Job Insecurity

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Abstract

We study a fixed-wage relationship between a firm and a worker in which neither knows how well-suited the worker is to the job. The worker decides how to allocate his time on the job, a choice that affects both learning and the firm's bottom line. The employer, seeing the worker's activity choices and outcomes, decides whether or not to continue employing the worker. Even with no private information, no hidden choices, and no cost of effort, a nontrivial agency problem arises. When the employer becomes pessimistic enough about the match quality, she cannot commit not to fire the worker. We show that, rather than aligning interests, this threat creates a perverse incentive not to attract attention: the worker strategically slows learning, harming productivity. As the firm anticipates this, job insecurity can be a self-fulfilling prophecy. We study the set of Markov perfect equilibria in our continuous-time, dynamic game with multiple forward-looking players, explicitly describing the unique Pareto optimum. We show that the firm necessarily employs ad hoc performance standards: small differences in early random outcomes can have long-lasting career consequences.

Keywords: career concerns, dynamic agency.

JEL Classification:

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1 Introduction

> *When it comes to job performance, be it in politics or in a company, perception becomes reality. This implies that you ought to manage your image and reputation as well as your actual work.*[1]

Individual reputation is a major factor in an employee's career within a firms. In most firms, employees retain some flexibility as to how they allocate their time. How they spend their time will affect how much their superiors can learn about them, and so too their standing in the firm. Potentially, a career minded employee's reputational motive and the firm's preferences may not be fully aligned. For example, as Arthur, Hall, and Lawrence (1989) say, *"If employees engage in these career strategies, what are the implications for organizations? . . . [E]mployees [may] make investments based on the perceived value of the investment. However, investments that hold perceived value for career success do not necessarily hold real economic value for a firm."*

This misalignment creates a conflict of interest in the dynamic relationship. For example, an employee might improve his reputation by delivering results on an assigned project or he might do so by actively talking to people across the firm suggesting ideas. The latter, while valuable, is potentially less valuable to the employer than direct results on an assigned project. A conflict of this nature is pervasive in many large organizations, employees can work both on deliverables and on their image in the workplace.

For example, as Barney and Lawrence report, a group of engineers in an R&D department of a firm were interviewed. It was found that the engineers got recognition for solving last minute problems under deadline pressure. Moreover, engineers feel such activities contribute more to career success than research and development innovation. While short-term problem solving is important, majority of a firm's economic success lies in innovation. Thus, from an employee's perspective, organization provides more perceived value for activities that have less real economic value.

We focus on studying what is the effect of such internal reputation of an employee on the relationship between the employee and the firm. In particular, we look at a situation where employee's effort in shaping a perception about himself is less productive from the firm's perspective. The motive behind perception management is to assure the firm that the match between the employer and the employee is in fact a good match. Since firms make profits out of synergies between the organization and the employees, firms wish to continue relationships with *good* employees

[1] J. Pfeffer, HBR, 2010 https://hbr.org/2010/10/shape-perceptions-of-your-work

and terminate the relationships with *bad* employees. In most firms, explicit contracts on effort allocation choices and reputation building are infeasible. Therefore, it is natural to study the dynamics of such relationships without commitment i.e. through a game. This lack of contractibility over time allocation across various tasks and employee's reputational motives introduce a wedge between the owner and the employee's incentives for allocation choice.

The key innovation of our paper is in introducing a separate, payoff irrelevant, channel to affect reputation for an employee. Specifically, our model consists of an employee allocating his share of time between working on a project that has tangible economic returns to the firm, and on activities related to improving his reputation within the firm. More importantly, both the performance on the project and the reputation activity directly affect the employee's reputation. The distinction, from the employee's perspective, between the two tasks is purely about the sensitivity of the evolution of beliefs based that the two choices induce. In particular, we assume that if the agent allocates one unit of time towards building his reputation then the expected change in his reputation is higher than the change were he to allocate a unit of time towards performing his actual work. This setting seems realisitc in many contexts where employees have the freedom to manage their time and their standing in the firm has consequences on their career within the firm.

We study an infinite horizon continuous time game in which, at every instant, the agent makes a decision about the share of his time that he wishes to allocate towards building his reputation and the remaining to his actual work. An agent's reputation is captured by a belief regarding the likelihood of the type of the employee. An employee more well-suited at the firm is more likely to be able to produce good performance in his actual work as well as in his attempt to manage a perception about himself. We assume that the reputation activity is payoff irrelevant for the firm. Therefore, the firm wishes the employee spent only so much time on the reputation activity so that it facilitates firm's learning about the quality of the employee. For example, if an employee devotes all his time towards creating his image, even if he is successful at that, the firm would wish to terminate the employee. Therefore, in equilibrium, the employee must, at some point of time, be spending some time on his actual work in order to stay employed. The employee's preferences are to remain employed for as long as possible. The firm, at every instant, decides whether to continue the relationship or fire the employee. The game ends if the firm fires the employee. We allow the principal to mix between continuing the interaction and firing at any instant.

We characterize the set of all *Markov Perfect Equilibria* of this game. One could potentially look at more equilibria that take into account the entire history of beliefs. However, in firms or elsewhere, such complicated strategies are probably unrealistic. Employee appraisals often are based on the employee's previous reputation and his performance after the most recent appraisal. Moreover, from the perspective of the model, one can think of MPEs as a restriction where both the players only respond to the reputation of the agent.

There always exists a trivial equilibrium in which the employee always exerts all the effort in building his reputation and the employer fires him at every belief. We show that besides this trivial equilibrium, every other equilibrium generates the same value to the principal. The equilibria are characterized by two cutoffs (z_L, z^*). The principal never fires the agent if the belief is above z^* that is given in equation 6. Between (z_L, z^*) the agent keeps the agent indifferent by keeping his flow payoffs zero. The principal also randomizes according to a particular rule to ensure that such a behaviour is indeed optimal for the agent. We show that, for a wide class of noise structures, above a belief z^* the agent puts a constant fraction of his effort across the two tasks. This is irrespective of how more or less informative the two technologies are. Statistically, the agent's incentives are to minimize learning. Our Proposition 3 shows that, in fact, there is a unique fraction of effort that minimizes learning for the principal. Given how general the noise structure is, we believe, that this is of independent interest from the perspective of learning.

2 Existing Literature

There is a huge body of literature on career concerns starting from Holmström (1999). However, much of that literature focuses on one task that affects both, the firm's profits as well as the agent's reputation. We believe that that in many organizations there are distinct activities that have no economic consequences directly except learning about the employee's match quality for the firm. Also, Dewatripont, Jewitt, and Tirole (1999), Dewatripont and Jewitt (1999) and Holmström (1999) have hidden action for the agent and career concerns for the agent affects his effort. We depart from this setting by introducing observable action to isolate the effect of pure reputation building where signalling based on actions is ruled out. Besides being realistic, this allows us to focus exclusively on the incentives arising out of learning that is driven by output and not by conjectured effort.

At a more fundamental level, however, models of career concern affect the employee's wage outside the firm as in Holmstrom(1982/99), Jovanovic (1979). This is the key difference between our paper and most literature on career concerns. We focus entirely on the dynamics *within the firm* that has no consequences *outside the firm*.

3 Model

3.1 The Environment

We study an infinite horizon, continuous time game between a principal (firm) and an agent (worker). At each instant, the principal decides whether or not to irreversibly fire the agent, while the agent decides how to allocate his efforts between two activities conditional on not being fired.

The worker-firm match quality $\theta \in \{0, 1\}$ is symmetrically unknown to the players, with prior $\mathbb{E}_0\theta = p_0 \in (0, 1)$. Of the two activities, one is productive on average if and only if the agent is well-matched with the firm (something neither principal nor agent knows). The other activity is unproductive, though its outcomes are still correlated with firm-employee match. How the agent allocates his efforts, along with the outcomes each activity generates, are public. The principal only finds it worthwhile to employ the agent if he will be productive enough; the agent wants to stay employed, and so cares about what his efforts reveal.

At each instant, the agent makes an allocation choice $k_t \in [0, 1]$, which indicates what proportion of his energy the agent allocates toward the productive activity; he allocates the remaining fraction $1 - k_t$ toward the unproductive activity. The principal observes this allocation. Over the course of $[t,\ t + \mathrm{d}t)$, the principal observes the allocation choice as well as the resulting (stochastic) output on each activity. The players update their beliefs about the match quality, taking into account the allocation choice. The principal decides whether to continue the relationship, at a flow cost of c, or to fire the agent.

We stress that, no matter how the agent allocates his energy, our two players always (on-path and off-path) have the same information about the match quality, so that there can be no private beliefs.

3.2 Outcomes

We model both the productive and unproductive activities as a diffusion technologies with drift depending on both on the match quality and the agent's allocation choices. Let X_{1t} denote the cumulative output on the productive activity and X_{2t} denote the cumulative outcome of the unproductive activity. The laws of motion of the two activities are:

$$\begin{aligned} \mathrm{d}X_{1t} &= k_t\mu\theta\,\mathrm{d}t + \sigma_1(k_t)\,\mathrm{d}B_{1t} \\ \mathrm{d}X_{2t} &= (1 - k_t)\beta\mu\theta\,\mathrm{d}t + \sigma_2(1 - k_t)\,\mathrm{d}B_{2t}, \end{aligned}$$

where $B_t := (B_{1t}, B_{2t})$ is a 2-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

The firm's accrued payoff over $[t,\ t + \mathrm{d}t)$ is $\mathrm{d}X_{1t} - c\,\mathrm{d}t$ while employing the worker, and the worker's accrued payoff is $\mathrm{d}t$. Both players get a continuation payoff of zero once the worker is fired.

