A TALE OF TWO LEMONS: MULTI-GOOD DYNAMIC ADVERSE SELECTION

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Abstract. This paper studies the role of cross-market information spillovers in a multi-good dynamic bargaining problem with interdependent values. More precisely, in an environment where a seller has two heterogeneous goods for sale in two markets and is better informed than the potential buyers about the qualities of the goods, we investigate how the information revealed through (non-)trade of one good affects the probability of trade of the other good, and its consequences to the trading dynamics and patterns of specialization. Our main finding is that when the qualities of the two goods are sufficiently negatively correlated and the seller is patient, then even if adverse selection precludes first-best efficiency for both goods, it is mitigated as sequential trade occurs quickly through the seller's endogenous signaling motive, as long as buyers in one market observe the (non-)trading outcome in the other market. As a consequence, sellers have an incentive to specialize in one of the two goods before playing the bargaining game with the buyers, in such a way to endogenously generate the required negative correlation between the qualities of the two goods. In contrast, without such cross-market observability and subsequent specialization, i.e., endogenous negative correlation, there is either bargaining delay or impasse in both markets as in the standard dynamic adverse selection problem.

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Contents

1. Introduction

Akerlof's (1970) seminal work on the "lemons" problem establishes that adverse selection can lead to severe market failure, even complete market unraveling. When sellers have private information about the quality of the product in question, sellers of peaches (that is, high quality products) will likely stay out of the market, as the market price is likely to be not high enough; but this forces the market price to be even lower, as only sellers of lemons (that is, low quality products) will likely stay in the market. In the extreme case, only the lowest quality products are brought to the market, and there is severe market failure. While Akerlof's results are derived in a competitive market setting and hence do not explain the determination of market prices, the subsequent literature confirms his findings using specific models of price formation (mostly non-cooperative bargaining) to study the price dynamics under adverse selection (e.g., Evans (1989), Deneckere and Liang (2006), Hörner and Vieille (2009), and Daley and Green (2012)). There is also a vast literature providing ways sellers could work around the adverse selection problem (e.g., engaging in costly signaling activities as in Spence (1973), Milgrom and Roberts (1986), etc.). In all these papers, the main focus is on sellers of a single product.

Strictly speaking, the assumption of a single product is a simplification, because typical products have several attributes, and as such even a single product is a bundle of attributes. This paper relaxes this assumption and studies the adverse selection problem in the context of multi-good sellers. The basic insight is that buyers, while still not able to observe the quality of the products, can make inferences about the quality of a product by observing the trading activities of the other products. That is, we consider the possibility of cross-market spillovers as a way around the adverse selection problem. To fix ideas, consider a situation with many sellers and buyers, where sellers sell two products, A and B, whose qualities are the seller's private information, and can be either high or low. Assume that buyers are divided into two groups, A and B, where a buyer from group i only demands product i, for $i = A, B$. Assume that sellers are specialized in the sense that if one of the products is of high quality, the other must be of low quality, and conversely. The following market arrangement works around the adverse selection problem: the sellers segment the market into two separate markets, 1 and 2; the sellers of high quality product A and low quality product B go to market 1 and the other sellers go to market 2; half of buyers A and half of buyers B go to market 1 and the other buyers go to market 2; the price of product A in market 1 is high, and that of product B is low, the opposite is true for market 2. That is, there's no market failure, and this is achieved solely because of cross-market information spillover: buyers that value good A are willing to go to market 1 as they can see that the prices of both goods in both markets and correctly infer that the quality of good A is high in market 1 and low in market 2. In other words, the quality of a certain product in a given market can be signaled by the price in the other market.

As a more concrete example, let sellers be car mechanics, and buyers be divided into one group that values careful and detailed service and another group that values speedy service. A seller specialized in detailed service is probably not particularly good in speedy service, and conversely. In particular, it is not atypical to encounter mechanics almost fully specialized in either detailed or speedy service. Also, the prices of services provided by car mechanics are likely to be consistent with the story of the previous paragraph: in equilibrium, a detailed car mechanic will perform speedy service at a low price to the (probably very low) residual demand of buyers that end up not served by the speedy ones; likewise, speedy car mechanics will perform detailed work at low price to the residual demand of buyers that end up not served by detailed ones. Buyers are then willing to pay a higher price to the service they value, as they correctly infer that the quality of the service must be good, given the underlying specialization of the sellers.

We build a game-theoretic model to examine price formation of the competitive equilibrium constructed above in which prices serve as signals of product qualities, and explain why negative correlation between product qualities emerges as a result of sellers' endogenous choices. Formally, we study two-good adverse selection in a dynamic bargaining model which is preceded by an investment phase where product qualities are endogenously chosen by sellers. In the investment phase, a seller chooses (at a cost) the qualities of two goods, 1 and 2, each of which can be either high (H) or low (L). Buyers observe an imperfect signal about the seller's choices which reveals whether the seller is bad at both goods or not, and use it to form their prior over the possible quality profiles of the seller: HH (high qualities for both goods), LL (low qualities for both goods), HL (high quality for good 1 and low quality for good 2), LH (low quality for good 1 and high quality for good 2). Then the bargaining game proceeds: the long-lived seller bargains sequentially with potential buyers until agreement is reached, if ever, and delay is costly. In each period, two short-lived buyers arrive, and each buyer makes a take-it-or-leave-it offer to the good she values, as long as the good has not been traded yet. The seller decides to accept or reject each offer. Buyers can observe all the previous (non-)trading activities in both markets: cross-market information spillover. To be specific, we study two information structures: public offers and private offers. With public offers, the current buyers know the previous offers in both markets and whether the offers were rejected or not. With private offers, the current buyers do not observe previous offers, but know whether the previous offers in both markets were accepted or rejected.

With public offers, we show that sellers choose to invest in such a way that the resulting prior of the buyers features negative correlation between the qualities of two goods, which then ensures that in the unique Perfect Bayesian equilibrium of the bargaining game with cross-market information spillover, trade of both goods occurs with positive probability. On the contrary, without cross-market information spillover, bargaining ends up in an impasses as illustrated by Evans (1989) and Hörner and Vieille (2009). As such, adverse selection is mitigated by cross-market spillovers. In addition, the investment phase provides a novel perspective on specialization patterns: even though it is feasible and not prohibitively costly to invest in high quality for both products, the sellers choose high quality for only one product to take advantage of the mitigated adverse selection going forward.

We fully characterize the unique Perfect Bayesian Equilibrium outcome of the bargaining game with public offers. If there is insufficiently negative correlation,[1] adverse selection is not mitigated by the information spillover: bargaining impasses exist for both goods as if buyers cannot observe (non-)trading activities in the other market. If there is sufficiently negative correlation, the introduction of multi-good bargaining results in several new phenomena:

- Information spillover. The observation that one good is accepted in the previous period is a signal that the remaining good is likely to have high quality, due to sufficiently negative correlation. The LL seller neither rejects nor accepts both offers since by trading one good and rejecting the other good, she gets a high offer from the other good.

[1]The case with independent distributions of qualities is covered. Note that the independent case is different from the unobservable case. Independent initial belief does not necessarily mean that the trading activities of the other good conveys no information, since the strategy of the seller could convey information.

- Mitigated adverse selection. Trade for good i takes places quickly if the other good is accepted previously by information spillover, which reduces the severity of adverse selection for good i. If both goods are rejected, the trade agreement of both goods is eventually reached. Otherwise, it is not profitable for LH and HL to mimic HH by rejecting both offers, which in turn makes the buyers believe that the qualities of both good are high in next period, a contradiction to no trade agreement happens.
- No skimming property. In dynamic adverse selection models with one good to sell, high-quality sellers are more willing to delay trade for a high price than low-quality sellers (skimming property), and low-quality sellers have an incentive to mimic the high-quality sellers, which relies heavily on one-dimensional types. In contrast, with multi-dimensional types, a seller LL has multiple types to mimic, and in equilibrium she chooses to mimic HL or LH and separates from HH.
- Bertrand competition with mixing offers. In period 1, two buyers compete for the LL seller by randomizing over a continuum of offers since LL seller only accepts one one of the two offers instead of accepting both offers. Given the other buyer's strategy, each buyer increases the probability of acceptance by making a higher offer since LL is more willing to accept a higher offer and reject a lower offer. In all, two buyers engage in a Bertrand competition indirectly by making offers for different goods.
- Non-monotonicity of mixing offers. For each good, the mixing offer is non-monotonic in the probability of LL seller. As the probability of LL increases, the randomizing offer increases initially and decreases eventually in the first-order stochastic dominance (FOSD) sense. The non-monotonicity comes from two opposite effects: namely" Bertrand competition effect", related to two buyers' incentives to bid up the prices to attract the seller, and "information spillover effect", related to the effectiveness of information spillover.

With private offers, all the results above still hold in the sense that the seller chooses to specialize to be good at one of the two goods, thus generates sufficiently negative correlation between the qualities of two goods, which again hastens trade agreements as in the public offers case. With sufficiently negative correlation, accepting one offers serves as a signal that the remaining good is more likely to have high quality, thus trade for the remaining good happens immediately. Without sufficiently negative correlation, trade in each market happens as if buyers cannot observe (non-)trading activities in the other market. In all, treating information structure as given, we show that sufficiently negative correlation mitigates adverse selection through information spillover, regardless of public offers or private offers.

2. Literature Review

Our analysis is related to the dynamic adverse selection models in three different aspects: market inefficiency, arrival of news, and transparency.

Market inefficiency. Standard adverse selection models a la Akerlof (1970) consider almost exclusively the case in which an informed seller trades one good with uninformed buyers. In a dynamic environment, adverse selection leads to market inefficiency, which typically takes the form of delay and, therefore, a central question is how quickly gains from trade are realized. (See for example Evans (1989), Vincent (1989, 1990), Janssen and Roy (2002), Deneckere and Liang (2006), Hörner and Vieille (2009), Moreno and Wooders (2002, 2010, 2015), Fuchs and Skrzypacz (2013, 2015), Kim (2015a) and Gerardi and Maestri (2015) for contributions.). Our closest precursor is Hörner and Vieille (2009). They study an interdependent-value bargaining model with a single long-run seller and a sequence of short-run buyers. They find that inefficiencies take different forms in the two opposing information structures. While highlighting market inefficiencies caused by information asymmetry, these models have largely overlooked the possibility that having multiple goods for sale could mitigate such inefficiencies.[2]

Arrival of news. There is a strand of literature in which information is gradually revealed to the uniformed players by the arrival of exogenous news (signals). Daley and Green (2012) show that exogenous news with Brownian noises leads to a unique equilibrium with "no-trade region", in which there are periods in which trade occurs with probability zero and the quality of the assets drifts up and down. In the version of Poisson arrival of news, "no-trade region" does not exist if no news is bad news. See also Kremer and Skrzypacz (2007), Zryumov (2014), Kaya and Kim (2014), Lauermann and Wolinsky (2013) and Zhu (2012) for exogenous arrival of news. Instead of exogenous arrival of news, Asriyan, Fuchs, and Green (2015) study endogenous arrival of news (information spillover) in a two-period

[2]Gerardi and Maestri (2015) consider adverse selection with multiple goods. However, in their model the qualities of the goods are the same.

adverse selection model in which two sellers sell two assets with correlated values. They show that this endogeneity of information leads to multiple equilibria when the positive correlation between asset values is sufficiently high.

We study information spillover in a infinite-horizon dynamic adverse selection setting where one seller sells two lemons. Information is endogenously conveyed by the (non-) trading activities in the other market. Whether the information from the other market is good, bad or neutral about the quality of the good depends on the initial belief of seller's type. With "enough" negative correlation, the news that the other goods is traded indicates that the quality of the remaining good is likely to be high. Without "enough" negative correlation, news from other market conveys the same information.

Transparency. There are papers that study the impact of information about past rejected offers (transparency) on efficiency of trade in dynamic markets with asymmetric information. Hörner and Vieille (2009) show that the observability of past price offers unambiguously reduces market efficiency.[3] Fuchs, Öry, and Skrzypacz (2015) reach similar results in a finite-horizon model with intra-temporal competition. Kim (2015b) demonstrates that market efficiency is not monotone in the amount of information available to buyers in a model with search friction.

In the multi-good setting, transparency indicates the observability of (non-)trading activities in other market. Asriyan, Fuchs, and Green (2015) parametrize the degree of observability of trading activities in other market, and show that total welfare is higher when markets are fully transparent than when the market is fully opaque. Our model assumes that each short-run uninformed buyer can observe the (non-)trading activities in other market, and show that adverse selection is mitigated with "enough" negative correlation.

3. A Motivating Example

Consider a measure one of dealers, each of whom sells two used vehicles, a fuel-efficient car (car 1) and a sports car (car 2), in two different markets. In particular, a measure one of buyers in market 1 only demand one fuel-efficient car and a measure one of buyers in market 2 only want to buy one sports car. The qualities of the cars, which determine both the dealer's costs and buyers' valuations, are the dealer's private information. Specifically,

[3] See Kaya and Liu (2015) for the study of price transparency in private-value settings.

let x_1 be the cost of car 1 to the dealer and x_2 the cost of car 2. The valuation of car 1 to buyers in the fuel-efficient car market is $3x_1/2$ and the valuation car 2 to buyers in the sports car market is $3x_2/2$.

Assume it is common knowledge that the qualities of the two cars are perfectly negatively correlated in the sense that dealers' costs of both cars are uniformed distributed on the line $\{(x_1, x_2) : x_1 + x_2 = 2, x_1 \geq 0, x_2 \geq 0\}$. Note that buyers in both markets observe neither x_1 nor x_2, but they think that either x_1 or x_2 is uniformly distributed on the interval $[0, 2]$. See Figure 1 for an illustration of the joint distribution F. That is, from any buyer's viewpoint, the average quality of the cars in his market is $E(x_1) = E(x_2) = 1$.

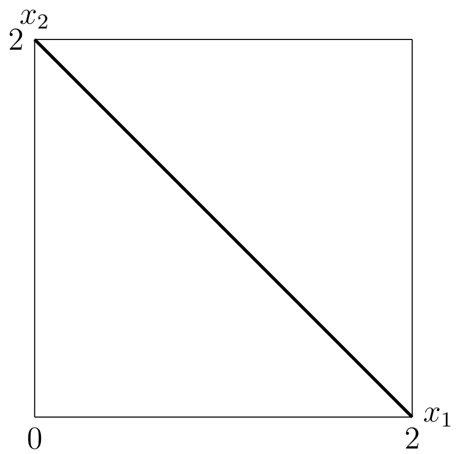

FIGURE 1. The joint distribution F: perfect negative correlation ($x_1 + x_2 \equiv 2$)

First suppose that buyers in one market do not observe the price nor quantity traded in the other market, that is, the two markets are separated. Then each market suffers from adverse selection as in the example of Akerlof (1970). It is easy to see that both markets break down as no buyer is willing to pay more than the expected valuation of cars that are actually traded in the market.

Now suppose that buyers in one market can see the price in the other market. If one requires that market i can only sell car i, then no market exists by the same logic described above. However, trade occurs efficiently under the following market arrangement with information spillover. There are two markets I and II, each of which sells both car 1 and car 2 at the same time. However, the quality of car 1(2) is higher in market I(II). Specifically, sellers $\{(x_1, 2 - x_1) : x_1 \in [1, 2]\}$ sell in market I, and the price of car 1 is 2 and the price of car 2 is 0.5. Likewise, sellers $\{(x_1, 2-x_1) : x_1 \in [0,1]\}$ sell in market II, and the price of car 1 is 0.5 and the price of car 2 is 2.[4] Therefore, each seller is indifferent between selling in market I and II, since the profit of any seller $\{(x_1, x_2) : x_1 + x_2 = 2\}$ from market I or II is $2 + 0.5 - x_1 - x_2 = 0.5$. Each buyer is also indifferent between buying from market I and II. Define $E(x_i|k)$ as the expected cost of car $i \in \{1, 2\}$ in market $k \in \{\text{I,II}\}$. Buyer 1 gets $\frac{3}{2}E(x_1|\text{I}) - 2 = 0.25$ in market I and $\frac{3}{2}E(x_1|\text{II}) - 0.5 = 0.25$ in market II; buyer 2 gets $\frac{3}{2}E(x_2|\text{I}) - 0.5 = 0.25$ in market I and $\frac{3}{2}E(x_2|\text{II}) - 2 = 0.25$ in market II. In equilibrium, each market has a measure 0.5 of the sellers, buyers 1 and buyers 2. See Figure 2 as a summary of the above equilibrium.

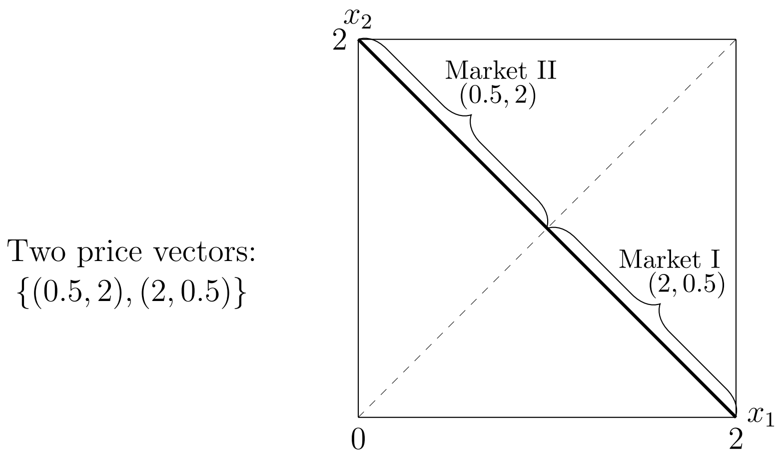

FIGURE 2. An efficient integrated markets under perfect negative correlation.

Finally, we would like to mention that negative correlation plays a key role for effective information spillover constructed above. For instance, in the opposite case with perfect positive correlation, that is, the dealer's costs of both cars are uniformly distributed on the line $\{(x_1, x_2) : x_1 = x_2 \in [0, 2]\}$, it follows from the same logic as in Akerlof (1970) that there is no trade in equilibrium even if buyers can observe price and trade in the other market.

The above example illustrates the driving force of our main insight. However, since it is formulated in a general equilibrium framework as in Akerlof (1970), it is silent about why

[4]Note that an implicit assumption in this market arrangement is that a seller cannot sell one (low quality) good at price 2 and then leave the market with the other good untraded. Alternatively, one could think that trade happens sequentially in both markets in the sense that any seller must first trade one good at price 0.5 in order to be able to trade the other good at price 2. The latter interpretation is closely related to the equilibrium outcome the bargaining model to be presented in the next section.

and how such price vectors can arise in equilibrium. In the next section, we shall consider a non-cooperative bargaining game to study the impact of information spillover on price formation and trading dynamics.

4. Model

A long-run seller has two goods, 1 and 2, to sell, one unit for each good. The quality of each good is either high (H) or low (L), which is the seller's private information. There are four types of sellers: HH, LL, HL and LH. HH has high valuation of both goods 1 and 2. LL has low valuation of both good 1 and 2. HL has high valuation of good 1 and low valuation of good 2. LH has low valuation of good 1 and high valuation of good 2. There are two groups of buyers: buyer 1 and 2. Buyer $i = 1, 2$ only buys good i. For each good i, the seller's cost and buyer i's valuation are interdependent, indicated by the following table. The seller's total cost is the sum of the costs for both goods. We assume that it is common knowledge that there is gain from trade: $\alpha v < v < \alpha < 1$.

i's quality	seller's value	buyer i's value
H	α	1
L	αv	v

Table 1. Seller's and buyers' valuations for good $i = 1, 2$.

