STATIONARY BAYESIAN-MARKOV EQUILIBRIA IN BAYESIAN STOCHASTIC GAMES WITH PERIODIC INFORMATION REVELATION

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Abstract

We show existence of a stationary Bayesian-Markov equilibrium in a Bayesian stochastic game when the type and action profiles are perfectly observed at the end of each stage. We call this condition periodic revelation. At each stage, common prior is specified by the previous type and action profiles, and thus, a stationary Bayesian-Markov strategy maps, at each subgame specified by the previous action and type profiles, the same current type to the same mixed actions. An interim expected continuation value function of a player maps the same previous type and action profiles and current type to the same real number. We illustrate an incomplete information version of an innovation race as a possible application of Bayesian stochastic games with periodic revelation.

KEYWORDS: stochastic game, periodic revelation, stationary Markov perfect equilibrium, equilibrium existence, innovation race.

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1 Introduction

Suppose the U.S. Justice department starts investigation on potential pricing collusion for four major airlines: American, Delta, United and Southwest.[1] A conventional stochastic game framework will be the first candidate to used for the investigation. As in practice, then, the heterogeneity of those airlines' cost structures, differences in their consumer bases and their idiosyncratic managing environment shocks will be assumed as public information. These characteristics, however, are mostly private information. So firms might appeal that dynamic Bayesian games or dynamic framework with asymmetric information will be a more realistic and pertinent model and rebut investigator's research. Moreover, those private information tends to persistent and intercorrelated. Hence, appropriate framework to capture these components will be Bayesian stochastic games. [2]

Bayesian stochastic games are a class of infinite horizon discounted stochastic games whose stage game is a static Bayesian game. The type profile evolves stochastically over time according to a first-order Markov process which depending on the previous type and action profiles. Given the previous type and actions profiles, the Markov process of type evolution will be prior which is not necessarily 'common' prior. If a player's type history is not revealed to the other players, then players cannot have common prior. Consequently, they cannot have consistent beliefs over the other players' type. Here we observe the facts that firms have to report their financial statements for their investors, and that upon their releases of quarter or annual reports or by analysts' research based on those reports, private information becomes public with time lag. So we consider Bayesian stochastic games with periodic revelation[3]; players observe the collections of realized types and actions at the end

[1]They were for real under investigation in the summer 2015 . The existence of common share holder problem has been covered by media, but that would be just one of factors. In order to see to root cause of pricing collusion, they need to analyze dynamic oligopoly. See Athey and Bagwell (2008).

[2]For example, Southwest flight attendants are expected to be more jokey on average than other airlines' flight attendants. Hiring humorous employees and/or educating them to do so are costly and these costs will be persistent. Then, a Markov process for characteristics evolution can be considered to incorporate persistent characteristics. Bayesian stochastic games possess desirable properties.

[3]This terminology corresponds to 'periodic full revelation', when period is 1 (aperiodic), of Fershtman and Pakes (2012). T-periodic full revelation means, every $T, 2T, \cdots$ period, players' private information are all revealed. Then, at the beginning of $nT-1$ period, each player observes (1) $(n-1)T$ period realized private and public information sets for all players, (2) their own private information realization from $(n-$

of each period. Therefore, the lagged information over the previous stage helps players have common prior for the current stage Bayesian game, and thus have consistent beliefs in each period. Then, a stage game can be indexed by the common prior (equivalently, previously realized type and action profiles) and the collection of current types.

Comparing with IO literature, our formulation departs from Independent Private Value (IPV) assumption. IPV assumptions with non-persistent types (Conditional Independence, Rust (1994)) play a major role in tractability in existing IO literature. In contrast, we allow for serially correlated and inter-correlated types.This generality would have led much complexity. But we assume periodic revelation of types, and thus, players have common prior and consistent beliefs. Then we regain tractability without IPV and CI conditions.

With this game structure, we consider stationary Bayesian-Markov strategies to take the same mixed action whenever players face the same previous type and action profiles and the same current type.[4] We show existence of a stationary Bayesian-Markov equilibrium in Bayesian stochastic games when the previous type and action profiles are perfectly observed. The type space is a complete separable metric space, and the action space is a compact metric space.

There have been literature from the same motivation to us. Especially, in applied theory side, Athey and Bagwell (2001, 2008) analyzed the collusive equilibria in dynamic Bertrand competition where costs are private information. Their action spaces are general; prices belong to the set of positive real numbers, market share proposals belong to the unit interval. But their type spaces are limited: their 2001 paper and the model 1 of 2008 paper have two cost types. Even though the model 2 of 2008 paper allowed for infinitely many cost types in the unit interval, they analyzed only the perfectly persistent (permanent) case that cost types do not change over time. Their 2001 paper analyzed the case in which types are drawn from the same fixed distribution over time (i.i.d). Their two models in 2008 paper allowed for serial correlation of types for each player, but still excluding influences of actions. [5] In their papers, they assumed action profiles are perfectly monitored, but type

$1)T+1$ period to $nT-1$ period, (3) all players (observable) action histories from $(n-1)T+1$ period to $nT-2$ period (since players have not chosen yet their actions for $nT-1$ period).

[4]Even if the previous type and action profiles are different, if they induces the same common prior, with the same current type, they are mapped into the same mixed action.

[5]Serial correlation means that player i's type in $(t+1)$ period $s_{i,t+1}$ is correlated to the previous period types, $s_{i,t}$, etc. Autoregression process is one of examples of serial correlations. In contrast, cross-

profiles are never revealed by the other players; instead, they allowed for communication among players. In repeated Bertrand competition with private cost shocks, their 2001 paper investigated the set of perfect public equilibrium (PPE) payoffs in which a public strategy maps a history of announcements, prices, and market share proposal profiles to actions,[6] while 2008 paper analyzed perfect public Bayesian equilibria (PPBE) in which a 'public strategy'[7] maps a history of the same profiles and the current cost type of a player to actions.

In more abstract setting, Cole and Kocherlakota (2001) considered a finite dynamic Bayesian game when actions and types are unobservable. In addition to hidden types and actions, they allowed for pubic signals and another hidden private shocks (noises). Both shocks are drawn stochastically based on action and type profiles of the current period and affect stage payoffs. They assumed that a player's type evolves stochastically depending on the previous type realization of the player, action of the player, public shock, and private shock of the player (IPV). They defined Markov private equilibria, focusing on a Markov private strategy which maps a history of public shocks and the current type of a player to actions (under Markov beliefs that depends on the history of public shocks and the current type of a player). They obtained a recursive algorithm that solves for the set of Markov private equilibrium payoff functions, similarly to Abreu, Pearce, and Stacchetti (1990).

In IO literature, computation and estimation techniques have been developed under the stochastic game framework with private shocks. These shocks are assumed as additively separable, action specific payoff shocks, and even unobservable to econometricians. In doing so, existence of a (Bayesian) Markov equilibrium was at interest. For stationary (Bayesian) Markov Perfect equilibria under finite-(public)-state/ finite-action Bayesian stochastic games with action-specific private payoff shocks, Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008), and Escobar (2013) [8] have shown

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correlation or inter-correlation means that given period, types of player i and other players $-i$ are correlated, so that beliefs over types of other players $-i$ changes in player i's type realization. Here the inter-correlation should satisfy the absolutely continuous information condition of Milgrom and Weber (1985) to guarantee existence of equilibrium.

[6]With helps of public randomization device, the set of PPE will be a subset of the set of PPBE.

[7]The usage of the term 'public strategy' in their 2008 paper is at issue. Cole and Kocherlakota called the same strategy as a Markov private strategy.

[8]Doraszelski and Satterthwaite (2010) formulated finite-(public)-state/continuous-action stochastic

existence of equilibrium in stationary Markov (distributional) strategies. [9] Even though they have introduced incomplete information, in their formulation, first, private shocks only serve to justify a certain selection of equilibrium action profile by the assumption that those private shocks are action specific payoff shocks. Second, private shocks does not influence evolution of the state of the world once actions are controlled ('Conditional Independence', CI). Moreover, each player's private shocks are independent from other player's ('Independent Private Value', IPV): that is, a player's beliefs about other players will be constant regardless of her type. Our structure, departing from IPV, allows for correlated type evolution. Notice that our structure allows for models with finite state/finite action spaces. If we were under that setting, researchers can avoid potential curse of dimensionality by the assumption that players observe the previous game's information perfectly, and thus have common prior for the current stage game. Bajari, Benkard, Levin (2007) also considered a Bayesian stochastic games with continuous (public) state space and continuous action space (model 2). Their private shocks are not necessarily separable in payoffs but still assumed that once public state is controlled, distributions are identical across time and independent across players (CI) and private shocks do not have influence to the public state evolution (IPV). Moreover, they assume that players' action is increasing in private shocks. They developed an algorithm to estimate stochastic games while they

games and established existence of a stationary Markov perfect equilibrium. Although the scrap values of incumbents and setup values of entrants are private information (drawn from i.i.d.), entry/exit decisions depend on the cutoff values which are public state dependent in their formulation. As a result, the probability of remaining industry as one of a player's strategies can be formulated a functions of public state. With this formulation, their equilibrium concept is a stationary Markov perfect equilibrium, rather than a stationary Bayesian Markov equilibrium. Aguirregabiria and Mira (2007) and Escobar (2013) had independent private shock distribution for each player across time and states. Pesendorfer and Schmidt-Dengler (2008) assumed that once public state is controlled, private shock distributions are identical across players and across time.

[9] See Milgrom and Weber (1982) for existence of Bayesian Nash equilibria in distributional strategies. Given a stationary Bayesian Markov strategy of player i, the corresponding distributional strategy is defined by integrating out player i's strategy with respect to her marginal type distribution. That is, for each public state, stationary Markov distributional strategy of player i gives each action the (probability) measure of types that will play the action ($P_i(a_i|x)$). Their approaches can be summarized as follows: first, compute a conditional choice probability $P_i(a_i|x)$, which can be interpreted as a mixed strategy of corresponding stochastic games without private shocks, and second, find coupled fixed points for the vector of conditional choice probability and value functions.

assumed (p.1337) existence of a stationary (Bayesian) Markov equilibrium in pure strategy. Our formulation includes the case of CI and IPV and does not need any restriction such that players' actions are increasing in private information. However, our existence theorem does not yet guarantee pure strategy equilibrium.

Recently, Fershtman and Pakes (2012) also dealt with a dynamic framework with asymmetric information. They considered a framework that can cover almost every case; previous actions are allowed for either unobservable or perfectly observed by competitors; similarly, firms' previous private characteristics realizations are allowed for either unobservable or perfectly observed by competitors. An information set (type) of player i in a stage game is defined as the finite public history plus the finite private history of player i. Their information set of finite histories are based on T-periodic full revelation. The state of the world or stage game is the collection of players' information sets. Player i's strategy maps her information set to actions[10]. Then, the evolution of the states of the world (the evolution of information sets for player i as well) follows a first-order (empirical) Markov chain. In this environment, they defined an experience based equilibrium (EBE). The experience based equilibrium concept seems inspired by the perfect Bayesian equilibrium concept: since posterior beliefs are not testable, instead, they suggested to use limiting empirical distribution from fictitious plays of stage games.[11]. Using the limiting empirical Markov chain of information sets plays an defining role in solution concept. With this limiting empirical transition matrix, players compute their expected continuation values (they used the term 'evaluations') and choose an action that maximizes their evaluation. Notice that they assumed that some limiting empirical Markov chain exists, and in their computation algorithm, the Markov transition matrix do not have to be specified. The set of recurrent states, strategies, and evaluations consist of an experience based equilibrium. Since the transition matrix itself presumes already some previously chosen actions, this solutions concept works only on equilibrium path.

In Athey and Bagwell, and Cole and Kocherlakota, and computation literature in IO-stochastic games, types are never revealed. In first two papers, players' beliefs over opponents' types are controlled by a beliefs updating operator, which is a part of the game structure. Theoretically, we can devise such an operator, but it is hard to identify the

[10]Their equilibrium concept is based on pure strategies but existence proof is based on mixed strategies.

[11]Since limiting distribution is defined on recurrent states, their EBE concept has limitations.

operator and track all updated beliefs with data empirically. In computation papers, the assumptions of CI (or i.i.d.) and IPV are introduced. CI condition in existing computation literature helps researchers not to concern about (posterior) beliefs because conditional on the public signal, private signals is neither affected by previous actions directly nor serially correlated. Under IPV, there is no cross correlated in types. Fershtman and Pakes relaxed the CI assumption and allowed for serial correlation of private signals. However, they needed to assume periodic full revelation of private histories. Here, we relax both CI, IPV assumptions while we assume periodic revelation. Moreover, we deal with infinite-state/action space Bayesian stochastic games: type and action spaces are complete separable metric spaces and compact metric spaces, respectively.

