Motivational Ratings*

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July 7, 2016

Abstract

Rating systems not only provide information to users but also motivate the rated agent. This paper solves for the optimal (effort-maximizing) rating system within the standard career concerns framework. It is a mixture two-state rating system. That is, it is the sum of two Markov processes, with one that reflects the belief of the rater and the other the preferences of the rated agent. The rating, however, is not a Markov process. Our analysis shows how the rating combines information of different types and vintages. In particular, an increase in effort may affect some (but not all) future ratings adversely.

Keywords: Career Concerns; Mechanism Design; Ratings.

JEL codes: C72, C73

1 Introduction

Helping users make informed decisions is only one of the goals of ratings. Another is to motivate the rated firm or agent. These two goals are not necessarily aligned.

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*We would like to thank Sümeyra Akin, Ian Ball, and Daria Khromenkova for excellent research assistance.

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Excessive information depresses career concerns and distorts the agent's choices.[1] The purpose of this paper is to examine this trade-off. In particular, we ask the following: how should different sources of information be combined? At what rate, if any, should past observations be discounted? Finally, how do standard rating mechanisms compare?

We demonstrate that the optimal rating system always confounds the different signals yet never adds any irrelevant noise. To maximize incentives for effort, the rater combines the entire history of signals in a one-dimensional statistic, which neither is a simple function of the rater's current belief (about the agent's type) nor enables the market to back out this belief from the rating history. It is not simply a function of her latest rating and signal either.[2] Furthermore, the time series of ratings fails to satisfy the Markov property.[3]

However, the optimal rating system has a remarkably simple structure: it is a linear combination of two processes, namely, the rater's underlying belief and an *incentive state* that reflects both the agent's preferences and the determinants of the signal processes. That is, the optimal rating process admits a simple decomposition as a two-dimensional Markov mixture model.

The agent's preferences determine the *impulse response* of the incentive state via his impatience. That is, past observations are discounted in the overall rating at a rate equal to the agent's discount rate. Instead, the characteristics of the signal processes determine the weights of the signal innovations in the incentive state; that is, these characteristics determine the role and relative importance of the signals in the overall rating. Hence, the optimal rating balances the rater's information, as summarized by the rater's belief, with some short-termism that is in proportion to the agent's impatience.[4] Signals that boost career concerns should see their weight

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[1] In the case of health care, Dranove, Kessler, McClellan, and Satterthwaite (2003) find that, at least in the short run, report cards decreased patient and social welfare. In the case of education, Chetty, Friedman, and Rockoff (2014a,b) argue that the benefits of value-added measures of performance outweigh the counterproductive behavior that it encourages—but gaming is also widely documented (see Jacob and Lefgren (2005) among many others).

[2] This contrasts with several algorithms based on the principle that the new rating is a function of the old ratings and the most recent review(s) (Jøsang, Ismail, and Boyd (2007)). However, there is also significant evidence that, in many cases, observed ratings (based on proprietary rules) cannot be explained by a simple (time-homogeneous) Markov model. See, among others, Frydman and Schuermann (2008), who precisely argue that two-dimensional Markov models provide a better explanation for actual credit risk dynamics. Such two-state systems are already well-studied under the name of mixture (multinomial) models. See, among others, Adomavicius and Tuzhilin (2005).

[3] In credit ratings, this failure has been widely empirically documented; see Section 3.2.

[4] The ineffectiveness of irrelevant conditioning also resonates with standard principal-agent theory; see, for instance, Green and Stokey (1983).

amplified, while those that stifle career concerns should be muted.

These findings are robust to the informational environment. They hold irrespective of whether past ratings can be hidden from the market (confidential vs. public ratings) and of whether the market has access to additional non-proprietary information (exclusive vs. non-exclusive ratings). However, these distinctions matter for the particulars of the rating mechanism. For instance, if past ratings are observable, then hiding information is only effective if the mechanism has access to diverse sources of information (*i.e.*, multidimensional signals). If it relies on a single source of information, then the best public rating is transparent. Non-exclusivity also matters. In the public case, the mechanism might release more information regarding its hidden sources when others are freely available. Instead, in the confidential case, the free information and that revealed by the rating can be substitutes. In fact, the Markov mixture rating system is canonical, in the sense that any effort that can be induced for some rating system can be induced by a Markov mixture rating system, possibly with some added noise. Hence, their emergence does not rely on our emphasis on effort maximization as objective, or the ability of the rater to commit.

Surprisingly, perhaps, we show that the rating system can count past performance against it. That is, performing well at some point can boost the rating in the short term but depress it in the long term. This is because the impact of a rating is proportional to its scale, the market adjusting for its variance. However, when the agent's ability is not too persistent (low mean-reversion), the variance of the rating is naturally high. By counting recent and older signals in opposing directions, the rating counteracts this. Of course, there is also a direct adverse impact on incentives, but this effect is smaller than the indirect positive effect if the agent is impatient.

Our analysis builds on the seminal model of Holmström (1999).[5] An agent exerts effort unbeknown to the market, which pays him a competitive wage. This wage is based on the market's expectation of the agent's productivity, which depends on instantaneous effort and his ability, a mean-reverting process. This expectation is based on the market's information. Rather than directly observing a noisy signal that reflects ability and effort, the market obtains its information via the rating set by some intermediary. The intermediary potentially has many sources of information about the agent and freely chooses how to convert these signals into the rating. In brief, we view a rating system as an information channel that must be optimally designed. We focus on a simple objective that in our environment is equivalent to social surplus: to maximize the agent's incentive to exert effort or, equivalently, to

[5]Modeling differences with Holmström (1999) include the continuous-time setting, mean-reversion in the type process, and a multidimensional signal structure. See Cisternas (2015) for a specification that is similar to ours in the first two respects.

solve for the range of effort levels that are implementable. (We also examine the trade-off between the level of effort and the precision of the market's information.)[6,7]

We allow for a broad range of mechanisms, imposing that the rating mechanism be time-invariant and its quality be history-independent.[8] As we show, a rating mechanism is equivalent to a time-invariant linear filter, mapping all the bits of information available to the intermediary into a (without loss) scalar rating. In general, such mechanisms are infinite-dimensional.

In Section 4, we study two extensions. First, we allow for ratings that are not exclusive. That is, the market has access to independent public information. We show how the optimal rating reflects the content of this free information. Second, we discuss how our results extend to the case of multiple actions.[9] We show that it can be optimal for the optimal rating system to encourage effort production in dimensions that are unproductive, if this is the only way to also encourage productive effort. Third, we apply our techniques to compare existing methods, showing that exponential smoothing dominates a moving window.

Related Literature. Foremost, our paper builds on Holmström (1999). (See also Dewatripont, Jewitt, and Tirole (1999).) His model elegantly illustrates why neither perfect monitoring nor a lack of oversight cultivates incentives. His analysis prompts the question raised and answered in our model: what type of feedback stimulates effort? Our interest in multifaceted information is similar to Holmstrom and Milgrom (1991), who consider multidimensional effort and output to examine

[6]These two objectives feature prominently in economic analyses of ratings according to practitioners and theorists alike. As Gonzalez *et al.* (2004) state, the rationale for ratings stems from their ability to gather and analyze information (information asymmetry) and affect the agents' actions (principal-agent). To quote Portes (2008), "Ratings agencies exist to deal with principal-agent problems and asymmetric information." To be sure, resolving information asymmetries and addressing moral hazard are not the only roles that ratings play. Credit ratings, for instance, play a role in a borrowing firm's default decision (Manso (2013)). Additionally, ratings provide information to the agent himself (*e.g.*, performance appraisal systems); see Hansen (2013). Moreover, whenever evaluating performance requires input from the users, ratings must account for their incentives to experiment and report (Kremer, Mansour, and Perry (2014), Che and Hörner (2015)).

[7]Throughout, we ignore the issues that rating agencies face in terms of possible conflict of interest and their inability to commit, which motivates a broad literature.

[8]As we show, this calls for ratings that are stationary Gaussian processes. Our focus on such mechanisms nonetheless abstracts from some interesting questions, such as the granularity of the rating (the rating's scale) or its periodicity (*e.g.*, yearly vs. quarterly ratings), as well as how ratings should be adjusted to account for the rated firm's age.

[9]With multidimensional product quality, information disclosure on one dimension may encourage firms to reduce their investments in others, harming welfare (Bar-Isaac, Caruana, and Cuñat (2012)).

optimal compensation. Their model has neither incomplete information nor career concerns. Our work is also related to the following strands of literature.

Reputation. The eventual disappearance of reputation in standard discounted models (as in Holmström (1999)) motivates the study of reputation effects when players' memory is limited. There are many ways to model such limitations. One is to simply assume that the market can only observe the last K periods (in discrete time), as in Liu and Skrzypacz (2014). This allows reputation to be rebuilt. Even more similar to our work is Ekmekci (2011), who interprets the map from signals to reports as ratings, as we do. His model features an *informed* agent. Ekmekci shows that, absent reputation effects, information censoring cannot improve attainable payoffs. However, if there is an initial probability that the seller is a commitment type that plays a particular strategy every period, then there exists a finite rating system and an equilibrium of the resulting game such that the expected present discounted payoff of the seller is approximately his Stackelberg payoff after every history. As in our paper, Pei (2015) introduces an intermediary in a model with moral hazard and adverse selection. The motivation is very similar to ours, but the modeling and the assumptions differ markedly. In particular, the agent knows his own type, and the intermediary can only choose between disclosing and withholding the signal, while having no ability to distort its content. Furthermore, in Pei (2015), the intermediary is not a mediator in the game-theoretic sense but a strategic player with her own payoff that she maximizes in the Markov perfect equilibrium of the game.

Design of reputation systems. The literature on information systems has explored the design of rating and recommendation mechanisms. See, among others, Dellarocas (2006) for a study of the impact of the frequency of reputation profile updates on cooperation and efficiency in settings with pure moral hazard and noisy ratings. Inspired by the health care market, Glazer and McGuire (2006) show how obfuscation (providing a rating that is an average of different measures) dominates full transparency, because of adverse selection. This literature abstracts from career concerns, the main driver here.

Design of information channels. There is a vast literature in information theory on how to design information channels, and it is impossible to do it justice. Restrictions on the channel's quality are derived from physical rather than strategic considerations (*e.g.*, limited bandwidth). See, among many others, Chu (1972), Ho and Chu (1972) and, more related to economics, Radner (1961). Design under incentive constraints is recently considered by Ely (2015) and Renault, Solan, and Vieille (2015). However, these are models in which information disclosure is distorted because of the incentives of the users of information; the underlying information process is exogenous.

2 The Model

2.1 Exogenous Information

The relationship involves a long-lived agent (he) and a competitive market (it), mediated by an intermediary (she). We first abstract from the intermediary's objective by treating the information transmitted by the intermediary to the market as exogenous. We then turn to optimizing over information structures in Section 3. Time is continuous, indexed by $t \geq 0$, and the horizon is infinite.

There is incomplete information. The agent's ability, or *type*, is $\theta_t \in \mathbf{R}$. We assume that θ_0 has a Gaussian distribution. It is drawn from $\mathcal{N}(0, \gamma^2/2)$. The law of motion of θ is mean-reverting, with increments

$$\mathrm{d}\theta_t = -\theta_t \, \mathrm{d}t + \gamma \, \mathrm{d}Z_{0,t}, \tag{1}$$

where Z_0 is an independent standard Brownian motion (BM), and $\gamma > 0$.[10] The unit rate of mean-reversion is a mere normalization, as is its zero mean.[11] Mean-reversion ensures that the variance of θ remains bounded, independent of the market information, thereby accommodating a large class of information structures.[12] The noise Z_0 ensures that incomplete information persists and that the stationary distribution is nontrivial. The specification of the initial variance precisely ensures that the process is stationary.

The type affects the distribution over output and signals. Specifically, given some real-valued process A_t (the action of the agent), cumulative output $X_t \in \mathbf{R}$ solves

$$\mathrm{d}X_t = (A_t + \theta_t) \, \mathrm{d}t + \sigma_1 \, \mathrm{d}Z_{1,t}, \tag{2}$$

with $X_0 = 0$. Here, Z_1 is an independent standard Brownian motion, and $\sigma_1 > 0$. We allow for but do not require additional signals of ability.[13] We model such sources of

[10] Throughout, when we refer to an independent standard Brownian motion, we mean a standard Brownian motion independent of all the other random variables and random processes of the model.

[11] Going from a mean-reversion rate of 1 to ρ requires the following changes of variables: $t \mapsto \rho t$, $\gamma \mapsto \gamma/\sqrt{\rho}$, $r \mapsto r/\rho$, $(\alpha_k, \beta_k, \sigma_k) \mapsto (\alpha_k/\rho, \beta_k/\rho, \sigma_k/\sqrt{\rho})$. The optimal confidential rating process, as described in Theorem 3.1, remains well-defined and non-degenerate as $\rho \to 0$. On the other hand, the optimal public rating process becomes transparent in that limit.

[12] An alternative approach that we leave unexplored is to allow for some background learning.

[13] In the case of a company, besides earnings, there is a large variety of indicators of performance (profitability, income gearing, liquidity, market capitalization, etc.). In the case of sovereign credit ratings, Moody's and Standard & Poor's list numerous economic, social, and political factors that underlie their rating (Moody's Investor Services (1991), Moody's Investor Services (1995), Standard & Poor's (1994)); similarly, workers are evaluated according to a variety of performance measures, both objective and subjective (see Baker, Gibbons, and Murphy (1994)).

information as processes $\{S_{k,t}\}$, $k = 2, \ldots, K$, which are solutions to

$$\mathrm{d}S_{k,t} = (\alpha_k A_t + \beta_k \theta_t)\,\mathrm{d}t + \sigma_k\,\mathrm{d}Z_{k,t}, \tag{3}$$

with $S_{k,0} = 0$. Here, $\alpha_k \in \mathbf{R}$, $\beta_k \geq 0$ (wlog), $\sigma_k > 0$ and Z_k is an independent standard Brownian motion. For convenience, we set $S_1 = X$ (and $\alpha_1 = \beta_1 = 1$), as output also plays the role of a signal. Alongside some initial (for now, exogenous) sigma-algebra $\mathcal{G}_0$, the random variables $\mathbf{S} := \{S_k\}_{k=1}^K$ are the only sources of information. We refer to the corresponding filtration as $\mathcal{G}$, where $\mathcal{G}_t$ is (the usual augmentation of) $\mathcal{G}_0 \vee \sigma(\{\mathbf{S}_s\}_{s \leq t})$. This is the information of the intermediary. The agent observes these signals but also knows his own past effort choices. Note that, like the intermediary, the agent learns about his type over time by observing $\mathcal{G}$. On path, his belief coincides with the intermediary's assessment.

The information available to the market at time t is modeled by a sigma-algebra $\mathcal{F}_t \subseteq \mathcal{G}_t$. We do not impose that $\mathcal{F}$ be a filtration, an important point for the sequel. An (agent) strategy is a bounded process A that is progressively measurable with respect to $\mathcal{G}$. Let $\mathcal{A}$ denote the collection of strategies.[14] Neither the market nor the intermediary observe the process A. Instead, the market forms a conjecture about A, denoted $A^* \in \mathcal{A}$, from which a "belief" P^{A^*} is derived. This belief defines the law of motion of the signals and output processes as if the agent exerted effort A_t^* at time t, instead of A_t. Expectations relative to this measure are denoted $\mathbf{E}^*[\cdot]$. This contrasts with the belief P^A of the agent, which captures the actual law of motion.

[14] This is intuitive, but heuristic. Formally, as in continuous-time principal-agent models, to avoid circularity problems where actions depend on the process that they define, (2) and (3) are to be interpreted in the weak formulation of stochastic differential equations (SDE), where Z is a BM that generally depends on A. Specifically, signal processes are defined for a reference effort level (say, 0); one defines S_k as the solution to

$$S_{k,t} = \beta_k \int_0^t \theta_s\,\mathrm{d}s + \sigma_k Z_{k,t},$$

and then $\mathcal{G}$ as the natural augmented filtration generated by the processes S_k alongside $\mathcal{G}_0$, with associated probability measure P^0. Thus, the agent actions do not define the signal process itself, which is fixed *ex ante*. Instead, they define the law of the process: given $A \in \mathcal{A}$, define Z_k^A by $Z_{k,t}^A = Z_{k,t} - \frac{\alpha_k}{\sigma_k^2} \int_0^t A_s\,\mathrm{d}s$. By Girsanov theorem, there exists a probability measure P^A such that the joint law of $(\theta, Z_1^A, \ldots, Z_K^A)$ under P^A is the same as the joint law of $(\theta, Z_1, \ldots, Z_K)$ under P^0. Given $A \in \mathcal{A}$, the signal S_k satisfies

$$\mathrm{d}S_{k,t} = (\alpha_k A_t + \beta_k \theta_t)\,\mathrm{d}t + \sigma_k\,\mathrm{d}Z_{k,t}^A,$$

with $Z_{k,t}^A$ a BM under P^A. These are the signals that the intermediary observes.

Expectations relative to P^A are denoted $\mathbf{E}[\cdot]$.[15]

We now turn to payoffs. Given a (cumulative) real-valued transfer process to the agent (a continuous, $\mathcal{F}$-adapted process) π, the market retains

$$\int_0^\infty e^{-rt}(\mathrm{d}X_t - \mathrm{d}\pi_t),$$

whereas the agent receives

$$\int_0^\infty e^{-rt}(\mathrm{d}\pi_t - c(A_t)\,\mathrm{d}t).$$

Here, $r > 0$ is the common discount rate.[16] The cost of effort $c(\cdot)$ is twice differentiable, with $c'(0) = 0$, and $c'' > 0$. The transfer π does not matter for efficiency (joint surplus maximization), which demands setting A_t at the constant solution of $c'(A_t) = 1$.

The equilibrium definition has three ingredients. The first is how transfers are set. We assume that the market is competitive and that there is no commitment, in the sense that output-contingent wages are not allowed. Given the market conjecture A^*, it pays a flow transfer $\mathrm{d}\pi_t$ equal to $\mathbf{E}^*[\mathrm{d}X_t \mid \mathcal{F}_t]$ "upfront" (note that this transfer can be negative). Second, the agent chooses his strategy A to maximize his expected payoff. Third, the market has rational expectations, and hence, its belief about A coincides with the optimal strategy. Because our focus will be on equilibria with deterministic effort, we assume throughout that A^* is a deterministic function of time.

Definition 2.1 *Fix an information structure $\mathcal{F}$. An equilibrium is a profile (A, A^*, π), $A, A^* \in \mathcal{A}$, such that:*

1. (Zero-profit) For all t,

$$\pi_t = \int_0^t \mathbf{E}^*[A_s^* + \theta_s \mid \mathcal{F}_s]\,\mathrm{d}s.$$

[15] Formally, we use the star notation when we refer to the law on $(\theta, S_1, \ldots, S_K)$ induced by P^{A^*} (see ft. 14), *i.e.*, the law of motion of the ability and signal processes from the perspective of the market. We use the no-star notation when we refer to the law on $(\theta, S_1, \ldots, S_K)$ induced by P^A, *i.e.*, the law of motion of the agent according to his own belief. The belief of the market is given by the law of the joint process $(\theta, S_1, \ldots, S_K)$, but as the mean ability is all that is payoff-relevant, we abuse language and often call belief the mean ability. The same remark holds for the agent's belief. We drop the star notation for variance and covariance, for which the distinction is irrelevant.

[16] Only the agent's impatience is relevant for equilibrium analysis, and this is how we interpret the parameter r. However, equal discounting is necessary for transfers to be irrelevant for efficiency.

2. (Optimal effort)

$$A \in \underset{\widehat{A} \in \mathcal{A}}{\operatorname{argmax}} \mathbf{E} \left[\int_0^\infty e^{-rt} (\mathrm{d}\pi_t - c(\widehat{A}_t)\, \mathrm{d}t) \right].$$

3. (Correct beliefs) It holds that

$$A^* = A.$$

This paper is concerned with the optimal design of the information structure. An important special case is obtained for $\mathcal{F} = \mathcal{G}$ such that the market observes all there is to observe, save for the actual effort. With some abuse, we refer to this case as the model of Holmström, or as *transparency*. However, many more structures are considered. The following are two important properties of information structures.

Definition 2.2 *An information structure $\mathcal{F}_t (\subseteq \mathcal{G}_t)$ is public if $\mathcal{F}$ is a filtration.*

Hence, an information structure is public if all information available to the market in the past remains available at later times. We say that the information structure is *confidential* to insist that we do not require, but do not rule out, that it is public.[17]

Definition 2.3 *An information structure $\mathcal{F}_t (\subseteq \mathcal{G}_t)$ is non-exclusive (w.r.t. signals $K' \subseteq \{1, \dots, K\}$) if*

$$\sigma\left(\{S_{k,t}\}_{k \in K'}\right) \subseteq \mathcal{F}_t.$$

Informally, non-exclusivity means that some signals are observed by the market over time. When such a restriction is not imposed, the information structure is *exclusive* (with further abuse, as non-exclusive information structures are instances of exclusive ones). Non-exclusivity with respect to X is a natural case to consider because this information can be backed out from the payoff process of the market.[18]

Because the payments received by the agent reflect the market belief concerning his type, the agent has incentives to affect this belief via his effort. Hence, given the equilibrium payment, the agent maximizes his discounted reputation, net of his cost of effort, as formalized below in (4). Fixing the conjecture A^*, a sufficient statistic for $\mathcal{F}$ is the conditional expectation $\mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$. This is all the information that matters for equilibrium analysis.[19] Unless stated otherwise, all proofs are in the appendices.

[17] Confidentiality can be defined in a stronger sense: different market participants at time t could receive different information. This makes no difference to the analysis.

[18] The relative importance of exclusive vs. non-exclusive information varies substantially across and within industries: in the credit rating industry, solicited ratings are based on both public and confidential information; unsolicited ratings, by contrast, rely exclusively on public information.

[19] Note that, unlike $\mathcal{F}$, the information structure $\{\mathbf{E}^*[\theta_t \mid \mathcal{F}_t]\}_t$ refers to the market conjecture.

Lemma 2.4

1. *Given a payment process that satisfies the zero-profit condition, the effort strategy A maximizes the agent's payoff if, and only if, it maximizes*

$$\mathbf{E}\left[\int_0^\infty e^{-rt}(\mu_t - c(A_t))\,\mathrm{d}t\right] \tag{4}$$

over $\mathcal{A}$, where $\mu_t := \mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$ is derived using A^* as the market conjecture.

2. *If $(\mathcal{A}, \pi)$ is an equilibrium given $\mathcal{F}$, then it is an equilibrium given $\{\sigma(\mathbf{E}^*[\theta_t \mid \mathcal{F}_t])\}_t$.*

2.2 Ratings

The intermediary selects an information structure $\mathcal{F}$. *A priori*, such a structure can be arbitrarily complex. However, given Lemma 2.4, the equilibrium effort when the market observes $\mathcal{F}_t$ at time t is identical to the equilibrium effort when the market observes the scalar $\mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$ only. Hence, without loss, it can be assumed that the intermediary releases a scalar rating to the market, Y_t, at time t. Figure 1 summarizes how participants interact.

We focus on stationary environments. This requires defining $\mathcal{G}_0$ such that the environment is *as if* time began at $-\infty$. One way to do so is to regard signals S_k and θ as two-sided processes.[20,21] *Rating processes* are a special class of scalar ratings, defined as follows.

Definition 2.5 *A (two-sided) process Y is a rating process if, for all $t \in \mathbf{R}$, Y_t is $\mathcal{G}_t$-measurable, and, when the agent's effort is constant over time,*

(1) for all $\tau > 0$, $(\bar{Y}_t, \mathbf{S}_t - \mathbf{S}_{t-\tau})$ is jointly stationary and Gaussian, where $\bar{Y}_t := Y_t - \mathbf{E}[Y_t]$ is the mean-normalized rating;

[20] A two-sided process is defined on the entire real line, as opposed to the nonnegative half-line. In particular, we call two-sided standard Brownian motion any process $Z(= \{Z_t\}_{t\in\mathbf{R}})$ such that both $\{Z_t\}_{t\geq 0}$ and $\{Z_{-t}\}_{t\geq 0}$ are standard Brownian motions. Let $\{\mathcal{G}_t\}_{t\in\mathbf{R}}$ be the natural augmented filtration generated by $\{S_k\}_k$, which induces the filtration $\mathcal{G}$ on the nonnegative real line.

