More on linear-potential values and extending the 'Shapley family' for TU-games*

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Abstract

We generalize the potentials of Hart & Mas-Colell [1989] inspired by an idea of 'taxing and redistributing'. To such a potential an additive efficient value is associated giving each player his linearly modified contribution to the potential of the grand coalition of a so-called taxed game, plus an equal share in the 'tax revenues'.

Egalitarian, discounted and weighted Shapley values are linear-potential values. We extend the 'Shapley family' with semi-egalitarian discounted weighted Shapley values and equal-coalitional-improvement Shapley values. We investigate connections between restrictions on linear-potential values and axioms. We characterize several subclasses of the 'Shapley family' by single axioms used before to axiomatize the egalitarian Shapley value.

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1 Introduction

We generalize the potentials of Hart & Mas-Colell [1989] in order to analyze certain values, some of which related to the Shapley value [1953], in a unified manner and at the same time be able to 'design' and analyze new ones. To each transferable utility game[1] (N, v) the (a, b, α)-potential $P^{a,b,\alpha}$ attributes a real number such that $P^{a,b,\alpha}(\emptyset, v) = 0$ and

$$v^{\alpha}(N) = \sum_{i\in N} a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N\backslash\{i\}, v).$$

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[1]A transferable utility game is an ordered pair (N, v) where $N \neq \emptyset$ is the set of players, and v is a map attributing to each coalition $S \subseteq N$ a real number such that $v(\emptyset) = 0$.

Here, $a, b \in \mathbb{R}^{|Z|}$ are vectors of weights, each component connected to one player in the set of possible players $Z \supseteq N$;[2] α is a real number, the right-hand side of the second equation adds up to $v^\alpha(N) \equiv (1-\alpha)v(N)$, i.e., the worth of (grand) coalition N for the α-taxed game (N, v^α) arising by raising a 'flat rate' proportional tax of α on every coalition in (N, v).

We connect the linear-potential value $\psi^{a,b,\alpha}$ to the (a, b, α)-potential as follows. For all transferable utility games (N, v), and all $i \in N$:

$$\psi_i^{a,b,\alpha}(N, v) = a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N \backslash \{i\}, v) + \alpha \tfrac{v(N)}{|N|}.$$

So, each player gets his marginal contribution to the potential of the α-taxed game plus an equal share in the total tax revenues. The former reflects the power or strength of each player in the taxed game, the latter reflects a social norm, i.e., the level of egalitarianism or solidarity in the society. Social norms on equality and solidarity exist in real life, and several authors aim to devise solutions incorporating them.[3] For $\alpha = 0$ each player gets a utility according to his 'raw' power, whereas $\alpha = 1$ ignores power aspects and redistributes the entire utility from cooperation equally.

We characterize subclasses of linear-potential values: α-egalitarian weighted Shapley values by α-egalitarianism, semi-egalitarian discounted Shapley values by symmetry, equal-coalitional-improvement Shapley values by α-consistency, semi-egalitarian discounted weighted Shapley values by egalitarianism with respect to reducing players (Van den Brink & Funaki [2015]).

We also present an overview of *connections among those axioms* characterizing α-egalitarian Shapley values in Joosten [1996], Joosten *et al.* [1994], and Van den Brink *et al.* [2007,2013]. For $\alpha \neq 0, 1$, e.g., Sobolev-consistency implies symmetry which is equivalent to λ-standardness, symmetry implies equal coalitional improvement which is equivalent to α-consistency in turn, whereas α-standardness implies α-egalitarianism and α-consistency.

2 Preliminaries

Let $\mathbb{R}$ denote the set of real numbers. Let Z be a nonempty set of natural numbers, representing the **set of potential players**. (Strict) inclusions are denoted by $(\subset)$ $\subseteq$. A **coalition** is a finite subset of Z. A **transferable utility game** is a pair (N, v) where $N \subseteq Z$ is a coalition and $v : 2^N \rightarrow \mathbb{R}$, with $v(\emptyset) = 0$. The function v is the **characteristic function**. We denote the set of all games with player set N by G^N, the set of all games is G.

Let $(N, v) \in G$, then:

- for $M \subseteq N$, the characteristic function of the game (M, v) is the map v restricted to 2^M;

[2] Similar to the vector of the weighted Shapley value cf., Hart & Mas-Colell [1989].

[3] E.g., Dutta & Ray [1989, 1991], Dutta [1990] and more explicitly Nowak & Radzik [1994], Ju *et al.* [2007], Van den Brink & Funaki [2009], Malawski [2013].

- the **marginal contribution** of $i \in S \subseteq N$ is given by $\Delta_i^v(S) = v(S) - v(S\backslash\{i\})$;

- player $i \in N$ is **δ-reducing** in (N, v) if $v(S) = \delta v(S\backslash\{i\})$ for all $S \subseteq N, S \ni i$; a 0-reducing player is called a **nullifying-player;** a 1-reducing player is called a **null-player**; $\mathcal{N}(N, v)$ is the **set of null-players** in (N, v);

- players $i, j \in N$ are **symmetric** in (N, v), if $\Delta_i^v(S) = \Delta_j^v(S)$ for all $S \subseteq N, S \supseteq \{i, j\}$.

Let $v, w : 2^N \to \mathbb{R}$, and $\alpha, \lambda \in \mathbb{R}$, then:

- $(\lambda(v + w))(S) = \lambda v(S) + \lambda w(S)$, for all $S \subseteq N$,

- $(v^\alpha)(S) = (1 - \alpha)v(S)$ for all $S \subseteq N$.

With the first operation G^N is a linear space; the game (N, v^α) is the **α-taxed game** of (N, v), i.e., the game remaining after a proportional tax of α is levied on the worth of each coalition in (N, v). For nonempty $T \subseteq N \subseteq Z$, the **T-unanimity game** (N, u_T) is the game with $u_T(S) = 1$ if $T \subseteq S \subseteq N$, and $u_T(S) = 0$ otherwise.

Shapley [1953] demonstrated that the collection of all T-unanimity games $(\emptyset \neq T \subseteq N)$ constitutes a basis of the linear space G^N. Hence, for every game (N, v) with nonempty player-set N, there exists a unique set of numbers $\{c_T \in \mathbb{R} \mid \emptyset \neq T \subseteq N\}$, satisfying $v = \sum_{\emptyset \neq T \subseteq N} c_T u_T$ where for given $T \subseteq N$, $c_T = \sum_{\emptyset \neq S \subseteq T} (-1)^{|T|-|S|} v(S)$.

A **value** is a map ψ assigning to each game (N, v), a vector in $\mathbb{R}^{|N|}$. The interpretation is that if the value is applied to a game (N, v), the i-th component of the vector represents the utility attributed to player $i \in N$ in the game (N, v). Let ψ be a value, then:

- ψ is **efficient** if $\sum_{i \in N} \psi_i(N, v) = v(N)$ for every $(N, v) \in G$;

- ψ is **symmetric** if $\psi_i(N, v) = \psi_j(N, v)$ whenever $i, j \in N$ are symmetric players in $(N, v) \in G$;

- ψ is **linear** if $\psi(N, \lambda v + \mu w) = \lambda\psi(N, v) + \mu\psi(N, w)$ for all $\lambda, \mu \in \mathbb{R}$, and all $(N, v), (N, w) \in G$;

- ψ is **trivial** if $\psi_i(N, v^1) = 0$ for all $N \subseteq Z$, $i \in N$ (Chun[1989]);

- ψ is **socially acceptable** if $\psi_j(N, u_T) \geq \psi_i(N, u_T) \geq 0$ for any $(N, u_T) \in G$ whenever $j \in T$, $i \in N\backslash T$ (Joosten *et al.* [1994]);

- ψ is **α-egalitarian** if $\psi_i(N, v) = \alpha\overline{\psi(N, v)}$ whenever $i \in \mathcal{N}(N, v)$ where $\alpha \in \mathbb{R}$, $\overline{\psi(N, v)} \equiv \frac{\sum_{j \in N} \psi_j(N,v)}{|N|}$ (Joosten *et al.* [1994]);

- ψ satisfies the **δ-reducing-player property** if $\psi_i(N,v) = 0$ whenever i is a δ-reducing player (Van den Brink & Funaki [2015]); the 0-reducing-player property is called the **nullifying-player property** (Van den Brink [2007]), the 1-reducing-player property is called the **null-player property**;

- ψ satisfies **α-marginality** if for any pair of games (N,v), $(N,w) \in G$, and any $i \in N$, it holds that if $\Delta_i^v(S) = \Delta_i^w(S)$ for all $S \subseteq N$, then $\psi_i(N,v) - \alpha\overline{\psi(N,v)} = \psi_i(N,w) - \alpha\overline{\psi(N,w)}$ (Joosten *et al.* [1994], generalizing Young [1985] who covered $\alpha = 0$);

- ψ is **λ-standard** if $\psi_i(\{i,j\},v) = \frac{v(\{i,j\})-(1-\lambda)v(\{j\})+(1-\lambda)v(\{i\})}{2}$ for all 2-person games $(\{i,j\},v) \in G$ (Joosten *et al.* [1994]);

- ψ satisfies **equal coalitional improvement** if for all $(N,v),(N,w)$ and any $\emptyset \neq T \subseteq N$ such that $w(S) - v(S) = c \in \mathbb{R}$ whenever $S \supseteq T$, and $w(S) - v(S) = 0$ otherwise, then $\psi_i(N,w) - \psi_i(N,v) = \widetilde{c} \in \mathbb{R}$ *for all* $i \in T$ (Joosten *et al.* [1994]).

For any game (N,v), the **egalitarian value** η is defined by $\eta_i(N,v) = \frac{v(N)}{|N|}$ for all $i \in N$, i.e., η distributes the worth of the grand coalition equally among the players. So, η is efficient, symmetric, linear, socially acceptable, trivial, 1-standard, and satisfies the nullifying property, 1-marginality and equal coalitional improvement.

Another example of a value is the **Shapley value** (cf., Shapley [1953], Roth [1988]) Sh. For every $(N,v) \in G$ and every $i \in N$, Sh is given by

$$Sh_i(N,v) = \sum_{S \subseteq N: i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \Delta_i^v(S).$$

The Shapley value assigns to each player his average marginal contribution in any game. The Shapley value is efficient, symmetric, linear, socially acceptable, trivial, and (0-)standard, moreover it satisfies strong monotonicity, (0-)marginality, the null-player property, and equal coalitional improvement.

For linear values, $\psi(N,v) = \sum_{\emptyset \neq T \subseteq N} \psi(N, c_T u_T)$ for all $(N,v) \in G$. Let $w \in \mathbb{R}^{|Z|}$ be a vector of exogenously given weights satisfying $w_i > 0$ for all $i \in Z$. Then, the **weighted Shapley value** (Shapley [1953]), denoted by Sh^w in the sequel, is for every $(N,v) \in G$ given by

$$Sh_i^w(N,v) = \sum_{T \subseteq N: T \ni i} c_T \left(\frac{w_i}{\sum_{j \in T} w_j} \right) \text{ for all } i \in N.$$

If $w_i = w_j > 0$ for all $i, j \in Z$, we have $Sh^w = Sh$. Each weighted Shapley value satisfies efficiency, linearity and the null-player property. Symmetric players *are not* treated equally in general, see e.g., Kalai & Samet [1987].

3 Linear potentials and associated values

We now come to one central purpose of this paper: to introduce a family of potentials generalizing those of Hart & Mas-Colell [1989], and associate with each potential a unique efficient and linear value.

The families of potentials and values depend on a tuple of parameters (a, b, α). The vectors $a, b \in \mathbb{R}^{|Z|}$ are exogenously given weights similar to the weights of the weighted Shapley values. As before, $\alpha \in \mathbb{R}$ is the level of taxation reflecting the norms on egalitarianism in the society.

Definition 1 *Let $a, b \in \mathbb{R}^{|Z|}$, $\alpha \in \mathbb{R}$ satisfy $\sum_{i \in S \subseteq Z, S \neq \emptyset} a_i \neq 0$. Then the (a, b, α)-potential is the unique map $P^{a,b,\alpha} : G \to \mathbb{R}$ given by $P^{a,b,\alpha}(\emptyset, v) = 0$,* and for $N \neq \emptyset$

$$\sum_{i \in N} \left[a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N \backslash \{i\}, v) \right] = v^{\alpha}(N), \text{ for all } (N, v) \in G.$$

The linear-potential value $\psi^{a,b,\alpha}$ is for all $(N, v) \in G$, $i \in N$ given by

$$\psi_i^{a,b,\alpha}(N, v) = a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N \backslash \{i\}, v) + \alpha \tfrac{v(N)}{|N|}.$$

An interpretation of the value $\psi^{a,b,\alpha}$ is that for game (N, v) it gives to player $i \in N$ the sum of the proportion α of the per-capita income of the grand coalition, and his linearly modified marginal contribution to the potential of the taxed game v^{α}, i.e., $a_i P^{a,b,\alpha}(N, v) - b_i P^{a,b,\alpha}(N \backslash \{i\}, v)$. As $\sum_{i \in S} a_i \neq 0$ for all $S \subseteq Z$, the definition implies

$$P^{a,b,\alpha}(N, v) = \frac{(1-\alpha)v(N) + \sum_{k \in N} b_k P^{a,b,\alpha}(N \backslash \{k\}, v)}{\sum_{k \in N} a_k}.$$

Two instances of α are special, for $\alpha = 0$ 'taxing and redistributing' becomes void, whereas for $\alpha = 1$ the potential of each coalition is zero, which follows easily by recursion.