Dealing with general type/action spaces gives us flexibility. First, our framework allows for continuous variable models as well as finite variable models. Second, when data itself is given in continuous state or action, discretization of them makes computation and estimation complicated. For this reason, having continuous variable models are better for practice. And as far as we know, there has been no framework which allows for infinite state spaces in discounted stochastic games with private information (as Bajari, Benkard, Levin (2007) assumed it).

Table 1: Models with explicit public states

No serial correlation of private shocks, No inter-correlation of private shocks

authors	state space (public) (x)	action space (a)	private shocks ϵ (payoff shock)	CI	IPV	state evolution (public)
Aguirregabiria and Mira	finite	finite	$\mathbb{R}^K$	$p(x^+, \epsilon^+ \vert x, a, \epsilon)$ $= p_\epsilon(\epsilon^+) f(x^+ \vert x, a)$	$p_\epsilon(\epsilon) = \prod_i g_i(\epsilon_i)$	$f(x^+ \vert x, a)$
Pesendorfer and Schmidt-Dengler	finite	finite	$\mathbb{R}^K$	$p(x^+, \epsilon^+ \vert x, a, \epsilon)$ $= p_\epsilon(\epsilon^+ \vert x^+) f(x^+ \vert x, a)$	$p_\epsilon(\epsilon \vert x) = \prod_i g(\epsilon_i \vert x)$ (identical g)	$f(x^+ \vert x, a)$
Escobar	countable	compact linear metric space	$\mathbb{R}^K$	$p(x^+, \epsilon^+ \vert x, a, \epsilon)$ $= p_\epsilon(\epsilon^+) f(x^+ \vert x, a)$	$p_\epsilon(\epsilon) = \prod_i g_i(\epsilon_i)$	$f(x^+ \vert x, a)$
Bajari, Benkard, and Levin (1)	$S \subset \mathbb{R}^L$	finite	$\mathbb{R}^K$	$p(x^+, \epsilon^+ \vert x, a, \epsilon)$ $= p_\epsilon(\epsilon^+ \vert x^+) f(x^+ \vert x, a)$	$p_\epsilon(\epsilon \vert x) = \prod_i g_i(\epsilon_i \vert x)$	$f(x^+ \vert x, a)$
Bajari, Benkard, and Levin (2)	$S \subset \mathbb{R}^L$	$A_i \subset \mathbb{R}$	$\mathbb{R}$	$p(x^+, \epsilon^+ \vert x, a, \epsilon)$ $= p_\epsilon(\epsilon^+ \vert x^+) f(x^+ \vert x, a)$	$p_\epsilon(\epsilon \vert x) = \prod_i g_i(\epsilon_i \vert x)$	$f(x^+ \vert x, a)$

CI : conditional independence (Rust 1994); IPV : independent private value (Aguirregabiria and Mira 2007);
The reason why IO literature generally assume IPV, see Pesendorfer and Schmidt-Dengler (2008, p.906, section 3.Model–state);
For the case (2) of Bajari, Benkard, Levin, private shocks are not necessarily payoff shocks but assumed that they are increasing in private shocks following IO practice.

Table 2: Models with explicit public states (continued)

Not necessarily ergodic Markov process. Serial correlation of private shocks, but no inter-correlation of private shocks, (including) hidden action						
authors	state space (public) (x)	action space (a)	private shocks ϵ (payoff shock)	CI	IPV	state evolution (public)
Fershtman and Pakes	finite (public) (history)	finite	finite (private) (history)	(No) $p^e(s^+\|s)$ $= \prod_i p^e(J_i^+\|J_i)$	(action profile dependent) $p_\omega(\omega^+\|\omega, a)$ $= \prod_i p_\omega(\omega_i^+\|\omega_i, a)$ (identical)	Not separated
(s the state of the world; $s = (J_i)_i$; J_i individual info set; $J_i = (\xi, z_i)$; ξ (finite dim) public history; z_i (finite dim) private history; ω_i private shock; $\omega_{i,t}$ and $a_{i,t}$ may not be observed forever; they can be either $\xi_{t+\tau}$ or $z_{i,t+\tau}$.)						
Cole and Kocherlakota	finite	finite	finite	(No) $p^e(s^+\|s)$ $= \prod_i p^e(J_i^+\|J_i)$	(Yes, action-dependent) $g(\omega^+\|\omega, a, \psi)$ $= \prod_i g_i(\omega_i^+\|\omega_i, a_i, \psi)$	Not separated
	($\longleftarrow$ infinite history $\longrightarrow$)					
(s the state of the world; $s = (J_i)_i$; J_i individual info set; $J_i = (\xi, z_i, \omega_i)$; ξ public history; z_i private history without ω_i; ω_i current private state; ω_i and $a_{i,t}$ are not observed forever; $a_{i,t}$ belongs to private history $z_{i,t+\tau}$; ω_i follows an individually independent law of motion.)						

Table 3: Models with no public states (state of the world: collection of types)

authors	state space (public) (x)	action space (a)	private shocks ϵ (payoff shock)	CI	IPV
i.i.d private shocks, no inter-correlation of private shocks					
Athey and Bagwell (1) (2001)	–	(report: finite) $\mathbb{R}$	finite	$p(\epsilon^+) = p(\epsilon^+ \| \epsilon, a)$ (i.i.d.)	$p(\epsilon) = \prod_i p(\epsilon_i)$
Type realizations are not observable forever; In this formulation, they only considered the strategies which never depend on private shocks. With help of public randomization device, the set of PPE will be a subset of to set of PPBE.					
Serial correlation of private shocks, but no inter-correlation of private shocks					
Athey and Bagwell (2) (2008)	–	(report: finite) $\mathbb{R}$	finite	$p(\epsilon^+ \| \epsilon) = p(\epsilon^+ \| \epsilon, a)$	$g(\epsilon^+ \| \epsilon) = \prod_i g_i(\epsilon_i^+ \| \epsilon_i)$ (identical)
Athey and Bagwell (3) (2008)	–	(report: $[\underline{\epsilon}, \overline{\epsilon}]$) $\mathbb{R}$	$[\underline{\epsilon}, \overline{\epsilon}]$	persistent types	
Type realizations are not observable forever. In this formulation, they considered the strategies which depend on public history and the current private shocks.					

The sketch of the proof is as follows: Let v be a profile of continuation value functions and M_v be the set of measurable selectors from the interim expected payoffs correspondence, that is from the previous type-action profiles and the current state of the world to the Cartesian product of the interim expected payoffs given v, $(s^-, a^-, s) \mapsto E_v^{(s^-,a^-)}(s)$. We want to show that there is a fixed point of a correspondence $v \mapsto M_v$ on the set of continuation value function profiles V. To this end, we need to show (i) V is compact, convex, non-empty subset of locally compact Hausdorff topological vector space and (ii) the correspondence $v \mapsto M_v$ is non-empty, convex valued and closed graph. For the convex valuedness, we show that at each (s^-, a^-, s), $E_v^{(s^-,a^-)}(s) = co\ E_v^{(s^-,a^-)}(s)$, using Hildenbrand's (1974, 2015, p.64) Theorem 4 (a version of Lyapunov's theorem). After we get a fixed point of $v \mapsto M_v$, we apply Filippov's implicit function theorem to find our stationary Bayesian-Markov equilibrium strategy profile.

Technically, stationary Bayesian-Markov equilibria in Bayesian stochastic games correspond to both of stationary correlated Markov equilibria (Nowak and Raghavan (1992)) and stationary (uncorrelated) semi-Markov equilibria (Chakrabarti (1999), Barelli and Duggan (2014)). The former follows that Bayesian structure on stochastic games helps us have convexity that is obtained by public correlation device in conventional stochastic games. The latter relationship is immediate in the sense that common prior depends on the previous type and action profiles.

In the sense that we can have existence of stationary equilibria in a variant of conventional stochastic games, noisy stochastic games (Duggan (2012)) and Bayesian stochastic games share the similar spirit. But there are structural differences. In both models, the state s has a product structure ($S = Q \times R$ and $S = S_i \times S_{-i}$ for each i, respectively), and thus, transition probability can be decomposed into two parts. The continuation value functions are defined at the interim stage in each period (Q and S_i for each i, respectively). However, a Markov equilibrium in a noisy stochastic game is closer to a conventional correlated equilibrium in a conventional stochastic game than a stationary Bayesian-Markov equilibrium. First of all, if we incorporate a realization of public correlation device to a state variable of each stage game in conventional stochastic games, then it can be interpreted as a version of noisy stochastic games ('noise' can be viewed that it plays a role as a public correlation device). Second, every player observes the same noise in a noisy stochastic game, whereas each player observes distinct noise in a Bayesian stochastic game. That is, the state s is decomposed into (q, r) in noisy stochastic games each of which are publicly perfectly observed, while the state s is decomposed into (s_i, s_{-i}) in Bayesian stochastic games and only s_i is observed by player i: interpreting s_{-i} as noise for player i, she cannot observe the noise. Third, since a Markov strategy in noisy stochastic games is a measurable function of state s, realized noise r affects players' actions. In contrast, a stationary Bayesian-Markov strategy of player i in our model cannot depend on the realized 'noise' s_{-i} in her point of view, since she cannot observe it. Fourth, players have the probability measure over realization of noise $\mu_q(\cdot|q) \in \Delta(R)$ as common knowledge at the interim stage q in a noisy stochastic game, while each player i has distinct beliefs about the other players' type realization at their interim stage s_i. Finally, Duggan (2012) assumed that once conditional on the interim state q, the distribution of r in the current period is independent of the previous state and actions [12]. However, in a Bayesian stochastic game, the beliefs of player i are specified by the common prior $\tau(\cdot|s^-, a^-) \in \Delta(S)$ which determined by each state-action pair from the previous period. We may assume that beliefs of each player i are memoryless, i.e. $\eta(\cdot|s_i)$ for each previous state-action pair, to make it comparable to Duggan (2012). But then, we cannot include the case where the common prior and beliefs change following state-action pair, i.e. $\tau(\cdot|s_1^-, a_1^-) \neq \tau(\cdot|s_2^-, a_2^-)$ and beliefs $\eta(\cdot|s_1^-, a_1^-, s_i) \neq \eta(\cdot|s_2^-, a_2^-, s_i)$ for player i.

On the other hand, we relate Bayesian stochastic games to conventional stochastic games with complete information as follows: intuitively, we view stationary Bayesian-Markov equilibria as stationary correlated Markov equilibria 'with asymmetric information' in conventional discounted stochastic games. Consider a Bayesian stochastic game where the previous type-action profiles are perfectly observed. Suppose we have a stationary Bayesian-Markov equilibrium σ. Now, newly define the previous type-action pair (s^-, a^-) as the current state of the world, then it is a conventional discounted stochastic game where each player has an additional exogenous private signal at each stage, which is generated by $\tau(\cdot|s^-, a^-)$. Consider the common prior $\tau(\cdot|s^-, a^-)$ as a (partial) correlation device.[13] An omniscient mediator, knowing the private signal (her type) s_i without

[12]This feature shows that 'noise' part inherits the nature of public correlation device in extensive form games. See Forges (1986) and Nowak and Raghavan (1992).

[13]For normal form correlated equilibria, Aumann (1974) mentioned a 'partially' correlated equilibrium. For extensive form correlated equilibria, see Forges (1986). He defined the equilibrium concept where

any reports, suggests player i to take the induced mixed action $\sigma_i(s^-, a^-, s_i) \in \Delta(X_i)$ by her stationary Bayesian-Markov equilibrium strategy σ_i. Assuming that the other players follow the symmetric indications, player i has no incentive to deviate from the suggested mixed action. This is because it is a stationary Bayesian-Markov equilibrium play of her realized type s_i and this satisfies the condition for correlated equilibria. Hence, stationary Bayesian-Markov equilibria can be viewed as stationary correlated Markov equilibria 'with asymmetric information'.

Hidden stochastic games have been studied recently by Renault and Ziliotto (2014) and Yamamoto (2015). In hidden stochastic games, players only observe public noisy signals about the true state. Meanwhile, there is a Markov process for the hidden states of a stage game. Bayesian stochastic games can be considered a case of hidden stochastic games, since the state of the world, the current type profile, is unobservable at the beginning of the period. In our formulation, however, since we assume that type profile is observed at the end of the period, so it is not a permanently hidden state. Also, in our formulation, each player's type is a private noisy signal for the state of the world and there are no other public signals.

The paper proceeds as follows: we introduce the model and specify our assumptions in section 2. The existence theorem and proof follow in section 3. As an application, an incomplete information version of an innovation race is shown in section 4. Section 5 concludes.

2 The Model

2.1 The primitives

We use superscript "$+$(resp. $-$)" to denote the next period (resp. the previous period). A discounted Bayesian stochastic game is a tuple,

$$(\mathcal{I}, ((S_i, \mathfrak{S}_i), X_i, A_i, u_i, \delta_i, \mu_i, \eta_i)_{i \in \mathcal{I}}, \tau),$$

such that

players observe 'private exogenous signals at every stage'. Notice that he allowed for the signals to be serially correlated (p.1376).

