies that $p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\} \leq b$. Then

$$\begin{aligned}&a^*(t)v - p^*(t)\\ =&a(t)v - p(t) - (1 - q(t))\theta(t, n) - q(t)\theta(t, b)\\ \geq &a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b)\\ =&a^*(\hat{t})v - p^*(\hat{t}) + q(\hat{t})\theta(\hat{t}, \hat{b}) - q(\hat{t})\theta(\hat{t}, b)\\ =&a^*(\hat{t})v - p^*(\hat{t}).\end{aligned}$$

If $\hat{b} \neq b$, then $b \geq p^*(\hat{t}) + c = p(\hat{t}) + (1 - q(\hat{t}))\theta(\hat{t}, n) + q(\hat{t})\theta(\hat{t}, \hat{b}) + c \geq p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\}$. Then

$$\begin{aligned}&a^*(t)v - p^*(t)\\ =&a(t)v - p(t) - (1 - q(t))\theta(t, n) - q(t)\theta(t, b)\\ \geq &a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b)\end{aligned}$$

$$
\begin{aligned}
&= a^*(\hat{t})v - p^*(\hat{t}) + q(\hat{t})\theta(\hat{t},\hat{b}) - q(\hat{t})\theta(\hat{t},b) \\
&\geq a^*(\hat{t})v - p^*(\hat{t}) - q^*(\hat{t})\theta^*(\hat{t},b).
\end{aligned}
$$

The last inequality holds since $\theta(\hat{t},\hat{b}) \geq 0$ and $\theta^*(\hat{t},b) = c \geq \theta(\hat{t},b)$. ■

B The Revelation Principle

Consider a general mechanism that consists of a message space $\mathscr{M}$ and a quadruplet (a, p, q, θ), where $a : \mathscr{M} \to [0,1]$ denotes the probability an agent obtains the good, $p : \mathscr{M} \to [0,1]$ denotes the payment an agent must make, $q : \mathscr{M} \to [0,1]$ denotes the probability of inspecting and $\theta : \mathscr{M} \times \{n, b_1, b_2\} \to [0,1]$ denotes the probability an agent is penalized. In particular, $\theta(m, n)$ denotes the probability an agent is penalized if he is not inspected and $\theta(m, b)$ denotes the probability an agent is penalized if he is inspected and his budget is revealed to be b. Given a mechanism, an agent of type $t = (v, b)$ chooses $m \in \mathscr{M}$ to maximize

$$
a(m)v - p(m) - (1 - q(m))\theta(m,n)c - q(m)\theta(m,b)c
$$

subject to the constraint that $p(m) \leq b$. Let $m^*(t)$ denote the solution to the agent's problem. For simplicity, I assume $m^*(t)$ is deterministic, but it is easy to accommodate mixed strategies. If the agent's problem has multiple solutions, then some deterministic selection rule is used. Consider a new mechanism with message space T. Let $a^*(t) = a(m^*(t))$, $p^*(t) = p(m^*(t))$, $q^*(t) = a(m^*(t))$ and $\theta^*(t,\cdot) = \theta(m^*(t),\cdot)$. Then the new mechanism is incentive compatible. Clearly, an agent has no incentive to report $\hat{t}$ such that $\theta^*(\hat{t}) > b$. For $\hat{t}$ such that $\theta^*(\hat{t}) \leq b$, we have

$$
\begin{aligned}
& a(m^*(t))v - p(m^*(t)) - (1 - q(m^*(t)))\theta(m^*(t),n)c - q(m^*(t))\theta(m^*(t),b)c \\
\geq & a(m^*(\hat{t}))v - p(m^*(\hat{t})) - (1 - q(m^*(\hat{t})))\theta(m^*(\hat{t}),n)c - q(m^*(\hat{t}))\theta(m^*(\hat{t}),b)c.
\end{aligned}
$$

The inequality simply follows from the fact that $m^*(t)$ is the solution to a type t agent's problem in the original mechanism.

C Common Knowledge Budgets

Proof of Theorem 1. Let (a, p) be a feasible mechanism. For each $b \in B$, $a(\cdot, b)$ is non-decreasing and $p(v, b) = va(v, b) - \int_0^v a(v, b)\mathrm{d}v - u(0, b)$. Assume without loss of generality that $a(0, b) = 0$. Consider another mechanism (a^*, p^*). Let $a^*(\cdot, b)$ be defined by

$$a^*(v, b) = \begin{cases} a(\overline{v}, b) & \text{if } v \geq v_b^* \\ 0 & \text{otherwise} \end{cases},$$

where v_b^* is such that

$$\int_0^{\overline{v}} a(v, b) f(v)\mathrm{d}v = a(\overline{v}, b)(1 - F(v_b^*)). \tag{11}$$

Let $p^*(v, b) = va^*(v, b) - \int_0^v a^*(v, b)\mathrm{d}v - u(0, b)$. Clearly, (a^*, p^*) satisfies constraints (IR), (IC) and (S) and improves welfare. The revenue obtained by (a^*, p^*) is

$$\mathbb{E}_t[p^*(t)] = \frac{1}{2}\left\{ -u(0, b_1) - u(0, b_2) + \int_0^{\overline{v}} \left[v - \frac{1 - F(v)}{f(v)} \right] [a^*(v, b_1) + a^*(v, b_2)]\mathrm{d}v \right\}$$

By Assumption 1, $v - [1 - F(v)]/f(v)$ is strictly increasing. Thus, (a^*, p^*) also improves revenue, and therefore satisfies the (BB) constraint. Finally, we show that the (BC) constraint holds:

$$\begin{aligned} p^*(\overline{v}, b) =& \overline{v}a(\overline{v}, b) - \int_0^{\overline{v}} a^*(v, b)\mathrm{d}v \\ \leq& \overline{v}a(\overline{v}, b) - \int_0^{\overline{v}} a(v, b)\mathrm{d}v \leq b. \end{aligned}$$

The inequality holds if and only if

$$\int_0^{\overline{v}} [a^*(v,b) - a(v,b)]\mathrm{d}v \geq 0,$$
$$\Longleftrightarrow \int_{v_b^*}^{\overline{v}} [a^*(v,b) - a(v,b)]\mathrm{d}v \geq \int_0^{v_b^*} [a(v,b) - a^*(v,b)]\mathrm{d}v.$$

The inequality holds since

$$\begin{aligned}
\int_{v_b^*}^{\overline{v}} [a^*(v,b) - a(v,b)]\mathrm{d}v &= \int_{v_b^*}^{\overline{v}} [a^*(v,b) - a(v,b)]f(v)\frac{1}{f(v)}\mathrm{d}v \\
&\geq \int_{v_b^*}^{\overline{v}} [a^*(v,b) - a(v,b)]f(v)\frac{1}{f(v_b^*)}\mathrm{d}v \\
&= \int_0^{v_b^*} [a(v,b) - a^*(v,b)]f(v)\frac{1}{f(v_b^*)}\mathrm{d}v \\
&\geq \int_0^{v_b^*} [a(v,b) - a^*(v,b)]f(v)\frac{1}{f(v)}\mathrm{d}v \\
&= \int_0^{v_b^*} [a(v,b) - a^*(v,b)]\mathrm{d}v,
\end{aligned}$$

where the second and fourth line holds since f is non-increasing by Assumption 2 and the third line holds by (11). Hence there exists v_1^* and v_2^* such that the optimal allocation rule satisfies $a(v,b_1) = \chi_{\{v \geq v_1^*\}}(v) \min\left\{\frac{u(0,b_1)+b_1}{v_1^*}, 1\right\}$ and $a(v,b_2) = \chi_{\{v \geq v_2^*\}}(v)$. ■

D Privately Known Budgets

D.1 No Inspection

Proof of Lemma 2. If $v = 0$, (14) reduces to $u(0,b_2) \geq u(0,b_1)$. Suppose $u(0,b_2) > u(0,b_1)$. Let

$$u^*(0,b_1) = u^*(0,b_2) = (1-\pi)u(0,b_1) + \pi u(0,b_2).$$

Let v^* be such that

$$v^* := \sup\left\{ v \left| \begin{array}{c} \int_0^v a(\nu, b_1)\mathrm{d}\nu + u(0, b_1) - \int_0^v \min\{a(\nu, b_1), a(\nu, b_2)\}\mathrm{d}\nu \\ -(1-\pi)u(0, b_1) - \pi u(0, b_2) \leq 0 \end{array} \right. \right\}.$$

Let $v^- := \sup\{v \leq v^* | a(v, b_2) \geq a(v, b_1)\}$ and $v^+ := \inf\{v \geq v^* | a(v, b_2) \geq a(v, b_1)\}$. Note that if $v^* = \overline{v}$, then $v^+ = v^*$. Note also that if $a(v^*, b_2) \geq a^*(v, b_1)$, then $v^+ = v^- = v^*$. Clearly, $a(v, b_1) > a(v, b_2)$ for all $v \in (v^-, v^+)$, $a(v^-, b_1) = a(v^-, b_2)$ and $a(v^+, b_1) = a(v^+, b_2)$. There exists $\alpha \in (0, 1)$ such that

$$\begin{aligned} &\int_0^{v^+} a(\nu, b_1)\mathrm{d}\nu + u(0, b_1) - \int_0^{v^-} \min\{a(\nu, b_1), a(\nu, b_2)\}\mathrm{d}\nu \\ &- \int_{v^-}^{v^+} [\alpha a(\nu, b_1) + (1-\alpha)a(\nu, b_2)]\mathrm{d}\nu - (1-\pi)u(0, b_1) - \pi u(0, b_2) = 0. \end{aligned}$$

Let

$$a^*(v, b_1) = \begin{cases} \min\{a(v, b_1), a(v, b_2)\} & \text{if } v < v^- \\ \alpha a(v, b_1) + (1-\alpha)a(v, b_2) & \text{if } v^- < v < v^+ \\ a(v, b_1) & \text{if } v > v^+ \end{cases},$$

and

$$a^*(v, b_2) = \begin{cases} \frac{(1-\pi)\left[a(v,b_1) - \min\{a(v,b_1), a(v,b_2)\}\right]}{\pi} + a(v, b_2) & \text{if } v < v^- \\ \frac{(1-\pi)\left[a(v,b_1) - \alpha a(v,b_1) - (1-\alpha)a(v,b_2)\right]}{\pi} + a(v, b_2) & \text{if } v^- < v < v^+ \\ a(v, b_2) & \text{if } v > v^+ \end{cases}.$$

Clearly, $a^*(v, b)$ is feasible and non-decreasing in v. By construction, we have $(1-\pi)u(0, b_1) + \pi u(0, b_2) = (1-\pi)u^*(0, b_1) + \pi u^*(0, b_2)$, $(1-\pi)a(v, b_1) + \pi a(v, b_2) = (1-\pi)a^*(v, b_1) + \pi a^*(v, b_2)$, and

$$u(0, b_1) + \int_0^{v^+} a(v, b_1)\mathrm{d}v = u^*(0, b_1) + \int_0^{v^+} a^*(v, b_1)\mathrm{d}v. \quad (12)$$

Hence

$$u(0,b_2)+\int_0^{v^+} a(v,b_2)\mathrm{d}v = u^*(0,b_2)+\int_0^{v^+} a^*(v,b_2)\mathrm{d}v. \tag{13}$$

Let $p^*(v,b) = va^*(v,b) - \int_0^v a(\nu,b)\mathrm{d}\nu - u^*(0,b)$. Then

$$\begin{aligned}
p^*(\overline{v},b_1) &= \overline{v}a^*(\overline{v},b_1) - \int_0^{\overline{v}} a^*(v,b_1)\mathrm{d}v - u^*(0,b_1)\\
&= \overline{v}a(\overline{v},b_1) - \int_0^{v^+} a^*(v,b_1)\mathrm{d}v - \int_{v^+}^{\overline{v}} a(v,b_1)\mathrm{d}v - u^*(0,b_1)\\
&= \overline{v}a(\overline{v},b_1) - \int_0^{v^+} a(v,b_1)\mathrm{d}v - \int_{v^+}^{\overline{v}} a(v,b_1)\mathrm{d}v - u(0,b_1) \le b_1,
\end{aligned}$$

where the third line follows from (12). Hence the (BC) constraint holds. For $v < v^-$, we have $a^*(v,b_2) \ge a^*(v,b_1)$ and $u^*(0,b_1) = u^*(0,b_2)$. Hence (3) holds. For $v > v^+$, we have

$$\begin{aligned}
u^*(0,b_1)+\int_0^v a^*(\nu,b_1)\mathrm{d}\nu &= u(0,b_1)+\int_0^v a(\nu,b_1)\mathrm{d}\nu\\
&\le u(0,b_2)+\int_0^v a(\nu,b_2)\mathrm{d}\nu\\
&= u^*(0,b_2)+\int_0^v a^*(\nu,b_2)\mathrm{d}\nu,
\end{aligned}$$

where the first line follows from (12) and the third line follows from (13). Finally, consider $v \in [v^-, v^+]$. Suppose $\alpha \le 1-\pi$, then $a^*(v,b_1) \le a^*(v,b_2)$ for $v \in [v^-, v^+]$ and we have

$$\begin{aligned}
u^*(0,b_1)+\int_0^v a^*(\nu,b_1)\mathrm{d}\nu &= u^*(0,b_1)+\int_0^{v^-} a^*(\nu,b_1)\mathrm{d}\nu + \int_{v^-}^v a^*(\nu,b_1)\mathrm{d}\nu\\
&\le u^*(0,b_2)+\int_0^{v^-} a^*(\nu,b_2)\mathrm{d}\nu + \int_{v^-}^v a^*(\nu,b_2)\mathrm{d}\nu\\
&= u^*(0,b_2)+\int_0^v a^*(\nu,b_2)\mathrm{d}\nu.
\end{aligned}$$

Suppose $\alpha > \pi$, then $a^*(v,b_1) > a^*(v,b_2)$ for $v \in [v^-, v^+]$ and we have

$$u^*(0,b_1)+\int_0^v a^*(\nu,b_1)\mathrm{d}\nu = u^*(0,b_1)+\int_0^{v^+} a^*(\nu,b_1)\mathrm{d}\nu - \int_v^{v^+} a^*(\nu,b_1)\mathrm{d}\nu$$

$$\leq u^*(0,b_2) + \int_0^{v^+} a^*(\nu,b_2)\mathrm{d}\nu - \int_v^{v^+} a^*(\nu,b_2)\mathrm{d}\nu$$
$$= u^*(0,b_2) + \int_0^{v} a^*(\nu,b_2)\mathrm{d}\nu.$$

Hence the (IC-b) constraint holds. Clearly, (a^*, p^*) also satisfies constraints (IR), (IC-v), (S) and (BB), and does not change welfare. ■

Proof of Lemma 3. Given Lemma 2, (3) becomes

$$\int_0^{v} a(\nu,b_2)\mathrm{d}\nu \geq \int_0^{v} a(\nu,b_1)\mathrm{d}\nu,\ \forall v. \tag{14}$$

For each $b \in \boldsymbol{B}$, we have

$$\int_0^{v} a(\nu,b)f(\nu)\mathrm{d}\nu = \int_0^{v} f(\nu)\mathrm{d}\int_0^{\nu} a(\nu',b)\mathrm{d}\nu'$$
$$= f(v)\int_0^{v} a(\nu',b)\mathrm{d}\nu' - \int_0^{v}\left[\int_0^{\nu} a(\nu',b)\mathrm{d}\nu'\right] f'(\nu)\mathrm{d}\nu.$$

Since $f \geq 0$ and $-f' \geq 0$, (4) follows from (14). ■

Proof of Theorem 2. We first solve the optimal mechanism of $\mathcal{P}'$ and then verify that the optimal mechanism also satisfies the (IC) constraint of low-budget agents. Let (a,p) be a feasible mechanism. For each $b \in \boldsymbol{B}$, $a(\cdot,b)$ is non-decreasing and $p(v,b) = va(v,b) - \int_0^v a(\nu,b)\mathrm{d}\nu - u(0,b)$. Assume without loss of generality that $a(0,b) = 0$. Consider another mechanism (a^*, p^*).