The following result may be proven straightforwardly.

Lemma 1 *For all admissible (a, b, α), the linear-potential value $\psi^{a,b,\alpha}$ satisfies efficiency, linearity, triviality, and homogeneity of degree 0 in (a, b).*

4 On egalitarian Shapley values

This section summarizes work on egalitarian Shapley values, and forms a stepping stone to the sequel where we examine links between axioms used.

Weighted Shapley values satisfy the null-player property, as do several others, e.g., the nucleolus (Schmeidler [1969]) and the τ-value (Tijs [1981]). Instead, Joosten *et al.* [1994] introduced α-egalitarianism stipulating that utility received by a null-player in any game is a fixed scalar multiple ('fraction') α of the per-capita income. Clearly, 0-egalitarianism is implied by the null-player property. Joosten *et al.* [1994] defined the family of *α-egalitarian Shapley values* given by

$$Sh^{\alpha}(N,v) = Sh(N,v^{\alpha}) + \eta(N,v^{1-\alpha}) = (1-\alpha)\cdot Sh(N,v) + \alpha\cdot\eta(N,v). \quad (1)$$

This value can be easily recovered by setting $a = b = (c, c, ...) \in \mathbb{R}^{|Z|}$ with $c > 0$ using the notations presented in the previous section.

Joosten *et al.* [1994] introduced social acceptability implying that null-players in a unanimity game share in the worth of the grand coalition but not to the extent that agents having marginal contributions which are at least as high as the null-players' marginal contribution, get less than the latter. This restricts the range of α to the unit interval. Chameni Nembua & Demsou [2013] call a pair of values ordinally equivalent, if the ordinal ranking of the utilities under both is the same for all possible games and players. The class of α-egalitarian Shapley values ordinally equivalent to the Shapley value is characterized by $\alpha \leq 1$.[4]

To continue, we present a characterization using α-egalitarianism, equal coalitional improvement and α-marginality. Part i. is a next of kin to a result due to Shapley [1953], Part iii. relates to Young [1985].

Proposition 2 *(Joosten [1996]) Let $\alpha \in \mathbb{R}$ and let ψ be a value. Then, the following statements are equivalent:*

i. ψ satisfies efficiency, linearity, symmetry, and α-egalitarianism;

ii. ψ satisfies efficiency, triviality, equal coalitional improvement, and α-marginality;

iii. ψ satisfies efficiency, symmetry, and α-marginality;

iv. $\psi = Sh^{\alpha}$.

Consistency is a reduced-game property, which may be described informally as follows. Let ψ be a value. For a group of players in a game, one defines a reduced game among them by giving the rest of the players payoffs according to ψ. Then, ψ is called consistent if, when applied to the reduced game, it yields the same payoffs as in the original game. The following formalizes such a property.

Definition 2 *(Hart & Mas-Colell [1989]) Let ψ be a value, $(N,v) \in G$, and $\emptyset \neq U \subset N$. Then the (U,ψ)-reduced game of v is the game $v^{U,\psi}$ satisfying* $v^{U,\psi}(S) = v(S \cup U) - \sum_{k\in U} \psi_k(S\cup U, v)$ *for all $\emptyset \neq S \subseteq N\backslash U$.*

[4] Other contributions restrict this value to these ranges: Casajus & Huettner [2013] provide interesting characterizations, one implying $\alpha \leq 1$, two others imply $\alpha \in [0,1]$. Van den Brink *et al.* [2007,2013] restrict α to the unit interval, albeit implicitly.

Sobolev [1973] introduced an alternative reduced game.

Definition 3 *(Sobolev [1973]) Given $(N,v) \in G$, player $j \in N$, and efficient payoff vector $x \in \mathbb{R}^{|N|}$, i.e., $\sum_{i\in N} x_i = v(N)$, the reduced game with respect to j and x is the game $(N\backslash\{j\}, v^x)$ for all $\emptyset \neq S \subseteq N\backslash\{j\}$ given by*

$$v^x(S) = \tfrac{|S|}{|N|-1}\left(v\left(S\cup\{j\}\right) - x_j\right) + \tfrac{|N|-1-|S|}{|N|-1} v(S).$$

Rewriting the latter shows that the worth of coalition S is decreased by x_j, and S is compensated by $\frac{|N|-1-|S|}{|N|-1}\left[v(S) - v(S\cup\{j\}) - x_j\right]$ where considerations on the likelihood that coalition $S\cup\{j\}$ forms versus the event that j remains alone, motivate number $\frac{|N|-1-|S|}{|N|-1}$ (Van den Brink *et al.* [2007]).

Yet another example of a reduced game is the following.

Definition 4 (*Joosten, Peters & Thuijsman [1994]) Let $(N,v) \in G$, $\alpha \in \mathbb{R}$, and let ψ be a value. For nonempty $U \subset N$, the (U,ψ,α)-reduced game $(N\backslash U, v^{U,\psi,\alpha})$ of v is given by $v^{U,\psi,\alpha}(\emptyset) = 0$* and

i. *$v^{U,\psi,\alpha}(S) = v(S\cup U) - \sum_{k\in U}\psi_k(S\cup U, v)$, if $\alpha = 1$ and $S \neq \emptyset$, or if $S = N\backslash U$;*

ii. *$v^{U,\psi,\alpha}(S) = v(S\cup U) - \sum_{k\in U}\psi_k(S\cup U, v) + \frac{\alpha}{1-\alpha}\left[\frac{|U|}{|S|+|U|}v(S\cup U) - \sum_{k\in U}\psi_k(S\cup U,v)\right]$, if $\alpha \neq 1$, $\emptyset \neq S \subset N\backslash U$.*

The interpretation is as follows. For any group of players in a game $S \subseteq N\backslash U$, one defines a reduced game among them by giving the rest of the players, i.e., U, the payoffs according to ψ in the game $(S\cup U, v)$. Then, the worth of S is compensated for the group 'leaving with the amount $\sum_{k\in U}\psi_k(S\cup U, v)$' by returning an amount $\frac{\alpha}{1-\alpha}\left[\frac{|U|}{|S|+|U|}v(S\cup U) - \sum_{k\in U}\psi_k(S\cup U,v)\right]$.

With these reduced games, we define specific forms of consistency.

Definition 5 *Let ψ be a value. Then,*

i. *ψ is HM-consistent if and only if for every $(N,v) \in G$ and all $\emptyset \neq U \subset N : \psi_i(N\backslash U, v^{U,\psi}) = \psi_i(N,v)$ for all $i \in N\backslash U$;*

ii. *ψ is S-consistent if and only if for every $(N,v) \in G$ with $|N| \geq 2$, $j \in N : \psi_i\left(N\backslash\{j\}, v^{\psi}\right) = \psi_i(N,v)$ for all $i \in N\backslash\{j\}$;*

iii. *ψ is α-consistent if and only if $\psi_i(N\backslash U, v^{U,\psi,\alpha}) = \psi_i(N,v)$ for all $(N,v) \in G$, all nonempty $U \subset N$, and all $i \in N\backslash U$.*

Capitals HM and S are mnemonic for Hart & Mas-Colell and Sobolev respectively, α is be linked to the parameter denoting the level of egalitarianism.

Other notions of consistency depend on idiosyncratic reduced games, cf., e.g., Driessen [1991], Yanovskaya [2003], as well.

The following result summarizes and merges findings of Joosten *et al.* [1994] and Van den Brink *et al.* [2007].

Proposition 3 *Let $\alpha \in \mathbb{R}$ and let ψ be a value. Then, the following statements are equivalent:*

i. *ψ is α-consistent and α-standard;*

ii. *ψ is S-consistent and α-standard;*

iii. *$\psi = Sh^{\alpha}$.*

Note that the case $\alpha = 0$ covers the characterizations of the Shapley value in Hart & Mas-Colell [1989] and Sobolev [1973].

5 Linear-potential values and axioms

Our goal here is to find relationships between restrictions on parameters (a, b, α) and axioms used in Propositions 2 and 3, and also possible connections among these axioms implied. We arrange our material around four themes defining one subclass of linear-potential values each. A graphical overview of results is to be found in the final section.

5.1 Egalitarianism

First, we present *asymmetric* generalizations of egalitarian Shapley values.

Definition 6 *Let $w \in \mathbb{R}^{|Z|}$ satisfy $\sum_{i \in S} w_i \neq 0$ for every $S \subseteq Z$, and let $\alpha \in \mathbb{R}$. The α-egalitarian w-weighted Shapley value $Sh^{w,\alpha}$ is for every $(N, v) \in G$ given by $Sh^{w,\alpha}(N, v) = Sh^{w}(N, v^{\alpha}) + \eta(N, v^{1-\alpha})$.*

We call the values obtained by taking all admissible w and α *egalitarian weighted Shapley values.* Clearly, they form α-dependent linear combinations of the weighted Shapley value and the egalitarian value. The next two intermediate results are used in the ensuing one.

Lemma 4 *For linear values α-egalitarianism is equivalent to α-marginality.*

Lemma 5 *$P^{w,w,\alpha}(N, v) = P^{w,w,\alpha}(N \backslash \{i\}, v)$ whenever $i \in \mathcal{N}(N, v)$.*

Next, we show links between axioms and restrictions on the parameters.

Proposition 6 *$\psi^{a,b,1}$ satisfies 1-egalitarianism and 1-marginality for all admissible (a, b). For $\alpha \neq 1$, the following statements are equivalent:*

i. $\psi^{a,b,\alpha}$ *satisfies* α*-egalitarianism.*

ii. $\psi^{a,b,\alpha}$ *satisfies* α*-marginality.*

iii. $\psi^{a,b,\alpha} = Sh^{w,\alpha}$ *with* $w = a = b$.

Moreover, $\psi^{a,b,\alpha}$ *satisfies* λ*-egalitarianism, implies* $\lambda = \alpha$.

The final statement stipulates that there can be at most one instance for which a constant proportion to the null players for all games with the same worth of the grand coalition is attributed by a linear-potential value, namely exactly linked to the α in the tuple (a, b, α).

5.2 Symmetry

First, we recall a subclass of linear-potential values in Joosten *et al.* [1994] to be seen as *symmetric* generalizations of the α-egalitarian Shapley values.

Definition 7 *For* $\alpha \in \mathbb{R}$, $\delta \in \mathbb{R}\backslash\{0\}$*, the semi* α*-egalitarian* δ*-discounted Shapley value* Sh_δ^α *is for all* $(N, v) \in G$, $i \in N$ *given by* $(Sh_\delta^\alpha)_i (N, v) = (1 - \alpha) \sum_{S \subseteq N\backslash\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \delta^{|N|-|S|-1} [v(S \cup \{i\}) - \delta v(S)] + \alpha \frac{v(N)}{|N|}$.

The class obtained by taking all admissible δ and α, will be called *semi-egalitarian discounted Shapley values.* Note that $\alpha = 0$, $\delta = 1$ yields the Shapley value, arbitrary α and $\delta = 1$, yields the α-egalitarian Shapley value. The term *semi* is used because Sh_δ^α is not α-egalitarian for $\alpha \neq 0$. Instead, any δ-reducing player in (N, v) receives $\alpha \frac{v(N)}{|N|}$.

Driessen & Radzik [2002] introduced the term of 'δ-discounted Shapley values' for the special instance that $\alpha = 0$ where the following interpretation for $\delta \in (0, 1)$ is given: '*...the worth of a coalition in an* n*-person game is weakly discounted whenever the size* s *of the coalition is relatively large (or strongly discounted, if the size of the coalition is relatively small) in comparison with the size* n *of the player set'.*

To give an elegant interpretation of these values, we present another notion to be used in the remainder.

Definition 8 *Let* $\delta \in \mathbb{R}\backslash\{0\}$*, then for* $(N, v) \in G$*, the* δ*-discounted game* (N, v_δ) *is given by* $v_\delta(S) = \delta^{|N|-|S|} v(S)$ *for all* $S \subseteq N$.

Any δ-discounted Shapley value of the game $(N, v) \in G$ is equivalent to the Shapley value of the δ-discounted game $(N, v_\delta) \in G$. The reader may confirm this by applying the Shapley value to the game (N, v_δ) followed by substituting $v_\delta(S) = \delta^{|N|-|S|} v(S)$ for all $S \subseteq N$. Similarly, the α-egalitarian δ-discounted Shapley value of the game (N, v) is the α-egalitarian Shapley value applied to the δ-discounted game (N, v_δ). The following result links the linear-potential value of a game to the one of a discounted game.

Lemma 7 *For all* $(N, v) \in G$, $\delta \in \mathbb{R} \backslash \{0\}$, $\psi^{a,\delta \cdot b,\alpha}(N, v) = \psi^{a,b,\alpha}(N, v_\delta)$.

In what follows, we use a notational convenience by writing ψ_δ defined by $\psi_\delta(N, v) = \psi(N, v_\delta)$ for every (N, v) and value ψ with the customary restriction $\delta \neq 0$.

The axiom of λ-standardness stipulates the utilities players in each 2-person game receive depending on the real number λ and the worths of the two-person grand coalition and the two 'stand alone' coalitions. Next, we show connections between restrictions on the parameters (a, b, α) and the axioms of λ-standardness and symmetry.

Proposition 8 *The linear-potential value* $\psi^{a,b,1}$ *satisfies symmetry and* 1-*standardness. For linear-potential values* $\psi^{a,b,\alpha}$ *with* $\alpha \neq 1$, *the following statements are equivalent:*

i. $\psi^{a,b,\alpha}$ *is symmetric;*

ii. $\psi^{a,b,\alpha}$ *satisfies* λ-*standardness;*

ii. $\exists_\delta$ *such that* $\psi^{a,b,\alpha} = Sh_\delta^\alpha$.

