April 15, 2016

PLANNING FOR THE LONG RUN: PROGRAMMING WITH PATIENT, PARETO RESPONSIVE PREFERENCES

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Abstract. Society is an aggregate of present and future generations. We study stochastic societal optimization problems in which similar treatment of generations in similar situations is possible. For such problems, all patient, inequality averse societal welfare functions that are perfectly Pareto responsive have the same optimal policies. When the outcomes of irreversible decisions are partially learnable, the optimal policies for patience preferences yield a variant of the precautionary principle. Under mild conditions, optimal policies exist and there is a single Bellman-like equation characterizing them.

Generations to come

> . . . intergenerational solidarity is not optional, but rather a basic question of justice, since the world we have received also belongs to those who will follow us. (Pope Francis [23])

> As we peer into society's future, we — you and I, and our government — must avoid the impulse to live only for today, plundering for our own ease and convenience the precious resources of tomorrow. We cannot mortgage the material assets of our grandchildren without risking the loss also of their political and spiritual heritage. We want democracy to survive for all generations to come . . . . (Eisenhower, 1953)

Timescales and choices

> With 500 million years left of acceptable habitat for humans on Earth, population being stable at 10 billion with an average length of life equal to 73 years, the ratio of people who will potentially live in the future to people living now is approximately 10 million to 1. (Asheim [3])

> If you are planning for a year, sow rice; if you are planning for a decade, plant trees; if you are planning for a lifetime, educate people. (Attributed to Confucius, 500 B.C.)

1. Introduction

Present societal choices may have an enormous impact on the number of future people and the world in which they live. Analyses of intergenerational allocations of costs and benefits have a long history. We take and generalize an approach from an early attempt to reason about them, an attempt that is ch is consonant with the values commonly expressed in analyses of equity, and which meets the more pragmatic criteria of tractability and applicability.

1.1. **Present Threats and an Old Analysis.** The looming possibility of drastic change to the climate equilibrium and the associated easy access resources from oceans and forests may be a threat to civilization as we know it. We still believe, or hope, that the expected duration of human society is much longer than the timescale of the decisions that affect this possibility. In the presence of decisions with extremely long-lasting effects and the associated mis-match of timescales, notions of patient preferences for long-run optimality become attractive criteria for decision problems that affect society, society being conceived of as an aggregate of the generations that make it up.

The analysis of intergenerational allocations and welfare has a long history, from which we take, as starting point, Sèbastien Le Prestre de Vauban's *Traité de la Culture des Forêts* (in e.g. [39] or [40]), written in the late 1600's. Vauban (Louis XIV's defense minister), noted that several aspects of the economics and ecology of forests complicate the analysis of good societal practices for foresty: first, forests, being a free or easy access resource, were systematically over-exploited; second, after replanting, forests start being productive in slightly less than 100 years but don't become fully productive for 200 years; and third, no private enterprise can have so long and multi-generational a time horizon. From these observations, Vauban concluded that the only institutions that could, and should, undertake such projects in society's interest were the government, in the form of the monarchy at the time, and the church.[1] The calculations behind his conclusion assumed that society would be around for at least the next 200 years to enjoy the net undiscounted benefits.

From Vauban's summing of undiscounted costs and benefits as a way of expressing concern for the welfare of future generations, we take the following: if τ represents the random time until the end of society and $\boldsymbol{u} = (u_0, u_1, u_2, \ldots)$ is a sequence of numerical measures of different generations, then

$$L_\tau(\boldsymbol{u}) := E\, \tfrac{1}{\tau+1} \textstyle\sum_{t=0}^{\tau} u_t \tag{1}$$

is a measure of society's welfare that, conditional on $\tau = t$, treats all generations equally. If $Prob(\tau \leq M)$ is small for large M, then $L_\tau(\cdot)$ is a measure of welfare for a patient society, one confident in its longevity. To capture the $10^7 : 1$ ratio between people in the present generation and people who may live in future generations cited above, we take limits of the $L_\tau(\cdot)$'s in (1) as $Prob(\tau \leq M)$ goes to 0 for all M.

This class of limits have representations as integrals against a subclass of strongly translation invariant, purely finitely additive measures, and this property will be crucial below. Having positive multiples of these limits as tangents is one of two offered characterizations of the concave social welfare functions studied here. The second characterization arises from a variant of the preferences used to study optimality while maintaining a concern for intergenerational equity.

1.2. **Intergenerational Ethics.** There is a long-held view that it is not acceptable to slight future generations (e.g. [37], [35]) and by taking limits, we guarantee, per Ramsey [36] that in our theory, "we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination." Ramsey's theory posited a "Bliss" point, reachable in finite time, and examined the optimality equations for this point. These optimality equations are less generally applicable than one might hope, Chakravarty [14] showed that, in quite simple examples, the divergence of infinite horizon integrals/sums can lead to the Ramsey optimality equations being satisfied by feasible plans with minimal long-run utility.

To overcome such problems, Weiszäcker [41] formulated the notion of overtaking optimality — a path is overtaking optimal if its accumulated benefits are eventually weakly higher than any other feasible path (see Brock [12] for an axiomatization of these preferences). The overtaking criterion captures a notion of patience and has been extensively used in studies of growth theory, especially the 'turnpike' properties

[1]As well as the government and the church, Vauban also argued for the possibility that, in some settings, a market-like solution to the various incentives problem might be found by making large enough stakes in a forest inheritable but not divisible.

of deterministic optimal paths. This literature was surveyed, extended and unified in McKenzie [31], and the first general existence result for overtaking optimality in convex problems was given by Brock and Haurie [11].

We are primarily interested in intergenerational choice in problems where uncertainty is central. The concavity of the social welfare functions we study has two roles: applied to distributions over streams of utility, it captures risk aversion; applied to streams of utility with no randomness it captures a preference for smoothing. The property of having tangents in the class described above can be formulated in terms of a variant of the overtaking criterion studied by Denardo and Miller [17], overtaking of the intergenerational *average* of accumulated benefits.

Our main axiom for these preferences is the following: for any two paths, $\boldsymbol{u}$ and $\boldsymbol{v}$, of measures of generational utilities, if the long-run average of the difference $\boldsymbol{u}-\boldsymbol{v}$ is above and strictly bounded away from 0, then $\boldsymbol{u}$ is strictly prefered to $\boldsymbol{v}$. Unlike the classical overtaking criterion, the average overtaking criterion is immune to bounded (and more) permutations of the "names" of the generations.[2] In the study of intergenerational allocations, indifference between permuted sequences of measures of generational well-being is called "equity," or "weak anonymity," or "intergenerational neutrality."

There is a set of results giving an "incompatibility" between this equity requirement and the Pareto principle.

1.3. **On the Pareto Criterion.** Diamond [18] showed that there is no continuous function on the space of sequences of utilities that is indifferent to uniformly bounded permutations and also satisfies a version of the Pareto principle. Basu and Mitra [7] and Fleurbaey and Michel [22] showed that this incompatibility extends to all real-valued social welfare functions, continuous or not, and Asheim [3] provides an extensive review of this literature. From the perspective of criteria based on average overtaking, this conflict arises from treating utility improvements accruing to null coalitions as being strict improvements in social welfare.

We normalize sequences of generational measures of well-being to be non-negative, suppose that they are bounded, and denote by $\mathbf{W}$ the result class of sequences.[3] An intergenerational allocation $\boldsymbol{u}$ in $\mathbf{W}$ strictly average overtakes an allocation $\boldsymbol{v}$ if $\liminf_T \frac{1}{T+1}\sum_{t=0}^{T}(u_t - v_t) > 0$ and average overtakes it if the lim inf is non-negative. The allocation $\mathbf{0} = (0,0,0,\ldots)$ can never strictly average overtake any element of $\mathbf{W}$, but there is an important subclass of allocations that it does weakly overtake, those with benefits accruing to a null coalition.

For a coalition of generations B and a stream of measures of well-being $\boldsymbol{u}$, $\boldsymbol{u} + r1_B$ represents giving a utility bump of r to every member of B. A coalition B is a null coalition if $\mathbf{0}$ average overtakes $r1_B$, equivalently, if $\boldsymbol{u}$ average overtakes $\boldsymbol{u} + r1_B$, and it is a non-null coalition if the allocation $r1_B$ strictly average overtakes $\mathbf{0}$. There is an intimate connection between this view of coalitions of generations and Hildenbrand's [24] foundational treatment of Pareto optimality with measure spaces of agents —

[2]If $\boldsymbol{u} = (5,8,0,8,0,\ldots)$ and $\boldsymbol{v} = (0,8,0,8,0,\ldots)$, then $\sum_{t\leq T}(u_t - v_t) \equiv 5$ so that $\boldsymbol{u}$ overtakes $\boldsymbol{v}$, but by permuting $\boldsymbol{v}$ to $\boldsymbol{v}^\pi = (8,0,8,0,8,0,\ldots)$, $\sum_{t\leq T}(u_t - v_t^\pi)$ is equal to -3 for $T = 0,2,4,\ldots$ and is equal to $+5$ for $T = 1,3,5,\ldots$. By contrast, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{v}^\pi$ all have the same long-run average, 4.

[3]See Blackorby et al. [9] for a discussion of this normalization.

there, as here, the appropriate Pareto criterion involves ignoring null coalitions and recognizing non-null coalitions.

Our main axiom requires that if $\boldsymbol{u}$ strictly overtakes $\boldsymbol{v}$ on average, then intergenerational preferences strictly prefer $\boldsymbol{u}$ to $\boldsymbol{v}$. This delivers Pareto responsiveness for our social welfare functions: for any non-null coalition B and any utility bump $r > 0$, $\boldsymbol{u} + r1_B$ strictly average-overtakes $\boldsymbol{u}$, hence $\boldsymbol{u} + r1_B$ is strictly prefered; for any null coalition, $\boldsymbol{u} + r1_B$ average-overtakes $\boldsymbol{u}$ which in turn average-overtakes $\boldsymbol{u} + r1_B$, hence $\boldsymbol{u}$ and $\boldsymbol{u} + r1_B$ are indifferent.

1.4. **Methodology.** Our methodological point of view is that one cannot fully understand a class of preferences without knowing their implications in the analysis of problems of interest. Axiomatic analyses are complementary, but not completely satisfactory.[4] For this reason, we study the implications of our social welfare functions in three classes of applications: general equilibrium models with infinite time horizons; Markovian decision problems; irreversible decisions with long-run implications; and the use of the Pareto criterion.

In general equilibrium models with an infinite time horizon, one typically makes assumptions — on preferences in exchange economy models and on preferences and technologies in production economy models — to guarantee that equilibria have prices that can be represented as integrals against countably additive probabilities. Bewley [8] identifies the requisite assumptions as "an asymptotic form of impatience," Brown and Lewis [13] gave a sharp characterization of the form of "myopia," and Araujo [2, Thm. 3] showed that combining the countably additive and purely finitely additive properties of the tangents in one preference relation leads to the non-existence of Pareto optima in these models. Working with preferences having tangents in the subclass of purely finitely additive measures identified above, we give an existence result as well as strong variants of the First and Second Welfare Theorems. In terms of dynasty interpretations of infinitely-lived economic agents, the different weights assigned to coalitions of future generations by our tangents give the tradeoffs between the welfare of different non-null subsets of the future generations within the dynasties.