- $\mathcal{I} = \{1, \ldots, n\}$ is a finite set of n players,

  and for each $i \in \mathcal{I}$,

- $(S_i, \mathfrak{S}_i)$ is a measurable space of player i's types, regardless of the calendar time, i.e. $S_i^+ = S_i$,
- $(X_i, \mathfrak{X}_i)$ is a measurable space of player i's actions,
- $A_i : S_i \rightrightarrows X_i$ is the feasible action correspondence,
- $u_i : S \times X \to \mathfrak{R}$ is the payoff function, where $S = \prod_{i \in \mathcal{I}} S_i$ and $X = \prod_{i \in \mathcal{I}} X_i$,
- $\delta_i \in [0, 1)$ is the discount factor,
- $\mu_i : S \times X \times \mathfrak{S}_i \to [0, 1]$ is a transition function[14],
- $\eta_i : S \times X \times S_i \times \mathfrak{S}_{-i} \to [0, 1]$ is a transition function,
- $\tau : S \times X \times \mathfrak{S} \to [0, 1]$ is a transition function,
- and it is an infinite horizon game.

Let $\Delta(\cdot)$ denote the set of probability measures. We assume that

($\mathcal{A}1$) For each $i \in \mathcal{I}$, S_i is a Borel subset of a complete separable metric space, and $\mathfrak{S}_i$ is its Borel σ-algebra. Endowed with the product topology, the Cartesian product S is a Borel subset of a complete separable metric space. A product of σ-algebras $\mathfrak{S} = \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_n$ is its Borel σ-algebra (M10, Billingsley (1999, p.254).

($\mathcal{A}2$) For each $i \in \mathcal{I}$, there is an atomless probability measure ν_i such that $(S_i, \mathfrak{S}_i, \nu_i)$ is a complete measure space of player i's types. ν is a product probability measure such that $\nu = \nu_1 \times \cdots \times \nu_n$. [15][16]

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[14]For the formal definition of a transition function, see Stokey and Lucas with Prescott (1989, p.212)

[15]Notice that the absolutely continuous information structure (Milgrom and Weber (1985). Also called, information diffuseness condition) is obviously satisfied: the product measure ν is absolutely continuous with respect to $\nu_1 \times \cdots \times \nu_n$.

[16]Precisely, we mean the Caratheodory extension of premeasure $\nu_1 \times \cdots \times \nu_n$ over the σ-algebra of $(\nu_1 \times \cdots \times \nu_n)^*$-measurable subsets of S. $(\cdot)^*$ means the outer measure induced by a set function.

($\mathcal{A}3$) For each $i \in \mathcal{I}$, X_i is a compact metric space. $\mathfrak{X}_i$ is its Borel-σ algebra. Endowed with the product topology, the finite Cartesian product X is a compact metric space. $\mathfrak{X}$ is its Borel-σ algebra. A typical element is denoted by a. There is a measure κ such that $(X, \mathfrak{X}, \kappa)$ is a complete measure space.

($\mathcal{A}4$) For each i, define $T_i \equiv S \times X \times S_i$. A typical element is denoted by (s^-, a^-, s_i) or t_i. Notice that T_i is a complete separable metric space. Let $\mathfrak{T}_i$ be its Borel-σ algebra. There is an atomless probability measure λ_i such that $(T_i, \mathfrak{T}_i, \lambda_i)$ is complete measure space. Endowed with the product topology, the Cartesian product T is also a complete separable metric space and a product of σ-algebras $\mathfrak{T}$ is its Borel-σ algebra. λ is a product probability measure such that $\lambda = \lambda_1 \times \cdots \lambda_n$.

($\mathcal{A}5$) For each $i \in \mathcal{I}$, A_i is nonempty, compact valued, and lower measurable.

($\mathcal{A}6$) $u_i(\cdot, \cdot)$ is bounded; there exists $C_i \in \mathfrak{R}_+$ such that for each (s, a), $|u_i(s, a)| \leq C_i$. For each $a \in X$, $u_i(\cdot, a)$ is measurable; for each $s \in S$, $u_i(s, \cdot)$, continuous.

($\mathcal{A}7$) For each (s^-, a^-), there is a common prior $\tau(\cdot|s^-, a^-) \in \Delta(S)$ about the current type distribution. For each $Z \in \mathfrak{S}$, $\tau(Z|\cdot, \cdot)$ is jointly measurable. $\tau(\cdot|s^-, a^-)$ is absolutely continuous with respect to the atomless measure ν.

($\mathcal{A}8$) For each (s^-, a^-, s_i), there are beliefs $\eta_i(\cdot|s^-, a^-, s_i) \in \Delta(S_{-i})$ about the other players' current types. This is the s_i-section of $\tau(\cdot|s^-, a^-)$. Given (s^-, a^-) and for each s_i, the mapping $(s^-, a^-, s_i) \mapsto \eta_i(\cdot|s^-, a^-, s_i)$ is a regular conditional probability on S_{-i}. For each Z_{-i}, $\eta_i(Z_{-i}|\cdot, \cdot, \cdot)$ is jointly measurable. $\eta_i(\cdot|s^-, a^-, s_i)$ is absolutely continuous with respect to the atomless product measure $\nu_1 \times \cdots \times \nu_{i-1} \times \nu_{i+1} \times \cdots \times \nu_n$.

($\mathcal{A}9$) For each (s, a), for each player i, there is an anticipation $\mu_i(\cdot|s, a) \in \Delta(S_i)$ about the future type of player i herself. This is a marginal distribution derived by $\tau(\cdot|s, a)$ (Note that $S_i^+ = S_i$). For each $Z_i \in \mathfrak{S}_i$, $\mu_i(Z_i|s, a)$ is jointly measurable in (s, a). $\mu_i(\cdot|s, a)$ is absolutely continuous with respect to the complete, atomless measure ν_i. For ν -almost all s, the mapping $a \mapsto \mu_i(\cdot|s, a)$ is norm-continuous.

($\mathcal{A}10$) For each i, τ is decomposed into μ_i and η_i:for each $(s,a) \in S \times X$, and all $Z^+ \in \mathfrak{S}$, we have the following:

$$\tau(Z^+|s,a) = \int_{s_i^+} \int_{s_{-i}^+} \mathcal{I}_Z(s_i^+, s_{-i}^+)\ \eta_i(ds_{-i}^+|s,a,s_i^+)\ \mu_i(ds_i^+|s,a),$$

2.2 Timing

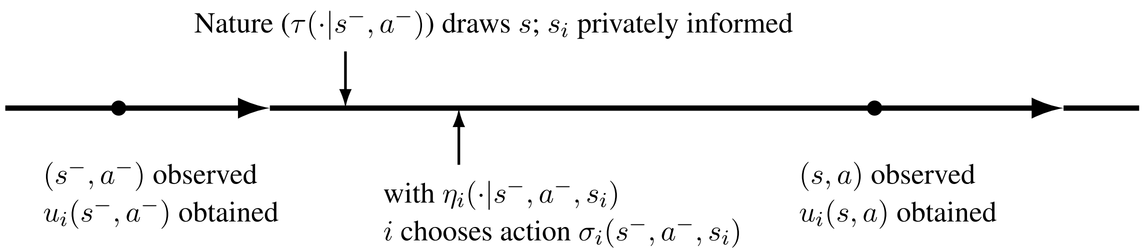

Figure 1: Timeline

1. At the end of the previous period, the type profile and action profile (s^-, a^-) are revealed. Nature moves to draw each player's type based on the Markov process $\tau(\cdot|s^-,a^-)$ for all i.

2. For each i, player i whose type is s_i chooses actions based on her beliefs $\eta_i(\cdot|s^-,a^-,s_i)$, with maximizing her discounted sum of expected payoffs.

3. At the end of the current period, the realized type profile and action profile (s,a) are revealed, and payoffs $u_i(s,a)$ are obtained. Nature moves to draw each player's next period type based on the Markov process $\tau(\cdot|s,a)$ for all i.

2.3 Stationary Bayesian-Markov equilibria

A stationary Bayesian-Markov strategy for player i is a measurable mapping $\sigma_i : S \times X \times S_i \to \Delta(X_i)$. For each (s^-,a^-,s_i), a probability measure $\sigma_i(s^-,a^-,s_i)$ assigns probability one to $A_i(s_i)$. Let Σ_i denote the set of stationary Bayesian-Markov strategies:

$$\Sigma_i = \{\sigma_i |\ \sigma_i \in \mathcal{M}(S \times X \times S_i, \Delta(X_i)),\ \sigma_i(s^-,a^-,s_i)(A_i(s_i)) = 1\}.$$

For each $s \in S$, let $\sigma(s^-, a^-, s)$ denote the product probability measure $\sigma_1(s^-, a^-, s_1)\times \cdots \times \sigma_n(s^-, a^-, s_n)$.[17] We let σ also denote a profile of mappings $(\sigma_1, \cdots, \sigma_n)$ and Σ denote the set of stationary Bayesian-Markov strategy profiles σ.

For each σ, player i's interim expected continuation value function $v_i(\cdot|\sigma) : S \times X \times S_i \to \mathfrak{R}$ is a measurable function defined as follows:

for each $(s_{\tau-1}, a_{\tau-1}, s_{i,\tau})$,[18]

$$v_i(s_{\tau-1}, a_{\tau-1}, s_{i,\tau}|\sigma) = (1-\delta_i)\sum_{t=\tau}^{\infty}\delta_i^{t-\tau}\int_{s_{-i,t}}\int_{a_t} u_i(s_t, a_t)\sigma(s_{t-1}, a_{t-1}, s_t)(da)\,\eta_i(ds_{-i,t}|s_{t-1}, a_{t-1}, s_{i,t}).$$

Or, by recursion, for each (s^-, a^-, s_i),

$$v_i(s^-, a^-, s_i|\sigma) = \int_{s_{-i}}\int_a \left[\begin{array}{l}(1-\delta_i)u_i(s,a)\\ +\delta_i\int_{s_i^+} v_i(s,a,s_i^+|\sigma)\mu_i(ds_i^+|s,a)\end{array}\right]\sigma(s^-, a^-, s)(da)\,\eta_i(ds_{-i}|s^-, a^-, s_i).$$

A profile of stationary Bayesian-Markov strategies σ is a stationary Bayesian-Markov equilibrium if for each (s^-, a^-, s), each player i's strategy σ_i maximizes i's interim expected continuation values. That is, given (s^-, a^-), for each s_i, $\sigma(s^-, a^-, s_i)$ puts probability one on the set of solution to

$$\max_{a_i\in A_i(s_i)}\int_{s_{-i}}\int_{a_{-i}} \left[\begin{array}{l}(1-\delta_i)u_i(s,a)\\ +\delta_i\int_{s_i^+} v_i(s,a,s_i^+|a_i,\sigma_{-i})\mu_i(ds_i^+|s,a)\end{array}\right]\sigma_{-i}(s^-, a^-, s_{-i})(da_{-i})\,\eta_i(ds_{-i}|s^-, a^-, s_i). \tag{1}$$

By one-shot deviation principle, every stationary Bayesian-Markov equilibrium is subgame perfect.

[17] Precisely, we mean the Caratheodory extension of premeasure $\sigma_1(s^-, a^-, s_1)\times\cdots\times\sigma_n(s^-, a^-, s_n)$ over the σ-algebra of $(\sigma_1(s^-, a^-, s_1)\times\cdots\times\sigma_n(s^-, a^-, s_n))^*$-measurable subsets of $A_1(s_1)\times\cdots\times A_n(s_n)$. $(\cdot)^*$ means outer measure induced by a set function.

[18] Fix (s^-, a^-). Then common prior $\tau(\cdot|s^-, a^-)$ is specified and so are each player's beliefs $\eta_i(\cdot|s^-, a^-, s_i)$. The absolutely continuous information structure that $\eta_i \ll \nu_1\times\cdots\times\nu_{i-1}\times\nu_{i+1}\times\cdots\times\nu_n$ allows us to express player i's interim expected payoffs in this manner. See Milgrom and Weber (1985) or Balder (1988) Theorem 2.5.