5.3 Equal coalitional improvement

We now introduce linear-potential values forming in some sense a class of hybrids between egalitarian Shapley values and egalitarian weighted Shapley values. The former are symmetric, the latter are not.

Definition 9 *Let* $w \in \mathbb{R}^{|Z|}$ *satisfy* $\sum_{i \in S} w_i \neq 0$ *for every* $S \subseteq Z$, *and let* $\alpha \in \mathbb{R}$, *then the ECI*(w, α)-*Shapley value* $Sh_{\mathbf{ECI}}^{\mathbf{w},\boldsymbol{\alpha}}$ *is given by*

$$\left(Sh_{ECI}^{w,\alpha}\right)_i(N, v) = P^{a,\tau \cdot w,\alpha}(N, v) - w_i P^{a,\tau \cdot w,\alpha}(N \backslash \{i\}, v) + \alpha \frac{v(N)}{|N|},$$

where $a_i = \tau$ *for all* $i \in Z$.

ECI is mnemonic for equal coalitional improvement, and we will refer to this class as ECI-Shapley values. Observe that equal coalitional improvement is implied by symmetry, but not vice versa. Therefore, the class just presented contains all semi-egalitarian discounted Shapley values. The following result establishes the relation between this axiom and the parameters (a, b, α).

Proposition 9 $\psi^{a,b,1}$ *satisfies equal coalitional improvement. For* $\alpha \neq 0, 1$, *the following statements are equivalent:*

i. $\psi^{a,b,\alpha}$ *satisfies equal coalitional improvement;*

ii. $\psi^{a,b,\alpha}$ *is* α*-consistent;*

iii. $\psi^{a,b,\alpha} = Sh^{w,\alpha}_{ECI}$.

5.4 Hart & Mas-Colell and Sobolev consistency

Recall that α-consistency and HM-consistency coincide for $\alpha \in \{0,1\}$. This does not necessarily imply that for instance every linear-potential value $\psi^{a,b,0}$ satisfies HM-consistency. We will first prove an intermediate result stating that if a linear-potential value is both HM-consistent and α-consistent, then $\alpha \in \{0,1\}$. We then proceed to make a much more precise statement about the connection between HM-consistency and α, for linear-potential values.

Lemma 10 *Let* ψ *be a linear-potential value. If* ψ *is HM-consistent and* α*-consistent, then* $\psi = \eta$*, or* $\psi = \psi^{a,b,0}$ *for some pair of vectors* $a, b \in \mathbb{R}^{|Z|}$.

We are now able to show the following.

Proposition 11 *Let* ψ *be a linear-potential value. Then, the following statements are equivalent:*

i. ψ *is HM-consistent;*

ii $\psi = \psi^{a,b,1} = \eta$*, or* $\psi = \psi^{a,b,0}$ *for some pair of vectors* $a, b \in \mathbb{R}^{|Z|}$.

So, even the rather large class of weighted Shapley values form merely a subclass of the HM-consistent linear-potential values, yet only two instances of α allow HM-consistency.

Note that by the result of Van den Brink *et al.* [2007,2013] all α-egalitarian Shapley values are S-consistent. Kleinberg [2015] addresses the issue of finding all S-consistent values in the realm of the efficient, linear and symmetric values. We address a similar issue, but do not assume symmetry, yet linearity and efficiency are intrinsically linked to linear-potential values.

Proposition 12 *Let* ψ *be a linear-potential value. Then, i) implies ii):*

i. ψ *is S-consistent;*

ii. $\psi = \psi^{a,b,\alpha}$ *with* $\alpha = 1$*, or with* $\alpha = 0$ *and* $b_i = b_j$ *for all* $i, j \in Z$*, or with* $a_i = a_j$ *and* $b_i = b_j$ *for all* $i, j \in Z$.

So, an S-consistent linear-potential value is HM-consistent or α-consistent.

6 Additional members of the 'Shapley family'

Now, we use Lemma 6 for an interesting result on asymmetric values. Observe that applying the α-egalitarian weighted Shapley value to the game (N, v_δ) yields another value for (N, v), the specifics are given next.

Definition 10 *Given $w \in \mathbb{R}^{|Z|}$ satisfying $\sum_{i\in S} w_i \neq 0$ for every $S \subseteq Z$, $\alpha \in \mathbb{R}$, and $\delta \neq 0$, the semi α-egalitarian δ-discounted w-weighted Shapley value for all (N, v) and all $i \in N$ is given by*

$$\left(Sh_\delta^{w,\alpha}\right)_i (N, v) = w_i \left[P^{w,\delta\cdot w,\alpha}(N, v) - \delta P^{w,\delta\cdot w,\alpha}(N\backslash\{i\}, v)\right] + \alpha \tfrac{v(N)}{|N|}.$$

Again, 'semi' as $Sh_\delta^{w,\alpha}$ gives $\alpha \frac{v(N)}{|N|}$ to δ-reducing players. Recall that α-egalitarianism allows to differentiate as to what to give to *null-players*. An interpretation of the reducing player property is that it helps to identify who exactly gets zero in a game. The following hybridizes these interpretations.

Definition 11 *Let $\alpha, \delta \in [0, 1]$, then the value ψ is α-egalitarian with respect to δ-reducing players if $\psi_i(N, v) = \alpha\overline{\psi(N, v)}$ whenever $v(S \cup \{i\}) = \delta v(S)$ for all $S \subseteq N\backslash\{i\}$, $i \in N$.*

So, a δ-reducing player gets a proportion α of the average income under such a value ψ. The null-player, δ-reducing player, and nullifying player properties are special examples of the above, i.e., by taking $\alpha = 0$, and $\delta = 1$, $\delta \in (0, 1)$ and $\delta = 0$ respectively. Also, α-egalitarianism is covered as a special case, i.e., $\delta = 1$. Now, we characterize[5] the semi-egalitarian discounted weighted Shapley values by this axiom within the class of linear-potential values.

Proposition 13 *Let $\alpha, \delta \in [0, 1]$, and ψ be a linear-potential value. Then, the following two statements are equivalent*

i. ψ is α-egalitarian with respect to δ-reducing players;

ii. $\psi = Sh_\delta^{w,\alpha}$ for some admissible $w \in \mathbb{R}^{|Z|}$.

Thus far, mostly symmetric linear-potential values seem to have been studied e.g., discounted Shapley values or egalitarian Shapley values. For the former asymmetric generalizations exist which satisfy the δ-reducing player property, and for the latter asymmetric generalizations satisfy α-egalitarianism.

7 Social acceptability

Joosten *et al.* [1994] started with the mission to let null-players share in the revenues of cooperation, without them receiving more than those that

[5] A remark by René van den Brink inspired this characterization.

'contribute more'. This led to the formulation of the axiom of social acceptability. Here, we examine which restrictions it places on the parameters.

To show the desired properties for a class rather than specific examples of values, the axiom of ordinal equivalence (cf., Chameni Nembua & Demsou [2013]) is quite useful. The following intermediate result pertains to this.

Lemma 14 *Linear-potential values ψ^{a,b,α_1} and ψ^{a,b,α_2} are ordinally equivalent, iff either $\alpha_1, \alpha_2 < 1$, or $\alpha_1, \alpha_2 > 1$.*

In the proof of following, ordinal equivalence is used fruitfully. However, first, we need a couple of additional notations. Let $A_S \equiv \sum_{k\in S} a_k$ for $S \subseteq Z$, $\underline{\mu}_j^+ \equiv \inf_{N\subseteq Z: j\in N}\left\{\frac{A_N}{|N|} | \frac{A_N}{|N|} > 0\right\}$ and $\overline{\mu}_j^- \equiv \sup_{N\subseteq Z: j\in N}\left\{\frac{A_N}{|N|} | \frac{A_N}{|N|} < 0\right\}$. So, $\underline{\mu}_j^+$ ($\overline{\mu}_j^-$) is the smallest positive (largest negative) average weight over all sets containing j.

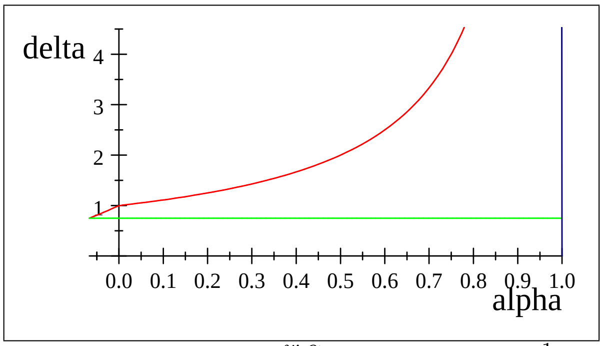
Proposition 15 *$\psi^{a,b,1}$ is socially acceptable for admissible (a, b). For $\alpha \neq 1$, if $\psi^{a,b,\alpha}$ is socially acceptable, then for all $i, j \in Z$, $i \neq j$, it holds that*

$$\begin{array}{rcccll} -\frac{\alpha}{(1-\alpha)}\underline{\mu}_j^+ & \leq & a_j - b_j & \leq & \left(1 + \frac{b_j}{A_{N\setminus\{j\}}}\right) a_i & \textit{if } A_N > 0, \quad \alpha < 1, \\ \left(1 + \frac{b_j}{A_{N\setminus\{j\}}}\right) a_i & \leq & a_j - b_j & \leq & -\frac{\alpha}{(1-\alpha)}\overline{\mu}_j^- & \textit{if } A_N < 0, \quad \alpha < 1, \\ \left(1 + \frac{b_j}{A_{N\setminus\{j\}}}\right) a_i & \leq & a_j - b_j & \leq & -\frac{\alpha}{(1-\alpha)}\underline{\mu}_j^+ & \textit{if } A_N > 0, \quad \alpha > 1, \\ -\frac{\alpha}{(1-\alpha)}\overline{\mu}_j^- & \leq & a_j - b_j & \leq & \left(1 + \frac{b_j}{A_{N\setminus\{j\}}}\right) a_i & \textit{if } A_N < 0, \quad \alpha > 1. \end{array}$$

The class of linear potential values seems too large to provide a ready-to-use set of restrictions implying social acceptability. Even focusing on the Shapley family does not yield an immediate help, as the following indicates that even more structure seems to be required. Let for $w \in \mathbb{R}^{|Z|}$, the asymmetry index be given by $AI^w = \frac{\max_{k\in Z} w_k - \min_{k\in Z} w_k}{\max_{k\in Z} w_k}$. An AI^w equal to zero implies that the agents are symmetric, AI^w close to one indicates that at least two agents are quite asymmetric.

Corollary 16 *Let $\alpha \neq 1$, $w_j > 0$ for all $j \in Z$, and let for every real number $K > 0$, set $Z_K \subset Z$ exist such that $\sum_{k\in Z_K} w_k \geq K$. If $Sh_\delta^{w,\alpha}$ is socially acceptable, then $AI^w \leq \delta \leq \frac{1}{1-\alpha} + \frac{\max\{0,\alpha\}}{1-\alpha}\frac{\max_{k\in Z} w_k}{\min_{k\in Z} w_k} AI^w$ and $\alpha \leq 1$.*

Graph 1 illustrates the corollary for $Sh_\delta^{w,\alpha}$, $\alpha < 1$ and $AI^w = \frac{3}{4}$. The red curve is kinked at the point $(0, 1)$, AI^w is the horizontal green line. An interesting notion is the wedge, i.e., $Wdg^w = \{(\alpha, \delta) \,|AI^w \leq \delta \leq \frac{1}{1-\alpha} + \frac{\max\{0,\alpha\}}{1-\alpha}\frac{\max_{k\in Z} w_k}{\min_{k\in Z} w_k} AI^w$ and $\alpha < 1\}$. The wedge Wdg^w increases (decreases) if the asymmetry index AI^w decreases (increases), as the green line drops (goes up) and the intersection point moves to the left (right). Observe that $\{(\alpha, \delta) \,|1 \leq \delta \leq \frac{1}{1-\alpha}$ and $\alpha < 1\} \subset Wdg^w$ for $w > 0$.



Graph 1: For socially acceptable $Sh_{\delta}^{w,\alpha}$ with $\min_k w_k = \frac{1}{4} \max_k w_k$ all pairs (α, δ) in the wedge between the red line and the green one (the asymmetry index) are allowed. The wedge increases as asymmetry decreases since the index (the horizontal line) goes down and the kink at $(0, 1)$ becomes flatter.

8 Conclusions and discussion

We introduced the class of linear-potential values containing all weighted, egalitarian and discounted Shapley values (Shapley [1953], Kalai & Samet [1987], Joosten *et al.* [1994], Driessen & Radzik [2002]). For this purpose we extended the potential of Hart & Mas-Colell [1988, 1989] in a linear fashion. We then attributed a value to each potential by giving each player in a game, his linearly modified marginal contribution to this generalized potential. Since the sum of these amounts over all players in the grand coalition does not necessarily add up to the worth of the grand coalition, we add an equal share of the difference between the latter and the former. An early motivation of our potential and the associated value in Joosten [1996] expanding work by Joosten *et al.* [1994], is based on taxing the worths of all coalitions by a flat tax rate, i.e., fixed over all coalitions, and redistributing the total tax revenues equally among all members of the grand coalition.