Time is discrete and infinite, $t \in \{0, 1, 2 \ldots, +\infty\}$. In period 0, there is an investment stage where the seller chooses the qualities for both goods endogenously. Specifically, the seller has four pure strategies: HH, HL, LH and LL, corresponding to four types of qualities. Define $C(H_i)$ and $C(L_i)$ as the cost of investing in high and low quality for good i. Assume that investing in H is more costly than investing in L for both goods: $C(H_i) > C(L_i)$. For simplicity, assume that $C(H_1) - C(L_1) = C(H_2) - C(L_2)$. The total investment cost for both goods is the sum of the cost of both goods.

In period 0, there is an imperfect signal of seller's qualities of two goods, which can be observed by all the future buyers. The signal structure is described as follows: the signal has two possible values g, b, which represent a good outcome and a bad outcome, respectively. $P(s|\omega)$ is the probability of observing signal $s \in \{g, b\}$ conditioning on the true state $\omega \in \{HH, HL, LH, LL\}$.

(1) $P(g|HL) = P(g|LH) = P(g|HH) = 1$.[5]

(2) $P(b|LL) = \rho > 0.5 > 1 - \rho = P(g|LL)$.

The signal reveals whether the seller choose low qualities for both goods or not. If a bad outcome b is observed, the buyers believe that both goods have low qualities. If a good outcome g is observed, the buyers are likely to believe that at least one of the two goods has high quality.

From period 1 onward, there is a dynamic bargaining game where the seller bargains sequentially with two sequences of potential buyers until agreements are reached, if ever, and delay is costly: the seller's discount factor is $\delta \in (0, 1)$. Specifically, in each period $t \geq 1$, if both goods are still left untraded, two short-run buyers arrive and make two take-it-or-leave-it offers simultaneously to the seller. Denote the offer made by buyer $i = 1, 2$ as p_i. After observing two offers (p_1, p_2), the seller decides whether to accept each of the two offers or not. There are four choices for the seller: rr (rejecting both offers), aa (accepting both offers), ar (accepting p_1 and rejecting p_2) and ra (rejecting p_1 and accepting p_2). If two offers are accepted (aa), the game is over. If two offers are rejected (rr), the seller stays with two goods and waits for another two offers in the next round. If offer i is accepted (ar or ra), the seller is left with the other good j. In the next period, buyer j arrives and makes a take-or-leave-it offer p_j, and the seller decides to accept p_j or reject. If p_j is accepted, the game is over. Otherwise, the game repeats with the seller selling good j in the next period.

The buyers can observe (non-)trading activities for both goods. Specifically, there are two information environments: public offers and private offers. With public offers, the current buyers know the offers made by previous buyers and whether the offers were rejected or not. With private offers, the current buyers do not observe previous offers, but know whether the offers made by previous buyers were accepted or rejected.

5. Dynamic Bargaining Game: Public offers

In this section, we study the dynamic bargaining game, taking the initial distribution of two qualities as given. Define μ_{HH}, μ_{LL}, μ_{HL} and μ_{LH} as the initial belief of HH, LL, HL and LH. Therefore, the initial belief of good 1 to be a high type is $\mu_{HL} + \mu_{HH}$ and the initial

[5]We can show that all the results in this section hold if we perturb the signal structure by changing 1 to $1 - \epsilon$, for small enough $\epsilon > 0$.

belief of good 2 to be a high type is $\mu_{LH} + \mu_{HH}$. Define μ^* as $\mu^* + (1 - \mu^*)v = \alpha$. In other words, if the probability of high quality good $i = 1, 2$ is μ^*, then buyer i's expected value is equal to the reservation value of high quality good i for the seller. We make the following assumptions.

Assumption 5.1 (severe adverse selection): $\mu^* > \mu_{HL} + \mu_{HH}$ and $\mu^* > \mu_{LH} + \mu_{HH}$.

Assumption 5.2 (full support): $\mu_{HH} > 0,\ \mu_{HL} > 0,\ \mu_{LH} > 0,\ \mu_{LL} > 0.$

Assumption 5.3 (high discount factor): $\delta > \frac{v - \alpha v}{\alpha - \alpha v}$.

Assumption 5.1 says that for each good, adverse selection is severe in the sense that there is not a price under which trade takes place for all quality level in a static problem, but only the low quality good can be traded. Assumption 5.2 requires that the seller's type space has full support. In Section 8, we extend the results to cases in which the seller's types do not have full support. Assumption 5.3 requires that discount factor δ is high enough.

With public offers, there are two types of equilibria: *delay equilibria I* and *impasse equilibria*.

5.1. **Sufficiently Negative Correlation.** In this section, we study the case where the initial belief of two goods' qualities are sufficiently negatively correlated.

Assumption 5.4: $\mu^* > \frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^* - 1} + \frac{\mu_{LL}}{1 - \mu^*} < 1$.

Assumption 5.4 implies that μ_{HH} and μ_{LL} are small, thus μ_{HL} and μ_{LH} are large. Therefore, Assumption 5.4 says that there is sufficiently-negative correlation.[6]

In the *delay equilibria I*, trade delays but happens eventually. The equilibrium outcome is described by

- In period 1, buyer $i = 1, 2$ mixes over a continuum of offers with distribution $F_i(p_i)$ on the support $[\alpha v, \bar{p}]$, where $\alpha v < \bar{p} < v$. Given offers (p_1, p_2), HH chooses rr; HL randomizes between rr and ra; LH randomizes between rr and ar; LL puts probability $p_{ar}(p_1, p_2)$ on ar and probability $1 - p_{ar}(p_1, p_2)$ on ra, with $p_{ar}(p_1, p_2) \in [0, 1]$.
- In period 2, given rr of (p_1, p_2), the belief of the seller's type being H in each market is μ^* and buyer i randomizes between a winning offer α and a losing offer. Given ar or ra of (p_1, p_2), the posterior belief of the remaining good is not less than μ^* and the buyer

[6]Independent distribution does not satisfy Assumption 5.4.

of the remaining good chooses between a winning offer α and a losing offer, with positive probability on the winning offer.

- In period $t \geq 3$, if good i is left untraded, the belief of good i is μ^* and buyer i randomizes between a winning offer α and a losing offer.

Theorem 5.1. *Under public offers and Assumptions 5.1-5.4, there is a unique Perfect Bayesian Equilibrium (PBE) outcome, which is described by delay equilibria I.*

The main implication of *delay equilibrium I* is cross-market information spillover. Under sufficiently negative correlation, with only one good left untraded in period 2, the updated quality of this good is perceived to be high by the buyers since only low quality good can be traded in period 1, which indicates that it is very likely that the remaining good has good quality, thus trade happens with probability not less than μ^* with a high offer α. As a result, adverse selection is mitigated through the "direct" channel of information spillover given only one offer is rejected in period 1.

Another new feature of *delay equilibrium I* is that there is no skimming property which holds in dynamic adverse selection models with one good to sell. To be specific, skimming property says that high-quality sellers are more willing to delay trade for a high price than low-quality sellers, and low-quality sellers have an incentive to mimic the high-quality sellers, which relies heavily on the well-ordering of sellers' types. In contrast, with multi-dimensional types as in this paper, although LH and HL mimic HH by rejecting both offers in period 1, LL seller chooses to mimic HL or LH and separates from HH, thus low-quality sellers does not necessarily mimic high-quality sellers. The intuition is that LL seller takes advantage of the information spillover by accepting only one offer to enjoy a high continuation payoff from the rejected good. In other words, instead of accepting both offers or mimicking HH by rejecting both offers, LL seller decides to mimic HL or LH by choosing ra or ar.[7]

In *delay equilibrium I*, cross-market information spillover creates intra-temporal Bertrand competition between the buyers in period 1. In equilibrium, two buyers randomize over a continuum of offers in period 1. Since LL seller only accepts one offer in period 1, then two buyers engage in Bertrand competition with two horizontally differentiated goods in order to attract LL seller. Corollary 5.2 says that the winner's offer is accepted with a higher

[7] A detailed characterization of LL seller's strategy is in Corollary 5.3.

probability than the loser's offer.[8] In other words, two buyers can increase the probability to be accepted by the seller by raising her own offer. However, the Bertrand competition does not have a pure equilibrium in which two buyer bids their reservation value v and get zero profit. The reason is that a lower offer, say offer 1, is not rejected with probability one since LH seller still chooses to accept offer 1 with a positive probability although LL seller rejects offer 1. In all, the unqiue equilibrium of the Bertrand competition is two randomizing offers.

Corollary 5.2. *Under the same assumptions as in Theorem 5.1,*

$$p_{ai}(p_1, p_2) = 1_{p_i > p_j} A_i + 1_{p_i < p_j} B_i + 1_{p_i = p_j} C_i,$$

where $0 < B_i < A_i < 1$ *are constants independent of the offers*[9] *and* $C_i \in [B_i, A_i]$*. Define* $p_{ai}(p_1, p_2)$ *as the expected probability of acceptance in market* i *in period 1 given offer* (p_1, p_2)*.*

In *delay equilibrium I*, given both offers are rejected in period 1, trade happens with positive probability from period 2 on, which serves as an "indirect" channel of mitigating adverse selection through cross-market information spillover. The logic is as follows: (1) In period 1, low quality seller for each good gets positive profit by accepting an intermediate offer $p \in (\alpha v, v)$ in period 1 due to Bertrand competition. (2) Assume by contradiction that trade takes place with probability zero, then only HH seller is willing to reject both offers in period 1 since by accepting an intermediate offer in period 1, LL, HL and LH get positive payoff. However, given only HH left untraded in period 2, trade happens with probability one since there is no adverse selection, a contradiction. In all, Bertrand competition restricts the inter-temporal negative externalities between buyers who arrive in different periods, which causes the bargaining impasses if the seller sells two goods separately in the sense that each short-lived buyer can only observes the previous offers of the good she values as illustrated by Hörner and Vieille (2009).

Next, we do comparative statics analysis to figure out how the equilibrium responds to the change of the initial distribution of the two qualities. As illustrated by Corollary 5.3, the randomizing offer for each good is non-monotonic in μ_{LL}: the probability of LL seller. As the probability of LL increases, the randomizing offer increases initially and decreases eventually in the first-order stochastic dominance (FOSD) sense. The non-monotonicity comes

[8] If there is a tie, one of the buyer gets a lower supply and the other buyer gets a higher supply.

[9] A_i and B_i depend on the initial belief of the seller type. See Lemma A.10 for details.

from two opposite effects: namely" Bertrand competition effect", related to two buyers' incentives to bid up the prices to attract the seller, and "information spillover effect", related to the effectiveness of information spillover. Define $\mu^{**} \equiv \frac{1-\mu^*}{1+\mu^*}(1 - \frac{\mu_{HH}}{2\mu^*-1})$, a threshold that determines equilibrium behavior qualitatively as described in Corollary 5.3.

Corollary 5.3. *Under the same assumptions as in Theorem 5.1 and $\mu_{LH} = \mu_{HL}$,*

(1) For $\mu_{LL} < \mu^{**}$, $F_i(p_i)$, the distribution of offer p_i, is decreasing in μ_{LL}. LL accepts the higher offer and rejects the lower offer.

(2) For $\mu_{LL} > \mu^{**}$, $F_i(p_i)$, the distribution of offer p_i, is increasing in μ_{LL}. LL randomizes between ra and ar. The probability of accepting the higher offer is decreasing in μ_{LL}.

If negative correlation is strong enough ($\mu_{LL} < \mu^{**}$), each of the two offers is increasing in μ_{LL} in the sense of FOSD and LL accepts only the higher offer since only "Bertrand competition effect" plays a role. With very strong negative correlation, accepting either of the two offers is a signal that the adverse selection is not severe for the rejected good, thus a winning offer α is made for the rejected good in the next period. In all, the continuation payoff from the rejected good has a constant value, thus LL seller only cares about the stage-game payoff and accepts the higher offer in period 1. As a result, both buyers compete by bidding up the offer to attract LL.

If negative correlation is not very strong ($\mu_{LL} > \mu^{**}$), each of the two offers is decreasing in μ_{LL} in the sense of FOSD and LL randomizes over ar and ra since "information spillover effect" dominates "Bertrand competition effect." Without strong negative correlation, the acceptance of the higher offer by LL seller cannot generate sufficient evidence that the good left untraded has high quality, thus LL seller gets zero continuation payoff from the remaining good. Therefore, it is a profitable deviation for LL seller to only accepts the lower offer, successfully pretends to be a high type for the remaining good and gets a high continuation payoff. Therefore, we have shown that LL randomizes over ra and ar. Note that the higher μ_{LL} is, the larger is the likelihood that the LL type seller needs to accept a lower offer in period 1. As a result, as μ_{LL} increases, "winning" LL seller becomes less attractive, thus two buyers have incentives to bid down the offers in the sense of FOSD.

5.2. **Insufficiently Negative Correlation.** In this section, we study the case where the initial belief of seller's type are not sufficiently negatively correlated.

Assumption 5.5: $\mu^* < \frac{1}{2}$ or both $\mu^* > \frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} > 1$.

Assumption 5.5 says there is insufficiently negative correlation. If $\mu^* < \frac{1}{2}$, then by Assumption 5.1, $\max\{\mu_{HL}, \mu_{LH}\} < \mu^* - \mu_{HH} < \frac{1}{2} - \mu_{HH}$. If $\mu^* > \frac{1}{2}$, Assumption 5.5 implies that $\mu_{HL} + \mu_{LH} < \mu^* + \frac{2-3\mu^*}{2\mu^*-1}\mu_{HH}$. In both cases, μ_{HL} and μ_{LH} are small.

In the *impasse equilibria*, there is bargaining-impasse for any remaining good from period 2 on. Specifically, the equilibrium outcomes are characterized as below:

- In period 1, both buyers offers αv. HH rejects in both markets; HL randomizes between rr and ra; LH randomizes between rr and ar; LL chooses among all four choices.
- In period $t \geq 2$, if only good i is left untraded, the updated belief of good i offers is μ^* and buyer i makes a losing offer.

Theorem 5.4 shows that without sufficiently negative correlation, there is bargaining impasse for any remaining good from period 2 on. Information spillover does not work well in the sense that the probability of high quality for good i is not so high even if only good i remains untraded in period 2. Consequently, LL seller has no strong incentive to accept only one offer in period in order to enjoy the high continuation payoff in period 2. Then, Bertrand competition does not exists between two buyers in period 1. Actually, the probability of acceptance for any offer p_i remains a constant for each $i = 1, 2$, thus each buyer's best choice is to make the lowest offer αv. We can show that trade does not take place from period 2 on, otherwise low quality seller in each market would reject the lowest offer αv, thus there would be no belief updating in period 2, a contradiction.

Theorem 5.4. *Under public offers and Assumptions 5.1-5.3 and 5.5, all Perfect Bayesian Equilibria (PBE) are payoff-equivalent and described by impasse equilibria.*

6. Dynamic Bargaining Game: Private Offers

In this section, we study the dynamic bargaining game, taking the initial distribution of two qualities as given. With private offers, there are two types of equilibria: *delay equilibrium II* and *delay equilibria III.*

In *delay equilibria II*, trade delays but happens eventually in the future. Specifically, the equilibrium outcomes are characterized as below:

- In period 1, both buyers offers v. HH rejects in both markets; HL randomizes between rr and ra; LH randomizes between rr and ar; LL chooses between all four choices.

- In period $t \geq 2$, if good i is left untraded, the updated belief of good i offers is μ^* and buyer $i = 1, 2$ randomizes between a winning offer α and a losing offer, the probability of the winning offer λ satisfies $v - \alpha v = \delta(\lambda\alpha + (1 - \lambda)v - \alpha v)$.

In the *delay equilibria III*, if both goods are left untraded, there is a delay of trade from period 2 on, but trade will eventually happen in the future. If only one good is left untraded in period 2, there is immediate trading for this good. Specifically, the equilibrium outcome is characterized as below:

- In period 1, both buyers offers v. HH rejects in both markets, HL randomizes between rr and ra; LH randomized between rr and ar; and LL chooses between ar and ra, with probability $p_{ar} \in [0, 1]$ on ar.
- In period 2, given rr of any offers, the updated beliefs of H in both markets are μ^* and buyer $i = 1, 2$ randomizes between offering a winning offer α and a losing offer, where the probability of the winning offer λ satisfies $v - \alpha v = \delta(\lambda\alpha + (1 - \lambda)v - \alpha v)$. Given ra and ar of any offers, the updated belief of the remaining good is no less than μ^*, and buyer i offers α and the seller accept it.
- In period $t \geq 3$, if good i is left untraded, buyer i randomizes between offering a winning offer α and a losing offer, and the probability of the winning offer $\lambda \in [0, 1]$ satisfies $v - \alpha v = \delta(\lambda\alpha + (1 - \lambda)v - \alpha v)$.

Theorem 6.1. *Under private offers and Assumptions 5.1-5.3, all the Perfect Bayesian Equilibria (PBE) are payoff-equivalent and characterized as below:*
(1) Under Assumption 5.5, all Perfect Bayesian Equilibria are delay equilibria II.
(2) Under Assumption 5.4, all Perfect Bayesian Equilibria are delay equilibria III.

With private offers, sufficiently negative correlation between the qualities of two goods hastens trade agreements as in the public offers case. With sufficiently negative correlation, accepting one offers serves as a signal that the remaining good is more likely to have high quality, thus trade for the remaining good happens immediately. Without sufficiently negative correlation, trade in each market happens as if buyers cannot observe (non-)trading activities in the other market. Notice that adverse selection is only mitigated through the "direct" channel if only one offer is rejected in period 1. The "direct" channel mentioned in Section 5 does not work since the inter-temporal negative externalities between buyers who arrive in different periods do not exist in privates offer environment without cross-market information spillover, thus leaving cross-market information spillover no room to improve efficiency.

7. Welfare Comparison

In this section, we compare the expected discounted surplus (gain from trade) under various information structures and correlation assumptions. Under public (private) offers , $V_s^{public}(V_s^{private})$ and $V_i^{public}(V_i^{private})$ represent the surplus with sufficiently negative correlation and insufficiently negative correlation. As a benchmark, we also study the surplus without cross-market observability: V_{no}^{public} indicates the surplus if each buyer $i = 1, 2$ only observes the past offers in market i and $V_{no}^{private}$ indicates the surplus if each buyer $i = 1, 2$ only observes whether previous offers in market i are rejected or not.

Theorem 7.1. *If δ is high enough, then*
(1) $V_s^{public} > V_s^{private}$, $V_i^{public} > V_i^{private}$.
(2) $V_s^{public} > V_{no}^{public}$, $V_s^{private} > V_{no}^{private}$.
(3) $V_i^{public} = V_{no}^{public}$, $V_i^{private} = V_{no}^{private}$.

Theorem 7.1 gives a complete comparison of the expected discounted surplus under different information and correlation structures. The first part of Theorem 7.1 says that the expected discounted surplus under private offers is higher that under public offers, independent of the correlation assumption. The second and third part of Theorem 7.1 says that observability improves expected discounted surplus if and only if there is sufficiently negative correlation, under public offers or private offers. In all, if we take the expected discounted total surplus as the standard of efficiency, cross-market observability as well as sufficiently negative correlation together mitigate adverse selection.