Let $\hat{v} := \inf\{v | a(v,b_2) \geq a(\overline{v},b_1)\}$. Note that $\hat{v} = \overline{v}$ if $a(\overline{v},b_1) > a(\overline{v},b_2)$ and $\hat{v} = 0$ if $a(\overline{v},b_1) \leq a(0,b_2)$. Let a^* be defined by

$$a^*(v,b_1) = \begin{cases} a(\overline{v},b_1) & \text{if } v \geq v_1^*, \\ 0 & \text{otherwise,} \end{cases}$$

where v_1^* satisfies $a(\overline{v}, b_1)[1 - F(v_1^*)] = \int_0^{\overline{v}} a(v, b_1) f(v) \mathrm{d}v$, and

$$a^*(v, b_2) = \begin{cases} 1 & \text{if } v \geq v_2^{**}, \\ a(\overline{v}, b_1) & \text{if } v_2^* \leq v < v_2^{**}, \\ 0 & \text{otherwise,} \end{cases}$$

where v_2^* satisfies $a(\overline{v}, b_1)[F(\hat{v}) - F(v_2^*)] = \int_0^{\hat{v}} a(v, b_2) f(v) \mathrm{d}v$ and v_2^{**} satisfies $1 - F(v_2^{**}) + a(\overline{v}, b_1)[F(v_2^{**}) - F(\hat{v})] = \int_{\hat{v}}^{\overline{v}} a(v, b_2) f(v) \mathrm{d}v$. Let $p^*(v, b) = v a^*(v, b) - \int_0^v a^*(\nu, b) \mathrm{d}\nu - u(0, b)$.

We show that $v_1^* \geq v_2^*$. If $v_1^* \geq \hat{v}$, then $v_1^* \geq v_2^*$. If $v_1^* < \hat{v}$, then

$$\begin{aligned} a(\overline{v}, b_1)[F(\hat{v}) - F(v_1^*)] &= \int_0^{\hat{v}} a(v, b_1) f(v) \mathrm{d}v + \int_{\hat{v}}^{\overline{v}} [a(v, b_1) - a(\overline{v}, b_1)] f(v) \mathrm{d}v \\ &\leq \int_0^{\hat{v}} a(v, b_1) f(v) \mathrm{d}v \\ &\leq \int_0^{\hat{v}} a(v, b_2) f(v) \mathrm{d}v \\ &= a(\overline{v}, b_1)[F(\hat{v}) - F(v_2^*)], \end{aligned}$$

where the third line holds by Lemma 3. In this case, it must be that $a(\overline{v}, b_1) > 0$ since otherwise $a(\overline{v}, b_1) = 0 \leq a(0, b_2)$, which implies $\hat{v} = 0 \leq v_1^*$. Hence, $v_2^* \leq v_1^*$. Thus, (a^*, p^*) satisfies the (IC-b) constraint.

Clearly, (a^*, p^*) also satisfies constraints (BC), (IR), (IC-v), (S) and (BB) and strictly improves welfare. Suppose $v_2^* < v_1^*$, then it is welfare improving to increase v_2^* and reduce v_1^* without affecting any constraint. Hence, it is optimal to set $v_1^* = v_2^* = v^*$. Let $u^* = u(0, b_1) = u(0, b_2)$. Then the optimal allocation rule satisfies $a(v, b_1) = \chi_{\{v \geq v^*\}} \min\left\{\frac{u^*+b_1}{v^*}, 1\right\}$ and $a(v, b_2) = \chi_{\{v \geq v^*\}} \min\left\{\frac{u^*+b_1}{v^*}, 1\right\} + \chi_{\{v \geq v_2^{**}\}}\left(1 - \min\left\{\frac{u^*+b_1}{v^*}, 1\right\}\right)$.

Clearly, if $u^* + b_1 > v^*$, we can reduce u^* such that $u^* + b_1 = v^*$ without affecting any constraint or the principal's objective function. This completes the characterization of the optimal mechanism of $\mathcal{P}'$. Finally, it is easy to see that the (IC) constraint of low-budget types is satisfied. This completes the proof. ■

D.2 The General Case

Proof of Lemma 4. Suppose not. Then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare.

Let $\hat{v}_2^m = \inf\{v|a(v, b_2) \geq a^m\}$ for $m = 1, \dots, M$, $\hat{v}_2^0 = 0$ and $\hat{v}_2^{M+1} = \overline{v}$. Given a, the optimal inspection rule satisfies $q(v, b_1) = q^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$, where

$$q^m = \frac{1}{c}\max\left\{0, u(0, b_1) - u(0, b_2) - \int_0^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v_1^{m-1}) + \int_0^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu\right\}.$$

For each $m = 1, \dots, M+1$, there exists $v_2^{m-1} \in [\hat{v}_2^{m-1}, \hat{v}_2^m]$ such that

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} a(v, b_2)f(v)\mathrm{d}v = a^{m-1}[F(v_2^{m-1}) - F(\hat{v}_2^{m-1})] + a^m[F(\hat{v}_2^m) - F(v_2^{m-1})]. \tag{15}$$

Consider $a^*(v, b_2)$ such that $a^*(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 1, \dots, M$, $a^*(v, b_2) = 0$ if $v < v_2^0$ and $a^*(v, b_2) = 1$ if $v > v_2^M$. Note that if $a^1 = 0$, then $v_2^0 = 0$. If $a^M = 1$, then v_2^M is in-determined and we assume $v_2^M = v_2^{M-1}$. Let $a^*(v, b_1) = a(v, b_1)$.

Let $p^*(v, b) = va^*(v, b) - \int_0^v a^*(\nu, b)\mathrm{d}\nu - u(0, b)$. Let $q^*(v, b_1) = q(v, b_1)$. We show that the (IC-b) constraint is satisfied. That is, for $m = 1, \dots, M$,

$$q^m c \geq u(0, b_1) - u(0, b_2) - \int_0^{v_2^{m-1}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v_1^{m-1}) + \int_0^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu.$$

Since $a^*(v, b_2) = a^m$ for $v \in (v_2^{m-1}, \hat{v}_2^m)$, we have

$$\begin{aligned}
&u(0, b_1) - u(0, b_2) - \int_0^{v_2^{m-1}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v_1^{m-1}) + \int_0^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu \\
=&u(0, b_1) - u(0, b_2) - \int_0^{\hat{v}_2^m} a^*(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v_1^{m-1}) + \int_0^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu \\
\leq&u(0, b_1) - u(0, b_2) - \int_0^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v_1^{m-1}) + \int_0^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu,
\end{aligned}$$

where the last inequality holds if and only if

$$\int_0^{\hat{v}_2^m} [a^*(v, b_2) - a(v, b_2)]\mathrm{d}v \geq 0.$$

To prove this, we prove that for $m = 1, \ldots, M$

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} [a^*(v, b_2) - a(v, b_2)]\mathrm{d}v \geq 0. \tag{16}$$

(16) holds if and only if

$$\int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(v, b_2) - a(v, b_2)]\mathrm{d}v \geq \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(v, b_2) - a^*(v, b_2)]\mathrm{d}v. \tag{17}$$

(17) follows from the construction of a^* and Assumption 2:

$$\begin{aligned}
\int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(v, b_2) - a(v, b_2)]\mathrm{d}v &\geq \int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(v, b_2) - a(v, b_2)] f(v) \frac{1}{f(v_2^{m-1})}\mathrm{d}v \\
&= \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(v, b_2) - a^*(v, b_2)] f(v) \frac{1}{f(v_2^{m-1})}\mathrm{d}v \\
&\geq \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(v, b_2) - a^*(v, b_2)]\mathrm{d}v.
\end{aligned}$$

By Assumption 1, $\mathbb{E}_t[p^*(t)] \geq \mathbb{E}_t[p(t)]$. Hence, constraint (BB) is satisfied. It is also clear that (a^*, p^*, q^*) satisfies constraints (IR), (IC-v), (BC) and (S), and strictly improves welfare. ∎

Proof of Lemma 5. The proof is by construction. Let (a, p, q) be a feasible mechanism, where a is a M-step allocation rule, p satisfies the envelope condition and q is given by (7). Suppose (a, p, q) satisfies neither (C1) nor (C2). We show that we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare and satisfies one of the two conditions. Furthermore, a^* is a M-step function. We break the proof into three steps.

Step 1. Suppose $u(0, b_1) - u(0, b_2) + a^1 v_2^0 < 0$. Let $m > 1$ be such that $v_2^{m'-1} - v_1^{m'-1} \leq 0$ for all $m' < m$ and $v_2^{m-1} - v_1^{m-1} > 0$. If there is no such m, then (a, p, q) satisfies (C2). Let $\hat{v}$ be defined by

$F(\hat{v}) = \pi F(v_2^{m-1}) + (1-\pi)F(v_1^{m-1})$ if $F(v_1^m) > \pi F(v_2^{m-1}) + (1-\pi)F(v_1^{m-1})$ and $\hat{v} = v_1^m$ otherwise. Consider two different cases.

Case 1

Suppose $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) \geq \pi[u(0,b_2) - u(0,b_1) - a^1 v_2^0]$, let $\tilde{v}_1^{m-1} \in [v_1^{m-1}, \hat{v}]$ be such that

$$(a^m - a^{m-1})(\tilde{v}_1^{m-1} - v_1^{m-1}) = \pi[u(0,b_2) - u(0,b_1) - a^1 v_2^0].$$

Let $\tilde{v}_2^{m-1} \in [\hat{v}, v_2^{m-1}]$ be such that $\pi[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] = (1-\pi)[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})]$. Let $\tilde{v}_i^{m'} = v_i^{m'}$ for $i = 1,2$ and $m' \neq m-1$. Let $a^*(v,b_1) = a^{m-1}$ if $v \in (v_1^{m-1}, \tilde{v}_1^{m-1})$ and $a^*(v,b_1) = a(v.b_1)$ otherwise. Let $a^*(v,b_2) = a^m$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m-1})$ and $a^*(v,b_2) = a(v,b_2)$ otherwise. Let $u^*(0,b_1) = (1-\pi)u(0,b_1) + \pi u(0,b_2) - \pi a^1 v_2^0$ and $u^*(0,b_2) = (1-\pi)u(0,b_1) + \pi u(0,b_2) + (1-\pi)a^1 v_2^0$. Let $p^*(v,b) = va^*(v,b) - \int_0^v a^*(\nu,b)\mathrm{d}\nu - u^*(0,b)$. By construction, $p^*(\overline{v},b_1) = p(\overline{v},b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v,b_1) = q(v,b_1)$. By Assumption 1, the (BB) constraint holds. For $v \in (\tilde{v}_1^{m'-1}, \tilde{v}_1^{m'})$, $m' = 1,\ldots,m-1$, (IC-b) holds since

$$u^*(0,b_1) - u^*(0,b_2) + \sum_{j=1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq 0 \leq q^*(v,b_1)c.$$

For $v \in (\tilde{v}_1^{m'-1}, \tilde{v}_1^{m'})$, $m' = m,\ldots,M$, we have $q^*(v,b_1) = q^m$. Then (IC-b) holds since

$$\begin{aligned}
&u^*(0,b_1) - u^*(0,b_2) + \sum_{j=1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=&\sum_{j=1}^{m'}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^m - a^{m-1})(\tilde{v}_2^{m-1} - \tilde{v}_1^{m-1} - v_2^{m-1} + v_1^{m-1}) - a^1 v_2^0 \\
\leq&\sum_{j=1}^{m'}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \frac{(a^m - a^{m-1})(v_1^{m-1} - \tilde{v}_1^{m-1})}{\pi} - a^1 v_2^0 \\
=&\sum_{j=1}^{m'}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + u(0,b_1) - u(0,b_2) \\
=&q^{m'}c,
\end{aligned}$$

where the third line holds since by Assumption 2

$$
\begin{aligned}
v_2^{m-1} - \tilde{v}_2^{m-1} \geq & \frac{1}{f(\tilde{v}_2^{m-1})}[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] \\
\geq & \frac{1-\pi}{\pi} \frac{1}{f(\tilde{v}_1^{m-1})}[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})] \\
\geq & \frac{1-\pi}{\pi} \left(\tilde{v}_1^{m-1} - v_1^{m-1}\right).
\end{aligned}
$$

Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Note also that the new mechanism satisfies $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$.

Suppose $\tilde{v}_1^{m-1} < v_1^m$, then continue with the argument in step 2.

Suppose $\tilde{v}_1^{m-1} = v_1^m < \tilde{v}_2^{m-1}$, then by the arguments in Lemma 4, we can construct a new mechanism which is feasible and strictly increases welfare, and whose allocation rule is a $(M-1)$-step allocation rule. Continue with the argument in step 2.

Case 2

Suppose $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) < \pi[u(0, b_2) - u(0, b_1) + a^1 v_2^0]$. Let $\tilde{v}_1^{m-1} = \hat{v}$. Let $\tilde{v}_2^{m-1} \in [\hat{v}, v_2^{m-1}]$ be such that $\pi[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] = (1-\pi)[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})]$. Let $\tilde{v}_i^{m'} = v_i^{m'}$ for $i = 1, 2$ and $m' \neq m-1$. Let $a^*(v, b_1) = a^{m-1}$ if $v \in (v_1^{m-1}, \tilde{v}_1^{m-1})$, and $a^*(v, b_1) = a(v.b_1)$ otherwise. Let $a^*(v, b_2) = a^m$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m-1})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(0, b_1) = u(0, b_1) + (a^m - a^{m-1})(\hat{v} - v_1^{m-1})$ and $u^*(0, b_2) = u(0, b_2) - (1-\pi)(a^m - a^{m-1})(\hat{v} - v_1^{m-1})/\pi$. Then $u^*(0, b_2) > u^*(0, b_1) + a^1 v_2^0 \geq 0$. Let $p^*(v, b) = v a^*(v, b) - \int_0^v a^*(\nu, b) \mathrm{d}\nu - u^*(0, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint is satisfied. By the same argument in Case 1, the (IC-b) constraint is satisfied. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare.