3 Existence Theorem

(Theorem). *For every Bayesian stochastic game with periodic revelation, there exists a stationary Bayesian-Markov equilibrium.*

Fix $i \in \mathcal{I}$. We choose $L^\infty(T_i, \lambda_i)$ and weak-* topology for the space of continuation value functions of player i, similarly to the standard approach in general stochastic games. $L^\infty(T_i, \lambda_i)$ is the collection of λ_i-equivalence classes of λ_i-essentially bounded, measurable extended real-valued functions from $S \times X \times S_i$ to $\mathfrak{R}$. $L^\infty(T_i, \lambda_i)$ is equipped with the usual norm $\| \cdot \|_{L^\infty}$ which gives the smallest essential upper bound; that is, $\| f \|_{L^\infty} = \inf M$ if $|f(s^-, a^-, s_i)| \leq M$ for λ_i-almost all (s^-, a^-, s_i). $L^1(T_i, \lambda_i)$ is the collection of λ_i-equivalence classes of integrable functions from $T_i = S \times X \times S_i$ to $\mathfrak{R}$. $L^1(T_i, \lambda_i)$ is equipped with $\| \cdot \|_{L^1}$ such that $\| g \|_{L^1} = \int_{T_i} |g| d\lambda_i = \int_{S_i} \int_X \int_S |g| d\nu d\kappa d\nu_i$ (Tonelli's theorem, Royden-Fitzpatrick (2010), hereafter RF, p.420) . By the Riesz representation theorem (RF, p.400), $L^\infty(T_i, \lambda_i)$ is the dual space of $L^1(T_i, \lambda_i)$, and it is endowed with weak-* topology. By Proposition 14.21 of RF p.287, $L^\infty(T_i, \lambda_i)$ is a locally convex Hausdorff topological vector space. Let $L^\infty(T, \lambda)$ denote the Cartesian product $L^\infty(T_1, \lambda_1) \times \cdots \times L^\infty(T_n, \lambda_n)$. Then endowed with product topology, $L^\infty(T, \lambda)$ is a locally convex Hausdorff topological vector space.

Let V_i consist of functions $v_i \in L^\infty(T_i, \lambda_i)$ with $\| v_i \|_{L^\infty} \leq C_i$; that is, $|v_i(s^-, a^-, s_i)| \leq C_i$ for λ_i-almost all (s^-, a^-, s_i). The constant $C_i \in \mathfrak{R}_+$ is of which $|u_i(s, a)| \leq C_i$ for all (s, a). Clearly, V_i is non-empty and convex. Then the Cartesian product $V = V_1 \times \cdots \times V_n$ is also non-empty and convex. By Alaoglu's theorem (RF, p.299), V_i is compact. The finite product V is also compact (Theorem 26.7, Munkres (2000), p.167).

Now we show V is metrizable in the weak-* topology. Since T_i is separable metric space, its Borel σ-algebra is countably generated. By Theorem 19.5 of RF p.398 and the fact that the set of rational numbers is countable and dense in $\mathfrak{R}$, the subspace of rational-valued simple functions with finite support is countable and dense in $L^1(T_i, \lambda_i)$. Hence $L^1(T_i, \lambda_i)$ is separable. By Corollary 15.11 of RF p.306, V_i is metrizable in the weak-* topology. The Cartesian product V is metrizable in the product weak-* topology, so it is compact if and only if sequentially compact. (Theorem 28.2, Munkres (2000), p.179)

In order to see if the correspondence $v \mapsto M_v$ is non-empty, closed graph, and convex valued, we consider the following induced game. Eventually we want to show that the set of interim expected payoffs profiles $E_v^{(s^-,a^-)}(s)$ from the induced game is in fact equivalent to $co\ E_v^{(s^-,a^-)}(s)$. Let $\Gamma_v^{(s^-,a^-)}(s)$ denote an induced game of a Bayesian stochastic game. Given a profile of continuation value functions v and the realized state-action profiles (s^-, a^-) in the previous period, the induced game is an interim stage of static Bayesian game,

$$\Gamma_v^{(s^-,a^-)}(s) = (\mathcal{I}, ((S_i, \mathfrak{S}_i), X_i, A_i(s_i), \widehat{U}_i^{(s^-,a^-)}(\cdot|v), \mu_i(\cdot|\cdot,\cdot), \eta_i(\cdot|s^-,a^-,s_i))_{i\in\mathcal{I}}),$$

where $\widehat{U}_i^{(s^-,a^-)}(\cdot|v) : S_i \times \Sigma \to \mathfrak{R}$ is defined as follows: for each s_i, each σ,

$$\widehat{U}_i^{(s^-,a^-)}(s_i, \sigma|v) = \int_{s_{-i}} \int_a \left[\begin{array}{l} (1-\delta_i)u_i(s,a) \\ +\delta_i \int_{s_i^+} v_i(s,a,s_i^+)\mu_i(ds_i^+|s,a) \end{array} \right] \sigma(s^-,a^-,s)(da)\, \eta_i(ds_{-i}|s^-,a^-,s_i).$$

In the interim stage of a general one-shot Bayesian game, suppose players use behavioral strategies. Knowing her realized type (s_i), player i exerts a mixed action ($\beta_i(s_i) \in \Delta(X_i)$), induced by her behavioral strategy (a measurable mapping $\beta_i : S_i \to \Delta(X_i)$). In the induced game of our Bayesian stochastic game, players behave in the similar way when they use stationary Bayesian-Markov strategies. The mixed actions induced by stationary Bayesian-Markov strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ of all players determine a product probability measure $\sigma(s^-,a^-,s) = \sigma_1(s^-,a^-,s_1) \times \cdots \times \sigma_n(s^-,a^-,s_n)$. The difference is that beliefs are given by the consequence of the previous stage game, and thus, mixed actions depend on the previously realized type and action profiles as well as the current own type of player i herself.[19]

The space $\Delta(X_1) \times \cdots \times \Delta(X_n)$ and $\Delta(A_1(s_1)) \times \cdots \times \Delta(A_n(s_n))$ are endowed with the product weak topology. By Theorem 2.8, Billingsley (1999), p.23, $\alpha^m \to \alpha$ in $\Delta(X_1) \times \cdots \times \Delta(X_n)$ if and only if $\alpha_i^m \to \alpha_i$ in $\Delta(X_i)$ for each i.

Lemma 1. *Given (s^-, a^-), for each v and each σ, $\widehat{U}_i^{(s^-,a^-)}(\cdot, \sigma|v)$ is measurable in s_i. Similarly, $\widehat{U}_i^{(\cdot,\cdot)}(\cdot, \sigma|v)$ is also measurable in (s^-, a^-, s_i). Given (s^-, a^-), for each s_i, $\widehat{U}_i^{(s^-,a^-)}(s_i, \cdot|\cdot)$ is jointly continuous in (σ, v).*

[19]We like to think that for the terms, "Bayesian" is about beliefs and "Markov" is about current state (type) regardless of the calendar time. If the common prior and the beliefs are given by the fixed probability measure (i.i.d.), then $T_i = S_i$ and $\eta(Z_{-i}|s^-,a^-,s_i) = \eta(Z_{-i}|s_i)$, and thus, our stationary Bayesian-Markov strategy is simply $\sigma_i : S_i \to \Delta(X_i)$.

Proof. Define

$$U_i^{(s^-,a^-)}(s,\sigma|v) = \int_a \Big[(1-\delta_i)u_i(s,a) + \delta_i \int_{s_i^+} v_i(s,a,s_i^+)\mu_i(ds_i^+|s,a)\Big]\sigma(s^-,a^-,s)(da)$$

and $\widehat{U}_i^{(s^-,a^-)}(s_i,\sigma|v) = \int_{s_{-i}} U_i^{(s^-,a^-)}(s,\sigma|v)\ \eta_i(ds_{-i}|s^-,a^-,s_i)$. For each $Z_i \in \mathfrak{S}_i$, $\mu_i(Z_i|s,a)$ is measurable in s_i. $v_i(s,a,s_i^+)$ is bounded, then Theorem 19.7 of Aliprantis and Border (2006), hereafter AB, p.627 implies $\int_{s_i^+} v_i(s,a,s_i^+)\mu_i(ds_i^+|s,a)$ is measurable in s_i. Recall that given (s^-,a^-), for each $Z_{-i} \in \mathfrak{S}_{-i}$, $\eta_i(Z_{-i}|s^-,a^-,\cdot)$ is measurable. Since $U_i^{(s^-,a^-)}(s,\sigma|v)$ is bounded, $\widehat{U}_i^{(s^-,a^-)}(\cdot,\sigma|v)$ is measurable. Similarly, since for each $Z_{-i} \in \mathfrak{S}_{-i}$, $\eta_i(Z_{-i}|\cdot,\cdot,\cdot)$ is measurable, $\widehat{U}_i^{(\cdot,\cdot)}(\cdot,\sigma|v)$ is also measurable.

Fix (s^-,a^-). Consider a sequence $\{(a^m,v^m)\} \to (a,v)$. For each i, we have

$$\begin{aligned}
&\left|\widehat{U}_i^{(s^-,a^-)}(s_i,a^m|v^m) - \widehat{U}_i^{(s^-,a^-)}(s_i,a|v)\right| \\
&= \left|\int_{s_{-i}} \begin{bmatrix} (1-\delta_i)u_i(s,a^m) - (1-\delta_i)u_i(s,a) \\ +\delta_i \int_{s_i^+} v_i^m(s,a^m,s_i^+)\mu_i(ds_i^+|s,a^m) \\ -\delta_i \int_{s_i^+} v_i(s,a,s_i^+)\mu_i(ds_i^+|s,a) \end{bmatrix} \eta_i(ds_{-i}|s^-,a^-,s_i)\right|.
\end{aligned}$$

Since $u_i(s,\cdot)$ is continuous, $|(1-\delta_i)u_i(s,a^m) - (1-\delta_i)u_i(s,a)| \to 0$, and

$$\begin{aligned}
&\left|\int_{s_i^+} v_i^m(s,a^m,s_i^+)\mu_i(ds_i^+|s,a^m) - \int_{s_i^+} v_i(s,a,s_i^+)\mu_i(ds_i^+|s,a)\right| \\
&\le \left|\int_{s_i^+} v_i^m(s,a^m,s_i^+)\mu_i(ds_i^+|s,a^m) - \int_{s_i^+} v_i^m(s,a^m,s_i^+)\mu_i(ds_i^+|s,a)\right| \\
&\quad + \left|\int_{s_i^+} v_i^m(s,a^m,s_i^+)\mu_i(ds_i^+|s,a) - \int_{s_i^+} v_i(s,a,s_i^+)\mu_i(ds_i^+|s,a)\right| \\
&\le C_i \left\|\mu_i(\cdot|s,a^m) - \mu_i(\cdot|s,a)\right\| \\
&\quad + \left|\int_{s_i^+} \left[v_i^m(s,a^m,s_i^+) - v_i(s,a,s_i^+)\right] \mu_i(ds_i^+|s,a)\right|
\end{aligned}$$

The first inequality is by the triangle inequality. The second inequality is by the fact that $v_i^{(s^-,a^-)}$ is essentially bounded by C_i and that $|\int f d\mu_i| \le \int |f| d\mu_i$.

By Radon-Nikodym theorem (RF, p.382), we can write $\mu(E^+|s,a) = \int_{E^+} f\, \nu(ds_i^+)$ for all $E^+ \in \mathfrak{S}^+$. Then $\int_{s_i^+} v_i^m(s, a^m, s_i^+)\, f\, \nu_i(ds_i) = \int_{S_i^+} \int_X \int_S v_i^m\, \delta_s \delta_{a^m} f\, \nu(ds)\kappa(da)\nu_i(ds_i)$, where δ_{x_0} means a Dirac delta density function concentrated at x_0[20]. Let $g^m = \delta_{(s,a^m)} f$ and $g = \delta_{(s,a)} f$. Clearly, $\|g^m - g\|_{L^1} \to 0$ as $a^m \to a$ in X. As $v^m \to v$ in weak-* topology, by Proposition 3.13 of Brezis (2010, p.63), $\int_{T_i} v_i^m g^m \lambda_i(dt_i) \to \int_{T_i} v_i g \lambda_i(dt_i)$ [21]. This gives us $\left| \int_{s_i^+} [v_i^m(s, a^m, s_i^+) - v_i(s, a, s_i^+)]\, \mu_i(ds_i^+|s,a) \right| \to 0$. Then, combined with norm continuity of $\mu_i(\cdot|s,a)$ in a, we have the RHS of the last inequality converges to 0. Now continuity of $\widehat{U}_i^{(s^-,a^-)}(s_i, \sigma|v)$ follows from that given s, the family of real valued functions $\mathcal{U} \equiv \left\{ \widehat{U}_i^{(s^-,a^-)}(s_i, \cdot|v^m) \right\}_{m \in N}$ is equicontinuous at each a and the result of Rao (1962)[22] under the absolutely continuous information structure. $\square$

Fix (s^-, a^-) and v. For each s, let $B_v^{(s^-,a^-)}(s)$ be the set of mixed action profiles induced by Bayesian Nash equilibria of $\Gamma_v^{(s^-,a^-)}$ (s). Define

$$\widehat{\xi}^{(s^-,a^-)}(s, \sigma; v) = \sum_{i=1}^{n} \left[\widehat{U}_i^{(s^-,a^-)}(s_i, \sigma; v) - \max_{a_i \in A_i(s_i)} \widehat{U}_i^{(s^-,a^-)}(s_i, a_i, \sigma_{-i}; v) \right].$$

Then

$$B_v^{(s^-,a^-)}(s) = \left\{ \sigma(s^-, a^-, s) \in \Delta(A_1(s_1)) \times \Delta(A_n(s_n)) \mid \sigma \in \Sigma \text{ and } \widehat{\xi}^{(s^-,a^-)}(s, \sigma; v) = 0 \right\}.$$

Lemma 2. *For each v, $(s^-, a^-, s) \mapsto B_v^{(s^-,a^-)}(s)$ is nonempty, compact valued, and lower measurable.*

Proof. A correspondence $s \mapsto \Delta(A_1(s_1)) \times \cdots \times \Delta(A_n(s_n))$ is lower measurable, nonempty, and compact valued. To see this, we want that for each open subset G of X, the lower inverse image of G to be measurable; $A^l(G) = \{s \in S \mid A(s) \cap G \neq \emptyset\} \in \mathfrak{S}$. Since $\mathfrak{S} = \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_n$ is generated by measurable rectangles, $A^l(G) \in \mathfrak{S}$ if and only

[20] A Dirac delta function defined on $(X, \mathcal{M}, \mu)$ satisfies $\int_X \delta_{x_0} d\mu = 1$.