8. Non-full Support

In this section, we deal with the case where the distribution of two qualities has no full support. We assume without loss of generality that $\mu_{HL} > 0$ and $\mu_{LH} > 0$ and study two cases: $\mu_{HH} = 0$ or $\mu_{LL} = 0$.

If $\mu_{LL} = 0$, $\mu_{HH} > 0$, then Assumption 5.1 implies that $0 < \mu_{HH} < 2\mu^* - 1$. There are two types of equilibria: *impasse equilibrium II* and *delay equilibrium IV*.

In *impasse equilibrium II*,

- In period 1, both buyers offer αv. Given (p_1, p_2), HH chooses rr; HL randomizes between rr and ra, with probability $p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}}$ on rr; LH mixes between rr and ar, with probability $p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}$ on rr.
- In period 2, if only good i is left untraded, then the belief of good i is 1, and the buyer i makes a winning offer α and the game is over. If both goods are left untraded, the updated beliefs for both goods are μ^* and two losing offers are made from period 2 on.

In *delay equilibrium IV*,

- In period 1, both buyers offer v. Given (p_1, p_2), HH chooses rr; HL randomizes between rr and ra, with probability $p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}}$ on rr; LH mixes between rr and ar, with probability $p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}$ on rr.
- In period 2, if only good i is left untraded, then the belief of H for good i is 1, and the buyer i makes a winning offer α and the game is over. If both goods are left untraded, the updated beliefs for both goods are μ^*, and two buyers randomizes between a winning offer α and a losing offer from period 2 on, with probability λ on the α. λ solves $v - \alpha v = \delta(\lambda\alpha + (1-\lambda)v - \alpha v)$.

Proposition 8.1. *Under Assumptions 5.1, 5.3 and $\mu_{LL} = 0$ and $\mu_{HH} > 0$, all the Perfect Bayesian Equilibria (PBE) are characterized as below:*
(1) With public offers, the unique PBE is impasse equilibrium II.
(2) With private offers, the unique PBE is delay equilibrium IV.

If $\mu_{LL} = 0$ and $\mu_{HH} > 0$, with public offers, adverse selection is mitigated only through "direct" channel if only one offer is rejected in period 1. However, the "indirect" channel of mitigating adverse selection given two offers are rejected in period 1 does not work, since $\mu_{LL} = 0$ implies that there is no Bertrand Competition between buyers in period 1, thus two lower offers αv are made in period 1, leading to bargaining impasse given both offers are rejected. With private offers, adverse selection is mitigated only through "direct" channel as illustrated in Section 6.

Proposition 8.2. *If $\mu_{HH} = 0$, then Theorems 5.1, 5.3, 6.1, and Proposition 8.1 hold.*

If $\mu_{HH} = 0$, we can show that all previous results for $\mu_{HH} > 0$ hold for $\mu_{HH} = 0$, except that the history rr is off the equilibrium.

If there is perfect negative correlation: $\mu_{LL} = \mu_{HH} = 0$, trade happens with probability one. In period 1, both buyers make offers αv; HL choose ra and LH chooses ar. In period 2, there is only one good left untraded and the buyer is certain that the remaining good has high quality and makes a winning offer α. In all, there is no bargaining impasse.

9. Pre-bargaining Investment Stage

In this section, we study the investment stage where the seller chooses the qualities of both goods traded in the future bargaining game.

Define $\underline{\rho} = (\frac{\delta(\alpha-\alpha v)}{v-\alpha v} + 1 + \frac{\mu^*}{1-\mu^*}\ln(\mu^*))^{-1}\frac{\delta(\alpha-\alpha v)}{v-\alpha v}$ as a lower bound for ρ, the informativeness of the signal s. Define $\bar{c} = \rho\big(1 + \frac{\mu^*}{1-\mu^*}\ln(\mu^*)\big)(v - \alpha v) - (1-\rho)\delta(\alpha - \alpha v)$ as an upper bound for the extra investment cost by choosing high quality.

Assumption 9.1: $\underline{\rho} < \rho < 1$.

Assumption 9.2 : $C(H_i) - C(L_i) < \min\{\bar{c}, \delta(1-\alpha)(1-v)\}$ for $i = 1, 2$.

Assumption 9.1 says that signal b is highly informative, although not perfectly informative. Assumption 9.2 means that the investment cost for high quality is not too high relative to the investment cost for low quality, otherwise, the seller would lose the incentive to invest in high quality. $C(H_i) - C(L_i) < \delta(1-\alpha)(1-v)$ says that investment in high quality is efficient: compared with low quality, the increase of gain from trade by choosing high quality, i.e., $\delta(1-\alpha)(1-v)$ outweighs the extra investment cost $C(H_i) - C(L_i)$.

Proposition 9.1 says that there is always a trivial equilibrium that the seller chooses LL and future buyers believe that the seller is LL type given signal b or g.

Proposition 9.1. *In stage 0, there exists an equilibrium such that the seller chooses LL and gets zero profit. Given signal g or b, the buyers are certain that the seller is LL, regardless of public offers or private offers in the bargaining game.*

9.1. **Public offers.** With public offers, there are two other equilibria constructed in Proposition 9.2, which provides a novel theory of specialization under adverse selection. Note that the seller never chooses HH because the future benefit of HH is always zero in the future bargaining process, not because it is very costly to choose HH. Actually, each seller randomizes over HL, LH and LL. In the equilibria, we can show that by investing in HL (LH), the seller gets positive payoff from good 2 (good 1). This is because signal g arrives for certain and the seller, with low quality good 2 (good 1) can enjoy information rents in the subsequent bargaining game with information spillover. To the contrary, by investing in LL, the seller is detected to be LL type by receiving the signal b with high probability and gets punished by receiving a zero payoff. In all, marginal benefit of choosing HL or LH is equal to marginal cost of choosing LL, thus the seller is indifferent among HL, LH and LL. Most important, in order to take advantage of the information spillover, the seller randomizes in such a way that there will be sufficiently negative correlation from future buyers' point of view since adverse selection is mitigated if and only if there is sufficiently negative correlation, as shown in the previous sections.

Proposition 9.2. *Under Assumptions 9.1-9.2 and $\mu^* > \frac{1}{2}$, in stage 0 before dynamic bargaining process with public offers, there are two equilibria described as below: there exists $q \in \{q_1, q_2\}$ such that*

(1) Seller's equilibrium strategy: the seller randomizes over HL, LH and LL, with probability $(q/2, q/2, 1-q)$ respectively.

(2) Belief updating: Given signal b, buyers are certain that the type is LL. Given signal g, the updated belief satisfies $P(HL|g) = P(LH|g) = \frac{q/2}{q+(1-q)(1-\rho)} < \mu^*$ and $P(LL|g) = \frac{(1-q)(1-\rho)}{q+(1-q)(1-\rho)} < 1-\mu^*$.

(3) The seller's expected payoffs are the same in two equilibiria $q = q_1, q_2$. The expected discount surplus in equilibrium $q = q_2$ is higher than that in equilibrium $q = q_1$.

The investment stage of endogenous quality choice has a population interpretation. If we add a private value component to each seller so that each seller has private information about her investment cost, then each seller will investment according to her comparative advantage. Assume that there are a continuum type of sellers (s_1, s_2) uniformly distributed on $[0, 1] \times [0, 1]$. Define $C_s(H_i)$ and $C_s(L_i)$ as the investment cost of high and low quality for good i by type $s \equiv (s_1, s_2)$. Assumption 9.3 says that investing in H_i is more costly than investing in L_i for all seller types, and higher s_i corresponds to lower cost for good i. The total cost for both goods is the sum of the cost of both goods. Signal structure is the same as before.

Assumption 9.3 : $0 < C_s(H_i) - C_s(L_i) < C_{s'}(H_i) - C_{s'}(L_i) < \delta(1-\alpha)(1-v)$ for $s_i > s'_i$.

Assumption 9.4 : $C_s(H_i) - C_s(L_i) < \bar{c}$ for any $s \in [0, 1] \times [0, 1]$.

Proposition 9.3 proposes a new interpretation of specialization. Sellers with higher investment cost for both goods choose low qualities for both goods. The remaining sellers choose high quality for the good with relatively lower investment cost. No sellers invest for both goods since there is investment cost but no future benefit. From buyers' point of view, specialization implies a sufficiently negative correlation between qualities of the goods, which, together with cross-market observability, mitigates the adverse selection in the future dynamic bargaining game. In all, the purpose of specialization is to take advantage of the information spillover, instead of avoiding the high investment cost of having high qualities for all goods.

If relative gain from trade of the high quality good is higher than the relative investment cost of the high quality good: $\delta(1-\alpha)(1-v) > C_s(H) - C_s(L)$, cross-market observability also improves efficiency in the investment stage by providing sellers incentives to invest in high quality good compared with the environment without cross-market observability in which sellers always choose low qualities for both goods.

Proposition 9.3. *Under Assumptions 9.1, 9.3, 9.4 and* $\mu^* > \frac{1}{2}$*, in stage 0 before dynamic bargaining process with public offers, there are two equilibria as below: there exists two different* $s^* \in \{s_1^*, s_2^*\}$ *such that*

(1) Seller's equilibrium strategy: the seller chooses LL *for* $s_1 \leq s^*$ *and* $s_2 \leq s^*$*; seller choose* HL *for* $s_1 \geq s^*$ *and* $s_1 > s_2$*; seller choose* LH *for* $s_2 \geq s^*$ *and* $s_2 > s_1$*.*

(2) Belief updating: Given b*, buyer is certain that the type is* LL*. Given* g*, the updated belief satisfies* $P(HL|g) = P(LH|g) = \frac{(1-s^{*2})}{2(1-\rho s^{*2})} < \mu^*$ *and* $P(LL|g) = \frac{s^{*2}(1-\rho)}{1-\rho s^{*2}} < 1-\mu^*$*.*

9.2. **Private offers.** With private offers, there are a continuum of equilibira constructed in Proposition 9.4. Each seller randomizes over HL and LH. By investing in HL (LH), the seller gets positive profit from good 2 (good 1) since signal g arrives for certain. By investing in LL, she is detected to be LL type by receiving the signal b with high probability and get zero payoff. With low investment cost in high quality, marginal benefit of HL and LH is larger than marginal cost of LL. Compared with the public offer cases, the seller is more likely to invest in HL and LH since the offers in the next period are higher if signal g is observed. In all, the seller chooses a complete specialization strategy by investing in only one of the goods.

Proposition 9.4. *If $\rho > \frac{\delta(\alpha-\alpha v)}{v-\alpha v+\delta(\alpha-\alpha v)}$, $\mu^* > \frac{1}{2}$, then for any $C(H_i) - C(L_i) < \rho(v - \alpha v) - (1-\rho)\delta(\alpha - \alpha v)$, then in stage 0 before dynamic bargaining process with private offers, there are a continuum of equilibria for any $q \in (1 - \mu^*, \mu^*)$ as below:*

(1) Seller's equilibrium strategy: the seller mixes between HL and LH, with probability $(q, 1-q)$.

(2) Belief updating: Given signal b, buyers are certain that the type is LL. Given signal g, the updated belief satisfies $P(HL|g) = q$, $P(LH|g) = 1 - q$ and $P(LL|g) = 0$.

(3) In any of the equilibria, the seller's expected payoff is the same; the expected discount surplus is the same.

10. A Continuum of Types: Perfect Negative Correlation

In previous sections, we assume that the quality of each good has two types: high quality H and low quality L. In this section, we generalize the model by assuming that the quality of each good $i = 1, 2$ has a continuum of types (x_1, x_2) distributed on $[\underline{x}, \bar{x}]^2$, where $\underline{x} > 0$. The equilibria can be fully characterized in a specific case that two qualities are perfectly negatively correlated as illustrated by Assumption 10.1. Denote $F(\cdot)$ as the distribution of x_1, which is enough to describe the distribution of (x_1, x_2).

Assumption 10.1 (Perfect Negative Correlation): $x_1 + x_2 = \underline{x} + \bar{x}$.

Denote $v(x_i)$ is buyer i's valuation of good i. Denote $c(x_i)$ as the seller's reservation value of good i. The total reservation value of the seller is the sum of the reservation values for both goods. We assume that $c(\cdot)$ is (strictly) positive, (strictly) increasing, and twice differentiable, with bounded derivatives. We also assume that $v(\cdot)$ is positive, increasing, and continuously differentiable. Assume that there is gain from trade: $v(x_i) > c(x_i)$.

For $n \geq 0$, we define x^n to satisfy $E(v(x_i)|x_i \in [x^{n-1}, x^n]) = c(x^n)$. For uniform distribution $F(x_i)$, $c(x_i) = x_i$ and $v(x_i) = \alpha x_i$, we can show that $x^n = (2\alpha^{-1} - 1)^n$. Define $y^n = \bar{x} + \underline{x} - x^n$. Assumption 10.2 is equivalent to $x^1 > \underline{x}$, which says that for each good, adverse selection is severe in the sense that there is not a price under which trade takes place for all quality levels in a static problem. Specifically, the good with highest quality cannot be trade.

Assumption 10.2 (Severe Adverse Selection): $E(v(x_i)|x_i \in [\underline{x}, \bar{x}]) < c(\bar{x})$ for $i = 1, 2$.

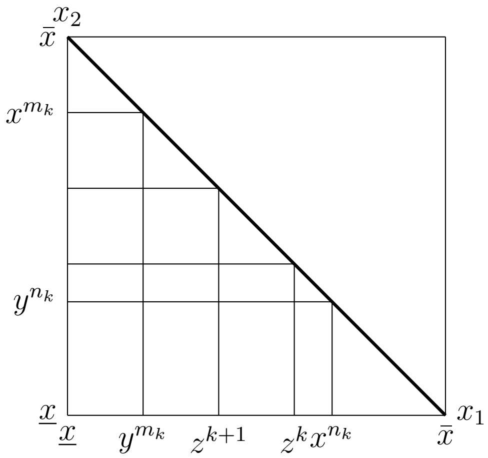

FIGURE 3. Perfect negative correlation ($x_1 + x_2 \equiv \bar{x} + \underline{x}$).

Assume that $x^{N+1} < \underline{x} < x^N$. In all, there are $N+1$ cutoffs on $[\underline{x}, \bar{x}]$: $\bar{x} = x^0 > x^1 > \cdots > x^{N-1} > x^N$, and there are $N+1$ corresponding cutoffs on $[\underline{x}, \bar{x}]$: $\underline{x} = y^0 < y^1 < \cdots < y^{N-1} < y^N$. We combine the two sequences and rank them: $1 = z^0 > z^1 > \cdots > z^{2N} > z^{2N+1} = \underline{x}$, where $\{z^k\}_{k=0}^{2N+1} \equiv \{x^n, y^n\}_{n=0}^{N}$.

For each $0 \le k \le 2N$, define $A_k = (1-\delta)(\bar{x} + \underline{x} - 2z^k) + \delta(x^{m_k} - x^{n_k})$ and $B_k = (1-\delta)(\bar{x} + \underline{x} - 2z^{k+1}) + \delta(x^{m_k} - x^{n_k})$, where m_k is the largest integer m such that $y^m \le z^{k+1}$ and n_k is the smallest integer n such that $z^k \le x^n$. See Figure 3 as an illustration of x^{n_k} and y^{m_k}. By definition, m_k is non-increasing in k and n_k is non-decreasing in k.

Define $x(p)$ is a non-increasing function as follows: for each $0 \le k \le 2N$,

$$x(p) = \begin{cases} \bar{x} & p_2 - p_1 < A_0. \\ \frac{1}{2}(\bar{x} + \underline{x} - \frac{p_2 - p_1 - \delta(x^{m_k} - x^{n_k})}{1-\delta}) & p_2 - p_1 \in [A_k, B_k], \\ z^{k+1} & p_2 - p_1 \in [B_k, A_{k+1}]. \\ \underline{x} & p_2 - p_1 > B_{2N}. \end{cases}$$

We consider the following *sequential trading equilibrium*:

(1) In period 1, both buyers mix over a continuum of offers $[\underline{x}, \bar{p}]$, where $\bar{p} < \bar{x}$. If $p_2 - p_1 \in [A_0, B_{2N}]$, seller $x_1 \in [x(p), \bar{x}]$ chooses ra; seller $x_1 \in [\underline{x}, x(p)]$ chooses ar. If $p_2 - p_1 > B_{2N}$, all sellers choose ra. If $p_2 - p_1 < A_0$, all sellers choose ar.

(2) In period 2, given ar of (p_1, p_2), the updated belief is $x_1 \in [\underline{x}, x(p)]$; given ra of (p_1, p_2), the updated belief is $x_1 \in [x(p), \bar{x}]$.

(a) If $p_2 - p_1 \in [A_k, B_k]$, then given ra, buyer 1 makes an offer x^{n_k}; given ar, buyer 2 makes an offer x^{m_k}.

(b) If $p_2 - p_1 \in [B_k, A_{k+1}]$, then

(i) If $z^{k+1} = x^{n_k}$, given ra, buyer 1 mixes between $x^{n_k - 1}$ and a losing offer; given ar, buyer 2 makes an offer x^{m_k}.

(ii) If $z^{k+1} = x^{m_k}$, given ra, buyer 1 makes an offer x^{n_k}; given ar, buyer 2 mixes between $x^{m_k - 1}$ and a losing offer.

(c) If $p_2 - p_1 \geq B_{2N}$, then given ra, buyer 1 offers x^N.

(d) If $p_2 - p_1 < A_0$, then given ar, buyer 2 offers x^N.

(3) In period 2, if there is only good i left untraded, faced with $x_i = x^{n_k}$, $x_i \in [x(p), x^{n_k}]$ accepts and $x_i \in [x^{n_k}, 1]$ rejects.

(4) From period 3 on, if there is only good i left untraded, the seller makes losing offer for good i.

Theorem 10.1. *Under Assumption 10.1, 10.2, public offers and δ is high enough, there is a Perfect Bayesian Equilibrium characterized by the sequential trading equilibrium .*

Theorem 10.1 says that there exists a *sequential trading equilibrium* in which two goods are sold sequentially: one of the two goods is sold in period 1 and the remaining good will be traded in period 2 with certain probability. $x(p)$ characterizes the seller's decision in period 1. For each price pair $p = (p_1, p_2)$, sellers with valuation of good 1 that is equal to $x(p)$ is indifferent between ar and ra; sellers with valuation of good 1 higher than $x(p)$ choose ra; the remaining types choose ar. In all, each seller only accepts one of the two offers and rejects the other offer. Moreover, by offering a higher offer p_2, buyer 2 can attract more sellers in period 1: $x(p)$ is decreasing in $p_2 - p_1$. By the same logic, by offering a higher offer p_1, buyer 1 can attract more sellers in period 1. Therefore, two buyers engage in a Bertrand competition with a mixed strategy equilibrium.

Adverse selection is mitigated in the following sense. In period 1, half of both goods is traded in period 1. To be specific, given a pair $p = (p_1, p_2)$, the probabilities of trading for good 1 and good 2 are $1 - F(x(p))$ and $F(x(p))$ respectively, where $F(\cdot)$ is the distribution of x_1. For the remaining good in period 2, trade happens for lower quality good and markets breaks down for higher quality good, as illustrated by Hörner and Vieille (2009).