General equilibrium theory is rarely focused on the analysis of externalities, our primary interest. To capture the one-way flow of externalities, both positive and negative, from earlier to later generations, we turn to a class of problems known as Markovian decision problems (MDPs). In these models: the utility at any given time is a function of the state of the system and the choice of present action; externalities are encoded as states of a system; and the present state and present choice of actions determine the distribution of future states. There is a vast literature on the existence and characterization of stationary solutions to MDPs that maximize the long-run

[4]This is also expressed in [5, p. 206], "By applying ethical criteria to concrete economic models, we learn about their consequences, and this may change our views about their attractiveness."

average reward, Arapostathis et al. [1] survey 200+ of the important works in this field.[5]

We will see that the patient welfare functions studied here are particularly well-behaved on the class of ergodic sequences, those that arise from the stationary policies that maximize the long run average payoff. Therefore, the set of utility functionals to which the MDP existence and characterization results apply include the concave, patients ones that we study. Further, the concavity of the social welfare functions allows for a more complete treatment of Markovian decision problems with irreversibilities, and this yields both a variant of the precautionary principle as well as a different foundation for sustainability analyses.

The insistence on social indifference to boons given to null subsets of a patient society has an implication that Chichilnisky [15] refers to as the "dictatorship of the future." The translation invariant, purely finitely additive character of the measures representing the tangents to our social welfare functions means that they are indifferent to, inter alia, ignoring the "mistreatment" of any finite number of generations, a "dictatorship of the future" if you will. Chichilnisky's proposal to alleviate this problem is to use the class of nonstationary preferences that Araujo [2, Thm. 3] used to show the non-existence of Pareto optimal allocations in general equilibrium models. We give an alternate, ethically sound method to circumvent the dictatorship, one for which a conditional equal treatment assumption can be made with no loss.

The existence of stationary solutions means that it is optimal for generations in similar situations to make similar decisions, and this is our conditional equal treatment assumption. With this, the only generations that can be "mistreated" are those that find themselves in situations that do not occur. For us, the existence of patient optima with conditional equal treatment properties outweighs the existence of alternative, non-stationary, ethically suspect optima.

1.5. **Outline.** (Outline TBD) Theorems are about our class of preferences. Propositions concern applications of our preferences to particular models.

2. Patient, Inequality Averse Social Welfare Functionals

The inequality averse intergenerational social welfare functions that we work with satisfy a strong form of patience/anonymity. One can develop this class of functions axiomatically or one can specify that the tangents have desirable properties. We begin with the axiomatics, but one could as easily start with the class of tangents.

2.1. **Notation and Setting.** Realizations of intergenerational streams of well-being belong to $\mathbf{W}$, the non-negative elements of ℓ_∞. By assumption, ℓ_∞ is equipped with the sup norm, $\|\boldsymbol{u}\| := \sup_{t\in\mathbb{N}_0} |u_t|$ where $\boldsymbol{u} = (u_0, u_1, \ldots)$, and the associated distance, $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$. The interior of $\mathbf{W}$ is denoted $\text{int}(\mathbf{W})$, and $\boldsymbol{u} \in \text{int}(\mathbf{W})$ if and only if $\inf_t u_t \geq r$ for some strictly positive r.

[5]Since the survey, Meyn [33] has used an operator theoretic framework to reduce the analysis of stability and approximability of infinite state space MDPs to the finite state space case, Borkar [10] has extended the generality of the convex analytic approach to MDPs, while Feinberg et al. [20] give a more general set of sufficient conditions for the existence and characterization of solutions.

To allow for the study of stochastic dynamic intergenerational problems, the domain for preferences is the mixture set, $\mathcal{M}$, of countably additive Borel probabilities on $\mathbf{W}$ having bounded support, $(\exists B)[p(\{\boldsymbol{u} \in \mathbf{W} : \|\boldsymbol{u}\| \leq B\}) = 1]$. By assumption, the set $\mathcal{M}$ is given the weak* topology.

2.1.1. *Resultants.* We study 'risk averse expected utility' preferences on $\mathcal{M}$. These are the preferences that can be represented by $p \succ q$ if and only if $\int_{\mathbf{W}} S(\boldsymbol{u})\, dp(\boldsymbol{u}) > \int_{\mathbf{W}} S(\boldsymbol{u})\, dq(\boldsymbol{u})$ for a continuous, concave intergenerational welfare function $S(\cdot)$. To give the axiomatic form of the inequality aversion contained in concavity, we will use the **resultant** or **expectation** of a $p \in \mathcal{M}$ is denoted $\boldsymbol{r}(p)$. This is the infinite dimensional version of the expectation of a vector in $\mathbb{R}^k$, and it is defined as the unique point $\mathbf{W}$ satisfying $\int \langle \boldsymbol{v}, \boldsymbol{y}\rangle\, dp(\boldsymbol{v}) = \langle \boldsymbol{r}(p), \boldsymbol{y}\rangle$ for all $\boldsymbol{y} \in \ell_1$.

2.1.2. *Tangents.* A concave function $S : \mathbf{W} \to [0, \infty)$ has a non-empty set of tangents, denoted $\mathbf{D}S(\boldsymbol{u})$, at any interior $\boldsymbol{u}$. The set of tangents determine properties of $S(\cdot)$ — if $L \in \mathbf{D}S(\boldsymbol{u})$, then concavity implies that for any $\boldsymbol{v} \in \mathbf{W}$, $S(\boldsymbol{u}) + L(\boldsymbol{v} - \boldsymbol{u}) \geq S(\boldsymbol{v})$. Knowing the class of differences, $\boldsymbol{v} - \boldsymbol{u}$, for which each $L(\cdot)$ is non-negative gives general property of $S(\cdot)$. '

Tangents are continuous linear functionals on ℓ_∞, and as such, each has an integral representation, $L(\boldsymbol{u}) = \int_{\mathbb{N}_0} u_t\, d\gamma(t)$, denoted $\langle \boldsymbol{u}, \gamma\rangle$, where γ is a signed, finitely additive measure on $\mathbb{N}_0$ having finite (variation) norm, $\|\gamma\| := \sup_{\|\boldsymbol{u}\| \leq 1} |L(\boldsymbol{u})| < \infty$. A net (generalized sequence) γ_α of measures $\mathbb{N}_0$ converges in the weak*-topology to γ if $\langle \boldsymbol{u}, \gamma_\alpha\rangle \to \langle \boldsymbol{u}, \gamma\rangle$ for all $\boldsymbol{u} \in \ell_\infty$. By Alaoglu's theorem, the weak* closure of a norm-bounded set of measures on $\mathbb{N}_0$ is weak* compact.

2.1.3. *Permutations.* We will define the patience of a social welfare functional using indifference to a class of permutations. The set of integers, negative and non-negative is denoted $\mathbb{Z}$ and defined as $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. A **permutation** is a 1-to-1 function $\pi : \mathbb{N}_0 \to \mathbb{Z}$ that is onto $\mathbb{N}_0$. Given $\boldsymbol{u} = (u_0, u_1, u_2, \ldots) \in \ell_\infty$ and a permutation π, define $\boldsymbol{u}^\pi$ as $(u_{\pi^{-1}(0)}, u_{\pi^{-1}(1)}, u_{\pi^{-1}(2)}, \ldots)$. A permutation is **finite** if $\pi(T) = T$ for all but finitely many generations, it is **bounded** if $|\pi(T) - T|$ is uniformly bounded, and it is an $\mathcal{O}(T)$ **permutation** if $\limsup_T \frac{|\pi(T)-T|}{T+1} = 0$. This last class of permutations are closely related to a variant of the overtaking criterion.

2.1.4. $\mathcal{O}(T)$*-Overtaking.* For $\boldsymbol{u}, \boldsymbol{v} \in \ell_\infty$, $\boldsymbol{u}$ $\mathcal{O}(T)$**-overtakes** $\boldsymbol{v}$, written $\boldsymbol{u} \succsim_{\mathcal{O}(T)} \boldsymbol{v}$, if

$$\liminf_{T\to\infty} \tfrac{1}{T+1} \textstyle\sum_{t=0}^{T} (u_t - v_t) \geq 0, \tag{2}$$

and $\boldsymbol{u}$ **strictly** $\mathcal{O}(T)$**-overtakes** $\boldsymbol{v}$, written $\boldsymbol{u} \succ_{\mathcal{O}(T)} \boldsymbol{v}$, if

$$\liminf_{T\to\infty} \tfrac{1}{T+1} \textstyle\sum_{t=0}^{T} (u_t - v_t) > 0. \tag{3}$$

The classical definition of $\boldsymbol{u}$ overtaking $\boldsymbol{v}$ requires the stronger, unaveraged condition $\liminf_{T\to\infty} \sum_{t=0}^{T} (u_t - v_t) \geq 0$, but the classical strict overtaking criterion, $\sum_{t=0}^{T} (u_t - v_t) \geq \epsilon$ for some strictly positive ϵ and for all sufficiently large T is not sufficient for $\mathcal{O}(T)$-strict overtaking.[6] The $\mathcal{O}(T)$-overtaking criterion is, for obvious reasons, called

[6]See [31] for a survey of models of long-run optimal economic growth using the overtaking criterion, [30] and [22] for axiomatic treatments of patience, [3] for a survey.

"average overtaking" in the literature, see e.g. [17]. Lemma 1 (below) gives our reason for using this name.

2.1.5. *Ergodicity.* The **ergodic** subclass of ℓ_∞ is denoted **Erg** and defined as the set of $\boldsymbol{u} \in \ell_\infty$ for which the long run average, $\mathbf{lra}(\boldsymbol{u}) := \lim_{T\to\infty} \frac{1}{T+1}\sum_{t=0}^{T} u_t$, exists. The $\mathcal{O}(T)$ permutations, overtaking and the ergodic class are tightly related.

Lemma 1. *If π is an $\mathcal{O}(T)$ permutation, then for all $\boldsymbol{u} \in \ell_\infty$, $\boldsymbol{u} \succsim_{\mathcal{O}(T)} \boldsymbol{u}^\pi \succsim_{\mathcal{O}(T)} \boldsymbol{u}$, and for all $\boldsymbol{u} \in \mathbf{Erg}$, $\mathbf{lra}(\boldsymbol{u}) = \mathbf{lra}(\boldsymbol{u}^\pi)$.*

There is useful geometry behind Lemma 1. Define $N = \{\boldsymbol{z} \in \ell_\infty : \mathbf{lra}(\boldsymbol{z}) = 0\}$ and for $\boldsymbol{v} \in \mathbf{W}$, and define $(\boldsymbol{v} + N) = \{\boldsymbol{v} + \boldsymbol{z} : \boldsymbol{z} \in N\}$. We have $\boldsymbol{u} \succsim_{\mathcal{O}(T)} \boldsymbol{v}$ iff $\boldsymbol{u} \geq (\boldsymbol{v} + N)$ and $\boldsymbol{u} \succ_{\mathcal{O}(T)} \boldsymbol{v}$ iff for some $r > 0$, $\boldsymbol{u} + r1_{\mathbb{N}_0} \geq (\boldsymbol{v} + N)$.