[21] Formally, for all $t \in \mathbf{R}$,

$$\theta_t = e^{-t}\bar{\theta} + \int_0^t e^{-(t-s)}\gamma\,\mathrm{d}Z_{0,s},$$

where $\bar{\theta} \sim \mathcal{N}(0, \gamma^2/2)$, and Z_0 is two-sided. Similarly, let $X = S_1$ and, given the two-sided BM Z_k, S_k be the two-sided process defined by (see ft. 14),

$$S_{k,t} = \beta_k \int_0^t \theta_s\,\mathrm{d}s + \sigma_k Z_{k,t}.$$

(2) for all k, $\tau \mapsto \mathbf{Cov}[Y_t, S_{k,t-\tau}]$ is absolutely continuous, with integrable and square-integrable generalized derivative.

We restrict attention to information structures induced by rating processes. A rating process Y induces an information structure $\mathcal{F}$ via $\mathcal{F}_t = \sigma(Y_t)$. We say that Y is a *confidential/public rating process* when $\mathcal{F}$ is a confidential/public information structure. Unless stated otherwise, Y is not deterministic and we assume throughout $\mathbf{Var}[Y_t] > 0$ for all t.

Rating processes preclude some interesting practices. Normality rules out coarse ratings, for instance.[22] Still, it encompasses a variety of rating practices. In the case of a one-dimensional signal, for instance, the process can involve *exponential smoothing* (as allegedly used by Business Week in its business school ranking), which involves setting

$$Y_t = \int_{s \le t} e^{-\delta(t-s)} \, \mathrm{d}X_s,$$

for some choice of $\delta > 0$. The rating system can be a *moving window* (as commonly used in consumer credit ratings or Better Business Bureau (BBB) grades) when

$$Y_t = \int_{t-T}^{t} \mathrm{d}X_s,$$

for some $T > 0$. (In both cases, the choice of $\mathcal{G}_0$ ensures that this is also well defined for $s \le 0$.) A comparison between these two ratings is given in Section 4.3.

Furthermore, it is not difficult to think of special cases in which there exists a scheme that would boost effort beyond what is possible with a rating process. Suppose, for instance, that one of the signals perfectly reveals the agent's effort.[23] Then, it suffices for the rating system to raise a "red flag" (ostensibly ceasing to provide any rating in the future) as soon as it detects a deviation from the desired effort level to ensure that any deviation is unattractive in the first place.

Yet, we view such schemes as unrealistic for many applications. In punishing the agent, the rating also "punishes" the market by worsening the information it provides. We believe it is both a realistic and desirable property for a rating system that the history affect the content of the rating, but not the quality of the information that it conveys.

[22] The restriction to adapted processes also rules out the use of extraneous noise by the intermediary. This is merely a modeling choice, as white noise can be included as a dummy signal.

[23] More sophisticated schemes can be devised that apply when there is some small noise in the signal regarding effort while inducing efficient effort at all times.

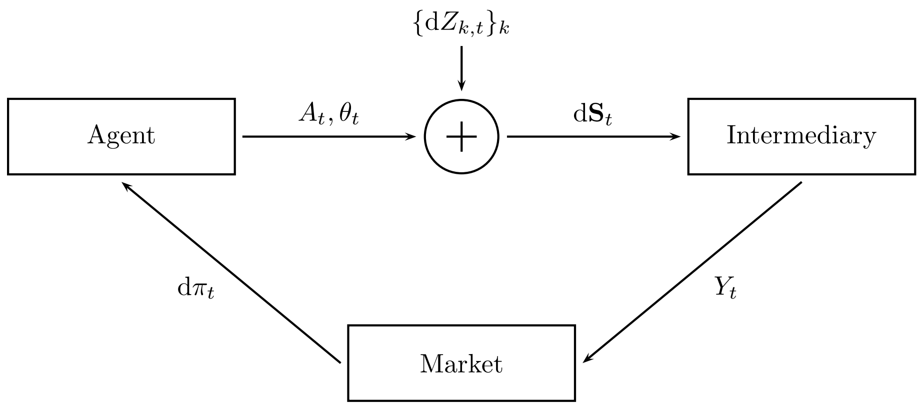

Figure 1: Flow of information and payments between participants.

Perhaps surprisingly, normality is a *consequence* of assuming that history does not matter for the quality of information: as we prove in Appendix D (see Lemma D.2), as long as the covariances (between rating and type, and rating and signal) are deterministic processes (with some smoothness and integrability properties), the non-stationary version of our Representation Lemma (Lemma 2.11) below must hold, and hence the rating *must* have a Gaussian distribution. We further impose stationarity, as the quality of the rating should not depend on calendar time either.

Gaussian processes make the model tractable. It allows to apply linear filtering techniques that we adapt to our framework. Stationarity ensures that equilibrium effort is a scalar, facilitating comparisons. From now on, A^* is taken to be constant on $\mathbf{R}_+$ and the equilibrium to be stationary (set by convention $A_s = A^*_s = 0$ if $s < 0$).

Rating processes can represent beliefs for both confidential and public information structures. They admit the following immediate and intuitive characterization.

Proposition 2.6 *Let Y be a rating process. Then, Y is as follows:*

1. *A belief for a confidential information structure if, and only if, for all t,*

$$\mathbf{E}^*[\theta_t \mid Y_t] = Y_t.$$

2. *A belief for a public information structure if, and only if, for all t,*

$$\mathbf{E}^*[\theta_t \mid \{Y_s\}_{s \le t}] = Y_t.$$

The following provides a simple criterion to decide whether a rating process is equal to a market belief of a confidential or a public information structure.[24]

Lemma 2.7 (Confidential Belief) *A rating process Y is a belief for a confidential information structure if, and only if, for all t,*

$$\mathbf{E}^*[Y_t] = 0 \text{ and } \mathbf{Cov}[Y_t, \theta_t] = \mathbf{Var}[Y_t].$$

Hence, the lemma implies that any rating process with mean zero is proportional to the mean belief that it induces.

Lemma 2.8 (Public Belief) *A rating process Y is a belief for a public information structure if, and only if, it is a belief for a confidential information structure and in addition, for all t and all $\tau \geq 0$,*

$$\mathbf{Corr}[Y_t, Y_{t+\tau}] = \mathbf{Corr}[\theta_t, \theta_{t+\tau}] \ (= e^{-\tau}).$$

Instead of focusing on beliefs, it is often convenient to work with a slighter broader class of rating processes. Scaling a rating process by a nonzero constant does not affect its informational content. Hence, we may select as convenient a rating process within the equivalence class that this constant of proportionality defines.

2.3 Characterization of Equilibrium

Lemma 2.9 *Fix a rating process Y. Under the information structure it induces, $\mathcal{F} = \{\sigma(Y_t)\}_{t\geq 0}$, there exists a unique equilibrium.*

We have assumed that the agent observes the signal realizations over time, as captured by the filtration $\mathcal{G}$. In turn, such information defines the set of agent strategies. But one may also want to consider an environment in which the agent observes different information, modeled by some arbitrary filtration $\mathcal{H}$, allowing for

[24] The characterization is helpful to compute the optimal ratings, as it allows us to restrict attention to ratings that are belief processes. The optimization is then performed under a set of constraints that we relax by internalizing them in the objective function.

As an alternative approach, we could optimize over the general, unconstrained family of rating processes Y, and compute the associated beliefs $\mathbf{E}[\theta_t \mid Y_t]$ or $\mathbf{E}[\theta_t \mid \{Y_s\}_{s\leq t}]$. This can be done for confidential information structures (with or without exclusive information). However, for public information structures, the computation of beliefs is not tractable. In particular, since Y is not required to have a Markovian structure, common linear filters such as the Kalman-Bucy filter cannot be used. Instead, the computation of beliefs involves the determination of a continuum of variables associated with conditional variances of ratings that solve a continuum of equations, the analytic solution to which can only be written in some special cases.

another set of strategies.[25] The concept of equilibrium, presented in Definition 2.1, depends on the agent information only through the set of agent strategies, and so remains valid without modification. The same observation applies to the proof of Lemma 2.9, which does not depend on the specific information available to the agent, because the optimal effort level specified by the equilibrium, turns out to be constant. Therefore, the following corollary immediately obtains.

Corollary 2.10 *Fix an arbitrary filtration $\mathcal{H}$. If the agent instead observes information captured by $\mathcal{H}$ (as opposed to $\mathcal{G}$) there continues to exist a unique equilibrium, and the equilibrium effort level does not depend on the chosen filtration $\mathcal{H}$.*

We now turn to the characterization. This is done in two stages. First, the restriction to rating processes leads to a convenient analytic representation.

Lemma 2.11 (Representation Lemma) *Fix a rating process Y. Given any market conjecture A^*, there exist essentially unique*[26] *integrable and square-integrable functions u_k, $k = 1, \ldots, K$, such that*

$$Y_t = \mathbf{E}^*[Y_t] + \sum_{k=1}^{K} \int_{s \le t} u_k(t-s)(\mathrm{d}S_{k,s} - \alpha_k A_s^* \,\mathrm{d}s). \tag{5}$$

The coefficient $u_k(s)$ is the weight that the current rating Y_t attaches to the innovation (the term $(\mathrm{d}S_{k,s} - \alpha_k A_s^* \,\mathrm{d}s)$) pertaining to the signal of type k and vintage s. Following information-theoretic terminology, we refer to $\{u_k\}_k$ as the *linear filter* defined by Y. When the filter is a sum of exponentials (*e.g.*, $u_k(t) = \sum_\ell c_\ell e^{-\delta_\ell t}$), the coefficients (resp., exponents) are the *weights* (resp., *impulse responses*) of the filter. Conversely, given some filter $\{u_k\}_k$, (5) uniquely defines a rating process.

The decomposition of Lemma 5 can be interpreted as a regression of Y_t on the continuum of signal increments $\mathrm{d}S_{k,s}$, $s \in (-\infty, t]$.[27] It is an infinite-dimensional version of the familiar result that a Gaussian variable that is a dependent function

[25] Formally, an agent strategy is then a bounded process progressively measurable with respect to $\mathcal{H}$.

[26] Unique up to measure zero sets.

[27] Determining the coefficients of such continuous-time regressions is often achieved via a linear filtering argument. Here, the lack of Markovian structure with the infinite fictitious history, together with the stationarity condition, makes the problem non-trivial because it prevents the use of the Kalman-Bucy filter and involves finding a continuum of terms of the form $\mathbf{Var}[Y_t \mid \mathcal{G}_{t-s}]$ that solve a continuum of equations. To obtain the closed-form solution for the coefficients u_k, we write the equations that link f to u_k; then, via algebraic manipulation and successive differentiation, we obtain a differential equation that u_k must satisfy, the solution of which is found explicitly.

of finitely many Gaussian variables is a linear combination thereof.[28] It turns the optimization problem into a deterministic one. There is an explicit formula for u_k in terms of the covariance of the rating, which shows how u_k captures not only the covariance between the rating and a weighted average of the signals of a given vintage but also how this covariance decays over time for signal k. For all $t \geq 0$,

$$u_k(t) = \frac{\beta_k \gamma^2}{\sigma_k^2 \kappa} \left(\frac{\sinh \kappa t + \kappa \cosh \kappa t}{1+\kappa} \int_0^\infty e^{-\kappa s} \, \mathrm{d}\bar{f}(s) - \int_0^t \sinh \kappa (t-s) \, \mathrm{d}\bar{f}(s) \right) - \frac{f_k'(t)}{\sigma_k^2},$$

with $\kappa := \sqrt{1 + \gamma^2 \sum_k \beta_k^2 / \sigma_k^2}$ (> 1), and

$$f_k(s) := \mathbf{Cov}[Y_t, S_{k,t-s}], \text{ and } \bar{f}(s) := \sum_{k=1}^K \frac{\beta_k}{\sigma_k^2} f_k(s).$$

Second, we express equilibrium effort in terms of the filter given by Lemma 5.

Lemma 2.12 *Let Y be a rating process with normalized variance,* $\mathbf{Var}[Y_t] = 1$.[29] *The unique equilibrium effort level A is constant and determined by*

$$c'(A) = \frac{\gamma^2}{2} \left[\sum_{k=1}^K \alpha_k \int_0^\infty u_k(t) e^{-rt} \, \mathrm{d}t \right] \left[\sum_{k=1}^K \beta_k \int_0^\infty u_k(t) e^{-t} \, \mathrm{d}t \right], \tag{6}$$

where u_k is defined by Lemma 2.11, given Y.

Hence, effort is proportional to the product of two covariances. The first pertains to the agent: the impact of effort and his discount rate. The other pertains to the type: the impact of ability and the mean-reversion rate. This formula assumes a normalized variance. Alternatively, we may write (6) in a compact way as

$$c'(A^*) = \left[\sum_{k=1}^K \alpha_k \int_0^\infty u_k(t) e^{-rt} \, \mathrm{d}t \right] \frac{\mathbf{Cov}[Y_t, \theta_t]}{\mathbf{Var}[Y_t]}. \tag{7}$$

The objective of Section 3 is to find the rating process that maximizes the right-hand side of (6), under the constraints imposed by Lemmas 2.7 and 2.8.

[28]Given the continuum, stronger assumptions are necessary. The restriction to stationary processes is key to obtaining a linear representation. See Jeulin and Yor (1979) for a counter-example otherwise.

[29]The somewhat unwieldy statement of this constraint in terms of $\{u_k\}_k$ is given in (15) below.

2.4 Transparency

Here, we consider the benchmark in which $\mathcal{F} = \mathcal{G}$. This case is close to the one considered by Holmström (1999) in discrete time (specifically, his Section 2.2 solves for the stationary equilibrium with one signal, and no mean-reversion). Define:

$$m_\alpha = \sum_{k=1}^{K} \frac{\alpha_k^2}{\sigma_k^2}, \qquad m_{\alpha\beta} = \sum_{k=1}^{K} \frac{\alpha_k \beta_k}{\sigma_k^2}, \qquad m_\beta = \sum_{k=1}^{K} \frac{\beta_k^2}{\sigma_k^2}. \tag{8}$$

The belief of the market μ_t (defined as $\mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$) is then equal to the intermediary's belief $\nu_t \coloneqq \mathbf{E}^*[\theta_t \mid \mathcal{G}_t]$. The latter is a Markov process that solves

$$\mathrm{d}\nu_t = (\kappa - 1) \sum_k \frac{1}{m_\beta} \frac{\beta_k}{\sigma_k^2} (\mathrm{d}S_{k,t} - \alpha_k A_t^* \,\mathrm{d}t) - \kappa \nu_t \,\mathrm{d}t, \tag{9}$$

where $A^* \in \mathbf{R}_+$ is equilibrium effort. Explicitly, the belief is equal to

$$\nu_t = (\kappa - 1) \int_{s \le t} e^{-\kappa(t-s)} \sum_k \frac{1}{m_\beta} \frac{\beta_k}{\sigma_k^2} \left(\mathrm{d}S_{k,s} - \alpha_k A_s^* \,\mathrm{d}s\right). \tag{10}$$

Innovations $(\mathrm{d}S_{k,s} - \alpha_k A^* \,\mathrm{d}s)$ are weighted according to their type and their vintage. A signal of type k is weighted by the signal-to-noise ratio β_k/σ_k^2.[30] If it is noisy (high σ) or insensitive to ability (low β), it matters little for inferences. Given that ability changes, older signals matter less than recent ones: a signal of vintage $t - s$ is discounted (in the belief at time t) at rate κ. The market "rationally forgets."

Theorem 2.13 *The unique equilibrium effort level when $\mathcal{F} = \mathcal{G}$ is the solution to*

$$c'(A) = \frac{1}{\kappa + r} \frac{m_{\alpha\beta}}{m_\beta} (\kappa - 1), \tag{11}$$

if the right-hand side of (11) *is nonnegative. Otherwise,* $A = 0$.

Equation (11) is a standard optimality condition for investment in productive capital. The market's belief is an asset. Effort is an investment in that asset. We interpret the three terms in (11) as persistence ($(\kappa + r)^{-1}$), substitutability ($m_{\alpha\beta}$), and sensitivity $(\kappa - 1)$. The asset depreciates at rate κ, to be added to the discount rate when evaluating the net present value of effort. Investment has productivity $m_{\alpha\beta}/m_\beta$,

[30] Specifically, β_k/σ_k^2 is the inverse of the Fano factor (the signal-to-noise ratio is β_k/σ_k).

which measures the increase in belief given a (permanent) unit increase in effort. In turn, sensitivity measures the increase in belief given a unit increase in the type.

Substitutability, sensitivity and persistence already appear in (9). Sensitivity is the first coefficient, scaling the impact of a surprise in the signal on the belief; substitutability appears in the sum, as the impact of effort on the surprise; and persistence enters via the last term, capturing the rate of decay of the belief. Only discounting is missing. The general formula given by (6) shows that, for an arbitrary rating process, effort depends also on a fourth term, the ratio $\mathbf{Cov}[Y,\theta]/\mathbf{Var}[Y]$, which is equal to one under transparency. Persistence, sensitivity and substitutability matter as well, and are all nested in the first term, $\int_{t\geq 0}(\sum_k \alpha_k u_k(t))e^{-rt}\,\mathrm{d}t$.

Effort can be too high or low, according to how (11) compares to one. If $m_{\alpha\beta}<0$, the agent has perverse career concerns: to impress the market, lower effort is better. As a result, equilibrium effort is 0. If $m_{\alpha\beta}=0$, effort has no impact on the market belief, and equilibrium effort is also 0. Hereafter, we assume that $m_{\alpha\beta}>0$. A signal for which $\alpha_k=0$ is not irrelevant, as it enters both sensitivity and persistence. With no signal beside output, effort is inefficiently low, even as discounting vanishes. This is in contrast to Holmström and is due to mean-reversion, which eventually erases the benefits from an instantaneous effort increase.[31] The proof of the following is immediate and omitted.

Lemma 2.14 *Effort increases in γ and α_k, $k=1,\ldots,K$. It decreases in σ_k if signals are homogenous ($\alpha_k,\beta_k,\sigma_k$ independent of k). It admits an interior maximum with respect to β_k.*

The comparative statics with respect to α_k,γ need no explanation.[32] The role of β_k is more interesting. If it is small, then the market dismisses signal k in terms of learning. If it is high, then the small variation in the signal caused by an effort increase is (wrongly) attributed to the type, but by an amount proportional to β_k^{-1}, which is small and hence not worth the effort cost: a higher β_k makes signal k more relevant but less manipulable. The "best" signals are those involving intermediate values of β_k. Adding a signal has ambiguous consequences for effort, as should be clear. Depending on parameters, it might reduce noise, and so bolster incentives, but it might help tell apart effort and ability, and so undermine career concerns. Either way, it improves the quality of the market's information.

[31] See Cisternas (2012) for the same observation in a model with human capital accumulation.
[32] Effort need not decrease in σ_k if signals are heterogeneous.

2.5 The Role of the Intermediary

The intermediary's objective is to maximize equilibrium effort A. Her instrument is information. Controlling directly the information available to the agent (an instrument considered by, for instance, Hansen (2013)) is useless: by Corollary 2.10, the equilibrium effort level is independent on the information provided to the agent.

On the other hand, the information provided to the market has an indirect impact on effort. Hence, we focus on her choice of the (market) information structure $\mathcal{F}$ via a rating process Y, which, by our earlier results, we can consider scalar and proportional to the market mean belief it induces. Recall from Lemma 2.9 that the stationary equilibrium is unique, and hence that her choice of Y determines A. She might face constraints: the information structure that the rating defines might be public, non-exclusive, or both. She has commitment, in the sense that Y is chosen once and for all, and it is common knowledge.[33]

Maximizing A does not always maximize efficiency. Even under transparency, equilibrium effort can be too high (cf. (11)). However, solving for the maximum effort is equivalent to solving for the range of implementable actions. If effort is excessive, a simple adjustment to the rating process (adding "white noise," for instance) scales it to any desired lower effort, including the efficient level.

Lemma 2.15 *Fix a confidential/public rating process Y such that stationary effort is A.*

For all $A' \in [0, A]$, there exists a confidential/public rating process Y' such that, under the information structure defined by Y', equilibrium effort is A'.

However, under non-exclusivity, there can be a strictly positive lower bound on the effort that the intermediary can induce. (The information structure that Y' defines in Lemma 2.15 might violate non-exclusivity constraints satisfied by Y.) Surprisingly, this lower bound is not typically achieved by silence (the intermediary disclosing no information). Maximum and minimum effort are dual problems. In the presence of non-exclusivity, an optimized information structure can depress effort below what silence achieves, just as it can motivate effort beyond what transparency achieves. See Section 4.1 for further discussion.

Hence, our goal is primarily normative: to identify the range of implementable actions. Yet, there are plausible scenarios in which a profit-maximizing rating agency

[33]This intermediary can be regarded as a "reputational intermediary," an independent professional whose purpose is to transmit a credible quality signal about the agent. Commitment, then, results from the professional's incentive to preserve his reputation. Reputational intermediaries not only include so-called rating agencies but also, in some of their roles, underwriters, accountants, lawyers, attorneys and investment bankers (see Coffee (1997)).

would find it optimal to induce the maximum effort level. For instance, if the agency charges the market a commission (a set percentage of the value of output), then maximizing effort is equivalent to maximizing revenue.

Depending on the context, it might be desirable to evaluate the performance of a rating process along other dimensions, for instance, the quality of the information it conveys (as measured by the variance of the type conditional on the belief), or its stability over time (as measured by the variance of the belief).[34] These properties satisfy a simple relationship.

Lemma 2.16 *Fix a rating process Y. It holds that*

$$\mathbf{Var}[\theta_t \mid \mu_t] + \mathbf{Var}[\mu_t] = \frac{\gamma^2}{2} \ (= \mathbf{Var}[\theta_t]).$$

Hence, precision and stability are perfect substitutes. If stability comes first, lower precision is desirable. This also means that we can restrict attention to one of these measures when evaluating the trade-off with maximum effort. A systematic analysis of this trade-off would take us too far afield, but it can be done, as illustrated in Section 3.5.

3 Main Results

3.1 Persistence vs. Sensitivity: Two Examples

To build intuition, let us begin with a simple example: exponential smoothing as a confidential rating. Suppose that the intermediary wishes to use the rating

$$Y_t = \sum_k \frac{\beta_k}{\sigma_k^2} \int_{s \le t} e^{-\delta\kappa(t-s)} \, \mathrm{d}S_{k,s},$$

where she freely selects $\delta > 0$. The choice $\delta = 1$ reveals her own belief, and transparency results, as in Section 2.4. Any other choice of δ implies that the market is less well-informed than she is. Using the formula from Lemma 2.12, we obtain

$$c'(A) = \frac{1}{\delta\kappa + r} \frac{m_{\alpha\beta}}{m_\beta} \frac{\delta(\kappa+1)(\kappa-1)}{\kappa+\delta}.$$

[34] These properties of ratings are often cited as being desirable (Cantor and Mann (2007)).

In terms of the effects introduced before, the first factor ($\frac{1}{\delta\kappa+r}$) is persistence. Future returns on effort are discounted both because of impatience and because future ratings discount past signals at rate $\delta\kappa$. Rating persistence increases the impact of current effort on future ratings. However, increasing persistence decreases sensitivity. This is clear from the last term, which increases in δ and goes to zero if δ does. If δ is small, then the rating is very persistent, which means that it treats old and recent innovations symmetrically. Because ability changes over time, this blunts the impact of a one-time innovation in the belief. If, instead, δ is large, the rating disproportionately reacts to recent innovations, heightening their relative importance.

What goes up must come down: in a stationary system, a blip in a signal cannot simultaneously jolt the belief and have its impact linger. The intermediary must trade off persistence with sensitivity. But she can do better than transparency. Taking derivatives (with respect to δ) yields as optimal solution

$$\delta = \sqrt{r}.$$

She chooses a rating process that is more or less persistent than Bayesian updating according to $r \lesseqgtr 1$, that is, depending on how the discount rate compares to the rate of mean-reversion. The best choice reflects agent preferences, which Bayes' rule ignores. If the agent is patient, it pays to amplify persistence, and δ is low.