11. Conclusion

This paper studies the role of cross-market information spillover if a seller has two different goods for sale with private information about the qualities of the goods. We find that adverse selection can be mitigated compared to the environment without cross-market observability if and only if the qualities of two goods are sufficiently negatively correlated. We also provide a new rationale for specialization in markets with adverse selection. If sellers, with heterogeneous investment costs, can choose the qualities of different goods for sale in the future, they will specialize by choosing higher quality for the good with lower investment cost and lower quality for the good with higher investment cost. From buyers' point of view, specialization implies a negative correlation between qualities of the goods, which, together with cross-market observability, mitigates adverse selection in the markets. In contrast, without cross-market observability, sellers always choose low qualities for both goods.

Based on this paper, there are several extensions. (1) We generalize the model to the case where there are a continuum of types of quality for each good in a special case where the qualities are perfectly negatively correlated in Section 10. Future work is need to study the equilibria under more general form of distribution of qualities to confirm the role of cross-market information spillover in mitigating adverse selection. (2) The model can be extended to the case with more than two goods. Take three goods as example, if the initial distribution is that given one good has low quality, the conditional probability of high qualities of the other two goods is high enough, then an early trade of one good is a signal that the remaining two goods are more likely to have high qualities, which facilitates trade. (3) We show in the online Appendix that cross-market information spillover helps to improve efficiency under other particular bargaining protocols. For example, there are many short-run buyers arriving in each period, forming a market price for each good, with buyers in each market earning zero profit under free entry condition. In equilibrium, market belief of the quality of each good will drift up and hastens trade if the other good is traded early. Future work can be done to confirm the above idea under a broader class of dynamic bargaining game. (4) Huangfu and Liu (2015) study another model where two sellers, with one good for each seller, sell two goods to two sequences of short-run buyers, and may not enjoy the full benefit of endogenous signaling due to possible strategic interaction between two sellers. We show that with negative correlation between qualities of two goods, two sellers are engaged in a war of attrition. In each period, expecting the other seller to accept with positive probability, each seller has an incentive to reject to convince the future buyers that she has a high quality good in next period. As a result, both sellers reject with high probability and trade is delayed.

Appendix A. Proofs of Section 5: Public offers

In this section, we prove a series of Lemmas: Lemma A.0-A.7, which are used in the proof of Theorem 5.1, Corollary 5.2, Corollary 5.3 and Theorem 5.4.

We define the following equations and inequalities which describe the belief updating in period 2:

$$\frac{\mu_{HL}p_{HL}+\mu_{HH}}{\mu_{HL}p_{HL}+\mu_{LH}p_{LH}+p_{rr}\mu_{LL}+\mu_{HH}}=\frac{\mu_{LH}p_{LH}+\mu_{HH}}{\mu_{LH}p_{LH}+\mu_{HL}p_{HL}+p_{rr}\mu_{LL}+\mu_{HH}}=\mu^*. \tag{A.1}$$

.

$$\frac{\mu_{HL}(1-p_{HL})}{\mu_{HL}(1-p_{HL})+p_{ra}\mu_{LL}}=\frac{\mu_{LH}(1-p_{LH})}{\mu_{LH}(1-p_{LH})+p_{ar}\mu_{LL}}=\mu^*. \tag{A.2}$$

$$\frac{\mu_{HL}(1-p_{HL})}{\mu_{HL}(1-p_{HL})+p_{ra}\mu_{LL}}\geq\mu^*,\ \frac{\mu_{LH}(1-p_{LH})}{\mu_{LH}(1-p_{LH})+p_{ar}\mu_{LL}}\geq\mu^*. \tag{A.3}$$

Lemma A.0: (i) If $\mu^*\leq\frac{1}{2}$, there are solutions to (A.1) and (A.2). (ii) If $\mu^*>\frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^*-1}+\frac{\mu_{LL}}{1-\mu^*}\geq 1$, there are solutions to (A.1) and (A.2). (iii) If $\mu^*>\frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^*-1}+\frac{\mu_{LL}}{1-\mu^*}<1$, there are solutions to (A.1) and (A.3).

Proof. **Step 1:** Show that there are solutions to (A.1) and (A.2) if $\mu^*\leq\frac{1}{2}$.

Since $\mu^*\leq\frac{1}{2}$, $\mu^*\geq\mu_{HH}+\mu_{HL}$ and $\mu^*\geq\mu_{HH}+\mu_{LH}$, then $1>2\mu^*\geq 2\mu_{HH}+\mu_{HL}+\mu_{LH}$. Therefore, $\mu_{HH}<\mu_{LL}$.

(A.2) implies that

$$p_{ra}=\frac{\mu_{HL}}{\mu_{LL}}\frac{1-\mu^*}{\mu^*}(1-p_{HL}). \tag{A.4}$$

$$p_{ar}=\frac{\mu_{LH}}{\mu_{LL}}\frac{1-\mu^*}{\mu^*}(1-p_{LH}). \tag{A.5}$$

(A.1) implies that

$$p_{HL}=\frac{\mu^*}{1-2\mu^*}\frac{\mu_{LL}}{\mu_{HL}}p_{rr}-\frac{1-\mu^*}{1-2\mu^*}\frac{\mu_{HH}}{\mu_{HL}}. \tag{A.6}$$

$$p_{LH}=\frac{\mu^*}{1-2\mu^*}\frac{\mu_{LL}}{\mu_{LH}}p_{rr}-\frac{1-\mu^*}{1-2\mu^*}\frac{\mu_{HH}}{\mu_{HL}}. \tag{A.7}$$

(A.4), (A.5), (A.6), (A.7) imply that

$$p_{ra} = \frac{\mu_{HL}}{\mu_{LL}}\frac{1-\mu^*}{\mu^*} - \frac{1-\mu^*}{1-2\mu^*}p_{rr} + \frac{(1-\mu^*)^2}{\mu^*(1-2\mu^*)}\frac{\mu_{HH}}{\mu_{LL}}. \tag{A.8}$$

$$p_{ar} = \frac{\mu_{LH}}{\mu_{LL}}\frac{1-\mu^*}{\mu^*} - \frac{1-\mu^*}{1-2\mu^*}p_{rr} + \frac{(1-\mu^*)^2}{\mu^*(1-2\mu^*)}\frac{\mu_{HH}}{\mu_{LL}}. \tag{A.9}$$

As $p_{ar} \geq 0$, $p_{ra} \geq 0$, $p_{HL} \geq 0$ and $p_{LH} \geq 0$, then (A.8), (A.9), (A.6), (A.7) imply that

$$\frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq p_{rr} \leq \frac{\mu_{HL}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

$$\frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq p_{rr} \leq \frac{\mu_{LH}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

The above two equations make sense since $\mu^* < \frac{1}{2}$.

As $p_{ar} + p_{ra} + p_{rr} \leq 1$, then (A.8), (A.9) imply that

$$p_{rr} \geq (\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1)(1-2\mu^*) + \frac{(1-\mu^*)^2}{\mu^*}\frac{2\mu_{HH}}{\mu_{LL}}.$$

We need to find $p_{rr} \in [0,1]$ to satisfy all above three inequalities.

First, check that there exists p_{rr} to satisfy satisfy all above three inequalities. It is equivalent to show that

$$\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1 < \frac{\min\{\mu_{HL},\mu_{LH}\}}{\mu_{LL}\mu^*} - \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

The above inequality holds if $\mu^* \geq \mu_{LH} + \mu_{HH}$ and $\mu^* \geq \mu_{HL} + \mu_{HH}$, which are true.

Next, show that there exists $p_{rr} \in [0,1]$, which is equivalent to show that the lower bound of p_{rr} is less than 1: $(\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1)(1-2\mu^*) + \frac{(1-\mu^*)^2}{\mu^*}\frac{2\mu_{HH}}{\mu_{LL}} \leq 1$ and $\frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} < 1$. The first inequality is equivalent to $(1-\mu^*)(\mu^* - \frac{\mu_{HL}+\mu_{LH}}{2} - \mu_{HH}) > 0$, thus $\mu^* \geq \frac{\mu_{HL}+\mu_{LH}}{2} + \mu_{HH}$, which is true. The second inequality is equivalent to $\mu^* > \frac{\mu_{HH}}{\mu_{HH}+\mu_{LL}}$. By the fact that $\mu_{HH} < \mu_{LL}$, we have $\mu_{HH} + \frac{\mu_{HL}+\mu_{LH}}{2} - \frac{\mu_{HH}}{\mu_{HH}+\mu_{LL}} = \frac{\mu_{HL}+\mu_{LH}}{2}\frac{\mu_{LL}-\mu_{HH}}{\mu_{HH}+\mu_{LL}} > 0$. Therefore, $\mu^* > \mu_{HH} + \frac{\mu_{HL}+\mu_{LH}}{2} > \frac{\mu_{HH}}{\mu_{HH}+\mu_{LL}}$.

Step 2: Show that there exist solutions to (A.1) and (A.2) if $\mu^* = \frac{1}{2}$.

$\mu^* = \frac{1}{2}$ implies that $\mu^* > \mu_{HH} + \mu_{HL}$ and $\mu^* > \mu_{HH} + \mu_{LH}$. Therefore, $1 = 2\mu^* > 2\mu_{HH} + \mu_{HL} + \mu_{LH}$, then $\mu_{HH} < \mu_{LL}$.

It is trivial that $p_{rr} = \frac{\mu_{HH}}{\mu_{LL}} < 1$, $p_{ra} = \frac{\mu_{HL}}{\mu_{LL}}(1-p_{HL})$, $p_{ar} = \frac{\mu_{LH}}{\mu_{LL}}(1-p_{LH})$, $\mu_{HL}p_{HL} = \mu_{LH}p_{LH}$. In order to satisfy $p_{ra} + p_{ar} + p_{rr} \leq 1$, we need $\frac{1}{2} \leq \mu_{LL} + \mu_{HL}p_{HL}$. Assume WLOG that $\mu_{HL} \leq \mu_{LH}$, then let $p_{HL} = 1$ and $p_{LH} = \frac{\mu_{HL}}{\mu_{LH}}$. Then, we only need to show that $\frac{1}{2} \leq \mu_{LL} + \mu_{HL}$, which holds since $\frac{1}{2} = \mu^* > \mu_{HH} + \mu_{LH}$.

In all, we construct a solution: $p_{rr} = \frac{\mu_{HH}}{\mu_{LL}}$, $p_{ra} = 0$, $p_{ar} = \frac{\mu_{LH}-\mu_{HL}}{\mu_{LL}}$, $p_{HL} = 1$ and $p_{LH} = \frac{\mu_{HL}}{\mu_{LH}}$.

Step 3: Show that there exist solutions to (A.1) and (A.2) if $\mu^* > \frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} \geq 1$.

Because $p_{ar} \geq 0$, $p_{ra} \geq 0$, $p_{HL} \geq 0$ and $p_{LH} \geq 0$, then (A.6), (A.7), (A.8) and (A.9) implies that

$$\frac{\mu_{HL}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq p_{rr} \leq \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$
$$\frac{\mu_{LH}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq p_{rr} \leq \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

As $\mu^* > \frac{1}{2}$, the above two equations make sense.

Because $p_{ar} + p_{ra} + p_{rr} \leq 1$, then (A.8) and (A.9) implies that

$$p_{rr} \leq (\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1)(1-2\mu^*) + \frac{(1-\mu^*)^2}{\mu^*}\frac{2\mu_{HH}}{\mu_{LL}}$$

We need to find $p_{rr} \in [0,1]$ to satisfy all above three inequalities.

First, check that there exists p_{rr} to satisfy the above three inequalities. It is equivalent to show that

$$\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1 < \frac{\min\{\mu_{HL},\mu_{LH}\}}{\mu_{LL}}\frac{1}{\mu^*} - \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

The above two equations hold if $\mu^* \geq \mu_{LH} + \mu_{HH}$ and $\mu^* \geq \mu_{HL} + \mu_{HH}$, which are true.

Next, show that there exists $p_{rr} \in [0,1]$. We need to show that the lower bound of p_{rr} is less than 1: $\frac{\mu_{HL}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq 1$ and $\frac{\mu_{LH}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} < 1$. It is equivalent to $\mu^* > \frac{\mu_{HH}+\mu_{HL}}{\mu_{LL}+\mu_{HH}+2\mu_{HL}}$ and $\mu^* > \frac{\mu_{HH}+\mu_{LH}}{\mu_{LL}+\mu_{HH}+2\mu_{LH}}$. Assume WLOG that $\mu_{HL} \geq \mu_{LH}$. Then, $\mu_{LL} + \mu_{HH} + 2\mu_{HL} > 1$, so $\mu^* \geq \mu_{HH} + \mu_{HL} > \frac{\mu_{HH}+\mu_{HL}}{\mu_{LL}+\mu_{HH}+2\mu_{HL}}$. Next, show that $\mu_{HH} + \mu_{HL} \geq \frac{\mu_{HH}+\mu_{LH}}{\mu_{LL}+\mu_{HH}+2\mu_{LH}}$, which is equivalent to $\mu_{LH} - \mu_{HL} \geq \frac{\mu_{LH}-\mu_{HL}}{\mu_{HH}+\mu_{HL}}$. This is true because $\mu_{HH} + \mu_{LH} < 1$ and $\mu_{LH} \leq \mu_{HL}$. In all, $\mu^* \geq \mu_{HH} + \mu_{HL} > \frac{\mu_{HH}+\mu_{LH}}{\mu_{LL}+\mu_{HH}+2\mu_{LH}}$.

Finally, show that the upper bound of p_{rr} is no less than 0: $(\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1)(1-2\mu^*) + \frac{(1-\mu^*)^2}{\mu^*}\frac{2\mu_{HH}}{\mu_{LL}} \geq 0$, which is equivalent to $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} \geq 1$.

Step 4: Show that there exist solutions to (A.1) and (A.3). if $\mu^* > \frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$.

Show that $p_{rr} = 0$. Given $p_{rr} = 0$ and (A.1),

$$p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}},\ p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}.$$

From the above equations, (A.3) and $p_{ar} + p_{ra} = 1$, we can show that $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$.

We can show that $p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}} < 1$ and $p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}} < 1$, which is implied by $\mu^* > \mu_{HL} + \mu_{HH}$, $\mu^* > \mu_{LH} + \mu_{HH}$ and $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$.

□

Lemma A.1: Under public offers, given any history with belief $(<\mu^*, \geq \mu^*)$, the offer is $(\alpha v, \alpha)$. The seller mixes between ra and aa so that given ra in the current period, the updated belief in the next period goes to μ^*.

Proof. For good 2, there is no severe adverse selection, thus buyer 2 make an offer $p_2 = \alpha$ and the seller accepts $p_2 = \alpha$ with probability one, independent of the quality of good 2. For good 1, by the same proof as Hörner and Vieille (2009), buyer 1 makes an offer $p_1 = \alpha v$, the seller of good 1 mixes between accepting and rejecting. In next period, given offer 1 is rejected, the updated belief is μ^*, and there is a bargaining impasse.

□

Lemma A.2: With public offers, given any history with the belief $(<\mu^*, <\mu^*)$, the belief in both markets goes to (μ^*, μ^*) given rr of (p_1, p_2), where $\alpha v \leq p_i \leq v$ and $p_1 + p_2 > 2\alpha v$.

Proof. Define μ_i as the updated belief of good $i = 1, 2$ in period 2 if good i is left untraded. Denote $\bar{p} > 2\alpha v$ as the supreme of $p_1 + p_2$ in all equilibria. For any $\epsilon > 0$, there exists p_1 and p_2 that $p_1 + p_2 > \bar{p} - \epsilon$.

Step 1: Show that given rr of (p_1, p_2) and the belief of one of the good is equal to μ^*, then the belief of both goods goes to (μ^*, μ^*).

Prove by contradiction and assume that given rr of (p_1, p_2), $\mu_2 < \mu^*$. By lemma A.1, the offer will be $(\alpha, \alpha v)$. Therefore, all seller types will reject the offer 1. Given rr of (p_1, p_2), LL needs to mix between ra and rr to support the increase of the belief of good 1. Therefore, given ra of (p_1, p_2), $\mu_1 \geq \mu^*$, a contradiction to the fact that $\mu_1 = \mu^*$ given rr of (p_1, p_2).

Given rr of (p_1, p_2), $\mu_2 < \mu^*$. Therefore, all seller type is to reject p_2 to get α in the next period. Given rr of (p_1, p_2), LL needs to mix between ar and rr to support the belief increase for good 2. Therefore, given ar, $\mu_2 \geq \mu^*$, a contradiction to the fact that $\mu_2 > \mu^*$ given rr.

Step 2: Show that given rr of (p_1, p_2), it is impossible that the belief of one of the goods is larger than μ^*.

Assume that given rr of (p_1, p_2), if $\mu_1 > \mu^*$, then LH will reject p_1 in the first period to get α in the next period. Therefore, LL needs to mix between ra and rr in order to support the increase of the belief of good 1. Therefore, given ra, $\mu_1 \geq \mu^*$. However, it is impossible that $\mu_1 > \mu^*$ given both rr and ra: $\frac{\mu_{HL}p_{HL}+\mu_{HH}}{\mu_{LH}+\mu_{HL}p_{HL}+\mu_{HH}} > \mu^*$ and $\frac{\mu_{HL}(1-p_{HL})}{\mu_{HL}(1-p_{HL})+\mu_{LL}} > \mu^*$ imply $\mu_{HL} + \mu_{HH} > \mu^*$, a contradiction.

Step 3: Show that given rr of (p_1, p_2), it is impossible that the belief of one of the goods is less than μ^*.

Assume that given rr of (p_1, p_2), $\mu_1 < \mu^*$. By Step 1, $\mu_2 < \mu^*$. Next, show that LL does not play rr in period 1. Assume by contradiction that LL plays rr in period 1.

Show that LL will not choose aa in period 2. If LL plays aa in period 2 given rr in period 1, then LL can get no more than $\delta(\bar{p} - 2\alpha v)$. By playing aa in period 1, LL gets more than $\bar{p} - \epsilon - 2\alpha v$, so LL strictly prefers aa to rr in period 1, a contradiction to the fact that LL plays rr in period 1.

Given rr of (p_1, p_2), if only HH and LL are left, the offer will be $(\alpha v, \alpha v)$ in the next period, a contradiction to the fact that LL plays rr in period 1.

Show that if HH, LL and LH are left given rr of (p_1, p_2) in period 1, then the updated belief in period 3 is $(< \mu^*, < \mu^*)$ given rr in period 2. Assume by contradiction that given rr in period 2, the updated belief is $(\geq \mu^*, \geq \mu^*)$. Then, either LL or LH plays ar, since otherwise given rr in period 2, the updated belief of good 1 is less than μ^*, a contradiction. If LL plays ar with positive probability, then given ar in period 2, $\mu_2 \geq \mu^*$, since otherwise ar is weakly dominated by aa for LL, a contradiction to the fact that LL chooses ar with positive probability and does not choose aa. If LL does not play ar and LH plays ar with positive probability, then given ar, $\mu_2 = 1$. In all, we have shown that given ar, $\mu_2 \geq \mu^*$. Moreover, LL does not play ra, which would reveal its type and gets a lower payoff than ar. However, given the fact that LL does not choose aa and ra in period 2, it is impossible that given both rr and ar in period 2, the updated belief of good 2 is not less than μ^*. In all, given rr in period 2, the updated belief is $(< \mu^*, < \mu^*)$.