We assume that $\mu > c > 0$. This tells us that the firm would strictly want to employ a well matched worker who puts enough energy toward productive activities, and that the firm would strictly want to fire an agent who is known to be poorly matched with the firm.

For most of the paper, we assume $\sigma_1 = \sigma_2 = 1$ and $\beta > 1$. This exact functional form is used for expositional convenience, but only three features—which could easily be ensured[2] by qualitative assumptions on primitives—actually matter for our results: (1) some learning always happens, (2) multitasking is bad for learning, and (3) the two activities aren't exactly equally informative about match quality. By making the unproductive activity more productive ($\beta > 1$), we leave open the possibility that the principal would sometimes like the unproductive technology to be used for informative purposes.

3.3 Evolution of Beliefs

We assume that the players (symmetrically) revise their beliefs about worker-firm match via Bayesian updating. We assume that players cannot signal what they do not know: nobody draws any direct inferences from the principal's stopping choice or the agent's activity choice. However, the agent's choice k_t does indirectly affect learning, as it affects the stochastic law of $(\mathrm{d}X_{1t}, \mathrm{d}X_{2t})$, and it affects the inferences (concerning θ) the players draw from $(\mathrm{d}X_{1t}, \mathrm{d}X_{2t})$.

Let $P_t \in (0, 1)$ be the time t public expectation of θ. i.e. P_t is the common belief at time t that the worker and firm are well matched. Sometimes, it is more convenient to work with the state variable $Z_t := \log \frac{P_t}{1-P_t}$. It is, of course, informationally equivalent: any z is associated with a probability $p(z) := \frac{1}{1+e^{-z}}$. As noted in Daley and Green (2012) the process Z_t is a linear transformation of the two output processes (X_{1t}, X_{2t}), and therefore is a Brownian diffusion process.

[2]For instance, we could assume (in addition to $\sigma_1, \sigma_2 > 0$ being measurable and of bounded variation) that:

1. σ_1, σ_2 are both bounded away from zero.
2. $\alpha(k) :=$ ... is strictly convex in k with interior minimum k^*.
3. $\frac{\sigma_2(1)}{\beta} \neq \sigma_1(1)$.

Lemma 1. *Whenever the principal has not yet stopped, the belief variable Z follows the law of motion:*

$$dZ_t = \left[\mu k_t \, dX_{1t} - \frac{\mu^2 k_t^2}{2} \, dt\right] + \left[\beta\mu(1-k_t) \, dX_{2t} - \frac{\beta^2\mu^2(1-k_t)^2}{2} \, dt\right] \tag{1}$$

Proof. Proof in appendix. □

3.4 Strategies

Let $\mathcal{H}_t$ be the σ-algebra generated by[3] $\{Z_s, X_{1s}, X_{2s} : 0 \leq s \leq t\}$. Referring to Equation 1, we can see that the law of motion for beliefs involves the agent's chose allocation, k_t. Thus, we define an **agent strategy** to be a $[0, 1]$-valued, pathwise left-continuous stochastic process $\{k_t\}_t$ that is measurable/predictable with respect to the filtration $\{\mathcal{H}_t\}_t$.

Principal, at every instant, chooses whether to continue the relationship with the agent or to irreversibly fire the agent. We model this by a left continuous stochastic process $s = \{s_t : 0 \leq t \leq \infty\}$ that is measurable/predictable with respect to $(\mathcal{H}_t)_{t\geq 0}$. The interpretation is that the principal uses a stopping rule s_t such that

- If $s_t < \infty$, then the principal's probability of firing the agent during $[t, t + \mathrm{d}t)$ is $s_t \, \mathrm{d}t + o(\mathrm{d}t)$.

- If $s_t = \infty$, then the principal fires the agent immediately.

We restrict our attention to Markov strategies where the state is captured by the belief about the match quality, Z. Therefore, our equilibrium concept is Markov Perfect Equilibria (MPE). The reasons for doing so are twofold. Maskin and Tirole (2001) present an excellent discussion on MPEs and their relevance. To quote them, *"Markov Strategies prescribe the simplest form of behaviour that is consistent with rationality"*. In our setting the natural state variable is the belief about the match quality. With beliefs and their law of motion defined, we can now describe Markov strategies with beliefs as the state variable.

Definition 1. *A **Markov strategy profile** is a pair* (s, k), *where*

1. $k : \mathbb{R} \longrightarrow [0, 1]$ *is Borel-measurable and of locally bounded variation. Let the space of admissible strategies for the agent be* $\mathcal{K}$

[3] We do not assume that Z_t is measurable with respect to $\{X_{1s}, X_{2s} : 0 \leq s \leq t\}$. This is for technical convenience, as the stochastic differential equation for the beliefs given by Equation 1 is only guaranteed to have a weak solution.

2. *$s : \mathbb{R} \longrightarrow [0, \infty]$ is Borel-measurable, and of locally bounded variation in some neighborhood of any z with $s(z) < \infty$. Let the space of admissible strategies for the principal be $\mathcal{S}$.*

The interpretation is that, before the principal has terminated the agent:

- Beliefs move according to equation 1, yielding Z_t.
- The agent currently allocates effort according to $k_t = k(Z_t)$.
- The principal uses stopping rule $s_t = s(Z_t)$.
  - If $s_t < \infty$, then the principal's probability of firing the agent during $[t, t + \mathrm{d}t)$ is $s_t\,\mathrm{d}t + o(\mathrm{d}t)$.
  - If $s_t = \infty$, then the principal fires the agent immediately.

Some quick remarks on how we've defined a strategy profile.

- By allowing $s \in (0, \infty)$, we have permitted the principal to mix.[4] That is, whenever $s > 0$, the instantaneous probability of stopping is $s_t\,\mathrm{d}t$, and correspondingly the probability of not stopping is $1 - s_t\mathrm{d}t$. This possibility is of great importance as principal's mixing can induce different behavior from the agent. As we will see, this mixing is a necessary component of Pareto optimal equilibrium.

- By forcing s to be bounded in any neighborhood of a point z with $s(z) < \infty$, we have ensured that whenever $s(Z_t) < \infty$, the probability of not being fired in the next $\epsilon > 0$ time is positive. This can be viewed as a normalization.

- Allowing the agent to choose $k(z) \in (0, 1)$ is not exactly mixing. Rather, we are allowing the agent to choose how to observably allocate his effort across two tasks.[5]

- Two agent Markov strategies $k, \tilde{k}$ are essentially the same (i.e. generate the same expected payoffs to both players for any given s and any given prior) if and only if $k(z) = \tilde{k}(z)$ for almost every $z \in \mathbb{R}$.

- Two principal Markov strategies $s, \tilde{s}$ are essentially the same if and only if $s(z) = \tilde{s}(z)$ for almost every $z \in \mathbb{R}$, *and, in addition,*

$$\{z \in \mathbb{R} : \; s(z) < \infty\} = \{z \in \mathbb{R} : \; \tilde{s}(z) < \infty\}.$$

[4] We haven't allowed the probability of stopping at some belief z to be exactly $q \in (0, 1)$. Notice: conditional on hitting z, the state hits it again in the next $\epsilon > 0$ time almost surely.

[5] We can easily allow for behavioral strategies $\kappa : \mathbb{R} \longrightarrow \Delta[0, 1]$ on the part of the agent—so that the agent tosses a private coin to choose public k_t via measure $\kappa(\cdot|Z_t)$—but little is gained in terms of insight, and the range of equilibrium value functions is the same.

3.5 Strategies

3.5.1 Principal

The principal chooses an $\mathcal{H}_t$ measurable stopping rule s_t. From time t to $t + \mathrm{d}t$ the principal fires the agent with probability $s_t\mathrm{d}t$. The principal chooses s_t to maximize,

$$\sup_s \mathbb{E}\left[\int_0^\infty e^{-rt} e^{-\int_0^t s(Z_\tau)\mathrm{d}\tau}(\mathrm{d}X_{1t}^k - c\mathrm{d}t)\right] \tag{2}$$

where k_t denotes the agent's allocation k_t. s is the principal's stopping strategy $\mathrm{d}X_{1t}^{k_t}$ is the associated controlled stochastic process X_{1t}.

3.5.2 Agent

The agent only cares about being employed for as long as he can. He earns a flow payoff of w (normalized to 1) so long as he is employed. Agent chooses a $k = (k_s)$, a $\mathcal{H}_t$ measurable function. Agent's problem is -

$$\sup_k \mathbb{E}\left[\int_0^\infty e^{-rt} e^{-\int_0^t s(Z_\tau)\mathrm{d}\tau}\mathrm{d}t)\right] \tag{3}$$

4 Equilibrium

Before formally defining an equilibrium, we will need a couple of definitions. Given any control $k \in \mathcal{K}$, and the principal's strategy $s \in \mathcal{S}$, the agent's value is defined as -

$$J(z, k, s) = \mathbb{E}_z\left[\int_0^\infty e^{-rt} e^{-\int_0^t s(Z_\tau)\mathrm{d}\tau}\ \mathrm{d}t\right] \tag{4}$$

Note that while k does not appear explicitly in the right hand side of the above equation, $\mathrm{d}Z$ is driven by the choice of k.