We focused on (connections between) axioms used to characterize the α-egalitarian Shapley values in Joosten *et al.* [1994] and Van den Brink *et al.* [2007,2013]. For linear-potential values, we found the axiom of α-egalitarianism to be equivalent to α-marginality, a variant of marginality (cf., Young [1985]); symmetry to imply α-consistency, and the latter to be equivalent to equal coalitional improvement. For given (a, b, α) determining the linear-potential value, λ-standardness implies symmetry for $\lambda \neq \alpha$ and vice versa; for $\lambda = \alpha$ it implies α-egalitarianism. Figure 1 presents an overview of connections between axioms, helpful for the remainder.

We distinguished subclasses of linear-potential values each characterized by a single axiom: α-egalitarian weighted Shapley values by α-*egalitarianism*, semi-egalitarian discounted Shapley values by *symmetry*, equal-coalitional-improvement Shapley values by α-*consistency*, and semi-egalitarian discounted weighted Shapley values by *egalitarianism with respect to reducing play-*

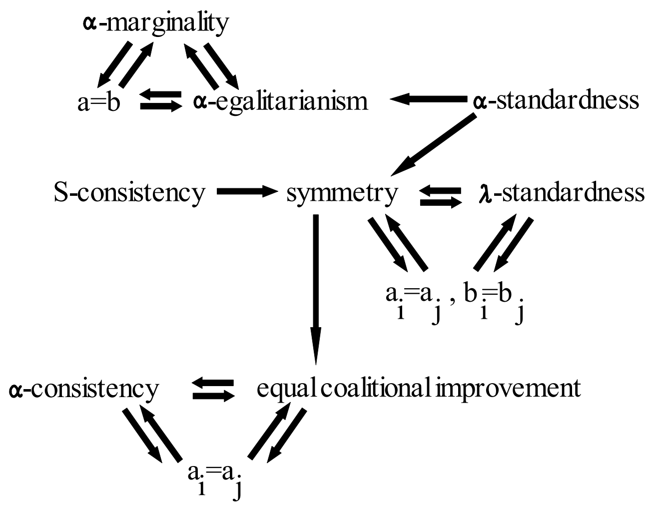


Figure 1: Arrows imply implications for all linear-potential values $\psi^{a,b,\alpha}$ satisfying $\alpha \neq 0, 1$. Note furthermore that $\lambda \neq \alpha$.

ers. This axiom 'unifies' *egalitarianism* (Joosten *et al.* [1994]) and the *reducing player property* (Van den Brink & Funaki [2015]). The first three subclasses are related to the three cycles in Figure 1.

We turned to (different notions of) consistency. We extended the class of values for which HM-consistency (Hart & Mas-Colell [1989]) is known by showing that the class of HM-consistent linear-potential values is characterized by $\alpha = 0, 1$. Moreover, S-consistency (Sobolev [1973]) coincides with HM- and 1-consistency for $\alpha = 1$; S-consistency implies HM-consistency for $\alpha = 0$, but requires b to possess identical components with no restrictions beyond admissibility on a; otherwise S-consistency implies α-consistency for $\alpha \neq 0, 1$, but requires symmetry which is more stringent than equal coalitional improvement which is equivalent to α-consistency in the class of linear-potential values.

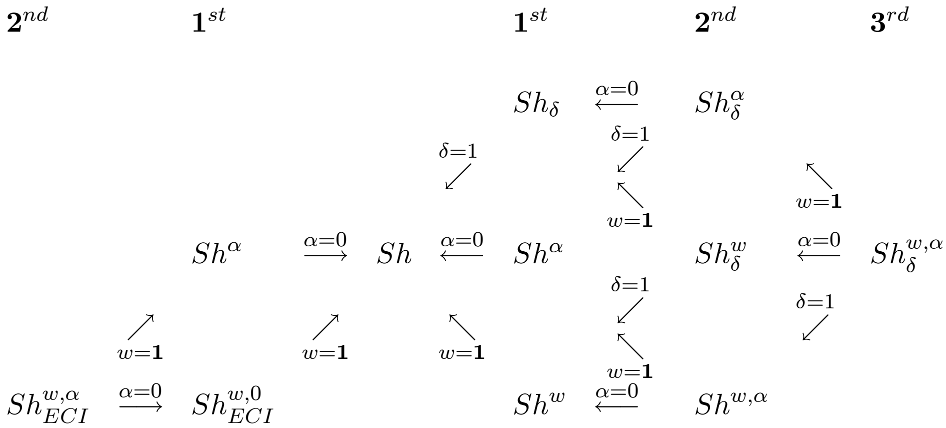
Van den Brink *et al.* [2007] characterize egalitarian Shapley values by α-standardness and Sobolov consistency. Joosten [1996] already showed that egalitarian Shapley values are characterized by α-standardness and the fact that they admit[6] an (a, b, α)-potential, as for linear-potential values α-standardness implies α-consistency. Both properties together, characterize egalitarian Shapley values. The present results prove Sobolev consistency to be much more stringent than α-consistency for the class of linear-potential values, and independent from α-standardness even within this class.

Our analysis complements Kleinberg [2015]. We obtained a complete picture of Sobolev consistency for linear-potential values without assuming

[6] This expression was coined by Calvo & Santos [1997], cf., e.g., Driessen & Calvo [2001].

symmetry. The original restriction of the author on symmetric values turns out to be not trivial, on the one hand it allows to deviate from $\alpha = 0$, explaining why the endeavors of Van den Brink [2007,2013] were successful, on the other, precisely for $\alpha = 0$ it can very well be dispensed with.

The following diagram visualizes connections within the 'Shapley family.'[7] In bold at the top, we depict the 'degrees of kinship' to the Shapley value. So, for instance, equal-coalitional-improvement Shapley values are 'cousins in the second degree' of the Shapley value. Paths end at the Shapley value, going in opposite directions induces generalizations. The 'Shapley family' is a strict subclass of the linear-potential values.



Naumova [2005] presents a class of consistent values originating from a potential, partially overlapping with ours. It is easy to find values belonging to Naumova's class and not ours (e.g., the proportional value of Ortmann [2000]), or vice versa (e.g., α-egalitarian Shapley values). Driessen & Radzik [2002] and Feng [2013] define, connected to sequences of reals $\alpha_{\mathbb{N}}, \beta_{\mathbb{N}}, \gamma_{\mathbb{N}}$ and a potential $P^{\alpha_{\mathbb{N}}, \beta_{\mathbb{N}}, \gamma_{\mathbb{N}}}$, a value as a weighted pseudo-gradient, i.e., a vector of marginal contributions to $P^{\alpha_{\mathbb{N}}, \beta_{\mathbb{N}}, \gamma_{\mathbb{N}}}$. These sequences are **not** connected to the players, but to the cardinality of the player set for a particular game. The class of values implied by Driessen & Radzik [2002] contains discounted Shapley values and the solidarity value of Nowak & Radzik [1994], but not any of the asymmetric generalizations of the Shapley value treated here.

Malawski [2013] introduces 'procedural' values. For the Shapley value the following story is well known. Suppose all players enter a room one by one, each receiving his marginal contribution to the coalition arising by his entering. Then, after the last player has entered, the vector of the players' utilities is an efficient division of the worth of the grand coalition, $v(N)$. To obtain the Shapley value, it is necessary to perform this procedure for each and every sequence of the players entering the room and then to determine

[7] Here e.g., $w = (\leq, <, \geq, >)\mathbf{1}$ for $w \in \mathbb{R}^{|Z|}$ is short-hand for the same component-wise comparison, a bold number is a vector of constants.

for each player the average utilities taken over all possible sequences. For a 'procedural' value only one aspect changes, namely the amount each player may keep to himself upon entering. These amounts are predetermined and depend on each player's marginal contribution and *on the order in which the players enter*. Egalitarian Shapley values and the solidarity value are procedural values, but asymmetric weighted Shapley values are not. All linear-potential values are however, pyramidal values (cf., Prop. 1 in Flores *et al.* [2014]) closely related to but more general than procedural values.

Van den Brink & Van der Laan [2007] define a μ-potential and a function giving each player a share in the worth of the grand coalition.[8] Admittedly, taking $\mu(N,v) = v^{\alpha}(N)$ for all $(N,v) \in G$, yields $P^{v^{\alpha}}$ i.e., the potential of the α-egalitarian Shapley value. However, the vector of marginal contributions to their potential

$$\sigma(N,v) = \left(P^{v^{\alpha}}(N,v) - P^{v^{\alpha}}(N\backslash\{1\},v), ..., P^{v^{\alpha}}(N,v) - P^{v^{\alpha}}(N\backslash\{n\},v)\right),$$

is 'projected' unto the efficient hyperplane quite differently: we use an orthogonal projection of $\sigma(N,v)$, whereas Van den Brink & Van der Laan [2007] use a ray projection[9] along the ray connecting $(0,...,0)$ and $\sigma(N,v)$. So, the latter yields the Shapley value for all $\alpha \neq 1$ and ours yields a unique value $Sh^{\alpha}(v)$ for each $\alpha \neq 1$.

Ordinal Shapley values (Pérez-Castrillo & Wettstein [2006]) do not belong to 'Shapley family', yet they originated from a desire to incorporate notions of consistency, fairness and other properties of the Shapley value into a surplus sharing concept for non-transferable utility games (NTU-games). So, one might say that ordinal Shapley values belong to a 'Shapley clan', as does the 'Shapley family'. Whether similar modifications of the 'Shapley family' for TU-games towards NTU-games may induce new members of the 'Shapley clan' must remain a subject of future research.

9 Appendix

Proof of Lemma 4: First, let $(N,v), (N,w) \in G$, then $i \in \mathcal{N}(N, v - w) \iff \Delta_i^{v-w}(S) = 0 \iff \Delta_i^v(S) = \Delta_i^w(S)$ for all $S \subseteq \overline{N\backslash\{i\}}$. For linear value ψ, we have the following equivalences $\psi_i(N, v-w) = \alpha\overline{\psi(N, v-w)} \iff \psi_i(N,v) - \psi_i(N,w) = \alpha\overline{\psi(N,v)} - \alpha\overline{\psi(N,w)} \iff \psi_i(N, v-w) - \alpha\overline{\psi(N,v)} = \psi_i(N, v-w) - \alpha\overline{\psi(N,w)}$. The first (last) part in the former sequence of

[8] μ is a real-valued function depending on the number of players and the characteristic function of an arbitrary game. The Shapley value and the egalitarian value can be formulated in terms of such a potential and share function, by simply taking $\mu(N,v) = v(N)$ respectively $\mu(N,v) = 0$ for all $(N,v) \in G$.

[9] This terminology follows Joosten & Roorda [2011]. Orthogonal projection on a particular hyperplane is also a feature of ordinary least squares estimators in statistics, but any connection with least squares values as in e.g., Ruiz *et al.* [1996,1998] is coicidental.

equivalences combined with the first (last) of the second series of equivalences defines α-egalitarianism (α-marginality).

Proof of Lemma 5: We prove the $P^{a,a,\alpha}(N,v) = P^{a,a,\alpha}(N\backslash\{i\},v)$ whenever $i \in \mathcal{N}(N,v)$, by induction on $|N|$.
If $N = \{i\}$, the statement holds trivially. Now, let $|N| \geq 2$, and assume for all games (S,v) with $|S| < |N|$ that $P^{a,a,\alpha}(S,v) = P^{a,a,\alpha}(S\backslash\{i\},v)$ whenever $i \in \mathcal{N}(S,v)$. Then,

$$\begin{aligned}
\textstyle\sum_{k\in N} a_k P^{a,a,\alpha}(N,v) &= (1-\alpha)v(N) + \textstyle\sum_{k\in N} a_k P^{a,a,\alpha}(N\backslash\{k\},v) \\
&= (1-\alpha)v(N\backslash\{i\}) + \textstyle\sum_{k\in N\backslash\{i\}} a_k P^{a,a,\alpha}(N\backslash\{i,k\},v) \\
&\quad + a_i P^{a,a,\alpha}(N\backslash\{i\},v) \\
&= \textstyle\sum_{k\in N\backslash\{i\}} a_k P^{a,a,\alpha}(N\backslash\{i\},v) + a_i P^{a,a,\alpha}(N\backslash\{i\},v) \\
&= \textstyle\sum_{k\in N} a_k P^{a,a,\alpha}(N\backslash\{i\},v).
\end{aligned}$$

Hence, $P^{a,a,\alpha}(N,v) = P^{a,a,\alpha}(N\backslash\{i\},v)$. The second equality follows from the induction assumption.

Proof of Prop 6: Lemmas 1 and 4 imply equivalence of (i) and (ii). We only prove the remaining part for $\alpha \neq 1$. **(iii) implies (i):** By Lemma 5: $\psi_i^{a,a,\alpha}(N,v) = a_i P^{a,a,\alpha}(N,v) - a_i P^{a,a,\alpha}(N\backslash\{i\},v) + \alpha\frac{v(N)}{|N|} = \alpha\frac{v(N)}{|N|}$ whenever $i \in \mathcal{N}(N,v)$. **(i) implies (iii):** Let $i,j \in Z$, α-egalitarianism of $\psi^{a,b,\alpha}$ implies

$$\begin{aligned}
0 &= a_i P^{a,b,\alpha}(\{i,j\},u_{\{j\}}) - b_i P^{a,b,\alpha}(\{j\},u_{\{j\}}) \\
&= (1-\alpha)\left[a_i \frac{1+b_i\frac{1}{a_j}}{a_i+a_j} - b_i\frac{1}{a_j}\right] = (1-\alpha)\left[\frac{a_i + b_i\frac{a_i}{a_j} - b_i\frac{a_i}{a_j} - b_i\frac{a_j}{a_j}}{a_i+a_j}\right] \\
&= (1-\alpha)\left[\frac{a_i-b_i}{a_i+a_j}\right].
\end{aligned}$$

Hence, $a_i = b_i$. This in turn implies $a = b$.