Let us turn to a richer example. Departing from our convention regarding output, assume that output is solely a function of ability, not of effort ($\beta := \beta_1 > 0, \alpha_1 = 0$), while the unique other signal purely concerns effort ($\alpha := \alpha_2 > 0, \beta_2 = 0$), and set $\sigma := \sigma_1 = \sigma_2$.[35] Consider the best rating system within the two-parameter family

$$u_1(t) = \frac{\beta}{\sigma^2} e^{-\kappa t}, \quad u_2(t) = d\frac{\beta}{\sigma^2}\sqrt{\delta} e^{-\delta t},$$

with $d \in \mathbf{R}, \delta > 0$. This family is special yet intuitive: because the agent cannot affect output, the intermediary does not distort the corresponding innovations. However, she adds to the resulting integral an integral over the innovations of the second signal. The parameter d scales the weight on this term and δ is its impulse response. The normalization constant $\sqrt{\delta}$ ensures that the choice of δ does not affect the variance of the market belief.[36] Using Lemma 2.12 here as well,

$$c'(A) = \frac{1}{\delta + r}\sqrt{\delta} m_{\alpha\beta} d \frac{2}{(1+d^2)(1+\kappa)}. \tag{12}$$

[35] In this example, efficiency requires $A = 0$: an efficient rating process should discourage effort, a trivial endeavor. We seek the effort-maximizing scheme.

[36] Plainly, once u_2 is squared and integrated over all $\tau \geq 0$, δ vanishes.

The first term is familiar by now: it is the impact on persistence of the choice of δ. An effort increase at time t is reflected in the rating at time $t+\tau > t$ but discounted twice: at a rate $e^{-r\tau}$ by the agent and $e^{-\delta\tau}$ by the market. Integrating over $\tau \geq 0$ yields a boost to incentives proportional to $1/(r+\delta)$, which is further amplified by the factor $\sqrt{\delta}$ that scales substitutability. The constant d increases substitutability. But increasing it also increases the belief variance, depressing sensitivity and, hence, effort. This is reflected by the denominator $1+d^2$. If d is too small, sensitivity disappears, as δ is useless if the second signal does not enter the rating; if too large, sensitivity vanishes because ratings no longer inform ability. An intermediate value is best. The maximization problem is separable, as is clear from (12); $d/(1+d^2)$ is maximized at $d=1$ and $\sqrt{\delta}/(r+\delta)$ at $\delta = r$. Independent of δ, the optimal weight on the second term is 1; independent of d, the choice of impulse response is r.

Hence, the intermediary does not only wish to influence persistence, by distorting via δ the weights assigned to signals of different vintages. Via d, she also manipulates the weights assigned to signals of different types to influence substitutability.

3.2 Optimal Ratings

This section solves for the optimal exclusive rating processes. We assume throughout that κ, κ^2, r, and 1 are all distinct. Define

$$\lambda = (\kappa-1)\sqrt{r}(1+r)m_{\alpha\beta} + (\kappa - r)\sqrt{\Delta}, \quad \Delta = (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2.$$

Theorem 3.1 *The optimal confidential rating process is unique and given by*[37,38]

$$u_k^c(t) = \frac{\beta_k}{\sigma_k^2}\left(d_k^c \frac{\sqrt{r}}{\lambda} e^{-rt} + e^{-\kappa t}\right),$$

with coefficients

$$d_k^c := (\kappa^2 - r^2)m_\beta \frac{\alpha_k}{\beta_k} - (\kappa^2 - 1)m_{\alpha\beta}.$$

Theorem 3.2 *The optimal public rating process is unique and given by*

$$u_k^p(t) = \frac{\beta_k}{\sigma_k^2}\left(d_k^p \frac{\sqrt{r}}{\lambda} e^{-\sqrt{r}t} + e^{-\kappa t}\right),$$

[37]Recall that we take ratings as proportional to the market mean belief. Throughout, uniqueness is to be understood as up to such a transformation.

[38]For convenience, the formula here also assumes that $\lambda \neq 0$. The proof gives the general formula.

with coefficients

$$d_k^p := \frac{\kappa - \sqrt{r}}{\kappa - r} d_k^c + \lambda \frac{\sqrt{r} - 1}{\kappa - r}. \tag{13}$$

Theorems 3.1 and 3.2 provide solutions that are remarkably similar. Because the linear filter is the sum of two exponentials, the rating can be written as a sum of two Markov processes. That is, in both cases, for some $\phi \in \mathbf{R}$,

$$Y_t = \phi I_t + (1 - \phi)\nu_t,$$

where

$$\mathrm{d}I_t = \frac{\beta_k}{\sigma_k^2} \frac{\sqrt{r}}{\lambda} \sum_k d_k (\mathrm{d}S_{k,t} - \alpha_k A_t^* \,\mathrm{d}t) - \delta I_t \,\mathrm{d}t,$$

with $(d, \delta) = (d^c, r)$ in the confidential case and $(d, \delta) = (d^p, \sqrt{r})$ in the public one. The intermediary combines her own belief ν_t with another Markov process, which we denote I (for "incentive"). Its impulse response reflects the agent's patience, as in the second example of Section 3.1. If he is patient, the rating is persistent. If not, performance is reflected in the rating more rapidly than under Bayes' rule.

This common representation has several consequences:

- The optimal rating is not a Markov process. This echoes a large empirical literature documenting that (bond and credit) ratings do not appear to satisfy the Markov property (Altman and Kao (1992), Altman (1998), Nickell, Perraudin, and Varotto (2000), Bangia *et al.* (2002), Lando and Skødeberg (2002), etc.).
- The optimal rating is not a function of the intermediary's belief alone.[39] At first glance, this might be surprising, as the intermediary's belief is the only payoff-relevant variable (in the confidential case). Hence, one might expect the solution to be Markov with respect to this belief. However, the effort-maximizing way of distributing the impact of an innovation over future ratings is not measurable with respect to the intermediary's beliefs at those future times.
- The optimal rating is a two-state mixture Markov rating—a combination of Markov chains moving at different speeds (Frydman (2005)). Using an EM algorithm, Frydman and Schuermann (2008) find that not only does such a two-state mixture Markov model outperform the Markov model in explaining credit ratings, but it also explains economic features of rating data.

[39]This property is distinct from the first. The rating can be Markov without being a function of the belief (this occurs in the first example of Section 3.1). The rating can be a function of ν without being Markov, as functions of Markov processes typically fail to inherit the Markov property.

What is most surprising is not that two Markov processes are needed to compute the rating, but that two suffice. The part of the proof establishing sufficiency, explained in Section 3.3, sheds light on this. When regarded as a principal-agent model (the principal is the intermediary), promised utility does not suffice as a state variable. Utility is meted out via the market's belief, and beliefs are correct on average. This imposes a constraint on an auxiliary variable and hence demands a second state.

The incentive state is an abstract construct. Another way of understanding what the intermediary does involves re-writing the rating in terms of another pair of states. Unsurprisingly, there is considerable leeway here. For instance, using (Y, ν) (the rating itself and the intermediary's beliefs) leads to a more concrete if less elegant prescription. Explicitly, in the confidential case,

$$\mathrm{d}Y_t = \frac{\beta_k}{\sigma_k^2}\frac{\sqrt{r}}{\lambda}\sum_k (d_k^c + 1)\left(\mathrm{d}S_{k,t} - \alpha_k A_t^* \,\mathrm{d}t\right) - rY_t \,\mathrm{d}t + \frac{(\kappa+1)(r-\kappa)}{\gamma^2}\nu_t \,\mathrm{d}t,$$

and hence, the intermediary continues to incorporate some of her private information (via her belief ν_t) into the rating. In terms of (Y, ν), Y is a hidden Markov process, with ν as the hidden state. This is the formulation occasionally considered for empirical purposes; see Giampieri, Davis, and Crowder (2005). Other representations of the rating process are possible, of course (*e.g.*, as a process with rating momentum; see Stefanescu, Tunaru, and Turnbull (2006)).

Consider now the specific coefficients of the optimal rating processes. The following holds for both the optimal confidential and public ratings. *White noise is harmful*: if $\alpha_k = \beta_k = 0$, then signal k's weight in the rating is zero. Irrelevant noise has no use, as it depresses effort. *All signals enter the rating*: except for a non-generic parameter set, the rating involves them all. Some might be weighted negatively, when innovations along that dimension adversely impact incentives. However, as long as a signal is informative of at least type or effort, the rating takes it into account.

Given the proof of Lemma 2.15, and given Theorems 3.1–3.2, it follows that any effort level $A' \in [0, A]$ (where A is maximum equilibrium effort) can be achieved by considering a rating that is an average of Y_t, the rating as defined by the theorems, and pure "white noise" W_t, an arbitrary independent Wiener process. Hence, two-state Markov mixture ratings are canonical, in the sense that they are optimal, independently of the exact objective of the intermediary, its commitment ability and market power.

Corollary 3.3 *Any stationary public/confidential equilibrium effort level can be achieved by a two-state mixture Markov model plus white noise.*

Among the differences between public and confidential ratings, two are notable. First, the impulse response on the incentive state conspicuously differs across the two environments: this state decays at the discount rate r in the confidential case, whereas in the public case, it does so at a rate equal to the geometric mean between discounting and mean-reversion: $\sqrt{r} = r^{\frac{1}{2}}1^{\frac{1}{2}}$ (the correct interpretation, using the change of variables in ft. 11). The reason is simple. As the second example in Section 3.1 suggests, the impulse response that best trades off persistence with sensitivity is r. Unfortunately, the resulting autocorrelation fails to align with that of a public belief, which decays at the mean-reversion rate (Lemma 2.8). The optimal public rating fixes this in two ways: it distorts the impulse response on the incentive state away from the discount rate toward the mean-reversion rate, and it skews the weight on the incentive term d_k^p away from its favorite weight d_k^c (see (13)).

The second difference is concealed in the definition of this weight d_k^p. If signals are identical (more generally, if, and only if, the ratio α_k/β_k is the same for all signals that are not white noise), then these weights are all zero, and transparency is obtained. While this condition is non-generic for $K > 1$, it is always true when output is the only signal. Instead, with confidential ratings, transparency is a non-generic phenomenon, independent of K. The problem with one signal only is that twisting a weight and an impulse response partially is insufficient to fix the autocorrelation. The weight must be taken all the way down to zero: the "continuum" of autocorrelation constraints determines the "one-dimensional continuum" of variables (the filter $u_1(\cdot)$), and hence the rating, up to some white noise that the intermediary does not wish to use.

To conclude this section, we note that the weights of different signals in the incentive term are ordered according to α_k/β_k.[40]

3.3 Proof Overview

Our problem has some unconventional features, so that applying dynamic programming or Pontryagin's maximum principle directly (as is usually done in principal-agent models) is difficult. Hence, our method of proof is somewhat non-standard. Hopefully, it may be useful in related contexts.

In the first part, we derive necessary conditions using calculus of variations. The necessary condition determines a unique candidate for the optimal rating (up to a factor), if it exists and is sufficiently regular. In the second part, we verify that the guess from the first part is optimal. This step introduces a parameterized family of auxiliary principal-agent models and takes limits in a certain way.

[40] However, whether the ranking increases or decreases in the ratio depends not only on the sign of m_β but also on whether $r \in [\kappa, \kappa^2]$ ($r \leq \kappa^2$) in the confidential (public) case.

Part I: Necessary Conditions

Recall that the ratings communicated to the market may be confidential or public, and the information generated by the signals exclusive or non-exclusive. Thus, there are four settings of interest. In all settings, we normalize the mean rating to zero, and the variance to one.

The Representation Lemma (Lemma 2.11) characterizes all rating processes in terms of a linear filter $\mathbf{u}$, which we use as a control variable. Lemma 2.12 expresses the equilibrium marginal cost of the agent as a function of the filter. Maximizing the equilibrium action is equivalent to maximizing the marginal cost. Thus, we seek to identify a control $\mathbf{u}$ that maximizes a product of two integrals over $\mathbf{u}$:

$$\frac{\gamma^2}{2}\left[\sum_{k=1}^{K}\alpha_k\int_0^\infty u_k(t)e^{-rt}\,\mathrm{d}t\right]\left[\sum_{k=1}^{K}\beta_k\int_0^\infty u_k(t)e^{-t}\,\mathrm{d}t\right]. \tag{14}$$

In this first part of the proof, we focus on controls that exhibit a sufficient degree of regularity, and we assume that a solution exists within that family.

The maximization is subject to the constraints that the rating process must satisfy. In the simplest case of confidential exclusive information structures, the only constraint is the variance normalization, which is written as follows:

$$\sum_{k=1}^{K}\sigma_k^2\int_0^\infty u_k(s)^2\,\mathrm{d}s+\frac{\gamma^2}{2}\int_0^\infty\int_0^\infty U(s)U(t)e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}t=1, \tag{15}$$

where $U := \sum_k \beta_k u_k$. The higher dimensionality of the problem is plain in (15). Maximizing (14) subject to (15) is a variational problem with an isoperimetric constraint. We form the Lagrangian and consider a relaxed, unconstrained problem that "internalizes" the variance normalization as part of the objective function. However, the problem is not standard: both objective (14) and constraint (15) include multiple integrals, yet the control has a one-dimensional input. Adapting standard arguments, we prove a version of the Euler-Lagrange necessary condition that covers our class of programs (see Appendix D). This condition takes the form of an integral equation in $\mathbf{u}$, which can be solved in closed form via successive differentiation and algebraic manipulation. The solution of the relaxed problem can be shown to be a solution of the original problem, which yields a candidate for the optimal rating (unique subject to regularity conditions).

In the more general public and/or non-exclusive settings (see Section 4.1), the objective (14) remains the same, but there are additional constraints on the rating process. These capture the restriction that market beliefs are linked to public or non-exclusive information structures. Lemmas 2.7 and 2.8 state these constraints in the exclusive case, and Lemma 4.2 does so for the non-exclusive cases. Then, we can apply the Representation Lemma (Lemma 2.11) to express these constraints in terms of the filter $\mathbf{u}$ directly.

There are two additional difficulties in these settings. First, there is no longer a finite number of constraints, but a continuum of them. Second, these constraints involve further integral equations with delay.[41] For example, in the public exclusive setting, the constraint (15) is replaced by

$$\sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(t) u_k(t+\tau)\, \mathrm{d}t \\ + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(s)U(t) e^{-|s+\tau-t|}\, \mathrm{d}s\, \mathrm{d}t = 1, \qquad \forall \tau \geq 0. \quad (16)$$

To address this, we reduce the continuum of constraints to a finite set of constraints, applying "educated" linear combinations. We solve the relaxed optimization problem with a finite number of constraints in a manner similar to that for the simplest setting just described. For instance, in the public exclusive setting, we replace (16) by

$$\begin{aligned} 1 &= \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(t) u_k(t)\, \mathrm{d}t + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(s)U(t) e^{-|s-t|}\, \mathrm{d}s\, \mathrm{d}t, \\ \int_0^\infty h(\tau)\, \mathrm{d}\tau &= \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty \int_0^\infty h(\tau) u_k(t) u_k(t+\tau)\, \mathrm{d}t\, \mathrm{d}\tau \\ &\quad + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty \int_0^\infty h(\tau) U(s) U(t) e^{-|s+\tau-t|}\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}\tau, \end{aligned}$$

where $h(\tau) := e^{-r\tau}$. Naturally, h can be interpreted as a continuum of Lagrange multipliers, but as opposed to the discrete Lagrange multipliers, deriving h via the Euler-Lagrange equations is not feasible. Instead, inspired by numerical simulations, we guess the functional form of h. Because two solutions satisfy the Euler-Lagrange conditions, corresponding to a minimum and maximum equilibrium action, we must

[41] There is a small literature on the calculus of variations with delayed arguments for single integrals. See Kamenskii (2007) and references therein. There is also a literature on multiple integrals without delayed arguments; see Morrey (1966) for a classical treatise. In both cases, the domain of the control is of the same dimension as the domain of integration.

select the maximizer using some form of second-order condition, which is, loosely, in our setting the analogue of the classical Legendre necessary condition.

Part II: Verification

The calculus of variations determines an essentially unique candidate for the filter **u** and thus a unique candidate rating. However, few sufficient conditions are known in the calculus of variations. Most are based on the Hilbert Invariant Integral. However, in the case of (even one-dimensional) integral equations with delayed argument, the method does not apply (Sabbagh (1969)).[42] Instead, we interpret the intermediary's optimization differently, as a principal-agent model. In this auxiliary model, the agent produces signals and outputs exactly as in the original model and obtains the same payoffs. However, there is no longer a market, nor an intermediary. Instead, the agent receives transfers from a principal, who observes all outputs and signals, as does the agent. The principal's information at time t is thus $\mathcal{G}_t$, as defined in the original model. To simplify the exposition, let us focus on the confidential exclusive case. There are already two difficulties to overcome here: the action must be constant (a constraint that is difficult to formalize in the principal-agent context) and the transfer must be equal to the "market" belief.

The principal chooses a transfer process μ, which is interpreted as the instantaneous payment flow from the principal to the agent. As in the original model, the agent chooses an action process A (the agent's strategy) that maximizes, at all t,

$$\mathbf{E}\left[\int_{s\geq t} e^{-r(s-t)}(\mu_s - c(A_s))\,\mathrm{d}s \,\middle|\, \mathcal{G}_t\right]. \tag{17}$$

In the principal-agent formulation, the transfer process μ is not constrained to be a belief nor to have a Gaussian form.

The principal has a discount rate $\rho < r$ and seeks to maximize the *ex ante* payoff

$$\mathbf{E}\left[\int_0^\infty \rho e^{-\rho t}(c'(A_t) + \phi\mu_t(\nu_t - \mu_t))\,\mathrm{d}t\right], \tag{18}$$

where ϕ is some scalar multiplier and ν_t, defined as the mean ability of the agent under transparency, $\mathbf{E}[\theta_t \mid \mathcal{G}_t]$. The maximization is performed over all strategies

[42] The Lagrangian can be interpreted as a bilinear quadratic form with a continuum of variables. Proving that the candidate control **u** is optimal is then equivalent to proving that the quadratic form has no saddle point. This involves a diagonalization of the quadratic form in an infinite-dimensional space, which in our case is not tractable.

A and transfer processes μ such that the action A is incentive compatible, *i.e.*, it maximizes (17).

To interpret the principal's objective, it is useful to consider the reward appearing in (18). The term $c'(A_t)$ is the agent's marginal cost, which the intermediary maximizes in the original model. If the payoff were reduced to this term, the principal might not choose a μ associated with a market belief. However, for the principal-agent and original models to be comparable, μ must be close to a market belief. The second term $\phi\mu_t(\nu_t - \mu_t)$ imposes a penalty on the principal to incite the principal to choose a μ close to a market belief. Indeed, observe that if μ_t and ν_t are jointly normal, then $\mathbf{E}[\mu_t(\nu_t - \mu_t)] = 0$ if, and only if, $\mathbf{Cov}[\mu_t, \nu_t] = \mathbf{Var}[\mu_t]$: this is the condition required for a Gaussian process μ to be a market belief, by Lemma 2.7.

If μ is a market belief process associated with a confidential information structure, then $\mathbf{Cov}[\mu_t, \nu_t] = \mathbf{Var}[\mu_t]$ and the principal's payoff is equal to

$$\mathbf{E}\left[\int_0^\infty \rho e^{-\rho t} c'(A_t)\right] = c'(A),$$

where $c'(A)$ refers to the stationary marginal cost. Thus, *the maximum payoff of the principal is never less than the marginal cost in the original model* for every ρ.

We find that there is no multiplier ϕ such that the principal maximizes his payoff by choosing a μ that is exactly a market belief. However, using the calculus of variations from Part I, we can "guess" a multiplier ϕ such that the payoff-maximizing μ approaches a market belief as $\rho \to 0$.

Note that in the original model, the intermediary must induce a constant equilibrium effort by the agent. In the principal-agent formulation, instead, the principal maximizes over all equilibrium action processes. Perhaps surprisingly, it is easier to solve this "fully dynamic" problem. Indeed, we are able to solve the principal-agent problem in closed form for every $\rho \in (0, r)$. Then, sending the principal's discount rate to zero leads to a solution that is constant in the limit, the optimal transfer tends to a market belief, and the principal's payoff becomes equal to the intermediary's objective in the original model (the agent's marginal cost). Formally, by sending ρ to 0, the maximum principal payoff converges to the conjectured maximum marginal cost from Part I. Because the principal's payoff cannot be lower than the intermediary's objective, this proves that the rating obtained in Part I is optimal.

In the public and non-exclusive cases, the methodology is similar, with a payoff specification that includes penalty terms reflecting the relevant constraints. In those cases, the principal's payoff includes additional state variables to induce the principal to choose a μ associated with public or non-exclusive market beliefs.

Note that, if we were able to properly internalize the constraint that the principal must choose transfer processes among what would correspond to market beliefs, the principal-agent formulation could, in principle, be used to obtain the necessary conditions of Part I. The difficulty is precisely that we cannot internalize these constraints, both with finite and infinite horizons, with a positive discount rate. This is why we consider a family of principal-agent problems and take limits as $\rho \to 0$. The calculus of variations then makes it possible to obtain the candidate optimal rating and the correct multipliers to be used in the principal-agent formulation.

3.4 The Incentive State as a Benchmark

To gain further insight into the role and structure of the incentive state, let us consider a special case. Suppose that signals are identical, namely, $\alpha_k = \beta_k = 1$, $\sigma_k = \sigma$ for all k. As discussed, transparency is obtained under public ratings. Let us instead consider confidential ratings. Theorem 3.1 immediately yields that, for all k,

$$u_k^c(t) = u^c(t) := \frac{1}{\sigma^2}\left[\frac{1-\sqrt{r}}{\kappa-\sqrt{r}}\sqrt{r}e^{-rt} + e^{-\kappa t}\right].$$

Hence, whether the incentive state is added or subtracted from the belief state depends on how $\sqrt{r}$ compares to 1 and κ. If $\sqrt{r}$ lies in $[1,\kappa]$, the sign of its coefficient is negative, meaning that it is subtracted. If it is outside this interval, it is added.

Plainly, which of the two impulse responses r and κ is largest depends on whether $\sqrt{r} \lesseqgtr \sqrt{\kappa} \in (1,\kappa)$, leading us to distinguish four intervals: $\sqrt{r} \in [0,1]$, $[1,\sqrt{\kappa}]$, $[\sqrt{\kappa},\kappa]$, and $[\kappa,\infty)$. The relative size of r vs. κ translates into how the negative sign affects the shape of $u^c(\cdot)$, as illustrated by Figure 2. If $\sqrt{r} \in [1,\sqrt{\kappa}]$, then $u(0) > 0$, but it is single-troughed and negative above some threshold t. Instead, if $\sqrt{r} \in [\sqrt{\kappa},\kappa]$, then $u(0) < 0$ and u is single-peaked and positive above some threshold t. To see why a negative weight on the incentive term can be optimal, consider the case of a patient agent ($r < \kappa$) with output as the only signal and a rating process from the family

$$u(t) = \frac{\beta}{\sigma^2}\left(de^{-\delta t} + e^{-\kappa t}\right),$$

for some $d \in \mathbf{R}, \delta > 0$. Applying Lemma 2.12 (see (7)) yields as effort

$$c'(A) = \frac{\alpha\int_0^\infty u(t)e^{-rt}\,\mathrm{d}t}{\sqrt{\mathbf{Var}[Y_t]}}\mathbf{Corr}[Y_t,\theta_t] \propto \underbrace{\frac{\frac{d}{\delta+r}+\frac{1}{\kappa+r}}{\sqrt{\frac{(1+d)^2+\delta(1+d\kappa/\delta)^2}{1+\delta}}}}_{\text{Term A}}\underbrace{\mathbf{Corr}[Y_t,\theta_t]}_{\text{Term B}}. \qquad (19)$$

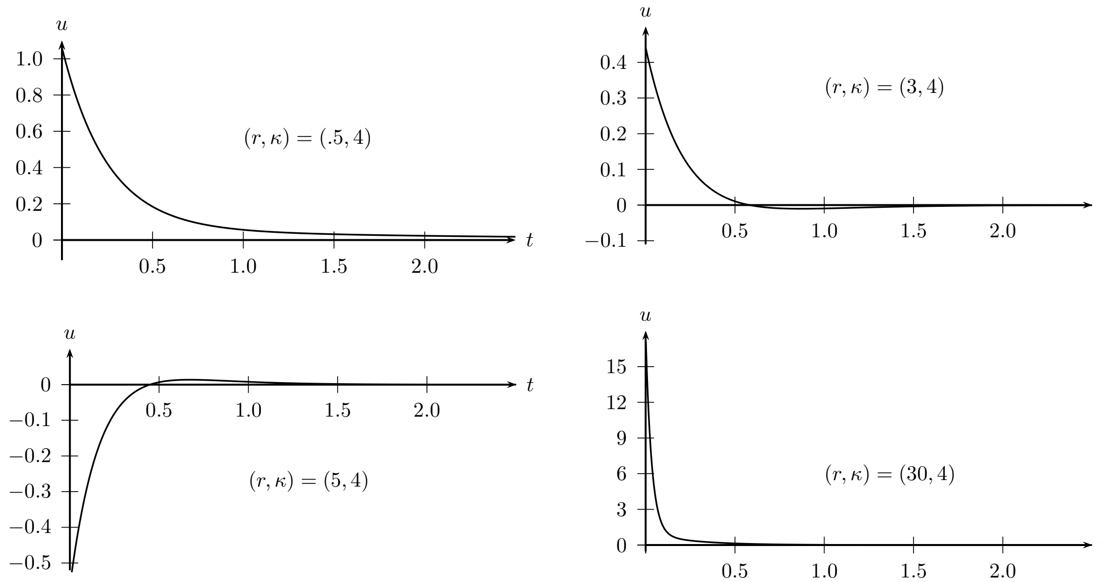

Figure 2: Rating in the case of homogenous signals (here, $\alpha = \beta = \sigma = 1$).