Similarly, the principal's value for a fixed control $s \in \mathcal{S}$ given an agent's strategy $k \in \mathcal{K}$ is defined as -

$$B(z, k, s) = \mathbb{E}\left[\int_0^\infty e^{-rt} e^{-\int_0^t s(Z_\tau)\mathrm{d}\tau}(\mathrm{d}X_{1t} - c\ \mathrm{d}t)\right] \tag{5}$$

Definition 2. *A Markov Perfect Equilibrium is a pair of strategies* (k, s) *such that -*

- (s, k) *are* $\{Z_t\}$ *measurable.*

- ***Agent Optimality*** *Given s, k solves the agent's problem (3). That is* [6],

$$J^*(Z, s) := \sup_{\hat{k} \in \mathcal{K}} J(Z, \hat{k}, s) = J(Z, k, s)$$

- ***Agent Sequential Rationality:*** *For all Z, the distance between $K_\Delta(Z, k, s)$ and $k(Z)$ tends to zero as Δ does, where, for any $\Delta > 0$,*

$$K_\Delta(Z, k, s) := \text{argmax}_{\hat{k} \in [0,1]} (1 - e^{-\Delta}) + e^{-\Delta} \mathbb{E} \left[J(Z_\Delta, \hat{k}, s) \;\middle|\; Z_0 = Z,\ k_s = \hat{k}\ \forall s \in [0, \Delta) \right].$$

If, at some time t, say $Z_t = z$ such that $s(Z) = \infty$, then,

$$k_t \in \arg\max e^{-\tau} \mathbb{E}[J(Z_{t+\tau}, k, s | Z_t = z]$$

for some $\tau > 0$. (TO-DO)

- ***Principal Optimality*** *Given k, s solves the principal's problem (2). That is,*

$$B^*(Z, k) := \sup_{\hat{s} \in \mathcal{S}} B(Z, k, \hat{s}) = B(Z, k, s)$$

- ***Belief Consistency:*** *Finally, the law of motion for the beliefs follows Equation 1 given any k.*

The requirement of sequential rationality in this framework is the only non-standard one, its form being tailored for continuous time. We want to require that the agent is behaving optimally at every history. In a discrete time analogue of our game, a currently hired agent knows his action will be able to affect the public belief until *the next period*; our definition of sequential rationality recovers this intuition, having an agent best-respond as though he will at least retain his job for *the next instant*.

4.1 Characterizing Equilibrium

Fix an equilibrium (s, k), and let $\pi, u : \mathbb{R} \longrightarrow \mathbb{R}$ be the following value functions:

- $\pi(z)$ is the expected discounted continuation profit of the principal, given public state Z and the agent's strategy k. In particular $\pi(Z) = B^*(Z, k)$ and we suppress k to save notation.

[6] Typically, there are finiteness requirements on these value functions that are trivially satisfied due to boundedness in our setting and hence are not specified.

- $u(z)$ is the expected discounted time spent unemployed, given public state z, and the principal's strategy s. Note that $J^*(Z, s) \leq 1$. Agent's payoff is his expected discounted time that he wishes to maximize. We can then turn it into a minimization problem by looking at the expected discounted unemployed time. That is, $U(z) = 1 - J^*(z, s)$. Again, we suppress s to save notation. This simple change of notation induces one important change in the agent's problem. While the agent was maximizing $J(Z, S)$ now, he minimizes $U(z)$. That is, the agent wants to minimize his expected discounted employment time.

When we want to talk about values as stochastic processes, we will use the notation $\Pi_t = \pi(Z_t)$, etc.

These value functions might be very wild. In particular, as one player's behavior might change with beliefs in an erratic way in equilibrium—and may be *required to* by his/her own best response property—neither value function is known to be the optimal value function to a well-behaved decision problem. This presents a challenge in deriving regularity properties. Even so, we are able to show the following, proved in the appendix.

Lemma 2. *π, u are continuous.*

Recall Equation 1 for the law of motion for beliefs. Let us note down the following important facts that come out of Equation 1.

$$\mathbb{E}(\mathrm{d}Z_t) = \left[\frac{\mu^2 k_t^2}{2} + \frac{\beta^2\mu^2(1-k_t)^2}{2}\right][2p(z) - 1]\ \mathrm{d}t$$
$$\mathbb{E}(\mathrm{d}Z_t)^2 = [\mu^2 k_t^2 + \beta^2\mu^2(1-k_t)^2]\ \mathrm{d}t$$

Let us define

$$\alpha(k) := \frac{\mu^2[k^2 + \beta^2(1-k)^2]}{2} \tag{6}$$

The following lemma is our first step towards characterizing the entire set of equilibria. Towards characterizing all the equilibria, one could potentially expect arbitrary agent behaviour inducing different equilibria. However, it is shown below that on an interval where the principal is assured not to fire the agent, the agent behaves in a particular manner. He always chooses a fraction $k = \frac{\beta^2}{1+\beta^2}$ irrespective of the belief about the match quality. This fraction has the property that it minimizes the speed of learning for the principal and the agent. The intuition behind this choice comes from agent's preferences. He wishes to be employed for as long as possible. Therefore, if there is an interval on which the principal is assured not to fire the agent, then ideally the agent would want to be in that interval forever. However, that is ruled out due to the technology we have which ensures that no matter what the agent does, some information about the match quality is conveyed. Therefore, the agent chooses a k such that the rate of this information is the slowest possible. Moreover, as can be seen, this k has the property that it is not a function of Z_t, the belief about the match quality.

Lemma 3. *If* $-\infty \leq z_0 < z_1 \leq \infty$, *and* (z_0, z_1) *is a maximal interval* I *such that* $\pi(z) > 0\ \forall z \in I$, *then* $k_{(z_0,z_1)} = k^* := \frac{\beta^2}{1+\beta^2}$ *almost everywhere, and* $z_1 = \infty$.

Proof. Principal best response implies $s|_{(z_0,z_1)}$ is finite-valued and is a.e. zero. Then, whenever $Z_t = z \in (z_0, z_1)$, agent best response requires

$$\begin{aligned}
U_t &= \min_{k\in[0,1]} (1 - r\,\mathrm{d}t)\mathbb{E}[U_{t+\mathrm{d}t}] + o(\mathrm{d}t) \\
\implies rU_t\,\mathrm{d}t &= \min_{k\in[0,1]} \mathbb{E}[\mathrm{d}U_t] + o(\mathrm{d}t) \\
&= \min_{k\in[0,1]} u'(Z_t)\mathbb{E}[\mathrm{d}Z_t] + \frac{1}{2}u''(Z_t)\mathbb{E}\left[(\mathrm{d}Z_t)^2\right] + o(\mathrm{d}t) \\
&= \min_{k\in[0,1]} \alpha(k)\left\{[2p(Z_t) - 1]u'(Z_t) + u''(Z_t)\right\}\,\mathrm{d}t + o(\mathrm{d}t) \\
u(z) &= \min_{k\in[0,1]} \alpha(k)\left\{[2p(Z_t) - 1]u'(z) + u''(z)\right\}.
\end{aligned}$$

Because $u > 0$ on (z_0, z_1) and $\alpha > 0$, it must be that the term on the right is also positive. Therefore, the agent's minimization problem amounts to minimizing $\alpha(k)$.

Recall,

$$\alpha(k) = \frac{1}{2}\mu^2[k^2 + \beta^2(1-k)^2] \tag{7}$$

It is easy to see that this is minimized at $k = k^*$, given by,

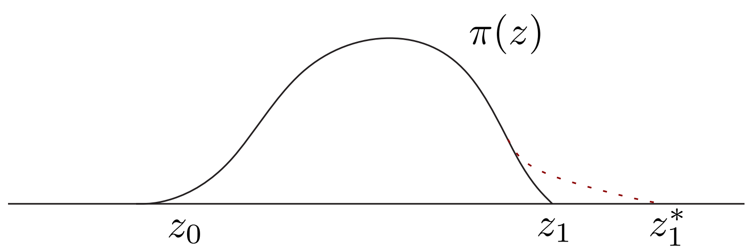
$$k^* := \frac{\beta^2}{1+\beta^2} \tag{8}$$

Now, the principal's flow payoff is $f(z) := \mu k^* p(z) - c$ for a.e. $z \in (z_0, z_1)$.
Fix $\tilde{z} \in (z_0, z_1)$. As π is zero at any finite endpoint (by maximality of the interval and Lemma 2), it must be that

$$\pi(\tilde{z}) = \int_{z_0}^{z_1} f\,\mathrm{d}\nu$$

for some positive, absolutely continuous measure ν on (z_0, z_1). As $\pi(\tilde{z}) > 0$, it must be that f is positive on a set of positive measure. Since f is increasing, it is positive in a neighborhood of z_1. Finally, applying Rüschendorf, Urusov et al. (2008)

(Prop.2.9, Pg.864), it cannot be optimal for the principal to stop at z_1 if z_1 is finite. In fact, his optimal stopping point on the right, say $z_1^* > z_1$, as shown in the figure below. That is, the principal would like to continue, despite a possibly negative payoff, to the right of z_1. The intuition behind this result is that for a diffusion process, volatility induces (roughly) a movement of $\sqrt{(\mathrm{d}t)}$ in either direction while the drift induces a movement of the order of dt. Since $\sqrt{\mathrm{d}t} >> \mathrm{d}t$, the volatility introduces an option value that forces the principal to stop a little further to the right of z_1. But then, if the principal is guaranteed to not stop between (z_1, z_1^*) then, by the earlier result, the agent uses k^* on this interval as well. But then, given that the flow payoff f is positive when the agent uses k^* at z_1, it is obviously positive for any $z > z_1$. Therefore, we have an unraveling effect as now it can no longer be optimal to stop at z_1^*. Therefore, $z_1 = \infty$.