Proof of Lemma 7: For $S \subseteq N$, we show $P^{a,\delta\cdot b,\alpha}(S,v) = \frac{1}{\delta^{|N|-|S|}} P^{a,b,\alpha}(S,v_\delta)$ by induction on the cardinality of the player sets $S' \subseteq S$. Observe that

$$\begin{aligned}
P^{a,\delta\cdot b,\alpha}(\{i\},v) &= \frac{(1-\alpha)v(\{i\})}{a_i} = \frac{1}{\delta^{|N|-1}}\frac{(1-\alpha)\cdot\delta^{|N|-1|}v(\{i\})}{a_i} \\
&= \frac{1}{\delta^{|N|-1}}\frac{(1-\alpha)\cdot v_\delta(\{i\})}{a_i} = \frac{1}{\delta^{|N|-1}} P^{a,b,\alpha}(\{i\},v_\delta).
\end{aligned}$$

Assume: $P^{a,\delta\cdot b,\alpha}(S',v) = \frac{1}{\delta^{|N|-|S'|}} P^{a,b,\alpha}(N,v_\delta)$ for all $S' \subsetneq S$. Then,

$$\begin{aligned}
P^{a,\delta\cdot b,\alpha}(S,v) &= \frac{(1-\alpha)v(S) + \sum_{k\in S}\delta\cdot b_k P^{a,\delta\cdot b,\alpha}(S\backslash\{k\},v)}{\sum_{k\in S} a_k} \\
&= \frac{(1-\alpha)\frac{1}{\delta^{|N|-|S|}}v_\delta(S) + \sum_{k\in S}\delta\cdot\frac{1}{\delta^{|N|-|S|+1}} b_k P^{a,b,\alpha}(S\backslash\{k\},v_\delta)}{\sum_{k\in S} a_k} \\
&= \frac{1}{\delta^{|N|-|S|}}\frac{(1-\alpha)v_\delta(S) + \sum_{k\in S} b_k P^{a,b,\alpha}(S\backslash\{k\},v_\delta)}{\sum_{k\in S} a_k} \\
&= \frac{1}{\delta^{|N|-|S|}} P^{a,b,\alpha}(S,v_\delta).
\end{aligned}$$

The second equality follows from the induction assumption and the definition of the discounted game v_δ. Hence, we have

$$\begin{aligned}\psi^{a,\delta\cdot b,\alpha}(N,v) &= a_i P^{a,\delta\cdot b,\alpha}(N,v) - \delta \cdot b_i P^{a,\delta\cdot b,\alpha}(N\backslash\{i\},v) + \alpha\tfrac{v(N)}{|N|} \\ &= a_i P^{a,b,\alpha}(N,v_\delta) - \delta\cdot \tfrac{1}{\delta} b_i P^{a,b,\alpha}(N\backslash\{i\},v_\delta) + \alpha\tfrac{v(N)}{|N|} \\ &= a_i P^{a,b,\alpha}(N,v_\delta) - b_i P^{a,b,\alpha}(N\backslash\{i\},v_\delta) + \alpha\tfrac{v(N)}{|N|} \\ &= \psi^{a,b,\alpha}(N,v_\delta).\end{aligned}$$

Proof of Prop 8: We prove the part for $\alpha \neq 1$.**(iii) implies (i):** Straightforward. **(i) implies (iii):** Take $i,j \in Z$, then by symmetry of $\psi^{a,b,\alpha}$

$$\begin{aligned}0 &= \psi_i^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) - \psi_j^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) \\ &= a_i P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) - b_i P^{a,b,\alpha}(\{j\},u_{\{i,j\}}) + \alpha\tfrac{u_{\{i,j\}}(\{i,j\})}{2} - \\ &\quad \left[a_j P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) - b_j P^{a,b,\alpha}(\{i\},u_{\{i,j\}}) + \alpha\tfrac{u_{\{i,j\}}(\{i,j\})}{2}\right] \\ &= (a_i - a_j) P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}).\end{aligned}$$

Since $P^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) \neq 0$, we obtain $a_i = a_j$. So take τ_1 such that $\tau_1 = a_i$ for all $i \in Z$. Take $(\{i,j\},v)$ with $v(\{i\}) = v(\{j\}) \neq 0$. Then, by symmetry of $\psi^{a,b,\alpha}$

$$\begin{aligned}0 &= \psi_i^{a,b,\alpha}(\{i,j\},v) - \psi_j^{a,b,\alpha}(\{i,j\},v) \\ &= \tau_1 P^{a,b,\alpha}(\{i,j\},v) - b_i P^{a,b,\alpha}(\{j\},v) + \alpha\tfrac{v(\{i,j\})}{2} - \\ &\quad \left[\tau_1 P^{a,b,\alpha}(\{i,j\},v) - b_j P^{a,b,\alpha}(\{i\},v) + \alpha\tfrac{v(\{i,j\})}{2}\right] \\ &= -b_i P^{a,b,\alpha}(\{j\},v) + b_j P^{a,b,\alpha}(\{i\},v) \\ &= -b_i\tfrac{(1-\alpha)v(\{j\})}{\tau_1} + b_j\tfrac{(1-\alpha)v(\{i\})}{\tau_1} = (b_j - b_i)\tfrac{(1-\alpha)}{\tau_1}v(\{i\}).\end{aligned}$$

Hence, $b_i = b_j$ for all $i,j \in Z$. Take τ_2 such that $b_i = \tau_2$ for all $i \in Z$.
Now, homogeneity in vectors (a,b) of linear potential values implies that one may divide both τ_1 and τ_2 by τ_1 and subsequently substituting $\delta = \frac{\tau_2}{\tau_1}$ in Def. 7 yields the reformulation $\psi^{a,b,\alpha} = Sh^{\alpha}_{\frac{\tau_2}{\tau_1}}$ whenever $a_i = \tau_1 \in \backslash\{0\}$ and $b_i = \tau_2 \in \backslash\{0\}$ for all $i \in Z$.
The part **(iii) implies (ii)** is confirmed easily by writing out the equations for 2-person games taking $\delta = \frac{1-\lambda}{1-\alpha}$. We now prove that **(ii) implies (iii)**. Let $\alpha \neq 1$ and let $\psi^{a,b,\alpha}$ satisfy λ-standardness. Then, apply $\psi^{a,b,\alpha}$ to the 2-person game $(\{i,j\},v)$. Furthermore, clearly

$$\begin{aligned}\psi_i^{a,b,\alpha}(\{i,j\},v) &= a_i\tfrac{(1-\alpha)v(\{i,j\}) + b_i P^{a,b,\alpha}(\{j\},v) + b_j P^{a,b,\alpha}(\{i\},v)}{a_i + a_j} \\ &\quad - b_i P^{a,b,\alpha}(\{j\},v) + \tfrac{\alpha v(\{i,j\})}{2}.\end{aligned}$$

As λ-standardness implies $\psi_i^{a,b,\alpha}(\{i,j\},v) = \frac{v(\{i,j\}) + (1-\lambda)v(\{i\}) - (1-\lambda)v(\{j\})}{2}$ and isolating parts depending on $v(\{i,j\})$ we obtain

$$\tfrac{a_i}{a_i + a_j}(1-\alpha)\,v(\{i,j\}) + \tfrac{\alpha}{2}v(\{i,j\}) = \tfrac{1}{2}v(\{i,j\}).$$

This implies $\frac{a_i}{a_i+a_j}(1-\alpha)\,v(\{i,j\}) = \frac{1}{2}(1-\alpha)\,v(\{i,j\})$. As $(\{i,j\},v)$ was chosen arbitrarily and $\alpha \neq 1$, $a_i = a_j$ for all $i,j \in Z$. Let τ denote the real number satisfying $a_i = \tau$ for all $i \in Z$. Then, substituting τ and isolating those parts depending on $v(\{i\})$,we obtain:

$$\frac{\tau}{\tau+\tau} b_j \frac{(1-\alpha)v(\{i\})}{\tau} = \frac{(1-\lambda)v(\{i\})}{2}.$$

Since $(\{i,j\},v)$ was chosen arbitrarily and $\alpha \neq 1$, $b_j = \tau\frac{1-\lambda}{1-\alpha}$ for all $j \in Z$. Finally, isolating the part depending on $v(\{j\})$ and substituting the real numbers obtained yields that the following equation must hold:

$$\left[\frac{\tau}{\tau+\tau} - 1\right]\tau\frac{1-\lambda}{1-\alpha}P^{a,b,\alpha}(\{j\},v) = -\frac{(1-\lambda)}{2}v(\{j\}).$$

This in turn implies that it must hold that

$$\begin{aligned} 0 &= -\tfrac{1}{2}\tau\tfrac{1-\lambda}{1-\alpha}P^{a,b,\alpha}(\{j\},v) + \tfrac{(1-\lambda)}{2}v(\{j\}) \\ &= -\tfrac{1}{2}\tau\tfrac{1-\lambda}{1-\alpha}\tfrac{(1-\alpha)v(\{j\})}{\tau} + \tfrac{(1-\lambda)}{2}v(\{j\}) \\ &= -\tfrac{(1-\lambda)}{2}\tfrac{\tau}{\tau}\tfrac{(1-\alpha)}{1-\alpha}v(\{j\}) + \tfrac{(1-\lambda)}{2}v(\{j\}) = 0. \end{aligned}$$

Since the latter did not yield a contradiction this completes the proof.

Proof of Prop 9: Part **(iii) implies (i)**: Straightforward. **(i) implies (iii)**: Observe that for all (N,v), $T \subseteq N$, $P^{a,b,\alpha}(N\backslash\{k\},cu_T) = 0$ for all $k \in T$, as $(N\backslash\{k\},cu_T)$ is a zero-game. So we obtain $\psi_k^{a,b,\alpha}(N,cu_T) = a_kP^{a,b,\alpha}(N,cu_T)+\frac{\alpha c}{|N|}$. Let $i,j \in Z$, clearly $\psi_i^{a,b,\alpha}(\{i,j\},v_0) = \psi_j^{a,b,\alpha}(\{i,j\},v_0) = 0$, and $(\{i,j\},u_{\{i,j\}}) = (\{i,j\},v_0+u_{\{i,j\}})$. By equal coalitional improvement $\psi_i^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) - \psi_j^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) = 0$, hence,

$$a_iP^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) + \frac{\alpha}{|N|} = a_jP^{a,b,\alpha}(\{i,j\},u_{\{i,j\}}) + \frac{\alpha}{|N|}$$

which proves $a_i = a_j$. **(i) is equivalent to (ii)**: Observe that for $(N,v) \in G$, and nonempty $U \subseteq N$,

$$v^{U,\psi^{a,b,\alpha},\alpha} = v^{U,\psi^{a,b,0}} + \alpha\left[\sum_{k\in U}\psi_k^{a,b,0}(N,v) - \frac{|U|}{|N|}v(N)\right]u_{N\backslash U}.$$

Let $\alpha \notin \{0,1\}$, let $\emptyset \neq U \subset N$, then for $i \in N\backslash U$

$$\begin{aligned} &\psi_i^{a,b,0}\left(N\backslash U, v^{U,\psi^{a,b,0},\alpha}\right) \\ =\ &\psi_i^{a,b,0}\left(N\backslash U, v^{U,\psi^{a,b,0}} + \alpha\left[\textstyle\sum_{k\in U}\psi_k^{a,b,0}(N,v) - \frac{|U|}{|N|}v(N)\right]u_{N\backslash U}\right) \\ =\ &\psi_i^{a,b,0}(N,v) + \alpha\left[\textstyle\sum_{k\in U}\psi_k^{a,b,0}(N,v) - \frac{|U|}{|N|}v(N)\right]\psi_i^{a,b,0}\left(N\backslash U, u_{N\backslash U}\right). \end{aligned}$$

The second equality follows from linearity, and from HM-consistency of $\psi^{a,b,0}$ applied to the first part between the brackets. Furthermore,

$$\eta_i\left(N\backslash U, v^{U,\psi^{a,b,\alpha},\alpha}\right) = \eta_i(N,v) - (1-\alpha)\frac{\left[\sum_{k\in U}\psi_k^{a,b,0}(N,v) - \frac{|U|}{|N|}v(N)\right]}{|N\backslash U|}.$$

It is now a matter of calculation to find that $\psi_i^{a,b,\alpha}\left(N\backslash U, v^{U,\psi^{a,b,\alpha},\alpha}\right) = \psi_i^{a,b,\alpha}(N,v)$ if and only if $\psi_i^{a,b,0}\left(N\backslash U, u_{N\backslash U}\right) = \frac{1}{|N\backslash U|}$. The latter holds if and only if $\psi^{a,b,0}$ (and hence, $\psi^{a,b,\alpha}$) satisfies equal coalitional improvement.