2.2. **An Axiomatic Formulation.** Our assumptions on social preferences are given in terms of a binary relation $\succ$ on $\mathcal{M}$. We always assume that $\succ$ is **asymmetric**, that is, if $p \succ q$, then it is not the case that $q \succ p$. Define $\sim$ and $\succsim$ on $\mathcal{M}$ by $p \sim q$ if neither $p \succ q$ nor $q \succ p$, and $p \succsim q$ by $p \succ q$ or $p \sim q$. As usual, $\succ$ is **negatively transitive** if for all $p, q, r \in \mathcal{M}$, $[p \succ r] \Rightarrow [\,[p \succ q] \vee [q \succ r]\,]$, and we call a negatively transitive $\succ$ an **asymmetric weak order**.

2.2.1. *Desiderata.* We have two desiderata for our intergenerational preferences: they should be patient, defined as indifference to $\mathcal{O}(T)$ permutations; and they should be exactly Pareto, increases in the utility of null coalitions has no effect on intergenerational preferences while uniform increases accruing to non-null coalitions have a strictly positive effect.

For $\boldsymbol{u} \in \mathbf{W}$ and $q \in \mathcal{M}$, we write "$\boldsymbol{u} \succ q$" for $p \succ q$ where $p(\{\boldsymbol{u}\}) = 1$, with the same convention for "$\succsim$" and "$\sim$." In particular, we write $\boldsymbol{u} \succ$ (resp. $\succsim$, resp. $\sim$)$\boldsymbol{v}$ for $p \succ$ (resp. $\succsim$, resp. $\sim$)q where $p(\{\boldsymbol{u}\}) = 1$ and $q(\{\boldsymbol{v}\}) = 1$. In this fashion, we restrict $\succ$ to $\mathbf{W}$ by identifying points $\boldsymbol{u}$ in $\mathbf{W}$ with the associated point masses/Dirac measures, $\delta_{\boldsymbol{u}}$, in $\mathcal{M}$.

Definition 2.1. *A preference relation $\succ$ on $\mathbf{W}$ is* ***patient*** *if for all $\mathcal{O}(T)$ permutations π and all $\boldsymbol{u} \in \mathbf{W}$, $\boldsymbol{u} \sim \boldsymbol{u}^\pi$.*

The vector $\mathbf{0} = (0, 0, 0, \ldots)$ can never strictly overtake any element of $\mathbf{W}$. There is an important subclass that $\mathbf{0}$ does overtake. We say that $B \subset \mathbb{N}_0$ is a **null coalition** if $\mathbf{0} \succsim_{\mathcal{O}(T)} 1_B$ and it is a **non-null coalition** if $1_B \succ_{\mathcal{O}(T)} \mathbf{0}$. The appropriate Pareto criterion for patient preferences involves ignoring null coalitions and recognizing non-null coalitions.

Definition 2.2. *A preference relation $\succ$ on $\mathbf{W}$ is* ***exactly Pareto*** *if*

(a) *for all $\boldsymbol{u} \in \mathbf{W}$, all null coalitions B and all $r > 0$, $\boldsymbol{u} + r1_B \sim \boldsymbol{u}$, and*
(b) *for all $\boldsymbol{u} \in \mathbf{W}$, all non-null coalitions B and all $r > 0$, $\boldsymbol{u} + r1_B \succ \boldsymbol{u}$.*

2.2.2. *Axioms.* We will invoke the following axioms on $\succ$.

Axiom I. **Weak Order**. $\succ$ is an asymmetric weak order.

Axiom II. **Independence**. For all $p, q, r \in \mathcal{M}$ and all $\alpha \in (0, 1)$, if $p \succ q$, then $\alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$.

Axiom III. **Continuity**. For all $q \in \mathcal{M}$, the sets $\{p \in \mathcal{M} : p \succ q\}$ and $\{p \in \mathcal{M} : p \prec q\}$ are open.

Axiom IV. **Risk and Inequality aversion**. For any $p \in \mathcal{M}$, $\boldsymbol{r}(p) \succsim p$.

Axiom V. **Respect for overtaking**. $[\boldsymbol{u} \succ_{\mathcal{O}(T)} \boldsymbol{v}] \Rightarrow [\boldsymbol{u} \succ \boldsymbol{v}]$.

The first three axioms are standard in the expected utility theory of choice under uncertainty: Axiom I is the usual ordering assumption for preference relations; Axiom II is the "linearity in probabilities" assumption for the existence of an expected utility representation for $\succ$; and Axiom III guarantees that the representation is continuous. For expected utility theory, Axiom IV guarantees risk aversion in the form of concavity of the expected utility function, here the concavity of the social expected utility function implies that social preferences are inequality averse on $\mathbf{W}$ and risk averse on $\mathcal{M}$. As noted, Axiom V loosens the classical overtaking criterion.

2.2.3. *Representation.* The next result gives the representation theorem for preferences satisfying the Axioms.

Theorem A. *A social ordering $\succ$ satisfies Axioms* I-V *if and only if there exists a continuous, concave $S : \mathbf{W} \to [0, \infty)$ such that $[p \succ q] \Leftrightarrow [\int S(\boldsymbol{u})\, dp(\boldsymbol{u}) > \int S(\boldsymbol{u})\, dq(\boldsymbol{u})]$ with $S(\cdot)$ satisfying the following properties,*

(1) *it is patient, for all $\boldsymbol{u} \in \mathbf{W}$ and all $\mathcal{O}(T)$ permutations, $S(\boldsymbol{u}) = S(\boldsymbol{u}^\pi)$, and*
(2) *it is exactly Pareto, for all $\boldsymbol{u} \in \mathbf{W}$ and all $r > 0$, if B is a null coalition, then $S(\boldsymbol{u} + r1_B) = S(\boldsymbol{u})$, and if B is a non-null coalition, then $S(\boldsymbol{u} + r1_B) > S(\boldsymbol{u})$.*

2.3. **Formulation by Tangents.** Let τ be a random variable with $\sum_{T \in \mathbb{N}_0} P(\tau = T) = 1$. If we interpret τ as the random time at which society ends and $P(\tau < M) < \epsilon$ for large M and small ϵ, then the mapping $\boldsymbol{u} \mapsto L_\tau(\boldsymbol{u}) := E\, \frac{1}{\tau+1} \sum_{t=0}^{\tau} u_t$ is a measure of welfare for a patient society, one confident in its longevity. The mappings $L_\tau(\cdot)$ are also continuous, linear, positive, and have norm 1. Therefore, for each distribution of τ, there exists a unique probability η_τ such that $L_\tau(\boldsymbol{u}) = \langle \boldsymbol{u}, \eta_\tau \rangle$ for all $\boldsymbol{u}$.

2.3.1. *The Class* $\mathbb{V}$. We denote by $p\mathbb{V}$ the set of weak* accumulation points of the linear functionals L_τ as $P(\tau < M) \to 0$ for all M,

$$p\mathbb{V} = \bigcap \{\mathrm{cl}\,(\{\eta_\tau : Prob(\tau \leq M) \leq \epsilon\}) : M \in \mathbb{N}_0, \epsilon > 0\}. \tag{4}$$

The class $p\mathbb{V}$ is non-empty (by the finite intersection property of the class of sets in equation (4)), as well as compact and convex.

A concave function on $\mathbf{W}$ has a non-empty set of tangents at every interior point of $\mathbf{W}$. Our tangents will belong to the following class.

Definition 2.3. *The class $\mathbb{V}$ is the closed convex cone generated by $p\mathbb{V}$.*

2.3.2. *Interior Points and* $\mathbb{V}$*-Concavity.* Continuous concave functions may fail to have tangent functions at the boundary points of their domain. The Cobb-Douglas functions $f(x_1, x_2) = x_1^\alpha x_2^{(1-\alpha)}$ for $0 < \alpha < 1$ and $x_1, x_2 \geq 0$ are a case in point. These considerations indicate the need form care in formulating a subset of the concave functions on $\mathbf{W}$ in terms of the existence of tangents.

Definition 2.4. *A function* $S : \mathbf{W} \to [0, \infty)$ *is* $\mathbb{V}$***-concave on*** $\text{int}(\mathbf{W})$ *if it is continuous and for all* $\boldsymbol{u} \in \text{int}(\mathbf{W})$*, the set of tangent functions at* $\boldsymbol{u}$ *is a closed, non-empty, norm bounded set of strictly positive elements of* $\mathbb{V}$*. A function is* $\mathbb{V}$***-concave*** *if the same condition holds for all* $\boldsymbol{u} \in \mathbf{W}$*.*

2.3.3. *Axiomatic Properties Arising from Tangents.* The following relates $\mathbb{V}$-concave functions and preferences satisfying our Axioms.

Theorem B. *If* $S : \mathbf{W} \to [0, \infty)$ *is* $\mathbb{V}$*-concave, then the expected utility preferences it represents satisfy Axioms* I-V*, and if a continuous concave* $S : \mathbf{W} \to [0, \infty)$ *represents preferences satisfying Axioms* I-V*, then it is* $\mathbb{V}$*-concave on* $\text{int}(\mathbf{W})$*.*

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{Erg}$, $\boldsymbol{u} \succ_{\mathcal{O}(T)} \boldsymbol{v}$ if and only if $\mathbf{lra}(\boldsymbol{u}) > \mathbf{lra}(\boldsymbol{v})$. This leads to the social welfare functions having a particularly simple structure when restricted to the class $\mathbf{Erg} \subset \mathbf{W}$.

Corollary B.1. *If a social ordering* $\succ$ *satisfies Axioms* I-V*, then there exists an increasing, concave* $\varphi : [0, \infty) \to [0, \infty)$ *such that for all* $p, q \in \mathcal{M}$ *satisfying* $p(\mathbf{Erg}) = q(\mathbf{Erg}) = 1$*,* $p \succ q$ *if and only if* $\int \varphi(\mathbf{lra}(\boldsymbol{v}))\, dp(\boldsymbol{v}) > \int \varphi(\mathbf{lra}(\boldsymbol{v}))\, dq(\boldsymbol{v})$*.*

This will play a large role in our analysis of optimization problems with ergodic solutions. In strongly ergodic models, it is a probability 1 event under the optimal policy that all paths yield the same long run average. In such models, Corollary B.1 implies that one maximizes the long run average if and only if one maximizes the expected value of $S(\cdot)$ for every social welfare function satisfying Axioms I-V. In models where optimal policies give a distribution over the long run average, the concavity of $\varphi(\cdot)$ determines how the social welfare function trades off different risky long run prospects.

3. Applications

We begin with a brief treatment of patient preferences in general equilibrium models with bounded sequence commodity spaces. We show that with our preferences, competitive equilibria exist and the First and Second Welfare Theorems hold. The contrast between this result and the results on the need for "myopic" preferences for the existence of Pareto optima clarifies some aspects of the structure of patient preferences.

General equilibrium theory has a difficult time with externalities, and intergenerational externalities are our major focus. Our second set of applications are models in which externalities are encoded as the states in a Markovian decision problem and present actions crucially influence the distribution of future states. In the context of models with a single long-run average payoff, we have the strong result that all of our patient preferences have the same optimality equations.

Irreversible decisions are the *ne plus ultra* of externalities in intergenerational dynamic problems. This is our third class of applications. For patient preferences, optimal policies in the presence of irreversible decisions embody a version of the precautionary principle.

3.1. **Patience and Myopia in General Equilibrium Theory.** Bewley [8] studies general equilibrium models with commodity spaces that are uniformly bounded sequences of non-negative consumption vectors in $\mathbb{R}^k$. Bewley's Theorem 1 gives sufficient conditions for the existence of a competitive equilibrium with prices that positive, finitely additive $\mathbb{R}^k$-valued measures, while his Theorems 2 and 3 study conditions for the existence of equilibria with prices in the set of summable $\mathbb{R}^k$-valued sequences. Brown and Lewis [13] and Araujo [2] study the same class of general equilibrium models and give results, respectively, on the role of/need for myopia in preferences for the existence of Pareto optima.