□

Lemma 3 established the agent's behaviour on an interval where the principal is assured not to fire the agent, and also correspondingly established that should such an interval exist, then its right endpoint must be ∞. Now, before we present our main result, let us get some structure on the principal's behaviour whenever the agent does something other than k^*. The following lemma shows that if there is an interval such that the agent uses a $k \neq k^*$ a.e., moreover on that interval the agent is not being fired with probability 1, then the principal's firing behaviour is guided by the agent's value function. The reason behind this is simple. The agent can be induced to use any k other than k^* only if he is indifferent. The agent can be kept indifferent provided the principal behaves in a certain way which the following lemma establishes.

Lemma 4. *Suppose $-\infty \leq z_0 < z_1 \leq \infty$, and suppose the interval $I = (z_0, z_1)$ has $k(z) \notin \{k^*, 0, 1\}$ for almost every $z \in I$ and $u(z) < 1$ for every $z \in I$. Then $u|_I$ is twice* *differentiable, and*

$$
\begin{aligned}
s(z) &= \frac{u(z)}{1-u(z)} \textit{ for almost every } z \in I, \\
s(z) &< \infty \textit{ for every } z \in I, \\
u''(z) &= -[2p(z)-1]u'(z) \textit{ for every } z \in I.
\end{aligned}
$$

Proof. TO-DO: show that u is twice-differentiable. The argument rests on essentially the economic intuition that to induce any $k(z) \notin \{k^*, 0, 1\}$, u must be affine in p, the beliefs. If not, the agent would have a strict preference in either minimizing information transmission or maximizing it. Given that neither is the case, u must be affine in p.

Suppose $Z_t = z \in I$. As $u(Z_t) < 1$, we know $s(Z_t) < \infty$, so that the agent's problem at $Z_t \in I$ solves

$$
u(Z_t) = \min_{k \in [0,1]} s(Z_t)\,\mathrm{d}t + [1 - s(Z_t)\,\mathrm{d}t](1-\mathrm{d}t)\mathbb{E}u(Z_t + \mathrm{d}Z_t) + o(\mathrm{d}t),
$$

which in turn rearranges to

$$
\begin{aligned}
[1 + s(Z_t)]u(Z_t)\,\mathrm{d}t - s(Z_t)\,\mathrm{d}t &= \min_{k \in [0,1]} \mathbb{E}u(Z_t + \mathrm{d}Z_t) + o(\mathrm{d}t) \\
&= \min_{k \in [0,1]} \mathbb{E}[\mathrm{d}U_t] + o(\mathrm{d}t) \\
&= \min_{k \in [0,1]} u'(Z_t)\mathbb{E}[\mathrm{d}Z_t] + \frac{1}{2}u''(Z_t)\mathbb{E}[(\mathrm{d}Z_t)^2] + o(\mathrm{d}t) \\
&= \min_{k \in [0,1]} \alpha(k)\left[(2P_t - 1)u'(Z_t)\mathrm{d}t + u''(Z_t)\mathrm{d}t + o(\mathrm{d}t)\right]
\end{aligned}
$$

The above implies the agent HJB,

$$
[1 + s(z)]u(z) - s(z) = \min_{k \in [0,1]} \alpha(k)\left\{[2p(z) - 1]u'(z) + u''(z)\right\}. \tag{9}
$$

If $[2p(Z_t)-1]u'(Z_t)+u''(Z_t)$ is anything other than zero, then the agent's objective will be to minimize $\alpha(k)$ or maximize it. As it's uniquely minimized by k^* and can only be maximized by[7] $k \in \{0, 1\}$. Since the agent (a.e.) doesn't use any such k, agent best response implies $(2p - 1)u' + u''$ on I. Then the above HJB yields $(1 + s)u - s =_{\text{a.e.}} 0$ on I as well. □

[7] If $\beta > 1$, it's only maximized by 0. If $\beta < 1$, it's only maximized by 1.

Now, for the following lemma, define:

$$\delta := \frac{\sqrt{1+\frac{4}{\alpha(k^*)}}-1}{2} > 0,$$
$$p^* := \frac{c\delta}{\mu k^*(1+\delta)-c}$$
$$z^* := \log\frac{p^*}{1-p^*}.$$

Proposition 1. *There exist* $z_H \in \{z^*, \infty\}$ *and* $z_L \in [\underline{z}, z_H) \cup \{z_H\}$ *such that, given* $z \in \mathbb{R}$*:*

- *For* $z < z_L$*:*

$$\pi(z) = 0,\ u(z) = 1,\ s(z) = \infty,\ k(z) =_{a.e.} \begin{cases} 0 & : \beta > 1 \\ 1 & : \beta < 1 \end{cases}.$$

- *For* $z \in (z_L, z_H)$*:*

$$\pi(z) = 0,\ u(z) < 1,\ s(z) < \infty,\ s(z) =_{a.e.} \frac{u(z)}{1-u(z)},\ k(z) =_{a.e.} \frac{c}{\mu p(z)}.$$

- *For* $z > z_H$*:*

$$\pi(z) > 0,\ u(z) < 1,\ s(z) < \infty,\ s(z) =_{a.e.} 0,\ k(z) =_{a.e.} k^*.$$

Proof. Let $z_H := \inf\{z \in \mathbb{R} :\ \pi(z) > 0\}$.
If π isn't globally zero, then:
Lemma 3 tells us (since, by Lemma 2, and point with positive profit is surrounded by an interval with positive profit) the form of k to the right of z_H and that $\pi > 0$ to the right of it. Principal best response then tells us the form of s to the the right of z_H, which then implies $u < 1$ there. Lemma 6 then tells us $z_H = z^*$.
If π is globally zero, then $z_H = \infty$, and the third bullet is then vaccuous.

Let $z_L := \inf\left[\{z \in (-\infty, z_H) :\ u(z) < 1\} \cup \{z_H\}\right]$. If $z_L < z_H$, then:
Lemma 8 then tells us (again, mediated by Lemma 2) that $u < 1$ on (z_L, z_H) and provides the form of k on that interval. The form of $\pi = 0$ on that interval follows from the definition of z_H. The function relating s and u there comes from Lemma 4. It implies $s < \infty$ on the interval, so that $u < 1$ there.
If $z_L = z_H$, then the second bullet is vaccuous.

By definition of z_L, we must have $u = 1$ to the left of z_L, which then implies

$s = \infty$ and so $\pi = 0$ in the same interval. The form of k in this interval follows from Lemma 4. □

Call a postulated equilbrium as above a (z_L, z_H)**-cutoff equilibrium.**

Theorem 1. *For any* $z_L \in \left[\underline{z},\ z^*\right]$, *there is a unique* (z_L, z^*)*-cutoff equilibrium.*

Proof. This is merely putting together all the lemmata above. □

5 Discussion

We have modeled a setting to isolate the effects of reputational concerns to study their effect on efficiency. As Holmström (1999) noted, achieving efficiency in the presence of reputational concerns is in general hard. Meanwhile, an ability to build a reputation can create useful information for the firm. We demonstrate that there is no tradeoff: the presence of the reputation-building technology unambiguously harms the firm, yielding lower productivity *and* slower learning. Our equilibria exhibit three distinct phases—one in which the employees work less on reputation building and produce output that justifies their being in the firm, a second in which they focus more on building reputation and are expected to contribute to the firm in future, and a third in which their contribution, despite their high investment in reputation building, is productive for the firm.

Reputation can be thought of as a standing in the firm that captures the kind of roles an employee is given. Often times, in firms, employees, as their career progresses, take up roles that do not directly benefit the firm but it is expected that this experience will in the future bring tangible benefits to the firm. This corresponds to the second phase of our equilibria.

One key assumption in our model is that wages cannot be conditioned on, say, the effort on the productive technology. Given that the effort is observable it is a natural extension. While interesting from a modeling perspective, such wage structures aren't observed in firms. Our setting of a constant wage is considerably more common. We show how the tradeoff between reputation and productivity plays out when incentivizing the correct allocation through money is not feasible,

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Appendix

Proof of Lemma 2

Proof. Let us first prove that π is right-continuous (the same argument can then be repeated exactly for left-continuity) at an arbitrary point z_1.

Consider any $z_2 > z > z_1$, and consider stochastic process $\{Z_t\}_t$ induced by k with initial condition $Z_0 = z$. Define the stopping time $\tau_{z_1,z_2} := \inf\{s : z_s \notin (z_1, z_2)\}$, the first exit time for the beliefs process from the interval (z_1, z_2).

Now,

$$V(z) = \mathbb{P}(z_\tau = z_1)\mathbb{E}[e^{-r\tau}V(z_1)] + \mathbb{P}(z_\tau = z_2)\mathbb{E}[e^{-r\tau}V(z_2)] + (1 - e^{-r\tau})a$$

where $a \in [-c, \mu]$ is the average flow payoff. Now, if we show that $\lim_{z \to z_1} \mathbb{E}(e^{-r\tau}) = 1$ and $\lim_{z \to z_1} \mathbb{P}(z_\tau = z_2) = 0$ then we are done.

To this end, recall that the belief process is given by the SDE -

$$dZ_t = (2p(Z_t) - 1)\frac{\mu^2}{2}\Big[k_t^2 + \beta^2(1 - k_t)^2\Big]\,dt + \mu\Big[k_t\,dB_{1t} + \beta(1 - k_t)\,dB_{2t}\Big] \tag{10}$$

Observe the above SDE has a bounded drift and bounded diffusion that is also bounded from below by a strictly positive number.

Therefore, by lemma 5,we know that $\lim_{z \to z_1} \mathbb{E}(e^{-r\tau}) = 1$ whenever z satisfies the SDE 10.