Correlation (Term B) is maximized by transparency, setting $d = 0$: the market is never as well informed as when the intermediary reveals her own belief. Hence, to understand whether $d \gtrless 0$, we focus on the first term of (19), Term A. Its numerator is a gross (non-adjusted) measure of incentives. It is decreasing in r and linearly increasing in d: the higher the rating scale is, the greater the impact of additional effort on the rating and hence, if the market does not account for the scale, the stronger the agent's incentives. However, the market adjusts for scaling via the denominator of Term A (the standard deviation of the rating). This standard deviation is decreasing in the rate of mean-reversion (see ft. 11) and nonlinear in d. The derivative of Term A evaluated at $d = 0$ is of the same sign as

$$\frac{\kappa + r}{\delta + r} - \frac{\kappa + 1}{\delta + 1}, \tag{20}$$

the sign of which when $\delta < \kappa$ (as when $\delta = r$, its optimal value) is determined by $r \gtrless 1$. Impatience dilutes the positive impact of a higher d (the first term of (20)) on the numerator of Term A, just as mean-reversion dilutes the negative impact of a higher d (the second term) via the denominator. If impatience outweighs mean-reversion

($r > 1$), it is better to opt for a lower standard deviation and select a negative d.[43]

For $(r, \kappa) = (3, 4)$, for instance, the negative weight for $t = 1$ implies that a positive surprise at time τ negatively impacts the rating at $\tau + 1$ (see the top-right panel of Figure 2). However the rating has a positive impact until then (or, rather, until $\sim \tau + .6$). The market accounts for the fact that the rating "understates" performance; the way it is done improves its quality.

This might be a subtle point, but it is robust. While it is easiest to see in the case of identical signals, it holds for a broad range of parameters (roughly, when r is close to κ) for confidential ratings. It also occurs under public ratings, for the same reasons (indeed, it can occur under public ratings in case where it does not with confidential ratings, and vice versa). Moreover, it resonates with some practices. Murphy (2001) documents the widespread practice of past-year benchmarking as an instrument to evaluate managerial performance, commenting on its seemingly perverse incentive to underperform with an eye on the long term. Ratcheting does not explain it, as the compensation systems under study involve commitment by the firm.

3.5 Public vs. Confidential Ratings: A Closer Look

In this subsection, we further develop the comparison between public and confidential ratings by examining performance (effort) and informativeness (variance of the market belief). Throughout, the superscripts p and c refer to the information structure. The explicit value of the objective is given first.

Lemma 3.4 *The marginal cost induced is*

$$c'(A^c) = \frac{\kappa - 1}{4(\kappa + r)m_\beta}\left(2m_{\alpha\beta} + \sqrt{\Delta/r}\right),$$

given the optimal confidential rating process, and, given the optimal public process,

$$c'(A^p) = \left(1 - \left(\frac{\sqrt{r} - 1}{\sqrt{r} + 1}\right)^2\right) c'(A^c).$$

The first factor in the formula for $c'(A^p)$ quantifies the extent to which the public rating fails to match the performance of the confidential rating. Because the discount rate is the only parameter that enters this wedge, the two effort levels vary in the same way with respect to all other parameters. A higher impact of effort on signals

[43]The case in which $r > \kappa$ can be interpreted similarly, but $c'(A)$ is not single-peaked in $d \leq 0$ in that case and the derivative at 0 is not informative.

(m_α) or noise in the type process (γ) increases effort. It is readily verified that effort is decreasing in r in the public case and that this need not be in the confidential case. In both cases, effort vanishes when $r \to \infty$ and is maximized when $r \to 0$. In the confidential case, effort then grows without bound, whereas it approaches a finite limit with a public rating. The informativeness of the rating is measured by the variance of the belief: the higher this variance is, the better informed the market.

Lemma 3.5 *The variance of the market belief is*

$$\mathbf{Var}\,\mu^c = \frac{(\kappa-1)^2}{4m_\beta}(1 + 2m_{\alpha\beta}\sqrt{r/\Delta}),$$

given the optimal confidential rating process, and

$$\mathbf{Var}\,\mu^p = \left(1 + \left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right)^2\right)\mathbf{Var}\,\mu^c,$$

given the optimal public rating process.

Hence, the market is better informed given public ratings, confirming a plausible but not foregone conclusion. Here also, the wedge is a function of the discount rate alone, implying that the degrees of informativeness vary alike in all other respects.[44]

Remarkably, despite the differences in the specification between public and confidential rating systems, variance and performance differ by a constant that only depends on the discount rate. However, with respect to the discount rate, the variation of accuracy could not be more different. As the left (right) panel of Figure 3 illustrates, variance is maximized (minimized) at an intermediate level of patience in the confidential (public) case. When ratings are confidential, an emphasis on the incentive state becomes dominant with extreme discounting. Thus, the rating becomes less accurate. Instead, given public ratings, transparency is obtained asymptotically, whether $r \to 0, \infty$. Publicness is a constraint that leaves the intermediary with little flexibility when only the long term matters ($r \approx 0$). When only the very short term matters, the incentive state decays too rapidly. As a result, under public ratings, the market backs out the belief state (the weight that the rating would have to assign to the incentive state to prevent this would transform the rating into *de facto* white noise).

This raises a natural question: is requiring ratings to be public equivalent to setting standards of accuracy? To answer this, we plot the solution (maximum

[44]Which is not to say that these comparative statics are foregone conclusions. For instance, *adding* a signal can lead to a less-informed market.

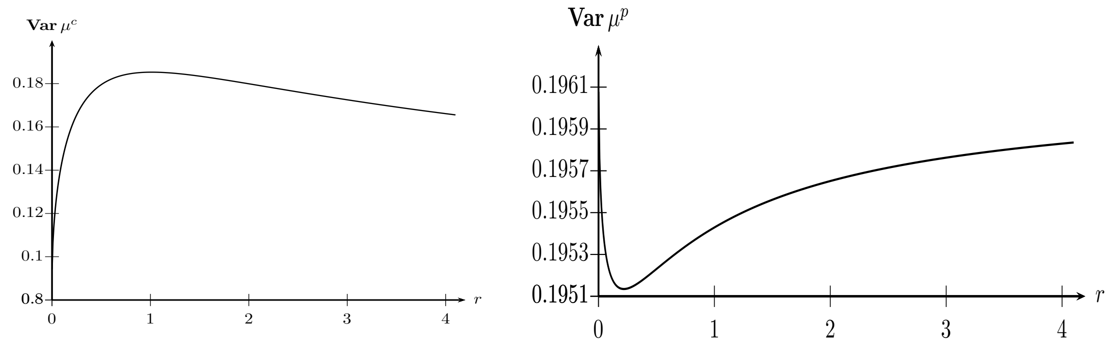

Figure 3: Confidential and public variance, as a function of r (here, $(\alpha_2, \beta_2, \sigma_2, \gamma, \sigma_1) = (3, 2, 1, 1, 2)$, $K = 2$).

marginal cost of effort) to the two problems—confidential and public ratings—*subject to* an additional constraint on the variance of the market belief. See Figure 4.[45]

Quality and effort are substitutes: transparency does not maximize effort. These substitutes are imperfect, as the effort-maximizing rating does not leave the market in the dark. Hence, there is a range of precision levels over which it conflicts with effort provision. Fixing precision, there is a maximum effort level that can be induced by the rating. (Curves are truncated at this maximum.) This maximum effort corresponds to a rating process qualitatively similar to the unconstrained one; only the weights on the exponentials vary. As is clear from the figure, effort is higher in the confidential case for any given level of variance. A confidential rating system is simultaneously able to incentivize more effort and provide better information than a public system.

4 Extensions

For brevity, we focus here on two generalizations. First, we allow some signals to be non-exclusive. That is, the intermediary cannot prevent the market from observing them publicly. Second, we consider the case in which the agent's actions are multidimensional, possibly differentially affecting signals and output. All proofs for this section are in Appendix D.

[45]This plot is based on necessary conditions, but we expect that Theorems 3.1–3.2 extend.

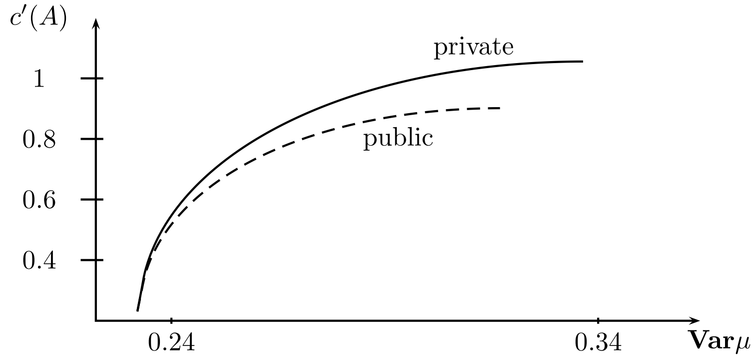

Figure 4: Marginal cost of effort as a function of maximum belief variance, public vs. confidential ratings (here, $(\beta_1, \beta_2, \alpha_1, \alpha_2, \gamma, r, \sigma_1, \sigma_2) = (3, 2, 1/3, 5, 1, 1/5, 1, 2)$).

4.1 Exclusivity

Not all information can be hidden. If the market represents long-run consumers that repeatedly interact with the agent, cumulative output is likely publicly observable. In credit ratings, solicited ratings are based on a mix of information that is widely available to market participants, as well as information that is exclusively accessible to the intermediary (see ft. 18). We refer to this distinction as *exclusive* vs. *non-exclusive* information. The intermediary does not ignore the fact that the market has direct access to this source of information. What she reveals about the exclusive signals that she can conceal also reflects the characteristics of those signals that she cannot.

Formally, all participants observe $\{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}$ in addition to the information provided by the intermediary (we consider the cases of both public and confidential structures, according to whether past information is publicly available).[46] Signals $S_{k,t}$, $k > K_0$, are only observed by the intermediary and the agent. If $K_0 = 0$, ratings are exclusive, as in Section 3. If $K_0 = K$, it is transparency, as in Section 2.4. The statements for $K_0 = 0, K$ require adjustments in the theorems given below; as they are already covered by earlier results, we rule them out.

The following proposition generalizes Proposition 2.6.

Proposition 4.1 *Let Y be a rating process. Then, Y is:*

1. A belief for a confidential information structure with non-exclusive signals

[46]By our ordering convention, output is observed whenever any signal is observed.

$S_1, \ldots, S_{K_0}$ if, and only if, for all t,

$$\mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \le t, k=1,\ldots,K_0}, Y_t] = Y_t.$$

2. *A belief for a public information structure with non-exclusive signals $S_1, \ldots, S_{K_0}$ if, and only if, for all t,*

$$\mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \le t, k=1,\ldots,K_0}, \{Y_s\}_{s \le t}] = Y_t.$$

The next lemma extends Lemmas 2.7 and 2.8 to account for the non-exclusive signals.

Lemma 4.2 (Confidential/Public Belief with Non-Exclusive Signals) *A rating process Y is a belief for a confidential/public information structure with non-exclusive signals $S_1, \ldots, S_{K_0}$ if, and only if, it is a belief for a confidential/public information structure and, for all $k = 1, \ldots, K_0$, all t, and all $\tau \ge 0$,*

$$\mathbf{Cov}[S_{k,t}, Y_{t+\tau}] = \mathbf{Cov}[S_{k,t}, \theta_{t+\tau}]. \tag{21}$$

Hence, the constraints that a rating process must satisfy to be a belief for a confidential or public information structure are the constraints given in Lemmas 2.7 and 2.8 respectively, and the covariance equalities (21) which capture the constraint of non-exclusivity for every signal that is publicly observed.

As in the exclusive setting of Section 3, we focus on ratings that are equal (or proportional) to beliefs, and we express the rating by its linear filter as in the Representation Lemma (Lemma 2.11). But a new choice arises: does the belief represent the interim belief based solely on the information communicated by the intermediary, to be combined with the non-exclusive signals into a posterior belief (in which case, $u_k = 0$ for $k \le K_0$), or this posterior belief itself? In other words, should the rating already incorporate the information conveyed by the non-exclusive signals? This is a matter of convention. We attempt to preserve as much as possible the analogy with the solution in the exclusive case. This demands an interim approach for confidential information structures and a posterior approach for public information structures. Note that the beliefs of Lemma 4.2 refer to the posterior beliefs.

The main results of this section require the following notation. First, we introduce the rate at which a belief based solely on public signals decays, namely,

$$\hat{\kappa} := \sqrt{1 + \gamma^2 \sum_{k=1}^{K_0} \frac{\beta_k^2}{\sigma_k^2}}.$$

Second, we generalize the sums (8) to the current framework, *e.g.*,

$$m_\alpha^n := \sum_{k=1}^{K_0} \frac{\alpha_k^2}{\sigma_k^2}, \quad m_{\alpha\beta}^n := \sum_{k=1}^{K_0} \frac{\alpha_k \beta_k}{\sigma_k^2}, \quad m_\beta^n := \sum_{k=1}^{K_0} \frac{\beta_k^2}{\sigma_k^2},$$

and

$$m_\alpha^e := \sum_{k=K_0+1}^{K} \frac{\alpha_k^2}{\sigma_k^2}, \quad m_{\alpha\beta}^e := \sum_{k=K_0+1}^{K} \frac{\alpha_k \beta_k}{\sigma_k^2} \quad m_\beta^e := \sum_{k=K_0+1}^{K} \frac{\beta_k^2}{\sigma_k^2}.$$

We assume throughout that either $m_{\alpha\beta}^n \geq 0$ or $m_{\alpha\beta} \geq 0$, ensuring that positive effort can be achieved in equilibrium (by either disclosing no or all exclusive information).[47] More generally, we add superscripts n, e (for non-exclusive and exclusive) whenever convenient, with the meaning being clear from the context.

We find that Theorem 3.1 holds *verbatim*, provided we redefine Δ. Let

$$\lambda = (\kappa - 1)\left(\sqrt{r}(1+r)m_{\alpha\beta} + (\kappa^2 - r^2)\sqrt{\Delta}\right),$$

where

$$\Delta := \frac{(\kappa+1)(\hat{\kappa}+1)}{2(\kappa-\hat{\kappa})}\left[\frac{m_\alpha^e m_\beta^e}{\kappa^2 - \hat{\kappa}^2} + \frac{(1+2r+\hat{\kappa})(m_{\alpha\beta}^n)^2}{(r+\hat{\kappa})^2(\hat{\kappa}+1)} - \frac{(1+2r+\kappa)m_{\alpha\beta}^2}{(r+\kappa)^2(\kappa+1)}\right].$$

With these slightly generalized formulas, we restate Theorem 3.1.

Theorem 4.3 *The optimal confidential rating process is unique and given by, for $f \leq K_0$, $u_k = 0$ and $k > K_0$,*

$$u_k(t) = \frac{\beta_k}{\sigma_k^2}\left(d_k \frac{\sqrt{r}}{\lambda} e^{-rt} + e^{-\kappa t}\right),$$

with coefficients

$$d_k := (\kappa^2 - r^2) m_\beta \frac{\alpha_k}{\beta_k} - (\kappa^2 - 1) m_{\alpha\beta}.$$

Theorem 4.4 *The optimal non-exclusive public rating process is unique and given by, for signals $k \leq K_0$,*

$$u_k^n(t) = \frac{\beta_k}{\sigma_k^2}\left(d^n e^{-\delta t} + e^{-\kappa t}\right),$$

[47] These assumptions are not necessary. The rating process defined in the theorem yields a candidate value for $c'(A)$. If it is positive, the rating system is optimal. If not, then effort is zero.

and for signals $k > K_0$,

$$u_k^e(t) = \frac{\beta_k}{\sigma_k^2}\left(\left(c^e\frac{\beta_k}{\sigma_k^2} + d^e\frac{\alpha_k}{\beta_k}\right)e^{-\delta t} + e^{-\kappa t}\right),$$

for some constants d^n, c^e, d^e and $\delta > 0$ given in Appendix C.

The parameters d^n, c^e, d^e are elementary functions of δ, where δ is a root of a polynomial of degree 6. This polynomial is irreducible. In fact, Galois theory can be used to show that it cannot be expressed in terms of radicals. It always admits exactly two positive roots, and we indicate how to select the correct one (see Lemma C.1 in Appendix C).

The differences in parameter values should not distract from the overarching commonalities. Most important, as in the exclusive case, the optimal process is expressed in terms of a two-state Markov process, with one state being the intermediary's belief. As before, it can be restated as a system in which the intermediary revises the rating by gradually incorporating her belief. As under exclusivity, with public ratings, the optimal rating reduces to transparency if the exclusive signals are redundant (*i.e.*, if α_k/β_k is independent of k, $k > K_0$), as is the case if there is only one such signal.

The intermediary does not need to observe the realized values of the non-exclusive signals to incentivize the agent.[48] Yet non-exclusivity affects the quality of the information available to the market. As an example, consider Figure 5, which describes variances under confidential ratings in a variety of cases. The market is better informed (*i.e.*, the variance of the market belief is highest) when information is non-exclusive (the higher solid line) than when it is not (the dotted line). However, this is only the case because the market can rely on the non-exclusive signal (the output) in addition to the rating. If (counterfactually) a market participant were to rely on the rating alone to derive inferences on ability (lower solid line), he would be worse off under non-exclusivity. This does not necessarily imply that the information conveyed by the rating is degraded because of the existence of another signal that the intermediary cannot hide. As is clear from Figure 5, variance could be even lower if the non-exclusive signal did not exist at all and we were considering the confidential rating process for the case of one signal only (dashed line). For nearly all discount rates, however, the presence of non-exclusive information depresses the intermediary's willingness to disclose information regarding her unshared signal—free information and the information conveyed by the rating are then strategic substitutes.[49]

[48] This is not obvious from the statement of Theorem 4.4 because we chose to state the optimal

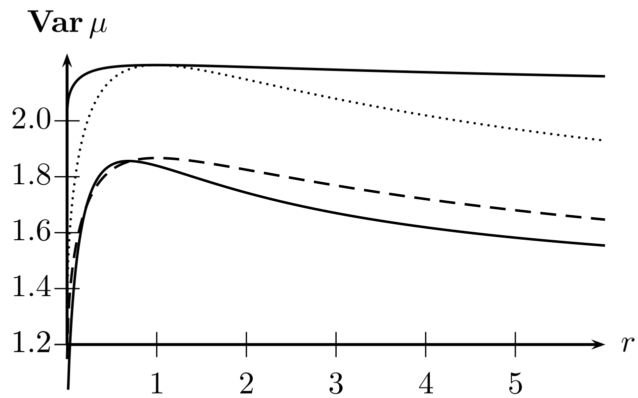

Figure 5: Belief variances (here, $K = 2$ and $(\alpha_k, \beta_k, \sigma_k, \gamma) = (1, 1, 1, 4)$, $k = 1, 2$).

4.2 Multiple Actions

Ratings are often criticized for biasing, rather than bolstering, incentives. When the agent engages in multiple tasks, a poorly designed system might distract attention from those actions that boost output and toward those that boost ratings.

Such moral hazard takes many forms. In credit rating, for instance, both shirking and risk-shifting by the issuer are costly moral hazard activities that rating systems might encourage (see Langohr and Langohr (2009), Ch. 3). Report cards in sectors such as health care and education are widely criticized for encouraging providers to "game" the system, leading doctors to inefficient selection behavior and teachers to concentrate their effort on developing those skills measured by standardized tests.[50]

Our model can accommodate such concerns. We illustrate how in the context of confidential ratings. Suppose that there is not one but L effort levels A_ℓ, $\ell = 1, \ldots, L$, with a cost of effort that is additively separable.[51] With some abuse of notation,

$$c(A_1, \ldots, A_L) := \sum_{\ell=1}^{L} c(A_\ell).$$

non-exclusive public rating process as a posterior belief.

[49]This is consistent with a large empirical literature in finance showing that (i) ratings do not summarize all the information that is publicly available and that (ii) the value-added of these ratings decreases in the quality of information otherwise available.

[50]See Porter (2015) for a variety of other examples.

[51]For a discussion of the restriction implied by separability, see Holmstrom and Milgrom (1991).

For concreteness, assume that $c(A_\ell) = cA_\ell^2$, $c > 0$, although the method applies more generally. Signals are now defined by their law

$$\mathrm{d}S_{k,t} = (\textstyle\sum_\ell \alpha_{k,\ell} A_{\ell,t} + \beta_k \theta_t)\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_{k,t},$$

for all $k = 1, \ldots, K$, with $\sum_\ell \alpha_{1,\ell} \neq 0$. The model is otherwise unchanged. The intermediary's objective is the maximization of the expected discounted output, as in the baseline model.

This model is solved as in Section 3.2 via a change of variables. Define a fictitious model with one-dimensional effort A, cost $c(A) = cA^2$ and signals $\widetilde{S}_k$ with law

$$\mathrm{d}\widetilde{S}_{k,t} = (\alpha_k A_t + \beta_k \theta_t)\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_{k,t},$$

for all $k = 1, \ldots, K$, where

$$\alpha_k := \frac{\sum_\ell \alpha_{1,\ell}\alpha_{k,\ell}}{\sum_\ell \alpha_{1,\ell}}.$$

Lemma 4.5 *The linear filter of the optimal confidential rating process is the same in both the original model and the fictitious model.*

In terms of the optimal linear filter $\{u_k\}_k$ for the fictitious model, each effort level in the original model is then given by

$$c'(A_\ell) = \frac{\mathbf{Cov}[Y_t, \theta_t]}{\mathbf{Var}[Y_t]} \int_0^\infty e^{-rt} \left(\sum_k \alpha_{k,\ell} u_{k,t} \right) \mathrm{d}t,$$

for $\ell = 1, \ldots, L$. The following example shows that the optimal rating remains opaque and does not seek to deter effort in unproductive tasks. Output is only a function of effort A_1; however, the signal S_2 reflects both effort A_2 and the agent's type; namely,

$$\mathrm{d}S_{1,t} = A_{1,t}\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_{1,t}, \text{ and}$$
$$\mathrm{d}S_{2,t} = (A_{2,t} + \theta_t)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_{2,t}.$$

Absent any rating, if either only the first signal or both signals are observed, the unique equilibrium involves $A_\ell = 0$, $\ell = 1, 2$. Action A_1 does not affect learning about the type, and the type does not enter output. The optimal rating is given by

$$u_1(t) = \frac{\sqrt{r}}{\sigma_1} e^{-rt}, \text{ and } u_2(t) = \frac{e^{-\kappa t}}{\sigma_2^2}.$$

The signal that is irrelevant for learning is not discarded. Rather, it is exclusively assigned to the incentive term; conversely, the signal that matters for learning matters only for the learning term. This leads to positive effort on both dimensions, namely,

$$c'(A_1) = \frac{\kappa - 1}{4\sqrt{r}\sigma_1}, \quad c'(A_2) = \frac{\kappa - 1}{2(r + \kappa)\sigma_2^2},$$

and market belief variance $\frac{1}{4}(\kappa - 1)^2\sigma_2^2$. Unproductive effort in the unobservable dimension that affects learning is the price to pay for effort in the productive activity.