3.1.1. *Notation and Assumptions.* Let ℓ_∞^k denote the set of $\boldsymbol{x} \in (\mathbb{R}^k)^{\mathbb{N}_0}$ such that $\sup_{t\in\mathbb{N}_0} \|x_t\| < \infty$. Feasible consumptions and endowments belong to $\mathbf{W}^k := \{\boldsymbol{x} \in \ell_\infty^k : \boldsymbol{x} \geq 0\}$. Each agent/dynasty i in a finite set I, has an endowment $\boldsymbol{\omega}_i \in \mathbf{W}^k$. We give $\mathbf{W}^k$ the sup norm.

Assumption A. *Each* $\boldsymbol{\omega}_i$ *is an interior point of* $\mathbf{W}^k$*.*

This implies that $\boldsymbol{\omega} := \sum_i \boldsymbol{\omega}_i$ is also an interior point of $\mathbf{W}^k$.

Feasible consumption streams are vectors $(\boldsymbol{x}_i)_{i\in I}$ with each $\boldsymbol{x}_i \in \mathbf{W}^k$, and $\sum_i \boldsymbol{x}_i \leq \boldsymbol{\omega}$. The preferences of i are given by a utility function $\boldsymbol{x} \mapsto U_i(\boldsymbol{x})$. We say that a $\mathbb{V}$-concave $S : \mathbf{W} \to \mathbb{R}$ is **$\mathbb{V}$-concave at the boundary** if $\mathbf{D}S(\boldsymbol{u}) \neq \emptyset$ for all $\boldsymbol{u} \in \mathbf{W}$ (not just all $\boldsymbol{u} \in \mathrm{int}(\mathbf{W})$).

Assumption B. $U_i(\boldsymbol{x}) = S_i(\boldsymbol{u}_i(\boldsymbol{x}))$ *where:* $\boldsymbol{u}_i = (u_{i,0}(x_0), u_{i,1}(x_1), \ldots)$*;* $S_i(\cdot)$ *is* $\mathbb{V}$*-concave at the boundary; and the* $u_{i,t}$*,* $t \in \mathbb{N}_0$ *are uniformly bounded, continuous, concave, strictly increasing period utility functions on* $[0, \boldsymbol{\omega}_t] \subset \mathbb{R}_+^k$ *with* $u_{i,t}(0) = 0$ *and* $\liminf_t\{\max u_{i,t}(\boldsymbol{\omega}_{i,t})\} > 0$*.*

It is the aggregation of period utilities by a $\mathbb{V}$-concave utility function that makes the preferences patient. The assumption that the range of the utilities $u_{i,t}$ does not disappear guarantees that the agents are non-trivial parts of the economy.

Definition 3.1. *An* ***equilibrium*** *for an exchange economy model* $\mathcal{E} = (\boldsymbol{\omega}_i, U_i)_{i\in I}$ *is a feasible consumption stream,* $(\boldsymbol{x}_i)_{i\in I}$ *and a price* π *in the topological dual of* ℓ_∞^k *such that for all* $i \in I$ *and all* $\boldsymbol{y} \in W^k$*,*

$$[U_i(\boldsymbol{y}) > U_i(\boldsymbol{x})] \Rightarrow [\langle \boldsymbol{y}, \pi\rangle > \langle \boldsymbol{x}, \pi\rangle]. \tag{5}$$

3.1.2. *Equilibrium Existence and the Welfare Theorems.* The following equilibrium existence result follows directly from Bewley [8, Theorem 1] and does not require $\mathbb{V}$-concavity at the boundary.

Proposition 1. *Under Assumptions* A *and* B*, an equilibrium exists.*

For finite dimensional production and exchange economies without externalities, the First Welfare Theorem states that competitive equilibria are Pareto optimal, and the Second Welfare Theorem states that all Pareto optima are competitive equilibria after appropriate re-arrangement of the initial endowments. Using preferences with tangents that are integrals against convex combinations of countably and purely finitely additive measures Araujo [2, Theorem 3] shows that Pareto optimal allocations may not exist.

Such preferences are not in the class of patient preferences we are using here, and this difference leads to very different results.

Because the tangents of $\mathbb{V}$-concave functions have representations as integrals against purely finitely measures, the price vector in Proposition 1 must be purely finitely additive. Our next result shows that, for such preferences, we have the First and Second Welfare Theorems. To complete the analysis, we present two examples. The first, due to Araujo [2], shows that for preferences with tangents that are a mix of countably and purely finitely additive measures, ϵ-Pareto optimal equilibria may not exist. The second example draws a parallel with optimal allocations of uncertainty: in the allocation of risk, if (say) agents i and j assign probability 1 and probability 0 to an event E, then the optimal allocations give i all of the consumption in the event E and give agent j all of the consumption in the event E^c; if dynasties i and j put mass 1 and mass 0 on the coalition E, we have the same pattern.

Proposition 2. *Under Assumptions* A *and* B*, every equilibrium is Pareto optimal, and every Pareto optimal allocation is an equilibrium for an appropriate re-arrangement of the initial endowments.*

The tangents to $\mathbb{V}$-concave preferences have representations as integrals against purely finitely measures. This is crucial to the existence of Pareto optimal points as the following example demonstrates.

Example 3.1 (Araujo)**.** *For* $I = \{1, 2\}$*, let* $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \mathbf{1}$ *be the constant endowment of one unit of the single good. For an allocation* $\boldsymbol{x}$*, let* $U_1(\boldsymbol{x}) = x_1 + \langle \boldsymbol{x}, \eta \rangle$ *where* η *is a non-negative, purely finitely additive measure that satisfies* $\langle \boldsymbol{\omega}_1, \eta \rangle > 1$*. Let* $U_2(\boldsymbol{y}) = \langle \boldsymbol{y}, \gamma \rangle + \langle \boldsymbol{y}, \eta \rangle$ *where* γ *is countably additive and strictly positive, say* $\gamma_t = (1 - \beta)\beta^t$ *so that* $\langle \boldsymbol{y}, \gamma \rangle = (1 - \beta) \sum_{t=0}^{\infty} y_t \beta^t$*.*

Suppose now that $(\boldsymbol{x}, \boldsymbol{y})$ *is an individually rational Pareto optimal allocation. Because* $\gamma_t > 0$ *for all* t*, Pareto optimality implies that* $x_{1,t} = 0$ *for all* $t \geq 2$*. Feasibility implies that* $x_1 \leq 2$*. Combining,* $U_1(\boldsymbol{x}) \leq 2 + 0$*. However,* $U_1(\boldsymbol{\omega}_1) = 1 + \langle \boldsymbol{\omega}_1, \eta \rangle > 2$*, a contradiction.*

Continuous linear preferences on $\mathbf{W}^k$ can be decomposed into a countably additive part and a purely finitely additive part. For any $\epsilon > 0$, the value of the countably additive parts are determined on $\{0, 1, \ldots, T\}$ for sufficiently large T, while the value of the purely finitely additive part is entirely determined on $\{T+1, T+2, \ldots\}$. For linear preferences, this implies that ϵ-individually rational and Pareto optimal allocations exist. We conjecture that the same is true for concave utility functions.

Example 3.2. *Let the agents and their endowments be as in the previous example. Suppose that* $U_i(\boldsymbol{y}) = \langle \boldsymbol{y}, \gamma_i \rangle$ *where* (γ_1, γ_2) *is any accumulation point of the set* $(Unif_{0,T^!}, Unif_{0,(T^!)^2})$*. It is possible to demonstrate*[7] *a set of generations,* E*, such that* $\langle 1_E, \gamma_1 \rangle = 1$ *and* $\langle 1_E, \gamma_2 \rangle = 0$*. All Pareto optimal allocations must assign dynasty* 1 *the entire economy's endowment in* E *and must assign dynasty* 2 *the entire economy's endowment in* E^c*.*

[7] General form of this result to be added to appendix.

3.2. **Patience in MDPs.** Example of MDP without overtaking optimal, but having average overtaking optimal solution from [17]. Simpler more general proof of existence.

3.2.1. *A Stark Model.* Suppose that the world's ecosystem can be in one of two states, damaged or undamaged: in the damaged state, the seas, forests and the biota that survive are unable to produce oxygen and foodstuff in the amounts humans are evolved to need: in the undamaged state, the seas and forests are able to produce oxygen concentrations supporting life as we currently know it. In the undamaged state, $x = G$, society chooses the transition probability, r to the damaged state, $x = B$, with $0 < \underline{r} \leq r \leq \overline{r} < 1$. The expected utility of choosing r is $u_G(r)$, and higher choices of r lead to a higher expected utility for the present generation, $u'_G(r) > 0$. In a parallel fashion, in the damaged state, society chooses the transition probability, s to the undamaged state with $0 < \underline{s} \leq s \leq \overline{s} < 1$, and higher choices of s lead to lower expected utility of the present generation, $u'_B(s) < 0$. A generation in a good state can sacrifice some present utility in order to lower the future probability of disastrous climate changes, a generation in a bad state must sacrifice some of their present utility in order to raise the the future probability of a return to a better world.

Starting from the present, $t = 0$, a **policy**, w, chooses an r and a s as a function of the present state. This choice gives rise to a Markov process, $\Phi^w = (\Phi^w_t)_{t\in\mathbb{N}_0}$, taking either the value G or B. A policy w is **S-optimal** if it maximizes $E\left(S(\Phi)|\Phi_0 = x\right)$ for each x. The easiest way to develop the optimality equations uses the expression of the Markov process Φ^w as a sequence of i.i.d. (independent and identically distributed) **sojourns**, here adapted to a two-state process.

If α is an atom of a Markov process, then from any recurrent state, $x \neq \alpha$, the process will find its way back to α in a random time with finite expectation. The time path of the process during periods between a departure from α and a first return to α is called a **sojourn from** α because the process is temporarily staying someplace other than α. In a the present model, either state can be regarded as an atom.

3.2.2. *The Long-Run Average Optimality Equations.* Because the probabilities r and s are interior, any policy $w = (r, s)$ leads to the process Φ^w having a well-defined long-run average, $\rho = \rho^w := E\ \lim_{T\to\infty} \frac{1}{T+1} u(\Phi^w_t)$.

Let E^w_x be the expectation operator when using a policy w and starting from $\Phi_0 = x$. To characterize an optimal w, we need to find numbers ρ, $h(G)$, and $h(B)$ such that

$$\rho + h(G) = \max_{r\in[\underline{r},\overline{r}]} \left[u_G(r) + E^w_G h(\Phi_1)\right], \text{ and} \tag{6}$$

$$\rho + h(B) = \max_{s\in[\underline{s},\overline{s}]} \left[u_B(s) + E^w_B h(\Phi_1)\right]. \tag{7}$$

In parallel with the Bellman equations for a discounted stochastic dynamic programming problem, the problems in (6) and (7) maximize a present utility plus an expected value of where the process will arrive one step into the future. The difference is the replacement of the discounted value function with the expected deviation from the long-run average during the sojourn.