□

Lemma 5. *Suppose X is a (strong or weak) solution to the SDE*

$$dX_t = \mu dt + \sigma(X_t)db_t \tag{11}$$

where $\mu \in \{-\gamma, \gamma\}, \gamma > 0$, $\sigma(x) \in \mathbb{R}^d$ *and* $\|\sigma(X_t)\| \in [m, M]$*Suppose* $X_0 = x$.

Define, $\tau_{ab} = \inf\{s : X_s \notin (a, b)\}$.

Then, $\lim_{x \to a} \mathbb{E}^x(e^{-r\tau_{ab}}) = 1$ *and* $\lim_{x \to a} \mathbb{P}^x(X_{\tau_{ab}} = a) = 1$.

The same statements hold for b.

Proof. Let us fix $\mu = \gamma$ wlog. Firstly,

$$X_t = \gamma t + \int_0^t \sigma(X_s)db_s$$

Therefore,

$$X_t - \gamma t = \int_0^t \sigma(X_s)db_s$$

is a $\mathcal{F}_t$ martingale. Therefore, we can perform a time change to say that

$$X_t - \gamma t = \int_0^t \sigma(X_s)db_s = W_{\alpha(t)}$$

where $W_{\alpha(s)}$ is a different brownian motion with a clock $\alpha(t) = \int_0^t \sum_i^d \sigma_i^2(X_s)ds$

It can be easily seen that $\alpha(t)$ is a strictly increasing and continuous function of t for any ω. Moreover, $\alpha(t) \in [m^2 t, M^2 t]$. Note that

$$\{X_t = a\} = \{X_t - x - \gamma t = a - x - \gamma t\} = \{\int_0^t \sigma(X_s)db_s = a - x - \gamma t\} = \{W_{\alpha(t)} = a - x - \gamma t\} \tag{12}$$

We know that $exp(\theta W_{\alpha(t)} - \frac{\theta^2}{2}\alpha(t))$ is a martingale for any $\theta > 0$.

Define,

$$\tau^{\alpha} = \inf\{\alpha : W_{\alpha} = a - x - \gamma\alpha \text{ or } W_{\alpha} = b - x - \gamma\alpha\} \tag{13}$$

Now, a straightforward application of the optional stopping theorem gives that

$$exp(\theta W_{\alpha(0)} - \frac{\theta^2}{2}0) = 1 = \mathbb{E}exp(\theta W_{\tau^{\alpha}} - \frac{\theta^2}{2}\tau^{\alpha})$$
$$= \mathbb{P}(W_{\tau^{\alpha}} = a - x - \gamma\tau^{\alpha})exp(\theta(a - x)\mathbb{E}(exp(-\tau^{\alpha}(\gamma\theta - \frac{\theta^2}{2}))$$
$$+\mathbb{P}(W_{\tau^{\alpha}} = b - x - \gamma\tau^{\alpha})exp(\theta(b - x)\mathbb{E}(exp(-\tau^{\alpha}(\gamma\theta - \frac{\theta^2}{2}))$$

It can be easily checked by Dynkin's formula that

$$\lim_{x\to a}\mathbb{P}(W_{\tau^{\alpha}} = a - x - \gamma\tau^{\alpha}) = 1.$$

Therefore, taking limits as $x \to a$, we have,

$$1 = \lim_{x\to a}\mathbb{E}(exp(-\tau^{\alpha}(\gamma\theta - \frac{\theta^2}{2})) \tag{14}$$

Choosing the value of θ appropriately, we have that $\mathbb{E}(exp(-r\tau^{\alpha})) \to 1$. Since $\alpha(t) \in [m^t, M^t]$, $\tau^{\alpha} \in [m^2\tau_{ab}, M^2\tau_{ab}]$. Hence, $\mathbb{E}(exp(-r\tau_{ab}) \to 1$.

□

Lemma 6. *If $-\infty < z_0 < \infty$ with $k|_{(z_0,\infty)} =_{a.e.} k^*$ and $z_0 = \inf\{z \in \mathbb{R} : \pi(z) > 0\}$, then $z_0 = z^*$ and*

Proof. By Lemma 3, $\pi = 0$ to the left of z_0. By Lemma 2, π is differentiable at z_0, i.e. $\pi(z_0) = 0$ (value matching), and the right-hand side derivative is $\pi'(z_0) = 0$ (smooth pasting).

Conditional on not stopping, the belief law of motion yields

$$\begin{aligned} \mathbb{E}\Big[\mathrm{d}Z_t\Big] &= \alpha(k^*)[2p(Z_t) - 1]\,\mathrm{d}t \\ \frac{1}{2}\mathbb{E}\Big[(\mathrm{d}Z_t)^2\Big] &= \alpha(k^*)\,\mathrm{d}t \end{aligned}$$

a.s. whenever $Z_t > z_0$. Moreover, the principal isn't stopping as $\Pi_t > 0$ there. Itō's Lemma then yields

$$\pi(z) = \left[p(z)\mu k^* - c\right] + \alpha(k^*)\left\{[2p(z) - 1]\pi'(z) + \pi''(z)\right\}$$

for $z \in (z_0, \infty)$.

Now, we temporarily switch back to the domain of probabilities with the function $\hat{\pi}$ given by $\hat{\pi}(p) := \pi\left(\log \frac{p}{1-p}\right)$ for $p \in (0, 1)$. The change of variables yields

$$\hat{\pi}(p) = (p\mu k^* - c) + p^2(1-p)^2\alpha(k^*)\hat{\pi}''(p)$$

for $p > p_0 := p(z_0)$.

Therefore, on this interval, $\hat{\pi}$ takes the form

$$\hat{\pi}(p) = (p\mu k^* - c) + C_1 p^{-\delta}(1-p)^{1+\delta} + C_2 p^{1+\delta}(1-p)^{-\delta}$$

for some constants C_1, C_2.

[IS THIS THE SOLUTION?]

Notice that $\delta > 0$. Applying boundedness of $\hat{\pi}$ at the right side of the interval then implies $C_2 = 0$. Therefore:

$$\begin{aligned}
\hat{\pi}(p) &= (p\mu k^* - c) + C_1 p^{-\delta}(1-p)^{1+\delta} \text{ for } p > p_0 \\
\hat{\pi}'(p) &= \mu k^* + \left(\frac{-\delta}{p} - \frac{1+\delta}{1-p}\right)[\hat{\pi}(p) - (p\mu k^* - c)] \text{ for } p > p_0.
\end{aligned}$$

Then, applying value matching and smooth pasting at p_0 yields

$$\begin{aligned}
0 &= \mu k^* + \left(\frac{-\delta}{p_0} - \frac{1+\delta}{1-p_0}\right)[0 - (p_0\mu k^* - c)] \\
\mu k^* &= (c - p_0\mu k^*)\left(\frac{\delta}{p_0} + \frac{1+\delta}{1-p_0}\right) \\
&= (c - p_0\mu k^*)\frac{\delta + p_0}{p_0(1-p_0)} \\
(p_0 - p_0^2)\mu k^* &= c(\delta + p_0) - p_0\mu k^*(\delta + p_0) \\
p_0\mu k^* &= c(\delta + p_0) - p_0\mu k^*\delta \\
p_0 &= \frac{c\delta}{\mu k^*(1+\delta) - c}.
\end{aligned}$$

So $p_0 = p^*$, and therefore $z_0 = z^*$. □

In passing, we pin down the principal's value function in any productive equilibrium.

For $z \geq z^*$,

$$\begin{aligned} \pi(z) &= p(z)\mu k^* - c + C_1 \left(\frac{p}{1-p} \right)^{-\delta} (1-p) \\ &= [p(z)\mu k^* - c] + C_1 \frac{e^{-\delta z}}{1+e^z} \\ 0 &= \pi(z^*) \\ &= [p^* \mu k^* - c] + C_1 \frac{e^{-\delta z^*}}{1+e^{z^*}} \\ C_1 &= -[p^* \mu k^* - c] \frac{1+e^{z^*}}{e^{-\delta z^*}}, \end{aligned}$$

so that

$$\pi(z) = \left\{ [p(z)\mu k^* - c] - [p^* \mu k^* - c] \frac{1+e^{z^*}}{1+e^z} e^{-\delta(z-z^*)} \right\} \mathbf{1}_{z \geq z^*} \tag{15}$$

$$= \left\{ [p(z) - p^*]\mu k^* + [p^* \mu k^* - c] \left(1 - \frac{1+e^{z^*}}{1+e^z} e^{-\delta(z-z^*)} \right) \right\} \mathbf{1}_{z \geq z^*} \tag{16}$$

Lemma 7. *On any interval I over which the agent a.e. chooses $k \neq 0, k^*, 1$, the function u takes the form*

$$u(z) = Lp(z) + M$$

for some constants L, M.

Proof. By Lemma 4, it must be that $(2p-1)u' + u'' = 0$ on the interval. That is, $(u')' = (1-2p)u'$. As $\frac{\mathrm{d}}{\mathrm{d}z}[z - 2\log(e^z+1)] = 1 - 2p(z)$, the solution to the given homogeneous first-order ODE for u' is:

$$u'(z) = \tilde{L} e^{z - 2\log(e^z+1)} = \tilde{L} \frac{e^z}{(1+e^z)^2} \text{ for some constant } \tilde{L}.$$

Integrating again yields $u(z) = \frac{\tilde{L}}{1+e^z} + \tilde{M} = \tilde{L}[1 - p(z)] + \tilde{M}$ for some constants $\tilde{L}, \tilde{M}$. Replacing the constants finishes the argument. □

Let us define

$$\underline{p} := \frac{c}{\mu} \tag{17}$$

We can define $\underline{z} := z(\underline{p}) = \log \frac{c}{\mu - c}$ accordingly.