4.3 Performance of Standard Policies

Many real-world systems do not use two-state mixtures. Here, we illustrate how our methods also allow us to compare some standard policies that are used in practice. As mentioned in Section 2.2, exponential smoothing and moving windows are two common systems. We argue that a properly calibrated exponential smoothing rating process outperforms any moving window rating process. For simplicity, we focus on confidential exclusive ratings with only one additional signal (simply denoted S_t).

Formally, in the case of exponential smoothing, the intermediary releases signal

$$Y_t = \int_{s \le t} e^{-\delta(t-s)} \left[c \, \mathrm{d}X_s + (1 - c) \, \mathrm{d}S_s\right]$$

at time t, where $\delta > 0$ is the coefficient of smoothing and c is the relative weight placed on the output. With a moving window, the intermediary releases a signal

$$Y_t = \int_{t-T}^{t} \left[c \, \mathrm{d}X_s + (1 - c) \, \mathrm{d}S_s\right],$$

where $T > 0$ is the size of the moving window. The *optimal* exponential smoothing (resp., moving window) system is defined by the choice of (c, δ) (resp., (c, T)) such that equilibrium effort is maximized. It is simple to show the following.

Lemma 4.6 *The optimal confidential exponential smoothing rating process yields higher effort than any moving window rating process.*

The proof establishes a stronger statement: for any weight c on the output, the best rating process using exponential smoothing with that weight outperforms the best moving window rating process with the same weight.

5 Concluding Comments

Our stylized model lays bare why one should not expect ratings to be Markovian and why, for instance, the same performance can have an impact on the rating that is either positive or negative according to its vintage. Richer versions might deliver more nuanced rating systems but will not overturn these insights.

Nonetheless, it is desirable to extend the analysis in several directions. First, in terms of technology, we have assumed that effort and ability are substitutes. While this follows Holmström (1999) and most of the literature on career concerns, it is limiting, as Dewatripont, Jewitt, and Tirole (1999) make clear. Building on Cisternas (2015), for instance, it might be possible to extend the analysis to cases in which effort and ability are complements. The absence of risk-aversion allows us to use effort as a yardstick for efficiency. Allowing for CARA preferences, for instance, would be useful to discuss the welfare implications of the informativeness of ratings.

Second, in terms of market structure, we have assumed a competitive market without commitment and a single agent. When the firm that designs the rating system is the same that pays the worker, one might wish to align its ability to commit along these two dimensions. Harris and Holmstrom (1982) offer an obvious framework. Relative performance evaluation requires introducing more agents but is also a natural extension, given the prevalence of the practice in performance appraisal.

Third, in terms of the rating process, stationarity, in particular, is an assumption that one might wish to relax. It is needed (among other uses) for the Representation Lemma (Lemma 2.11), one of the premises of our analysis. Nonetheless, one can bypass this difficulty by simply asserting that the rating process admits a (possibly non-stationary) linear filter. This might be hard to interpret in terms of primitives (the random process of the rating), but it certainly includes the class considered here.

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Appendix A: Proofs of Section 2

A.1 Proof of Lemma 2.4

1. If the cumulative payment process satisfies the zero-profit condition, then the agent who chooses effort strategy A makes (*ex ante*) payoff

$$\mathbf{E}\left[\int_0^\infty \left(A_t^* + \mu_t - c(A_t)\right) e^{-rt}\,\mathrm{d}t\right],$$

where A^* denotes the market conjectured effort level. The agent has no impact on A^*. Thus, the agent's strategy is optimal if, and only if, it maximizes

$$\mathbf{E}\left[\int_0^\infty \left(\mu_t - c(A_t)\right) e^{-rt}\,\mathrm{d}t\right].$$

2. If $\mathcal{F}' = \{\mathcal{F}_t'\}_{t\geq 0}$, with $\mathcal{F}_t' = \sigma(\mu_t)$, then $\mathbf{E}^*[\theta_t|\mathcal{F}_t'] = \mathbf{E}^*[\theta_t|\mu_t] = \mu_t$, hence for a given conjectured effort level A^*, the market's transfers and the agent's optimal action are the same under both information structures $\mathcal{F}$ and $\mathcal{F}'$.

A.2 Proof of Proposition 2.6

1. If Y is a belief for a confidential information structure $\mathcal{F}$, then $Y_t = \mu_t$, where, by definition, $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{F}_t] = \mathbf{E}^*[\theta_t \mid \mu_t]$, and where the second equality follows from the law of iterated expectations. Conversely, if $Y_t = \mathbf{E}^*[\theta_t \mid Y_t]$, then Y is the belief μ for the confidential information structure induced by Y.

2. If Y is a belief for a public information structure structure $\mathcal{F}$, then $Y_t = \mu_t$, where, by definition, $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{F}_t] = \mathbf{E}^*[\theta_t \mid \{\mu_s\}_{s\leq t}]$, where the second equality follows from the law of iterated expectations, using that $\mathcal{F}$ is a filtration and thus $\mathcal{F}_t$ includes all information about $\{\mu_s\}_{s\leq t}$. Conversely, if $Y_t = \mathbf{E}^*[\theta_t \mid \{Y_s\}_{s\leq t}]$, then Y is the belief μ for the public information structure that is the filtration generated by Y.

A.3 Proof of Lemma 2.7

The lemma is immediate by applying Proposition 2.6, observing that, by the projection formula for jointly normal random variables,

$$\mathbf{E}^*[\theta_t \mid Y_t] = \frac{\mathbf{Cov}[\theta_t, Y_t]}{\mathbf{Var}[Y_t]}(Y_t - \mathbf{E}^*[Y_t]).$$

A.4 Proof of Lemma 2.8

The correlation between θ_t and $\theta_{t+\tau}$ satisfies

$$\mathbf{Corr}[\theta_t, \theta_{t+\tau}] = \frac{\mathbf{Cov}[\theta_t, \theta_{t+\tau}]}{\sqrt{\mathbf{Var}[\theta_t]}\sqrt{\mathbf{Var}[\theta_{t+\tau}]}} = e^{-\tau},$$

where we note that, as θ is a stationary Ornstein-Uhlenbeck process with mean-reverting rate 1 and scale γ,

$$\mathbf{Cov}[\theta_t, \theta_{t+\tau}] = \frac{\gamma^2}{2}e^{-\tau}, \text{ and } \mathbf{Var}[\theta_t] = \mathbf{Var}[\theta_{t+\tau}] = \frac{\gamma^2}{2}.$$

Let μ be the market belief process induced by some public information structure $\mathcal{F}$. We have $\mathbf{E}^*[\mu_t] = \mathbf{E}^*[\theta_t] = 0$. As $\mathcal{F}$ is also a confidential information structure, μ is also a belief for a confidential information structure.

Conditionally on μ_t, the random variable θ_t is then independent from every $\mu_{t-\tau}$, $\tau \geq 0$, because μ_t carries all relevant information about θ_t. Thus, $\mathbf{Cov}[\theta_t, \mu_{t-\tau} \mid \mu_t] = 0$. Let $\tau \geq 0$. The projection formulas for jointly normal random variables yield

$$\mathbf{Cov}[\theta_t, \mu_{t-\tau} \mid \mu_t] = \mathbf{Cov}[\theta_t, \mu_{t-\tau}] - \frac{\mathbf{Cov}[\theta_t, \mu_t]\,\mathbf{Cov}[\mu_{t-\tau}, \mu_t]}{\mathbf{Var}[\mu_t]}.$$

Hence,

$$\mathbf{Cov}[\mu_{t-\tau}, \mu_t] = \mathbf{Var}[\mu_t]\frac{\mathbf{Cov}[\theta_t, \mu_{t-\tau}]}{\mathbf{Cov}[\theta_t, \mu_t]} = \mathbf{Var}[\mu_{t-\tau}]\frac{\mathbf{Cov}[\theta_t, \mu_{t-\tau}]}{\mathbf{Cov}[\theta_{t-\tau}, \mu_{t-\tau}]}, \tag{22}$$

where we used the stationarity of the pair (μ, θ). Besides, by Lemma 2.11, there exist $u_1^\mu, \ldots, u_K^\mu$, such that μ_t can be written as

$$\mu_t = \sum_{k=1}^{K} \int_{s \leq t} u_k^\mu(t-s)[\mathrm{d}S_{k,s} - \alpha_k A_s^*\,\mathrm{d}s].$$

Hence, as $\mathbf{Cov}[\theta_t, \theta_{t-\tau}] = \gamma^2 e^{-\tau}/2$,

$$\mathbf{Cov}[\mu_{t-\tau}, \theta_{t-\tau}] = \frac{\gamma^2}{2} \sum_{k=1}^{K} \beta_k \int_0^\infty u_k^\mu(s) e^{-s} \, \mathrm{d}s,$$

and

$$\mathbf{Cov}[\mu_{t-\tau}, \theta_t] = \frac{\gamma^2}{2} \sum_{k=1}^{K} \beta_k \int_0^\infty u_k^\mu(s) e^{-(\tau+s)} \, \mathrm{d}s = e^{-\tau} \, \mathbf{Cov}[\mu_{t-\tau}, \theta_{t-\tau}].$$

Hence, plugging these last two expressions into (22), we have

$$\mathbf{Cov}[\mu_t, \mu_{t+\tau}] = \mathbf{Cov}[\mu_{t-\tau}, \mu_t] = \mathbf{Var}[\mu_{t-\tau}] e^{-\tau} = \mathbf{Var}[\mu_t] e^{-\tau}.$$

Now, we prove the converse. Let Y be a rating process that is a belief for a confidential information structure, and satisfies

$$\mathbf{Cov}[Y_{t+\tau}, Y_t] = \mathbf{Var}[Y_t] e^{-\tau},$$

for every $\tau \geq 0$. By Lemma 2.11, as $\mathbf{E}^*[Y_t] = 0$, there exist $u_1^Y, \ldots, u_K^Y$, such that Y_t can be written as

$$Y_t = \sum_{k=1}^{K} \int_{s \leq t} u_k^Y(t-s) [\mathrm{d}S_{k,s} - \alpha_k A_s^* \, \mathrm{d}s],$$

so that, as above, we get

$$\mathbf{Cov}[Y_{t-\tau}, \theta_t] = e^{-\tau} \, \mathbf{Cov}[Y_{t-\tau}, \theta_{t-\tau}] = e^{-\tau} \, \mathbf{Cov}[Y_t, \theta_t],$$

using the stationarity of (Y, θ), and we have by assumption on Y that

$$e^{-\tau} = \frac{\mathbf{Cov}[Y_t, Y_{t-\tau}]}{\mathbf{Var}[Y_{t-\tau}]} = \frac{\mathbf{Cov}[Y_t, Y_{t-\tau}]}{\mathbf{Var}[Y_t]}.$$

Therefore,

$$\mathbf{Cov}[\theta_t, Y_{t-\tau} \mid Y_t] = \mathbf{Cov}[\theta_t, Y_{t-\tau}] - \frac{\mathbf{Cov}[\theta_t, Y_t] \, \mathbf{Cov}[Y_{t-\tau}, Y_t]}{\mathbf{Var}[Y_t]} = 0.$$

As θ and Y are jointly normal, it implies that θ_t and $Y_{t-\tau}$ are independent conditionally on Y_t for every $\tau \geq 0$, so the market belief associated to the public information structure that is the filtration generated by Y satisfies

$$\mathbf{E}^* \left[\theta_t \mid \{Y_s\}_{s \leq t}\right] = \mathbf{E}^* \left[\theta_t \mid Y_t\right] = Y_t.$$

The conclusion follows from Proposition 2.6.

A.5 Proof of Lemma 2.9 and Lemma 2.12

We prove the existence and uniquess of the equilibrium, and give the closed-form expression of the equilibrium action.

We have, following the projection formulas for jointly normal random variables, using the Representation Lemma (Lemma 2.11),

$$\begin{aligned}
\mu_t &= \mathbf{E}^*[\theta_t \mid Y_t] \\
&= \mathbf{Cov}[Y_t, \theta_t](Y_t - \mathbf{E}^*[Y_t]) \\
&= \mathbf{Cov}[Y_t, \theta_t] \sum_{k=1}^{K} \int_{s \leq t} u_k(t-s) \left[\mathrm{d}S_{k,s} - \alpha_k A_s^* \,\mathrm{d}s\right],
\end{aligned}$$

where A^* is the effort level conjectured by the market. Observe that by stationarity, $\mathbf{Cov}[Y_t, \theta_t]$ is constant.

We prove that, given the (unique) cumulative payment process that satisfies the zero-profit condition, there exists an optimal effort strategy for the agent, and that it is unique (up to measure zero sets) and pinned down by the first-order condition given in Lemma 2.12. This, in turn, yields existence of a unique equilibrium.

Let us fix the cumulative payment process that satisfies the zero-profit condition, and suppose that the agent follows effort strategy A. The agent's time-0 (*ex post*) payoff is then

$$\int_0^\infty \left[A_t^* + \mu_t - c(A_t)\right] e^{-rt} \,\mathrm{d}t. \tag{23}$$

Maximizing the agent's *ex ante* payoff is equivalent to maximizing the agent's *ex post* payoff, up to probability zero events. Hence, we seek conditions on A that characterize when it is a maximizer of (23).

Therefore, as

$$\mathrm{d}S_{k,s} = (\alpha_k A_s + \beta_k \theta_s) \,\mathrm{d}s + \sigma_k \,\mathrm{d}Z_{k,s},$$

maximizing (23) is equivalent to maximizing

$$\mathbf{Cov}[Y_t, \theta_t] \int_0^\infty \int_0^t \sum_{k=1}^K \alpha_k u_k(t-s) A_s e^{-rt} \,\mathrm{d}s \,\mathrm{d}t - \int_0^\infty c(A_t) e^{-rt} \,\mathrm{d}t. \tag{24}$$

Let us re-write

$$\begin{aligned}
&\mathbf{Cov}[Y_t, \theta_t] \int_0^\infty \int_0^t \sum_{k=1}^K \alpha_k u_k(t-s) A_s e^{-rt} \,\mathrm{d}s \,\mathrm{d}t \\
&\qquad = \mathbf{Cov}[Y_t, \theta_t] \int_0^\infty \int_s^\infty \sum_{k=1}^K u_k(t-s) \alpha_k A_s e^{-rt} \,\mathrm{d}t \,\mathrm{d}s \\
&\qquad = \mathbf{Cov}[Y_t, \theta_t] \int_0^\infty A_s e^{-rs} \int_s^{+\infty} \sum_{k=1}^K u_k(t-s) \alpha_k e^{-r(t-s)} \,\mathrm{d}t \,\mathrm{d}s \\
&\qquad = \mathbf{Cov}[Y_t, \theta_t] \int_0^\infty A_s e^{-rs} \int_0^\infty \sum_{k=1}^K \alpha_k u_k(\tau) e^{-r\tau} \,\mathrm{d}\tau \,\mathrm{d}s.
\end{aligned}$$

Maximizing (24) is then the same as maximizing

$$\mathbf{Cov}[Y_t, \theta_t] \int_0^\infty A_s e^{-rs} \int_0^\infty \sum_{k=1}^K \alpha_k u_k(\tau) e^{-r\tau} \,\mathrm{d}\tau \,\mathrm{d}s - \int_0^\infty c(A_t) e^{-rt} \,\mathrm{d}t,$$

which is the same as maximizing

$$\mathbf{Cov}[Y_t, \theta_t] A_s \int_0^\infty \sum_{k=1}^K \alpha_k u_k(\tau) e^{-r\tau} \,\mathrm{d}\tau - c(A_s),$$

for (almost) every s. By strict convexity of the agent's cost, (24), and thus (23), is maximized if, and only if,

$$c'(A_t) = \mathbf{Cov}[Y_t, \theta_t] \int_0^\infty \sum_{k=1}^K \alpha_k u_k(\tau) e^{-r\tau} \,\mathrm{d}\tau,$$

for (almost) every t.

We note that $\mathbf{Cov}[Y_t, \theta_t]$ is constant and equal to

$$\mathbf{Cov}[Y_t, \theta_t] = \frac{\gamma^2}{2} \sum_{k=1}^{K} \beta_k \int_0^\infty u_k(s) e^{-s} \, \mathrm{d}s.$$

Hence, (23) is maximized if, and only if,

$$c'(A_t) = \frac{\gamma^2}{2} \left[\sum_{k=1}^{K} \beta_k \int_0^\infty u_k(t) e^{-t} \, \mathrm{d}t \right] \left[\sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(t) e^{-rt} \, \mathrm{d}t \right],$$

for every t, up to measure zero sets. Thus, the optimal effort strategy exists for the agent. It is unique (up to measure zero events and times), as well as constant and pinned down by the last equation.

A.6 Proof of Lemma 2.11

The proof proceeds in three parts. In the first part, we make additional regularity assumptions to derive necessary conditions so as to pin down a unique candidate for the coefficients u_k. In the second part, we prove that the candidate obtained is integrable and square-integrable. In the third part, we relax the regularity assumptions and show that the conjectured coefficients obtained in the first part are valid for the rating process being considered. The first part is useful to get an educated guess of the candidate weight functions. Given the educated guess, the second and third parts are self-contained and sufficient to prove the lemma.

Guess of the coefficients. Assume that Y has the linear representation

$$Y_t = \mathbf{E}^*[Y_t] + \sum_{k=1}^{K} \int_{s \le t} u_k(t-s)(\mathrm{d}S_{k,s} - \alpha_k A_s^* \, \mathrm{d}s),$$

with u measurable, integrable and square-integrable. Further, let $B > 0$, and assume that every u_k is twice continuously differentiable on $[0, B]$, and that $u_k(s) = 0$ if $s > B$. Later, we will relax the bounded-support assumption. We define $U \coloneqq \sum_k \beta_k u_k$.

We have

$$\begin{aligned}
f_k(\tau) &= \mathbf{Cov}[Y_t, S_{k,t-\tau}] \\
&= \sum_{i=1}^{K} \int_0^\infty u_i(s)\,\mathbf{Cov}[\mathrm{d}S_{i,t-s}, S_{k,t-\tau}] \\
&= \sigma_k^2 \int_\tau^\infty u_k(s)\,\mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_0^\infty \int_\tau^\infty U(s) e^{-|j-s|}\,\mathrm{d}j\,\mathrm{d}s.
\end{aligned}$$

Successive differentiations yield

$$f_k'(\tau) = -\sigma_k^2 u_k(\tau) - \frac{\beta_k \gamma^2}{2} \int_0^\infty U(s) e^{-|\tau-s|}\,\mathrm{d}s, \tag{25}$$

$$f_k''(\tau) = -\sigma_k^2 u_k'(\tau) + \frac{\beta_k \gamma^2}{2} \int_0^\tau U(s) e^{-(\tau-s)}\,\mathrm{d}s - \frac{\beta_k \gamma^2}{2} \int_\tau^\infty U(s) e^{+(\tau-s)}\,\mathrm{d}s,$$

$$f_k'''(\tau) = -\sigma_k^2 u_k''(\tau) + \beta_k \gamma^2 U(\tau) - \frac{\beta_k \gamma^2}{2} \int_0^\tau U(s) e^{-(\tau-s)}\,\mathrm{d}s - \frac{\beta_k \gamma^2}{2} \int_\tau^\infty U(s) e^{+(\tau-s)}\,\mathrm{d}s.$$

Thus,

$$f_k' - f_k''' = \sigma_k^2 u'' - \sigma_k^2 u - \beta_k \gamma^2 U. \tag{26}$$

Multiplying (26) by β_k/σ_k^2 and summing over k yields an ordinary differential equation (ODE) for U:

$$\bar{f}' - \bar{f}''' = U'' - U - \gamma^2 m_\beta U = U'' - \kappa^2 U, \tag{27}$$

where we recall that

$$\bar{f}(s) := \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} f_k(s).$$

Integrating by parts the general solution of (27) gives

$$U(\tau) = C_1 e^{\kappa\tau} + C_2 e^{-\kappa\tau} - \bar{f}'(\tau) - \frac{\kappa^2 - 1}{\kappa} \int_0^\tau \sinh(\kappa(\tau - s)) \bar{f}'(s)\,\mathrm{d}s, \tag{28}$$

for some constants C_1 and C_2. Multiplying the expression for f_k' by β_k/σ_k^2 and summing over k gives

$$\bar{f}'(\tau) = -U(\tau) - \frac{\kappa^2 - 1}{2} \int_0^\infty U(s) e^{-|\tau-s|}\,\mathrm{d}s, \tag{29}$$

for every $\tau \geq 0$. Together, and after simplification, (28) and (29) yield an equation that C_1 and C_2 should satisfy, for every τ:

$$\begin{aligned}
\bar{f}'(\tau) = \bar{f}'(\tau) - C_1 e^{\kappa\tau} - C_2 e^{-\kappa\tau} \\
&- \frac{\kappa^2-1}{2} C_1 \left[\frac{e^{B(\kappa-1)+\tau}}{\kappa-1} - \frac{e^{-\tau}}{\kappa+1} + \frac{e^{\kappa\tau}}{\kappa+1} - \frac{e^{\kappa\tau}}{\kappa-1} \right] \\
&- \frac{\kappa^2-1}{2} C_2 \left[-\frac{e^{-B(\kappa+1)+\tau}}{\kappa+1} + \frac{e^{-\tau}}{\kappa-1} + \frac{e^{-\kappa\tau}}{\kappa+1} - \frac{e^{-\kappa\tau}}{\kappa-1} \right] \\
&+ \frac{\kappa^2-1}{2} \frac{\kappa^2-1}{\kappa} \frac{e^{-B+\tau}}{\kappa^2-1} \int_0^B \bar{f}'(j) \left[\kappa \cosh(\kappa(B-j)) + \sinh(\kappa(B-j)) \right] \mathrm{d}j.
\end{aligned}$$

After further simplification, we obtain a system of two equations in C_1 and C_2:

$$\begin{aligned}
&- C_1 \frac{1}{\kappa-1} e^{B(\kappa-1)} + C_2 \frac{1}{\kappa+1} e^{-B(\kappa+1)} \\
&\quad + \frac{e^{B(\kappa-1)}}{2\kappa} (\kappa+1) \int_0^B \bar{f}'(j) e^{-\kappa j} \, \mathrm{d}j + \frac{e^{-B(\kappa+1)}}{2\kappa} (\kappa-1) \int_0^B \bar{f}'(j) e^{\kappa j} \, \mathrm{d}j = 0,
\end{aligned}$$

and

$$\frac{C_1}{\kappa+1} = \frac{C_2}{\kappa-1}.$$

Therefore, solving these two equations,

$$C_1 = \frac{1}{2\kappa} \frac{e^{B(\kappa-1)}(\kappa+1)^2(\kappa^2-1) \int_0^B \bar{f}'(j) e^{-\kappa j} \, \mathrm{d}j + e^{-B(\kappa+1)}(\kappa+1)^2(\kappa-1)^2 \int_0^B \bar{f}'(j) e^{\kappa j} \, \mathrm{d}j}{(\kappa+1)^2 e^{B(\kappa-1)} - (\kappa-1)^2 e^{-B(\kappa+1)}},$$

and

$$C_2 = \frac{1}{2\kappa} \frac{e^{B(\kappa-1)}(\kappa+1)^2(\kappa-1)^2 \int_0^B \bar{f}'(j) e^{-\kappa j} \, \mathrm{d}j + e^{-B(\kappa+1)}(\kappa^2-1)(\kappa-1)^2 \int_0^B \bar{f}'(j) e^{\kappa j} \, \mathrm{d}j}{(\kappa+1)^2 e^{B(\kappa-1)} - (\kappa-1)^2 e^{-B(\kappa+1)}}.$$

To get candidate coefficients whose support is not necessarily bounded, we send B to infinity and get

$$C_1 \to C_1^\infty := \frac{\kappa^2-1}{2\kappa} \int_0^\infty \bar{f}'(j) e^{-\kappa j} \, \mathrm{d}j, \tag{30}$$

and

$$C_2 \to C_1^\infty := \frac{(\kappa-1)^2}{2\kappa} \int_0^\infty \bar{f}'(j) e^{-\kappa j} \, \mathrm{d}j. \tag{31}$$

Thus, a candidate for U is

$$U(\tau) = C_1^{\infty} e^{\kappa\tau} + C_2^{\infty} e^{-\kappa\tau} - \bar{f}'(\tau) - \frac{\kappa^2 - 1}{\kappa} \int_0^{\tau} \sinh(\kappa(\tau - s))\bar{f}'(s)\,\mathrm{d}s.$$

We plug in the expression of U in (25), which yields the candidate for u_k:

$$u_k(\tau) = C_1^{\infty} \frac{\beta_k\gamma^2}{\sigma_k^2(\kappa^2-1)} e^{\kappa\tau} + C_1^{\infty} \frac{\beta_k\gamma^2}{\sigma_k^2(\kappa^2-1)} e^{-\kappa\tau} - \frac{f_k'(\tau)}{\sigma_k^2} - \frac{\beta_k\gamma^2}{\sigma_k^2\kappa} \int_0^{\tau} \sinh(\kappa(\tau - s))\bar{f}'(s)\,\mathrm{d}s, \tag{32}$$

and after simplification,

$$u_k(\tau) = \frac{\beta_k\gamma^2}{\sigma_k^2\kappa} \left(\frac{\sinh\kappa\tau + \kappa\cosh\kappa\tau}{1+\kappa} \int_0^{\infty} e^{-\kappa s}\,\mathrm{d}\bar{f}(s) - \int_0^{\tau} \sinh\kappa(t-s)\,\mathrm{d}\bar{f}(s) \right) - \frac{f_k'(\tau)}{\sigma_k^2}, \tag{33}$$

Proof of integrability. We show that every u_k defined by Equation (33) is integrable and square-integrable. To do so, we have to show that

$$(\sinh\kappa t + \kappa\cosh\kappa t) \int_0^{\infty} e^{-\kappa s} h(s)\,\mathrm{d}s - (1+\kappa) \int_0^{t} \sinh\kappa(t-s) h(s)\,\mathrm{d}s \tag{34}$$

is integrable and square-integrable whenever h and h^2 are. We note that (34) is linear in h, so that it suffices to show that its positive and negative parts are integrable. Hence, without loss, we assume that $h \geq 0$.