[21] There is another proof based on Hölder's inequality. $\left| \int_{T_i} (v_i^m g^m - v_i g) \lambda_i(dt_i) \right| = \left| \int_{T_i} (v_i^m g^m - v_i g^m) + (v_i g^m - v_i g) \lambda_i(dt_i) \right| \leq \left| \int_{T_i} [v_i^m g^m - v_i g^m] \lambda_i(dt_i) \right| + \left| \int_{T_i} [v_i g^m - v_i g] \lambda_i(dt_i) \right|$ by triangle inequality. Then $\left| \int_{T_i} [v_i g^m - v_i g] \lambda_i(dt_i) \right| \leq \int_{T_i} |v_i g^m - v_i g| \lambda_i(dt_i) \leq \|v_i\|_{L^\infty} \|g^m - g\|_{L^1} \to 0$ as $\|g^m - g\|_{L^1} \to 0$. As $v_i^m \to v_i$ in weak-*, for all $g^m \in L^1(T_i, \lambda_i)$, $\left| \int_{T_i} [v_i^m g^m - v_i g^m] \lambda_i(dt_i) \right| \to 0$.

[22] See Exercise 8 of Billingsley (1968), p.17.

if $A_i^l(G_i) \in \mathfrak{S}_i$, where $proj_i^{-1}(G_i)$ is a subbasis element for the product topology on X. From $(\mathcal{A}4)$, A_i is lower measurable for each i, it is clear that $s \mapsto (A_1(s_1)) \times \cdots \times (A_n(s_n))$ is lower measurable. And it is also nonempty, and compact valued because so is each A_i. Then $s \mapsto \Delta(A_1(s_1)) \times \cdots \times \Delta(A_n(s_n))$ is lower measurable, nonempty, and compact valued by Himmelberg and Van Vleck (1975).

Notice that

$$\begin{aligned}
&\widehat{\xi}^{(s^-,a^-)}(s,\sigma;v) \\
&= \sum_{i=1}^{n} \left[\widehat{U}_i^{(s^-,a^-)}(s_i,\sigma;v) - \max_{a_i \in A_i(s_i)} \widehat{U}_i^{(s^-,a^-)}(s_i,a_i,\sigma_{-i};v) \right]. \\
&= \sum_{i=1}^{n} \left[\begin{array}{l} \int_{s_{-i}} U_i(s,\alpha;v)\, \eta(ds_{-i} \mid s^-,a^-,s_i) \\ -\max_{a_i \in A_i(s_i)} \int_{s_{-i}} U_i(s,a_i,\alpha_{-i});v)\, \eta_i(ds_{-i} \mid s^-,a^-,s_i) \end{array} \right] \\
&= \sum_{i=1}^{n} \int_{s_{-i}} \left[U_i(s,\alpha;v) - \max_{a_i \in A_i(s_i)} U_i(s,a_i,\alpha_{-i});v) \right] \eta_i(ds_{-i} \mid s^-,a^-,s_i) \\
&= \sum_{i} \int_{S_i} \int_{s} \sum_{i=1}^{n} \left[U_i(s,\alpha;v) - \max_{a_i \in A_i(s_i)} U_i(s,a_i,\alpha_{-i};v) \right] \tau(ds \mid s^-,a^-)\, d\delta_{s_i},
\end{aligned}$$

where given (s^-,a^-,s), $\alpha = \sigma(s^-,a^-,s)$, and $U_i(s,\alpha;v) = \left[\begin{array}{l} (1-\delta_i)u_i(s,\alpha) \\ +\delta_i \int_{s_i^+} v_i(s,\alpha,s_i^+)\mu_i(ds_i^+ \mid s,\alpha) \end{array} \right]$. Similarly, $U_i(s,a_i,\alpha_{-i};v) = \left[\begin{array}{l} (1-\delta_i)u_i(s,a_i,\alpha_{-i}) \\ +\delta_i \int_{s_i^+} v_i(s,a_i,\alpha_{-i},s_i^+)\mu_i(ds_i^+ \mid s,a_i,\alpha_{-i})) \end{array} \right]$, and δ_{s_i} is the Dirac delta measure concentrated at s_i. Define $\xi(s,\alpha;v) = \sum_{i=1}^{n} \Big[U_i(s,\alpha) - \max_{a_i \in A_i(s_i)} U_i(s,a_i,\alpha_{-i}) \Big]$. Then $\xi(s,\alpha;v) = 0$ implies $\widehat{\xi}^{(s^-,a^-)}(s,\sigma;v) = 0$.[23] By Lemma 2 of Duggan (2012) and measurability of $\tau(Z|\cdot,\cdot)$ for each $Z \in \mathfrak{S}$, $(s^-,a^-,s) \mapsto B_v^{(s^-,a^-)}(s)$ is lower measurable. Given (s^-,a^-), Balder (1988) gives us nonemptiness of $B_v^{(s^-,a^-)}(s)$ for each s. Recall that the interim expected payoff function $\widehat{U}_i^{(s^-,a^-)}(s_i,\cdot;v)$ is continuous and the finite product of $\times_i \Delta(A_i(s_i))$ is compact. By the theorem of maximum (Theorem 17.31, AB, p.570), $B_v^{(s^-,a^-)}(s)$ is compact subset of $\Delta(A_1(s_1)) \times \cdots \times \Delta(A_n(s_n))$ for each s. □

[23]Necessary and sufficient condition is that $\xi(s,\alpha;v) = 0$ for τ-almost all s.

Lemma 3 (**AB 18.10**)**.** *Let (S, Σ) be a measurable space, let X be a separable metrizable space, and let $\varphi : S \rightrightarrows X$ be a correspondence with nonempty compact values. Then the following statements are equivalent.*

1. *The correspondence φ is lower measurable.*
2. *The correspondence φ is measurable.*
3. *The function $f : S \to \mathcal{K}$, defined by $f(s) = \varphi(s)$ is Borel measurable. ($\mathcal{K}$ is a compact subset of S)*

Lemma 4 (**Hildenbrand's (1974, 2015, p.64) Theorem 4**)**.** *Let φ be a correspondence of the measure space $(X, \mathcal{M}, \mu)$ into $\Re^m_+$ with a measurable graph. Then $co \int \varphi d\mu = \int co\ \varphi d\mu$. In particular, if the measure space is atomless, then $\int \varphi d\mu = \int co\ \varphi d\mu$.*

Lemma 5 (**Kuratowski–Ryll-Nardzewski : AB 18.13**)**.** *A lower measurable correspondence with non-empty closed values from a measurable space into a Polish space admits a measurable selector.*

Given s as realized type profile, define the set of realized payoffs for player i from $B_v^{(s^-,a^-)}(s)$ as $P_{v,i}^{(s^-,a^-)}(s) = U_i(s, B_v^{(s^-,a^-)}(s); v)$. Then $(s^-, a^-, s) \mapsto P_{v,i}^{(s^-,a^-)}(s)$ is nonempty, compact valued and lower measurable since $U_i(s, \cdot; v)$ is continuous. By Lemma 5 above, it admits a measurable selector. Then the correspondence is integrable. The set of interim expected payoffs for player i is denoted as $E_{v,i}^{(s^-,a^-)}(s_i) = \int_{s_{-i}} P_{v,i}^{(s^-,a^-)}(s)\, \eta_i(ds_{-i}|s^-, a^-, s_i)$. Let $E_v^{(s^-,a^-)}(s)$ denote the Cartesian product $E_{v,1}^{(s^-,a^-)}(s_1) \times \cdots \times E_{v,n}^{(s^-,a^-)}(s_n)$.

Lemma 6. *For each v, each (s^-, a^-, s), $E_v^{(s^-,a^-)}(s) = co\ E_v^{(s^-,a^-)}(s)$*

Proof. By Lemma 3 (Theorem 18.10, AB, p.598), the correspondence for each player i's realized payoffs $(s^-, a^-, s) \mapsto P_{v,i}^{(s^-,a^-)}(s)$ is in fact measurable. For any $(\tilde{s}^-, \tilde{a}^-, \tilde{s}_i)$, consider the $(\tilde{s}^-, \tilde{a}^-, \tilde{s}_i)$-section of the correspondence, $(\tilde{s}^-, \tilde{a}^-, \tilde{s}_i, s_{-i}) \mapsto P_{v,i}^{(\tilde{s}^-,\tilde{a}^-)}(\tilde{s}_i, s_{-i})$; it is clearly measurable. By Lemma 4 (Theorem 4, Hildenbrand (1974), p.64),

$$\int_{s_{-i}} P_{v,i}^{(\tilde{s}^-,\tilde{a}^-)}(\tilde{s}_i, s_{-i})\ \ \eta_i(ds_{-i}|\tilde{s}^-, \tilde{a}^-, \tilde{s}_i) = \int_{s_{-i}} co\ P_{v,i}^{(\tilde{s}^-,\tilde{a}^-)}(\tilde{s}_i, s_{-i})\ \ \eta_i(ds_{-i}|\tilde{s}^-, \tilde{a}^-, \tilde{s}_i).$$

Hence, for each (s^-, a^-, s_i), $E_{v,i}^{(s^-,a^-)}(s_i) = co\ E_{v,i}^{(s^-,a^-)}(s_i)$. Since Cartesian product of convex sets of $\Re$ is convex in $\Re^n$, $E_v^{(s^-,a^-)}(s) = co\ E_v^{(s^-,a^-)}(s)$.

□

Lemma 7 (**Proposition 2.3 of Himmelberg (1975, p.67)**). *Let J be an at most countable set and let $F_n : T \to X$ be a relation for each $n \in J$. Then,*

(i) If each F_n is measurable (lower measurable, etc.), so is the relation $\cup_n F_n : T \to X$ defined by $(\cup_n F_n)(t) = \cup_n F_n(t)$, and

(ii) if X is second countable and each F_n is lower measurable, then so is the relation $\times_n F_n : T \to X^J$ defined by $(\times_n F_n)(t) = \times_n F_n(t)$.

Lemma 8. *The mapping $(s^-, a^-, s) \mapsto E_v^{(s^-,a^-)}(s)$ is lower measurable, nonempty, compact, and convex valued.*

Proof. Notice that for each (s^-, a^-, s_i), $E_{v,i}^{(s^-,a^-)}(s_i)$ is nonempty and convex. Recall that $(s^-, a^-, s) \mapsto P_{v,i}^{(s^-,a^-)}(s)$ admits a measurable selector. Let $\{f_{i,k}\}_k$ denote the set of measurable selectors from the correspondence. Note that $\{f_{i,k}(s^-, a^-, s)\}_k \subset \mathfrak{R}$. Since $P_{v,i}^{(s^-,a^-)}(s)$ is compact, for each $f_{i,k}$, there is a family of functions that converge pointwise to $f_{i,k}$ at each (s^-, a^-, s). Notice that $U_i(s, \sigma(s); v)$ is bounded, so a family of functions that converges pointwise to $f_{i,k}$ is uniformly integrable. Recall that each λ_i is a probability measure. Then $S \times X \times S_i$ is a set of finite measure, the finite Cartesian product $S \times X \times S$ is also of finite measure. Obviously, the aforementioned family of functions is tight. Applying Vitali convergence theorem (RF, p.377), we obtain compactness of $\int_{s_{-i}} P_{v,i}^{(s^-,a^-)}(\bar{s}_i, s_{-i})\ \eta_i(ds_{-i}|s^-, a^-, \bar{s}_i)$ for each i. The finite product $E_v^{(s^-,a^-)}(s)$ is therefore compact at each (s^-, a^-, s). Applying Tonelli's theorem (RF, p.420), we have $\int_{s_{-i}} f_{i,k}(s^-, a^-, \bar{s}_i, s_{-i})\ \eta_i(ds_{-i}|\bar{s}^-, a^-, s_i)$ measurable. Thus we also get the lower measurability of $(s^-, a^-, s) \mapsto \int_{s_{-i}} P_{v,i}^{(s^-,a^-)}(\bar{s}_i, s_{-i})\ \eta_i(ds_{-i}|\bar{s}^-, a^-, s_i)$. Then, by Lemma 7 (ii) (Proposition 2.3 of Himmelberg (1975)), $(s^-, a^-, s) \mapsto E_v^{(s^-,a^-)}(s)$ is lower measurable. □

By above Lemma 5 (Kuratowski–Ryll-Nardzewski theorem, AB, p.600), $(s^-, a^-, s) \mapsto E_v^{(s^-,a^-)}(s)$ has a measurable selector. Given v, define M_v be the set of all λ-equivalence classes of measurable selectors of $(s^-, a^-, s) \mapsto E_v^{(s^-,a^-)}(s)$.