Lemma 8. *If $-\infty \leq \underline{z} \leq z_0 < z_1 \leq \infty$, and (z_0, z_1) is a maximal interval I such that $\pi(z) = 0$ & $u(z) < 1$ $\forall z \in I$, then $\mu p(z) k(z) = c$ for almost every $z \in (z_0, z_1)$, and $z_1 = \inf\{z \in \mathbb{R} : \pi(z) > 0\}$.*

Proof. As $u|_{(z_0,z_1)} < 1$, it must be that $s|_{(z_0,z_1)}$ is finite-valued. So principal optimality at $z \in (z_0, z_1)$ tells us that

$$\begin{aligned} 0 = \pi(z) &= \left[\mu p(z)k(z) - c\right] \mathrm{d}t + (1 - r\,\mathrm{d}t)\mathbb{E}U(z + \mathrm{d}Z) + o(\mathrm{d}t) \\ &= \left[\mu p(z)k(z) - c\right] \mathrm{d}t + o(\mathrm{d}t). \end{aligned}$$

Therefore, $\mu p(z)k(z) - c = 0$ at almost every z on the interval.
The remaining point, that $z_1 = \inf\{z \in \mathbb{R} : \pi(z) > 0\}$, is immediate if $z_1 = \infty$. So suppose now that $z_1 < \infty$.
We've shown that a.e. on the interval, the agent is using $k \neq 0, k^*, 1$. Therefore, by Lemma 7, u is constant, strictly increasing, or strictly decreasing on the interval.
That $\lim_{z \searrow z_0} \mu p(z_0) - c \geq 0$ implies $z_0 > -\infty$. Then, Lemma 3 implies π cannot be positive anywhere to the left of z_0. Then, the definition of (z_0, z_1) as maximal (together with Lemma 2) implies $u(z_0) = 1$, so that u is strictly decreasing on the interval. Therefore, $u(z_1) < 1$. Maximality of (z_0, z_1) then implies that $z_1 \geq \inf\{z \in \mathbb{R} : \pi(z) > 0\}$. Lastly, notice that we can't have $z_1 > \inf\{z \in \mathbb{R} : \pi(z) > 0\}$ by Lemma 3, since $k \neq k^*$ a.e. on the interval. □

Lemma 9. *In a (z_L, ∞)-cutoff equilibrium, it must be that $z_L = \infty$.*

Proof. Assume $z_L < \infty$ for a contradiction. By Lemma 7, we have some constants L, M such that $u(z) = L(e^z + 2z - e^{-z}) + M$ for every $z > z_L$. As u is bounded, it must then be that $L = 0$, so that u is constant on (z_L, ∞). But then $u = M < 1$ above z_L, and $u = 1$ above z_L, contradicting Lemma 2. □

So an equilibrium is either a (z_L, z^*)-cutoff equilibrium for some $z_L \in [\underline{z}, z^*]$, or it has $s = \infty$ globally.

Lemma 10. *On the interval (z_H, ∞), u takes the form*

$$u(z) = D_1 \frac{e^{-\delta z}}{e^z + 1}$$

for some constant D_1.

Proof. On this interval, s is finite-valued and a.e. zero. Here, Itō's Lemma yields

$$u = \alpha(k^*)\left\{[2p - 1]u' + u''\right\}$$

This DE is the homogeneous version of the one for π in the proof of Lemma 6. It follows readily that, when $p \geq p(z_H)$:

$$u\left(\log \frac{p}{1-p}\right) = D_1 p^{-\delta}(1-p)^{1+\delta} + D_2 p^{1+\delta}(1-p)^{-\delta}$$

for some constants D_1, D_2. Boundedness of u at the right then gives us $D_2 = 0$, yielding

$$u\left(\log \frac{p}{1-p}\right) = D_1 p^{-\delta}(1-p)^{1+\delta} \tag{18}$$

That is, for $z \geq z_H$,

$$\begin{aligned} u(z) &= D_1 p(z)^{-\delta}[1-p(z)]^{1+\delta} \\ &= D_1 \left(\frac{p(z)}{1-p(z)}\right)^{-\delta} [1-p(z)] \\ &= D_1 (e^z)^{-\delta} \frac{1}{1+e^z}, \end{aligned}$$

□

Let us put the following lemmata together in order to proceed towards the characterization of a (p_L, p_H)-cutoff equilibrium.

Lemma 11. *Every (p_L, p_H)-cutoff equilibrium has $\hat{u}(p_L) = 1$, $\hat{u}$ continuous and $\hat{u}|_{[p_L, p_H]}$ affine in p.*

The following result is perhaps of general interest from the perspective of dynamic games. Consider a two-player dynamic game in which both the players' strategy space is a measurable function from a state variable, say, $X \subseteq \mathbb{R}$ to $\mathbb{R}$. Let X be a diffiusion process. Moreover, imagine that player 1 is playing a strategy that is piecewise continuous in X. Then, can we say anything about the best response of player 2? In particular, is it also piecewise continuous and how smooth is the value function of player 2?

Proposition 2. *If s is piecewise continuous, bounded over (p_0, p_1), and is of bounded total variation, then $\hat{u}$ is differentiable over (p_0, p_1).*

We want the following from a (z_L, z_H)-cutoff equilibrium.

1. Given the principal's strategy s, specified k is IC for the agent. Above z^*, we have that from Lemma 4. Between (z_L, z_H), IC holds if and only if $(2p - 1)u'(z) + u''(z) = 0$. This implies, from Lemma 7, u takes the form $u(z) = A\frac{e^z}{1+e^z} + B$.

2. Such a form of u on (z_L, z_H) also has a boundary condition that $u(z_L) = 1$. Moreover, due to Lemma 10, u to the right of z_H must be of the form given by equation 18. It must be the case that the left and the right solutions *join smoothly* at z_H.

3. On the other hand, u is described by the DE 9. The solution to that DE, taking as given the $s(z)$, must satisfy the same boundary conditions. Our task is to show that there is a unique u that satisfies the boundary conditions and it takes the required form.

To this end, we first establish that the there is a unique triple (L, M, c_1) that satisfies the boundary conditions at z_L and z_H.

Lemma 12. *If*

$$u(z) = A\frac{e^z}{1 + e^z} + B$$

on the interval (z_L, z^*) and $u(z) = u^R(z)$ to the right of z^* then there is a unique triple (A, B, c_1) that satisfies the necessary boundary conditions.

Proof. On the left, we have $u^L(z) = A\frac{e^z}{1+e^z} + B$.

The necessary boundary conditions for $u(z)$ on the interval (z_L, z_H), denoted by $u^L(z)$ are-

$$\begin{aligned} u^L(z_L) &= 1 \\ u^L(z_H) &= u^R(z_H) \\ u^{L'}(z_H) &= u^{R'}(z_H) \end{aligned}$$

We have three unknowns, A, B, c_1 and three equations. The three equations are linearly indepedent (Done but not shown here). Hence, $\exists$ a unique solution (A, B, c_1) to the above system. □

Using the solution (A^*, B^*) obtained above, define,

$$u^*(z) = A\frac{e^z}{1 + e^z} + B \tag{19}$$

Accordingly, let us define,

$$s^*(z) = \begin{cases} 0 & \text{if } z \geq z_H \\ \frac{u^*(z)}{1-u^*(z)} & \text{if } z \in [z_L, z_H) \\ \infty & \text{otherwise} \end{cases} \tag{20}$$

Due to the lemma above, we know that in equilibrium the principal must be using s^* on (z_L, z_H).

The HJB equation for the agent's value function is given in equation 9, which is reproduced below for convenience.

$$(1 + s(z))u(z) = s(z) + \min_{k} \alpha(k) \left[(2p-1)u'(z) + u''(z)\right] \tag{21}$$

However, a key issue that we have not addressed so far is that in writing the HJB equation it is assumed that the value function $u(z)$ is C^2 a.e. and C^1 everywhere. In one-dimensional diffusion problems, this enables us to use Ito's formula towards the standard verification results. The verification approach entails proving that a candidate value function satisfies the HJB equation and the relevant boundary conditions. To this end, we first prove below that given an s^*, there exists a unique solution to the HJB equation that satisfies the boundary conditions, is C^2 a.e. and C^1 everywhere.

Proposition 3. *Given an s^* as defined in Equation 20, the agent's best response $k^*(z)$ is given by*

$$k^*(z) = \begin{cases} k^* & \text{if } z \geq z_H \\ \frac{c}{\mu p(z)} & \text{if } z \in [z_L, z_H) \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

and the associated value function of the agent is given by $u^*(z)$.

Proof. The proof is an application of the standard verification arguments. We will need a minor tweak to the verification theorems as in Pham (2009) to incorporate a discontinuity in $s^*(z)$. □

Lemma 13. *Given any (z_L, z_H) with $z_L < z_H$, and $s^*(z)$, there is a unique function $u(z)$ satisfying the following equations -*

$$(1 + s^*(z))u(z) = s^*(z) + \alpha(k(z)) \left[(2p-1)u'(z) + u''(z)\right] \quad \text{if } z \in (z_L, z_H) \tag{23}$$

$$u(z) = \alpha(k^*) \left[(2p-1)u'(z) + u''(z)\right] \quad \text{if } z > z_H \tag{24}$$

$$u(z_L) = 1 \tag{25}$$

$$u(\infty) = 0 \tag{26}$$

$$\lim_{z \to z_H-} u(z) = \lim_{z \to z_H+} u(z) \tag{27}$$

$$\lim_{z \to z_H-} u'(z) = \lim_{z \to z_H+} u'(z) \tag{28}$$

Proof. Let us begin with the solution to the right of z_H and in $p - space$ instead of $z - space$.