In this model, $E^w_G h(\Phi_1) = (1-r)h(G) + rh(B)$ and $E^w_B h(\Phi_1) = sh(G) + (1-s)h(B)$. Therefore, the first order equations (FOCs) for an interior solution to (6) and (7) are

$$u'_G(r) = [h(G) - h(B)] \text{ and } u'_B(s) = [h(B) - h(G)]\,. \tag{8}$$

In this particular model, $h(G) = (u_G(r) - \rho) \cdot E\,\tau_B$, $E\,\tau_B = (1-r)/r$, $h^w(B) = (u_B(s) - \rho) \cdot E\,\tau_G$, $E\,\tau_G = (1-s)/s$ because τ_B and τ_G are geometric distributions. We assume that the payoffs in the good state are higher than those in the bad state, which leads to $u_G(r) - \rho > 0 > u_B(s) - \rho$. When the expected values of the times until transitions between states are large, we expect $[h^w(G) - h^w(B)]$ to be a large positive number. From this, one expects the right-hand sides of these to be too large (in absolute value) for interior solutions. This would imply that the optimal policy is as careful as possible in the good state and works as hard as possible to return to the good states when in the bad state, that is, $w^* = (\underline{r}, \overline{s})$.[8]

By contrast, consider the policies that myopically maximize utility in the damaged state, the policies of the form $(r, \underline{s})$. Any such policy maximizes the expected value of the Rawlsian social welfare function, $S_{Rawls}(\boldsymbol{u}) = \liminf_t u_t$ because $S_{Rawls}(u) = u_B(s)$ with probability 1 for *any* policy. It is the failure of the Rawlsian ordering, $S_{Rawls}(\cdot)$, to respect the Pareto ordering that is at work here. With probability 1, along any path, there is a non-negligible portion of the generations in the good state, their utility does not enter in the Rawlsian ordering, and this precludes making tradeoffs between the welfare of different proportions of the generations that make up society.

3.2.3. *The Long-Run $S(\cdot)$-Optimal Equations.* As we argued above, one does not fully understand a class of preferences or the set of assumptions behind them until one knows their implications in the analysis of problems of interest. One of the two main results in this paper shows that a policy for a Markovian decision problem is S-optimal for a $\mathbb{V}$-concave societal welfare function if and only if the policy maximizes the long-run average utility. For the analysis of Markovian models, this means that we can appeal to all of the techniques that have been developed to insure the existence of solutions to equations such as (6) and (7) [32, See especially] to find S-optimal policies. Here, we give some intuitions for this result.

Following a policy $w = (r, s)$, gives rise, with probability 1, to a sequence of utilities $u = (u_0, u_1, u_2, \ldots)$ that is ergodic/Cesaro summable, $\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} u_t$ exists. Further, with probability 1, the limit is the same along all paths, and it was denoted $\rho = \rho^w$ above. Every L in the class $\mathbb{V}$ has the property that for all u, u' with the same Cesaro sum/long run average, $L(u) = L(u')$. Putting the pieces together, with probability 1, all of tangent functionals treat all of the realizations the same. This in turn means that, restricted to a probability 1 set of realizations, $S(u) = \varphi(\rho)$ where $\varphi(\cdot)$ is concave and strictly increasing. As a result, maximizing $\rho = \rho^w$ is necessary and sufficient for finding an S-optimal policy.

[8] In this simple model, it is easy to verify that each $w = (r, s)$ gives rise to the long-run average $\rho = \rho^w(r, s) = \frac{s}{r+s} u_G(r) + \frac{r}{r+s} u_B(s)$. Verifying that the FOCs $\partial\rho/\partial r = 0$ and $\partial\rho/\partial s = 0$ reduce to (8) is routine.

3.3. **Patience and Sustainability.** Following Krautkraemer [27], sustainability "basically gets at the issue of whether or not future generations will be at least as well off as the present generation." In systems potentially subject to large shocks such as dramatic climate change or collapses of civilizations, a definition must take into account the variability. One can do that using stochastic state variables as in MDPs, and in that context, our patient preferences suggest that "at least as well off as the present generation" should be replaced by "at least as well of in expectation as past generations have been." By handling variability, the definitions of patience and Pareto optimality advanced in this paper provide normative justifications of sustainability complementary to those in the literature. A careful analysis of these throws further light on our definitions.

Previous analyses of patience and intergenerational equity have centrally used weak anonymity, understood as social indifference to finite or bounded permutations. This property links patience to purely finitely additive measures on $\mathbb{N}_0$, the space of generations. A measure η on $\mathbb{N}_0$ is purely finitely additive if and only if for all $\boldsymbol{u} = (u_0, u_1, \ldots) \in \mathbf{W}$ with $u_t \to 0$, $\int_{\mathbb{N}_0} u_t \, d\eta(t) = 0$. If the u_t are strictly positive, this means that patient and equitable preferences must be indifferent to the addition of $\boldsymbol{u}$. This violates the strong Pareto optimality criterion, the one that insists that vanishingly small increases in the well-being of what we call null coalitions must be respected.

Domain restrictions have been used to circumvent such difficulties. For example, in the Dasgupta and Heal [16] and Solow [38] model of capital accumulation and resource depletion, discounted utilitarianism, ranking elements in $\mathbf{W}$ using $L_\beta(\boldsymbol{u}) := (1-\beta)\sum_{t\in\mathbb{N}_0} u_t\beta^t$, has the property that for all $\beta < 1$, the long-run well-being converges to 0. One relatively simple approach is to restrict attention to sustainable paths of generational well-being, i.e. the non-decreasing paths, and to look within that subset for paths that maximize the discounted sum of intergeneration well-being, or that satisfy other equity conditions from the social choice literature.

In the first approach, the optima among the non-decreasing sequences converge to steady state levels of well-being that are higher for discount factors β closer to 1. Taking limits of the $L_\beta(\cdot)$ as $\beta \uparrow 1$ gives a subset of $p\mathbb{V}$. The bounded monotonic sequences belong to **Erg**, and restricted to **Erg**, the rankings of all of the preferences in $\mathbb{V}$ agree, picking the stream with the highest long-run average, i.e. the Golden Rule optimum. In the second approach, a minimal set of ethical conditions for judging time paths of intergenerational utility is the Suppes-Sen Grading principle. This is a very incomplete ranking, but, under some mild conditions on the technology linking the well-being of generations, Asheim et al. [4] show that the set of Suppes-Sen maximal paths are the set of nondecreasing, technologically efficient paths. Such paths are the sustainable ones in the sense of non-diminishing paths of well-being, which means that the unsustainable paths are not maximal, hence cannot be justified.

If one takes seriously the possibility of large intergenerational swings in utility, then there are drastically non-convex technologies that make inequality socially optimal with out utility functions. For example, if the generational utilities can be either 1 or 5 or 2, and the technology is such that: every generation with utility 5 must be followed by a generation with utility of 1; every 1 can be followed either by a 2 or a 5; and every 2 can be followed by a 2 or a 1. The feasible path $(1, 5, 1, 5, 1, 5, \ldots)$ is much more unequal but has a higher long-run average than the feasible path $(1, 2, 2, 2, \ldots)$.

3.4. **Patience and Irreversibility.** We look at a simple model of a fishery where species extinction is possible, but avoidable at a cost. We will see the distinction between the long-run behavior of discount optimal policies and optimal policies for patient preferences at work.

Suppose that there are two states, 1 and 2, that the sets of available actions are $A(1) = [0, 1]$, $A(2) = \{1\}$, that utilities $u(a, s)$ are given by $u(1, 2) = 0$, that using a higher action in state 1 is more profitable, $\partial u(a, 1)/\partial a > 0$, but that higher actions make it more likely that you will move to state 2 and that state 2 is absorbing, $p_{2,2}(1) = 1$.

Specifically,

- $u(a, s = 1) = 10(1 + \sqrt{a})$,
- $p_{1,2}(a) = \begin{cases} 0 & \text{if } a \le a^\circ \\ \frac{1}{2}(a - a^\circ)^2 & \text{if } a > a^\circ \end{cases}$

3.4.1. *The Patient Solution.* Running any risk of the absorbing state means that $\mathbf{lra}(u) = 0$, therefore one sets the patient optimum as $a^P = a^\circ$ and $\mathbf{lra}(u) = 10(1 + \sqrt{a^\circ})$.

3.4.2. *The Discounted Solution.* Define $q_{1,2}(a) = 1 - p_{1,2}(a)$. The Bellman equation for discount factor β has $V_\beta(2) \equiv 0$, and

$$V_\beta(1) = \max_{a \in [0,1]} 10(1 + \sqrt{a}) + \beta \left[q_{1,2}(a) V_\beta(1) + p_{1,2}(a) 0 \right].$$

The FOCs are

$$\frac{5}{\sqrt{a}} = \beta V_\beta \cdot (a - a^\circ)$$

where $V_\beta = V_\beta(1)$.

We now argue that $(a^* - a^\circ) \propto \frac{(1-\beta)}{10(1+\sqrt{a^*})}$, which means that as $\beta \uparrow 1$, the optimal action converges downwards to a°. This is continuous policy convergence, but the long-run occupation measure always puts mass 1 on the absorbing state for $\beta < 1$.

To see why this works, note that if a^* is the solution, then

$$V_\beta(1) = V_\beta = u(a^*)/(1 - \beta \cdot q_{1,2}(a^*))$$

where $u(a^*) = 10(1 + \sqrt{a^*})$. Plugging this into the FOCs yields

$$\beta \cdot 10(1 + \sqrt{a^*})(a^* - a^\circ) = 1 - \beta(1 - \frac{1}{2}(a^* - a^\circ)) = 1 - \beta + \frac{1}{2}\beta(a^* - a^\circ).$$

Dividing both sides by $(a^* - a^\circ)$ yields

$$\beta \cdot 10(1 + \sqrt{a^*}) = \left(\frac{1 - \beta}{a^* - a^\circ} \right) + \frac{1}{2}\beta(a^* - a^\circ).$$

As $\beta \uparrow 1$, the only way to arrange this to stay true is to have $(a^* - a^\circ) \propto \frac{(1-\beta)}{10(1+\sqrt{a^*})}$.

3.4.3. *The Previous in a Marine Reserve Fishery Model.* Suppose that by putting aside p of the fishing area into a marine reserve, the probability that the fish go extinct is

$$\varphi(p) = \begin{cases} 0 & \text{if } p \geq p^\circ \\ \frac{1}{2}(p - p^\circ)^2 & \text{if } p < p^\circ \end{cases}.$$

Thus, having a marine reserve of less than p° in a stationary policy means that the fish go extinct, eventually, with probability 1, while a marine reserve larger than p° guarantees fish forever.

If $(x - a)$ of the fish are left to breed in the non-protected part of the fishery, the new fish available in the non-protected part are $(1 - p)f(x - a)$ with probability $\varphi(p)$ and 0 fish forever with probability $1 - \varphi(p)$, $f(\cdot)$ the usual fish growth function, strictly increasing, strictly concave, slope at 0 strictly larger than 1, but the slope is eventually strictly less than 1.

In the patient solutions, marine rine reserve size is p° and the policy involves golden rule fishing on the remaining part. By contrast, in the discounted solution, $p^*(\beta) \uparrow p^\circ$ as $\beta \uparrow 1$, $a^*(\beta) \uparrow$golden rule as $\beta \uparrow 1$.