Lemma 9. *The mapping $v \mapsto M_v$ is nonempty, closed-graph, and convex valued.*

Proof. By construction, for each v, M_v is nonempty, closed, and convex. Recall that the interim expected payoff function $\widehat{U}_i^{(s^-,a^-)}(s_i,\sigma;\cdot)$ is continuous. By the theorem of maximum, $v \mapsto B_v^{(s^-,a^-)}(s)$ is upper hemicontinuous. Suppose a sequence $(v^m, \sigma^m(s^-,a^-,s)) \to (v, \sigma(s^-,a^-,s)$, where $\sigma^m(s^-,a^-,s) \in B_{v^m}^{(s^-,a^-)}(s)$ and $\sigma(s^-,a^-,s) \in B_v^{(s^-,a^-)}(s)$. Lemma 1 tells us that for each player i [24],

$$d_i^m(s_i)^{(s^-,a^-)} \equiv \max_{\alpha \in \Delta(A(s))} \mid \widehat{U}_i^{(s^-,a^-)}(s_i,\alpha;v^m) - \widehat{U}_i^{(s^-,a^-)}(s_i,\alpha;v) \mid \to 0.$$

Now we proceed similarly to Lemma 7 of Nowak and Raghavan (1992). Suppose $g^m \in M_{v^m}$ and $g^m \to g$ in the product weak-* topology on V. We want to show that $g \in M_v$. By Mazur theorem on top of Alaoglu theorem (Brezis, p.61 and p.66), there is sequence $\{y^m\}$ be made up of convex combinations of the g^m's which satisfies $\|y^m - g\|_{L^\infty} \to 0$ as $m \to \infty$. This implies that $\{y^m\}$ converges to g pointwise almost everywhere on T. Let $y^m(t) \to g(t)$ for all $t \in T - T_0$ where $\lambda(T_0) = 0$. Recall that $T \equiv S \times X \times S$. Then for each $t \in T - T_0$, for each m, if $y^m(s^-,a^-,s) \in E_{v^m}^{(s^-,a^-)}(s)$, then $g(s) \in E_{v^m}^{(s^-,a^-)}(s)$; since $d_i^m(s_i)^{(s^-,a^-)} \to 0$, $E_{v^m}^{(s^-,a^-)}(s) \to E_v^{(s^-,a^-)}(s)$; thus, $g(s^-,a^-,s) \in E_v^{(s^-,a^-)}(s)$ for each $(s^-,a^-,s) = t \in T - T_0$ and clearly $g \in M_v$. Hence, $v \mapsto M_v$ is upper-hemicontinuous. In addition, we observe that for each m, M_{v^m} is closed and M_v is closed. By the closed graph theorem for correspondences (Theorem 17.11, AB, p.561), $v \mapsto M_v$ is closed-graph. □

Lemma 10 (Filippov: AB 18.17). *Let $(T, \mathfrak{T})$ be a measurable space and let X and Y be separable metrizable space. Suppose that $f : T \times X \to Y$ is a Caratheodory function and that $\Phi : T \rightrightarrows X$ is lower measurable with non-empty compact values. Assume also that $g : T \to Y$ is a measurable selector from the range of f on T in the sense that g is measurable and for each t, there exists $x \in \Phi(t)$ with $g(t) = f(t,x)$. Then the correspondence $\Gamma : T \rightrightarrows X$, defined by $\Gamma(t) = \{x \in \Phi(t) : f(t,x) = g(t)\}$, is measurable and admits a measurable selector. That is, there exists a measurable function $\pi : T \to X$ with $\pi(t) \in \Phi(t)$ and $g(t) = f(t,\pi(t))$ for each $t \in T$.*

Proof of Existence Theorem. Obviously $M_v \subset V$. By Kakutani-Fan-Glicksberg (Theorem 17.55, AB, p.583) theorem, we have a fixed point of $v \mapsto M_v$. Then we have

[24] $E_{v^m}^{(s^-,a^-)}(s)$ is the $\frac{1}{m}$-neighborhood of $E_v^{(s^-,a^-)}(s)$ as $v^m \to v$.

$w : S \times X \times S \to \mathfrak{R}^n$ such that given (s^-, a^-), $w(s^-, a^-, s) \in E_v^{(s^-,a^-)}(s)$ for all $s \in S - S_0$, $\nu(S_0) = 0$. Recall that $\widehat{U}_i^{(s^-,a^-)}(s_i, \alpha; v)$ is measurable in (s^-, a^-, s_i), continuous in α (Lemma 1). Now by Lemma 10 (Filippov's implicit function theorem), there exists a measurable mapping $f : S \times X \times S \to \Delta(X_1) \times \cdots \times \Delta(X_n)$ such that given (s^-, a^-), for each s, $f(s^-, a^-, s) \in B_v^{(s^-,a^-)}(s)$ and for all i, each $s_i \in S_i - proj_i(S_0)$,

$$w_i(s^-, a^-, s_i) = \int_{s_{-i}} \left[\begin{array}{l} (1-\delta_i)u_i(s, f(s^-, a^-, s)) \\ +\delta_i \int_{s_i^+} w_i(s, a, s_i^+)\mu_i(ds_i^+|s, f(s^-, a^-, s)) \end{array} \right] \eta(ds_{-i}|s^-, a^-, s_i).$$

For each i, $(s^-, a^-, s) \in S \times X \times (S - S_0)$, $(s^-, a^-, s) \mapsto B_v^{(s^-,a^-)}(s)$ admits a measurable selector by Lemma 5. Let f_0 be any measurable selector of $(s^-, a^-, s) \mapsto B_v^{(s^-,a^-)}(s)$. Given (s^-, a^-), put

$$f^*(s^-, a^-, s) = \begin{cases} f_0(s^-, a^-, s) & \text{if } s \in S_0 \\ f(s^-, a^-, s) & \text{if } s \in S - S_0 \end{cases}.$$

Then $f^*(s^-, a^-, s) \in B_v^{(s^-,a^-)}(s)$ for all s, and

$$w_i^*(s^-, a^-, s_i) = \int_{s_{-i}} \left[\begin{array}{l} (1-\delta_i)u_i(s, f^*(s^-, a^-, s)) \\ +\delta_i \int_{s_i^+} w_i^*(s, a, s_i^+)\mu_i(ds_i^+|s, f^*(s^-, a^-, s)) \end{array} \right] \eta(ds_{-i}|s^-, a^-, s_i).$$

Therefore f is our stationary Bayesian-Markov equilibrium strategy profile. □

4 Application

The patent/innovation race literature has been studied extensively. A usual assumption of the literature is that once discovery or invention is made, then the race is over. However, in real life, the other firms do not give up innovation of their products even if there is a patent-awarded leader. There are innovations which are differentiated serving for the same purpose. In addition, consumers' tastes may vary over time. Firms keep doing innovation to defense their loyal customers, too.[25] So we want to generalize the structure

[25]Recall that big screen cell phones such as Galaxy note 2 were popular but had not monopolized the market. Apple has become against Steve Jobs with iPhone 6s to keep their loyal customers from leaving for "a cell phone with a larger screen".

of an innovation race using the notion of 'loyal consumer'. In order to do this, we use our Bayesian stochastic game structure. We will exemplify pharmaceutical industry, but this model can be applied to any industry where a firm can be dependent on its loyal consumer base.

The notable literature which used a discrete time stochastic game is Judd, Schmedders, and Yeltekin (2012). They showed an innovation race using a discounted stochastic game in complete information: Heterogeneous firms compete for a patent while they know their competitors' current states. The one which obtains the patent the earliest monopolizes the market and the race ends. They dealt with detailed multistage innovation process and investigated optimal patent policy balancing overinvestment and quicker innovation. They used a discounted stochastic game with finite states and assumed that the winner of innovation takes all the market share, enjoying monopoly profits. In contrast, we assume that one-stage innovations which are immediately marketable. We deal with a continuum of states. We assume that competitor's type is not observed during the innovation process, and there is no 'winner' who monopolizes the market after the race; instead, there is a firm (or firms) which having gained new consumer base(s).

Suppose there are two pharmaceutical companies i and j that produce a certain category of medication for particular uses. For example, i is Bayer which producing Aspirin, and j is Johnson & Johnson McNeil of Tylenol. Hereafter, $k \in \{i, j\}$. Each of them uses special ingredients for its own medication (aspirin and acetaminophen). Each medication is effective (or non-allergic) for a certain set of patients. We assume that potential patients are on the product of sets of positive real numbers by their biological characteristics, and denote the set of patients for whom the medication k works well as $E_k \subset \mathfrak{R}^n$.[26] Among E_k, there are some patients who cannot use the other medication because of the ingredients. That is, $E_k \backslash E_{-k} \in \mathfrak{R}^n$ is nonempty. We define the measure of this group of patients $s_k = m(E_k \backslash E_{-k})$ as the type of the firm k. For simplicity, assume that $s_k \in [0, 1]$. The patients are perfectly inelastic to the price of the medication: $\frac{\partial \log s_k}{\partial \log p_k} = 0$. In contrast, the rest of patients in E_k can use both of medications: $E_k \cap E_{-k}$ is nonempty and those patients' price elasticity is nonzero. For simplicity, we normalize the measure of this group

[26]Technically, we assume that E_k belongs to σ-algebra of m-measurable set in $\mathfrak{R}^n$ which is the space of patients (characteristics). We implicitly construct a measure m on the set of patients, and obtain the type space S_k of firm k.

of patients as $m(E_i \cap E_j) = 1$ for every period.[27]

At each period, firms launch a new product line of their medication. As ingredients of each medication change, we assume that E_i, E_j both stochastically evolve according to these new medications. Given their newly developed chemical compound before market launching, they know how many patients, i.e. $m(E_k \backslash E_{-k})$, its medication will exclusively work. But they do not know what kind of chemical compound the competitor has developed. So, each firm knows its own type s_k but do not know the competitor's. However, they observe the previous market performances of their own and competitors overall, so each firm has beliefs about competitor's type. Given its realized type, considering competitor's type, firms are required to choose the price for the newly launched medication such that $p_i, p_j \in [c, \bar{P}]$, where $0 < c < \bar{P} < \infty$, and to decide how many researchers they hire for the new development for next period (R & D investment). We normalize the absolute size of investment such that $h_i, h_j \in [0, 1]$ (We assume that the action space and admissible action correspondence for each type are equivalent). Depending on the size of the investment, the type evolves stochastically. The law of motion for the next period type of firm k is given by

$$s_k^+ = (1 - \rho_k)s_k + \rho_k h_k.$$

That is, the current type s_k is depreciated by ρ_k, and it is replaced by the investment. This follows a typical assumption for the law of motion for capital in macroeconomics. Given ρ_k, increases in h_k has a proportional effect on s_k^+. This means the larger investment size h_k, the higher chances of "today's underdog to be tomorrow's champion (leapfrog)", given the competitor's investment h_j and the depreciation rate ρ_k.[28] However, we assume that the depreciation/replacement rate ρ_k is a random variable, which follows a uniform distribution $\rho_j^- \sim \mathfrak{u}[0, 1]$ and unobservable by competitors. This random depreciation rate assumption is based on the randomness of innovation: Increases in h_k has a random effect

[27] We do market growth rate adjustment. And by doing so, we restrict the measure of $E_k \backslash E_{-k}$ not to exceed the measure of $E_i \cap E_j$.

[28] The following is a description about transition probability itself. Given h_{-k}, $E\rho_k = \frac{1}{2}$. Therefore $\frac{\partial E\rho_k}{\partial h_k} = 0$. Then $\frac{\partial E s_k^+}{\partial h_k} = \frac{1}{2}$. That means, assuming that h_{-k} is given, as the size of current investment h_k increases, the next period type s_k^+ is more likely be larger ('monotone likelihood transition'). In our game, notice that when player k consider her next period possible type realization s_k^+, the competitor's investment decision h_{-k} will be integrated out with respect to player k's beliefs on $h_{-k}(s_{-k})$.

on s_k^+. Even if there were a lot of investment for innovation, the result might be meager, the consumer base does not increase. In contrast, it can be the case that the investment was little, but the result becomes a huge success and the firm takes a big market share at once.