Show that if all four types are left given rr of (p_1, p_2), then the updated belief in period 3 is $(< \mu^*, < \mu^*)$ given rr in period 2. Assume by contradiction that given rr in period 2, the updated belief in period 3 is $(\geq \mu^*, \geq \mu^*)$. If both LH and HL play rr for sure, then LL can only play rr since ar and ra is dominated by aa. Therefore, given rr in period 2, the updated belief remains to be $(<\mu^*,<\mu^*)$, a contradiction. If LH plays rr for sure and HL mixes between rr and ra, then it is trivial to show that given ra, the belief in market 1 is no less than μ^*. However, it is impossible that given rr and ra, the belief in market 1 is larger than or equal to μ^*. If LH mixes between rr and ar and HL mixes between rr and ra, then show that it is impossible that all four types are left given rr of (p_1, p_2). Since the sum of the two offer is at most $\bar{p}$, then there is at least one offer i in the next period after discounting is less than the offer i in period 1, a contradiction to the fact that both LH and HL plays rr with positive probability in period 1. In all, given rr in period 2, the updated belief in period 3 is $(<\mu^*,<\mu^*)$.

Next, by induction, given rr in period $t \geq 1$, the updated belief is always $(<\mu^*,<\mu^*)$. Therefore, in order for LL seller to choose rr in period 1, there must be a period $n \geq 2$ such that LL chooses neither rr nor aa. Since given rr in period n, the updated belief is that only HH, HL and LH are left untraded in period $n+1$. By Proposition 9.1, HH, HL and LH get zero profit from period 2 on. (1) If neither offer is αv in period n, then HL plays ra or LH plays ar in period n. Assume WLOG that HL plays ra, thus given rr in period n, the buyers believe that good 2 is H in period $n+1$, a contradiction. (2) If both offers are αv, the belief of HL or LH is zero at the beginning of period n since HL or LH would accept the offer in period 1 instead of getting zero profit by rejecting both offer until period n. Assume WLOG the belief of HL is zero, then given rr in period n, the buyers believe that good 2 is H in period $n+1$, a contradiction.

We have shown that LL does not choose rr in period 1. Therefore, given rr, only HH, HL and LH are left. By Proposition 8.1, HH, HL and LH get zero profit from period 2 on. Since $\min\{p_1, p_2\} > \alpha v$, HL plays ra or LH plays ar in period 1. Assume WLOG that HL plays ra, thus given rr, the buyers believe that good 2 is H, a contradiction to the assumption that given rr of (p_1, p_2), the updated belief is $(<\mu^*,<\mu^*)$.

□

Lemma A.3: Under Assumption 5.4 and public offers, for any history with the belief $(<\mu^*,<\mu^*)$, the the belief updating given ar or ra is as below:

Define $\bar{\mu} = \frac{\mu^*}{1-\mu^*}\mu_{LL} + \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}$.

(1) $p_1 > p_2$.

If $\mu_{LH} \leq \bar{\mu}$, then $p_{ar} = \frac{1-\mu^*}{\mu^* \mu_{LL}}(\mu_{LH} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}) < 1$ and $p_{ra} = 1 - p_{ar}$. In period 2, $\mu_2 = \mu^*$ given ar and $\mu_1 > \mu^*$ given ra.

If $\mu_{LH} > \bar{\mu}$, then $p_{ar} = 1$, $p_{ra} = 0$. In period 2, $\mu_2 > \mu^*$ given ar and $\mu_1 = 1$ given ra.

(2) $p_1 < p_2$.

If $\mu_{HL} \leq \bar{\mu}$, then $p_{ra} = \frac{1-\mu^*}{\mu^* \mu_{LL}}(\mu_{HL} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}) < 1$ and $p_{ar} = 1 - p_{ra}$. In period 2, $\mu_1 = \mu^*$ given ra and $\mu_2 > \mu^*$ given ar.

If $\mu_{HL} > \bar{\mu}$, then $p_{ra} = 1$, $p_{ar} = 0$. In period 2, $\mu_1 > \mu^*$ given ra and $\mu_2 = 1$ given ar.

(3) $p_1 = p_2$.

$\mu_1 \geq \mu^*$ given ra and $\mu_2 \geq \mu^*$ given ar.[10]

Proof. **Step 1:** Show that given ra and ar of (p_1, p_2), he belief of the remaining good is no less μ^*.

By Assumption 5.4, if $\mu_2 < \mu^*$ given ar, then $\mu_1 > \mu^*$ given ra. Therefore, ra dominates aa and ar. If LH chooses ar with positive probability, then given ar, $\mu_2 = 1$, a contradiction. Therefore, no seller type chooses ar. As a result, we can show that $\mu_{HH} + \mu_{HL} > \mu^*$, a contradiction.

Step 2: Show that LL never chooses rr or aa.

By Step 1 and Lemma A.0, $\mu_1 > \mu^*$ or $\mu_2 > \mu^*$. Assume WLOG that $\mu_2 > \mu^*$. By choosing aa, LL seller gets $p_1 + p_2 - 2\alpha v$. By choosing ar, LL seller gets $p_1 - \alpha v + \delta(\alpha - \alpha v) > p_1 + p_2 - 2\alpha$. Therefore, aa is never chosen by LL. By Lemma A.2, the belief goes to (μ^*, μ^*) given rr of (p_1, p_2). Therefore, rejection in the market i brings the low seller in market i the same payoff as acceptance. Therefore, rr brings LL seller the same payoff as aa, which is dominated by ar.

Step 2: Check the case $p_1 > p_2$.

Case 1: $\mu_{LH} < \bar{\mu}$.

Show that LL mixes between ar and ra, given $p_1 > p_2$. Otherwise, there are two cases. First, LL chooses ra for sure, given $p_1 > p_2$. Then, given ar, the belief of good 2 goes to 1. ar is a profitable deviation for LL, a contradiction. Second, LL chooses ar. Given ra, the belief of good 1 is 1. Therefore, given ar, $\mu_2 = \frac{\mu_{LH}(1-p_{LH})}{\mu_{LH}(1-p_{LH})+\mu_{LL}} > \mu^*$, a contradiction to $\mu_{LH} < \bar{\mu}$.

[10] $\mu_1 = \mu_2 = \mu^*$ does not hold.

Show that given ar of $p_1 > p_2$, $\mu_2 = \mu^*$. Otherwise, LL strictly prefers ar, a contradiction to LL mixes between ar and ra, given $p_1 > p_2$. By Lemma A.0, given ra of $p_1 > p_2$, $\mu_1 > \mu^*$.
Case 2: $\mu_{LH} > \bar{\mu}$.

Given $p_1 > p_2$, LL chooses ar. Otherwise, there are two cases. First, LL chooses ra, given $p_1 > p_2$. Then, given ar, the belief of good 2 goes to 1. ar is a profitable deviation for LL, a contradiction. Second, given $p_1 > p_2$, LL mixes between ar and ra. Given ar of $p_1 > p_2$, $\mu_2 = \mu^*$, otherwise LL would strictly prefer ar. thus $p_{ar} = \frac{1-\mu^*}{\mu^*\mu_{LL}}(\mu_{LH} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}) < 1$, a contradiction to $\mu_{LH} > \bar{\mu}$.

Since LL chooses ar, given $p_1 > p_2$, then only HL chooses ra. Then, given ra of $p_1 > p_2$, $\mu_1 = 1$. Given ar of $p_1 > p_2$, $\mu_2 = \frac{\mu_{LH}(1-p_{LH})}{\mu_{LH}(1-p_{LH})+\mu_{LL}} > \mu^*$, by $\mu_{LH} > \bar{\mu}$.

Step 3: Check the case $p_1 < p_2$.

The result is symmetric to the case that $p_1 > p_2$.

Step 4: Check the case $p_1 = p_2 > \alpha v$.

Assume that given ra of $p_1 = p_2$, the belief of good 1 is less than μ^*. Lemma A.0 implies that the belief of good 2 is larger than μ^* given ar of $p_1 = p_2$ (Otherwise, $p_{rr} < 0$). Therefore, LL strictly prefer ar to ra and rr. Then, no type chooses ra. We can show that given rr of $p_1 > p_2$, the belief of good 2 is less than μ^*, a contradiction to Lemma A.2. In all, we have show that given ra of $p_1 = p_2$, the belief of good 1 is larger than μ^*. In all, we have shown that given ra or ar of $p_1 = p_2$, the belief of the remaining good is larger than or equal to μ^*. It is impossible that both belief is μ^*.

□

Corollary A.3: Under the same assumptions as in Lemma A.3, if there is only one good left untraded in period 1, the equilibrium behavior from period 2 on is described as below:

(1) $p_1 > p_2$.

- (a) $\mu_{LH} \leq \bar{\mu}$. Given ar of $p_1 > p_2$, buyer 2 mixes between offers α and a losing offer, with probability $1 - \frac{p_1 - p_2}{\delta(\alpha - \alpha v)}$ on α that all sellers accept. Given ra of $p_1 > p_2$, buyer 1 offers the winning offer α.
- (b) $\mu_{LH} > \bar{\mu}$. Given ra of $p_1 > p_2$, buyer 1 offers the winning offer α. Given ar of $p_1 > p_2$, buyer 2 offers the winning offer α.

(2) $p_1 < p_2$.

(a) $\mu_{HL} \leq \bar{\mu}$. Given ra of $p_1 < p_2$, buyer 1 mixes between offers α and a losing offer, with probability $1 - \frac{p_2 - p_1}{\delta(\alpha - \alpha v)}$ on α that all sellers accept. Given ar of $p_1 < p_2$, buyer 2 offers the winning offer α.

(b) $\mu_{HL} > \bar{\mu}$. Given ar of $p_1 < p_2$, buyer 2 offers the winning offer α. Given ra of $p_1 < p_2$, buyer 1 offers the winning offer α.

(3) $p_1 = p_2$.

Lemma A.4: Under Assumption 5.4 and public offers, for any history with the belief $(< \mu^*, < \mu^*)$, define P_i as the set of equilibrium offers by buyer i. Show that $\inf P_1 = \inf P_2 = \alpha v$.

Proof. Define $\underline{p}_i = \inf P_i$. Show that $\underline{p}_1 = \underline{p}_2$. Assume that $\underline{p}_1 < \underline{p}_2$. By Lemma A.3, given ar of $(p_1, \underline{p}_2)$ (any $p_1 < \underline{p}_2$), the belief of good 2 is larger than μ^* and is a constant, which means that the probability of acceptance of good 1 is a constant. Therefore, buyer 1's optimal strategy is $\{\underline{p}_1\}$. Then, buyer 2 wants to deviate to $p_2 \in (\underline{p}_1, \underline{p}_2)$, a contradiction. In all, we have show that $\underline{p}_1 = \underline{p}_2$. Define $\underline{p} \equiv \underline{p}_1 = \underline{p}_2$.

Prove by contradiction and assume that $\underline{p} > \alpha v$. Firstly, show that it is impossible that the mixed strategies of both buyers have atoms at the lower bound $\underline{p}$. At $p_1 = p_2 = \underline{p}$, by Lemma A.3, one of the buyer can increase the offer by a small ϵ to get strictly more probability to be accepted, which is a profitable deviation. Assume the mixed strategy of buyer i has no atom at the lower bound $\underline{p}$. By choosing $p_i = \alpha v$, buyer i gets a higher payoff than the offer p_i^* that slightly larger than $\underline{p}$ since the probability of acceptance does not change as long as $p_i < p_j$ and the probability of $p_i^* < p_j$ is close to 1, a contradiction.

□

Lemma A.5: Under Assumption 5.4 and public offers, for any history with the belief $(< \mu^*, < \mu^*)$, it is impossible that $p_1 = p_2 = \alpha v$ are both atoms for both $F_1(p_1)$ and $F_2(p_2)$.

Proof. Assume that both $p_1 = \alpha v$ and $p_2 = \alpha v$ are atoms. Show that it is a profitable deviation for buyer 1 to offer $p_1 = \alpha v + \epsilon$ or for buyer 2 to offer $p_2 = \alpha v + \epsilon$ for small $\epsilon > 0$.

Case 1: Given rr of $(\alpha v, \alpha v)$, the belief goes to (μ^*, μ^*).

By the same argument as in Step 4 of Lemma A.3, we can show that given ra or ar of $(\alpha v, \alpha v)$, the belief of the remaining good is larger than or equal to μ^*. By Lemma A.3, there exists a buyer who is accepted by a higher probability if he raises the offer, and we call this buyer is not favored at the $p_1 = p_2 = \alpha v$. Assume WLOG that buyer 1 is not favored at the tie $p_1 = p_2 = \alpha v$. Buyer 1 could choose a price $\alpha v + \epsilon$ slightly higher than p such that the probability of acceptance will be higher at $\alpha v + \epsilon$ than p by a significant amount. Thus, $p_1 = \epsilon + \epsilon$ is a profitable deviation.

Case 2: Given rr of $(\alpha v, \alpha v)$, the updated belief is $(< \mu^*, < \mu^*)$.

Show that LL chooses rr. Prove by contradiction and assume that LL does not choose rr, then given rr, only HH, HL and LH are left and get zero profit. Thus, HL choose ra or LH choose ar. Assume WLOG that HL choose ra, then $\mu_2 = 1$ given rr, a contradiction.

Assume that given ar of $(\alpha v, \alpha v)$, the belief of one of the goods is larger than μ^*. WLOG, assume that given ar of $(\alpha v, \alpha v)$, the belief of good 2 is larger than μ^*. If HL chooses ra with positive probability, then buyer 1 will offer losing offer from period 2. Hence, in period 1, LL seller gets $\delta(\alpha - \alpha v)$, thus LL seller prefers ar to rr, a contradiction. Therefore, HL chooses rr. By the same logic, we can show that HL and LH play rr. Consequently, LL does not choose ar, which reveals the type. In all, good 1 is rejected for sure if $p_1 = p_2 = \alpha v$. Therefore, buyer 1 can make a profit by deviating to $p_1 = \alpha v + \epsilon$, which is accepted with a positively probability by Lemma A.2.

Assume that given ar and ra of $(\alpha v, \alpha v)$, the updated belief of the remaining good is less than or equal to μ^*. By Lemma A.2, given rr of $(\alpha v + \epsilon, \alpha v)$, the updated belief is (μ^*, μ^*). By Lemma A.3, given ra of $(\alpha v + \epsilon, \alpha v)$, the updated belief of good 1 is larger than μ^*. Therefore, faced with $(\alpha v + \epsilon, \alpha v)$, the probability of rejection by L type in market 1 is less than that faced with $(\alpha v, \alpha v)$. Therefore, it is a profitable deviation for buyer 1 to deviate to $p_1 = \alpha v + \epsilon$.

□

Lemma A.6: Under Assumption 5.4 and public offers, for any history with the belief $(< \mu^*, < \mu^*)$, (i) The supports of the two buyers' distributions must be the same; (ii) The equilibrium distributions of both buyers cannot possess any atoms for $p > \alpha v$.

Proof. **Step 1:** The supports of the two buyers' distributions must be the same.

Assume there is a point x in P_1 but not in P_2. From using the definition of support, there must exist an $\epsilon > 0$ such that $F_1(x + \epsilon) - F_1(x - \epsilon) > 0$ and $[x - 2\epsilon, x + 2\epsilon]$ is not in P_2.

If no $p \leq x-2\epsilon$ belongs to P_2, buyer 1 has a zero of probability of winning on $[x-2\epsilon, x+2\epsilon]$. In this case, buyer 1 could strictly increase its expected profits by moving the probability assigned to $[x-\epsilon, x+\epsilon]$ to a lower price close to αv. This is in contradiction to buyer 1 playing an equilibrium strategy.

Assume then there is a point $p \leq x-2\epsilon$ in P_2. The profit at p if buyer 2 wins must be strictly greater than the profit on $[x-\epsilon, x+\epsilon]$. If it were not, buyer 2 could move some probability mass into $[x-\epsilon, x+\epsilon]$. This would increase its probability of winning without giving up profits and generate higher expected profits, in contradiction to buyer 2 playing an equilibrium strategy. Let $\hat{p}$ be the highest price in P_2 to the left of $x-2\epsilon$ (note that $\hat{p}$ exists since we assumed there are points in the support of buyer 2 to the left of $x-2\epsilon$, and the support is a closed set). It is trivial that winning profits are strictly higher at $\hat{p}$ than on $[x-\epsilon, x+\epsilon]$, there must exist a point between $\hat{x}$ and $x-\epsilon$ that yields higher profits than any point in $[x-\epsilon, x+\epsilon]$. Buyer 1 could increase its expected profits by moving the probability assigned to $[x-\epsilon, x+\epsilon]$ to this point, in contradiction to buyer 1 playing an equilibrium strategy. Thus, P_1 is contained in P_2. Similarly, P_2 is contained in P_1 and hence we have shown that in equilibrium both supports coincide.

Step 2: An equilibrium distribution of both buyers cannot possess any atoms for $p > \alpha v$.

Assume that buyer 1 placed a probability mass of η on a point p ($\alpha v < p < v$). Since $p \in P_1$, from Step 1 it must also be that $p \in P_2$. We will now show that buyer 2 cannot place a positive mass η' on p, and that prices in $(p-\epsilon, p)$ do not belong to P_2 for a small enough $\epsilon > 0$.

First, we show that buyer 2 does not place a positive mass on p. Prove by contradiction and assume that buyer 2 places a positive mass on p. We have shown in Lemma A.3 that if there is a tie, there exists a buyer who is accepted by a higher probability if he raises the offer, and we call this buyer is not favored at the tie. Assume WLOG that buyer 1 is not favored at the tie $p_1 = p_2 = p$. Buyer 1 could choose a price p' slightly higher than p such that the winning profit at p' only decreases a little, but the probability of acceptance will be higher at p' than p by a significant amount. Thus, buyer 1 could increase its expected profits by moving the probability mass η to a point slightly larger than p, in contradiction to buyer 1 playing an equilibrium strategy. In all, buyer 2 does not place a positive mass on p.

Second, we show that prices in $(p - \epsilon, p)$, for a small enough $\epsilon > 0$, do not belong to P_2. There exists an ϵ such that for any $p'' \in (p - \epsilon, p)$, the probability of acceptance is strictly less than $p + \epsilon$. Thus, buyer 2 can increase its expected profits by moving probability mass from around p'' to $p + \epsilon$, in contradiction to buyer 2 playing an equilibrium strategy.

If no price $p' < p$ belongs to P_2, then buyer 1's offer p is larger than buyer 2's offer with probability 1. Then, buyer 1 can increase its expected profits by moving the probability mass η from p to a price larger than αv and strictly less than p to make a profit. If there is a $p' \in P_2$ with $p' < p - \epsilon$, we proceed as in Step 1 to show that buyer 1 could increase its expected profits by moving the mass η currently at p to a lower price. Thus in any case, we reach a contradiction to buyer 1 playing an equilibrium strategy.

Hence, the equilibrium distribution of buyer 1 cannot contain any atoms and similarly for buyer 2.

□

Lemma A.7: Under Assumption 5.4 and public offers, the offer of buyer $i = 1, 2$ is characterized by distribution $F_i(p)$ on the support $[\alpha v, \bar{p}]$: $F_i(p) = f_i \frac{p-\alpha v}{v-p} + \frac{v-\alpha v}{v-p} F_i(\alpha v)$, where $f_1 = \frac{B_2}{A_2 - B_2}$, $f_2 = \frac{B_1}{A_1 - B_1}$, $\bar{p} = \frac{1+\alpha \max(f_1, f_2)}{1+\max(f_1, f_2)} v$ and $F_i(\alpha v) = \min\{\frac{f_j - f_i}{1+f_2} \frac{v-\alpha v}{\alpha - \alpha v}, \ 0\}$. A_i is the probability of acceptance in market i if $p_i > p_j$; B_i is the probability of acceptance in market i if $p_i < p_j$.