In this case, by construction, we have $\tilde{v}_1^{m-1} = \min\left\{\tilde{v}_2^{m-1}, v_1^m\right\}$.

Suppose $\tilde{v}_1^{m-1} = \tilde{v}_2^{m-1} < v_1^m$, then let $m^* > m$ be such that $\tilde{v}_2^{m'-1} - \tilde{v}_1^{m'-1} \leq 0$ for all $m' < m^*$ and $\tilde{v}_2^{m^*-1} - \tilde{v}_1^{m^*-1} > 0$. If there is no such m^*, (a^*, p^*, q^*) then satisfies (C2). Otherwise repeat the argument in step 1 for m^*.

Suppose $\tilde{v}_1^{m-1} = v_1^m \leq \tilde{v}_2^{m-1}$, then by the argument in Lemma 4, we can construct a new mechanism which is feasible and strictly increases welfare, and whose allocation rule is a $(M-1)$-step allocation rule. Repeat the arguments in step 1 for m.

Since M is finite, in finite steps we can construct a feasible mechanism (a, p, q) that either satisfies (C2) or $u(0, b_1) - u(0, b_2) + a^1 v_2^0 \geq 0$. In the latter case, continue with the argument in step 2.

Step 2. Suppose $u(0, b_1) - u(0, b_2) + a^1 v_2^0 \geq 0$. Consider $m \geq 2$. Suppose (8) holds for all $m' < m$ and

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0.$$

If there is no such m, then (a, p, q) satisfies (C1). It must be the case that $v_2^{m-1} < v_1^{m-1}$. Let $m^* \geq m$ be the smallest m' such that $v_2^{m'} > v_2^{m-1}$. That is, $v_2^{m^*} > v_2^{m-1}$ and $v_2^{m'} = v_2^{m-1}$ for $m' = m, \dots, m^* - 1$. Let $\hat{v} \in [v_2^{m-1}, v_1^{m-1}]$ be such that

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m-1} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^m - a^{m-1})(\hat{v} - v_1^{m-1}) = 0.$$

We consider two different cases.

Case 1

Suppose $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] \leq (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$. Let $\tilde{v}_2^{m^*} \in [\hat{v}, v_2^{m^*})$ be such that

$$(a^{m^*} - a^{m-1})[F(\tilde{v}_2^{m-1}) - F(v_2^{m-1})] = (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\tilde{v}_2^{m^*})]. \tag{18}$$

Let $\tilde{v}_2^{m'} = \hat{v}$ for $m' = m-1, \dots, m^* - 1$ and $\tilde{v}_2^{m'} = v_2^{m'}$ if $m' < m-1$ or $m' > m^*$. Let $\tilde{v}_1^{m'} = v_1^{m'}$ for all m'. Let $a^*(v, b_1) = a(v, b_1)$. Let $a^*(v, b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v, b_2) = a^{m^*+1}$ if $v \in (\tilde{v}_2^{m^*}, v_2^{m^*})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_0^v a^*(\nu, b) d\nu -$

$u(0, b)$. Clearly, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. That is, for $m' = 1, \dots, M$

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq q^{m'} c.$$

This is trivial for $m' \leq m$. For $m' = m+1, \dots, m^*$, we have $\tilde{v}_2^{m'-1} = \tilde{v}_2^{m-1} \leq v_1^{m-1} < v_1^{m'-1}$. Hence

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m-1} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \sum_{j=m}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - v_1^{j-1}) < 0 \leq q^{m'} c.$$

Finally, consider $m' \geq m^* + 1$. It suffices to show that

$$\begin{aligned} & u(0, b_1) - u(0, b_2) + \sum_{j=m}^{m^*+1} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\ \leq & u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m^*+1} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}), \end{aligned}$$

which holds if and only if

$$(a^{m^*} - a^{m-1})(\tilde{v}_2^{m-1} - v_2^{m-1}) \leq (a^{m^*+1} - a^{m^*})(v_2^{m^*} - \tilde{v}_2^{m^*}).$$

The last inequality holds by (18) and Assumption 2. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Let $m'' > m$ be such that (8) holds for all $m' < m''$ and is violated for m''. If there is no such m'', then (a^*, p^*, q^*) satisfies (C1). Otherwise repeat the argument in step 2 for m''.

Case 2

Suppose $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] > (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$. Let $\tilde{v}_2^{m-1}$ be such that

$$(a^{m^*} - a^{m-1})[F(\tilde{v}_2^{m-1}) - F(v_2^{m-1})] = (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\tilde{v}_2^{m-1})].$$

Let $\tilde{v}_2^{m'} = \tilde{v}_2^{m-1}$ for $m' = m, \dots, m^*$ and $\tilde{v}_2^{m'} = v_2^{m'}$ if $m' < m - 1$ or $m' > m^*$. Let $\tilde{v}_1^{m'} = v_1^{m'}$ for all m'. Let $a^*(v, b_1) = a(v, b_1)$. Let $a^*(\cdot, b_2)$ such that $a^*(v, b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v, b_2) = a^{m^*+1}$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m^*})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_0^v a^*(v, b)\mathrm{d}v - u(0, b)$. Clearly, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds. By the same argument in Case 1, the (IC-b) constraint is satisfied. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Note that for (a^*, p^*, q^*) we have $\tilde{v}_2^{m^*} = \tilde{v}_2^{m-1}$. Repeat the argument in step 2 for m with m^* replaced by $m^* + 1$.

Since M is finite, in finite steps we can construct a feasible mechanism (a, p, q) that either satisfies (C1), or $v_2^M = v_2^{m-1} < v_1^{m-1}$ and

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0.$$

In the latter case, continue with the argument in step 3.

Step 3. Suppose $u(0, b_1) - u(0, b_2) + a^1 v_2^0 \geq 0$,

$$u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \geq 0$$

for all $m' = 1, \dots, m-1$, $v_2^M = v_2^{m-1} < v_1^{m-1}$, and $u(0, b_1) - u(0, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0$.

Let $\tilde{v}_1^{m-1} = v_1^{m-1} - \varepsilon$ for some $\varepsilon > 0$ and $\tilde{v}_2^{m'} = v_2^{m-1} + \delta$ for $m' = m-1, \dots, M$, where $\delta > 0$ is such that

$$(1-\pi)(a^m - a^{m-1})\left[F(v_1^{m-1}) - F(\tilde{v}_1^{m-1})\right] = \pi(1-a^{m-1})\left[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})\right]. \tag{19}$$

Let $\tilde{v}_i^{m'} = v_i^{m'}$ if $m' \neq m-1$ for $i = 1, 2$. Let $\varepsilon > 0$ be such that

$$\min\left\{\tilde{v}_1^{m-1} - v_1^{m-2}, u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1})\right\} = 0. \tag{20}$$

Since $\sum_{j=1}^{m-1}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \geq 0$, we have $\tilde{v}_2^{m'} \leq \tilde{v}_1^{m'}$ for all $m' \geq m-1$. Let $a^*(v, b_i) = a^m$ if $v \in (\tilde{v}_i^{m-1}, \tilde{v}_i^m)$ for $i = 1, 2$ and $m = 1, \ldots, M$, $a^*(v, b_2) = 0$ if $v < \tilde{v}_2^0$ and $a^*(v, b_2) = 1$ if $v > \tilde{v}_2^M$. Let $p^*(v, b) = va^*(v, b) - \int_0^v a^*(\nu, b)\mathrm{d}\nu - u(0, b)$. Since $a^*(\overline{v}, b_1) = a(v, b_1)$ and $a^*(v, b_1) \geq a(v, b_1)$ for all v, we have $p^*(\overline{v}, b_1) \leq p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q^m$ if $v \in (\tilde{v}_1^{m-1}, \tilde{v}_1^m)$ for $m = 1, \ldots, M$. Then the change of the inspection cost is

$$k(q^m - q^{m-1})[F(v_1^{m-1}) - F(\tilde{v}_1^{m-1})].$$

Since $q^m = 0 \leq q^{m-1}$, the inspection cost is reduced. Furthermore, by Assumption 1, the revenue increases. Hence, the (BB) constraint holds. Finally, we show that the (IC-b) constraint is satisfied. That is, for $m' = 1, \ldots, M$

$$u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq q^{m'}c.$$

This is trivial for $m' < m$. For $m' \geq m$, this holds since

$$u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq 0 = q^{m'}c.$$

Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare.

If the first term of (20) reaches zero first, then by the argument in Lemma 4, we can construct a new mechanism which is feasible and strictly increases welfare, and whose allocation rule is a $(M-1)$-step allocation rule. Then repeat the argument in step 3 for $m-1$. If the second term of (20) reaches zero first and $m < M$, then repeat the argument in step 3 for $m+1$. If the second term of (20) reaches zero first and $m = M$, then (a^*, p^*, q^*) satisfies (C1).

Since M is finite, in finite steps we can construct a feasible mechanism (a^*, p^*, q^*), which strictly improves welfare and satisfies (C1). Furthermore, a^* is a M'-step allocation rule for some $M' \leq M$. ■

Proof of Theorem 3. Let (a, p, q) denote an optimal mechanism of $\mathcal{P}'$. Then $p(v, b) = va(v, b) - \int_0^v a(v, b)\mathrm{d}v - u(0, b)$ for all $(v, b) \in T$ and q is defined by (6). Fix $M \geq 2$. Let $a^0 = 0$, $a^{M+1} = 1$ and $a^m = (m-1)a(\overline{v}, b_1)/M$ for $m = 1, \dots, M$. Let $v_1^0 = 0$, $v_1^M = \overline{v}$ and for $m = 0, \dots, M-1$

$$v_1^m = \inf \left\{ v \,\middle|\, a(v, b_1) \geq a^{m+1} \right\}.$$

Then $0 = v_1^0 \leq v_1^1 \leq \cdots \leq v_1^M = \overline{v}$ and $0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1$. Let $a^*(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$. Then $a(v, b_1) - 1/M \leq a^*(v, b_1) \leq a(v, b_1)$. Let $\hat{v}_2^m = \inf\{v | a(v, b_2) \geq a^m\}$ for $m = 1, \dots, M$, $\hat{v}_2^0 = 0$ and $\hat{v}_2^{M+1} = \overline{v}$. For each $m = 1, \dots, M+1$, there exists $v_2^{m-1} \in [\hat{v}_2^{m-1}, \hat{v}_2^m]$ such that

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} a(v, b_2) f(v)\mathrm{d}v = a^{m-1}[F(v_2^{m-1}) - F(\hat{v}_2^{m-1})] + a^m[F(\hat{v}_2^m) - F(v_2^{m-1})]. \tag{21}$$

Consider $a^*(v, b_2)$ such that $a^*(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 1, \dots, M$, $a^*(v, b_2) = 0$ if $v < v_2^0$ and $a^*(v, b_2) = 1$ if $v > v_2^M$. Note that since $a^1 = 0$, we have $v_2^0 = 0$. Clearly, a^* satisfies constraint (S). Let $p^*(v, b) = va^*(v, b) - \int_0^v a^*(v, b)\mathrm{d}v - u(0, b)$ for $b \in B$. Let q^* be such that

$$cq^*(v, b_1) = cq(v, b_1) + \frac{v}{M}.$$

We show that the (IC-b) constraint is satisfied, i.e., for all $v \in (v_1^{m-1}, v_1^m)$, $m = 1, \dots, M$,

$$cq^*(v, b_1) \geq u(0, b_1) - u(0, b_2) - \int_0^{v_2^{m-1}} a(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v) + \int_0^{v} a^*(\nu, b_1)\mathrm{d}\nu.$$

Recall that for $v \in (v_1^{m-1}, v_1^m)$, we have

$$cq(v, b_1) \geq u(0, b_1) - u(0, b_2) - \int_0^{\hat{v}_2^{m}} a(\nu, b_2)\mathrm{d}\nu + a(v, b_1)(\hat{v}_2^m - v) + \int_0^{v} a(\nu, b_1)\mathrm{d}\nu.$$

Then for $v \in (v_1^{m-1}, v_1^m)$

$$\begin{aligned}
cq^*(v, b_1) &= cq(v, b_1) + \frac{v}{M} \\
&\geq u(0, b_1) - u(0, b_2) - \int_0^{\hat{v}_2^{m}} a(\nu, b_2)\mathrm{d}\nu + a(v, b_1)\hat{v}_2^m - \left(a(v, b) - \frac{1}{M}\right) v + \int_0^{v} a(\nu, b_1)\mathrm{d}\nu \\
&\geq u(0, b_1) - u(0, b_2) - \int_0^{\hat{v}_2^{m}} a(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v) + \int_0^{v} a(\nu, b_1)\mathrm{d}\nu \\
&\geq u(0, b_1) - u(0, b_2) - \int_0^{\hat{v}_2^{m}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v) + \int_0^{v} a^*(\nu, b_1)\mathrm{d}\nu \\
&= u(0, b_1) - u(0, b_2) - \int_0^{v_2^{m-1}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v) + \int_0^{v} a^*(\nu, b_1)\mathrm{d}\nu,
\end{aligned}$$

where the third line holds since $a(v, b) - 1/M \leq a^*(v, b) \leq a(v, b)$ and the fourth line holds by the same argument in the proof of Lemma 4. Then

$$\begin{aligned}
&\mathbb{E}_t[p^*(t) - q^*(t)k] - \mathbb{E}_t[p(t) - q(t)k] \\
=&\pi \int_0^{\bar{v}} \left[v - \frac{1 - F(v)}{f(v)}\right] [a^*(v, b_2) - a(v, b_2)] f(v)\mathrm{d}v \\
&+ (1 - \pi) \int_0^{\bar{v}} \left[v - \frac{1 - F(v)}{f(v)}\right] [a^*(v, b_1) - a(v, b_1)] f(v)\mathrm{d}v - (1 - \pi) \int_0^{\bar{v}} k[q^*(v, b_1) - q(v, b_1)] f(v)\mathrm{d}v \\
\geq& - \frac{\mathbb{E}[v]}{M} - (1 - \pi)\frac{\mathbb{E}[v]}{M}\frac{k}{c}.
\end{aligned}$$

For any $D > 0$, there exists $\overline{M}(D)$ such that for all $M > \overline{M}(D)$, we have $\frac{\mathbb{E}[v]}{M} + (1 - \pi)\frac{\mathbb{E}[v]}{M}\frac{k}{c} < D$.