Proof of Lemma 10: Let $\psi = \psi^{a,b,\alpha}$ be a linear potential value and let ψ be HM-consistent and α-consistent. Furthermore, let $(\{i,j,k\},v) \in G$. Then, consider the two games w, w' given by

$$\begin{array}{lcll} w\{i,j\} & = & v(\{i,j,k\}) - \psi_k(\{i,j,k\},v) & \equiv X; \\ w(\{i\}) & = & v(\{i,k\}) - \psi_k(\{i,k\},v) & \equiv Y; \\ w(\{j\}) & = & v(\{j,k\}) - \psi_k(\{j,k\},v) & \equiv Z; \\ w'(\{i,j\}) & = & v(\{i,j,k\}) - \psi_k(\{i,j,k\},v); & \\ w'(\{i\}) & = & v(\{i,k\}) - \psi_k(\{i,k\},v) + & \\ & & \frac{\alpha}{1-\alpha}\left(\frac{1}{2}v(\{i,k\}) - \psi_k(\{i,k\},v)\right) & \equiv X + \frac{\alpha}{1-\alpha}A \;; \\ w'(\{j\}) & = & v(\{j,k\}) - \psi_k(\{j,k\},v) + & \\ & & \frac{\alpha}{1-\alpha}\left(\frac{1}{2}v(\{j,k\}) - \psi_k(\{j,k\},v)\right) & \equiv Y + \frac{\alpha}{1-\alpha}B \;. \end{array}$$

Observe that w is the reduced game associated with HM-consistency and that w' is the reduced game associated with α-consistency. Note that $\psi = \psi^{a,b,\alpha}$ satisfies equal coalitional improvement and that therefore $\tau \neq 0$ exists such that $a_j = \tau$ for all $j \in Z$, and by Lemma 1, $\psi^{a,b,\alpha} = \psi^{\frac{a}{\tau},\frac{b}{\tau},\alpha}$. Define $\widetilde{b} = \frac{b}{\tau}$, then

$$\begin{array}{l} P^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i\},w) = (1-\alpha)Y;\; P^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{j\},w) = (1-\alpha)Z \\ P^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,j\},w) = (1-\alpha)X + \widetilde{b}_i(1-\alpha)Z + \widetilde{b}_j(1-\alpha)Y \\ P^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i\},w') = (1-\alpha)Y + \alpha A;\; P^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{j\},w') = (1-\alpha)Z + \alpha B \\ P^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,j\},w') = (1-\alpha)X + \widetilde{b}_i(1-\alpha)Z + \widetilde{b}_j(1-\alpha)Y + \widetilde{b}_i\alpha B + \widetilde{b}_j\alpha A. \end{array}$$

This in turn implies

$$\begin{array}{lcl} \psi_i^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,j\},w) & = & \frac{1}{2}\left[(1-\alpha)X + \widetilde{b}_i(1-\alpha)Z + \widetilde{b}_j(1-\alpha)Y\right] - \\ & & \widetilde{b}_i(1-\alpha)Z + \frac{\alpha}{2}w(\{i,j\}) \\ \psi_i^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,j\},w') & = & \frac{1}{2}\left[(1-\alpha)X + \widetilde{b}_i(1-\alpha)Z + \widetilde{b}_j(1-\alpha)Y\right] - \\ & & \widetilde{b}_i(1-\alpha)Z + \frac{\alpha}{2}w'(\{i,j\}) + \\ & & \frac{1}{2}(1-\alpha)\left[\widetilde{b}_i\alpha B + \widetilde{b}_j\alpha A\right] - \widetilde{b}_i(1-\alpha)\alpha B. \end{array}$$

Applying both forms of consistency and noting $w(\{i,j\}) = w'(\{i,j\})$ yields:

$$\begin{array}{lcl} 0 & = & \psi_i^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,j,k\},v) - \psi_i^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,j,k\},v) \\ & = & \psi_i^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,j\},w') - \psi_i^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,j\},w) \\ & = & \frac{1}{2}(1-\alpha)\left[\widetilde{b}_i\alpha B + \widetilde{b}_j\alpha A\right] - \widetilde{b}_i(1-\alpha)\alpha B \\ & = & \frac{1}{2}(1-\alpha)\alpha\left[\widetilde{b}_jA - \widetilde{b}_iB\right]. \end{array}$$

So, $\alpha = 0, 1$, or

$$\begin{aligned}
0 &= \widetilde{b}_j A - \widetilde{b}_i B \\
&= \widetilde{b}_j \left[\tfrac{1}{2} v(\{i,k\}) - \psi_k^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{i,k\},v) \right] - \widetilde{b}_i \left[\tfrac{1}{2} v(\{j,k\}) - \psi_k^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(\{j,k\},v) \right] \\
&= \widetilde{b}_j \left[\tfrac{1}{2} v(\{i,k\}) - \left(\tfrac{1}{2} v(\{i,k\}) + \widetilde{b}_i (1-\alpha) v(\{k\}) + \widetilde{b}_k (1-\alpha) v(\{i\}) \right) \right] - \\
&\quad \widetilde{b}_i \left[\tfrac{1}{2} v(\{j,k\}) - \left(\tfrac{1}{2} v(\{j,k\}) + \widetilde{b}_j (1-\alpha) v(\{k\}) + \widetilde{b}_k (1-\alpha) v(\{j\}) \right) \right] \\
&= \widetilde{b}_i \widetilde{b}_k (1-\alpha) v(\{j\}) - \widetilde{b}_j \widetilde{b}_k (1-\alpha) v(\{i\}) \\
&= \widetilde{b}_k (1-\alpha) \left[\widetilde{b}_i v(\{j\}) - \widetilde{b}_j v(\{i\}) \right].
\end{aligned}$$

This implies $\alpha = 1$ or $\widetilde{b}_k = 0$. The former equality implies $\psi^{\frac{a}{\tau},\frac{b}{\tau},\alpha} = \eta$. The latter equality implies $\widetilde{b}_j = 0$ for all $j \in Z$ as the indices were chosen arbitrarily, and this implies

$$\psi_j^{\frac{a}{\tau},\frac{b}{\tau},\alpha}(N,v) = \tfrac{(1-\alpha)v(N)}{|N|} + \tfrac{\alpha v(N)}{|N|} = \tfrac{v(N)}{|N|} \quad \text{for all } j \in Z,\ (N,v) \in G.$$

Hence, $\psi^{\frac{a}{\tau},\frac{b}{\tau},\alpha} = \eta$. Wrapping up: $\alpha = 0$, or $\psi^{a,b,\alpha} = \psi^{\frac{a}{\tau},\frac{b}{\tau},\alpha} = \eta$.

Proof of Prop 11: For $\alpha = 1$, HM-consistency follows immediately as $\psi^{a,b,1} = \eta$. We prove the part for $\alpha = 0$. Let $(N,v) \in G$. For $U \subset N$ satisfying $|U| = |N|-1$, the proposition follows immediately. Now fix $U \subset N$ satisfying $0 \neq |U| < |N|-1$. To prove the proposition for $\alpha = 0$, we show

$$P^{a,b,0}(N,v) = P^{a,b,0}\left(N \backslash U, v^{U,\psi^{a,b,0}}\right) + \textstyle\prod_{k \in N \backslash U} \frac{b_k}{a_k} P^{a,b,0}(U,v). \tag{2}$$

We do this by induction on the cardinality of nonempty subsets S of $N \backslash U$. Note that for all $N \subseteq Z$, and $\emptyset \neq U \subset N$, we have

$$P^{a,b,0}(N,v) = \frac{v(N) - \sum_{k \in U} \psi_k^{a,b,0}(N,v) + \sum_{k \in N \backslash U} b_k P^{a,b,0}(N \backslash \{k\}, v)}{\sum_{k \in N \backslash U} a_k}.$$

Hence, taking $i \in N \backslash U$, we have

$$\begin{aligned}
P^{a,b,0}(\{i\} \cup U, v) &= \frac{v(\{i\} \cup U) - \sum_{k \in U} \psi_k^{a,b,0}(\{i\} \cup U, v) + b_i P^{a,b,0}(U,v)}{a_i} \\
&= \frac{v^{U,\psi^{a,b,0}}(\{i\})}{a_i} + \frac{b_i}{a_i} P^{a,b,0}(U,v) \\
&= P^{a,b,0}\left(\{i\}, v^{U,\psi^{a,b,0}}\right) + \frac{b_i}{a_i} P^{a,b,0}(U,v).
\end{aligned}$$

Let $S \subseteq N \backslash U$, $|S| \geq 2$, and assume for all $D \subset S$:

$$P^{a,b,0}(D \cup U, v) = P^{a,b,0}\left(D, v^{U,\psi^{a,b,0}}\right) + \textstyle\prod_{i \in D} \frac{b_i}{a_i} P^{a,b,0}(U,v). \tag{3}$$

Then,

$$\begin{aligned}
P^{a,b,0}(S \cup U, v) &= \frac{v(S \cup U) - \sum_{k \in U} \psi^{a,b,0}(S \cup U, v) + \sum_{k \in S} b_k P^{a,b,0}((S \cup U) \backslash \{k\}, v)}{\sum_{k \in S} a_k} \\
&= \frac{v^{U,\psi^{a,b,0}}(S) + \sum_{k \in S} b_k P^{a,b,0}((S \cup U) \backslash \{k\}, v)}{\sum_{k \in S} a_k}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{v^{U,\psi^{a,b,0}}(S)+\sum_{k\in S} b_k P^{a,b,0}\left(S\backslash\{k\}, v^{U,\psi^{a,b,0}}\right)}{\sum_{k\in S} a_k} + \\
&\quad \frac{\sum_{k\in S} b_k \left[\prod_{i\in S\backslash\{k\}} \frac{b_i}{a_i} P^{a,b,0}(U,v)\right]}{\sum_{k\in S} a_k} \\
&= P^{a,b,0}\left(S, v^{U,\psi^{a,b,0}}\right) + \frac{\sum_{k\in S} b_k \left[\prod_{i\in S\backslash\{k\}} \frac{b_i}{a_i} P^{a,b,0}(U,v)\right]}{\sum_{k\in S} a_k} \\
&= P^{a,b,0}\left(S, v^{U,\psi^{a,b,0}}\right) + \frac{\sum_{k\in S} a_k \left[\prod_{i\in S} \frac{b_i}{a_i} P^{a,b,0}(U,v)\right]}{\sum_{k\in S} a_k} \\
&= P^{a,b,0}\left(S, v^{U,\psi^{a,b,0}}\right) + \prod_{i\in S} \frac{b_i}{a_i} P^{a,b,0}(U,v).
\end{aligned}
$$

The second equality follows from Def. 4, the third one follows from (3). Now, let $k \in N\backslash U$, then by definition of $\psi^{a,b,0}$ applied to the game (N,v)

$$
\begin{aligned}
& \psi_k^{a,b,0}(N,v) \\
=\; & a_k P^{a,b,0}(N,v) - b_k P^{a,b,0}(N\backslash\{k\}, v) \\
=\; & a_k P^{a,b,0}\left(N\backslash U, v^{U,\psi^{a,b,0}}\right) - b_k P^{a,b,0}\left(N\backslash(U\cup\{k\}), v^{U,\psi^{a,b,0}}\right) + \\
& a_k \left[\prod_{i\in N\backslash U} \frac{b_i}{a_i} P^{a,b,0}(U,v)\right] - b_k \left[\prod_{i\in N\backslash(U\cup\{k\})} \frac{b_i}{a_i} P^{a,b,0}(U,v)\right] \\
=\; & \psi_k^{a,b,0}\left(N\backslash U, v^{U,\psi^{a,b,0}}\right) + \left[\frac{\prod_{i\in N\backslash U} b_i}{\prod_{i\in N\backslash(U\cup\{k\})} a_i} - \frac{\prod_{i\in N\backslash U} b_i}{\prod_{i\in N\backslash(U\cup\{k\})} a_i}\right] P^{a,b,0}(U,v) \\
=\; & \psi_k^{a,b,0}\left(N\backslash U, v^{U,\psi^{a,b,0}}\right).
\end{aligned}
$$

The second equality follows from (3), the third one follows by definition of the value $\psi^{a,b,0}$ applied to the game $\left(N\backslash U, v^{U,\psi^{a,b,0}}\right)$.