The DE for the agent is -

$$u(z) = \frac{s^*(z)}{1 + s^*(z)} + \frac{\alpha(k)}{1 + s^*(z)} \left[(2p-1)u'(z) + u''(z)\right] \tag{29}$$

In the $p-$space this becomes,

$$u(p) = \frac{s^*(p)}{1+s^*(p)} + \frac{\alpha(k)}{1+s*(p)}p^2(1-p)^2u''(p) \tag{30}$$

With $s(p) = 0$ and $k = k^*$ the solution to this DE was obtained in equation **??** in $z - space$ and in equation 18 in $p - space$.

$$u(p) = c_1 p^{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{4}{\alpha(k^*)}+1}}(1-p)^{\frac{1}{2}\left(\sqrt{\frac{4}{\alpha(k^*)}+1}+1\right)} \tag{31}$$

Or,

$$u(p) = Cp^{\delta(k^*)}(1-p)^{1-\delta(k^*)} \tag{32}$$

Recall that δ was defined in equation **??**. Therefore,

$$u'(p) = c_1\left[\delta(k^*)\frac{1-p}{p}^{1-\delta(k^*)} - (1-\delta(k^*))\frac{p}{1-p}^{\delta(k^*)}\right] \tag{33}$$

Since $\delta(k^*) < 0, \Rightarrow 1-\delta(k^*) > 0, \Rightarrow u'(p) < 0$.

Let us denote by $u_C(p)$ a particular solution of the form 32 with $c_1 = C$.

For any two constants $C > D > 0$, we have,

1. $u_C(p_H) > u_D(p_H)$
2. And $u'_C(p_H) < u'_D(p_H)$

We know that $u_C(p_H) - u_D(p_H) > 0$. Suppose $u_C(p) - u_D(p) > 0 \forall p \in [p_1, p_H]$ Then,

$$u_C(p_1) - u_D(p_1) = \frac{\alpha(k)}{1+s(p_1)}[u''_C(p_1) - u''_D(p_1)] \tag{34}$$

$$\Rightarrow [u''_C(p_1) - u''_D(p_1)] > 0 \tag{35}$$

$$\Rightarrow [u'_C(p_1) - u'_D(p_1)] = [u_C - u_D]'(p_1) \uparrow \forall p \in (p_1, p_H] \tag{36}$$

But since $(u^C - u^D)' < 0$ at $p_H \Rightarrow (u^C - u^D)' < 0\ \forall p \in (p_1, p_H]$. Therefore, $u^C - u^D$ is strictly decreasing over $(p_1, p_H]$, and is positive at p_H. Therefore, $u^C - u^D$ is strictly positive at p_1 and, in fact, the difference is strictly increasing as we move to the left.

Hence, there is a unique constant for the solution on the right that can satisfy the left boundary condition of $u(p_L) = 1$ at p_L. Therefore, the solution constructed has to be unique.

□

Having proved that there is a unique $u(z)$ that satisfies the requisite boundary conditions, and the HJB equation, we still need to prove that the given $s(z)$ and the agent's behaviour actually induce $u(z)$.

6 Appendix - Proof of Differentiability of value functions

6.1 Preliminaries on the viscosity solution approach

A well-known object in stochastic control problems is the Hamiltonian.

$$\text{Let } dX_t = \mu(x,\alpha)\mathrm{d}t + \sigma(x,\alpha)\mathrm{d}b_t H(x,a,M) := \sup_{\alpha\in A}\left[\mu(x,\alpha)a + \frac{\sigma(x,\alpha)^2}{2}M + f(x,\alpha)\right] \tag{37}$$

where $f(x,\alpha)$ is the flow payoff. Typically, $Dom(H)$ is the domain of the Hamiltonian that ensures that the Hamiltonian is less than ∞.

In our control problem for the agent,

$$H(p,a,M) := \sup_k [1 + p^2(1-p)^2\alpha(k)M]$$

Since this is always finite, $dom(H)$ for our problem is $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$

6.2 Viscosity Solution

This subsection borrows substantively from Pham (Chapter 4). Consider the following nonlinear second order PDE,

$$F(x, w(x), w'(x), w''(x)) = 0, x \in \mathcal{O} \tag{38}$$

$\mathcal{O}$ is an open subset of $\mathbb{R}$ and F is a continuous funcion of $\mathcal{O}\times\mathbb{R}\times\mathbb{R}\times real$ taking values in $\mathbb{R}$. The function F is also assumed to satisfy the ellipticity condition: for all $x \in \mathcal{O}, r \in \mathbb{R}, p \in \mathbb{R}, M, M' \in \mathbb{R}$,

$$M \le M' \Rightarrow F(x,r,p,M) \ge F(x,r,p,M')$$

As will be clear later, this condition is always satisfied in typical stochastic control problems and our problem is no exception.

Definition 3. *Let $w : \mathcal{O} \to \mathbb{R}$ be continuous and locally bounded.*

1. *w is a viscosity subsolution of 38 on $\mathcal{O}$ if*

$$F(x, w(x), \psi'(x), \psi''(x)) \le 0$$

for all $x \in \mathcal{O}$ and for all $\psi \in C^2(\mathcal{O})$ such that x is a maximum point of $w - \psi$.

2. *w is a viscosity supersolution of* 38 *on* $\mathcal{O}$ *if*

$$F(x, w(x), \psi'(x), \psi''(x)) \geq 0$$

for all $x \in \mathcal{O}$ *and for all* $\psi \in C^2(\mathcal{O})$ *such that x is a minimum point of* $w - \psi$.

3. *We say that w is a viscosity solution of* 38 *if it is both a subsolution and supersolution.*

Coming back to a problem where one player controls stopping and the other controls an action, define

$$\begin{aligned}
J(x, \alpha, \gamma) &:= \mathbb{E} \int_0^\infty e^{-\beta t} e^{-\int_0^t \gamma_s(x_s) \mathrm{d}s} f(x_t, \alpha_t) \mathrm{d}t + \mathbb{E} \int_0^\infty e^{-\beta t} (1 - e^{-\int_0^t \gamma_s(x_s) \mathrm{d}s}) \underline{f} \mathrm{d}t \\
R(x, \alpha, \gamma) &:= \mathbb{E} \int_0^\infty e^{-\beta t} e^{-\int_0^t \gamma_s(x_s) \mathrm{d}s} g(x_t, \alpha_t) \mathrm{d}t + \mathbb{E} \int_0^\infty e^{-\beta t} (1 - e^{-\int_0^t \gamma_s(x_s) \mathrm{d}s}) \underline{g} \mathrm{d}t \\
v(x, \gamma^*) &:= \sup_\alpha J(x, \alpha, \gamma^*) \\
\pi(x, \alpha^*) &:= \sup_\gamma R(x, \alpha^*, \gamma)
\end{aligned}$$

Let us fix player 1 to be the one who controls the stopping decision, often the Principal (P), and player 2 to be the agent (A) controlling the process $\{X_t\}$. Correspondingly, $v(x, \gamma^*)$ is the value function for the agent. We will suppress the dependence of v on γ^* and just call it $v(x)$ for the purpose of this section.

Let $\bar{\gamma}(x)$ be a number large enough so that $\gamma(y) < \bar{\gamma}(x)$ for all $y \in [x - \epsilon, x + \epsilon]$ for some $\epsilon > 0$.

6.3 Supersolution

Theorem 2. *Suppose the value function v is locally bounded and the function f has a quadriatic growth. Then, for all* $\beta > 0$ *large enough, v is a viscosity supersolution to the HJB equation,*

$$(\beta + \bar{\gamma}(x)) v(x) - H(x, v'(x), v''(x)) = 0 \tag{39}$$

Remark 1. *The proof closely mimics the argument from Pham.*

Proof. Let $\psi \in C^2(\mathbb{R})$ be a test function such that,

$$0 = (v - \psi)(x) = \min_y (v - \psi)(y)$$

Pick a sequence $x_m \to x$. By continuity of v, $v(x_m) \to v(x)$. Let $\epsilon > 0$ be a number such that $\gamma(y) \leq \bar{\gamma}(x)$ for all $y \in [x-\epsilon, x+\epsilon]$. Let the sequence x_m be such that $x_m \in [x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]$ for all m. This is wlog.

Let $\rho := \frac{\epsilon}{2}$. Also, let $\bar{f}(x) := \max_y \{f(y) : y \in [x-\epsilon, x+\epsilon]\}$.