Then (a^*, p^*, q^*) is a feasible solution to $\mathcal{P}'(M, D)$ for $M > \overline{M}(D)$. Furthermore, by Lemma 6,

$$\begin{aligned} V - V(2, D) =& V - V(M, D) \\ \leq& (1 - \pi) \left[\int_0^{\overline{v}} v[a(v, b_1) - a^*(v, b_1)] f(v) \mathrm{d}v - \int_0^{\overline{v}} \left[q(v, b_1) - q^*(v, b_1) \right] k f(v) \mathrm{d}v \right] \\ \leq& (1 - \pi) \left(1 + \frac{k}{c} \right) \frac{\mathbb{E}[v]}{M}. \end{aligned}$$

Let M goes to infinity and we have $V(2, 0) \leq V \leq V(2, D)$ for all $D > 0$. By the maximum theorem, $\lim_{D \to 0} V(2, D) = V(2, 0)$. Hence, $V = V(2, 0)$. ■

Proof of Corollary 2. By Theorem 3, there exists $u(0, b_1) \geq 0$, $u(0, b_2) \geq 0$, $0 \leq v_1^1 \leq \overline{v}$, $0 \leq v_2^0 \leq v_2^1 \leq v_2^2 \leq \overline{v}$ and $0 \leq a^1 \leq a^2 \leq \overline{v}$ such that $u(0, b_1) - u(0, b_2) + a^2 v_2^0 \geq 0$ and the optimal mechanism of $\mathcal{P}'$ is given by

$$\begin{aligned} a(v, b_1) &= a^1 + \chi_{\{v \geq v_1^1\}} \left(a^2 - a^1 \right), \\ a(v, b_2) &= \chi_{\{v \geq v_2^0\}} a^1 + \chi_{\{v \geq v_2^1\}} \left(a^2 - a^1 \right) + \chi_{\{v \geq v_2^2\}} (1 - a^2), \\ p(v, b_1) &= -u(0, b_1) + \chi_{\{v \geq v_1^1\}} \left(a^2 - a^1 \right) v_1^1, \\ p(v, b_2) &= -u(0, b_2) + \chi_{\{v \geq v_2^0\}} a^1 v_2^0 + \chi_{\{v \geq v_2^1\}} \left(a^2 - a^1 \right) v_2^1 + \chi_{\{v \geq v_2^2\}} (1 - a^2) v_2^2, \\ q(v, b_1) &= \frac{1}{c} \left[u(0, b_1) - u(0, b_2) + a^1 v_2^0 + \chi_{\{v \geq v_1^1\}} \left(a^2 - a^1 \right) (v_2^1 - v_1^1) \right], \\ q(v, b_2) &= 0. \end{aligned}$$

It is easy to verify that the above mechanism satisfies the (IC) constraint of type b_2 agents and therefore is also the optimal mechanism of $\mathcal{P}$. It follows from the proof of Lemma 6 that $v_2^1 \geq v_1^1$. We show below that $v_2^0 = 0$ and $a^1 = 0$.

First, we show that $v_2^0 = 0$. We consider two different cases: $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$ and $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$.

Suppose $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$**.**

Suppose to the contradiction that $v_2^0 > 0$. Then we can construct another feasible mechanism

(a^*, p^*, q^*), which strictly improves welfare. Since $v_2^0 > 0$, we have $u(0, b_2) > u(0, b_1)$ and, by construction, $a^1 > 0$.

Let $\varepsilon > 0$ be sufficiently small. Let $\tilde{v}_1^0 = \varepsilon$ and $\tilde{v}_2^0 < v_2^0$ be such that $\pi[F(v_2^0) - F(\tilde{v}_2^0)] = (1-\pi)F(\varepsilon)$. For $\varepsilon > 0$ sufficiently small, $\tilde{v}_1^0 < \min\{v_1^1, \tilde{v}_2^0\}$. Let $\tilde{v}_i^1 = v_i^1$ for $i = 1, 2$. Let $u^*(0, b_1) = u(0, b_1) + a^1\varepsilon$ and $u^*(0, b_2) = u(0, b_2) - (1-\pi)a^1\varepsilon/\pi$. For $\varepsilon > 0$ sufficiently small, $u^*(0, b_2) \geq u^*(0, b_1) > 0$. Let $a^*(v, b_1) = 0$ if $v < \tilde{v}_1^0$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^1$ if $v \in (\tilde{v}_2^0, v_2^0)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_0^v a(\nu, b)\mathrm{d}\nu - u(0, b)$. Since $u^*(0, b_1) - a^1\tilde{v}_1^0 = u(0, b_1)$, we have $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. (a^*, p^*, q^*) satisfies (BB) by Assumption 1.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. First, for $v < \varepsilon$, we have $u^*(0, b_1) - u^*(0, b_2) \leq 0 \leq q^*(v, b_1)c$. Next, we show that for $m = 1, 2$

$$q^m c \geq u^*(0, b_1) - u^*(0, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}).$$

Since

$$\begin{aligned} v_2^0 - \tilde{v}_2^0 &= \int_{\tilde{v}_2^0}^{v_2^0} f(v)\frac{1}{f(v)}\mathrm{d}v \\ &\geq \frac{1}{f(\tilde{v}_2^0)}[F(v_2^0) - F(\tilde{v}_2^0)] \\ &\geq \frac{1-\pi}{\pi}\frac{F(\varepsilon)}{f(\varepsilon)} \\ &\geq \frac{1-\pi}{\pi}\varepsilon, \end{aligned}$$

where the inequalities hold by Assumption 2, we have

$$u^*(0, b_1) - u^*(0, b_2) + \sum_{j=1}^{1}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1})$$

$$=u(0,b_1)-u(0,b_2)+a^1 v_2^0+a^1\varepsilon+a^1(\tilde{v}_2^0-v_2^0)$$
$$\leq u(0,b_1)-u(0,b_2)+a^1 v_2^0.$$

Furthermore, $\tilde{v}_i^1 = v_i^1$ for $i = 1, 2$ and $m \geq 1$. Hence, the (IC-b) constraint is satisfied. This contradicts to that (a, p, q) is optimal. Hence $v_2^0 = 0$.

Suppose $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$**.**

Suppose to the contradiction that $v_2^0 > 0$. In this case, $\gamma_2^0 = 0$. By construction, we have $a^1 > 0$ and therefore $\alpha_1 = 0$. Furthermore, since $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$ and $v_2^1 \geq v_1^1$, we have $\mu^1 = \mu^2 = 0$. Then v_2^0 satisfies

$$\pi\left[(\beta-(1+\lambda)v_2^0)f(v_2^0)+\lambda[1-F(v_2^0)]\right]-(1-\pi)(1+\lambda)\rho[1-F(v_1^0)]=0, \tag{22}$$

$$\pi\int_{v_2^0}^{v_2^M}[v+\lambda\varphi(v)-\beta]\,f(v)\mathrm{d}v \tag{23}$$

$$+(1-\pi)\left[\int_{v_1^0}^{v_1^M}[v+\lambda\varphi(v)-\beta]\,f(v)\mathrm{d}v-(1+\lambda)\rho(v_2^0-v_1^0)[1-F(v_1^0)]\right]$$

$$-\eta v_1^0-\alpha^{M+1}=0. \tag{24}$$

Since $v_2^0 \geq v_1^0$, it follows from Claims 3 and 13, and (24) that $\int_{v_1^0}^{v_1^M} \varphi(v,\lambda) f(v)\mathrm{d}v \geq \beta[1 - F(v_1^0)]$, i.e., $\hat{v}(\beta) = 0$.

Given β, η and λ, (31) and (32) define v_2^1 as functions of v_1^1, denoted by g_1 and g_2, respectively. By a similar argument in Claim 6, $g_1'(v) > 1$, and $g_2'(v) < 1$ if $v > \hat{v}(\beta)$ and $g(v) \geq v$. Let Δ_3 denote the left-hand side of (27) or (24), then

$$\frac{\partial\Delta_3}{\partial v_1^1}=(1-\pi)\left[(\beta-v_1^1-\lambda\varphi(v_1^1))f(v_1^1)+(1+\lambda)\rho(v_2^1-v_1^1)f(v_1^1)+(1+\lambda)\rho[1-F(v_1^1)]\right]-\eta,$$
$$\frac{\partial\Delta_3}{\partial v_2^1}=\pi(\beta-v_2^1-\lambda\varphi(v_2^1))f(v_2^1)-(1-\pi)(1+\lambda)\rho[1-F(v_1^1)].$$

Clearly, $\partial\Delta_3(v_1, g_2(v_1))/\partial v_2 = 0$ by (30). Since $v_2^1 \geq v_1^1$, then $g_2(v) > g_1(v)$ for all $v < v_1^1$.

Then $\partial\Delta_3(v_1, g_2(v_1))/\partial v_1 > \Delta_3(v_1, g_1(v_1))/\partial v_1 = 0$ for all $v_1 < v_1^1$. Then

$$0 = \Delta_3(v_1^1, v_2^1) = \Delta_3(v_1^0, v_2^0) + \int_{v_1^0}^{v_1^1} \frac{\partial\Delta_3(v_1, g_2(v_1))}{\partial v_1} \mathrm{d}v_1 > \Delta_3(v_1^0, v_2^0) = 0,$$

a contradiction. Hence, $v_2^0 = 0$.

Next, we show that $a^1 = 0$. Suppose $a^1 > 0$, then $\alpha^1 = 0$. Then v_2^0 satisfies

$$a^1 \left\{ \pi \left[\beta - v_2^0 - \lambda\varphi(v_2^0)\right] f(v_2^0) - (1-\pi)(1+\lambda)\rho + \sum_{j=1}^{2} \mu^j \right\} + \gamma_2^0 = 0, \tag{25}$$

$$\pi \int_{v_2^0}^{v_2^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v + (1-\pi) \int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v - \alpha^{M+1} = 0. \tag{26}$$

By Claims 3 and 13, it follows from (26) that $\int_{v_1^0}^{v_1^M} \varphi(v, \lambda) f(v)\mathrm{d}v - \beta \geq 0$, i.e., $\hat{v}(\beta) = 0$. Since $g_2'(v) \leq 1$ if $v \geq \hat{v}(\beta)$ and $g_2(v) \geq v$, and $g_2(v_1^1) = v_2^1 \geq v_1^1$, we have $v_2^0 = g_2(v_1^0) > v_1^0 = 0$, a contradiction. Hence, $a^1 = 0$.

Let $a^* = a^2$, $v_1^* = v_1^1$, $v_2^* = v_2^1$ and $v_2^{**} = v_2^2$. Let $u_1^* = u(0, b_1)$ and $u_2^* = u(0, b_2)$. This completes the proof. ■

D.3 The Proof of Lemma 6

Let $M \geq 3$ be an integer. We note that if a mechanism is a feasible solution to $\mathcal{P}'(M-1, D)$, then it is also a feasible solution to $\mathcal{P}'(M, D)$. Clearly, $V(M-1, D) \leq V(M, D)$. We show that $V(M-1, D) = V(M, D)$. Suppose to the contradiction that $V(M-1, D) < V(M, D)$, then in an optimal solution to $\mathcal{P}'(M, D)$ the allocation rule must be a M-step allocation rule, i.e.,

$$0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1,$$
$$0 = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}.$$

Hence $\alpha^2 = \cdots = \alpha^M = 0$ and $\gamma_1^1 = \cdots = \gamma_1^M = 0$. Let $\rho := k/c$. Then the first-order conditions of $\mathcal{P}'(M, D)$ are

$$\pi \left[\int_{v_2^{m-1}}^{v_2^m} \left[(1+\lambda)v - \lambda \frac{1-F(v)}{f(v)} \right] \mathrm{d}v - \beta [F(v_2^m) - F(v_2^{m-1})] \right]$$
$$+ (1-\pi) \left[\int_{v_1^{m-1}}^{v_1^m} \left[(1+\lambda)v - \lambda \frac{1-F(v)}{f(v)} \right] f(v) \mathrm{d}v \right]$$
$$- (1-\pi)(1+\lambda)\rho(v_2^{m-1} - v_1^{m-1})[F(v_1^m) - F(v_1^{m-1})] - (1-\pi)\beta[F(v_1^m) - F(v_1^{m-1})]$$
$$+ (1-\pi)(1+\lambda)\rho(v_2^m - v_1^m - v_2^{m-1} + v_1^{m-1})[1 - F(v_1^m)] + \eta(v_1^m - v_1^{m-1}) + \mu^m(v_2^{m-1} - v_1^{m-1})$$
$$- (v_2^m - v_1^m - v_2^{m-1} + v_1^{m-1}) \sum_{j=m+1}^{M} \mu^j + \alpha^m - \alpha^{m+1} = 0, \qquad (a^m, 1 \le m \le M-1)$$

$$\pi \left[\int_{v_2^{M-1}}^{v_2^M} \left[(1+\lambda)v - \lambda \frac{1-F(v)}{f(v)} \right] f(v) \mathrm{d}v - \beta [F(v_2^M) - F(v_2^{M-1})] \right]$$
$$+ (1-\pi) \int_{v_1^{M-1}}^{v_1^M} \left[(1+\lambda)v - \lambda \frac{1-F(v)}{f(v)} \right] f(v) \mathrm{d}v$$
$$- (1-\pi)(1+\lambda)\rho(v_2^{M-1} - v_1^{M-1})[F(v_1^M) - F(v_1^{M-1})] - (1-\pi)\beta[F(v_1^M) - F(v_1^{M-1})]$$
$$- \eta v_1^{M-1} + \mu^M(v_2^{M-1} - v_1^{M-1}) + \alpha^M - \alpha^{M+1} = 0, \qquad (a^M)$$

$$(a^{m+1} - a^m) \left\{ (1-\pi) \left[(\beta - (1+\lambda)v_1^m) f(v_1^m) + (\lambda + \rho + \lambda\rho)\left[1 - F(v_1^m)\right] + (1+\lambda)\rho(v_2^m - v_1^m) f(v_1^m) \right] \right.$$
$$\left. - \sum_{j=m+1}^{M} \mu^j - \eta \right\} = 0, \qquad (v_1^m, 1 \le m \le M-1)$$

$$a^1 \left\{ \pi \left[(\beta - (1+\lambda)v_2^0) f(v_2^0) + \lambda[1 - F(v_2^0)] \right] - (1-\pi)(1+\lambda)\rho + \sum_{j=1}^{M} \mu^j \right\} + \gamma_2^0 - \gamma_2^1 = 0, \quad (v_2^0)$$

$$(a^{m+1} - a^m) \left\{ \pi \left[(\beta - (1+\lambda)v_2^m) f(v_2^m) + \lambda[1 - F(v_2^m)] \right] - (1-\pi)(1+\lambda)\rho[1 - F(v_1^m)] + \sum_{j=m+1}^{M} \mu^j \right\}$$
$$+ \gamma_2^m - \gamma_2^{m+1} = 0, \qquad (v_2^m, 1 \le m \le M-1)$$

$$\pi(a^{M+1} - a^M) \left[(\beta - (1+\lambda)v_2^M) f(v_2^M) + \lambda[1 - F(v_2^M)] \right] + \gamma_2^M - \gamma_2^{M+1} = 0, \qquad (v_2^M)$$

$$\eta + \sum_{m=1}^{M} \mu^m - (1-\pi)(\lambda + \rho + \lambda\rho) + \xi_1 = 0, \qquad (u(0, b_1))$$

$$- \sum_{m=1}^{M} \mu^m - \pi\lambda + (1-\pi)(1+\lambda)\rho + \xi_2 = 0. \qquad (u(0, b_2))$$

The variables in the parentheses denote with respect to which variables the first-order conditions are taken. For later use, we note here that the summation of FOCs of $a^{m'}$, $m+1 \leq m' \leq M$, $m = 0, \dots, M-1$, gives:

$$\pi \left[\int_{v_2^m}^{v_2^M} \left[(1+\lambda)v - \lambda \frac{1-F(v)}{f(v)} \right] f(v)\mathrm{d}v - \beta[F(v_2^M) - F(v_2^m)] \right]$$
$$+ (1-\pi) \left[\int_{v_1^m}^{v_1^M} \left[(1+\lambda)v - \lambda \frac{1-F(v)}{f(v)} \right] f(v)\mathrm{d}v - (1+\lambda)\rho(v_2^m - v_1^m)[1-F(v_1^m)] - \beta[1-F(v_1^m)] \right]$$
$$- \eta v_1^m + (v_2^m - v_1^m) \sum_{j=m+1}^{M} \mu^j + \alpha^{m+1} - \alpha^{M+1} = 0. \tag{27}$$

Recall that $\alpha^2 = \cdots = \alpha^M = 0$. We break the proof into several claims. In all claims, we assume, without explicitly repeating this, that Assumptions 1 and 2 hold, $u(0, b_1)$, $u(0, b_2)$, $\{a^m\}_{m=1}^M$, $\{v_1^m\}_{m=1}^{M-1}$ and $\{v_2^m\}_{m=0}^M\}$ define an optimal mechanism of $\mathcal{P}'(M, D)$ and β, η, λ, ξ_1, ξ_2, $\{\mu^m\}_{m=1}^M$, $\{\alpha^m\}_{m=1}^{M+1}$, $\{\gamma_1^m\}_{m=1}^M$ and $\{\gamma_2^m\}_{m=0}^{M+1}$ are the associated Lagrangian multipliers.