After re-arranging the terms, (34) is equal to

$$\frac{1}{2}(\kappa+1) \left(e^{\kappa t} \int_t^{\infty} e^{-\kappa s} h(s)\,\mathrm{d}s + e^{-\kappa t} \int_0^{t} e^{\kappa s} h(s)\,\mathrm{d}s \right) + \frac{1}{2}(\kappa - 1) e^{-\kappa t} \int_0^{\infty} e^{-\kappa s} h(s)\,\mathrm{d}s. \tag{35}$$

Thus, (34) is nonnegative, and showing the integrability of (34) reduces to showing that the integral of (34) converges on $[0, +\infty)$.

It is readily verified by differentiation that (34) is the derivative of

$$\frac{\cosh\kappa t + \kappa\sinh\kappa t}{\kappa} \int_0^{\infty} e^{-\kappa s} h(s)\,\mathrm{d}s - \frac{1+\kappa}{\kappa} \int_0^{t} \cosh\kappa(t-s) h(s)\,\mathrm{d}s + \frac{1+\kappa}{\kappa} \int_0^{t} h(s)\,\mathrm{d}s.$$

We must show that this expression converges as $t \to \infty$. Since by assumption, the last term is convergent, it suffices to show that

$$(\cosh \kappa t + \kappa \sinh \kappa t) \int_0^\infty e^{-\kappa s} h(s)\, \mathrm{d}s - (1+\kappa) \int_0^t \cosh \kappa(t-s) h(s)\, \mathrm{d}s$$

converges. Further, since

$$\begin{aligned} \cosh \kappa t + \kappa \sinh \kappa t &= \frac{\kappa+1}{2} e^{\kappa t} - \frac{\kappa-1}{2} e^{-\kappa t}, \text{ and} \\ \cosh \kappa(t-s) &= \frac{e^{-\kappa(t-s)}}{2} + \frac{e^{\kappa(t-s)}}{2}, \end{aligned}$$

it suffices to show that

$$(\kappa+1) e^{\kappa t} \int_0^\infty e^{-\kappa s} h(s)\, \mathrm{d}s - (1+\kappa) \int_0^t e^{\kappa(t-s)} h(s)\, \mathrm{d}s = (\kappa+1) \int_t^\infty e^{-\kappa(s-t)} h(s)\, \mathrm{d}s$$

converges, which is immediate from the integrability of h. Thus, (34) is integrable. Next, to show that (35) is square-integrable, we show that

$$e^{\kappa t} \int_t^\infty e^{-\kappa s} h(s)\, \mathrm{d}s \tag{36}$$

is square-integrable. As square-integrable functions are closed under additivity, and h is integrable, (36) is the only non-trivial term of (35) for which we must show square-integrability. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(\int_t^\infty e^{-\kappa s} h(s)\, \mathrm{d}s \right)^2 &\le \left(\int_t^\infty e^{-\kappa s} h^2(s)\, \mathrm{d}s \right) \left(\int_t^\infty e^{-\kappa s}\, \mathrm{d}s \right) \\ &= \kappa^{-1} e^{-\kappa t} \int_t^\infty e^{-\kappa s} h^2(s)\, \mathrm{d}s. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^\tau \left(e^{\kappa t} \int_t^\infty e^{-\kappa s} h(s)\, \mathrm{d}s \right)^2 \mathrm{d}t &\le \kappa^{-1} \int_0^\tau e^{\kappa t} \int_t^\infty e^{-\kappa s} h^2(s)\, \mathrm{d}s\, \mathrm{d}t \\ &= \frac{1}{\kappa^2} \int_\tau^\infty e^{-\kappa(s-\tau)} h^2(s)\, \mathrm{d}s - \frac{1}{\kappa^2} \int_0^\infty e^{-\kappa s} h^2(s)\, \mathrm{d}s \\ &\quad + \frac{1}{\kappa^2} \int_0^\tau h^2(t)\, \mathrm{d}t, \end{aligned}$$

where the equality follows from integration by parts. Convergence is immediate by square-integrability of h.

Proof that the educated guess is correct. In this part, we show that the candidate for $\{u_k\}_k$ derived in the first step defines valid coefficients for the rating process.

Let u_k be defined by (33), or, equivalently, by (32). Let

$$\widetilde{Y}_t = \mathbf{E}^*[Y_t] + \sum_{k=1}^{K} \int_{s \le t} u_k(t-s)(\mathrm{d}S_{k,s} - \alpha_k A^*_s \,\mathrm{d}s).$$

Note that, if we have $\mathbf{Cov}[Y_t - \widetilde{Y}_t, S_{k,t-\tau}] = 0$ for every τ and k, then Y_t and $S_{k,t-\tau}$ are independent for every τ and k. As $Y_t - \widetilde{Y}_t$ is measurable with respect to the information generated by the past signals $S_{k,t-\tau}$, $\tau \ge 0$, $k = 1, \dots, K$, it implies that $\mathbf{Var}[Y_t - \widetilde{Y}_t] = 0$ and thus $Y_t = \widetilde{Y}_t$.

In the remainder of the proof, we show that $\mathbf{Cov}[Y_t - \widetilde{Y}_t, S_{k,t-\tau}] = 0$ for every $\tau \ge 0$ and every $k = 1, \dots, K$.

Let $g_k(\tau) = \mathbf{Cov}[\widetilde{Y}_t, S_{k,t-\tau}]$. Then we have:

$$\begin{aligned} g_k(\tau) &= \sum_{i=1}^{K} \int_0^\infty u_i(s)\, \mathbf{Cov}[\mathrm{d}S_{i,t-s}, S_{k,t-\tau}] \\ &= \sigma_k^2 \int_\tau^\infty u_k(s)\,\mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_0^\infty \int_\tau^\infty U(s) e^{-|s-j|}\,\mathrm{d}j\,\mathrm{d}s, \end{aligned}$$

and so

$$g_k'(\tau) = -\sigma_k^2 u_k(\tau) - \frac{\beta_k \gamma^2}{2} \int_0^\infty U(s) e^{-|\tau - s|}\,\mathrm{d}s.$$

So, replacing u_k by its definition in (32),

$$\begin{aligned} g_k'(\tau) = f_k'(\tau) &- C_1 \frac{\beta_k \gamma^2}{\kappa^2 - 1} e^{\kappa\tau} - C_2 \frac{\beta_k \gamma^2}{\kappa^2 - 1} e^{-\kappa\tau} \\ &+ \frac{\beta_k \gamma^2}{\kappa} \int_0^\tau \sinh(\kappa(\tau - s)) \bar{f}'(s)\,\mathrm{d}s \\ &- \frac{\beta_k \gamma^2}{2} \int_0^\infty U(s) e^{-|\tau - s|}\,\mathrm{d}s. \end{aligned} \tag{37}$$

Further, multiplying (32) by β_k and summing over k, we have

$$U(\tau) = C_1^\infty e^{\kappa\tau} + C_2^\infty e^{-\kappa\tau} - \bar{f}'(\tau) - \frac{\kappa^2-1}{\kappa}\int_0^\tau \sinh(\kappa(\tau-s))\bar{f}'(s)\,\mathrm{d}s.$$

It holds that

$$\int_0^\infty U(s)e^{-|\tau-s|}\,\mathrm{d}s = \lim_{B\to\infty}\int_0^B U(s)e^{-|\tau-s|}\,\mathrm{d}s.$$

Thus,

$$\begin{aligned}\int_0^B U(s)e^{-|\tau-s|}\,\mathrm{d}s = C_1^\infty \int_0^B e^{\kappa s}e^{-|\tau-s|}\,\mathrm{d}s + C_2^\infty \int_0^B e^{-\kappa s}e^{-|\tau-s|}\,\mathrm{d}s - \int_0^B \bar{f}'(s)e^{-|\tau-s|}\,\mathrm{d}s \\ - \frac{\kappa^2-1}{\kappa}\int_0^B\int_0^s \sinh(\kappa(s-j))\bar{f}'(j)e^{-|\tau-s|}\,\mathrm{d}j\,\mathrm{d}s.\end{aligned}$$

Then, for any $B > \tau$, we write

$$\begin{aligned}&\int_0^B\int_0^s \sinh(\kappa(s-j))\bar{f}'(j)e^{-|\tau-s|}\,\mathrm{d}j\,\mathrm{d}s \\ &\qquad = -\frac{\kappa}{\kappa^2-1}\int_0^B \bar{f}'(j)e^{-|\tau-j|}\,\mathrm{d}j \\ &\qquad\quad + \frac{e^{-B+\tau}}{\kappa^2-1}\int_0^B \bar{f}'(j)\left[\kappa\cosh(\kappa(B-j)) + \sinh(\kappa(B-j))\right]\mathrm{d}j \\ &\qquad\quad - \frac{2}{\kappa^2-1}\int_0^\tau \sinh(\kappa(\tau-j))\bar{f}'(j)\,\mathrm{d}j.\end{aligned}$$

Using the expressions for C_1^∞ and C_2^∞ given by (30) and (31), we get that

$$\begin{aligned}&C_1^\infty\left[\frac{e^{B(\kappa-1)+\tau}}{\kappa-1} - \frac{e^{-\tau}}{\kappa+1}\right] \\ &\quad + C_2^\infty\left[-\frac{e^{-B(\kappa+1)+\tau}}{\kappa+1} + \frac{e^{-\tau}}{\kappa-1}\right] \\ &\quad + \frac{\kappa^2-1}{\kappa}\frac{\kappa}{\kappa^2-1}\int_0^B \bar{f}'(j)e^{-|\tau-j|}\,\mathrm{d}j \\ &\quad - \frac{\kappa^2-1}{\kappa}\frac{e^{-B+\tau}}{\kappa^2-1}\int_0^B \bar{f}'(j)\left[\kappa\cosh(\kappa(B-j)) + \sinh(\kappa(B-j))\right]\mathrm{d}j\end{aligned}$$

converges to 0 as $B \to \infty$. Therefore,

$$\int_0^{\infty} U(s) e^{-|\tau-s|}\,\mathrm{d}s = \frac{2}{\kappa}\int_0^{\tau} \sinh(\kappa(\tau-j))\bar{f}'(j)\,\mathrm{d}j + C_1^{\infty}\left[\frac{e^{\kappa\tau}}{\kappa+1} - \frac{e^{\kappa\tau}}{\kappa-1}\right] + C_2^{\infty}\left[\frac{e^{-\kappa\tau}}{\kappa+1} - \frac{e^{-\kappa\tau}}{\kappa-1}\right]. \tag{38}$$

Plugging the expression of (38) in (37) yields

$$\begin{aligned} g_k'(\tau) &= f_k'(\tau) - C_1^{\infty}\frac{\beta_k\gamma^2}{\kappa^2-1}e^{\kappa\tau} - C_2^{\infty}\frac{\beta_k\gamma^2}{\kappa^2-1}e^{-\kappa\tau} \\ &\quad + \frac{\beta_k\gamma^2}{\kappa}\int_0^{\tau}\sinh(\kappa(\tau-s))\bar{f}'(s)\,\mathrm{d}s \\ &\quad - \frac{\beta_k\gamma^2}{\kappa}\int_0^{\tau}\sinh(\kappa(\tau-j))\bar{f}'(j)\,\mathrm{d}j \\ &\quad - \frac{\beta_k\gamma^2}{2}C_1^{\infty}\left[\frac{e^{\kappa\tau}}{\kappa+1} - \frac{e^{\kappa\tau}}{\kappa-1}\right] \\ &\quad - \frac{\beta_k\gamma^2}{2}C_2^{\infty}\left[\frac{e^{-\kappa\tau}}{\kappa+1} - \frac{e^{-\kappa\tau}}{\kappa-1}\right] \\ &= f_k'(\tau). \end{aligned}$$

So $g_k' = f_k'$. As $f_k(0) = g_k(0) = 0$, it follows that $f = g$. Uniqueness of the coefficients (up to measure zero sets) is immediate by linearity, as different coefficients on a set of positive measure yield a different joint distributions over ratings and signals.

A.7 Proof of Theorem 2.13

Following Equation (10) of Section 2.4, we have the following linear representation of ν,

$$\nu_t = \sum_{k=1}^{K}\int_{s\le t} u_k(t-s)\left(\mathrm{d}S_{k,s} - \alpha_k A_s^*\,\mathrm{d}s\right),$$

with

$$u_k(\tau) = (\kappa-1)\frac{\beta_k}{\sigma_k^2 m_\beta}e^{-\kappa\tau}.$$

We apply Lemma 2.12 and Equation (7), and get equilibrium action A given by

$$\begin{aligned} c'(A_t) &= (\kappa - 1) \sum_{k=1}^{K} \frac{\alpha_k \beta_k}{\sigma_k^2 m_\beta} \int_0^\infty e^{-\kappa\tau} e^{-r\tau} \, \mathrm{d}\tau \\ &= (\kappa - 1) \frac{m_{\alpha\beta}}{m_\beta} \frac{1}{\kappa + r}. \end{aligned}$$

A.8 Proof of Lemma 2.15

Let $\mathcal{F}$ be a public or confidential information structure. Let Y be a rating process proportional to the market belief, and let A be the (stationary) effort level it induces.

To show that any action in the range $[0, A]$ can be attained in the equilibrium of an alternative public/confidential information structure, we modify the rating process that achieves A to depress incentives to any desired extent. To do so, we use a source of independent noise. In addition to the K signals described in the model, we include one additional signal indexed by $K + 1$ that is entirely uninformative about both the agent's action and the agent's ability. Let us assume S_{K+1} is a two-sided standard Brownian motion.

Consider the two-sided process

$$\xi_t = \int_{s \le t} e^{-(t-s)} \, \mathrm{d}S_{K+1,s}.$$

From Proposition 2.12, if Y has linear filter $\{u_k\}_k$, the equilibrium action A in both the public and confidential cases is the solution to

$$c'(A) = \frac{\mathbf{Cov}[Y_t, \theta_t]}{\mathbf{Var}[Y_t]} \sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(\tau) e^{-r\tau} \, \mathrm{d}\tau.$$

Consider the alternative rating process $\widehat{Y} = (1 - a)Y + a\xi$, for some constant $a \in [0, 1]$. Note that $\widehat{Y}$ is a well-defined rating process for the information generated by the $K + 1$ signals.

Consider the information structure generated by the rating process $\widehat{Y}$, and the induced equilibrium action, $\widehat{A}$. We have

$$\begin{aligned} c'(\widehat{A}) &= \frac{\mathbf{Cov}[\widehat{Y}_t, \theta_t]}{\mathbf{Var}[\widehat{Y}_t]} \sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(\tau) e^{-r\tau} \, \mathrm{d}\tau \\ &= \frac{(1-a)\,\mathbf{Cov}[Y_t, \theta_t]}{(1-a)^2\,\mathbf{Var}[Y_t] + a^2\,\mathbf{Var}[\xi]} \sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(\tau) e^{-r\tau} \, \mathrm{d}\tau \\ &= \frac{1-a}{(1-a)^2 + a^2\,\mathbf{Var}[\xi]/\mathbf{Var}[Y_t]} c'(A). \end{aligned}$$

By varying a over the interval $[0, 1]$, $c'(\widehat{A})$ covers the entire interval $[0, c'(A)]$, and thus $\widehat{A}$ covers the interval $[0, A]$.

Besides, as Y and ξ are independent, for $\tau \geq 0$,

$$\mathbf{Cov}[\widehat{Y}_t, \widehat{Y}_{t+\tau}] = (1-a)^2\,\mathbf{Cov}[Y_t, Y_{t+\tau}] + a^2\,\mathbf{Cov}[\xi_t, \xi_{t+\tau}].$$

By Itô's isometry, we get

$$\mathbf{Cov}[\xi_t, \xi_{t+\tau}] = \int_0^\infty e^{-s} e^{-(s+\tau)} \, \mathrm{d}s = \frac{1}{2} e^{-\tau} = \mathbf{Var}[\xi_t] e^{-\tau}.$$

By Lemma 2.8,

$$\mathbf{Cov}[Y_t, Y_{t+\tau}] = \mathbf{Var}[Y_t] e^{-\tau}.$$

Thus,

$$\mathbf{Cov}[\widehat{Y}_t, \widehat{Y}_{t+\tau}] = ((1-a)^2\,\mathbf{Var}[Y_t] + a^2\,\mathbf{Var}[\xi_t]) e^{-\tau} = \mathbf{Var}[\widehat{Y}_t] e^{-\tau},$$

and invoking Lemma 2.8 a second time, we get that $\widehat{Y}$ is proportional to the market belief associated with a public information structure. Hence, $\widehat{A}$ also denotes the equilibrium action under that public information structure.

It follows that under both the public and the confidential information structure, any action in the interval $[0, A]$ can be induced in equilibrium.

A.9 Proof of Lemma 2.16

We note that θ_t and μ_t are jointly normal, and as μ_t is the market belief, $\mathbf{Cov}[\theta_t, \mu_t] = \mathbf{Var}[\mu_t]$ by Lemma 2.7, so applying the projection formulas:

$$\begin{aligned}\mathbf{Var}[\theta_t \mid \mu_t] &= \mathbf{Var}[\theta_t] - \frac{\mathbf{Cov}[\theta_t, \mu_t]^2}{\mathbf{Var}[\mu_t]} \\ &= \frac{\gamma^2}{2} - \mathbf{Var}[\mu_t].\end{aligned}$$

Appendix B: Proofs of Section 3

B.1 Proof of Theorem 3.1

In this section, we prove Theorem 3.1. The proof proceeds in two parts. In the first part, we provide a candidate optimal rating by deriving first-order conditions using a variational argument. In the second part, we verify the optimality of the candidate.

B.1.1 Part I: First-Order Conditions

Throughout this subsection, we use the following shorthand notation:

$$
\begin{aligned}
U(t) &:= \sum_{k=1}^{K} \beta_k u_k(t), \\
V(t) &:= \sum_{k=1}^{K} \alpha_k u_k(t), \\
U_0 &:= \int_0^\infty U(t) e^{-t}\, \mathrm{d}t, \\
V_0 &:= \int_0^\infty V(t) e^{-rt}\, \mathrm{d}t.
\end{aligned}
$$

We seek to maximize $c'(A)$ (where A is the stationary equilibrium action of the agent) among confidential information structures generated by rating processes with mean zero and with linear filter $\mathbf{u} := \{u_k\}_k$, which in addition satisfy the normalization condition that the rating has variance one.

Any such rating process Y can be written as

$$
Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \left[\mathrm{d}S_{k,s} - \alpha_k A_s^* \, \mathrm{d}s \right].
$$

We note that, by Itô's isometry,

$$
\mathbf{Var}[Y_t] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)^2 \, \mathrm{d}s + \sum_{k=1}^{K} \sum_{k'=1}^{K} \int_{j \le t} \int_{i \le t} \beta_k \beta_{k'} u_k(t-i) u_{k'}(t-j) \, \mathbf{Cov}[\theta_i, \theta_j] \, \mathrm{d}i \, \mathrm{d}j,
$$

and since θ is a stationary Ornstein-Uhlenbeck process with mean-reversion rate 1 and scale σ, we have $\mathbf{Cov}[\theta_t, \theta_s] = \gamma^2 e^{-|t-s|}/2$, so that

$$\mathbf{Var}[Y_t] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)^2 \,\mathrm{d}s + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j-i|} \,\mathrm{d}i \,\mathrm{d}j.$$

Applying Lemma 2.12, the problem of maximizing $c'(A)$ among rating processes that satisfy the normalization condition thus reduces to choosing a linear filter $\mathbf{u}$ that maximizes

$$\left[\int_0^\infty V(t)e^{-rt} \,\mathrm{d}t\right] \left[\int_0^\infty U(t)e^{-t} \,\mathrm{d}t\right],$$

subject to

$$\frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j-i|} \,\mathrm{d}i \,\mathrm{d}j + \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(t)^2 \,\mathrm{d}t = 1.$$

This optimization problem is a problem of calculus of variations with isoperimetric constraint. Assume there exists a solution $\mathbf{u}^*$ to this optimization problem, where $\mathbf{u}^*$ is twice differentiable, integrable, and square-integrable.

Let

$$L(\mathbf{u}, \lambda_0) = F(\mathbf{u}) + \lambda_0 G(\mathbf{u}),$$

where F and G are defined as

$$F(\mathbf{u}) = \left[\int_0^\infty V(t)e^{-rt} \,\mathrm{d}t\right] \left[\int_0^\infty U(t)e^{-t} \,\mathrm{d}t\right],$$

and

$$G(\mathbf{u}) = \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j-i|} \,\mathrm{d}i \,\mathrm{d}j + \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(t)^2 \,\mathrm{d}t.$$

The function L defines an unconstrained maximization problem for every given λ_0. It corresponds to the Lagrangian of the constrained optimization problem up to an additive, $\mathbf{u}$-independent term, where the coefficient λ_0 is a Lagrangian multiplier. However, we do not need to invoke the Theorem of Lagrange Multipliers and its extensions to isoperimetric problems in the calculus of variations. Instead, we will look for a constant λ_0 that yields a unique candidate of the unconstrained maximization problem that satisfies the Euler-Lagrange first-order conditions, and, in addition, satisfies the original constraint. In the remainder of this proof, we refer to the unconstrained optimization problem as the *relaxed* optimization problem, as opposed to the *original* (constrained) maximization problem.

Observe that we can write both F and G as a double integral:

$$F(\mathbf{u}) = \int_0^\infty \int_0^\infty V(i)U(j)e^{-ri}e^{-j}\,\mathrm{d}i\,\mathrm{d}j,$$

while

$$G(\mathbf{u}) = \int_0^\infty \int_0^\infty \left(\frac{\gamma^2}{2}U(i)U(j)e^{-|j-i|}\,\mathrm{d}i\,\mathrm{d}j + \sum_{k=1}^K \sigma_k^2 u_k(j)^2 e^{-i}\right)\mathrm{d}i\,\mathrm{d}j.$$

This allows us to apply the results of Proposition D.1. Assume there exists a $\lambda_0^* < 0$ such that $\mathbf{u} = \mathbf{u}^*$ maximizes $\mathbf{u} \mapsto L(\mathbf{u}, \lambda_0^*)$.[52]

Proposition D.1 gives the first-order condition derived from the Euler-Lagrange equations: if $\lambda_0 = \lambda_0^*$ and $\mathbf{u} = \mathbf{u}^*$, then for all k and all t, we have $L_k(t) = 0$, where

$$L_k(t) := \alpha_k U_0 e^{-rt} + \beta_k V_0 e^{-t} + \lambda_0\gamma^2\beta_k \int_0^\infty U(j)e^{-|t-j|}\,\mathrm{d}j + 2\lambda_0\sigma_k^2 u_k(t) = 0, \quad (39)$$

and where U_0, V_0, U and V are defined as above as an implicit function of $\mathbf{u}$.