Define,

$$\eta_m := v(x_m) - \psi(x_m) \to 0$$

Let $a \in A$ and α the control identically equal to a. Therefore, it is an admissible control. Denote by $X_s^{x_m}$ the associated controlled process. Let τ_m be the stopping time given by $\tau_m = \inf\{s \geq 0 : |X_s^{x_m} - x_m| \geq \rho\}$. Let (h_m) be a strictly positive sequence such that,

$$h_m \to 0 \text{ and } \frac{\eta_m}{h_m} \to 0$$

Let the agent's outside option, $\underline{f} = 0$. By applying the dynamic programming principle for $v(x_m)$ to $\theta_m := \tau_m \wedge h_m$, we get,

$$v(x_m) \geq \mathbb{E}\left[\int_0^{\theta_m} e^{-\beta s}\left(e^{-\int_0^s \gamma(x_u)\mathrm{d}u} f(X_s^{x_m}) + (1 - e^{-\int_0^s \gamma(x_u)\mathrm{d}u})\underline{f}\right)\mathrm{d}s + e^{-\beta\theta_m} e^{-\int_0^{\theta_m} \gamma(x_u)\mathrm{d}u} v(X_{\theta_m}^{x_m})\right]$$

$$\psi(x_m) + \eta_m \geq \mathbb{E}\left[\int_0^{\theta_m} e^{-\beta s} e^{-\int_0^s \gamma(x_u)\mathrm{d}u} f(X_s^{x_m})\mathrm{d}s + e^{-\beta\theta_m} e^{-\int_0^{\theta_m} \gamma(x_u)\mathrm{d}u} \psi(X_{\theta_m}^{x_m})\right]$$

$$\geq \mathbb{E}\left[\int_0^{\theta_m} e^{-\beta s} e^{-\int_0^s \gamma(x_u)\mathrm{d}u} f(X_s^{x_m})\mathrm{d}s + e^{-\beta\theta_m} e^{-\bar{\gamma}(x)\theta_m} \psi(X_{\theta_m}^{x_m})\right]$$

Using Ito's formula, we get,

$$e^{-\beta\theta_m} e^{-\bar{\gamma}(x)\theta_m}\psi(X_{\theta_m}^{x_m}) = \psi(x_m) - (\beta + \bar{\gamma}(x))\int_0^{\theta_m} e^{-(\beta+\bar{\gamma}(x))s}\psi(X_s^{x_m})\mathrm{d}s + \int_0^{\theta_m} e^{-(\beta+\bar{\gamma}(x))s}\mathcal{L}^a\psi(X_s^{x_m})\mathrm{d}s + \int_0^{\theta_m} e^{-(\beta+\bar{\gamma}(x))s}\psi'(X_s^{x_m})\mathrm{d}B_s$$

The stochastic integral term will cancel out due to boundedness of ψ when we take expectations. Substituting and dividing by h_m,

$$\frac{\eta_m}{h_m} + \mathbb{E}\left[\frac{1}{h_m}\int_0^{\theta_m} e^{-(\beta+\bar{\gamma}(x))s}\left((\beta + \bar{\gamma}(x))\psi(X_s^{x_m}) - \mathcal{L}^a\psi(X_s^{x_m})\right) - e^{-\beta s} e^{-\int_0^s \gamma(x_u)\mathrm{d}u}\underline{f}(x)\mathrm{d}s\right] \geq 0$$

where $\underline{f}(x)$ is some sort of lim inf of $f(x)$. The idea is that since I want the inequality to be $\geq$, I just need to take the least value of f around x. In particular, if f is bounded, we can replace $\underline{f}(x)$ with the lower bound. (**LEFT FOR ELLIOT TO TAKE CARE OF THIS**).

As $m \to \infty$, $\theta_m = h_m$ which itself is tiny (**LEFT FOR ELLIOT TO TAKE CARE OF THIS**). Therefore, the integral converges to

$$h_m \left((\beta + \bar{\gamma}(x))\psi(x_m) - \mathcal{L}^a \psi(x_m) - \underline{f}(x) \right)$$

Hence, we have,

$$\frac{\eta_m}{h_m} + \left((\beta + \bar{\gamma}(x))\psi(x_m) - \mathcal{L}^a \psi(x_m) - \underline{f}(x) \right) \geq 0$$

Taking limits as $m \to \infty$, we get,

$$(\beta + \bar{\gamma}(x))\psi(x) - \mathcal{L}^a \psi(x) - \underline{f}(x) \geq 0 \tag{40}$$

By the arbitrariness of a, we conclude the result.

□

6.4 Subsolution

Here, we assume that $H(x, a, M)$ define in Equation 37 is finite for all x. In most of the control problems that economists encounter, this will be true.

Let us specialize to our problem for now and define,

$$H(x, w, M) := \sup_k [\mu(x, k)w + x^2(1 - x)^2 \alpha(k) M + 1]$$

In typical HJB equations, w above involves to the first derivative of the value function and 1 is the flow payoff. In our case, since beleifs are a martingale, $\mu(x, k) = 0$ but we have chosen to present it nonetheless for the sake of making it look similar to standard problems in control theory.

Note that $H(x, w, M)$ is continuous in each of its arguments in case of our problem.

Theorem 3. *Suppose the value function v is locally bounded and the function f has a quadriatic growth. Then, for all $\beta > 0$ large enough, v is a viscosity supersolution to the HJB equation,*

$$\beta v(x) - H(x, v'(x), v''(x)) = 0 \tag{41}$$

Proof.

Remark 2. *This is nearly a reproduction of the argument in Pham except that we start with* $e^{-\beta s} e^{-\int_0^s \gamma(x_u)du}$ *and then replace it with* $e^{-\beta s}$*. We can do this because we need to get a* $\leq$ *inequality and since* $e^{-\beta s} e^{-\int_0^s \gamma(x_u)du} \leq e^{-\beta s}$*, and the value function is positive we are entitled to make that change. Thereafter, the argument is mimicking Pham.*

Let $x \in \mathbb{R}$ be an arbitrary point and let $\psi \in C^2(\mathbb{R})$ be a test function such that,

$$0 = (v - \psi)(x) = \max_y (v - \psi)(y)$$

Suppose v is not a subsolution to Equation 3. By continuity of H, $\exists \epsilon > 0$ such that

$$\beta v(y) - H(y, v'(y), v''(y)) > \epsilon$$

$\forall y \in [x - \eta, x + \eta]$.

Let $x_m \to x$ be a sequence. As before, define,

$$\gamma_m := v(x_m) - \psi(x_m)$$

As in the argument on supersolution, let (h_m) be a strictly positive sequence such that,

$$h_m \to 0 \text{ and } \frac{\gamma_m}{h_m} \to 0$$

Let, as before, $\theta_m := \tau_m \wedge h_m, \tau_m := \inf\{s \geq 0 : |X_s^{x_m} - x_m| \geq \eta'\}$ for some $0 < \eta' < \eta$.

By the dynamic programming principle, $\exists$ a control k^m that is admissible (in our case $k(p) \in [0, 1]$) such that,

$$\begin{aligned}
\psi(x_m) + \gamma_m - \frac{\epsilon h_m}{2} \leq & \mathbb{E}\left[\int_0^{\theta_m} e^{-\beta s} e^{-\int_0^s \gamma(x_u)\mathrm{d}u} \mathrm{d}s + e^{-\beta\theta_m} e^{-\int_0^{\theta_m} \gamma(x_u)\mathrm{d}u} \psi(X_{\theta_m}^{x_m})\right] \\
\leq & \mathbb{E}\left[\int_0^{\theta_m} e^{-\beta s} \mathrm{d}s + \psi(x_m) + \int_0^{\theta_m} (-\beta e^{-\beta t} \psi(X_t^{x_m}) + e^{-\beta t} \mathcal{L}^{k^m} \psi(X_s^{x_m}))\mathrm{d}t + e^{-\beta t} \psi'(X_t^{x_m})\mathrm{d}b_t\right] \\
0 \geq & \frac{\gamma_m}{h_m} - \frac{\epsilon}{2} + \mathbb{E}\left[\frac{1}{h_m} \int_0^{\theta_m} L(X_t^{x_m}, k_s^m)\right]
\end{aligned}$$

where $L(x, k) := \beta\psi(x) - \mathcal{L}^k \psi(x) - 1$

By the usual reasoning (**Needs to be made precise**) the expectation of the stochastic integral involving db will be 0, and hence the last inequality.

Now the argument runs exactly like Pham and we get the desired contradiction.

□

6.5 Supersolution to Differentiability

Having obtained that $v(x)$ satisfies Equation 40 in the sicosity supersolution sense, we can now prove differentiability of the value function if $v'_-(x) \leq v'_+(x)$.

Suppose the inequality is strict. Then, consider the function,

$$\psi_\epsilon(y) = v(x) + p(y - x) + \frac{1}{2\epsilon}(y - x)^2$$

for some $\rho \in (v'_-(x), v'_+(x))$. From the supersolution property,

$$(\beta + \bar{\gamma}(x)) - \sup_\alpha[b(x, \alpha)p + \underline{f}(x) + \frac{1}{2\epsilon}\sigma(x, \alpha)^2] \geq 0$$

Sending ϵ to 0 gets the desired contradiction as $\sigma > 0$

6.6 Subsolution to differentiability

Since $v(x)$ satisfies Equation 3 in the viscosity sense, now we can prove differentiability of $v(x)$ if $v'_-(x) \geq v'_+(x)$.

Suppose the inequality is strict. Then, consider the test function,

$$\psi_\epsilon(y) = v(x) + \rho(y - x) - \frac{1}{2\epsilon}(y - x)^2$$

for some $\rho \in (v'_+(x), v'_-(x))$.

From the subsolution property,

$$\beta\psi(x) - \sup_k[b(x, k)\rho + 1 - \frac{1}{2\epsilon}\sigma(x, k)^2] \leq 0$$

Sending ϵ to 0 gets the desired contradiction.

6.7 Principal

For the principal, the value function on the interval (p_L, p^*) satisfies,

$$\pi(p) = \alpha(k)p^(1 - p)^2\pi''(p)$$

On $(p^*, 1)$, we have ,

$$\pi(p) = (pk^*\mu - c) + \alpha(k^*)p^(1 - p)^2\pi''(p)$$

By optimal