Claim 1 $\gamma_2^m = 0$ *for* $m = 2, \dots, M-1$.

Proof. Since $a^{m+1} > a^m$ for $m = 1, \dots M-1$, the FOCs of v_1^m become

$$(1-\pi) \left[(\beta - (1+\lambda)v_1^m) f(v_1^m) + (\lambda + \rho + \lambda\rho) [1 - F(v_1^m)] + (1+\lambda)\rho(v_2^m - v_1^m) f(v_1^m) \right]$$
$$- \sum_{j=m+1}^{M} \mu^j - \eta = 0,$$

for $m = 1, \dots, M-1$. Then for $m = 1, \dots, M-1$

$$v_2^m = \frac{1+\rho}{\rho} v_1^m - \frac{\lambda + \rho + \lambda\rho}{(1+\lambda)\rho} \frac{1-F(v_1^m)}{f(v_1^m)} - \frac{\beta}{(1+\lambda)\rho} + \frac{\eta + \sum_{j=m+1}^{M} \mu^j}{(1-\pi)(1+\lambda)\rho f(v_1^m)}, \tag{28}$$

which is strictly increasing in v_1^m by Assumptions 1 and 2. Let $m = 1, \dots, M-2$. If $\mu^{m+1} = 0$, then $v_2^{m+1} > v_2^m$ since $v_1^{m+1} > v_1^m$ and (28). If $\mu^{m+1} > 0$, then $v_2^{m+1} \geq v_1^{m+1} > v_1^m \geq v_2^m$ since (8) holds for m and $m+2$ and (8) holds with equality for $m+1$. Hence, $\gamma_2^m = 0$ for $m = 2, \dots, M-1$. ■

Let

$$\varphi(v) := v - \frac{1 - F(v)}{f(v)},$$

denote the "virtual" value, which is strictly increasing in v by Assumption 1. In the rest of the proof we consider two cases: $v_2^{M-1} > v_1^{M-1}$ and $v_2^{M-1} \leq v_1^{M-1}$. In each case, we consider two subcases: $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$ and $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$.

Case 1: $v_2^{M-1} > v_1^{M-1}$**.** In this case, $\mu^M = 0$.

Claim 2 *Suppose* $v_2^{M-1} > v_1^{M-1}$*, then* $\overline{v} + \lambda\varphi(\overline{v}) > \beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$.

Proof. Since $\mu^M = 0$, the FOC of v_2^{M-1} implies that $\beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$. Since $v_2^{M-1} > v_1^{M-1}$ and $\mu^M = 0$, the FOC of a^M implies that

$$\pi \int_{v_2^{M-1}}^{v_2^M} [v + \lambda\varphi(v) - \beta] f(v) \mathrm{d}v + (1 - \pi) \int_{v_1^{M-1}}^{v_1^M} [v + \lambda\varphi(v) - \beta] f(v) \mathrm{d}v \geq 0.$$

Hence, it must be the case that $\beta < \overline{v} + \lambda\varphi(\overline{v})$. ■

Claim 3 *Suppose* $v_2^{M-1} > v_1^{M-1}$*, then* $\gamma_2^M = \gamma_2^{M+1} = 0$ *and* $v_2^M + \lambda\varphi(v_2^M) \leq \beta$.

Proof. Suppose $v_2^M + \lambda\varphi(v_2^M) > \beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$, then $v_2^M > v_2^{M-1}$ and therefore $\gamma_2^M = 0$. Suppose $v_2^M + \lambda\varphi(v_2^M) \leq \beta < \overline{v} + \lambda\varphi(\overline{v})$, then $v_2^M < \overline{v}$ and therefore $\gamma_2^{M+1} = 0$. Since $\gamma_2^{M+1} = 0$ and $v_2^M + \lambda\varphi(v_2^M) \leq \beta$, the FOC of v_2^M implies that $\gamma_2^M = 0$. Hence, $\gamma_2^M = 0$.

Suppose $a^{M+1} > a^M$, then the FOC of v_2^M implies that $\beta \geq v_2^M + \lambda\varphi(v_2^M)$. Suppose $a^{M+1} = a^M$, then by construction $v_2^M = v_2^{M-1}$ and therefore $v_2^M + \lambda\varphi(v_2^M) \leq \beta$. Hence, $v_2^M + \lambda\varphi(v_2^M) \leq \beta < \overline{v} + \lambda\varphi(\overline{v})$, which implies that $v_2^M < \overline{v}$ and therefore $\gamma_2^{M+1} = 0$. ■

Case 1.1. $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$**.**

Claim 4 *Suppose* $v_2^{M-1} > v_1^{M-1}$ *and* $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$*, then* $\gamma_2^1 = 0$.

Proof. Suppose $\gamma_2^0 > 0$, then $v_2^0 = 0$. Since (8) holds for $m = 2$, we have $v_2^1 \geq v_1^1 > 0 = v_2^0$. Hence, $\gamma_2^1 = 0$.

Suppose $\gamma_2^0 = 0$. Suppose $a^1 = 0$, then the FOC of v_2^0 implies that $\gamma_2^1 = 0$. Suppose $a^1 > 0$. Suppose to the contradiction that $\gamma_2^1 > 0$, then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Since $\gamma_2^1 > 0$, we have $v_2^0 = v_2^1 \geq v_1^1$. We consider two different cases: (1) $v_2^0 = v_2^1 = v_1^1$ and (2) $v_2^0 = v_2^1 > v_1^1$.

Suppose $v_2^0 = v_2^1 = v_1^1$.

Let $\tilde{v}_1^1$ be such that $a^2(v_1^1 - \tilde{v}_1^1) = a^1 v_1^1$. Then, by Assumption 2, we have

$$
\begin{aligned}
a^2\left[F(v_1^1) - F(\tilde{v}_1^1)\right] &= (a^2 - a^1 + a^1)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] \\
&\leq a^1\left[F(v_1^1) - F(\tilde{v}_1^1)\right] + (a^2 - a^1) f(\tilde{v}_1^1)(v_1^1 - \tilde{v}_1^1) \\
&= a^1\left[F(v_1^1) - F(\tilde{v}_1^1)\right] + a^1 f(\tilde{v}_1^1)\tilde{v}_1^1 \\
&\leq a^1 F(v_1^1).
\end{aligned}
$$

Let $\tilde{v}_2^0 = 0$ and $\tilde{v}_2^1$ be such that $\pi\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = (1-\pi)\left[a^1 F(v_1^1) - a^2\left[F(v_1^1) - F(\tilde{v}_1^1)\right]\right]$. Let $\tilde{v}_1^m = v_1^m$ and $\tilde{v}_2^m = v_2^m$ for all $m \geq 1$. Let $a^*(v, b_i) = a^m$ if $v \in (\tilde{v}_i^{m-1}, \tilde{v}_i^m)$ for $m \geq 2$ and $i = 1, 2$ and $a^*(v, b_i) = 0$ if $v \in (0, \tilde{v}_i^1)$ for $i = 1, 2$. Let $p^*(v, b) = va^*(v, b) - \int_0^v a^*(\nu, b)\mathrm{d}\nu - u(0, b)$. Then, by construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, (a^*, p^*, q^*) improves revenue and therefore satisfies the (BB) constraint. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. For $v \in (0, \tilde{v}_1^1)$, we have

$$u(0, b_1) - u(0, b_2) < u(0, b_1) - u(0, b_2) + a^1 v_2^0 = q^1 c.$$

For $v \in (\tilde{v}_1^1, v_1^1)$, we have

$$u(0,b_1) - u(0,b_2) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) \leq u(0,b_1) - u(0,b_2) + a^1 v_2^0 = q^1 c.$$

The first inequality holds since $a^2(\tilde{v}_2^1 - \tilde{v}_1^1) \leq a^2(v_1^1 - \tilde{v}_1^1) = a^1 v_1^1 = a^1 v_2^0$. For $v \in (v_1^{m-1}, v_1^{m-2})$, $m \geq 2$, we have

$$\begin{aligned}
&u(0,b_1) - u(0,b_2) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=&u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) - (a^2 - a^1)(v_2^1 - v_1^1) - a^1 v_2^0 \\
=&u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + a^2\tilde{v}_2^1 - a^2\tilde{v}_1^1 - a^2 v_2^1 + (a^2 - a^1)v_1^1 \\
\leq&u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c,
\end{aligned}$$

where the last inequality holds by construction. Hence, the (IC-b) constraint is satisfied. Thus, (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal.

Suppose $v_2^0 = v_2^1 > v_1^1$.

Let $a^*(v, b_1) = a^1 - \varepsilon$ for some $\varepsilon > 0$ sufficiently small if $v < v_1^1$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $\tilde{v}_2^0 < v_2^0$ be such that $\pi(a^1 - \varepsilon)\left[F(v_2^0) - F(\tilde{v}_2^0)\right] = (1-\pi)\varepsilon F(v_1^1)$. For $\varepsilon > 0$ sufficiently small, $v_1^1 < \tilde{v}_2^0$. Let $\tilde{v}_2^m = v_2^m$ for $m \geq 1$. Let $a^*(v, b_2) = a^1 - \varepsilon$ if $v \in (\tilde{v}_2^0, \tilde{v}_2^1)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(0, b_1) = u(0, b_1) + \varepsilon v_1^1$ and $u^*(0, b_2) = u(0, b_2) - (1 - \pi)\varepsilon v_1^1/\pi$. For $\varepsilon > 0$ sufficiently small, $u^*(0, b_2) \geq u^*(0, b_1) > 0$. Let $p^*(v, b) = va^*(v, b) - \int_0^v a(\nu, b)\mathrm{d}\nu - u(0, b)$. Then, by construction, we have $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. Then (a^*, p^*, q^*) satisfies (BB) by Assumption 1. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. Note that, by Assumption 2, we have

$$
\begin{aligned}
(a^1-\varepsilon)(v_2^0-\tilde v_2^0) &= (a^1-\varepsilon)\int_{\tilde v_2^0}^{v_2^0} f(v)\frac{1}{f(v)}\mathrm{d}v \\
&\geq (a^1-\varepsilon)\frac{1}{f(\tilde v_2^0)}[F(v_2^0)-F(\tilde v_2^0)] \\
&\geq \frac{1-\pi}{\pi}\varepsilon\frac{1}{f(v_1^1)}F(v_1^1) \\
&\geq \frac{1-\pi}{\pi}\varepsilon v_1^1.
\end{aligned}
$$

Then, for $v < v_1^1$, we have

$$
\begin{aligned}
& u^*(0,b_1)-u^*(0,b_2)+(a^1-\varepsilon)\tilde v_2^0 \\
=& u(0,b_1)-u(0,b_2)+a^1v_2^0+\frac{\varepsilon v_1^1}{\pi}+(a^1-\varepsilon)(\tilde v_2^0-v_2^0)-\varepsilon v_2^0 \\
\leq& u(0,b_1)-u(0,b_2)+a^1v_2^0+\frac{\varepsilon v_1^1}{\pi}-\frac{(1-\pi)\varepsilon v_1^1}{\pi}-\varepsilon v_2^0 \\
\leq& u(0,b_1)-u(0,b_2)+a^1v_2^0+\varepsilon(v_1^1-v_2^0) \\
<& u(0,b_1)-u(0,b_2)+a^1v_2^0=q^1c.
\end{aligned}
$$

For $v \in (v_1^{m-1}, v_1^m)$ for $m = 2, \dots, M$, we have

$$
\begin{aligned}
& u^*(0,b_1)-u^*(0,b_2)+(a^1-\varepsilon)\tilde v_2^0+(a^2-a^1+\varepsilon)(v_2^1-v_1^1)+\sum_{j=3}^{m}(a^j-a^{j-1})(\tilde v_2^{j-1}-\tilde v_1^{j-1}) \\
\leq& u(0,b_1)-u(0,b_2)+\sum_{j=1}^{m}(a^j-a^{j-1})(v_2^{j-1}-v_1^{j-1})+\varepsilon(v_1^1-v_2^0)+\varepsilon(v_2^1-v_1^1), \\
=& u(0,b_1)-u(0,b_2)+\sum_{j=1}^{m}(a^j-a^{j-1})(v_2^{j-1}-v_1^{j-1})=q^mc.
\end{aligned}
$$

Hence, the (IC-b) constraint is satisfied. Thus, (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal.

Hence, it must be that $\gamma_2^1 = 0$. ■

By Claims 1, 3 and 4, we have $\gamma_2^m = 0$ for $m = 1, \ldots, M+1$. Thus, for $m = 1, \ldots, M-1$, v_1^m and v_2^m satisfies

$$(1-\pi)\left[(\beta - v_1^m - \lambda\varphi(v_1^m))f(v_1^m) + (1+\lambda)\rho[1-F(v_1^m)] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right]$$
$$- \sum_{j=m+1}^{M} \mu^j - \eta = 0, \tag{29}$$
$$\pi(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) - (1-\pi)(1+\lambda)\rho[1-F(v_1^m)] + \sum_{j=m+1}^{M} \mu^j = 0. \tag{30}$$

Recall that (29) and (30) are the first-order conditions of v_1^m and v_2^m, respectively, for $m = 1, \ldots, M-1$.