We differentiate the above equation in the variable t twice, and get, for all k and all t:

$$\alpha_k U_0 r^2 e^{-rt} + \beta_k V_0 e^{-t} - 2\lambda_0\gamma^2\beta_k U(t) + \lambda_0\gamma^2\beta_k \int_0^\infty U(j)e^{-|t-j|}\,\mathrm{d}j + 2\lambda_0\sigma_k^2 u_k''(t) = 0. \quad (40)$$

The difference between (39) and (40) is

$$(1 - r^2)\alpha_k U_0 e^{-rt} + 2\lambda_0\gamma^2\beta_k U(t) + 2\lambda_0\sigma_k^2(u_k(t) - u_k''(t)) = 0. \quad (41)$$

In particular, multiplying (41) by β_k/σ_k^2 and summing over k, we get a linear differential equation that $U(t)$ must satisfy, namely,

$$(1 - r^2)m_{\alpha\beta}U_0 e^{-rt} + 2\lambda_0\gamma^2 m_\beta U(t) + 2\lambda_0(U(t) - U''(t)) = 0,$$

[52] Insofar as we find a coefficient λ_0^* that yields a unique candidate which is shown to solve the original problem, we need not prove uniqueness of the coefficient. However, it is easily seen that $\lambda_0^* < 0$ is a necessary second-order condition. The optimum marginal cost, if it exists, is strictly positive, *i.e.*, $F(\mathbf{u}^*) > 0$, since the optimal solution does at least as well as transparency (giving all information included in all signals to the market) and transparency induces a positive equilibrium effort by our assumption that $m_{\alpha\beta} > 0$. This implies $\lambda_0^* < 0$, because $F(\mathbf{u}^*) > 0$ and $G(\mathbf{u}^*) = 1$.

where we recall that $m_\beta = \sum_k \beta_k^2/\sigma_k^2$, $m_{\alpha\beta} = \sum_k \alpha_k\beta_k/\sigma_k^2$, and $m_\alpha = \sum_k \alpha_k^2/\sigma_k^2$.

The characteristic polynomial has roots $\pm\sqrt{1+\gamma^2 m_\beta} = \pm\kappa$. A particular solution is Ce^{-rt}, for some constant C. If the solution is admissible, it is bounded, hence we get

$$U(t) = C_1 e^{-rt} + C_2 e^{-\kappa t},$$

for some constants C_1 and C_2.

For such U, u_k satisfies the linear differential equation (41), whose characteristic polynomial has roots ± 1. A particular solution is a sum of scaled time exponentials e^{-rt} and $e^{-\kappa t}$. As every u_k is bounded, we must consider the negative root of the characteristic equation, and we get that

$$u_k(t) = D_{1,k} e^{-rt} + D_{2,k} e^{-\kappa t} + D_{3,k} e^{-t}, \tag{42}$$

for some constants $D_{1,k}, D_{2,k}, D_{3,k}$.

Determination of the constants. We have established that the solution belongs to the family of functions that are sums of scaled time exponentials. We now solve for the constant factors.

We plug in the general form of u_k from (42) in the expression for L_k, and get:

$$L_k = L_{1,k} e^{-rt} + L_{2,k} e^{-\kappa t} + L_{3,k} e^{-t},$$

where the coefficients $L_{1,k}$, $L_{2,k}$, $L_{3,k}$ depend on the primitives of the model and the constants $D_{1,k}, D_{2,k}, D_{3,k}$. The condition that $L_k = 0$ implies that $L_{1,k} = L_{2,k} = L_{3,k} = 0$.

First, note that $U(t)$ does not include a term of the form e^{-t}, which implies that

$$\sum_{k=1}^{K} \beta_k D_{3,k} = 0. \tag{43}$$

We also observe that

$$L_{2,k} = 2\lambda_0 \sigma_k^2 D_{2,k} - \frac{2\gamma^2 \lambda_0 \beta_k \sum_{i=1}^{K} \beta_i D_{2,i}}{\kappa^2 - 1},$$

so that $L_{2,k} = 0$ for all k implies

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2}, \tag{44}$$

for some multiplier a. Next, we use (44) together with (43) to show that

$$L_{3,k} = \frac{\beta_k}{2r}\sum_{i=1}^{K}\alpha_i D_{1,i} + \frac{\beta_k}{r+1}\sum_{i=1}^{K}\alpha_i D_{3,i} + \frac{\gamma^2\lambda_0\beta_k}{r-1}\sum_{i=1}^{K}\beta_i D_{1,i} + 2\lambda_0\sigma_k^2 D_{3,k} + \frac{a\gamma^2\lambda_0\beta_k m_\beta}{\kappa-1} + \frac{a\beta_k m_{\alpha\beta}}{\kappa+r},$$

and $L_{3,k}=0$ for every k implies that $D_{3,k}=0$ for all k. The equation $L_{3,k}/\beta_k = 0$ is linear in λ_0, and then simplifies to:

$$\lambda_0\left(\frac{\gamma^2}{r-1}\sum_{i=1}^{K}\beta_i D_{1,i} + \frac{a\gamma^2 m_\beta}{\kappa-1}\right) + \frac{1}{2r}\sum_{i=1}^{K}\alpha_i D_{1,i} + \frac{a m_{\alpha\beta}}{\kappa+r} = 0. \tag{45}$$

Next, we use (44) together with (43) to show that

$$L_{1,k} = 2\lambda_0\sigma_k^2 D_{1,k} + \frac{a\alpha_k m_\beta}{\kappa+1} + \frac{((r-1)\alpha_k - 2\gamma^2\lambda_0\beta_k)}{r^2-1}\sum_{i=1}^{K}\beta_i D_{1,i},$$

and, since $L_{1,k}=0$ must hold for every k, we get, since $\lambda_0 \neq 0$,

$$\sigma_k^2 D_{1,k} = \left(\frac{\gamma^2\beta_k}{r^2-1} - \frac{\alpha_k}{2\lambda_0 + 2\lambda_0 r}\right)\sum_{i=1}^{K}\beta_i D_{1,i} - \frac{a\alpha_k m_\beta}{2\kappa\lambda_0 + 2\lambda_0}. \tag{46}$$

We multiply (46) by β_k/σ_k^2, and sum over k to get

$$\left[(\kappa+1)\left((r-1)\left(m_{\alpha\beta} + 2\lambda_0(r+1)\right) - 2\gamma^2\lambda_0 m_\beta\right)\right]\sum_{i=1}^{K}\beta_i D_{1,i} = -a\left(r^2-1\right)m_{\alpha\beta}m_\beta.$$

As by assumption $r \neq 1$, the right-hand side is non-zero, which implies

$$\left((r-1)\left(m_{\alpha\beta} + 2\lambda_0(r+1)\right) - 2\gamma^2\lambda_0 m_\beta\right) \neq 0, \tag{47}$$

and thus

$$\sum_{i=1}^{K}\beta_i D_{1,i} = \frac{-a\left(r^2-1\right)m_{\alpha\beta}m_\beta}{(\kappa+1)\left((r-1)\left(m_{\alpha\beta} + 2\lambda_0(r+1)\right) - 2\gamma^2\lambda_0 m_\beta\right)}. \tag{48}$$

Similarly, if we multiply (46) by α_k/σ_k^2 and sum over k, we get

$$\begin{aligned}\sum_{i=1}^{K}\alpha_i D_{1,i} &= \left(\frac{\gamma^2 m_{\alpha\beta}}{r^2-1}-\frac{m_\alpha}{2\lambda_0+2\lambda_0 r}\right)\sum_{i=1}^{K}\beta_i D_{1,i}-\frac{a m_\alpha m_\beta}{2\kappa\lambda_0+2\lambda_0}\\ &= \frac{a m_\beta\left(m_\alpha\left(\gamma^2 m_\beta - r^2+1\right)-\gamma^2 m_{\alpha\beta}^2\right)}{(\kappa+1)\left((r-1)\left(m_{\alpha\beta}+2\lambda_0(r+1)\right)-2\gamma^2\lambda_0 m_\beta\right)}.\end{aligned}$$

Putting together (45), (48) and (B.1.1) yields a quadratic equation in λ_0 of the form

$$A\lambda_0^2+B\lambda_0+C=0, \tag{49}$$

which, after simplification and using that $\kappa^2=1+\gamma^2 m_\beta$, gives

$$\begin{aligned}A &= m_\beta\frac{\kappa+r}{1-\kappa},\\ B &= \frac{m_{\alpha\beta}\left(\gamma^2 m_\beta\left(-2\kappa^2+r^2+1\right)+\left(\kappa^2-1\right)\left(r^2-1\right)\right)}{\gamma^2\left(\kappa^2-1\right)\left(\gamma^2 m_\beta-r^2+1\right)}\\ &= -\frac{2}{\gamma^2}m_{\alpha\beta},\\ C &= \frac{m_\alpha m_\beta(\kappa+r)\left(r^2-\kappa^2\right)+m_{\alpha\beta}^2\left(\gamma^2 m_\beta(\kappa+r)-2(\kappa+1)(r-1)r\right)}{4\gamma^2(\kappa+1)r\left(r^2-\kappa^2\right)}\\ &= \frac{(\kappa-1)m_\alpha(\kappa+r)^2-\gamma^2 m_{\alpha\beta}^2(\kappa+2r-1)}{4\gamma^4 r(\kappa+r)}.\end{aligned}$$

As $\kappa>1$, we immediately have $A<0$. Also, C has the sign of

$$\begin{aligned}&(\kappa-1)m_\alpha(\kappa+r)^2-m_{\alpha\beta}^2(\kappa-1+2r)\gamma^2\\ &\qquad= (\kappa-1)m_\alpha(\kappa+r)^2-m_{\alpha\beta}^2(\kappa-1+2r)m_\beta^{-1}(\kappa^2-1).\end{aligned}$$

By the Cauchy-Schwarz inequality, $m_\alpha m_\beta \geq m_{\alpha\beta}^2$, so:

$$\begin{aligned}&(\kappa-1)m_\alpha(\kappa+r)^2-m_{\alpha\beta}^2(\kappa-1+2r)m_\beta^{-1}(\kappa^2-1)\\ &\qquad\geq m_\alpha\left\{(\kappa-1)(\kappa+r)^2-(\kappa-1+2r)(\kappa^2-1)\right\}\\ &\qquad= m_\alpha(\kappa-1)(1-r)^2>0.\end{aligned}$$

Hence C is positive, $A\cdot C$ is negative, and Equation (49) has two roots, one positive and one negative. Besides, as $m_{\alpha\beta}>0$ by assumption, $B<0$. As we have already established that λ_0 must be negative, we conclude that

$$\lambda_0 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

Pulling out the term $\sum_i \beta_i D_{1,i}$ in (46) using (48), we express $D_{1,k}$ as a solution of the linear equation. It follows that

$$D_{1,k} = a \frac{m_\beta \left[\gamma^2 m_{\alpha\beta} \frac{\beta_k}{\sigma_k^2} - (\kappa^2 - r^2) \frac{\alpha_k}{\sigma_k^2} \right]}{(1+\kappa) \left[2\lambda_0 (\kappa^2 - r^2) + (1-r) m_{\alpha\beta} \right]},$$

where the denominator is non-zero by (47). We can simplify those expressions further. We define

$$\lambda = (\kappa - 1)\sqrt{r}(1+r) m_{\alpha\beta} + (\kappa - r)\sqrt{\Delta},$$

where

$$\Delta \coloneqq (r+\kappa)^2 (m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2.$$

Then, $D_{1,k} = a\sqrt{r} c_k / \lambda$ with

$$c_k \coloneqq (\kappa^2 - r^2) m_\beta \frac{\alpha_k}{\sigma_k^2} + (1 - \kappa^2) m_{\alpha\beta} \frac{\beta_k}{\sigma_k^2}.$$

Note that, as a rating process induces the same effort level up to a scaling of the rating process, any multiplier a yields the same equilibrium action. Thus, a candidate optimal rating process for the original optimization problem is given by the linear filter

$$u_k(t) = c_k \frac{\sqrt{r}}{\lambda} e^{-rt} + \frac{\beta_k}{\sigma_k^2} e^{-\kappa t}, \ \forall k.$$

If

$$a = \frac{(\kappa - 1)\Big((\kappa - 1) m_{\alpha\beta} (r+1) \sqrt{r} + \sqrt{\Delta}(\kappa - r) \Big)}{2\sqrt{\Delta} m_\beta (\kappa - r)},$$

then the conditions of Lemma 2.7 are satisfied, so that the associated rating process is a market belief for a confidential information structure.

B.1.2 Part II: Verification

We now verify that the candidate rating process of Section B.1.1 is optimal. To so so, we consider an auxiliary principal-agent setting. We refer to the principal-agent setting as the auxiliary setting, and to the main setting detailed in the main body of the paper as the original setting.

Auxiliary setting. In the auxiliary setting, there is a principal (she) and an agent (he). Time $t \geq 0$ is continuous and the horizon infinite. The agent is as in the original model. He exerts private effort (his action), has an exogenous random ability, produces output X and generates signals $S_1 = X$, $S_2, \ldots, S_K$ over time. The various laws of motion, for the agent's ability, output, signals, are as in the original setting. The filtration $\mathcal{G}$ captures all information of the signal processes as in the original setting. The agent's information at time t continues to be $\mathcal{G}_t$, as defined in Section 2.1. The agent's strategy, which specifies his private action at every time as a function of his information, continues to be a bounded $\mathcal{G}$-adapted process A.

However, the agent's payoff is not defined as in Section 2.1. In the auxiliary setting, the agent is not paid by a market, but by a principal. Informally, over interval $[t, t + \mathrm{d}t)$, the principal transfers the amount $Y_t \, \mathrm{d}t$ to the agent. Here Y is the stochastic process that determines the transfer rate (payments may be negative). The agent is risk-neutral, he discounts future payoffs at rate $r > 0$, and his instantaneous cost of effort is $c(\cdot)$, as in the original setting. The agent's total payoff is

$$\int_0^\infty e^{-rt} \left(Y_t - c(A_t)\right) \mathrm{d}t.$$

Given Y, the agent chooses a strategy A that maximizes his expected discounted payoff:

$$A \in \operatorname*{argmax}_{\widehat{A}} \mathbf{E} \left[\int_0^\infty e^{-rt} \left(Y_t - c(\widehat{A}_t) \right) \mathrm{d}t \;\middle|\; \mathcal{G}_0 \right], \tag{50}$$

where the expectation is under the law of motion defined by strategy $\widehat{A}$. A strategy that satisfies (50) is called a *best-response* to the transfer process Y.

In the auxiliary setting, the principal combines features of both the market and the intermediary in the original setting. As the market, the principal sets the transfer to the agent, and as the intermediary, she observes all the signals the agent generates over time, *i.e.*, she knows $\mathcal{G}_t$ at time t. The principal also recommends a strategy for the agent, denoted A^*—the analogue of the market conjecture in the original setting. She is risk-neutral and has discount rate $\rho \in (0, r)$. Her total payoff is

$$\int_0^\infty e^{-\rho t} H_t \, \mathrm{d}t.$$

For now, we do not need to specify the instantaneous payoff process H. We specialize H below as we discuss the principal's optimization program.

A *contract* for the principal is a pair (A^*, Y). The contract is *incentive compatible* if A^* is a best-response to Y.

For the most part, we focus on *stationary linear contracts.* These are contracts whose transfer processes Y are affine in the past signal increments, and are stationary: there exist u_k, $k = 1, \ldots, K$, such that, up to an additive constant,

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s)\, \mathrm{d}S_{k,s}.$$

The principal wants to maximize her own payoff over all contracts that are incentive compatible. This implies that there are two optimal control problems, one embedded into the other. First, we solve the agent's problem, and then turn to the principal's problem.

The Agent's Problem. We first state conditions of incentive compatibility as in the main body of the paper. The proof follows the same arguments used in Lemma 2.12.

Lemma B.1 *Let (A, Y) be a stationary linear contract. The contract is incentive compatible if, and only if,*

$$c'(A) = \sum_{k=1}^{K} \alpha_k \int_0^{\infty} u(t) e^{-rt}\, \mathrm{d}t.$$

As common in principal-agent problems, to solve the principal's problem using a dynamic programming approach, we express incentive compatibility in terms of the evolution of the agent's continuation value, or equivalently, the agent's continuation transfer.

In the sequel, as in the main body of the paper, $\nu_t = \mathbf{E}[\theta_t \mid \mathcal{G}_t]$ is the agent's best current estimate about his ability.

Lemma B.2 *Let (A, Y) be a stationary linear contract. If the contract is incentive compatible, then there exists constants $C_1, \ldots, C_K$ such that the agent's continuation transfer process J defined by*

$$J_t = \mathbf{E}\left[\int_{s \ge t} e^{-r(s-t)} Y_s\, \mathrm{d}s \,\middle|\, \mathcal{G}_t\right],$$

(where the expectation is taken with respect to the law of motion induced by strategy A) satisfies the SDE

$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\xi_\beta \frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + C_k\right)[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\,\mathrm{d}t]\,,$$

and the two transversality conditions

$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau} \mid \mathcal{G}_t] = 0, \text{ and}$$
$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau}^2 \mid \mathcal{G}_t] = 0,$$

where $\xi_\beta := \sum_{k=1}^{K} \beta_k C_k$. In addition, the equilibrium action is defined by $c'(A_t) = \xi_\alpha := \sum_{k=1}^{K} \alpha_k C_k$.

Note that transversality is with respect to the principal's discount rate, not the agent's.

Proof. Consider a stationary linear contract (A, Y), where

$$Y_t = \sum_{k=1}^{K} \int_{s\le t} u_k(t-s)\,[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s]\,.$$

Let

$$J_T = \mathbf{E}\left[\int_{t\ge T} e^{-r(t-T)} Y_t\,\mathrm{d}t \;\middle|\; \mathcal{G}_T\right].$$

We compute

$$\int_{t\ge T} e^{-r(t-T)} Y_t\,\mathrm{d}t = \sum_{k=1}^{K} \int_{t\ge T}\int_{s\le T} e^{-r(t-T)} u_k(t-s)\,[\mathrm{d}S_{k,s} - A_s\,\mathrm{d}s]\,\mathrm{d}t$$
$$+ \sum_{k=1}^{K} \int_{s\ge T}\int_{t\ge s} e^{-r(t-T)} u_k(t-s)\,\mathrm{d}t\,[\mathrm{d}S_{k,s} - A_s\,\mathrm{d}s]\,.$$

Note that, for $t \ge T$, $\mathbf{E}[\theta_t \mid \mathcal{G}_T, \theta_T] = \mathbf{E}[\theta_t \mid \theta_T] = e^{-(t-T)}\theta_T$, so using the law of iterated expectations, $\mathbf{E}[\theta_t \mid \mathcal{G}_T] = \mathbf{E}[\mathbf{E}[\theta_t \mid \mathcal{G}_T, \theta_T] \mid \mathcal{G}_T] = \mathbf{E}[e^{-(t-T)}\theta_T \mid \mathcal{G}_T] =$

$e^{-(t-T)}\nu_T$. Hence, we can compute J_T as

$$\begin{aligned} J_T &= \sum_{k=1}^{K} \int_{t\geq T} \int_{s\leq T} e^{-r(t-T)} u_k(t-s) \left[\mathrm{d}S_{k,s} - A_s \, \mathrm{d}s \right] \mathrm{d}t \\ &\quad + \sum_{k=1}^{K} \beta_k \int_{s\geq T} \int_{t\geq s} e^{-r(t-T)} u_k(t-s) e^{-(s-T)} \nu_T \, \mathrm{d}t. \\ &= \int_{t\geq T} \int_{s\leq T} e^{-r(t-T)} u(t-s) \left[\mathrm{d}S_{k,s} - A_s \, \mathrm{d}s \right] \mathrm{d}t \\ &\quad + \frac{\nu_T}{1+r} \sum_{k=1}^{K} \beta_k \int_{\tau\geq 0} e^{-r\tau} u_k(\tau) \, \mathrm{d}\tau. \end{aligned}$$

Now, let us define the constants $C_1, \ldots, C_K$ as

$$C_k = \int_{\tau\geq 0} e^{-r\tau} u_k(\tau) \, \mathrm{d}\tau.$$

Then

$$\begin{aligned} \mathrm{d}J_T &= \frac{\xi_\beta}{1+r} \, \mathrm{d}\nu_T - Y_T \, \mathrm{d}T + \sum_{k=1}^{K} C_k \left[\mathrm{d}S_{k,T} - \alpha_k A_T \, \mathrm{d}T \right] + r J_T \, \mathrm{d}T - \frac{r}{1+r} \xi_\beta \nu_T \, \mathrm{d}T \\ &= \frac{\xi_\beta}{1+r} \, \mathrm{d}\nu_T + (r J_T - Y_T) \, \mathrm{d}T + \sum_{k=1}^{K} C_k \left[\mathrm{d}S_{k,T} - \left(\alpha_k A_T + \frac{r}{1+r} \beta_k \nu_t \right) \mathrm{d}T \right]. \end{aligned}$$

After simplification and using $\mathrm{d}\nu_t = -\kappa\nu_t \, \mathrm{d}t + \frac{\gamma^2}{1+\kappa} \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} \left[\mathrm{d}S_{k,t} - \alpha_k A_t \, \mathrm{d}t \right]$, we get

$$\mathrm{d}J_t = (r J_t - Y_t) \, \mathrm{d}t + \sum_{k=1}^{K} \left(\xi_\beta \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + C_k \right) \left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t) \, \mathrm{d}t \right].$$

That $c'(A_t) = \xi_\alpha$ follows from Lemma B.1. ■

Lemma B.3 *Let (A, Y) be a stationary linear contract. Suppose J and $\widehat{C}_1, \ldots, \widehat{C}_K$*

are $\mathcal{G}$-adapted processes, and that J satisfies the SDE

$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\widehat{\xi}_{\beta,t}\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\,\mathrm{d}t\right], \tag{51}$$

and the two transversality conditions

$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau}J_{t+\tau} \mid \mathcal{G}_t] = 0, \text{ and} \tag{52}$$

$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau}J_{t+\tau}^2 \mid \mathcal{G}_t] = 0, \tag{53}$$

where $\widehat{\xi}_\beta := \sum_k \beta_k \widehat{C}_k$.

Then, J_t is the agent's continuation transfer $\mathbf{E}\left[\int_{s\geq t} e^{-r(s-t)}Y_s\,\mathrm{d}s \,\middle|\, \mathcal{G}_t\right]$, the contract is incentive compatible, and the agent's equilibrium action satisfies $c'(A_t) = \sum_k \alpha_k \widehat{C}_k$.

Proof. We fix a stationary linear contract (A, Y). Let J and $\widehat{C}_1, \ldots, \widehat{C}_K$ be $\mathcal{G}$-adapted processes such that J satisfies (51) subject to (52) and (53).