Proof. By Lemma A.5, it is impossible that $p_1 = p_2 = \alpha v$ are atoms for both $F_1(p)$ and $F_2(p)$. Therefore, $\min\{F_1(\alpha v), \ F_2(\alpha v)\} = 0$. By Lemma A.6, the distributions of two offers have the same support without atoms for $p_i > \alpha v$. Define $\bar{p}$ as the supremum of the supports of both $F_1(p)$ and $F_2(p)$.

Given $p_i > p_j$, the probability of acceptance in market i is A_i. Given $p_i < p_j$, the probability of acceptance in market i is B_i. For any $p \in [\alpha v, \bar{p}]$, the profit of offering p by buyer i is $(v - p)[F_j(p)A_i + (1 - F_j(p))B_i]$, which is equal to the profit of offering αv: $(v - \alpha v)[F_j(\alpha v)A_i + (1 - F_j(\alpha v))B_i]$. Define $f_j \equiv \frac{B_i}{A_i - B_i}$, then

$$F_j(p) = f_j \frac{p - \alpha v}{v - p} + \frac{v - \alpha v}{v - p} F_j(\alpha v).$$

Therefore, $f_2 \frac{\bar{p}-\alpha v}{v-\bar{p}} + \frac{v-\alpha v}{v-\bar{p}} F_2(\alpha v) = F_2(\bar{p}) = 1 = F_1(\bar{p}) = f_1 \frac{\bar{p}-\alpha v}{v-\bar{p}} + \frac{v-\alpha v}{v-\bar{p}} F_1(\alpha v)$. If $f_2 > f_1$, then $F_1(\alpha v) > 0 = F_2(\alpha v)$. If $f_1 > f_2$, then $F_2(\alpha v) > 0 = F_1(\alpha v)$. If $f_2 = f_1$, then $F_1(\alpha v) = F_2(\alpha v) = 0$.

We can show that if $f_2 > f_1$, then $\bar{p} = \frac{v(1+\alpha f_2)}{1+f_2}$, $F_1(\alpha v) = \frac{f_2-f_1}{1+f_2}\frac{v-\alpha v}{\alpha-\alpha v} > 0$ and $F_2(\alpha v) = 0$. Similary, if $f_1 > f_2$, then $\bar{p} = \frac{v(1+\alpha f_1)}{1+f_1}$, $F_2(\alpha v) = \frac{f_1-f_2}{1+f_1}\frac{v-\alpha v}{\alpha-\alpha v} > 0$ and $F_1(\alpha v) = 0$. In all, $\bar{p} = \frac{1+\alpha \max(f_1,f_2)}{1+\max(f_1,f_2)}v$ and $F_i(\alpha v) = \min\{\frac{f_j-f_i}{1+f_j}\frac{v-\alpha v}{\alpha-\alpha v},\ 0\}$.

Next, we need to solve f_1, f_2 by calculating A_1, B_1, A_2 and B_2.

(1) $\max\{\mu_{LH}, \mu_{HL}\} < \bar{\mu}$.

By Lemma A.3, given *ar* of $p_1 > p_2$, $\mu_2 = \mu^*$; given *ra* of $p_1 < p_2$, $\mu_1 = \mu^*$. Therefore, Bayes' rule implies that $A_1 = \frac{1}{\mu^*}(\mu_{LH} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH})$ and $B_1 = 1 - \frac{\mu_{HH}+\mu_{HL}}{\mu^*}$, $A_2 = \frac{1}{\mu^*}(\mu_{HL} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH})$ and $B_2 = 1 - \frac{\mu_{HH}+\mu_{LH}}{\mu^*}$. Therefore,

$$f_1 = \frac{\mu^* - (\mu_{HH} + \mu_{LH})}{(1-\mu^*)(1 - \frac{\mu_{LL}}{1-\mu^*} - \frac{\mu_{HH}}{2\mu^*-1})},\ f_2 = \frac{\mu^* - (\mu_{HH} + \mu_{HL})}{(1-\mu^*)(1 - \frac{\mu_{LL}}{1-\mu^*} - \frac{\mu_{HH}}{2\mu^*-1})}.$$

(2) $\min\{\mu_{LH}, \mu_{HL}\} > \bar{\mu}$.

By Lemma A.3, given *ra* of $p_1 > p_2$, the belief of good 1 is 1 and given *ar* of $p_1 < p_2$, the belief of good 2 is 1. Then, $A_1 = \mu_{LH} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH} + \mu_{LL}$, $B_1 = \mu_{LH} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}$, $A_2 = \mu_{HL} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH} + \mu_{LL}$, $B_2 = \mu_{HL} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}$. Therefore,

$$f_1 = \frac{\mu_{HL} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}}{\mu_{LL}},\ f_2 = \frac{\mu_{LH} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}}{\mu_{LL}}.$$

(3) $\mu_{LH} > \bar{\mu} > \mu_{HL}$.

By Lemma A.3, given *ra* of $p_1 > p_2$, $\mu_2 = \mu^*$ and given *ar* of $p_1 < p_2$, $\mu_2 = 1$. Then, $A_1 = \mu_{LH} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH} + \mu_{LL}$ and $B_1 = 1 - \frac{\mu_{HH}+\mu_{HL}}{\mu^*}$, $A_2 = \frac{1}{\mu^*}(\mu_{HL} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH})$ and $B_2 = \mu_{HL} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH}$. Therefore,

$$f_1 = \frac{\mu^*}{1-\mu^*},\ f_2 = \frac{\mu^* - (\mu_{HH} + \mu_{HL})}{(1-\mu^*)(\mu_{HL} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH})}.$$

(4) $\mu_{HL} > \bar{\mu} > \mu_{LH}$.

Similarly,

$$f_1 = \frac{\mu^* - (\mu_{HH} + \mu_{LH})}{(1-\mu^*)(\mu_{LH} - \frac{1-\mu^*}{2\mu^*-1}\mu_{HH})},\ f_2 = \frac{\mu^*}{1-\mu^*}.$$

□

Proof of Theorem 5.1

Proof. Lemma A.2 and Step 1 of Lemma A.5 describes the belief updating of two goods' qualities given rr in period 1. HH seller always chooses rr. Lemma A.5 implies that

$$p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}},\ p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}.$$

Lemma A.3 describes the belief updating of two goods' qualities in period 2 given ar or ra in period 1 as well as LL seller's strategy in period 1, which is unique if $p_1 \neq p_2$. Since the probability of $p_1 = p_2$ is zero, then the equilibrium outcome is unique.

Lemma A.7 characterized buyers' equilibrium offers in period 1. Corollary A.3 characterizes the equilibrium from period 2 on. □

Proof of Corollary 5.2

Proof. See Lemma A.7. □

Proof of Corollary 5.3

Proof. If $\mu_{LL} < \mu^{**}$, then $\mu_{LH} = \mu_{HL} > \bar{\mu}$ and by Lemma A.7,

$$F_i(p) = \big(\frac{1}{2\mu_{LL}}(1 - \frac{\mu_{HH}}{2\mu^*-1}) - \frac{1}{2}\big)\frac{p-\alpha v}{v-p},$$

which is decreasing in μ_{LL}. Furthermore, given $p_1 > p_2$, $p_{ar} = 1$.

If $\mu_{LL} > \mu^{**}$, then $\mu_{LH} = \mu_{HL} < \bar{\mu}$ and

$$F_i(p) = \frac{\mu^* - \frac{1}{2} + \frac{\mu_{LL}}{2} - \frac{\mu_{HH}}{2}}{(1-\mu^*)(1 - \frac{\mu_{LL}}{1-\mu^*} - \frac{\mu_{HH}}{2\mu^*-1})}\frac{p-\alpha v}{v-p},$$

which is increasing in μ_{LL}. Furthermore, given $p_1 > p_2$, $p_{ar} = \frac{1-\mu^*}{2\mu^*}(\frac{1}{\mu_{LL}}(1 - \frac{\mu_{HH}}{2\mu^*-1}) - 1)$ is decreasing in μ_{LL}.

□

Proof of Theorem 5.4

Proof. The following steps describe the equilibria given any history with the belief $(<\mu^*, <\mu^*)$.

Step 1: The belief of good 1 is μ^* given ra of (p_1, p_2), where $p_1 > \alpha v$; the belief of good 2 is μ^* given ar of (p_1, p_2), where $p_2 > \alpha v$.

WLOG, assume that $p_1 > \alpha v$, then show that given ra of (p_1, p_2), the belief of good 1 must be μ^*.

(1) Given ra, the belief in market 1 is less than μ^*. Then, LL will not choose ra since aa dominates ra. If only HL plays ra, the the updated belief of good 1 is 1, a contradiction to the belief of good 1 is less than μ^*. If no type plays ra, then the belief of good 2 remains to be less than μ^* given ar. Therefore, LL strictly prefer rr to ar, thus only LH chooses ar and the belief of good 2 goes to 1 given ar, a contradiction.

(2) Given ra, the belief in market of good 1 is larger than μ^*. (i) $\mu^* \leq \frac{1}{2}$. Then, LL will not choose rr, then it is impossible that the belief is updated to (μ^*, μ^*) given rr, a contradiction; (ii) $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} \geq 1$ and $\mu^* > \frac{1}{2}$. Then, the belief of good 2 is less than μ^* given ar. LH strictly prefer rr to ar. As a result, given ra, the belief in market of good 1 remains to be less than μ^*, a contradiction.

Symmetrically, if $p_2 > \alpha v$, then given ar of (p_1, p_2), the belief of good 2 must be μ^*; given ra of (p_1, p_2), the belief of good 1 must be μ^*.

Step 2: $p_1 = p_2 = \alpha v$.

Assume by contradiction that $p_1 + p_2 > 2\alpha v$ and denote $\bar{p}$ as the supreme of such $p_1 + p_2$. For each $\epsilon > 0$, there exists (p_1, p_2) such that $p_1 + p_2 > \bar{p} - \epsilon$. Consider the deviation of buyer 1 from p_1 to $p_1 - \epsilon > \alpha v$ for small $\epsilon > 0$. By Lemma A.2, given rr of $(p_1 - \epsilon, p_2)$ or (p_1, p_2), the belief is updated to (μ^*, μ^*). By Step 1, given ra of (p_1, p_2) or $(p_1 - \epsilon, p_2)$, the belief of good 1 must be μ^*. Therefore, the probability of acceptance in market 1 is $\mu_{LH}(1 - p_{LH}) + p_{ar}\mu_{LL} = 1 - \frac{\mu_{HL}+\mu_{HH}}{\mu^*}$.

Step 3: Given rr of $(\alpha v, \alpha v)$, the updated belief is (μ^*, μ^*). Given ra or ar of $(\alpha v, \alpha v)$, the updated belief of the remaining good is μ^*.

(1) The updated belief is (μ^*, μ^*) given rr of $(\alpha v, \alpha v)$.

Assume that the updated belief is $(< \mu^*, < \mu^*)$ given rr of $(\alpha v, \alpha v)$. By Lemma A.0 under Assumption 5.5, the updated belief of the remaining good given ar or ra is not larger than μ^*. In all, given $p_1 = p_2 = \alpha v$, the probability of acceptance in market 1 is less than $1 - \frac{\mu_{HL}+\mu_{HH}}{\mu^*}$. Consider the deviation of buyer 1 to $\alpha v + \epsilon$. By Lemma A.2, given rr of $(\alpha v + \epsilon, \alpha v)$, the updated belief is (μ^*, μ^*). By Step 1, given ra of $(\alpha v + \epsilon, \alpha v)$, the updated belief of good 1 is μ^*. Therefore, faced with $(\alpha v + \epsilon, \alpha v)$, the probability of acceptance in market 1 is $1 - \frac{\mu_{HL}+\mu_{HH}}{\mu^*}$. Therefore, $(\alpha v + \epsilon, \alpha v)$ is a profitable deviation since buyer 1 is more likely to be accepted by deviating to a slightly higher offer $p_1 = \alpha v + \epsilon$.

(2) The updated belief of the remaining good is μ^* given ra of $(\alpha v, \alpha v)$

Otherwise, the updated belief of good 1 is less than μ^* given ra of $(\alpha v, \alpha v)$. Therefore, faced with $(\alpha v, \alpha v)$, the probability of acceptance in market 1 is less than $1 - \frac{\mu_{HL}+\mu_{HH}}{\mu^*}$. Consider a deviation of buyer 1 to $\alpha v + \epsilon$. By Lemma A.2, given rr of $(\alpha v + \epsilon, \alpha v)$, the updated belief is (μ^*, μ^*). If given ra of $(\alpha v + \epsilon, \alpha v)$, the belief in market 1 is less than μ^*, then LL will not choose ra. It is impossible that HL chooses ra with positive probability, which means that no type chooses ra. We can show that the probability of rejection by L type in market 1 is $\frac{1-\mu^*}{\mu^*}(\mu_{HH} + \mu_{HL})$. If given ra of $(\alpha v + \epsilon, \alpha v)$, the belief in market 1 is μ^*, then the probability of rejection by L type in market 1 is $\frac{1-\mu^*}{\mu^*}(\mu_{HH} + \mu_{HL})$. In all, $p_1 = \alpha v + \epsilon$ is a profitable deviation when ϵ is small enough since buyer 1 is more likely to be accepted if the offer is $(\alpha v + \epsilon, \alpha v)$.

Step 4: In period $t \geq 2$, if good i is left untraded, a losing offer is made. Otherwise, given offers $p_1 = p_2 = \alpha v$, all seller types reject both offers, a contradiction to belief updating in period 2.

Step 5: Given the belief updating given rr, ra and ar. The seller's strategy is essentially unique since the unconditional probability of acceptance for good 1 and 2 in period 1 is $1 - \frac{\mu_{HL}+\mu_{HH}}{\mu^*}$ and $1 - \frac{\mu_{LH}+\mu_{HH}}{\mu^*}$, respectively.

□

Appendix B. Proofs of Section 6: Private offers

In this section, we prove a series of Lemmas: Lemma B.1-B.3, which are used in the proof of Theorem 5.2.

Lemma B.1 : Under private offer, given any history with $(< \mu^*, < \mu^*)$, the buyers can only offer $p_1 = p_2 = v$. Given rr, the belief of high type is updated to be (μ^*, μ^*). Buyer i randomizes between offering v and α and all types of seller accept α and reject v. Specifically, the probability of offering v is λ such that $v - \alpha v = \delta(\lambda v + (1-\lambda)\alpha - \alpha v)$.

Proof. **Step 1 :** It is impossible that one offer is less than v and another offer is v.

Assume that $p_1 < v$ and $p_2 = v$. It is impossible for LH or LL to mix between accepting p_1 and rejecting p_1, since otherwise buyer 1 can increase the offer by small ϵ to attract LH or LL to buy good 1 for sure so that buyer 1 can make a profit.

Case 1: LH chooses ar.

Firstly, consider that LL accepts p_1 for sure. If LL accepts p_1 for sure and HL accepts $p_2 = v$, then by deviating to ra, LL can get $(v - \alpha v) + \delta(\alpha - \alpha v)$, which is a profitable deviation. If LL accepts p_1 for sure and HL rejects $p_2 = v$, then LH will deviate to rr to get $\delta(\alpha - \alpha v)$ instead of $p_1 - \alpha v$.

Secondly, consider that LL rejects p_1 for sure. (1) HL plays ra for sure. Then, we can show that LL plays rr for sure. Otherwise, LL will play ra with positive probability, but rr is a profitable deviation for LL. If LL deviate to ra, then he gets $v - \alpha v + \delta(\alpha - \alpha v)$. In order for HL not to deviate to rr, $\frac{\mu_{HH}}{\mu_{HH}+\mu_{LL}} < \mu^*$, then give rr, LL gets $2(v-\alpha v)$. Therefore, (r, a) is a profitable deviation for LL, a contradiction.

(2) HL plays rr for sure. Then, LL needs to play rr. Show that ar is a profitable deviation for LL. Given rr, it is trivial that the belief of high type in the second market decreases. If the belief of high type in the first market is still less than μ^*, then let $p_1 = \bar{p} - \epsilon$, where $\bar{p}$ is the supreme of offer by buyer 1. Therefore, ar is strictly better than rr for LH, so buyer 1 has incentive to lower p_1 a little to make a profit, a contradiction. If the belief of high type in the first market is μ^*, then LH can be indifferent between ar and rr. Furthermore, LL get at most $p_1 - \alpha v + \delta(v - \alpha v)$, which is less than $p_1 - \alpha v + \delta(\alpha - \alpha v)$, the payoff of LL by deviating to ar. Therefore, LL will deviate to ar, a deviation.

(3) HL mixes between ra and rr. Given rr, the second market needs μ^*. Consider the case that given rr, there is still adverse selection in market 1. LL can not choose rr for sure, since otherwise LL can deviate to ra to make a profit. Furthermore, LL can not choose ra for sure, because given rr, buyer 1 believes that the good has high type for sure, a contradiction to the fact that LL chooses ar. Therefore, LL mixes between ra and rr. Given ra, only good 2 is left untraded. If the updated belief in market 2 is larger than μ^* given ra, then ra is better than rr for LL, a contradiction. If the updated belief in market 2 is less than μ^* given ra, then ra gives LL the payoff $v - \alpha v + \delta(v - \alpha v)$. By deviating to ar, LL can get $p_1 - \alpha v + \delta(\alpha - \alpha v)$. Since LH prefers ar to ra, then $p_1 - \alpha v \geq v - \alpha + \delta(\alpha - \alpha v)$. Therefore, $p_1 - \alpha v + \delta(\alpha - \alpha v) > v - \alpha v + \delta(v - \alpha v)$ for large δ, thus ar is better than ra for LL, a contradiction. What about $= \mu^*$ given ra?

Case 2: LH chooses rr.

Consider the case that LL accepts p_1 for sure. If LL plays ar, then he will reveal himself. Thus, LL plays aa. If HL plays ra with positive probability, then LL can deviate to ra to mimic HL, which is a profitable deviation. Therefore, HL plays rr. It is impossible that LH is indifferent between rr and ar since otherwise buyer 1 can increase p_1 a little to attract LH to accept for sure. Therefore, ar gives LH better than rr. LL has an incentive to deviate to rr in order to get higher than $p_1 - \alpha v$ in market 1 and at least $v - \alpha v$ in market 2, which is higher than the payoff of aa. This is a contradiction.

The remaining case is that LL rejects p_1 for sure. Given rr, the first market cannot be severe, since buyer 1 can increase the offer very close to v to attract LH to earn a positive profitable. We need that given rr and ra, the first market is larger or equal to μ^*. Because $\frac{\mu_{HL}(1-p_{HL})}{\mu_{LL}(1-p_{rr})+\mu_{HL}(1-p_{HL})} \geq \mu^*$ and $\frac{p_{HL}\mu_{HL}+\mu_{HH}}{\mu_{LH}+p_{HL}\mu_{HL}+p_{rr}\mu_{LL}+\mu_{HH}} \geq \mu^*$, then $\frac{\mu_{HL}+\mu_{HH}}{\mu_{LL}+\mu_{HL}+\mu_{LH}+\mu_{HH}} \geq \mu^*$, a contradiction to $\mu^* > \mu_{HL} + \mu_{HH}$.

Step 2 : It is impossible that both offers are less than v.

It is impossible for LH or LL to mix between accepting p_1 and rejecting p_1, since otherwise buyer 1 can increase the offer by small ϵ to attract LH or LL to buy good 1 for sure so that buyer 1 can make a profit. By the same argument, it is impossible for HL or LL to mix between accepting p_2 and rejecting p_2. In all, all four types play pure strategy.