3.5. **Irreversibility and the Precautionary Principle.** In many stochastic dynamic decision problems, the decision maker has only partial information about the state variables or the consequences of actions. Let us again consider a specific example to heighten the intuitions about maximizing patient preferences in such contexts. A hidden state X takes the two values $x_l < 0 < x_h$, with strictly positive probabilities $(1 - g)$ and g, and we assume that $EX < 0$. When the action $a = 1$ is taken, utility will go up/down by X forever thereafter and no further actions are available. When the action $a = 0$ is taken, utility will be unchanged forever thereafter and no further actions are available. Until either $a = 0$ or $a = 1$ is chosen, the action s is available. When $a = s$ is chosen, a signal that is perhaps related to X will be observed. There is a random number of signals informative about the value of X. Every attempt to observe an informative signal costs c.

The random number of informative signals is another hidden state, M, distributed with probability p_m, $m = 0, 1, \ldots$ where $\sum_m p_m = 1$. The signals $S_0, S_1, \ldots, S_M$ are independent and informative Bernoulli's, that is, they are distributed iid $Bern(q)$ if $X = x_h$ and distributed iid $Bern(1 - p)$ if $X = x_l$, where $\frac{1}{2} < p, q < 1$. The signals $S_{M+1}, S_{M+2}, \ldots$ are distributed iid $Bern(\pi)$ for some $\pi \in (0, 1)$.

> For any $\mathbb{V}$-concave preferences we have the following version of the precautionary principle: if there is a positive probability that beliefs will converge to $E^\beta X > 0$, then in any optimal path, signals will be observed until beliefs β have converged to either $E^\beta X > 0$ or $E^\beta X \leq 0$; and it cannot be optimal to never make a decision.

For $s \in \{0, 1\}^n$, define $Bin(r, n)(s)$ as the probability that a binomial with parameters r and n takes the value s. The posterior beliefs after making m observations and observing $s \in \{0,1\}^m$ are

$$\beta(x_h|s,m) = P(M > m)\frac{g \cdot Bin(q,m)(s)}{g \cdot Bin(q,m)(s) + (1-g) \cdot Bin(1-p,m)(s)} + \sum_{n=0}^{M} p_n \frac{g \cdot R(q,n,m)(s)}{g \cdot R(q,n,m)(s) + (1-g) \cdot R(1-p,n,m)(s)} \tag{9}$$

where $R(\rho,n,m) = Bin(\rho,n)((s_i)_{i=1}^n) \cdot Bin(\pi, m-n)((s_i)_{i=n+1}^m)$.

Observation: picking M observations of s followed either by $a=0$ or $a=1$ so as to maximize $E\sum_{t\geq 0} u_t\delta^t$ has the property that for $\delta \uparrow 1$, the optimal $M^*(\delta)\uparrow$. If $p_m = 0$ for all $m \geq \overline{M}$, then $M^*(\delta) \uparrow \overline{M}$. If $p_m > 0$ for infinitely many m, then $M^*(\delta) \uparrow \infty$. The proof comes from the observation that for any $b > 0$,

$$\frac{c \cdot \sum_{t=0}^{M} \delta^t}{b \cdot \sum_{t>M} \delta^t} \downarrow 0 \text{ as } \delta \uparrow 1.$$

4. Discussion

We now discuss several issues.

- One could expand the set $p\mathbb{V}$ to include all of the Banach-Mazur limits. Using such utility functions would make it difficult to evaluate the social utility of stationary policies in "most" MDPs and would simultaneously make it possible for optimal policies to be strictly non-stationary, violating conditional equal treatment in a strong fashion.
- The conditional equal treatment property seems to have no implications for the value of optimal policies when a generation is in a transient state. However, the average reward optimality equations (AROEs) tell a very different story — one picks the action along the stochastic path to a recurrent state that maximizes the summed deviation from the long-run average payoff. We conjecture that the policies that solve AROEs in transient states have a property like perfect equilibria in games — they are the policies picked in nearby perturbations of the MDP.
- Restricted to the streams of well-being generated by stationary policies in MDPs, our preferences all agree, a form of "linearity" which rather defeats the idea of concavity as inequality aversion. This is mis-leading.
- One could have generation t's utility depend positive on the utility of generation $t+1$. This pushes solutions toward smoothing of intergenerational utilities.
- One could and should add a resilience analysis to the climate change model given above.

4.1. **Banach-Mazur Limits.** Our definition of patience requires that $S(u) = S(u^\pi)$ for all $o(t)$ permutations. This is a more stringent requirement than the requirement usually found in the literature, that $S(u) = S(u^\pi)$ only for the bounded permutations, those with $|\pi(t) - t|$ uniformly bounded. The class of tangent functionals allowed by this looser requirement grows from $\mathbb{V}$ to the the set of all Banach-Mazur limits. The use of this larger class of tangents allows social welfare functions for which optimality *requires* violations of the equal treatment property.

A continuous linear $L : \ell_\infty \to \mathbb{R}$ with the properties $L(u) \geq 0$ for all $u \geq 0$, $L((1,1,1,\ldots)) = 1$, and $L(u^\pi) = L(u)$ for all u and bounded permutations π is called a **Banach-Mazur limit** [6, Ch. II.3]. Every Banach-Mazur limit can be represented as an integral against a purely finitely additive probability, $L(u) = \langle u, \eta \rangle$, and the set of all such functionals is denoted $\mathbb{BM}$.

4.1.1. *Violations of Conditional Equal Treatment.* Elements of $\mathbb{BM} \setminus \mathbb{V}$ include accumulation points of uniform distributions on $\{T', \ldots, T\}$ where $(T - T')$ becomes unboundedly large and $\frac{T'}{T}$ becomes arbitrarily close to 1. Let η be such an accumulation point. Because η ignores all but the "tail" of the consumption streams, it simultaneously fails to respect the Pareto criterion and fails to be immune to $o(t)$ permutations.[9] As a result, there are well-behaved stationary Markovian decision problems for which the optima of e.g. the social welfare function $S(u) = \langle u, \eta \rangle$ are non-stationary in a particularly disturbing fashion. The optima involve "feasts" during a sequence of intervals $\{T'_n, \ldots, T_n\}$ at the expense of "starvation" of the generations between T_n and T'_{n+1}. Further, the set B of starving generations has $\underline{\ell}(B) = 1$ because $\frac{T'}{T}$ becomes arbitrarily close to 1.

4.1.2. *The Almost Convergent Sequences.* Starting with $U^0 = (u_0, u_1, u_2, \ldots)$, for each $j \in \mathbb{N}_0$, define the points $U^j \in \ell_\infty$ as follows.

U^0	u_0	u_1	u_2	u_3	$\cdots$
U^1	$\frac{u_0+u_1}{2}$	$\frac{u_1+u_2}{2}$	$\frac{u_2+u_3}{2}$	$\frac{u_3+u_4}{2}$	$\cdots$
U^2	$\frac{u_0+u_1+u_2}{3}$	$\frac{u_1+u_2+u_3}{3}$	$\frac{u_2+u_3+u_4}{3}$	$\frac{u_3+u_4+u_5}{3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
U^j	$\frac{u_0+\cdots+u_j}{j+1}$	$\frac{u_1+\cdots+u_{j+1}}{j+1}$	$\frac{u_2+\cdots+u_{j+2}}{j+1}$	$\frac{u_3+\cdots+u_{j+3}}{j+1}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Definition 4.1. $u = (u_0, u_1, u_2, \ldots) \in \ell_\infty$ *is **almost convergent to** r if for every $\epsilon > 0$, there exists J such that for all $j \geq J$, every element of U^j is within ϵ of r.*

It is clear that any almost convergent u belongs to **Erg**, however, the set of almost convergent utilities is too small to contain the outcomes of even the simplest non-degenerate Markov processes.

Example 4.1. *Suppose that for all $t \in \mathbb{N}_0$, u_t is either 0 or 1 and that u_t follows a Markov process: if $u_t = 0$ then $u_{t+1} = 1$ with probability $\alpha \in (0,1)$; and if $u_t = 1$ then $u_{t+1} = 0$ with probability $\beta \in (0,1)$. With probability 1, the realizations of this Markov chain belong to **Erg** and **lra**$(u) = \frac{\alpha}{\alpha+\beta}$. However, with probability 1, for each $j \in \mathbb{N}_0$, each realization contains infinitely many length $(j+1)$ sequences of 0's and contain infinitely many length $(j+1)$ sequences of 1's. Therefore each U^j contains infinitely many elements equal to 0 and infinitely many elements equal to 1, hence cannot be almost convergent.*

[9] The Polya index of [30] has this kind of tail-only sensitivity.

The following characterization of almost convergent sequences is [29, Theorem 1].

Theorem C (Lorentz)**.** *u is almost convergent if and only if for every $\eta, \eta' \in \mathbb{BM}$, $\langle u, \eta\rangle = \langle u, \eta'\rangle$.*

One can show that concave social welfare functions with Banach-Mazur tangents have the property that for all u that are almost convergent, $S(u) = \varphi(\mathbf{lra}(u))$ for some concave, strictly increasing $\varphi(\cdot)$. Thus, defining patience using tangents immune only to bounded permutations results in a theory that can be fruitfully applied, but only to models in which the optima involve almost convergent sequences of utilities with probability 1. This is a slightly larger class than e.g. deterministic growth models with convergent sequences of utilities, but seems smaller than one would wish.

4.2. **Concavity and Inequality Aversion.** The result that $S(u) > S(v)$ iff $\mathbf{lra}(u) > \mathbf{lra}(v)$ for $u, v \in \mathbf{Erg}$, could be interpreted as a linearity result, a repudiation of the inequality and risk aversion built into the concavity of $S(\cdot)$. An analysis of the programming applications, those with $u_t = v(a_t, \Phi_t)$ where a_t represents the actions taken by generation t, Φ_t represents the state of the system faced by generation t, shows that this is misleading in at least two senses.

The first, and perhaps most obvious, way in which this is misleading comes from the observation that Φ_t is stochastic. Variability in Φ_t and concavity of $v(a_t, \cdot)$ makes the expected value of u_t lower. It is the long-run expected value of the u_t that determines $S(u)$ for patient preferences, and riskier paths yield lower expected utility.

The second way in which this is misleading involves entrainment and/or hysteresis [34, For a review of these are other concepts related to irreversibilities, see]. In decision problems with "urn-like" components, early decisions and early stochastic events determine the long run path of the system.[10] When different long run paths have different long run utilities and these are stochastic, the concavity of $S(\cdot)$ induces risk aversion over the choice of paths.

4.3. **State Doubling, or Caring for the Next Generation.** The social welfare functions under study are equivalent to a monotonic transformation of the **linear** functional $u \mapsto L(u)$ on **Erg**. Therefore, the concavity of $S(\cdot)$ can have no bite on determistic elements of this domain. For example, the following two sequences of utilities belong to **Erg** and are indifferent for all $\mathbb{V}$-concave social welfare functions,

$$u = (1, 2, 3, 4, 4, 3, 2, 1, 1, 2, 3, 4, 4, 3, 2, 1, \ldots) \text{ and} \tag{10}$$

$$u^{\pi} = (4, 1, 3, 1, 4, 2, 3, 2, 4, 1, 3, 1, 4, 2, 3, 2, \ldots). \tag{11}$$

[10] Some decisions clearly set history on different paths. For example, in 1953, after Stalin's death, Dwight D. Eisenhower argued that the world found itself at "... one of those times in the affairs of nations when the gravest choices must be made, if there is to be a turning toward a just and lasting peace." He talked of the long-run consequences of present choices, "Every gun that is made, every warship launched, every rocket fired signifies, in the final sense, a theft from those who hunger and are not fed, those who are cold and are not clothed. This world in arms is not spending money alone. It is spending the sweat of its laborers, the genius of its scientists, the hopes of its children. ... This, I repeat, is the best way of life to be found on the road the world has been taking. This is not a way of life at all, in any true sense. Under the cloud of threatening war, it is humanity hanging from a cross of iron."