The law of motion is common knowledge, so firm i's beliefs $\eta_i(\cdot|s^-, (p^-, h^-), s_i)$ about the type of firm j today is as follows:

$$s_j = (1 - \rho_j^-)s_j^- + \rho_j^- h_j^-,$$

where $\rho_j^- \sim \mathfrak{u}[0, 1]$. ρ_j^- is realized at the beginning of the current period, but unknown to firm i until at the end of current period. Even if $\rho_j^- = 0$ or 1, firm i's beliefs does not degenerate because it does not know that until the period reaches its end. The beliefs η_i for s_j is given by

$$\begin{cases} h_j^- > s_j^- : & s_j \sim \mathfrak{u}[s_j^-, h_j^-] \\ h_j^- < s_j^- : & s_j \sim \mathfrak{u}[h_j^-, s_j^-] \end{cases} \tag{2}$$

(Notice that in any case, if $s_j^- = h_j^-$, beliefs degenerate to $s_j = s_j^-$. Since we want to illustrate a Bayesian stochastic game, we will focus on non-degenerate cases.) Firm j's beliefs $\eta_j(\cdot|s^-, (p^-, h^-), s_j)$ about the type of firm i today is symmetrically defined.

The period profit function of firm i is given by

$$\begin{aligned} &\pi_i(E_i, E_j, p, h) \\ &\quad = [m(E_i \backslash E_j) + m(E_i \cap E_j)L_i(p_i(s_i), p_j(s_j))] \cdot (p_i(s_i) - c) - f(h_i(s_i), h_j(s_j)). \end{aligned}$$

Replacing $m(E_i \backslash E_j)$ with s_i and $m(E_i \cap E_j) = 1$,

$$\pi_i(s, p, h) = [s_i + L_i(p_i(s_i), p_j(s_j))] \cdot (p_i(s_i) - c) - f(h_i(s_i), h_j(s_j)).$$

We assume that period payoff function π is differentiable in p_i, p_j, h_i, h_j. As we mentioned before, the exclusive users of medication i, elements of $E_i \backslash E_j$, are perfectly price-inelastic; we assume that firm i knows only the measure of them; the firm cannot implement price discrimination individually. Recall that $m(E_i \cap E_j)$ is normalized to 1. $L_i(p_i, p_j)$ is a partial demand function which reflects the price elasticity of the patients who can use both of medications. We assume that $\frac{\partial \log L_i(p_i,p_j)}{\partial \log p_i} < -1$ when $p_i > p_j$, and if $p_i \leq p_j$, $-1 < \frac{\partial \log L_i(p_i,p_j)}{\partial \log p_i} < 0$. That means, if there is a substitution of which price is lower, the demand of the medication is elastic; otherwise, inelastic. c is production cost. For simplicity, it is assumed as a constant for medications of any firm. $f_i(h_i, h_j)$ is the costs of R & D investment for firm i: $\frac{\partial f_i(h_i,h_j)}{\partial h_i} > 0$. Since R & D investment involves in hiring researchers and buying equipment, given h_i of firm i's investment, as competitor invests more, the factor market prices rise. This causes the costs of investment $f_i(h_i, h_j)$ for firm i depends on h_j as well as h_i and moreover, convex: $\frac{\partial^2 f_i}{\partial h_i^2} > 0$ and $\frac{\partial^2 f_i}{\partial h_i \partial h_j} > 0$. Firm j has a symmetric period profit function. All assumptions are symmetrically applied to firm j, too.

We call this model an incomplete information version of "innovation race" in the pharmaceutical industry.[29] In our innovation race model, there is always a stationary Bayesian-Markov equilibrium.

Proposition 1. *In an incomplete information version of innovation race in the pharmaceutical industry, there exists a stationary Bayesian-Markov equilibrium.*

Proof. The type space for each player $[0, 1]$ is complete separable metric space with usual metric. Equipped with the uniform probability measure, denoted by ν_k, it is a complete measure space. The action space for price $[c, \bar{P}]$ and for investment $[0, 1]$ are compact metric space. The action correspondence is equivalent to the action space for each type realization; nonempty, compact and lower measurable. The period profit function is continuous (even twice differentiable) in price and investment. It is measurable (even continuous and differentiable) in type. The transition probability is given by the uniform distribution generated from the law of motion $s_k^+ = (1 - \rho_k)s_k + \rho_k h_k$ where $\rho_k \sim \mathfrak{u}[0, 1]$ for each $k \in \{i, j\}$. Then it is clearly absolutely continuous with respect to ν_k and norm-continuous in (p, h). Since the law of motion is common knowledge, the beliefs are given by the corresponding law of motion for the competitor's inherited from the previous stage. Notice that for each k, s_{-k} is independent from the realization of s_k, since ρ_i^- and ρ_j^- are independently drawn. Then these satisfy the conditions for existence of stationary Bayesian-Markov equilibrium by our existence theorem. □

Now we additionally assume that when competitor starts offering the lowest price ever possible c in the industry, for any $\epsilon > 0$, firms which charge a higher price $c + \epsilon$ will suffer

[29] Even though we assume the pharmaceutical industry, this model can be applied to any industries which involve in continuous innovation and which are based on a group of loyal customers.

from the loss in marginal profits with respect to its price. That is , for each $k \in \{i,j\}$, $L_k(p_k, p_{-k})$ satisfies that $0 \geq 1 + L_k(c,c)\ (1 + \frac{\partial \log L_k(p_k,c)}{\partial \log p_k})\Big|_{p_k=c} - c \cdot \frac{\partial L_k(p_k,c)}{\partial p_k}\Big|_{p_k=c}$. Recall that we assume for each $k \in \{i,j\}$, $p_k > p_{-k}$, $\frac{\partial \log L_k(p_k,p_{-k})}{\partial \log p_k} < -1$ and $p_k \leq p_{-k}$, $-1 < \frac{\partial \log L_k(p_k,p_{-k})}{\partial \log p_k} < 0$.[30] We also assume that the functional forms of marginal costs of investment are symmetric for firms, i.e. $\frac{\partial f_i(h_i,h_j)}{\partial h_i} = \frac{\partial f_j(h_j,h_i)}{\partial h_j}$.[31] We show below that based on the aforementioned three assumptions, we have a stationary Bayesian-Markov equilibrium in pure strategy. Then, from our theorem, we know that for each i, there is a continuation value function v_i which satisfies the following incursion: first, define U_i

$$U_i(s,(p,h)) = (1-\delta_i)\left[s_i + L_i(p_i,p_j)(p_i(s_i) - c) - f(h_i(s_i), h_j(s_j))\right] \\ + \delta_i \int_{s_i^+} v_i(s,(p,h),s_i^+)\mu(ds_i^+|s,(p,h)).$$

Then, v_i is given by

$$v_i(s^-,(p^-,h^-),s_i) = \int_{s_{-i}} U_i(s,(p,h))\eta(ds_{-i}|s^-,(p^-,h^-)).$$

Note that since pricing does not affect the type evolution for each firm in our innovation race, in fact $\mu(ds_i^+|s,p,h) = \mu(ds_i^+|s,h)$ and $\eta(ds_{-i}|s^-,p^-,h^-) = \eta(ds_{-i}|s^-,h^-)$.

Proposition 2. *Let $k \in \{i,j\}$. Assume that $L_k(p_k,p_{-k})$ satisfies that*[32]

$$1 + L_k(c,c)\ (1 + \frac{\partial \log L_k(p_k,c)}{\partial \log p_k})\Big|_{p_k=c} - c \cdot \frac{\partial L_k(p_k,c)}{\partial p_k}\Big|_{p_k=c} \leq 0,$$

and that $\frac{\partial f_i(h_i,h_j)}{\partial h_i} = \frac{\partial f_j(h_j,h_i)}{\partial h_j}$. In an incomplete version of innovation race in pharmaceutical industry, there is a symmetric stationary Bayesian-Markov equilibrium such that $p_k(s^-,h^-,s_k) = \bar{P}$ for $s_k \geq \theta_k(s^-,h^-)$ and $p_k(s^-,h^-,s_k) = c$, otherwise. The changes in cutoff θ_k has perfect positive correlation with firm k's own previous state s_k^- and the

[30] This simply means that if a substitute is available, consumers tent to be price sensitive.

[31] For example, if $f_i(h_i,h_j) = (\frac{1}{2}h_i + h_j)h_i$ and $f_j(h_j,h_i) = (\frac{1}{2}h_j + h_i)h_j$, then $\frac{\partial f_i(h_i,h_j)}{\partial h_i} = \frac{\partial f_j(h_i,h_j)}{\partial h_j} = h_i + h_j$.

[32] If $\left| \frac{\partial \log L_k(p_k,c)}{\partial \log p_k}\Big|_{p_k=c} \right|$ is sufficiently large, this assumption is easily satisfied. For example, the demand function of typical Bertrand price competition model gives us $\left| \frac{\partial \log L_k(p_k,c)}{\partial \log p_k}\Big|_{p_k=c} \right| = \infty$.

previous investment decision h_k^-. And in this equilibrium, each player increases their investment $h_k(s^-, h^-, s_k)$ as s_k rises if their type is not greater than their own threshold of investment $s_k \leq \xi_k(s^-, h^-)$. They decreases the investment as s_k rises otherwise. The threshold $\xi_k(s^-, h^-)$ shows perfect positive correlation with s_k^- and h_k^-.

Proof. First, we deal with a pricing strategy. Note that price does not affect the transition, and thus, neither does the continuation function. Recall that we assume $p_i > p_j$, $\frac{\partial \log L_i(p_i,p_j)}{\partial \log p_i} < -1$ and $p_i \leq p_j$, $-1 < \frac{\partial \log L_i(p_i,p_j)}{\partial \log p_i} < 0$. Then $L_i(p_i, \bar{P})(1 + \frac{\partial \log L_i(p_i,\bar{P})}{\partial \log p_i}) - c \cdot \frac{\partial L_i(p_i,\bar{P})}{\partial p_i} > 0$ if $p_i \leq \bar{P}$. This implies that for any $s_i \in [0, 1]$,

$$s_i + L_i(p_i, \bar{P})(1 + \frac{\partial \log L_i(p_i, \bar{P})}{\partial \log p_i}) - c \cdot \frac{\partial L_i(p_i, \bar{P})}{\partial p_i} > 0.$$

We denote the LHS as $\mathrm{M}P_{s_i,\bar{P}}$. Also, for any $s_i \in [0, 1]$, our additional assumption on $L_i(p_i, p_j)$ implies that for $p_i \in [c, \bar{P}]$ and $s_i \in [0, 1)$ or $p_i \in (c, \bar{P}]$ and $s_i \in [0, 1]$,

$$s_i + L_i(p_i, c)(1 + \frac{\partial \log L_i(p_i, c)}{\partial \log p_i}) - c \cdot \frac{\partial L_i(p_i, c)}{\partial p_i} < 0.$$

Let the LHS be $\mathrm{M}P_{s_i,c}$. Without loss of generality, suppose that $h_i^- > s_i^-$ and $h_j^- > s_j^-$. Then the beliefs of firm i about s_j is $s_j \sim \mathfrak{u}[s_j^-, h_j^-]$. Given firm j's stationary Bayesian-Markov pricing strategy that if $s_j \geq \theta_j(s^-, h^-)$, then $p_j = \bar{P}$; otherwise, $p_j = c$ where $(\theta_j(s^-, h^-) \in (s_j^-, h_j^-) \subset [0, 1])$, firm i solves the following equation to obtain her cutoff $\theta_i(s^-, h^-)$:

$$\frac{1}{h_j^- - s_j^-} \cdot (h_j^- - \theta_j) \cdot \mathrm{M}P_{\theta_i,\bar{P}} + \frac{1}{h_j^- - s_j^-} \cdot (\theta_j - s_j^-) \cdot \mathrm{M}P_{\theta_i,c} = 0. \tag{3}$$

Notice that when firm i's realized type is s_i, $\mathrm{M}P_{s_i,\bar{P}} > 0$ and $\mathrm{M}P_{s_i,c} < 0$. So there is $\theta_i(s^-, h^-) \in (s_i^-, h_i^-)$ which satisfies the above equation by the intermediate value theorem. Symmetrically, firm j also solves her corresponding equation to obtain θ_j:

$$\frac{1}{h_i^- - s_i^-} \cdot (h_i^- - \theta_i) \cdot \mathrm{M}P_{\theta_j,\bar{P}} + \frac{1}{h_i^- - s_i^-} \cdot (\theta_i - s_j^-) \cdot \mathrm{M}P_{\theta_i,c} = 0. \tag{4}$$

By solving the system of equations (3) and (4), we get $\theta_i(s^-, h^-)$ and $\theta_j(s^-, h^-)$. Then for $k \in \{i, j\}$ if $s_k \leq \theta_k(s^-, h^-)$, the expected profits of player k is maximized when

$p_k(s^-, h^-, s_k) = \bar{P}$. Also, if $s_k < \theta_k(s^-, h^-)$, it is maximized when $p_k(s^-, h^-, s_k) = c$. By equation (1) of definition of stationary Bayesian-Markov equilibrium, we have a stationary Bayesian-Markov equilibrium pricing strategy which is in pure action. Notice that for each $k \in \{i, j\}$ if s_k^- or h_k^- increases, then θ_k rises to compensate the changes in probabilities. We consider the case that probabilities are perfectly compensated by new θ'_k, and thus, θ_{-k} is not affected by the changes in s_k^- or h_k^-.