Case 1: LH plays ar and HL plays ra.

LL can not play rr or aa since he can mimic LH by playing ar to make a profit. Assume WLOG that LL plays ar. If $p_1 < p_2$, then by deviating to ra, LL can get $p_2 - \alpha v + \delta(\alpha - \alpha v)$, which is larger than the best payoff if ar is played. If $p_1 = p_2$, then buyer 2 can increase p_2 a little to attract LL to accept the lower offer, a contradiction. Therefore, $p_1 > p_2$.

Show that $p_1 - p_2 > \frac{\mu_{LL}}{\mu_{HL}}(v - p_1)$ if $\frac{\mu_{LH}}{\mu_{LH}+\mu_{LL}} > \mu^*$. ar gives LL the payoff $p_1 - \alpha v + \delta(\alpha - \alpha v)$. If LL deviates to ra, he gets $p_2 - \alpha v + \delta(\alpha - \alpha v)$. If buyer 2 increases the offer from p_2 to $p_1 + \epsilon$ so that LL will choose ra. For buyer 2, the loss from HL is $(p_1 + \epsilon - p_2)\mu_{HL}$ and the gain is $(v - p_1 - \epsilon)\mu_{LL}$. To prevent this deviation by buyer 2, the loss is larger than the gain: $(p_1 + \epsilon - p_2)\mu_{HL} > (v - p_1 - \epsilon)\mu_{LL}$. For small ϵ, we have show that $p_1 - p_2 > \frac{\mu_{LL}}{\mu_{HL}}(v - p_1)$.

Show that $p_1 - p_2 > \delta(\alpha - v)$ if $\frac{\mu_{LH}}{\mu_{LH}+\mu_{LL}} < \mu^*$. ar gives LL the payoff $p_1 - \alpha v + \delta(v - \alpha v)$. If LL deviates to ra, he gets $p_2 - \alpha v + \delta(\alpha - \alpha v)$. Therefore, we need $p_1 - \alpha v + \delta(v - \alpha v) > p_2 - \alpha v + \delta(\alpha - \alpha v)$, thus $p_1 - p_2 > \delta(\alpha - v)$.

In order to prevent HL to mimic LH, we need $p_2 - \alpha v \geq p_1 - \alpha + \delta(\alpha - \alpha v)$, so $p_1 - p_2 < (1-\delta)(\alpha - v)$, a contradiction if δ is large enough.

Case 2: LH rejects p_1 for sure.

The same argument as in Case 2 of Step 1. □

Lemma B.2: Under private offer, if $\mu^* \leq \frac{1}{2}$ or $\mu^* > \frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} \geq 1$, then given ra and ar, the belief of the remaining good i is μ^* and buyer i randomizes between offering v and α and all types of seller accept α and reject v. Specifically, the probability of offering v is λ such that $v - \alpha v = \delta(\lambda v + (1-\lambda)\alpha - \alpha v)$. It is possible that ra or ar is off the equilibrium and the updated belief of the remaining good is less than or equal to μ^* given the off-the-equilibrium actions.

Proof. Show that given ra and ar, the belief of the remaining good is less than or equal to μ^*. Prove by contradiction. Assume that given ra, the belief of the market 1 is larger than μ^*, then LL will not choose rr and ar since ra gives LL the payoff $v - \alpha v + \delta(\alpha - \alpha v)$. If given ar, the belief of the market 2 is more than or equal to μ^*, then by step 4, we have $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$, a contradiction. Therefore given ar, the belief of the market 2 is less than μ^*. Therefore, LL will not choose ar and the belief of the market 2 is 1 given ar, a contradiction. By symmetry, it is impossible that given ar, the belief of the market 1 is larger than μ^*.

If both ra and ar is on the equilibrium, then the belief of the remaining good is μ^* given ar and ra. Otherwise, assume that given ra, the belief of the remaining good is less than μ^*, then LL will not choose ra and given ra, the remaining good is high type, a contradiction.

If both ra and ar is off the equilibrium, then we need $\mu_{HL} = \mu_{LH}$. If ra is off the equilibrium and ar is on the equilibrium, then the belief of market 2 goes to μ^* given ar and the belief of market 1 is less than or equal to μ^* given ra. In all, $p_{HL} = 1$ and from (A.1) and from (A.2),

$$p_{rr} = \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} + \frac{1-2\mu^*}{\mu^*}\frac{\mu_{HL}}{\mu_{LL}},\ p_{ar} = \frac{1-\mu^*}{\mu^*}\frac{\mu_{LH}-\mu_{HL}}{\mu_{LL}}.$$

Therefore, $\mu_{HL} \leq \mu_{LH}$ and $\mu^* \leq \frac{\mu_{HH}+\mu_{HL}}{\mu_{HH}+2\mu_{HL}}$. In all, the probability of acceptance in market 1 satisfies $1 - \frac{\mu_{HH}+\mu_{HL}}{\mu^*}$ and the probability of acceptance in market 2 is $1 - \frac{\mu_{HH}+\mu_{LH}}{\mu^*}$.

□

Lemma B.3: Under private offer, if $\frac{\mu_{HH}}{2\mu^*-1}+\frac{\mu_{LL}}{1-\mu^*}<1$, then given ra and ar, the belief of the remaining good i is larger or equal to μ^* and buyer i offers α and the seller accept it. It is not true that given both ra and ar, the belief of the remaining good i is equal to μ^*.

Proof. If both ra and ar is off the equilibrium, then we need $p_{rr}=\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}}{\mu_{LL}}$, a contradiction to $\frac{\mu_{HH}}{2\mu^*-1}+\frac{\mu_{LL}}{1-\mu^*}<1$. If ra is off the equilibrium and ar is on the equilibrium, then the belief of market 2 goes to μ^* given ar and the belief of market 1 is less than or equal to μ^* given ra. In all, $p_{HL}=1$ and from (1) and from (2),

$$p_{rr}=\frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}+\frac{1-2\mu^*}{\mu^*}\frac{\mu_{HL}}{\mu_{LL}},\ p_{ar}=\frac{1-\mu^*}{\mu^*}\frac{\mu_{LH}-\mu_{HL}}{\mu_{LL}},$$

which is impossible since $\frac{\mu_{HH}}{2\mu^*-1}+\frac{\mu_{LL}}{1-\mu^*}<1$ implies that $p_{rr}<0$, a contradiction.

Therefore, ra and ar are on the equilibrium. Then, show that the belief of the remaining good is no less than μ^* given ra and ar. Assume by contradiction that the belief of market 1 is less than μ^* given ra. Therefore, LL will not choose ra, and given ra, the remaining good is high type, a contradiction. If $\frac{\mu_{HH}}{2\mu^*-1}+\frac{\mu_{LL}}{1-\mu^*}<1$, it is impossible that the belief of the remaining good i is μ^* given both ra and ar. Assume WLOG that given ra, the belief of the market 1 is larger than μ^* and buyer 1 will offer α. As a result, LL will not choose rr. If given ar, the belief of the market 2 is larger than μ^*, then buyer 2 will offer α. If given ar, the belief of the market 2 is μ^* and buyer 2 will offer α for sure, then LL will not choose ar, a contradiction. □

Proof of Theorem 5.2

Proof. By Lemma B.1-B.3, the result follows.

□

Appendix C. Welfare Comparison

Proof of Proposition 7.1

Proof. **Step 1**: V_s^{public}.

We study the surplus under public offers and sufficiently negative correlation. The probability of ar or ra and the remaining good is low is μ_{LL}. The probability of ar or ra and the remaining good has high quality is $\mu_{HL}(1-p_{HL})+\mu_{LH}(1-p_{LH})=1-\frac{\mu_{HH}}{2\mu^*-1}-\mu_{LL}$. The probability of rr is $\frac{\mu_{HH}}{2\mu^*-1}$. Under offers (p_1,p_2), the surplus under ar or ra is

$$(1-\frac{\mu_{HH}}{2\mu^*-1})(v-\alpha v)+(1-\frac{\mu_{HH}}{2\mu^*-1}-\mu_{LL})\delta(1-\alpha)+\mu_{LL}\delta(v-\alpha v).$$

Under offers (p_1,p_2), the payoff under rr is

$$\frac{\mu_{HH}}{2\mu^*-1}\Big(\frac{\delta\lambda_1}{1-\delta(1-\lambda_1)}+\frac{\delta\lambda_2}{1-\delta(1-\lambda_2)}\Big)(\mu^*+(1-\mu^*)v)(1-\alpha)$$

$$=\frac{\mu_{HH}}{2\mu^*-1}\frac{p_1+p_2-2\alpha v}{\alpha-\alpha v}(\mu^*+(1-\mu^*)v)(1-\alpha).$$

The above equation holds since $\frac{\delta\lambda_1}{1-\delta(1-\lambda_1)}=\frac{p_1-\alpha v}{\alpha-\alpha v}$ and $\frac{\delta\lambda_2}{1-\delta(1-\lambda_2)}=\frac{p_2-\alpha v}{\alpha-\alpha v}$, where λ_i is the probability of offering winning offer by buyer $i=1,2$ from period 2 on. Therefore, under offers (p_1,p_2), the surplus $V_s^{public}(p_1,p_2)$ is

$$(1-\alpha)\big((1-\frac{\mu_{HH}}{2\mu^*-1}+\delta\mu_{LL})v+\delta(1-\frac{\mu_{HH}}{2\mu^*-1}-\mu_{LL})+\frac{\mu_{HH}}{2\mu^*-1}\frac{p_1+p_2-2\alpha v}{\alpha-\alpha v}(\mu^*+(1-\mu^*)v)\big).$$

By Theorem 5.1, in period 1, buyer $i=1,2$ mixes over a continuum of offer p_i. Denote the expectation over offers (p_1,p_2) as $Ep_1,p_2(\cdot)$. Therefore, the expected surplus V_s^{public} is $E_{p_1,p_2}V_s^{public}(p_1,p_2)$.

Step 2: $V_i^{public}=V_{no}^{public}$.

By Theorem 5.4, under public offers and insufficiently negative correlation, the surplus is exactly the sum of surpluses from both goods if each buyer cannot observe the trading activities in the other market: no cross-market observability.

The probability of accepting in market 1 is $1-\frac{\mu_{HL}+\mu_{HH}}{\mu^*}$. The probability of accepting in market 2 is $1-\frac{\mu_{LH}+\mu_{HH}}{\mu^*}$. The surplus in market 1 is

$$(1-\alpha)v(1-\frac{\mu_{HL}+\mu_{HH}}{\mu^*})=(1-\alpha)v(1-(\mu_{HL}+\mu_{HH})\frac{1-v}{\alpha-v}).$$

The surplus in market 2 is

$$(1-\alpha)v(1-\frac{\mu_{LH}+\mu_{HH}}{\mu^*})=(1-\alpha)v(1-(\mu_{LH}+\mu_{HH})\frac{1-v}{\alpha-v}).$$

In all, the surplus $V_i^{public}=V_{no}^{public}$ is

$$(1-\alpha)v(2-\frac{\mu_{HL}+\mu_{HH}+\mu_{LH}+\mu_{HH}}{\mu^*})=(1-\alpha)v(2-(1+\mu_{HH}-\mu_{LL})\frac{1-v}{\alpha-v}).$$

Step 3: $V_s^{private}$.

By Theorem 6.1, under private offers and sufficiently negative correlation, the offers in period are $p_1 = p_2 = v$. By similar argument as in Step 1, the surplus $V_s^{private}$ is

$$(1-\alpha)\big((1-\frac{\mu_{HH}}{2\mu^*-1}+\delta\mu_{LL})v+\delta(1-\frac{\mu_{HH}}{2\mu^*-1}-\mu_{LL})+\frac{\mu_{HH}}{2\mu^*-1}\frac{2(v-\alpha v)}{\alpha-\alpha v}(\mu^*+(1-\mu^*)v)\big).$$

Step 4: $V_i^{private} = V_{no}^{private}$.

By Theorem 6.1, under private offers and sufficiently negative correlation, the surplus is exactly the sum of surpluses from both goods if each buyer cannot observe the trading activities in the other market: no cross-market observability. The payoff in market 1 is

$$(1-\alpha)v(1-\frac{\mu_{LH}+\mu_{HH}}{\mu^*})+\frac{\mu_{LH}+\mu_{HH}}{\mu^*}\frac{\delta\lambda}{1-\delta(1-\lambda)}(\mu^*+(1-\mu^*)v)(1-\alpha)=(1-\alpha)v(1-\mu_{LH}-\mu_{HH}).$$

The payoff in market 2 is

$$(1-\alpha)v(1-\frac{\mu_{HL}+\mu_{HH}}{\mu^*})+\frac{\mu_{HL}+\mu_{HH}}{\mu^*}\frac{\delta\lambda}{1-\delta(1-\lambda)}(\mu^*+(1-\mu^*)v)(1-\alpha)=(1-\alpha)v(1-\mu_{HL}-\mu_{HH}).$$

In all , the payoff $V_i^{private} = V_{no}^{private}$ is $(1-\alpha)v(1-\mu_{HH}+\mu_{LL})$.

Step 5: $V_s^{public} > V_s^{private}$, $V_i^{public} > V_i^{private}$.

Since $\mu^* < 1$, then $V_i^{public} > V_i^{private}$. Since $v > p_1$ and $v > p_2$, then $V_s^{public} > V_s^{private}$.

Step 6: $V_s^{public} > V_{no}^{public}$, $V_s^{private} > V_i^{private} = V_{no}^{private}$.

As $\delta \to 1$, $V_s^{public} > (1-\alpha)\big((1-\frac{\mu_{HH}}{2\mu^*-1}+\delta\mu_{LL})v+\delta(1-\frac{\mu_{HH}}{2\mu^*-1}-\mu_{LL})\big) > 2(1-\alpha)v(1-\frac{\mu_{HH}}{2\mu^*-1}) > (1-\alpha)v(2-\frac{1+\mu_{HH}-\mu_{LL}}{\mu^*}) = V_{no}^{public}$. The above inequality is equivalent to $1 > \frac{\mu_{HH}}{2\mu^*-1}+\mu_{LL}$, which is implied by sufficiently negative correlation.

As $\delta \to 1$, $V_s^{private} > V_{no}^{private}$ is equivalent to $(1-A)\frac{1+\mu_{HH}-\mu_{LL}}{\mu^*} > (1-A)\frac{2\mu_{HH}}{2\mu^*-1}$, where $A \equiv \frac{(v-\alpha v)}{\alpha-\alpha v}(\mu^*+(1-\mu^*)v) < 1$. Therefore, we only need to show $\frac{1+\mu_{HH}-\mu_{LL}}{\mu^*} > \frac{2\mu_{HH}}{2\mu^*-1}$, which holds by sufficiently negative correlation.

□

Appendix D. Non-Full Support

Proof of Proposition 8.1

Proof. **Case 1:** Public offers.

Step 1: Given rr of (p_1, p_2), the updated belief given rr of (p_1, p_2) is (μ^*, μ^*) or $(<\mu^*, <\mu^*)$.

Show that given rr of (p_1, p_2) and the belief of good i is equal to μ^*, then the belief of both goods goes to (μ^*, μ^*). Assume that the belief of good 1 given rr of (p_1, p_2) is equal to μ^*.

Prove by contradiction that the belief of good 2 given rr of (p_1, p_2) is not μ^*. (1) Given rr of (p_1, p_2), the belief of good 2 is less than μ^*. By lemma A.1, the offer will be $(\alpha, \alpha v)$. Therefore, all seller types will reject p_1 to get α in the next period. Therefore, $p_{LH} = 1$ and the updated belief of good 1 given rr of (p_1, p_2) is equal to $\frac{\mu_{HL}p_{HL}+\mu_{HH}}{\mu_{HL}p_{HL}+\mu_{HH}+\mu_{LH}} < \mu_{HL} + \mu_{HH} < \mu^*$, a contradiction. (2) Given rr of (p_1, p_2), the belief of good 2 is larger than μ^*. Therefore, all seller type is to reject p_2 to get α in the next period. By the same argument as in the last paragraph, the updated belief of good 2 given rr of (p_1, p_2) is less than μ^*, a contradiction.

Show that given rr of (p_1, p_2), it is impossible that the belief of one of the goods is larger than μ^*. Assume that given rr of (p_1, p_2), if the belief of good 1 is larger than μ^*, then LH will reject p_1 in the first period to get α in the next period. Then, the updated belief of good 1 given rr of (p_1, p_2) is equal to $\frac{\mu_{HL}p_{HL}+\mu_{HH}}{\mu_{HL}p_{HL}+\mu_{HH}+\mu_{LH}} < \mu_{HL} + \mu_{HH} < \mu^*$, a contradiction.

Step 2: Suppose that in period n with the belief $(< \mu^*, < \mu^*)$ and that the equilibrium calls for two offers leading to a posterior belief (μ^*, μ^*). Then all future offers must be losing.

By Step 1, if the updated belief given rr of (p_1, p_2) is (μ^*, μ^*) or $(< \mu^*, < \mu^*)$. Therefore, we only need to consider history with belief (μ^*, μ^*) or $(< \mu^*, < \mu^*)$.

Suppose they were not and assume WLOG that offer 1 is not a losing offer. Then, in period n, buyer 1 must offer a price p_1^n strictly larger than αv (indeed, αv is LH's reservation value for good 1, and by rejecting, LH gets an offer α for good 1 in the future with positive probability).

Denote $\bar{p}_1 > \alpha v$ as the supreme of p_1 in equilibrium for any history with belief $(< \mu^*, < \mu^*)$. For any $\epsilon > 0$, pick a history such that $p_1^n > \bar{p}_1 - \epsilon$. Since given rr of (p_1^n, p_2) the updated belief is $(< \mu^*, < \mu^*)$, then $p_{LH} > \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}$. If buyer 1 deviates by offering $\bar{p}_1 - 2\epsilon$ instead, then either the posterior is (μ^*, μ^*) or $(< \mu^*, < \mu^*)$. In the first case, the probability of acceptance faced with $\bar{p}_1 - 2\epsilon$ would be $\frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}$, which is higher than that faced with offer p_1^n. Therefore, $\bar{p}_1 - 2\epsilon$ would be a profitable deviation, since the price would be lower and the probability of acceptance higher. In the second case, by accepting the offer, LH gets $\bar{p}_1 - 2\epsilon - \alpha v$, while by rejecting, he cannot hope for more than $\delta(\bar{p}_1 - \alpha v)$. Since he is supposed to be willing to reject, this is a contradiction for small enough ϵ.

Step 3: Given a history with belief $(< \mu^*, < \mu^*)$, then two buyers offers αv and the posterior is (μ^*, μ^*).

It follows that starting from a belief strictly below $(<\mu^*, <\mu^*)$, two offers is $(\alpha v, \alpha v)$. We only need to that the posterior is (μ^*, μ^*). By Step 1, the posterior can only be (μ^*, μ^*) or $(<\mu^*, <\mu^*)$. If the posterior is $(<\mu^*, <\mu^*)$, consider a deviation by buyer 1 in to an offer $\alpha v + \epsilon$. Again, if such an offer led to a posterior belief (μ^*, μ^*) , then it would be profitable since the probability of acceptance for good 1 increases, thus buyer i's expected payoff increases for small enough ϵ. But if not, LH should accept it for sure, since he cannot hope for more than αv in the future, a contradiction.