In the u, the utility differences between subsequent generations are either 0 or 1, in u^π, the utility differences range from 1 to 3. Thus, smoothing of inter-generational consumption patterns does not matter.

A solution is to make generation t's well-being depend on both their state and actions as well as the state and actions taken by generation $t+1$ and then to maximize $S(\cdot)$. In particular, if each generation dislikes their offspring doing worse than they do, this pushes the solutions toward smoothing.

4.4. **Resilience in the Climate Change Model.** One could interpret the states G and B in the climate change model as indicating two basins of attraction in a more complex model. In such a model, the "depth" of the basins is known as "resilience." By adding measures of closeness to the boundary of the basins to the simple model, one can capture more subtle aspects of optimal policies — for example, near a "tipping point" from the good state to the bad state, efforts to avert the change are more valuable and the optimal sacrifices larger.

5. Summary and Conclusions

A Pareto criterion that ignores vanishingly small rewards or positive rewards to vanishingly small portions of society is compatible with patience. For the purposes of applicability, the definition of patience should be indifference between $\mathcal{o}(t)$ permutations rather than the more commonly used notion of indifference between bounded permutations. For large classes of stationary problems [32], the optimal long run policies can be found by solving a functional equation similar to the Bellman equation for discounted dynamic programming. Further, this can be done by the appropriate variant of the policy iteration algorithm.

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Appendix A. Proofs

For the proofs using nonstandard analysis, we work in a κ-saturated, nonstandard enlargement of a superstructure $V(Z)$ where the base set, Z, contains $\mathbb{R}$ and ℓ_∞, and κ is a cardinal greater than the cardinality of $V(Z)$. For nearstandard $r \in {}^*\mathbb{R}^k$, ${}^\circ r \in \mathbb{R}^k$ denotes the standard part of r [25, §II.1 and II.8] or [28, Ch. 3]. The essential result that we use is [26, Theorem 3.1]: if η is an extreme point in the set of Banach-Mazur limits, then there exists in interval subset of ${}^*\mathbb{N}_0$, $\{T', T'+1, \dots, T\}$ with $(T-T') \simeq \infty$ such that $\langle \boldsymbol{u}, \eta\rangle = {}^\circ\langle {}^*\boldsymbol{u}, \eta_{T',T}\rangle$ where $U_{T',T}$ is the *-uniform distribution on $\{T', \dots, T\}$.

For bounded sequences, the Hardy-Littlewood Tauberian theorem tells us that $\lim_{T\to\infty} \frac{1}{T+1}\sum_{t=0}^{T} u_t = c$ if and only if $\lim_{\beta\uparrow 1}(1-\beta)\sum_{t=0}^{\infty} u_t\beta^t = c$. We record the nonstandard formulation of this and include a proof for completeness.

Lemma 2. *The following are equivalent:*

(1) $\boldsymbol{u} \in \mathbf{Erg}$*;*
(2) *for all* $\eta, \eta' \in p\mathbb{V}$, $\langle \boldsymbol{u}, \eta\rangle = \langle \boldsymbol{u}, \eta'\rangle$*; and*
(3) *for all* $\beta, \gamma \simeq 1$, $\beta, \gamma < 1$, $(1-\beta)\sum_{t=0}^{\infty} u_t\beta^t \simeq (1-\gamma)\sum_{t=0}^{\infty} u_t\gamma^t$.

Proof. (1) $\Leftrightarrow$ (2). Let η_T denote the uniform distribution on $\{0, 1, \dots, T\}$ so that $\langle \boldsymbol{u}, \eta_T\rangle = \frac{1}{T+1}\sum_{t=0}^{T} u_t =: Ave_T(\boldsymbol{u})$.

By definition, $\boldsymbol{u} \in \mathbf{Erg}$ iff $\lim_T Ave_T(\boldsymbol{u})$ exists. The existence of this limit is equivalent to the statement that for all infinite T, T', $Ave_T(\boldsymbol{u}) \simeq Ave_{T'}(\boldsymbol{u})$. From [26, Theorem 3.1], the extreme points of $\mathbb{V}$ have an expression as $L(\boldsymbol{u}) = {}^\circ\langle {}^*\boldsymbol{u}, \eta_T\rangle$ where η_T is the uniform distribution on $\{0, 1, \dots, T\}$, T infinite. Integrating to the same value on all extreme points implies integrating to the same on all points.

(2) $\Rightarrow$ (3) Suppose, without loss, that for all infinite t, $\langle \boldsymbol{u}, \eta_t\rangle \simeq 1$ where η_t is the uniform distribution on $\{0, 1, \dots, t\}$. For $\beta \simeq 1$, the density $y_t = (1-\beta)\beta^t$ is, to within an infinitesimal, a convex combination of uniform distributions on infinite intervals $\{0, 1, \dots, t\}$. To see this, calculation shows that

$$R := (1-\beta)\sum_{t=0}^{\infty} u_t\beta^t = (1-\beta)^2\sum_{t=0}^{\infty} \beta^t(t+1)\langle \boldsymbol{u}, \eta_t\rangle. \tag{12}$$

There exists infinite $\tau < \tau 1$ such that $\sum_{t=0}^{\tau} \beta^t(t+1) \simeq \sum_{t=\tau'}^{\infty} \beta^t(t+1) \simeq 0$. Therefore, because $\|\boldsymbol{u}\|$ is finite,

$$R \simeq (1-\beta)^2 \sum_{t=\tau}^{\tau'} \beta^t(t+1)\langle \boldsymbol{u}, \eta_t\rangle. \tag{13}$$

For each infinite t, $\langle \boldsymbol{u}, \eta_t\rangle \simeq 1$, and the sum of the weights $(1-\beta)^2\beta^t(t+1)$, $t \in \{\tau, \tau+1, \ldots, \tau'\}$ is infinitesimally close to 1. Therefore, $R \simeq 1$.

(2) $\Leftarrow$ (3) Now suppose that for all $\beta < 1, \beta \simeq 1$, ${}^\circ(1-\beta)\sum_{t=0}^{\infty} u_t\beta^t = 1$, where again the "1" is without loss. Let $\mathcal{G}$ denote the set of standard functions, $g : [0,1] \to \mathbb{R}$ such that for all $\beta < 1, \beta \simeq 1$,

$$^\circ(1-\beta)\sum_{t=0}^{\infty} u_t\beta^t {}^*g(\beta^t) = \int_0^1 g(x)\,dx. \tag{14}$$

We detour to develop properties of the class of "good" functions $\mathcal{G}$.

It is clear that $\mathcal{G}$ is a vector space of functions containing the constants. Therefore, to show that $\mathcal{G}$ contains the polynomials, it is sufficient to show that it contains the monomials, $g(x) = x^k$. For $g(x) = x^k$, we have $\int_0^1 g(x)\,dx = \frac{1}{k+1}$. We now show that for $g(x) = x^k$, $(1-\beta)\sum_{t=0}^{\infty} u_t\beta^t g(\beta^t) \simeq \frac{1}{k+1}$. For $\beta \simeq 1, \beta < 1$, we have $\beta^{k+1} \simeq 1$ and $\beta^{k+1} < 1$. By assumption, ${}^\circ(1-\beta^{k+1})\sum_{t=0}^{\infty} u_t(\beta^{k+1})^t = 1$. Therefore,

$$(1-\beta)\sum_{t=0}^{\infty} u_t\beta^t g(\beta^t) = (1-\beta)\sum_{t=0}^{\infty} u_t(\beta^{k+1})^t \tag{15}$$

$$= \frac{(1-\beta)}{(1-\beta^{k+1})}(1-\beta^{k+1})\sum_{t=0}^{\infty} u_t(\beta^{k+1})^t \tag{16}$$

$$\simeq \frac{(1-\beta)}{(1-\beta^{k+1})} \simeq \frac{1}{k+1} \tag{17}$$

where the last "$\simeq$" follows from l'Hôpital's rule.

Therefore, $\mathcal{G}$ contains all of the polynomials. It is also closed under uniform convergence, so by the (Stone-)Weierstrass theorem, it contains all of the continuous functions. Consider the function

$$g(x) = \begin{cases} 0 & \text{if } x < 1/e \\ 1/x & \text{if } 1/e \le x \le 1. \end{cases} \tag{18}$$

Note that $\int_0^1 g(x) = 1$, and that, if $g \in \mathcal{G}$, then taking $\beta = e^{1/N}$ for infinite N, $(1-\beta)\sum_{t=0}^{\infty} u_t\beta^t g(\beta^t) = \frac{1}{N}\sum_{t=0}^{N} u_t \simeq 1$.

The function g is not continuous, but for every $\epsilon > 0$, it is sandwiched between continuous functions that are within ϵ of g outside of the interval $(1/e - \epsilon, 1/e + \epsilon)$, and that integrate to within the interval $(1-\epsilon, 1+\epsilon)$. Since ϵ is arbitrary and the continuous functions belong to $\mathcal{G}$, for any $\beta \simeq 1$, $\beta < 1$, $(1-\beta)\sum_{t=0}^{\infty} u_t\beta^t \simeq 1$. □

Proof of Lemma 1. Suppose that π is $\mathcal{o}(T)$ and fix $\boldsymbol{u} \in \ell_\infty$. For any infinite $T \in {}^*\mathbb{N}_0$, let $t^\dagger$ be the largest $t \in \{0, 1, \ldots, T\}$ such that $\pi(t) < 0$ and let $t^\ddagger$ be the minimum $t \in \{0, 1, \ldots, T\}$ such that $\pi(t) > T$. We show that $t^\dagger/(T+1) \simeq 0$ and $t^\ddagger/(T+1) \simeq 1$. If ${}^\circ t^\dagger/T = \alpha > 0$, then $t^\dagger$ is infinite and $|\pi(t^\dagger) - t^\dagger|/t^\dagger = 1$, contradicting π being $\mathcal{o}(T)$. If ${}^\circ t^\ddagger/T = (1-\beta) < 0$, then $t^\ddagger$ is infinite and ${}^\circ|\pi(t^\dagger) - t^\dagger|/t^\dagger \ge \beta$, contradicting

π being $o(T)$. This is sufficient to show that $\boldsymbol{u} \succsim_{o(T)} \boldsymbol{u}^\pi$ because, letting $T' = \{t \in \{0,1,\ldots,T\} : 0 \leq \pi(t) \leq T\}$,

$$\begin{aligned}\left|\frac{1}{T+1}\sum_{t=0}^{T}(u_t - u_t^\pi)\right| &= \left|\frac{1}{T+1}\sum_{t=0}^{T}(u_t - u_{\pi^{-1}(t)})\right| \\ &\leq \left|\frac{1}{T+1}\sum_{t\in T'}(u_t - u_t)\right| + \frac{1}{T+1}\sum_{t\notin T', t\leq T}|u_t - u_t^\pi| \\ &\leq 0 + 2\|u\| \cdot \frac{\#\{t \notin T'\}}{T+1} \simeq 0\end{aligned}$$

The arguments for $\boldsymbol{u}^\pi \succsim_{o(T)} \boldsymbol{u}$ are essentially identical.