Claim 5 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$, then $\mu^{m+1} > 0$ implies that $\mu^{m'+1} > 0$ for all $1 \le m' < m$.*

Proof. Suppose not. Let $1 \le m^* < m-1$ be such that $\mu^{m^*+1} > 0$ and $\mu^{m'} = 0$ for $m' = m^*+2, \ldots, m$. Then (29) implies that

$$v_2^{m'} = \frac{1+\rho}{\rho} v_1^{m'} - \frac{\lambda+\rho+\lambda\rho}{(1+\lambda)\rho}\frac{1-F(v_1^{m'})}{f(v_1^{m'})} - \frac{\beta}{(1+\lambda)\rho} + \frac{\eta + \sum_{j=m+1}^{M}\mu^j}{(1-\pi)(1+\lambda)\rho f(v_1^{m'})}$$

for all $m' = m^*+1, \ldots, m$. It follows from Assumptions 1 and 2 that $v_2^{m'} - v_1^{m'}$ is strictly increasing in m' for $m' = m^*+1, \ldots, m$. Since $\mu^{m+1} > 0$, we have $v_2^m \le v_1^m$. Thus $v_2^{m'} < v_1^{m'}$ for $m' = m^*+1, \ldots, m-1$. However, $\mu^{m^*+1} > 0$ implies (8) holds with equality for m^*+1. Combining this and $v_2^{m^*+1} < v_1^{m^*+1}$ implies that (8) is violated for m^*+2, a contradiction. Suppose $\mu^{m'} = 0$ for all $m' = 2, \ldots, m$. By a similar argument, $v_2^{m'} < v_1^{m'}$ for $m' = 1, \ldots, m-1$. Since $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$, then (8) is violated for $m = 2$, a contradiction. ■

Claim 6 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$, then $M = 2$.*

Proof. Let $\hat{m}$ be such that $\mu^m > 0$ for all $2 \le m \le \hat{m}$ and $\mu^m = 0$ for $m > \hat{m}$. Then $v_2^m = v_1^m \equiv v^m$

for $m = 1, \dots, \hat{m} - 1$. Combining (29) and (30) yields

$$(\beta - (1 + \lambda)v^m) f(v^m) + \lambda[1 - F(v^m)] - \eta = 0.$$

The left-hand side of the above equation is strictly decreasing in v^m. Hence, given β, η and λ, there exists at most one v^m that satisfies this equation, i.e, $\hat{m} \leq 2$.

For $m = \hat{m}, \dots, M - 1$, (29) and (30) becomes

$$(1 - \pi)\left[(\beta - v_1^m - \lambda\varphi(v_1^m)) f(v_1^m) + (1 + \lambda)\rho[1 - F(v_1^m)] + (1 + \lambda)\rho(v_2^m - v_1^m) f(v_1^m)\right] - \eta = 0, \tag{31}$$

$$\pi(\beta - v_2^m - \lambda\varphi(v_2^m)) f(v_2^m) - (1 - \pi)(1 + \lambda)\rho[1 - F(v_1^m)] = 0, \tag{32}$$

Given β, η and λ, (31) and (32) define v_2^m as functions of v_1^m, denoted by g_1 and g_2, respectively. Clearly, by Assumptions 1 and 2, $g_1'(v_1^m) > 1$. Since $\mu^{\hat{m}} > 0$, (8) holds by equality for $\hat{m}$, which implies that $v_2^{\hat{m}} \geq v_1^{\hat{m}}$. Furthermore, since $g_1'(v_1^m) > 1$, $v_2^m \geq v_1^m$ for all $\hat{m} \leq m \leq M - 1$. Since $v + \lambda\varphi(v) < \beta$ for all $v < v_2^M$, $v_2^m \geq v_1^m \geq 0$, $\sum_{j=m+1}^{M} \mu^j = 0$, $\eta \geq 0$, $\alpha^{m+1} = 0$ and $\alpha^{M+1} \geq 0$, (27) implies that

$$\int_{v^m}^{\bar{v}} [v + \lambda\varphi(v) - \beta]\, f(v) \mathrm{d}v \geq 0,$$

which holds if and only if $v^m \geq \hat{v}(\beta)$, where

$$\hat{v}(\beta) := \inf \left\{ \hat{v} \,\middle|\, \int_{v^m}^{\bar{v}} [v + \lambda\varphi(v) - \beta]\, f(v) \mathrm{d}v \geq 0 \right\}.$$

By the implicit function theorem, we have

$$g_2'(v_1^m) = \frac{1 - \pi}{\pi} \frac{(1 + \lambda)\rho f(v_1^m)}{-(\beta - (1 + \lambda)v_2^m) f'(v_2^m) + (1 + 2\lambda) f(v_2^m)} > 0. \tag{33}$$

Note that Assumption 1 implies that for all $v \geq v_1^m$, we have

$$f(v) \geq f(v_1^m)\frac{1 - F(v)}{1 - F(v_1^m)}. \tag{34}$$

Then for $v_1^m \geq \hat{v}(\beta)$ we have

$$\begin{aligned}
1 - F(v_1^m) &\geq \frac{f(v_1^m)}{1 - F(v_1^m)} \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v \\
&= \frac{f(v_1^m)}{1 - F(v_1^m)} \left[(1 + \lambda) \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v - \lambda \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v\right] \\
&= \frac{f(v_1^m)}{1 - F(v_1^m)} \left[-(1 + \lambda)v_1^m[1 - F(v_1^m)] + \int_{v_1^m}^{\bar{v}} \left[(1 + \lambda)v - \lambda\frac{1 - F(v)}{f(v)}\right] f(v)\mathrm{d}v\right] \\
&\geq (\beta - (1 + \lambda)v_1^m)f(v_1^m),
\end{aligned}$$

where the first line holds by (34), the third line holds by integration by parts, and the last line holds since $v_1^m \geq \hat{v}(\beta)$. Combining this and (32) yields

$$\begin{aligned}
(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) &= \frac{1 - \pi}{\pi}(1 + \lambda)\rho[1 - F(v_1^m)] \\
&= \frac{1 - \pi}{\pi}\rho\left[[1 - F(v_1^m)] + \lambda[1 - F(v_1^m)]\right] \\
&\geq \frac{1 - \pi}{\pi}\rho\left[(\beta - (1 + \lambda)v_1^m)f(v_1^m) + \lambda[1 - F(v_1^m)]\right] \\
&= \frac{1 - \pi}{\pi}\rho\left[\beta - v_1^m - \lambda\varphi(v_1^m)\right] f(v_1^m).
\end{aligned}$$

Hence,

$$\frac{\rho f(v_1^m)}{f(v_2^m)} \leq \frac{\pi}{1 - \pi}\frac{\beta - v_2^m - \lambda\varphi(v_2^m)}{\beta - v_1^m - \lambda\varphi(v_1^m)}.$$

Since $v_2^m \geq v_1^m \geq \hat{v}(\beta)$, we have

$$g_2'(v_1^m) = \frac{1 - \pi}{\pi}\frac{(1 + \lambda)\rho f(v_1^m)}{-(\beta - (1 + \lambda)v_2^m)f'(v_2^m) + (1 + 2\lambda)f(v_2^m)} \leq \frac{\beta - v_2^m - \lambda\varphi(v_2^m)}{\beta - v_1^m - \lambda\varphi(v_1^m)} \leq 1.$$

Note that $g_2'(v_1^m) < 1$ if $v_1^m > \hat{v}(\beta)$ or $v_1^m < v_2^m$.

Thus, there exists at most one $v_1^m \geq \hat{v}(\beta)$ such that $g_1(v_1^m) = g_2(v_1^m) \geq v_1^m$, i.e., (31) and (32) has at most one solution such that $v_2^m \geq v_1^m \geq \hat{v}(\beta)$. Hence, $M = \hat{m} + 1$.

Suppose $\hat{m} = 2$ or $M = 3$. By Claim 3, $v + \lambda\varphi(v) < \beta$ for all $v \leq v_2^M$. Furthermore, $\eta \geq 0$ and $\alpha^{M+1} \geq 0$. Hence, it follows from (27) that

$$\int_{v^1}^{\overline{v}} [v + \lambda\varphi(v) - \beta]\, f(v)\mathrm{d}v \geq 0,$$

i.e., $v^1 \geq \hat{v}(\beta)$. Then we have $v_1^{\hat{m}} > v^{\hat{m}-1} \geq \hat{v}(\beta)$, and $g_2(v_1^{\hat{m}}) = v_2^{\hat{m}} \geq v_1^{\hat{m}}$ since $\mu^{\hat{m}} > 0$. Since $g_2'(v) < 1$ if $v > \hat{v}(\beta)$ and $g(v) \geq v$, we have $v^{\hat{m}-1} = g_2(v^{\hat{m}-1}) > v^{\hat{m}-1}$, a contradiction. Hence, $M = 2$ and $v_2^1 > v_1^1$. ■

Case 1.2. $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$**.** In this case, $\mu^1 = 0$.

Claim 7 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$, then $\gamma_2^1 = \gamma_2^0 = 0$.*

Proof. Since $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$, $\mu^1 = 0$. Suppose $\gamma_2^0 > 0$, then $v_2^0 = 0$. The FOC of v_2^0 implies that

$$\begin{aligned} 0 =& a^1 \left[\pi\beta f(0) + \pi\lambda - (1-\pi)(1+\lambda)\rho + \sum_{m=1}^{M} \mu^m\right] + \gamma_2^0 - \gamma_2^1 \\ \geq& \pi a^1 \beta f(0) + \gamma_2^0 - \gamma_2^1, \end{aligned}$$

where the inequality follows from the FOC of $u(0, b_2)$. Hence, $\gamma_2^1 > 0$. Then $v_2^1 = v_2^0 = 0$ and the FOC of v_2^1 implies that

$$\begin{aligned} 0 =& (a^2 - a^1)\left[\pi\beta f(0) + \pi\lambda - (1-\pi)(1+\lambda)\rho\left[1 - F(v_1^1)\right] + \sum_{m=1}^{M} \mu^m\right] + \gamma_2^1 \\ \geq& (a^2 - a^1)\left[\pi\beta f(0) + (1-\pi)(1+\lambda)\rho F(v_1^1)\right] + \gamma_2^1 > 0, \end{aligned}$$

a contradiction. Here the first line holds since $\mu^1 = 0$, and the second line follows from the FOC of $u(0, b_2)$. Hence, $\gamma_2^0 = 0$.

Suppose $a^1 = 0$, then the FOC of v_2^0 implies that $\gamma_2^1 = 0$. Suppose $a^1 > 0$. Then

$$\begin{aligned}
&\pi(\beta - (1+\lambda)v_2^0) f(v_2^0) + \pi\lambda[1 - F(v_2^0)] \\
\geq &(1-\pi)(1+\lambda)\rho - \sum_{j=2}^{M} \mu^j \\
> &(1-\pi)(1+\lambda)\rho[1 - F(v_1^1)] - \sum_{j=2}^{M} \mu^j \\
\geq &\pi(\beta - (1+\lambda)v_2^1) f(v_2^1) + \pi\lambda[1 - F(v_2^1)].
\end{aligned}$$

Since $(\beta - (1+\lambda)v) f(v) + \lambda[1 - F(v)]$ is strictly decreasing in v when $v + \lambda\varphi(v) < \beta$, we have $v_2^1 > v_2^0$ and therefore $\gamma_2^1 = 0$. ■

By Claims 1, 3 and 7, we have $\gamma_2^m = 0$ for $m = 1, \dots, M$. Thus, for $m = 1, \dots, M-1$, v_1^m and v_2^m satisfies (29), (30) and (27).

Claim 8 *Suppose* $v_2^{M-1} > v_1^{M-1}$ *and* $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$, *then* $v_2^m \leq v_1^m$ *for all* $m = 1, \dots, M-2$.

Proof. We prove the claim by considering two different cases: (1.2.1) $\mu^m = 0$ for all $m = 2, \dots, M$; and (1.2.2) there exists $m \geq 2$ such that $\mu^m > 0$.

Case 1.2.1. $\mu^m = 0$ **for all** $m = 2, \dots, M$**.**

Suppose $M > 2$ and there exists $1 \leq m < M-1$ such that $v_2^m \geq v_1^m$. It follows from (27) that $\int_{v_1^m}^{v_1^M} \varphi(v, \lambda) f(v) \mathrm{d}v \geq \beta[1 - F(v_1^m)]$, i.e., $v_1^m \geq \hat{v}(\beta)$. Both (v_1^m, v_2^m) and (v_1^{M-1}, v_2^{M-1}) are solutions to (31) and (32), and satisfy $v_2 \geq v_1 \geq \hat{v}(\beta)$. However, by a similar argument in Claim 6, (31) and (32) have at most one solution satisfying $v_2 \geq v_1 \geq \hat{v}(\beta)$, a contradiction. Hence, it must be $M = 2$, or $M > 2$ and $v_2^m < v_1^m$ for all $m = 1, \dots, M-2$.

Case 1.2.2. Suppose there exists $m \geq 2$ **such that** $\mu^m > 0$**.**

By a similar argument in Claim 5, there exists $m^* \leq m^{**}$ such that $\mu^m = 0$ for $m \leq m^*$ and

$m > m^{**}$, and $\mu^m > 0$ for all $m = m^* + 1, \dots, m^{**}$. Then

$$v_2^m = \frac{1+\rho}{\rho} v_1^m - \frac{\lambda + \rho + \lambda\rho}{(1+\lambda)\rho} \frac{1 - F(v_1^m)}{f(v_1^m)} - \frac{\beta}{(1+\lambda)\rho} + \frac{\eta + \sum_{j=m^*+1}^{M} \mu^j}{(1-\pi)(1+\lambda)\rho f(v_1^m)},$$

for all $m = 1, \dots, m^*$. By Assumptions 1 and 2, $v_2^m - v_1^m$ is strictly increasing for $m = 1, \dots, m^*$. Furthermore, $v_2^m \le v_1^m$ for $m = m^*, \dots, m^{**} - 1$ since $\mu^m > 0$ for $m = m^* + 1, \dots, m^{**}$. Hence, $v_2^m \le v_1^m$ for all $m \le m^{**} - 1$.