The converse statement (HM-consistency implies $\alpha = 0$ or $\alpha = 1$) is implied by the following. Take $\{i,j,k\} \in Z$ and define game v on player set $\{i,j,k\}$, and game w on player set $\{i,j\}$ as follows:

$$
\begin{aligned}
v(\{i,j,k\}) &= 1, & v(S) = 0 \text{ for all } S \subsetneq \{i,j,k\}; \\
w(\{i,j\}) &= 1 - \frac{a_k}{a_i+a_j+a_k}(1-\alpha) - \frac{\alpha}{3}, & w(S) = 0 \text{ for all} S \subsetneq \{i,j\}.
\end{aligned}
$$

Note that w is the reduced game of v connected to the HM-consistency for linear potential value $\psi^{a,b,\alpha}$ where player k leaves. Now,

$$
\begin{aligned}
\psi_i^{a,b,\alpha}(\{i,j,k\}, v) &= \frac{a_i}{a_i+a_j+a_k}(1-\alpha) + \frac{\alpha}{3}, \\
\psi_i^{a,b,\alpha}(\{i,j\}, w) &= \frac{a_i}{a_i+a_j}(1-\alpha)\left[1 - \frac{a_k}{a_i+a_j+a_k}(1-\alpha) - \frac{\alpha}{3}\right] \\
&\quad + \frac{\alpha}{2}\left[1 - \frac{a_k}{a_i+a_j+a_k}(1-\alpha) - \frac{\alpha}{3}\right].
\end{aligned}
$$

In the interest of the proof, we want to establish when the difference between those two amounts equals zero:

$$
\begin{aligned}
0 &= \psi_i^{a,b,\alpha}(\{i,j\}, w) - \psi_i^{a,b,\alpha}(\{i,j,k\}, v) \\
&= \frac{a_i}{a_i+a_j}(1-\alpha)\left[1 - \frac{a_k}{a_i+a_j+a_k}(1-\alpha) - \frac{\alpha}{3}\right] + \\
&\quad \frac{\alpha}{2}\left[1 - \frac{a_k}{a_i+a_j+a_k}(1-\alpha) - \frac{\alpha}{3}\right] - \frac{a_i}{a_i+a_j+a_k}(1-\alpha) - \frac{\alpha}{3} \\
&= (1-\alpha)\left[\frac{a_i}{a_i+a_j}\left[1 - \frac{a_k}{a_i+a_j+a_k}(1-\alpha) - \frac{\alpha}{3}\right] - \frac{a_i}{a_i+a_j+a_k}\right] + \\
&\quad \frac{\alpha}{2}\left[1 - \frac{a_k}{a_i+a_j+a_k}(1-\alpha) - \frac{\alpha}{3}\right] - \frac{\alpha}{3}
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\alpha)\left[\tfrac{a_i(a_i+a_j+a_k)-a_ia_k-a_i(a_i+a_j)}{(a_i+a_j)(a_i+a_j+a_k)} - \tfrac{\alpha}{3}\tfrac{a_i}{a_i+a_j} + \tfrac{\alpha a_i a_k}{(a_i+a_j)(a_i+a_j+a_k)}\right] \\
&\quad + \tfrac{\alpha}{2}\left[1 - \tfrac{a_k}{a_i+a_j+a_k}(1-\alpha) - \tfrac{\alpha}{3}\right] - \tfrac{\alpha}{3} \\
&= (1-\alpha)\,\alpha\left[-\tfrac{1}{3}\tfrac{a_i}{a_i+a_j} + \tfrac{a_ia_k}{(a_i+a_j)(a_i+a_j+a_k)}\right] + \\
&\quad \tfrac{\alpha}{2}\left[1 - \tfrac{a_k}{a_i+a_j+a_k}(1-\alpha) - \tfrac{\alpha}{3}\right] - \tfrac{\alpha}{3}.
\end{aligned}
$$

Clearly, $\psi_i^{a,b,\alpha}(\{i,j\},w) = \psi_i^{a,b,\alpha}(\{i,j,k\},v)$ if $\alpha = 0$ or if

$$
\begin{aligned}
0 &= (1-\alpha)\left[-\tfrac{1}{3}\tfrac{a_i}{a_i+a_j} + \tfrac{a_ia_k}{(a_i+a_j)(a_i+a_j+a_k)}\right] + \\
&\quad \tfrac{1}{2}\left[1 - \tfrac{a_k}{a_i+a_j+a_k}(1-\alpha) - \tfrac{\alpha}{3}\right] - \tfrac{1}{3} \\
&= (1-\alpha)\left[-\tfrac{1}{3}\tfrac{a_i}{a_i+a_j} + \tfrac{a_ia_k}{(a_i+a_j)(a_i+a_j+a_k)} - \tfrac{1}{2}\tfrac{a_k}{a_i+a_j+a_k}\right] + \tfrac{1}{6} - \tfrac{\alpha}{6} \\
&= (1-\alpha)\left[-\tfrac{1}{3}\tfrac{a_i}{a_i+a_j} + \tfrac{a_ia_k}{(a_i+a_j)(a_i+a_j+a_k)} - \tfrac{1}{2}\tfrac{a_k}{a_i+a_j+a_k} + \tfrac{1}{6}\right].
\end{aligned}
$$

So, another instance for which $\psi_i^{a,b,\alpha}(\{i,j\},w) = \psi_i^{a,b,\alpha}(\{i,j,k\},v)$ is clearly $\alpha = 1$. The final possibility is

$$
\begin{aligned}
0 &= -\tfrac{1}{3}\tfrac{a_i}{a_i+a_j} + \tfrac{a_ia_k}{(a_i+a_j)(a_i+a_j+a_k)} - \tfrac{1}{2}\tfrac{a_k}{a_i+a_j+a_k} + \tfrac{1}{6} \\
&= -2\tfrac{a_i}{a_i+a_j} + 6\tfrac{a_ia_k}{(a_i+a_j)(a_i+a_j+a_k)} - 3\tfrac{a_k}{a_i+a_j+a_k} + 1 \\
&= \tfrac{-2a_i(a_i+a_j+a_k)+6a_ia_k-3a_k(a_i+a_j)}{(a_i+a_j)(a_i+a_j+a_k)} + 1 \\
&= \tfrac{-2a_i^2-2a_ia_j+a_ia_k-3a_ja_k}{(a_i+a_j)(a_i+a_j+a_k)} + 1.
\end{aligned}
$$

Recall that indices i,j,k were chosen randomly, now this combined with the requirement $\frac{-2a_i^2-2a_ia_j+a_ia_k-3a_ja_k}{(a_i+a_j)(a_i+a_j+a_k)} = -1$ implies that $a_i = a_j = a_k$ for all $i,j,k \in Z$. Let $a_i = a_j = a_k = a \neq 0$, then

$$
\frac{-2a_i^2-2a_ia_j+a_ia_k-3a_ja_k}{(a_i+a_j)(a_i+a_j+a_k)} = \frac{a^2(-2-2+1-3)}{6a^2} = -1.
$$

So, for all $a \in R^{|Z|}$ such that $a_j = a_k$ for all $j,k \in Z$, $\psi_i^{a,b,\alpha}(\{i,j\},w) = \psi_i^{a,b,\alpha}(\{i,j,k\},v)$.

Proof of Prop 12 Let ψ be a linear potential value, and arbitrary three-player set $\{i,j,k\} \subset Z$. Then, $\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right) = \frac{a_k}{a_i+a_j+a_k}(1-\alpha) + \frac{\alpha}{3}$. Consider the 2-person game w given by

$$
\begin{aligned}
w(\{i\}) = w(\{j\}) &= -\tfrac{1}{2}\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right) \\
w(\{i,j\}) &= 1 - \psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right).
\end{aligned}
$$

Observe that w is the Sobolev-reduced game of 3-person unanimity game $\left(\{i,j,k\},u_{\{i,j,k\}}\right)$. Then,

$$
\begin{aligned}
P^{a,b,\alpha}(\{r\},w) &= \frac{-\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{a_r},\ r = i,j; \\
P^{a,b,\alpha}(\{i,j\},w) &= \frac{(1-\alpha)\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)-\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\left[\frac{a_ib_i+a_jb_j}{a_ia_j}\right]}{a_i+a_j}.
\end{aligned}
$$

So, $\psi_i(\{i,j\},w)$

$$
\begin{aligned}
&= a_i \frac{(1-\alpha)\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)-\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\left[\frac{a_ib_i+a_jb_j}{a_ia_j}\right]}{a_i+a_j}+\\
&\quad b_i\frac{-\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{a_j}+\frac{\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)}{2}\\
&= \frac{a_i}{a_i+a_j}(1-\alpha)\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)-\\
&\quad \frac{a_i}{a_i+a_j}\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\left[\frac{a_ib_i+a_jb_j}{a_ia_j}\right]-\\
&\quad b_i\frac{-\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{a_j}+\frac{\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)}{2}\\
&= \frac{a_i}{a_i+a_j}(1-\alpha)\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)+\frac{\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)}{2}+\\
&\quad -\frac{a_i}{a_i+a_j}\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\left[\frac{a_ib_i+a_jb_j}{a_ia_j}\right].
\end{aligned}
$$

Note that for the first term behind the equality sign it holds that

$$
\begin{aligned}
&(1-\alpha)\frac{a_i}{a_i+a_j}\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)\\
=\;&(1-\alpha)\frac{a_i}{a_i+a_j}\left(1-\frac{a_k}{a_i+a_j+a_k}(1-\alpha)-\frac{\alpha}{3}\right)\\
=\;&(1-\alpha)\frac{a_i}{a_i+a_j}\left(\frac{a_i+a_j}{a_i+a_j+a_k}+\alpha\left[\frac{a_k}{a_i+a_j+a_k}-\frac{1}{3}\right]\right)\\
=\;&(1-\alpha)\frac{a_i}{a_i+a_j+a_k}+(1-\alpha)\alpha\frac{a_i}{a_i+a_j}\left[\frac{a_k}{a_i+a_j+a_k}-\frac{1}{3}\right]
\end{aligned}
$$

and for next three terms depending on $\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)$ together we have

$$
\begin{aligned}
-&\;\frac{a_i}{a_i+a_j}\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\left[\frac{a_ib_i+a_jb_j}{a_ia_j}\right]+b_i\frac{\frac{1}{2}(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{a_j}\\
=&\;\frac{(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{2}\left[\frac{-b_j}{a_i+a_j}+\frac{-a_ib_i}{a_j(a_i+a_j)}+\frac{b_i}{a_j}\right]\\
=&\;\frac{(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{2}\left[\frac{-a_jb_j-a_ib_i+b_i(a_i+a_j)}{a_j(a_i+a_j)}\right]\\
=&\;\frac{(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{2}\left[\frac{a_j(b_i-b_j)}{a_j(a_i+a_j)}\right]\\
=&\;\frac{(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{2}\left[\frac{b_i-b_j}{a_i+a_j}\right].
\end{aligned}
$$

Hence, $\psi_i(\{i,j\},w) = (1-\alpha)\frac{a_i}{a_i+a_j+a_k}+(1-\alpha)\alpha\frac{a_i}{a_i+a_j}\left[\frac{a_k}{a_i+a_j+a_k}-\frac{1}{3}\right]+\frac{(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{2}\left[\frac{b_i-b_j}{a_i+a_j}\right]+\frac{\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)}{2}$.

Then, S-consistency implies

$$
\begin{aligned}
0 &= \psi_i(\{i,j\},w)-\psi_i\left(\{i,j,k\},u_{\{i,j,k\}}\right)\\
&= (1-\alpha)\alpha\frac{a_i}{a_i+a_j}\left[\frac{a_k}{a_i+a_j+a_k}-\frac{1}{3}\right]+\frac{(1-\alpha)\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)}{2}\left[\frac{b_i-b_j}{a_i+a_j}\right]+\\
&\quad \alpha\frac{\left(1-\psi_k\left(\{i,j,k\},u_{\{i,j,k\}}\right)\right)}{2}-\frac{\alpha}{3}\\
&= \frac{a_i}{a_i+a_j}(1-\alpha)\alpha\left[\frac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right]+\frac{(1-\alpha)}{2}\left[\frac{a_i}{a_i+a_j+a_k}(1-\alpha)+\frac{\alpha}{3}\right]\left[\frac{b_i-b_j}{a_i+a_j}\right]+\\
&\quad \frac{\alpha}{2}\left(1-\frac{a_k}{a_i+a_j+a_k}(1-\alpha)-\frac{\alpha}{3}\right)-\frac{\alpha}{3}\\
&= \frac{a_i}{a_i+a_j}(1-\alpha)\alpha\left[\frac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right]+\frac{(1-\alpha)}{2}\left[\frac{a_i}{a_i+a_j+a_k}(1-\alpha)+\frac{\alpha}{3}\right]\left[\frac{b_i-b_j}{a_i+a_j}\right]+\\
&\quad \frac{\alpha}{2}-\frac{\alpha}{2}\left(\frac{a_k}{a_i+a_j+a_k}(1-\alpha)\right)+\frac{\alpha}{2}\left(-\frac{\alpha}{3}\right)-\frac{\alpha}{3}
\end{aligned}
$$

$$\begin{aligned} = \quad & \tfrac{a_i}{a_i+a_j}\,(1-\alpha)\,\alpha\left[\tfrac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right]+\tfrac{(1-\alpha)}{2}\left[\tfrac{a_i}{a_i+a_j+a_k}(1-\alpha)+\tfrac{\alpha}{3}\right]\left[\tfrac{b_i-b_j}{a_i+a_j}\right] \\ & -\tfrac{\alpha}{2}\,(1-\alpha)\left(\tfrac{a_k}{a_i+a_j+a_k}\right)+\tfrac{(1-\alpha)\alpha}{6}. \end{aligned}$$

This holds clearly if $\alpha = 1$ or if

$$0=\tfrac{a_i}{a_i+a_j}\alpha\left[\tfrac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right]+\tfrac{1}{2}\left[\tfrac{a_i}{a_i+a_j+a_k}(1-\alpha)+\tfrac{\alpha}{3}\right]\left[\tfrac{b_i-b_j}{a_i+a_j}\right]-\tfrac{\alpha}{2}\left(\tfrac{a_k}{a_i+a_j+a_k}\right)+\tfrac{\alpha}{6}. \tag{4}$$

In (4), $a_i = a_j = a_k$ implies

$$0=\tfrac{1}{2}\left[\tfrac{1}{3}(1-\alpha)+\tfrac{\alpha}{3}\right]\left[\tfrac{b_i-b_j}{a_i+a_j}\right]-\tfrac{\alpha}{2}\left(\tfrac{1}{3}\right)+\tfrac{\alpha}{6}=\tfrac{1}{6}\left[\tfrac{b_i-b_j}{a_i+a_j}\right].$$

This can only hold if $b_i = b_j$.
Moreover, in Eq. (4) $\alpha = 0$ implies $0 = \tfrac{1}{2}\tfrac{a_i}{a_i+a_j+a_k}\left[\tfrac{b_i-b_j}{a_i+a_j}\right]$. So, $b_i = b_j$.
Also, in Eq. (4) $b_i = b_j$ implies

$$0=\tfrac{a_i}{a_i+a_j}\alpha\left[\tfrac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right]-\tfrac{\alpha}{2}\left(\tfrac{a_k}{a_i+a_j+a_k}\right)+\tfrac{\alpha}{6}.$$

Then, $\alpha = 0$ or

$$\begin{aligned} 0 &= \tfrac{a_i}{a_i+a_j}\left[\tfrac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right]-\tfrac{1}{2}\left(\tfrac{a_k}{a_i+a_j+a_k}-\tfrac{1}{3}\right) \\ &= \tfrac{a_i}{a_i+a_j}\left[\tfrac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right]-\tfrac{1}{2}\left(\tfrac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right) \\ &= \left[\tfrac{a_i}{a_i+a_j}-\tfrac{1}{2}\right]\left[\tfrac{2a_k-a_i-a_j}{a_i+a_j+a_k}\right]. \end{aligned}$$

The latter implies $a_i = a_j$. **Summarizing:** S-consistency induces $b_i = b_j \iff (\alpha = 0 \text{ or } a_i = a_j)$. So, an additional candidate for Eq. (4) must be looked for under the condition $b_i \neq b_j$ and $\alpha \neq 0$ and $a_i \neq a_j$, but this does not yield a solution (tedious calculations available from author).