It follows that $p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}$ and $p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}}$ so that the posterior belief given rr of $(\alpha v, \alpha v)$ is (μ^*, μ^*).

Case 2: Private offers.

Step 1 : Given any history $(<\mu^*, \mu^*)$, the two equilibrium offers are (v, v).

Given any history $(<\mu^*, \mu^*)$, it is impossible that one equilibrium offer is less than v and another equilibrium offer is v. Prove by contradiction. Assume that $p_1 < v$ and $p_2 = v$. It is impossible for LH to mix between accepting p_1 and rejecting p_1, since otherwise buyer 1 can increase the offer by small ϵ to attract LH to buy good 1 for sure so that buyer 1 can make a profit. There are two cases left. (1) LH chooses ar. Then LH gets $p_1 - \alpha v$ since given ar in the next period, it will known that the second good is H. It is a profitable deviation for LH to choose rr and get α in the next period. (2) LH chooses rr. Therefore, buyer 1 gets zero profit and given rr, the first market remains to have adverse selection. Then, it would be profitable for buyer 1 to increase the offer very close to v so that LH would choose ar and buyer 1 will earn a positive profitable, a contradiction.

Given any history $(<\mu^*, \mu^*)$, it is impossible that both equilibrium offers are less than v. Otherwise, it is impossible for LH to mix between accepting p_1 and rejecting p_1, since otherwise buyer 1 can increase the offer by small ϵ to attract LH to buy good 1 for sure so that buyer 1 can make a profit. By the same argument, it is impossible for HL to mix between accepting p_2 and rejecting p_2. In all, all three types play pure strategy. By Step 1, it is impossible for LH or HL to choose rr. Therefore, LH plays ar and HL plays ra. Then LH has an incentive to deviate to rr and get a better profit: $\delta(\alpha - \alpha v)$, a contradiction.

Step 2: Given rr of (v, v), the updated belief is (μ^*, μ^*).

In order for LH to be indifferent between rr and ar, it is impossible that the updated belief of good 1 is larger than μ^* since otherwise LH can get $\delta(\alpha - \alpha v)$ by choosing rr in period 1, a contradiction. By the same argument, it is impossible that the updated belief of good 2 is larger than μ^*. Therefore, the updated belief has to be $(\leq \mu^*, \leq \mu^*)$. Assume that the updated belief is $(< \mu^*, < \mu^*)$. By Step 1, the continuation payoff is no larger than $\alpha(v - \alpha v)$ for both LH and HL, then LH will accept v in period 1 and gets $v - \alpha v$, a contradiction to the fact that LH is indifferent between rr and ar. Assume that the updated belief is $(\mu^*, < \mu^*)$. By Lemma A.1, LH can get an offer α by choosing rr in period 1, a contradiction to the fact that LH is indifferent between rr and ar.

It follows that $p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}$ and $p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}}$ so that the posterior belief given rr of $(\alpha v, \alpha v)$ is (μ^*, μ^*).

□

Appendix E. Pre-bargaining Investment Stage

Proof of Proposition 9.2

Proof. Define $U(HH)$, $U(HL)$,$U(LH)$, and $U(LL)$ as the continuation payoff of the seller if the investment decision is HH, HL, LH and LL, respectively.

Belief updating: Given g, the belief satisfies severe adverse selection for both goods and negative correlation. The updated belief satisfies $P(HL|g) = P(LH|g) = \frac{q/2}{q+(1-q)(1-\rho)} < \mu^*$ (which always holds since $\mu^* > \frac{1}{2}$) and $P(LL|g) = \frac{(1-q)(1-\rho)}{q+(1-q)(1-\rho)} < 1 - \mu^*$. Therefore, $q > \frac{(1-\rho)\mu^*}{1-\rho\mu^*}$. Given b, the future buyers are certain that seller's type is LL.

Seller's optimal condition: It is not a best choice to choose HH since there is higher investment cost and the future benefit from selling two goods are zero. By Theorem 5.1, given b, seller's payoff is zero since it is known that the seller is LL type. Given g, $U(HL) = U(LH) = E(p_1) - \alpha v > 0$ and $U(LL) = E(\min\{p_1, p_2\}) + \alpha - 2\alpha v > 0$. Since the seller is indifferent between HL, LH and LL, then $U(HL) - U(LL) = U(LH) - U(LL) = E(p_1) - \alpha v - (1 - \rho)(E(\min\{p_1, p_2\}) + \alpha - 2\alpha v) = C(H_1) - C(L_1)$.

Choose the correct q to satisfy the indifference condition. WLOG, we only need to check that $U(LH) - U(LL) = C(H_1) - C(L_1)$. By Lemma A.7, $F_1(p) = \frac{2\mu^*-1+\mu_{LL}}{2(1-\mu^*-\mu_{LL})}\frac{p-\alpha v}{v-p}$ if $\frac{1-\mu^*}{1+\mu^*} \leq \mu_{LL} \leq 1 - \mu^*$; $F_1(p) = \frac{1-\mu_{LL}}{2\mu_{LL}}\frac{p-\alpha v}{v-p}$ if $0 < \mu_{LL} \leq \frac{1-\mu^*}{1+\mu^*}$. Therefore, $F_1(p)$ reaches the lowest level if $\mu_{LL} = \frac{1-\mu^*}{1+\mu^*}$. If $\mu_{LL} = \frac{1-\mu^*}{1+\mu^*}$ ($q = \frac{2(1-\rho)\mu^*}{1+\mu^*-2\rho\mu^*}$), the maximum of $E(p_1) - \alpha v$ is attained:

$E(p_1) - \alpha v = (1 + \frac{\mu^*}{1-\mu^*}\ln(\mu^*))(v - \alpha v)$. In all, for $q = \frac{2(1-\rho)\mu^*}{1+\mu^*-2\rho\mu^*}$, $U(HL) - U(LL) = E(p_1) - \alpha v - (1-\rho)(E(\min\{p_1,p_2\}) - \alpha + \delta(\alpha - \alpha v)) > \rho(E(p_1) - \alpha v) - (1-\rho)\delta(\alpha - \alpha v) \geq \rho\big(1 + \frac{\mu^*}{1-\mu^*}\ln(\mu^*)\big)(v - \alpha v) - (1-\rho)\delta(\alpha - \alpha v)$.

If $q \to 1$, then $\mu_{LL} \to 0$ and the minimum of $E_1(p) - \alpha v$ is attained: $E(p_1) - \alpha v = 0$. If $q \to \frac{(1-\rho)\mu^*}{1-\rho\mu^*}$, then $\mu_{LL} \to 1 - \mu^*$ and the minimum of $E(p_1) - \alpha v$ is attained: $E(p_1) - \alpha v = 0$. In all, for $q \to \frac{(1-\rho)\mu^*}{1-\rho\mu^*}$ and $q \to 1$, $U(LH) - U(LL) = -(1-\rho)\delta(\alpha - \alpha v) < 0$.

By Assumption 9.2 and intermediate value theorem, there are q_1 and q_2 to satisfy the indifference condition, where $\frac{(1-\rho)\mu^*}{1-\rho\mu^*} < q_1 < \frac{2(1-\rho)\mu^*}{1+\mu^*-2\rho\mu^*} < q_2 < 1$.

Next, show that the seller's expected payoff is the same under $q = q_1$ and $q = q_2$. By the third paragraph, $E(p_1) - (1-\rho)E(\min\{p_1,p_2\}$ is a constant, thus the distribution $F_1(p)$ is the same under $\mu_{LL} = 1 - q_1, 1 - q_2$. Therefore, the seller's payoff if LH is chosen is $E(p_1) - \alpha v - C(L_1) - C(H_2)$ is the same under $q = q_1$ and $q = q_2$.

At last, show that the expected surplus under $q = q_2$ is higher than that under $q = q_1$. By Proposition 7.1, the expected surplus without considering the investment cost is $(1-\alpha)\big(1 + \delta - \delta(1-v)\mu_{LL}\big)$. Define $c = C(H_1) - C(L_1)$. The expected surplus is $(1-\alpha)\big(1 + \delta - \delta(1-v)\mu_{LL}\big) - (1-\mu_{LL})c = (1-\delta)(1+\delta) - (\delta(1-\alpha)(1-v) - c)\mu_{LL}$, which is decreasing in μ_{LL}. In all, the expected surplus is higher under $\mu_{LL} = 1 - q_2$ than under $\mu_{LL} = 1 - q_1$. □

Proof of Proposition 9.3

Proof. Define $U_s(HH)$,$U_s(HL)$, $U_s(LH)$, and $U_s(LL)$ as the continuation payoff of seller type s if the investment decision is HH, HL, LH and LL, respectively.

Belief updating: Given g, the belief satisfies severe adverse selection for both goods and negative correlation. Define $q = 1 - s^{*2}$. The updated belief satisfies $P(HL|g) = P(LH|g) = \frac{q/2}{q+(1-q)(1-\rho))} < \mu^*$ (which always holds since $\mu^* > \frac{1}{2}$) and $P(LL|g) = \frac{(1-q)(1-\rho)}{q+(1-q)(1-\rho)} < 1 - \mu^*$. Therefore, $q > \frac{(1-\rho)\mu^*}{1-\rho\mu^*}$. Given b, the future buyers are certain that the seller chooses LL.

Seller's optimal condition: It is not a best choice to choose HH since there is higher investment cost and the future benefit from selling two goods are zero. Given b, seller's payoff is zero since it is known that the seller is LL type. Given g, $U_s(LH) = E(p_1) - \alpha v > 0$ and $U_s(LL) = E(\min\{p_1,p_2\}) + \alpha - 2\alpha v > 0$. Show that seller $s_1 < s_2 = s^*$ is indifferent between LH and LL. By Assumption 9.3, seller $s_1 < s_2$ and $s_2 > s^*$ chooses LH and seller $s_1 < s_2$ and $s_2 < s^*$ chooses LL.

Choose the correct s^* to satisfy $U_s(LH)-U_s(LL) = C_{s^*}(H_2)-C_{s^*}(L_2)$ for type $s_1 < s_2 = s^*$ We can show that $F_1(p) = \frac{2\mu^*-1+\mu_{LL}}{2(1-\mu^*-\mu_{LL})}\frac{p-\alpha v}{v-p}$ if $\frac{1-\mu^*}{1+\mu^*} \leq \mu_{LL} \leq 1-\mu^*$; $F_1(p) = \frac{1-\mu_{LL}}{2\mu_{LL}}\frac{p-\alpha v}{v-p}$ if $0 < \mu_{LL} \leq \frac{1-\mu^*}{1+\mu^*}$. Therefore, $F_1(p)$ reaches the lowest level if $\mu_{LL} = \frac{1-\mu^*}{1+\mu^*}$. If $\mu_{LL} = \frac{1-\mu^*}{1+\mu^*}$ ($q = \frac{2(1-\rho)\mu^*}{1+\mu^*-2\rho\mu^*}$), the maximum of $E(p_1)-\alpha v$ is attained: $E(p_1)-\alpha v = (1+\frac{\mu^*}{1-\mu^*}\ln(\mu^*))(v-\alpha v)$. In all, for $q = \frac{2(1-\rho)\mu^*}{1+\mu^*-2\rho\mu^*}$, $U_s(LH)-U_s(LL) = E(p_1)-\alpha v-(1-\rho)(E(\min\{p_1,p_2\})-\alpha v+\delta(\alpha-\alpha v)) > \rho(E(p_1)-\alpha v)-(1-\rho)\delta(\alpha-\alpha v) \geq \rho\big(1+\frac{\mu^*}{1-\mu^*}\ln(\mu^*)\big)(v-\alpha v)-(1-\rho)\delta(\alpha-\alpha v)$.

If $q \to 1$, then $\mu_{LL} \to 0$ and the minimum of $E(p_1)-\alpha v$ is attained: $E(p_1)-\alpha v = 0$. If $q \to \frac{(1-\rho)\mu^*}{1-\rho\mu^*}$, then $\mu_{LL} \to 1-\mu^*$ and the minimum of $E(p_1)-\alpha v$ is attained: $E(p_1)-\alpha v = 0$. In all, for $q \to \frac{(1-\rho)\mu^*}{1-\rho\mu^*}$ and $q \to 1$, $U_s(LH)-U_s(LL) = -(1-\rho)\delta(\alpha-\alpha v) < 0$.

By Assumption 9.4 and intermediate value theorem, there are q_1 and q_2 to satisfy the indifference condition, where $\frac{(1-\rho)\mu^*}{1-\rho\mu^*} < q_1 < \frac{2(1-\rho)\mu^*}{1+\mu^*-2\rho\mu^*} < q_2 < 1$. Therefore, $s_1 = \sqrt{1-q_1}$ and $s_2 = \sqrt{1-q_2}$.

□

Proof of Proposition 9.4

Proof. Define $U(HH)$, $U(HL)$,$U(LH)$, and $U(LL)$ as the continuation payoff of the seller if the investment decision is HH, HL, LH and LL, respectively.

Belief updating: Given g, the belief satisfies severe adverse selection for both goods and negative correlation. Given the equilibrium strategy of the seller, the updated belief satisfies $P(HL|g) = q < \mu^*$, $P(LH|g) = 1-q < \mu^*$ (which always holds since $\mu^* > \frac{1}{2}$) and $P(LL|g) = 0$. Given signal b, the future buyers are certain that seller's type is LL.

Seller's optimal condition: It is not a best choice to choose HH since there is higher investment cost and the future benefit from selling two goods are zero. Given b, seller's payoff is zero since it is known that the seller is LL type. By Theorem 6.1, for any $q \in (1-\mu^*, \mu^*)$, given signal g, $U(HL) = U(LH) = v-\alpha v > 0$ and $U(LL) = v+\alpha-2\alpha v > 0$. By $U(HL)-U(LL) = U(LH)-U(LL) = \rho(v-\alpha v)-(1-\rho)(\alpha-\alpha v) > C(H_1)-C(L_1)$, the seller is indifferent between HL and LH.

Next, show that the seller's expected payoff remains the same for any $q \in (1-\mu^*, \mu^*)$. By the third paragraph, the seller's payoff by choosing LH is $v-\alpha v-C(L_1)-C(H_2)$, which is a constant. It is also trivial that the expected discounted surplus is also a constant, independent of q.

□

Appendix F. A continuum of Types

Proof of Theorem 10.1

Proof. Without loss of generality, assume that $\bar{x} = 1$ and $c(\bar{x}) = 1$.

Step 1: Seller's optimal choice if $p_2 - p_1 \in (A_k, B_k)$, where $0 \leq k \leq 2N$.

In period 1, there exist $x(p) \in (z^{k+1}, z^k)$ and $y(p) \equiv 1 + \underline{x} - x(p)$ such that seller $x_1 = x(p)$ is indifferent between ar and ra: $p_1 - x(p) + \delta(x^{m_k} - y(p)) = p_2 - y(p) + \delta(x^{n_k} - x(p))$. Then, $p_2 - p_1 = (1-\delta)(y(p) - x(p)) + \delta(x^{m_k} - x^{n_k}) \in [A_k, B_k]$. We can verify that $x(p) = \frac{1}{2}(1 + \underline{x} - \frac{p_2 - p_1 - \delta(x^{m_k} - x^{n_k})}{1-\delta})$.

In period 2, given ra, buyer 1 makes an offer x^{n_k}; given ar, buyer 2 makes an offer x^{m_k}.

Step 2: Seller's optimal choice if $p_2 - p_1 \in [B_k, A_{k+1}]$.

In period 1, seller $x_1 = z^{k+1}$ is indifferent between ar and ra. There are two cases:

(1) $z^{k+1} = x^{n_k}$.

$p_1 - x^{n_k} + \delta(x^{m_k} - y^{n_k}) = p_2 - y^{n_k} + \delta V_1$, where $V_1 = \frac{1}{\delta}\big(p_1 - p_2 + y^{n_k} - x^{n_k} + \delta(x^{m_k} - y^{n_k})\big) \in [0, x^{n_k - 1} - x^{n_k}]$ by $p_2 - p_1 \in [B_k, A_{k+1}]$.

To be specific, $p_2 - p_1 \geq B_k$ implies that $V_1 \geq 0$. In order to show $V_1 \leq x^{n_k-1} - x^{n_k}$, we need $p_2 - p_1 \geq (1-\delta)(y^{n_k} - x^{n_k}) + \delta(x^{m_k} - x^{n_k - 1})$. By $p_2 - p_1 \leq A_{k+1}$, a sufficient condition is $x^{m_{k+1}} - x^{n_{k+1}} \geq x^{m_k} - x^{n_k - 1}$. Since $z^{k+1} = x^{n_k}$, it is true that that $x^{m_{k+1}} = x^{m_k}$ and $x^{n_k} = x^{n_{k+1}}$. Combined with the fact that $x^{n_k - 1} \geq x^{n_k}$, we can show that $x^{m_{k+1}} - x^{n_{k+1}} \geq x^{m_k} - x^{n_k - 1}$.

In period 2, given ra, buyer 1 mixes between $x^{n_k - 1}$ and a losing offer ; given ar, buyer 2 makes an offer x^{m_k}.

(2) $z^{k+1} = x^{m_k}$.

$p_1 - x^{m_k} + \delta V_2 = p_2 - y^{m_k} + \delta(x^{n_k} - y^{m_k})$, where $V_1 = \frac{1}{\delta}\big(p_2 - p_1 + x^{m_k} - y^{m_k} + \delta(x^{n_k} - y^{m_k})\big) \in [0, x^{m_k - 1} - x^{m_k}]$ by $p_2 - p_1 \in [B_k, A_{k+1}]$.

In period 2, given ra, buyer 1 makes an offer x^{n_k}; given ar, buyer 2 mixes between $x^{mn_k - 1}$ and a losing offer.

Step 3: Seller's optimal choice if $p_2 - p_1 > B_{2N}$.

By choosing ar, each seller can get at most $p_1 - \underline{x}$. By choosing ra, any seller $x_1 \geq \underline{x}$ gets no less than $p_2 - 1 + \delta(x^N - \underline{x}) > p_1 - \underline{x}$, by $p_2 - p_1 > B_{2N}$. Then, each seller prefers ra.

Step 4: Seller's optimal choice if $p_2 - p_1 < A_0$.

By choosing ra, each seller can get at most $p_2 - \underline{x}$. By choosing ar, any seller $x_1 \leq 1$ gets no less than $p_1 - 1 + \delta(x^N - \underline{x}) > p_2 - \underline{x}$, by $p_2 - p_1 < A_0$. Then, each seller prefers ar.

Step 5: Buyers' optimal offers.

Denote $V_i(p_1, p_2)$ as the expected payoff of buyer $i = 1, 2$, given two offers $p = (p_1, p_2)$.

$$V_i(p) = \begin{cases} P(x_i \in [\underline{x}, x(p)])\big(E(v(x_i)|x_i \in [\underline{x}, x(p)]) - p_i\big) & p_i < p_j, \\ P(x_i \in [x(p), 1])\big(E(v(x_i)|x_j \in [x(p), 1]) - p_i\big) & p_i > p_j. \end{cases}$$

Given buyer j's strategy $F(p_j)$, by making any offer $p_i \in [\underline{x}, \bar{p}]$, buyer i gets expected payoff

$$\int_{\underline{x}}^{\bar{p}} V_i(p_i, p_j) dF(p_j) = \int_{\underline{x}}^{\bar{p}} V_i(\underline{x}, p_j) dF(p_j).$$

Therefore, there exists $F(p_j)$ satisfying the above equation.

□

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