We show that $m^{**} = M - 1$. Clearly, $m^{**} \le M - 1$ since otherwise we have $v_2^{M-1} \le v_1^{M-1}$, a contradiction. Suppose $m^{**} < M - 1$, then $v_2^{m^{**}} \ge v_1^{m^{**}}$. Furthermore, (30) implies that for $m = 1, \dots, M - 1$

$$(1-\pi)(1+\lambda)\rho[1 - F(v_1^m)] - \sum_{j=m+1}^{M} \mu^j = \pi(\beta - \varphi(v_2^m, \lambda)) f(v_2^m) \ge 0.$$

Hence, it follows from (27) that $\int_{v_1^{m^{**}}}^{v_1^M} \varphi(v, \lambda) f(v) \mathrm{d}v \ge \beta[1 - F(v_1^{m^{**}})]$, i.e., $v_1^{m^{**}} \ge \hat{v}(\beta)$. Both $(v_1^{m^{**}}, v_2^{m^{**}})$ and (v_1^{M-1}, v_2^{M-1}) are solutions to (31) and (32), and satisfy $v_2 \ge v_1 \ge \hat{v}(\beta)$. However, by a similar argument in Claim 6, (31) and (32) have at most one solution satisfying $v_2 \ge v_1 \ge \hat{v}(\beta)$, a contradiction. Hence, it must be that $m^{**} = M - 1$. In this case, we have $v_2^m \le v_1^m$ for all $m = 1, \dots, M - 2$.

■

Claim 9 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$, then $v_1^m = v_2^m$ for $m = 1, \dots, M - 2$.*

Proof. Suppose not. Then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare.

Let $m^* > 1$ be the smallest m such that $v_2^m < v_1^m$. Let $\varepsilon > 0$ sufficiently small. Let $\tilde{v}_1^{m^*} = v_1^{m^*} - \varepsilon$. Let $\tilde{v}_2^{m^*} > v_2^{m^*}$ be such that $\pi \left[F(\tilde{v}_2^{m^*}) - F(v_2^{m^*}) \right] = (1-\pi) \left[F(v_1^{m^*}) - F(\tilde{v}_1^{m^*}) \right]$. For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^{m^*} \le \tilde{v}_1^{m^*}$. Let $\tilde{v}_i^m = v_i^m$ for $m \ne m^*$ and $i = 1, 2$. Let $u^*(0, b_1) = u(0, b_1) - (a^{m^*+1} - a^{m^*})\varepsilon$ and

$u^*(0,b_2) = u(0,b_2) + \frac{1-\pi}{\pi}(a^{m^*+1} - a^{m^*})\varepsilon$. For $\varepsilon > 0$ sufficiently small, $u^*(0,b_1) - u^*(0,b_2) + a^1 v_2^0 \geq 0$. Let $a^*(v,b_1) = a^{m^*+1}$ if $v \in (\tilde{v}_1^{m^*}, v_1^{m^*})$ and $a^*(v,b_1) = a(v,b_1)$ otherwise. Let $a^*(v,b_2) = a^{m^*}$ if $v \in (v_2^{m^*}, \tilde{v}_2^{m^*})$ and $a^*(v,b_2)$ otherwise. Let $p^*(v,b) = va^*(v,b) - \int_0^v a^*(\nu,b)\mathrm{d}\nu - u^*(0,b)$. Then

$$
\begin{aligned}
p^*(\overline{v}, b_1) =& a^M \tilde{v}_1^{M-1} - \sum_{j=1}^{M-1} a^j(\tilde{v}_1^j - \tilde{v}_1^{j-1}) - u^*(0,b_1) \\
=& a^M v_1^{M-1} - \sum_{j=1}^{M-1} a^j(v_1^j - v_1^{j-1}) - u(0,b_1) = p(\overline{v}, b_1) \leq b_1.
\end{aligned}
$$

Hence, then (BC) constraint is satisfied. Let $q^*(v,b_1) = q^{m^*+1}$ if $v \in (\tilde{v}_1^{m^*}, v_1^{m^*})$ and $q^*(v,b_1) = q(v,b_1)$ otherwise. Then the total inspection cost is reduced by

$$
k(q^{m^*} - q^{m^*+1})[F(v_1^{m^*}) - F(\tilde{v}_1^{m^*})] = -\rho(a^{m^*+1} - a^{m^*})(v_2^{m^*} - v_1^{m^*})[F(v_1^{m^*}) - F(\tilde{v}_1^{m^*})] \geq 0.
$$

Clearly, (a^*, p^*, q^*) improves revenue and therefore satisfies the (BB) constraint. It is also clear that (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S).

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. That is, for $m = 1, \dots, M$,

$$
u^*(0,b_1) - u^*(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq q^m c.
$$

For $m \leq m^*$, we have

$$
\begin{aligned}
& u^*(0,b_1) - u^*(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=& u(0,b_1) - u(0,b_2) - \frac{(a^{m^*+1} - a^{m^*})\varepsilon}{\pi} + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\
\leq& u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c.
\end{aligned}
$$

Since

$$\begin{aligned}
\tilde{v}_2^{m^*} - v_2^{m^*} &= \int_{v_2^{m^*}}^{\tilde{v}_2^{m^*}} f(v)\frac{1}{f(v)}\mathrm{d}v, \\
&\leq \frac{1}{f(\tilde{v}_2^{m^*})}[F(\tilde{v}_2^{m^*}) - F(v_2^{m^*})] \\
&\leq \frac{1-\pi}{\pi}\frac{1}{f(v_1^{m^*})}[F(v_1^{m^*}) - F(\tilde{v}_1^{m^*})] \\
&\leq \frac{1-\pi}{\pi}\left(v_1^{m^*} - \tilde{v}_1^{m^*}\right),
\end{aligned}$$

for $m > m^*$, we have

$$\begin{aligned}
&u^*(0,b_1) - u^*(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=&u(0,b_1) - u(0,b_2) + \frac{(a^{m^*+1} - a^{m^*})(\tilde{v}_1^{m^*} - v_1^{m^*})}{\pi} + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\
&+ (a^{m^*+1} - a^{m^*})((v_1^{m^*} - \tilde{v}_1^{m^*}) - (v_2^{m^*} - \tilde{v}_2^{m^*})) \\
=&u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^{m^*+1} - a^{m^*})\left[\frac{1-\pi}{\pi}(\tilde{v}_1^{m^*} - v_1^{m^*}) - (v_2^{m^*} - \tilde{v}_2^{m^*})\right] \\
\leq&u(0,b_1) - u(0,b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c.
\end{aligned}$$

Hence, the (IC-b) constrained is satisfied. Hence, (a^*, p^*, q^*) is feasible. Clearly, (a^*, p^*, q^*) strictly improves welfare. However, this contradicts to that (a, p, q) is optimal. ■

Claim 10 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(0,b_1) - u(0,b_2) + a^1 v_2^0 > 0$, then $M \leq 3$.*

Proof. Suppose $M > 3$. Then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Let $\hat{v}$ be such that

$$\sum_{m=2}^{M-2} a^m[F(v_1^m) - F(v_1^{m-1})] = a^{M-2}\left[F(v_1^{M-2}) - F(\hat{v})\right].$$

Let $a^*(v, b_1) = a^1$ if $v < \hat{v}$, $a^*(v, b_1) = a^{M-2}$ if $v \in (\hat{v}, v_1^{M-1})$ and $a^*(v, b_1) = a^{M-1}$ if $v > v_1^{M-1}$.

Let $a^*(v, b_2) = a^1$ if $v_2^0 < v < \hat{v}$, $a^*(v, b_2) = a^{M-2}$ if $\hat{v} < v < v_2^{M-1}$, and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_0^v a^*(\nu, b)\mathrm{d}\nu - u(0, b)$. Then the (BC) constraint is satisfied by Assumption 2. Let $q^*(v, b_1) = q(v, b_1)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v), (IC-b), (S) and (BB), and strictly improves welfare. ■

Claim 11 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$, then $M = 2$.*

Proof. Suppose $M = 3$. Since $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$, $v_2^1 = v_1^1$ and $v_2^2 > v_1^2$, we have $\mu^m = 0$ for $m = 1, 2, 3$. Since $v_2^1 = v_1^1$, it follows from (27) that $\int_{v_1^1}^{v_1^M} \varphi(v, \lambda) f(v)\mathrm{d}v > \beta[1 - F(v_1^1)]$, i.e., $v_1^1 > \hat{v}(\beta)$. Note that both (v_1^1, v_2^1) and (v_1^2, v_2^2) are solutions to (31) and (32). However, by a similar argument in Claim 6, (31) and (32) have at most one solution satisfying $v_2 \geq v_1 \geq \hat{v}(\beta)$, a contradiction. Hence $M = 2$. ■

Case 2. $v_2^{M-1} \leq v_1^{M-1}$.

Claim 12 *Suppose $v_2^{M-1} \leq v_1^{M-1}$, then $\beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$.*

Proof. Suppose $\beta < v_2^{M-1} + \lambda\varphi(v_2^{M-1})$. Add the FOCs of v_1^{M-1} and v_2^{M-1} yields

$$(1-\pi)(\beta - v_1^{M-1} - \lambda\varphi(v_1^{M-1}))f(v_1^{M-1}) + (1-\pi)(1+\lambda)\rho(v_2^{M-1} - v_1^{M-1})f(v_1^{M-1})$$
$$+ \pi(\beta - v_2^{M-1} - \lambda\varphi(v_2^{M-1}))f(v_2^{M-1}) \geq 0.$$

Since $v_2^{M-1} \leq v_1^{M-1}$ and $v + \lambda\varphi(v)$ is strictly increasing, the left-hand side of the above inequality strictly less than zero, a contradiction. Hence, $\beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$. ■

Claim 13 *Suppose $v_2^{M-1} \leq v_1^{M-1}$, then $\gamma_2^1 = \gamma_2^M = 0$ and $\beta \geq v_2^M + \lambda\varphi(v_2^M)$.*

Proof. By the same arguments in Claim 4 and Claim 7, we have $\gamma_2^1 = 0$.

Suppose $\gamma_2^M > 0$. Then $v_2^M = v_2^{M-1} \leq v_1^{M-1} < \bar{v}$. Hence, $\gamma_2^{M+1} = 0$ and $v_2^M + \lambda\varphi(v_2^M) = v_2^{M-1} + \lambda\varphi(v_2^{M-1}) \leq \beta$. Then the FOC of v_2^M implies that

$$0 = \pi(a^{M+1} - a^M)(\beta - v_2^M - \lambda\varphi(v_2^M))f(v_2^M) + \gamma_2^M > 0,$$

a contradiction. Hence, $\gamma_2^M = 0$.

Suppose $a^{M+1} > a^M$, then the FOC of v_2^M implies that $\beta \geq v_2^M + \lambda\varphi(v_2^M)$. Suppose $a^{M+1} = a^M$, then by construction $v_2^M = v_2^{M-1}$ and therefore $v_2^M + \lambda\varphi(v_2^M) \leq \beta$. ■

By Claims 1 and 13, we have $\gamma_2^m = 0$ for $m = 1, \ldots, M$. Thus, for $m = 1, \ldots, M-1$, v_1^m and v_2^m satisfies (29), (30) and (27).

Claim 14 *Suppose $v_2^{M-1} \leq v_1^{M-1}$, then $v_2^m \leq v_1^m$ for all $m = 1, \ldots, M-1$.*

Proof. First, $v_2^{M-1} \leq v_1^{M-1}$ by assumption. If $\mu^{M-1} = 0$, then

$$v_2^{m'} = \frac{1+\rho}{\rho} v_1^{m'} - \frac{\lambda + \rho + \lambda\rho}{(1+\lambda)\rho} \frac{1 - F(v_1^{m'})}{f(v_1^{m'})} - \frac{\beta}{(1+\lambda)\rho} + \frac{2\eta + 2\mu^M}{(1+\lambda)\rho f(v_1^{m'})}$$

for $m' = M-2, M-1$. Hence, $v_2^{m'} - v_1^{m'}$ is strictly increasing for $m' = M-2, M-1$. Hence, $v_2^{M-2} < v_1^{M-2}$. If $\mu^{M-1} > 0$, then $v_2^{M-2} \leq v_1^{M-2}$ by (8). Repeat this argument and we have $v_2^m \leq v_1^m$ for all $m = 1, \ldots, M-1$. ■

Claim 15 *Suppose $v_2^{M-1} \leq v_1^{M-1}$, then $v_1^m = v_2^m$ for $m = 1, \ldots, M-1$.*

Proof. We prove the claim by considering two different cases. Suppose $u(0, b_1) - u(0, b_2) + a^1 v_2^0 = 0$, then $v_2^m = v_1^m$ for all $m = 1, \ldots, M-1$ by (8). Suppose $u(0, b_1) - u(0, b_2) + a^1 v_2^0 > 0$. By the same argument in the proof of Claim 9, we have $v_1^m = v_2^m$ for all $m = 1, \ldots, M-1$. ■

Claim 16 *Suppose $v_2^{M-1} \leq v_1^{M-1}$, then $M = 2$.*

Proof. Suppose $M > 2$. Then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Let $\hat{v}$ be such that

$$\sum_{m=2}^{M-1} a^m [F(v_1^m) - F(v_1^{m-1})] = a^{M-1}[F(v_1^{M-1}) - F(\hat{v})].$$

Let $a^*(v, b_1) = a^{M-1}$ if $v > \hat{v}$ and $a^*(v, b_1) = a^1$ otherwise. Let $a^*(v, b_2) = a^1$ if $v_2^0 < v < \hat{v}$, $a^*(v, b_2) = a^{M-1}$ if $\hat{v} < v < v_2^M$, and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) -$

$\int_0^v a^*(\nu, b)\mathrm{d}\nu - u(0, b)$. Then the (BC) constraint is satisfied by Assumption 2. Let $q^*(v, b_1) = q(v, b_1)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v), (IC-b), (S) and (BB), and strictly improves welfare. ■

To summarize, we have shown in all four cases that $M = 2$. However, this contradicts to the assumption that $M \geq 3$. Hence, it must be that $V(M, D) = V(M-1, D)$ for all $M \geq 3$. This completes the proof.

D.4 Properties of the Optimal Mechanism

Let $a^* = a^2$, $v_1^* = v_1^1$, $v_2^* = v_2^1$, $v_2^{**} = v_2^2$, $u_1^* = u(0, b_1)$ and $u_2^* = u(0, b_2)$ denote an solution to $\mathcal{P}'(2, 0)$. Let $\beta, \eta, \lambda, \mu^1, \mu^2, \alpha^3, \xi_1$ and ξ_2 denote the corresponding Lagrangian multipliers.