Now we want to show that for $k \in \{i, j\}$, $h_k(s^-, h^-, s_k)$ increases as s_k increases if their type is not greater than their own threshold of investment $s_k \leq \xi_k(s^-, h^-)$. They decreases the investment as s_k rises otherwise. Suppose the competitor $-k$ plays a stationary Bayesian-Markov investment strategy and firm k knows that. Note that if firm k's type is realized as s_k, then as h_k increases, it incurs higher marginal costs by $(1 - \delta_k) \int_{s_{-k}} \frac{\partial f_k(h_k, h_{-k})}{\partial h_k} \eta(ds_{-k}|s^-, h^-, s_k)$. On the other hand, the possibility of higher type s_k^+ tomorrow rises as h_k increases. The higher type tomorrow potentially gives us higher expected continuation values: increases in the marginal expected continuation values are $\delta_k \int_{s_{-k}} \frac{\partial v_k(s,(p,h),s_k^+)}{\partial s_k^+} \cdot \frac{\partial E(s_k^+)}{\partial h_k} \eta(ds_{-k}|s^-, h^-, s_k)$. [33] In order to analyze the marginal expected continuation values, we change the view point to *ex post* and will come back to player k's interim stage again. Using the envelop theorem, we have[34]

$$\frac{\partial U_k(s,(p,h))}{\partial s_k} = (1 - \delta_k) \begin{bmatrix} 1 + \frac{\partial L_k(p_k, p_{-k})}{\partial p_k} \cdot \frac{\partial p_k}{\partial s_k} \cdot (p_k - c) \\ + L_k(p_k, p_{-k}) \cdot \frac{\partial p_k}{\partial s_k} - \frac{\partial f_k(h_k, h_{-k})}{\partial h_k} \cdot \frac{\partial h_k}{\partial s_k} \end{bmatrix}.$$

Reflecting our stationary Bayesian-Markov pricing strategy, the RHS of the above equation is simplified into $(1 - \delta_k) \left[1 - \frac{\partial f_k(h_k, h_{-k})}{\partial h_k} \cdot \frac{\partial h_k}{\partial s_k} \right]$. By stationarity of v_k and the indepen-

[33] This is because $\frac{d}{ds_k^+} \int_{s_k^+} v_k \mu(ds_k^+|s, h) = v_k(s_k^+|s, h)$ (an indefinite integral is an absolutely continuous function when it is a real-valued function) and using the chain rule, $\frac{\partial \left(\frac{d}{ds_k^+} \int_{s_k^+} v_k \mu(ds_k^+|s,h) \right)}{\partial h_k} = \frac{\partial v_k(s_k^+|s,h)}{\partial h_k} = \frac{\partial v_k(s_k^+)}{\partial s_k^+} \cdot \frac{\partial s_k^+}{\partial h_k}$. Since s_k^+ is random variable of which law of motion depends on h_k, we do not need to take partial derivative of s_k^+ with respect to h_{-k}, and we take expectation with respect to s_k^+.

[34] Note that here we take the "partial derivative" with respect to s_i, given other player's stationary Bayesian-Markov strategy: s_i realization does not affect j's strategy. So we can apply a technique from macroeconomics for the representative agent case. See 3.2 Stochastic control problem, Sargent and Ljungqvist (2012) (2012, p.110–111).

dent structure of law of motion for types[35] in this problem, we can put $\frac{\partial v_k(s,(p,h),s_k^+)}{\partial s_k^+} = \int_{s_{-k}} \frac{\partial U_k(s,(p,h))}{\partial s_k} \eta_k(ds_{-k}|s^-, h^-, s_k)$. Then we have the expected marginal benefit from increases in h_k as follows:

$$\frac{\partial v_k(s,(p,h),s_k^+)}{\partial s_k^+} \cdot \frac{\partial E(s_k^+)}{\partial h_k} = \int_{s_{-k}} \frac{1}{2} \cdot (1-\delta_k) \left[1 - \frac{\partial f_k(h_k, h_{-k})}{\partial h_k} \cdot \frac{\partial h_k}{\partial s_k} \right] \eta_k(ds_{-k}|s^-, h^-, s_k).$$

Given s_k, we find h_k which equals the expected marginal costs to the expected marginal benefits maximizes the continuation value, given the competitor's stationary Bayesian-Markov strategy. That is, $(1-\delta_k) \int_{s_{-k}} \frac{\partial f_k(h_k,h_{-k})}{\partial h_k} \eta(ds_{-k}|s^-, h^-, s_k) = \delta_k \int_{s_{-k}} \frac{1}{2} \cdot (1 - \delta_k) \left[1 - \frac{\partial f_k(h_k,h_{-k})}{\partial h_k} \cdot \frac{\partial h_k}{\partial s_k} \right] \eta(ds_{-k}|s^-, h^-, s_k)$. This is equivalent to the following: For almost all $s_k \in S_k$, $(1-\delta_k) \frac{\partial f_k(h_k,h_{-k})}{\partial h_k} = \frac{1}{2}\delta_k \cdot (1-\delta_k) \left[1 - \frac{\partial f_k(h_k,h_{-k})}{\partial h_k} \cdot \frac{\partial h_k}{\partial s_k} \right]$. Since $\delta_k \in [0,1)$, we have

$$\frac{\partial h_k}{\partial s_k} = \frac{1 - \frac{2}{\delta_k} \cdot \frac{\partial f_k(h_k,h_{-k})}{\partial h_k}}{\frac{\partial f_k(h_k,h_{-k})}{\partial h_k}}.$$

Recall that we assume $\frac{\partial f_k(h_k,h_{-k})}{\partial h_k} > 0$. Then, if $\frac{\partial f_k(h_k,h_{-k})}{\partial h_k} \leq \frac{\delta_k}{2}$, then $\frac{\partial h_k}{\partial s_k} \geq 0$, so h_k increases along with s_k rises. Otherwise, $\frac{\partial f_k(h_k,h_{-k})}{\partial h_k} > \frac{\delta_k}{2}$, and thus, $\frac{\partial h_k}{\partial s_k} < 0$, h_k decreases along with s_k rises.

Now we compute the threshold ξ_k which bifurcates the value of $\frac{\partial h_k}{\partial s_k}$. Without loss of generality, suppose $s_j^- < h_j^-$ and $s_i^- < h_i^-$. Then again, beliefs $\eta_i(\cdot|s^-, h^-)$ become $s_j \sim \mathfrak{u}[s_j^-, h_j^-]$. Given firm j's stationary Bayesian-Markov investment strategy that if $s_j > \xi_j(s^-, h^-)$, decrease h_j as s_j rises; otherwise, increase h_j along with s_j, firm i decide the threshold ξ_i as follows:

$$\begin{aligned} &\frac{1}{h_j^- - s_j^-} \cdot (h_j^- - \xi_j) \cdot \left(\left. \frac{\partial f_i(h_i, h_j(s_j|s_j > \xi_j))}{\partial h_i} \right|_{h_i=\xi_i} - \frac{\delta_i}{2} \right) \\ &+ \frac{1}{h_j^- - s_j^-} \cdot (\xi_j - s_j^-) \cdot \left(\left. \frac{\partial f_i(h_i, h_j(s_j|s_j < \xi_j))}{\partial h_i} \right|_{h_i=\xi_i} - \frac{\delta_i}{2} \right) = 0. \end{aligned} \tag{5}$$

Here firm j's stationary Bayesian-Markov investment strategy implies that $\left. \frac{\partial f_j(h_i,h_j(s_j|s_j>\xi_j))}{\partial h_j} \right|_{h_i=\xi_i} - \frac{\delta_i}{2} > 0$ and $\left. \frac{\partial f_j(h_i,h_j(s_j|s_j<\xi_j))}{\partial h_j} \right|_{h_i=\xi_i} - \frac{\delta_i}{2} < 0$. By our assumption of symmetric functional

[35] In our model, we have $\int_{s_{-k}} \frac{\partial U_k(s,(p,h))}{\partial s_k} \eta_k(ds_{-k}|s^-, h^-, s_k) = \frac{\partial \int_{s_{-k}} U_k(s,(p,h)) \eta_k(ds_{-k}|s^-,h^-,s_k)}{\partial s_k}$. This is because independence of s_i with $s_j = (1-\rho_j^-)s_j^- + \rho_j^- h_j^-$.

forms of marginal costs, we have $\frac{\partial f_i(h_i,h_j(s_j|s_j>\xi_j))}{\partial h_i}\Big|_{h_i=\xi_i} - \frac{\delta_i}{2} > 0$ and $\frac{\partial f_i(h_i,h_j(s_j|s_j<\xi_j))}{\partial h_i}\Big|_{h_i=\xi_i} - \frac{\delta_i}{2} < 0$. Then there is $\xi_i(s^-, h^-) \in (s_i^-, h_i^-)$ which satisfies the above equation by the intermediate value theorem. Symmetrically, firm j also solves her corresponding equation to obtain ξ_j:

$$\begin{aligned} &\frac{1}{h_i^- - s_i^-} \cdot (h_i^- - \xi_i) \cdot \left(\frac{\partial f_j(h_j, h_i(s_i|s_i > \xi_i))}{\partial h_j}\bigg|_{h_j=\xi_j} - \frac{\delta_j}{2} \right) \\ &+ \frac{1}{h_i^- - s_i^-} \cdot (\xi_i - s_i^-) \cdot \left(\frac{\partial f_j(h_j, h_i(s_i|s_i < \xi_i))}{\partial h_j}\bigg|_{h_j=\xi_j} - \frac{\delta_j}{2} \right) = 0. \end{aligned} \tag{6}$$

By solving the above system of equations (5) and (6), we obtain $\xi_i(s^-, h^-)$ and $\xi_j(s^-, h^-)$. Then each player increases their investment when their type is not greater than their own threshold of investment and decreases otherwise. Notice that for each $k \in \{i, j\}$, the threshold ξ_k shows perfect positive correlation with s_k^- and h_k^- for the same reason of our stationary Bayesian-Markov pricing strategies.

□

5 Conclusion

We consider a Bayesian stochastic game with periodic revelation: the case where the previous type-action profiles are perfectly observed at the beginning of each period. Types evolve stochastically depending on one-period previously realized type-action profiles (a first-order Markov process), so these previous stage action and type profiles are defining factors of common prior (and thus, so are for beliefs, on top of each player's own current type) in the current stage. We define a stationary Bayesian-Markov strategy as a measurable mapping which maps the same previous action-type profile (s^-, a^-) and the same type realization s_i to the same mixed action. We establish a stationary Bayesian-Markov equilibrium where type and action spaces are complete separable metric space and compact metric space, respectively. We illustrate an incomplete information version of innovation race in the pharmaceutical industry as a possible application. Imposing some assumptions on price elasticity and marginal costs of investment, we show existence of a stationary Bayesian-Markov equilibrium in which an equilibrium pricing strategy takes either the lower bound or the upper bound, depending on the previous type-investment profiles and the current type of the player; an equilibrium investment strategy is that to increase investment when the player's type is not greater than a threshold and to decrease otherwise.

With this basic modeling framework, we may extend to a Bayesian stochastic game where the previous actions and/or types are imperfectly observed.[36] Especially, it would be interesting to consider Bayesian stochastic game structure when revelation is made by K-period intervals, i.e. the case where each period t, players observe $t - K$ period's state of the world and afterwards, $t - K + 1, \dots, t$, only their own type and actions are observed. Moreover, it would also be interesting to look at, as Athey and Bagwell, Cole and Kocherlakota explored, the case that types are never revealed. Since transition depends on both action and type profiles, if either or them is unobservable, then there will be only hidden prior and players will have inconsistent beliefs. In this situation, usual approach may not work whereas assuming ambiguity averse players, we can restrict the set of players' possible beliefs. In this environment, a stationary Markov equilibrium in generalized Bayesian stochastic games will be able to explain interesting phenomena of real life better. Comparison the set of such equilibria with folk theorem result will be also worth exploring.

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[36] In fact, our model is directly applied to the case of imperfectly monitored i.i.d. Bayesian stochastic game in which types are drawn according to the fixed distribution . That is because in the i.i.d. case, previously realized action and type profiles are irrelevant to common prior of the current stage game, and thus, beliefs only depend on each player's current type.

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