Proof of Prop 13: That (ii) implies (i) is easily verified. To prove (i) implies (ii), note that any δ-reducing player in (N, v) is a null-player in the game (N, v_δ) and vice versa. Let $\psi^{a,b,\alpha}$ be a linear-potential value and $\delta \neq 0$, then $\psi^{a,b,\alpha}(N, v) = \psi^{a,\frac{b}{\delta},\alpha}(N, v_\delta)$ for all games (N, v). If $\psi^{a,b,\alpha}$ is α-egalitarian with respect to δ-reducing players, then due to Prop. 5, we must have $a = \frac{b}{\delta}$, i,e., $b = \delta a$. So, $\psi^{a,b,\alpha} = Sh_\delta^{a,\alpha}$.

Proof of Lemma 14: First, we relate linear-potential values to a natural reference point. Let $\psi^{a,b,\alpha}$ be a linear-potential value, then for an arbitrary game (N, v), and players $i, j \in N$

$$\begin{aligned} & \psi_i^{a,b,\alpha}(N,v) > \psi_j^{a,b,\alpha}(N,v) \iff \\ & a_i P^{a,b,\alpha}(N,v) - b_i P^{a,b,\alpha}(N\backslash\{i\},v) > \\ & a_j P^{a,b,\alpha}(N,v) - b_j P^{a,b,\alpha}(N\backslash\{j\},v) \iff \\ & (1-\alpha)\left[a_i P^{a,b,0}(N,v) - b_i P^{a,b,0}(N\backslash\{i\},v)\right] > \\ & (1-\alpha)\left[a_j P^{a,b,0}(N,v) - b_j P^{a,b,0}(N\backslash\{j\},v)\right] \iff \\ & (1-\alpha)\psi_i^{a,b,0}(N,v) > (1-\alpha)\psi_j^{a,b,0}(N,v). \end{aligned}$$

Hence, the order of the utilities i and j receive for $\alpha < 1$ is the identical to the one for $\alpha = 0$, and is reversed to the order for $\alpha = 0$ if $\alpha > 1$. So, taking $\alpha_1, \alpha_2 < 1$ yields ordinal equivalence between ψ^{a,b,α_1} and ψ^{a,b,α_2} as the order between $\psi_i^{a,b,0}(N,v)$ and $\psi_j^{a,b,0}(N,v)$ depends on (N,v) but is independent from α, and by the same argument so does $\alpha_1, \alpha_2 > 1$.

Proof of Prop 15: Let $(N,u_T) \in G$, $i \in T, j \in N\backslash T$ and let $\psi^{a,b,0}$ be socially acceptable, $A_S \equiv \sum_{k\in S} a_k$ for any $S \subseteq Z$, then

$$\begin{array}{lcl} \psi_i^{a,b,0}(N,u_{N\backslash\{j\}}) & \geq & \psi_j^{a,b,0}(N,u_{N\backslash\{j\}}) \Longleftrightarrow \\ a_i \frac{1+b_j/A_{N\backslash\{j\}}}{A_N} & \geq & a_j \frac{1+b_j/A_{N\backslash\{j\}}}{A_N} - \frac{b_j}{A_{N\backslash\{j\}}} \Longleftrightarrow \\ (a_i - a_j) \frac{A_{N\backslash\{j\}}+b_j}{A_{N\backslash\{j\}}A_N} & \geq & -\frac{b_j\left(A_{N\backslash\{j\}}+a_j\right)}{A_{N\backslash\{j\}}A_N} \Longleftrightarrow \\ \frac{(a_i-a_j)A_{N\backslash\{j\}}}{A_{N\backslash\{j\}}A_N} & \geq & -\frac{b_j\left(A_{N\backslash\{j\}}+a_j+a_i-a_j\right)}{A_{N\backslash\{j\}}A_N} \Longleftrightarrow \\ \frac{(a_i-a_j)}{A_N} & \geq & -\frac{b_j\left(A_{N\backslash\{j\}}+a_i\right)}{A_{N\backslash\{j\}}A_N}. \end{array}$$

By Lemma 14, the latter inequality must hold for all socially acceptable $\psi^{a,b,\alpha}$ with $\alpha < 1$, yet the converse must hold for $\alpha > 1$ leading to:

$$\begin{array}{lcll} a_j - b_j & \leq & \left(1 + \frac{b_j}{A_{N\backslash\{j\}}}\right) a_i & \text{if } A_N > 0, \quad \alpha < 1, \\ a_j - b_j & \geq & \left(1 + \frac{b_j}{A_{N\backslash\{j\}}}\right) a_i & \text{if } A_N < 0, \quad \alpha < 1, \\ a_j - b_j & \geq & \left(1 + \frac{b_j}{A_{N\backslash\{j\}}}\right) a_i & \text{if } A_N > 0, \quad \alpha > 1, \\ a_j - b_j & \leq & \left(1 + \frac{b_j}{A_{N\backslash\{j\}}}\right) a_i & \text{if } A_N < 0, \quad \alpha > 1. \end{array}$$

Second, we must have

$$a_j P^{a,b,\alpha}(N,u_T) - b_j P^{\alpha,b,\alpha}(N\backslash\{j\},u_T) + \frac{\alpha}{|N|} \geq 0.$$

In particular, for $T = N\backslash\{j\}$. Therefore,

$$\begin{array}{lcl} a_j P^{a,b,\alpha}(N,u_{N\backslash\{j\}\}}) - b_j P^{\alpha,b,\alpha}(N\backslash\{j\},u_{n\backslash\{j\}}) + \frac{\alpha}{|N|} & \geq & 0 \Longleftrightarrow \\ a_j \frac{(1-\alpha)+(1-\alpha)b_j/A_{N\backslash\{j\}}}{A_N} - (1-\alpha)\, b_j/A_{N\backslash\{j\}} + \frac{\alpha}{|N|} & \geq & 0 \Longleftrightarrow \\ (1-\alpha)\frac{a_jA_{N\backslash\{j\}}+a_jb_j-b_j\left(a_j+A_{N\backslash\{j\}}\right)}{A_{N\backslash\{j\}}A_N} + \frac{\alpha}{|N|} & \geq & 0 \Longleftrightarrow \\ (1-\alpha)\frac{(a_j-b_j)A_{N\backslash\{j\}}}{A_{N\backslash\{j\}}A_N} + \frac{\alpha}{|N|} & \geq & 0 \Longleftrightarrow \\ (1-\alpha)\frac{(a_j-b_j)}{A_N} + \frac{\alpha}{|N|} & \geq & 0. \end{array}$$

Let $\underline{\mu}^+ \equiv \inf_{N\subseteq Z: j\in N} \left\{\frac{A_N}{|N|} | \frac{A_N}{|N|} > 0\right\}$ and $\overline{\mu}^- \equiv \sup_{N\subseteq Z: j\in N} \left\{\frac{A_N}{|N|} | \frac{A_N}{|N|} < 0\right\}$, the latter implies.

$$\begin{array}{lcll} a_j - b_j & \geq & -\frac{\alpha}{(1-\alpha)}\underline{\mu}^+ & \text{if } A_N > 0, \quad \alpha < 1, \\ a_j - b_j & \leq & -\frac{\alpha}{(1-\alpha)}\overline{\mu}^- & \text{if } A_N < 0, \quad \alpha < 1, \\ a_j - b_j & \leq & -\frac{\alpha}{(1-\alpha)}\underline{\mu}^+ & \text{if } A_N > 0, \quad \alpha > 1, \\ a_j - b_j & \geq & -\frac{\alpha}{(1-\alpha)}\overline{\mu}^- & \text{if } A_N < 0, \quad \alpha > 1. \end{array}$$

Proof of Corollary 16: Because $w_j > 0$ for all $j \in Z$, Prop. 15 implies that social acceptability of $Sh_\delta^{w,\alpha}$ implies

$$\begin{array}{rcccll} -\frac{\alpha}{(1-\alpha)}\underline{\mu}_j^+ & \leq & (1-\delta)\, w_j & \leq & \left(1+\frac{(1-\delta)w_j}{W_{N\setminus\{j\}}}\right) w_i & \text{if } \alpha < 1, \\ \left(1+\frac{(1-\delta)w_j}{W_{N\setminus\{j\}}}\right) w_i & \leq & (1-\delta)\, w_j & \leq & -\frac{\alpha}{(1-\alpha)}\underline{\mu}_j^+ & \text{if } \alpha > 1, \end{array}$$

where $W_{N\setminus\{j\}} = \sum_{k\in N\setminus\{j\}} w_k$. Therefore, for $\alpha < 1$, we have

$$-\frac{\alpha}{(1-\alpha)}\underline{\mu}_j^+ \leq (1-\delta)\, w_j \leq \left(1+\frac{(1-\delta)w_j}{W_{N\setminus\{j\}}}\right) w_i.$$

Since $\min_{k\in Z} w_k \leq \underline{\mu}_j^+ \leq \max_{k\in Z} w_k$ and $W_{N\setminus\{j\}}$ can be made arbitrarily large, we have

$$-\frac{\alpha}{(1-\alpha)}\underline{\mu}_j^+ \leq (1-\delta)\min_{k\in Z} w_k \text{ and } (1-\delta)\max_{k\in Z} w_k \leq \min_{k\in Z} w_k$$

The latter leads to

$$\begin{array}{l} (1-\delta)\max_{k\in Z} w_k \leq \min_{k\in Z} w_k \implies \\ \max_{k\in Z} w_k - \min_{k\in Z} w_k \leq \delta \max_{k\in Z} w_k \implies \frac{\max_{k\in Z} w_k - \min_{k\in Z} w_k}{\max_{k\in Z} w_k} \leq \delta. \end{array}$$

The former leads for $\alpha > 0$ to

$$\begin{array}{l} -\frac{\alpha}{1-\alpha}\min_{k\in Z} w_k \leq (1-\delta)\min_{k\in Z} w_k \implies -\frac{\alpha}{1-\alpha} \leq (1-\delta) \implies \\ -\alpha \leq 1-\alpha-\delta+\alpha\delta \implies \delta \leq \frac{1}{1-\alpha}. \end{array}$$

The former leads for $\alpha \leq 0$ to

$$\begin{array}{l} -\frac{\alpha}{1-\alpha}\max_{k\in Z} w_k \leq (1-\delta)\min_{k\in Z} w_k \implies \\ -\alpha \max_{k\in Z} w_k \leq (1-\alpha-\delta+\alpha\delta)\min_{k\in Z} w_k \implies \\ -\alpha \max_{k\in Z} w_k - (1-\alpha)\min_{k\in Z} w_k \leq -(1-\alpha)\,\delta \min_{k\in Z} w_k \implies \\ \frac{-\alpha \max_{k\in Z} w_k - (1-\alpha)\min_{k\in Z} w_k}{-(1-\alpha)\min_{k\in Z} w_k} \geq \delta \implies \\ \delta \leq \frac{\alpha \max_{k\in Z} w_k + (1-\alpha)\min_{k\in Z} w_k}{(1-\alpha)\min_{k\in Z} w_k} = \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha}\left(\frac{\max_{k\in Z} w_k - \min_{k\in Z} w_k}{\min_{k\in Z} w_k}\right). \end{array}$$

So, for $0 \leq \alpha < 1$, $\frac{\max_{k\in Z} w_k - \min_{k\in Z} w_k}{\max_{k\in Z} w_k} \leq \delta \leq \frac{1}{1-\alpha}$, and for $\alpha \leq 0$, $\frac{\max_{k\in Z} w_k - \min_{k\in Z} w_k}{\max_{k\in Z} w_k} \leq \delta \leq \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha}\left(\frac{\max_{k\in Z} w_k - \min_{k\in Z} w_k}{\min_{k\in Z} w_k}\right)$.

For $\alpha > 1$, the following must hold

$$\left(1+\frac{(1-\delta)w_j}{W_{N\setminus\{j\}}}\right) w_i \leq (1-\delta)\, w_j \leq -\frac{\alpha}{(1-\alpha)}\underline{\mu}_j^+.$$

Now, the first inequality yields a contradiction as for $N = \{i, j\}$, we have

$$\begin{array}{l} \left(1+\frac{(1-\delta)w_j}{w_i}\right) w_i \leq (1-\delta)\, w_j \implies \\ w_i + (1-\delta)\, w_j \leq (1-\delta)\min_{k\in Z} w_k \implies \\ w_i + (1-\delta)\max_{k\in Z} w_k \leq (1-\delta)\min_{k\in Z} w_k \implies \\ w_i \leq (1-\delta)\left(\min_{k\in Z} w_k - \max_{k\in Z} w_k\right) \leq 0. \end{array}$$

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