Proof of Proposition 1. Let $S' := (1-\pi)a^*\left[1 - F(v_1^*)\right] + \pi a^*[F(v_2^{**}) - F(v_2^*)] + \pi[1 - F(v_2^{**})]$. Suppose to the contradiction that $S' < S$. Let $\kappa \in (0, 1)$ be such that $\kappa + (1-\kappa)S' = S$. Consider a new mechanism (a^*, p^*, q^*). Let $a^*(v, b) = \kappa + (1-\kappa)a(v, b)$ and $p^*(v, b) = va^*(v, b) - \int_0^v a^*(\nu, b)\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b)$ for all v and b. Finally, let $q(v, b_2) = 0$ for all v, $q(v, b_1) = (1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right]/c$ if $v < v_1^*$ and $q(v, b_1) = (1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^*(v_2^* - v_1^*)\right]/c$ if $v > v_1^*$. Clearly, (a^*, p^*, q^*) strictly improves welfare upon (a, p, q). Now we show that (a^*, p^*, q^*) is also feasible. By construction, (IR) and (IC-v) hold. Note that

$$
\begin{aligned}
p^*(v, b) &= va^*(v, b) - \int_0^v a^*(\nu, b)\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)va(v, b) + \kappa v - \int_0^v \left[\kappa + (1-\kappa)a(\nu, b)\right]\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)va(v, b) - (1-\kappa)\int_0^v a(\nu, b)\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)p(v, b).
\end{aligned}
$$

Hence, $\mathbb{E}\left[p^*(v, b) - kq^*(v, b)\right] = (1-\kappa)\mathbb{E}\left[p(v, b) - kq(v, b)\right] \geq 0$. That is, (BB) holds. Since $p^*(\overline{v}, b_1) = (1-\kappa)p(\overline{v}, b_1) \leq b_1$, (BC) holds. Since $\mathbb{E}[a^*(v, b)] = \kappa + (1-\kappa)\mathbb{E}[a(v, b)] = \kappa + (1-$

$\kappa)S' = S$, (S) holds. Finally, we show that (IC-b) holds. If $v \le v_1^*$, then

$$(1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right] + \kappa(0-0) \le q(v, b_1)c.$$

If $v > v_1^*$, then

$$(1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right] + \kappa(0-0) + (\kappa + (1-\kappa)a^* - \kappa)(v_2^* - v_1^*) \le q(v, b_1)c.$$

Thus, we can conclude that (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal. Hence, (S) holds with equality. ■

Proof of Proposition 2. First-best is achieved if the allocation rule satisfies $v^* := v_1^* = v_2^* = F^{-1}(1-S)$ and $a^* = 1$, and inspection is zero. Hence, $u_1^* = u_2^* = v^* - b_1$ and (BB) holds if and only if

$$b_1 - v^* F(v^*) \ge 0. \tag{35}$$

Since $v^* = F^{-1}(1-S)$, there exists $\hat{S}(b_1) < 1$ such that (35) holds if and only if $S \ge \hat{S}(b_1)$. Clearly, $\hat{S}(b_1)$ is strictly decreasing in b_1. ■

For the rest of this section, I assume that $S < \hat{S}(b_1)$.

Proof of Proposition 3. First, we show that (BC) holds with equality. Suppose to the contradiction that (BC) holds with strict inequality. We consider four different cases: (1) $v_2^* > v_1^*$, (2) $v_2^{**} > v_2^* = v_1^*$, (3) $v_2^{**} = v_2^* = v_1^*$ and $a^* < 1$ and (4) $v_2^{**} = v_2^* = v_1^*$ and $a^* = 1$.

Suppose $v_2^* > v_1^*$.

Let $\varepsilon > 0$ and $\delta > 0$ be such that $(1-\pi)\left[F(v_1^* + \varepsilon) - F(v_1^*)\right] = \pi[F(v_2^*) - F(v_2^* - \delta)]$. For $\varepsilon > 0$ sufficiently small, we have $v_2^* - v_1^* - \varepsilon - \delta \ge 0$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$a^*(v, b_1) = \chi_{\{v \ge v_1^* + \varepsilon\}} a^*, \;\; p(v, b_1) = \chi_{\{v \ge v_1^*\}} a^*(v_1^* + \varepsilon) - u_1^*,$$
$$q(v, b_1) = \frac{1}{c}\left[\chi_{\{v \ge v_1^*\}} a^*(v_2^* - v_1^* - \varepsilon - \delta) + u_1^* - u_2^*\right],$$

$$a(v, b_2) = \chi_{\{v \geq v_2^* - \delta\}} a^* + \chi_{\{v \geq v_2^{**}\}} (1 - a^*),$$

$$p(v, b_2) = \chi_{\{v \geq v_2^* - \delta\}} a^*(v_2^* - \delta) + \chi_{\{v \geq v_2^{**}\}} (1 - a^*) v_2^{**} - u_2^*,$$

$$q(v, b_2) = 0.$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

Suppose $v_2^{**} > v_2^* = v_1^*$.

Let $\varepsilon > 0$ and $\delta > 0$ be such that $a^* \left[F(v_1^* + \varepsilon) - F(v_1^*)\right] = \pi[F(v_2^{**}) - F(v_2^{**} - \delta)]$. For $\varepsilon > 0$ sufficiently small, we have $v_2^{**} - v_1^* - \varepsilon - \delta \geq 0$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$a^*(v, b_1) = \chi_{\{v \geq v_1^* + \varepsilon\}} a^*, \; p(v, b_1) = \chi_{\{v \geq v_1^*\}} a^*(v_1^* + \varepsilon) - u_1^*,$$

$$q(v, b_1) = \frac{1}{c} \left[\chi_{\{v \geq v_1^*\}} a^*(v_2^* - v_1^*) + u_1^* - u_2^*\right],$$

$$a(v, b_2) = \chi_{\{v \geq v_2^* + \varepsilon\}} a^* + \chi_{\{v \geq v_2^{**} - \delta\}} (1 - a^*),$$

$$p(v, b_2) = \chi_{\{v \geq v_2^* + \varepsilon\}} a^*(v_2^* + \varepsilon) + \chi_{\{v \geq v_2^{**} - \delta\}} (1 - a^*)(v_2^{**} - \delta) - u_2^*,$$

$$q(v, b_2) = 0.$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

Suppose $v_2^{**} = v_2^* = v_1^*$ and $a^* < 1$.

Let $\varepsilon > 0$ and $\delta > 0$ be such that $[(1 - \pi)a^* + \pi] \left[F(v_1^* + \varepsilon) - F(v_1^*)\right] = (1 - \pi)\delta[1 - F(v_1^* + \varepsilon)]$. For $\varepsilon > 0$ sufficiently small, we have $\delta \leq 1 - a^*$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$a^*(v, b_1) = \chi_{\{v \geq v_1^* + \varepsilon\}} (a^* + \delta), \; p(v, b_1) = \chi_{\{v \geq v_1^*\}} (a^* + \delta)(v_1^* + \varepsilon) - u_1^*,$$

$$q(v, b_1) = \frac{1}{c} \left(u_1^* - u_2^*\right),$$

$$a(v, b_2) = \chi_{\{v \geq v_2^* + \varepsilon\}}, \ p(v, b_2) = \chi_{\{v \geq v_2^* + \varepsilon\}} - u_2^*,$$
$$q(v, b_2) = 0.$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

Suppose $v_2^{**} = v_2^* = v_1^*$ and $a^* = 1$.

In this case, the first-best allocation rule is achieved. Hence, it must be the case that the total inspection cost is strictly positive, i.e., $u_1^* > u_2^* \geq 0$. Let $u_2^* - u_1^* \geq \varepsilon > 0$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$a^*(v, b_1) = \chi_{\{v \geq v_1^*\}}, \ p(v, b_1) = \chi_{\{v \geq v_1^*\}} v_1^* - u_1^* + \varepsilon,$$
$$q(v, b_1) = \frac{1}{c}\left(u_1^* - u_2^* - \varepsilon\right),$$
$$a(v, b_2) = \chi_{\{v \geq v_2^*\}}, \ p(v, b_2) = \chi_{\{v \geq v_2^*\}} - u_2^*,$$
$$q(v, b_2) = 0.$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

Next, we show that (BB) holds with equality. Suppose not. Then we can increase u_1^* and u_2^* by the same amount. The resulting new mechanism is feasible and gives the same welfare. In particular, (BC) holds with strict inequality in the new mechanism. Then we can repeat the above argument and construct another feasible mechanism which strictly improves welfare upon (a, p, q), a contradiction. ■

By Lemma 6 and Corollary 2, $v_1^*, v_2^*, v_2^{**}, a^2, u_1^*, u_2^*, \beta, \eta, \lambda, \mu^1, \mu^2, \alpha^3, \xi_1$ and ξ_2 satisfy the following first-order conditions:

$$(1 - \pi)\left[(\beta - v_1^* - \lambda\varphi(v_1^*))f(v_1^*) + (1 + \lambda)\rho[1 - F(v_1^*)] + (1 + \lambda)\rho(v_2^* - v_1^*)f(v_1^*)\right]$$

$$-\eta - \mu^2 = 0, \tag{36}$$

$$\pi(\beta - v_2^* - \lambda\varphi(v_2^*))f(v_2^*) - (1-\pi)(1+\lambda)\rho[1 - F(v_1^*)] + \mu^2 = 0, \tag{37}$$

$$(1 - a^*)(\beta - v_2^{**} - \lambda\varphi(v_2^{**}))f(v_2^{**}) = 0, \tag{38}$$

$$\pi \int_{v_2^*}^{v_2^{**}} [v + \lambda\varphi(v) - \beta]\, f(v)\mathrm{d}v$$
$$+ (1-\pi)\left[\int_{v_1^*}^{\overline{v}} [v + \lambda\varphi(v) - \beta]\, f(v)\mathrm{d}v - (1+\lambda)\rho(v_2^* - v_1^*)[1 - F(v_1^*)]\right]$$
$$-\eta v_1^* + \mu^2(v_2^* - v_1^*) - \alpha^3 = 0, \tag{39}$$

$$\eta + \mu^1 + \mu^2 - (1-\pi)(\lambda + \rho + \lambda\rho) + \xi_1 = 0, \tag{40}$$

$$-\mu^1 - \mu^2 - \pi\lambda + (1-\pi)(1+\lambda)\rho + \xi_2 = 0. \tag{41}$$

Furthermore, (S) and (BB) become:

$$(1-\pi)a^*[1 - F(v_1^*)] + \pi a^*[F(v_2^{**}) - F(v_2^*)] + \pi[1 - F(v_2^{**})] = S, \tag{42}$$

$$-(1-\pi)u_1^* + (1-\pi)a^* v_1^*[1 - F(v_1^*)] - \pi u_2^* + \pi a^* v_2^*[1 - F(v_2^*)] + \pi(1-a^*)v_2^{**}[1 - F(v_2^{**})]$$
$$-(1-\pi)\rho(u_1^* - u_2^*) - (1-\pi)\rho a^*(v_2^* - v_1^*)[1 - F(v_1^*)] = 0. \tag{43}$$

Case 1: $u_1^* > u_2^* \geq 0$. In this case, $\xi_1 = \mu^1 = \mu^2 = 0$. (40) implies that $\eta = (1-\pi)(\lambda + \rho + \lambda\rho)$. (41) implies $\xi_2 = \pi\lambda - (1-\pi)(1+\lambda)\rho$. Since $\xi_2 \geq 0$, we have $\lambda[\pi - \rho(1-\pi)] \geq \rho(1-\pi)$ which implies that $\rho < \pi/(1-\pi)$.

Case 1.1: $u_2^* > 0$. In this case, $\xi_2 = 0$ and $\eta = \lambda = \rho(1-\pi)/[\pi - \rho(1-\pi)]$.

Case 1.1.1: $a^* = 1$. In this case, $v_2^{**} = v_2^*$. Finally, u_2^*, v_1^*, v_2^*, β and α^3 are solutions to (36), (37), (39), (42) and (43).

Case 1.1.2: $a^* < 1$. In this case, $\alpha^3 = 0$. Finally, u_2^*, v_1^*, v_2^*, v_2^{**}, a^* and β are solutions to (36), (37), (38), (39), (42) and (43).

Case 1.2: $u_2^* = 0$. In this case, $u_1^* = a^* v_1^* - b_1$ and $u_2^* = 0$.

Case 1.2.1: $a^* = 1$. In this case, $v_2^{**} = v_2^*$. Finally, v_1^*, v_2^*, β, λ and α^3 are solutions to (36), (37), (39), (42) and (43).

Case 1.2.2: $a^* < 1$. In this case, $\alpha^3 = 0$. Finally, v_1^*, v_2^*, v_2^{**}, a^*, β, and λ are solutions to (36), (37), (38), (39), (42) and (43).

Case 2: $u_1^* = u_2^* \geq 0$. Since (BB) holds with equality, $u_1^* = u_2^* = a^* v_1^* - b_1$. Combining (40) and (41) yields $\eta - \lambda + \xi_1 + \xi_2 = 0$.

Case 2.1: $\mu^2 > 0$. In this case, $v_1^* = v_2^*$. Since (BB) holds with equality, it must be the case that $u_1^* = u_2^* > 0$, which implies that $\xi_1 = \xi_2 = 0$. Hence, $\eta = \lambda$. (41) implies that $\mu^1 = (1-\pi)(1+\lambda)\rho - \pi\lambda - \mu^2$. Since $\mu^1, \mu^2 \geq 0$, it must be that $(1-\pi)(1+\lambda)\rho - \pi\lambda \geq \mu^2 \geq 0$. Since $S < \hat{S}(b_1)$, i.e., first-best cannot be achieved, we have $a^* < 1$. Hence, $\alpha^3 = 0$. Finally, v_2^*, v_2^{**}, a^*, β, λ and μ^2 are solutions to (36), (37), (38), (39), (42) and (43).

Case 2.2: $\mu^2 = 0$, $u_1^* = u_2^* = 0$. In this case, $v_1^* = b_1/a^*$. Furthermore, (40) and (41) implies that

$$\xi_1 = (1-\pi)(\lambda + \rho + \lambda\rho) - \eta - \mu^1,$$
$$\xi_2 = -(1-\pi)(1+\lambda)\rho + \pi\lambda + \mu^1.$$

There exists $\mu^1 \geq 0$ such that $\xi_1 \geq 0$ and $\xi_2 \geq 0$ if and only if $\min\{\lambda, (1-\pi)(\lambda+\rho+\lambda\rho)\} \geq \eta$.

Case 2.2.1: $a^* = 1$. In this case, $v_2^{**} = v_2^*$. Finally, v_2^*, β, λ, η and α^3 are solutions to (36), (37), (39), (42) and (43).

Case 2.2.2: $a^* < 1$. In this case, $\alpha^3 = 0$. Finally, v_2^*, v_2^{**}, a^*, β, λ and η are solutions to (36), (37), (38), (39), (42) and (43).

Case 2.3: $\mu^2 = 0$, $u_1^* = u_2^* > 0$. In this case, $\xi_1 = \xi_2 = 0$. Hence, $\eta = \lambda$. (41) implies that $\mu^1 = (1-\pi)(1+\lambda)\rho - \pi\lambda$. Since $\mu^1 \geq 0$, it must be that $(1-\pi)(1+\lambda)\rho - \pi\lambda \geq 0$.

Case 2.3.1: $a^* = 1$. In this case, $v_2^{**} = v_2^*$. Finally, v_1^*, v_2^*, β, λ and α^3 are solutions to (36), (37), (39), (42) and (43).

Case 2.3.2: $a^* < 1$. In this case, $\alpha^3 = 0$. Finally, v_1^*, v_2^*, v_2^{**}, a^*, β, and λ are solutions to (36), (37), (38), (39), (42) and (43).

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