For second part of the claim, note that the arguments just given show that for any infinite T and $\boldsymbol{u} \in \mathbf{Erg}$, $Ave_T(\boldsymbol{u}) = Ave_T(\boldsymbol{u}^\pi)$. □

Proof of Theorem A. Suppose that $\succ$ satisfies Axioms I-V. We draw heavily on Theorem 4, Ch. 3 in [21] which characterizes expected utility preference relations $\succ$. Our Axioms I and II are Fishburn's A1 and A2, our Axiom III is a strong form of his A3 and it directly implies his A4* and A5*.

Our mixture set, $\mathcal{M}$, the set of measures with bounded support is closed under finite convex combinations. As we work with the Borel σ-field on $\mathbf{W}$, our Axiom III implies that the domain for our probabilities contains all preference intervals, which then implies that $\mathcal{M}$ is closed under taking conditional measures on preference intervals, completing the verification of Fishburn's A0.2.

Having verified A1-A5 and A0.2, Fishburn's result shows that there exists an integrable $S : \mathbf{W} \to \mathbb{R}$ such that $p \succ q$ iff $\int S(\boldsymbol{v})\, dp(\boldsymbol{v}) > \int S(\boldsymbol{v})\, dq(\boldsymbol{v})$. Restricted to the closed set of point masses, $\{\delta_{\boldsymbol{u}} : \boldsymbol{u} \in \mathbf{W}\}$, Axiom III implies that $S(\cdot)$ is continuous. By considering measures with two point supports and their resultants, Axiom IV implies that $S(\cdot)$ is concave. For the normalization that $S : \mathbf{W} \to [0,\infty)$, note that there is no loss in setting $S(\mathbf{0}) = 0$ — for all $\boldsymbol{u} \in \mathbf{W}$, $\boldsymbol{u} \succsim_{o(T)} \mathbf{0}$ so that Axiom V implies $S(\boldsymbol{u}) \geq S(\mathbf{0})$.

We now show that $S(\cdot)$ is exactly Pareto. If B is a null coalition, then for any $\boldsymbol{u} \in \mathbf{W}$ and any $r > 0$, $\boldsymbol{u} \succsim_{o(T)} (\boldsymbol{u} + r1_B) \succsim_{o(T)} \boldsymbol{u}$ so that Axiom V implies $S(\boldsymbol{u}) = S(\boldsymbol{u} + r1_B)$. If B is a non-null coaltion, then for any $\boldsymbol{u} \in \mathbf{W}$ and any $r > 0$, $(\boldsymbol{u} + r1_B) \succ_{o(T)} \boldsymbol{u}$ so that Axiom V implies $S(\boldsymbol{u} + r1_B) > S(\boldsymbol{u} + r1_B)$.

We now show that $S(\cdot)$ is patient. From Lemma 1, if π is an $o(T)$ permutation, then for all $\boldsymbol{u} \in \ell_\infty$, $\boldsymbol{u} \succsim_{o(T)} \boldsymbol{u}^\pi \succsim_{o(T)} \boldsymbol{u}$ so that Axiom V implies $S(\boldsymbol{u}) \geq S(\boldsymbol{u}^\pi) \geq S(\boldsymbol{u})$.

Now suppose that there exists a continuous, concave $S : \mathbf{W} \to [0,\infty)$ such that $[p \succ q] \Leftrightarrow [\int S(\boldsymbol{u})\, dp(\boldsymbol{u}) > \int S(\boldsymbol{u})\, dq(\boldsymbol{u})]$ with $S(\cdot)$ satisfying the properties (1) and (2). Verification of Axioms I through IV is routine. To verify Axiom V, suppose that $\boldsymbol{u} \succ_{o(T)} \boldsymbol{v}$ and let $r' = \liminf_{T\to\infty} \frac{1}{T+1}\sum_{t=0}^{T}(u_t - v_t)$. By definition, $r' > 0$. Set $r = r'/2$ and note that $\boldsymbol{u} \succ_{o(T)} (\boldsymbol{v} + r1_{\mathbb{N}_0}) \succ_{o(T)} \boldsymbol{v}$ and that $\mathbb{N}_0$ is a non-null coaltion. By property (2), $S(\boldsymbol{u}) > S(\boldsymbol{v} + r1_{\mathbb{N}_0}) > S(\boldsymbol{v})$. □

Proof of Theorem B. We first show that any $\mathbb{V}$-concave $S(\cdot)$ is perfectly Pareto. Pick an arbitrary $\boldsymbol{u} \in \mathbf{W}$.

- If B is a null coalition, we must show that $S(\boldsymbol{u}+r1_B) \geq S(\boldsymbol{u}) \geq S(\boldsymbol{u}+r1_B)$ for all $r > 0$. For all $\eta \in p\mathbb{V}$, $\langle 1_B, \eta\rangle = 0$. Because $\mathbb{V}$ is the closed cone containing $p\mathbb{V}$, for all $L \in \mathbb{V}$ and all $r \in \mathbb{R}$, $L(r1_B) = 0$. For any $L \in \mathbf{D}S(\boldsymbol{u})$, $S(\boldsymbol{u}) + L((\boldsymbol{u}+r1_B) - \boldsymbol{u}) \geq S(\boldsymbol{u}+r1_B)$ so that $S(\boldsymbol{u}) \geq S(\boldsymbol{u}+r1_B)$. Similarly, for any $L' \in \mathbf{D}S(\boldsymbol{u}+r1_B)$, $S(\boldsymbol{u}+r1B) + L'(\boldsymbol{u} - (\boldsymbol{u}+r1_B)) \geq S(\boldsymbol{u})$ so that $S(\boldsymbol{u}+r1_B) \geq S(\boldsymbol{u})$.
- If B is a non-null coalition, we must show that $S(\boldsymbol{u}+r1_B) > S(\boldsymbol{u})$ for all $r > 0$. For all $\eta \in p\mathbb{V}$, $\langle 1_B, \eta\rangle > 0$. Because $\mathbb{V}$ is the closed cone containing $p\mathbb{V}$ and any $L' \in \mathbf{D}S(\boldsymbol{u}+r1_B)$ is a strictly positive element of $\mathbb{V}$, $L'(r1_B) > 0$. Since $S(\boldsymbol{u}+r1_B) + L'(\boldsymbol{u} - (\boldsymbol{u}+r1_B)) \geq S(\boldsymbol{u})$, we have $S(\boldsymbol{u}+r1_B) \geq S(\boldsymbol{u}) + L'(r1_B)$ so that $S(\boldsymbol{u}+r1_B) > S(\boldsymbol{u})$.

We now show that any $\mathbb{V}$-concave $S(\cdot)$ is patient. Pick an arbitrary $\boldsymbol{u} \in \mathbf{W}$ and an arbitrary $\mathcal{O}(T)$ permutation. We must show that $S(\boldsymbol{u}) = S(\boldsymbol{u}^\pi)$. From Lemma 1, we know that $\boldsymbol{u} \succsim_{\mathcal{O}(T)} \boldsymbol{u}^\pi \succsim_{\mathcal{O}(T)} \boldsymbol{u}$ and that for every $\eta \in \mathbb{V}$ and every $\boldsymbol{z} \in N$, $\langle \boldsymbol{z}, \eta\rangle = 0$. Therefore, for any $L \in \mathbf{D}S(\boldsymbol{u})$ and any $L' \in \mathbf{D}S(\boldsymbol{u}^\pi)$, $L(\boldsymbol{u} - \boldsymbol{u}^\pi) = L'(\boldsymbol{u} - \boldsymbol{u}^\pi) = 0$. Therefore $S(\boldsymbol{u}) + 0 \geq S(\boldsymbol{u}^\pi)$ and $S(\boldsymbol{u}^\pi) + 0 \geq S(\boldsymbol{u})$.

For the second part, assume that $\succ$ satisfies Axioms I-V and that integrals against $S(\cdot)$ represent the preferences on $\mathcal{M}$. We must show that $S(\cdot)$ is $\mathbb{V}$-concave on $\text{int}(\mathbf{W})$. Pick an arbitrary $\boldsymbol{u} \in \text{int}(\mathbf{W})$ and $L \in \mathbf{D}S(u)$. If $L \notin \mathbb{V}$, then there exist $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbf{Erg}$ with $\mathbf{lra}(\boldsymbol{v}_1) = \mathbf{lra}(\boldsymbol{v}_2)$ and $L(\boldsymbol{v}_1 - \boldsymbol{v}_2) \neq 0$. Reversing the role of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ if necessary, $L(\boldsymbol{v}_1 - \boldsymbol{v}_2) < 0$. Because $\boldsymbol{u}$ is interior, for some $r > 0$, $\boldsymbol{v} := \boldsymbol{u} + r(\boldsymbol{v}_1 - \boldsymbol{v}_2)$ is interior.

We first prove the intermediate claim that $\boldsymbol{u} \succsim_{\mathcal{O}(T)} \boldsymbol{v} \succsim_{\mathcal{O}(T)} \boldsymbol{u}$. For any infinite T, $\frac{1}{T+1}\sum_{t=0}^T v_t = \frac{1}{T+1}\sum_{t=0}^T u_t + r\frac{1}{T+1}\sum_{t=0}^T (v_{1,t} - v_{2,t}) = \frac{1}{T+1}\sum_{t=0}^T u_t + 0$. Therefore, $\liminf_T \frac{1}{T+1}\sum_{t=0}^T v_t = \liminf_T \frac{1}{T+1}\sum_{t=0}^T u_t$.

Returning to the argument, by Axioms V and continuity, $S(\boldsymbol{u}) = S(\boldsymbol{v})$. However, by the properties of tangents for concave functions, $S(\boldsymbol{u}) + L(\boldsymbol{v} - \boldsymbol{u}) \geq S(\boldsymbol{v})$. However, $L(\boldsymbol{v} - \boldsymbol{u}) = rL(\boldsymbol{v}_1 - \boldsymbol{v}_2) < 0$, a contradiction. □

Proof of Proposition 1. As this is an exchange economy model, we need only verify (i)-(iv) in Bewley's Theorem 1. (i) is the assumption that the consumption sets are convex and Mackey closed, which is immediate. (ii) is the assumption that the preference relations are transitive, reflexive and complete, which is satisfied because the preferences are given by utility functions. (iii) is the assumption that for all $i \in I$ and all consumption vectors, $\boldsymbol{x}$, the set $\{\boldsymbol{z} \in W^k : U_i(\boldsymbol{z}) \geq U_i(\boldsymbol{x})\}$ is convex and Mackey closed. Convexity follows from the concavity of $U_i(\cdot)$, and a convex subset of the dual of a Banach space is closed for all the topologies between the weak*-topology and the norm topology if and only if it is norm closed [19, Cor. V.2.14]. Therefore, norm continuity of the $U_i(\cdot)$ delivers the necessary closure. (iv) is the the assumption that for all $i \in I$ and all consumption vectors, $\boldsymbol{x}$, the set $\{\boldsymbol{z} \in W^k : U_i(\boldsymbol{z}) \leq U_i(\boldsymbol{x})\}$ is norm closed, which follows directly from the norm continuity of $U_i(\cdot)$. □

Proof of Proposition 2. For convex sets, weak* and norm closure are equivalent. □