Integrating J yields

$$J_t - e^{-r\tau}J_{t+\tau} = \int_t^{t+\tau} e^{-r(s-t)}\left[Y_s - \sum_{k=1}^{K}\left(\widehat{\xi}_{\beta,t}\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\,\mathrm{d}t\right]\right],$$

and using that J is $\mathcal{G}$-adapted, together with the law of iterated expectations, we get

$$\begin{aligned} J_t &- \mathbf{E}\left[e^{-r\tau}J_{t+\tau} \,\middle|\, \mathcal{G}_t\right] \\ &= \mathbf{E}\left[\int_t^{t+\tau} e^{-r(s-t)}Y_s \,\middle|\, \mathcal{G}_t\right] \\ &\quad + \sum_{k=1}^{K}\mathbf{E}\left[\int_t^{t+\tau} e^{-r(s-t)}\left(\widehat{\xi}_{\beta,t}\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\,\mathrm{d}t\right] \,\middle|\, \mathcal{G}_t\right] \\ &= \mathbf{E}\left[\int_t^{t+\tau} e^{-r(s-t)}Y_s \,\middle|\, \mathcal{G}_t\right]. \end{aligned}$$

Taking the limit as $\tau \to +\infty$ and applying the transversality condition (52), we get

$J = V$, where V is the agent's continuation transfer:

$$V_t := \mathbf{E}\left[\int_t^\infty e^{-r(s-t)} Y_s \,\middle|\, \mathcal{G}_t \right].$$

As in the proof of Lemma B.2, for any stationary linear contract—incentive compatible or not—and an arbitrary strategy A of the agent, we have that

$$\mathrm{d}V_t = [rV_t - Y_t]\,\mathrm{d}t + \sum_{k=1}^{K} \left(\xi_\beta \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + C_k \right) [\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\,\mathrm{d}t],$$

with $C_k := \int_{\tau \geq 0} e^{-r\tau} u_k(\tau)\,\mathrm{d}\tau$. That $J = V$ implies $\widehat{C}_k = C_k$, and thus by Lemma B.1, the contract is incentive compatible. ■

The Principal's Problem. The problem for the principal is to choose a contract (A, Y) such that two conditions are satisfied:

1. The process Y maximizes

$$\mathbf{E}\left[\int_0^\infty e^{-\rho t} H_t \,\mathrm{d}t \,\middle|\, \mathcal{G}_0 \right].$$

2. The contract is incentive compatible.

In the remainder of this proof, we consider the following instantaneous payoff for the principal:

$$H_t := c'(A_t) - \phi Y_t (Y_t - \nu_t), \tag{54}$$

where

$$\phi := \frac{\sqrt{\Delta}}{\sqrt{r}(\kappa - 1)(r + \kappa)} > 0, \tag{55}$$

and $\Delta = (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2$, as defined in Section 3.2.

Remarks on the choice of the principal's payoff: In the original setting, the intermediary seeks to maximize the agent's discounted output. In a stationary setting, it is equivalent to maximizing the agent's discounted marginal cost. The marginal cost is the first term in the right-hand side of (54). However, in the original setting, the agent's incentives are driven by the market's belief process. By Proposition 2.6, the market belief process μ satisfies

$$\mu_t = \mathbf{E}[\theta_t \mid \mu_t] = \mathbf{E}[\nu_t \mid \mu_t] = \frac{\mathbf{Cov}[\mu_t, \nu_t]}{\mathbf{Var}[\mu_t]}\mu_t,$$

using the law of iterated expectations and the projection formula for jointly normal random variables. Thus $\mathbf{Cov}[\mu_t, \nu_t] = \mathbf{Var}[\mu_t]$. To make the principal's payoff in the auxiliary setting and the intermediary's objective of the original setting comparable, we include a penalty term $\phi\mu_t(\nu_t - \mu_t)$ in the principal's payoff. Note that $\mathbf{E}\left[Y_t(\nu_t - Y_t)\right] = \mathbf{Cov}[Y_t, \nu_t] - \mathbf{Var}[Y_t]$. As a Lagrangian multiplier, the parameter ϕ captures the tradeoff between the maximization of the agent's marginal cost and the penalty term, so as to constrain the transfer to be close to a market belief. Its specific value (given in (55)) is picked using the conjectured optimal rating derived in the first part of the proof.

The principal's problem is an optimal control problem with two natural state variables: the agent's estimate of his ability, ν, and the agent's continuation transfer J. The state ν appears explicitly in the principal's payoff. Recall that ν can be expressed in closed form, namely,

$$\nu_t = \frac{\gamma^2}{1+\kappa}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\int_{s\le t} e^{-\kappa(t-s)}\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$

Thus, for $t \geq 0$, the state variable ν is determined by its initial value,

$$\nu_0 = \frac{\gamma^2}{1+\kappa}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\int_{s\le 0} e^{\kappa s}\,\mathrm{d}S_{k,s},$$

and the equation of evolution of ν,

$$\mathrm{d}\nu_t = -\kappa\nu_t\,\mathrm{d}t + \frac{\gamma^2}{1+\kappa}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$

The other state J does not appear explicitly in the principal's payoff, but must be controlled to ensure that the transversality conditions are satisfied—by the lemmas B.2 and B.3, these transversality conditions are necessary and sufficient to ensure that the contract is incentive compatible.

The principal's problem can then be restated as follows: the principal seeks to find a stationary linear contract (A, Y), along with processes $\widehat{C}_k$, $k = 1, \ldots, K$ such that, for all t, the principal maximizes

$$\mathbf{E}\left[\int_t^{\infty} \rho e^{-\rho(s-t)}(c'(A_t) - \phi Y_t(Y_t - \nu_t))\,\mathrm{d}s \,\middle|\, \mathcal{G}_t\right]$$

subject to:

1. Incentive compatibility: $c'(A_t) = \widehat{\xi}_\alpha$, where we recall that $\widehat{\xi}_\alpha = \sum_k \alpha_k \widehat{C}_k$.
2. The evolution of the agent's belief ν, given by

$$\mathrm{d}\nu_t = -\kappa\nu_t\,\mathrm{d}t + \frac{\gamma^2}{1+\kappa}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$

3. The evolution of the agent's continuation transfer J, given by

$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\widehat{\xi}_{\beta,t}\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],$$

   where we recall that $\widehat{\xi}_\beta = \sum_k \beta_k \widehat{C}_k$.

4. The transversality conditions, given by

$$\lim_{\tau\to+\infty}\mathbf{E}[e^{-\rho\tau}J_{t+\tau} \mid \mathcal{G}_t] = 0, \text{ and}$$
$$\lim_{\tau\to+\infty}\mathbf{E}[e^{-\rho\tau}J_{t+\tau}^2 \mid \mathcal{G}_t] = 0.$$

To solve the principal's problem, we use dynamic programming. The principal maximizes

$$\mathbf{E}\left[\int_t^{\infty} \rho e^{-\rho(s-t)}(\xi_{\alpha,t} - \phi Y_t(Y_t - \nu_t))\,\mathrm{d}s \,\middle|\, \mathcal{G}_t\right]$$

for every t, subject to the evolution of the state variables ν and J, and the transversality conditions on J. Without the restriction to stationary linear transfer processes, the dynamic programming problem is standard. We solve the principal's problem without imposing that restriction, and verify *ex post* that the optimal transfer in this relaxed problem is indeed stationary linear.

Assume the principal's value function V as a function of the two states J and ν is $\mathcal{C}^2(\mathbf{R}^2)$. By standard arguments, an application of Itô's Lemma yields the

Hamilton-Jacobi-Bellman (HJB) equation for V:

$$\begin{aligned}\rho V = \sup_{y,c_1,\ldots,c_K} & \rho\widehat{\xi}_\alpha - \rho\phi y(y-\nu) + (rJ-y)V_J - \nu_t V_\nu + \gamma^2\frac{\kappa-1}{\kappa+1}V_{\nu\nu} \\ & + \frac{(\kappa+r)\gamma^2\widehat{\xi}_\beta}{(1+\kappa)(1+r)}V_{\nu J} + \sum_k\left(\widehat{\xi}_\beta\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k} + \sigma_k c_k\right)^2 V_{JJ},\end{aligned} \tag{56}$$

where to shorten notation we have used the subscript notation for the (partial) derivatives of V, and have abused notation by using $\widehat{\xi}_\alpha$ and $\widehat{\xi}_\beta$ to denote $\sum_k \alpha_k c_k$ and $\sum_k \beta_k c_k$, respectively.

We conjecture a quadratic value function V of the form

$$V(J,\nu) = a_0 + a_1 J + a_2\nu + a_3 J\nu + a_4 J^2 + a_5\nu^2. \tag{57}$$

Using the general form of the conjectured value function (57), we can solve for $y, c_1, \ldots, c_K$ using the first-order condition. We can then plug these variables expressed as a function of the coefficients a_i's back into (56), which allows to uniquely identify the coefficients.[53] We obtain

$$\begin{aligned}a_0 &= -\frac{m_{\alpha\beta}^2(\kappa-1)(1+2r+\kappa)}{4m_\beta(\kappa+r)^2(2r-\rho)\phi} + \frac{m_{\alpha\beta}(\kappa-1)}{2m_\beta(\kappa+r)} + \frac{(\kappa-1)^2\phi}{2m_\beta(2+\rho)} + \frac{m_\alpha}{4(2r-\rho)\phi}, \\ a_1 &= 0, \\ a_2 &= 0, \\ a_3 &= \frac{(2r-\rho)\rho\phi}{1+r}, \\ a_4 &= -(2r-\rho)\rho\phi, \\ a_5 &= \frac{\rho(1-r+\rho)^2\phi}{4(1+r)^2(2+\rho)}.\end{aligned}$$

It is readily verified that the second-order condition is equivalent to $a_5 < 0$, and so it is satisfied for all $\rho < r$.

[53] The details of the identification are lengthy and omitted.

After simplification, we obtain the following expressions for y and c_k:

$$y(J,\nu) = (2r-\rho)J + \frac{1-r+\rho}{2(1+r)}\nu, \text{ and} \tag{58}$$
$$c_k(J,\nu) = \frac{\alpha_k}{2\sigma_k^2(2r-\rho)\phi} - \frac{\beta_k(\kappa-1)\left(m_{\alpha\beta}(1+2r+\kappa)-(r+\kappa)(2r-\rho)\phi\right)}{2\sigma_k^2 m_\beta(r+\kappa)^2(2r-\rho)\phi}.$$

Thus, we obtain that the optimal processes $\widehat{C}_k$ are constant, and we obtain the optimal transfer at time t, Y_t as a linear function of the state variables J_t, ν_t:

$$Y_t = \begin{bmatrix} 2r-\rho \\ \frac{1-r+\rho}{2(1+r)} \end{bmatrix} \cdot \begin{bmatrix} J_t \\ \nu_t \end{bmatrix}. \tag{59}$$

We insert the expression of the optimal control Y_t back into the equations that determine the evolution of the state variables. Doing so yields a linear two-dimensional stochastic differential equation for the state variables, namely

$$\mathrm{d}\begin{bmatrix} J_t \\ \nu_t \end{bmatrix} = M \begin{bmatrix} J_t \\ \nu_t \end{bmatrix} + \sum_{k=1}^{K} \begin{bmatrix} \widehat{\xi}_{\beta,t}\dfrac{\gamma^2}{(1+\kappa)(1+r)}\dfrac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t} \\ \dfrac{\kappa-1}{m_\beta}\dfrac{\beta_k}{\sigma_k^2} \end{bmatrix} \left[\mathrm{d}S_{k,t} - \alpha_k A_t \, \mathrm{d}t\right],$$

where

$$M = \begin{bmatrix} -r+\rho & -\widehat{\xi}_\beta \dfrac{\kappa+r}{1+r} - \dfrac{1-r+\rho}{2(1+r)} \\ 0 & -\kappa \end{bmatrix}.$$

The matrix M has two eigenvalues, $-(r-\rho)$ and $-\kappa$, which are generically distinct, and negative for $\rho < r$. We can write

$$\begin{bmatrix} J_t \\ \nu_t \end{bmatrix} = \sum_{k=1}^{K} \int_{s\le t} \left(\mathbf{f}_k e^{-(r-\rho)(t-s)} + \mathbf{g}_k e^{-\kappa(t-s)}\right) \left[\mathrm{d}S_{k,t} - \alpha_k A_t \, \mathrm{d}t\right],$$

where $\mathbf{f}_k$ and $\mathbf{g}_k$ are two-dimensional vectors that can be expressed in closed form as:

$$\mathbf{f}_k = \begin{bmatrix} \frac{m_\beta(r+\kappa)(r-\kappa-\rho)}{2m_\beta(r+\kappa)(2r-\rho)(r-\kappa-\rho)\phi}\frac{\alpha_k}{\sigma_k^2} + \frac{m_{\alpha\beta}(\kappa-1)(1+\kappa+\rho)}{2m_\beta(r+\kappa)(2r-\rho)(r-\kappa-\rho)\phi}\frac{\beta_k}{\sigma_k^2} \\ 0 \end{bmatrix}, \text{ and}$$

$$\mathbf{g}_k = \begin{bmatrix} -\frac{(\kappa-1)\left(m_{\alpha\beta}(1+r)^2-(r+\kappa)(2r-\rho)(r-\kappa-\rho)\phi\right)}{2m_\beta(1+r)(r+\kappa)(2r-\rho)(r-\kappa-\rho)\phi}\frac{\beta_k}{\sigma_k^2} \\ \frac{\kappa-1}{m_\beta}\frac{\beta_k}{\sigma_k^2} \end{bmatrix}.$$

Moreover, when we plug the expressions of the state variables into (58), we get a stationary linear transfer process

$$Y_t = \sum_k \int_{s\leq t} u_k(t-s)\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right],$$

with linear filter

$$u_k(\tau) = F_k e^{-(r-\rho)\tau} + G_k e^{-\kappa\tau},$$

with

$$F_k = \frac{m_\beta(r+\kappa)(r-\kappa-\rho)}{2m_\beta(r+\kappa)(r-\kappa-\rho)\phi}\frac{\alpha_k}{\sigma_k^2} + \frac{m_{\alpha\beta}(\kappa-1)(1+\kappa+\rho)}{2m_\beta(r+\kappa)(r-\kappa-\rho)\phi}\frac{\beta_k}{\sigma_k^2}, \text{ and}$$
$$G_k = \frac{(\kappa-1)\left(m_{\alpha\beta}(1+r)+(\kappa+r)(\kappa-r+\rho)\phi\right)}{2m_\beta(r+\kappa)(\kappa-r+\rho)\phi}\frac{\beta_k}{\sigma_k^2}.$$

The equilibrium action for the agent is stationary and given by

$$c'(A_t) = \frac{\Delta + m_{\alpha\beta}(\kappa-1)(r+\kappa)(2r-\rho)\phi}{2m_\beta(r+\kappa)^2(2r-\rho)\phi}.$$

Thus, the contract (A, Y) just defined is an optimal stationary linear contract for the principal.

Note that, as $\rho \to 0$,

$$c'(A_t) \to \frac{\kappa-1}{4(\kappa+r)m_\beta}\left(2m_{\alpha\beta} + \sqrt{\Delta/r}\right),$$

and $\{u_k\}_k$ converges to the linear filter associated with the market belief of the conjectured optimal rating in the first part of this proof.

Back to the original model. We now make the connection between the auxiliary model and the original model, and conclude the verification. We prove by contradiction that the candidate rating obtained in the first part of this proof is indeed optimal.

We continue to consider the auxiliary model defined in this section. Let (A^*, Y^*) be the incentive-compatible contract defined by

$$c'(A^*) = \frac{\kappa - 1}{4(\kappa + r)m_\beta}\left(2m_{\alpha\beta} + \sqrt{\Delta/r}\right),$$

and

$$Y_t^* = \frac{(\kappa-1)\left((\kappa-1)m_{\alpha\beta}(r+1)\sqrt{r} + \sqrt{\Delta}(\kappa - r)\right)}{2\sqrt{\Delta}m_\beta(\kappa - r)} \cdot \sum_{k=1}^{K}\int_{s\le t} u_k^c(t-s)\left[\mathrm{d}S_{k,s} - \alpha_k A_s^*\,\mathrm{d}s\right].$$

Note that Y_t^* is defined as the market belief of the conjectured optimal rating of the original setting, obtained in Section B.1.1, while A^* is the corresponding equilibrium action. Consider an information structure $\widehat{\mathcal{F}}$, generated by some rating process, that induces a stationary action $\widehat{A}$. Let $\widehat{Y} := \mathbf{E}[\theta_t \mid \widehat{\mathcal{F}}_t]$. Note that $(\widehat{A}, \widehat{Y})$ is a well-defined incentive-compatible stationary linear contract. We show that $c'(A^*) \ge c'(\widehat{A})$. Let $(A^{(\rho)}, Y^{(\rho)})$ be the optimal incentive-compatible stationary linear contract defined above, as a function of the discount rate of the principal ρ. Let $V^{(\rho)}$ be the corresponding principal's expected payoff.

Note that, for every confidential exclusive information structure $\mathcal{F}$ generated by a rating process, the equilibrium market belief of the original setting, $\mu_t = \mathbf{E}[\theta_t \mid \mathcal{F}_t]$, satisfies $\mathbf{Cov}[\mu_t, \nu_t] = \mathbf{Var}[\mu_t]$, and thus the principal's expected payoff for contract (A^*, Y^*) is $V^* := c'(A^*)/\rho$, while the principal's expected payoff for contract $(\widehat{A}, \widehat{Y})$ is $\widehat{V} := c'(\widehat{A})/\rho$.

Then, for every $\rho \in (0, r)$, the inequalities $\rho V^{(\rho)} \ge \rho\widehat{V} = c'(\widehat{A})$ must hold. However, as $\rho \to 0$, $c'(A^{(\rho)}) \to c'(A^*)$, and the linear filter of $Y^{(\rho)}$ converges pointwise to the linear filter of Y^*. Thus, $\mathbf{Cov}[Y^{(\rho)}, \nu_t] - \mathbf{Var}[Y^{(\rho)}] \to 0$, which in turn implies that $\rho V^{(\rho)} \to c'(A^*)$. Hence, $c'(A^*) \ge c'(\widehat{A})$.

B.2 Proof of Theorem 3.2

We prove Theorem 3.2. As in the proof of Theorem 3.1, we proceed in two parts. In the first part, we compute a candidate optimal rating using calculus of variations, while in the second part, we verify the optimality of the candidate using an auxiliary principal-agent model.

B.2.1 Part I: First-Order Conditions

Recall the shorthand notation of Section B.1.1 that will be used throughout this proof as well:

$$\begin{aligned} U(t) &:= \sum_{k=1}^{K} \beta_k u_k(t), \\ V(t) &:= \sum_{k=1}^{K} \alpha_k u_k(t), \\ U_0 &:= \int_0^\infty U(t) e^{-t} \, \mathrm{d}t, \\ V_0 &:= \int_0^\infty V(t) e^{-rt} \, \mathrm{d}t. \end{aligned}$$

We want to maximize $c'(A)$, with A the stationary equilibrium action of the agent, among all *public* information structures generated by some rating process Y that satisfies the variance normalization $\mathbf{Var}[Y_t] = 1$ and that is proportional to the market belief. Such rating processes can be described by their linear filter $\mathbf{u} := \{u_k\}_k$, and are written as

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \left[\mathrm{d}S_{k,s} - \alpha_k A_s \, \mathrm{d}s \right].$$

As in Section B.1.1, we note that, by Itô's isometry, for $\tau \ge 0$,

$$\begin{aligned} \mathbf{Cov}[Y_t, Y_{t+\tau}] = &\sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s) u_k(s+\tau) \, \mathrm{d}s \\ &+ \sum_{k=1}^{K} \sum_{k'=1}^{K} \int_{i \le t} \int_{j \le t+\tau} \beta_k \beta_{k'} u_k(t-i) u_{k'}(t+\tau-j) \, \mathbf{Cov}[\theta_i, \theta_j] \, \mathrm{d}j \, \mathrm{d}i. \end{aligned}$$

Hence, as $\mathbf{Cov}[\theta_i, \theta_j] = \gamma^2 e^{-|i-j|}/2$, after a change of variables in the last term, we get

$$\mathbf{Cov}[Y_t, Y_{t+\tau}] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s) u_k(s+\tau)\,\mathrm{d}s + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j) e^{-|j+\tau-i|}\,\mathrm{d}i\,\mathrm{d}j.$$

By Lemma 2.8, the rating process Y is proportional to the belief of a public information structure if, and only if, $\mathbf{Cov}[Y_t, Y_{t+\tau}] = e^{-\tau}$ for every $\tau \geq 0$.

Using the expression for $\mathbf{Cov}[Y_t, Y_{t+\tau}]$ just obtained, and applying Lemma 2.12, our optimization problem is thus that of maximizing

$$\frac{\gamma^2}{2} \left[\int_0^\infty U(t) e^{-t}\,\mathrm{d}t \right] \left[\int_0^\infty V(t) e^{-rt}\,\mathrm{d}t \right],$$

subject to the continuum of constraints

$$\sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(j) u_k(j+\tau)\,\mathrm{d}j + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j) e^{-|j+\tau-i|}\,\mathrm{d}i\,\mathrm{d}j = e^{-\tau},$$

for every $\tau \geq 0$.

The continuum of constraints makes it difficult to solve this optimization problem directly by forming the Lagrangian, as we have done in the proof of Theorem 3.1. Instead, we solve a relaxed optimization problem with a single constraint: we maximize $F(\mathbf{u})$, defined as

$$F(\mathbf{u}) = \left[\int_0^\infty U(t) e^{-t}\,\mathrm{d}t \right] \left[\int_0^\infty V(t) e^{-rt}\,\mathrm{d}t \right],$$

(as before, the original objective without the constant factor $\gamma^2/2$), subject to $G(\mathbf{u}) = \frac{2}{1+r}$, where

$$G(\mathbf{u}) \coloneqq g(\mathbf{u}, 0) + (1-r) \int_0^\infty e^{-r\tau} g(\mathbf{u}, \tau)\,\mathrm{d}\tau,$$

with

$$g(\mathbf{u}, \tau) \coloneqq \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(j) u_k(j+\tau)\,\mathrm{d}j + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j) e^{-|j+\tau-i|}\,\mathrm{d}i\,\mathrm{d}j.$$

Assume there exists a solution $\mathbf{u}^*$ to this optimization problem, where $\mathbf{u}^*$ is twice differentiable, integrable, and square-integrable. As will be shown, the solution of this relaxed constrained problem satisfies the original continuum of constraints.

As in the confidential setting, we work with an unconstrained problem that internalizes the above constraint. Thus, we relax the problem a second time, and we let

$$L(\mathbf{u}, \lambda_0) = F(\mathbf{u}) + \lambda_0 G(\mathbf{u})$$

be the Lagrangian, from which we remove the additive terms that do not depend on $\mathbf{u}$. Assume there exists some $\lambda_0^* < 0$ such that $\mathbf{u}^*$ maximizes $\mathbf{u} \mapsto L(\mathbf{u}, \lambda_0^*)$.[54] We apply Proposition D.1 to get first-order conditions: if $\lambda_0 = \lambda_0^*$ and $\mathbf{u} = \mathbf{u}^*$, then for all $k = 1, \ldots, K$ and all t, $L_k(t) = 0$, with

$$L_k(t) \coloneqq F_k(t) + \lambda_0 G_k(t),$$

$$F_k(t) \coloneqq \alpha_k U_0 e^{-rt} + \beta_k V_0 e^{-t},$$

and

$$\begin{aligned} G_k(t) \coloneqq\ & 2\sigma_k^2 u_k(t) + \gamma^2 \beta_k \int_0^\infty U(j) e^{-|j-t|}\,\mathrm{d}j \\ & + (1-r)\sigma_k^2 \int_0^\infty e^{-r\tau} \left[u_k(t+\tau) + u_k(t-\tau)\right] \mathrm{d}\tau \\ & + (1-r)\frac{\gamma^2\beta_k}{2} \int_0^\infty e^{-r\tau} \int_0^\infty U(j) e^{-|j+\tau-t|}\,\mathrm{d}j\,\mathrm{d}\tau \\ & + (1-r)\frac{\gamma^2\beta_k}{2} \int_0^\infty e^{-r\tau} \int_0^\infty U(i) e^{-|t+\tau-i|}\,\mathrm{d}i\,\mathrm{d}\tau. \end{aligned}$$

Throughout the proof, any function h defined on the nonnegative real line is extended to the entire real line with the convention that these functions assign value zero to any negative input. By convention, the derivative of h at 0 is defined to be the right-derivative of h at 0, which is well-defined for h twice differentiable. Let some function $h : \mathbf{R}_+ \to \mathbf{R}$ be twice differentiable and such that h, h', h'' are all integrable. Throughout the proof, to compute derivatives of integral functions, we use the following arguments.

First, if $H(t) = \int_0^\infty h(i) e^{-|t+\tau-i|}\,\mathrm{d}i$ for some $\tau \geq 0$, then

$$H(t) = \int_0^{t+\tau} h(i) e^{-(t+\tau-i)}\,\mathrm{d}i + \int_{t+\tau}^\infty h(i) e^{t+\tau-i}\,\mathrm{d}i,$$

so that

$$H''(t) = H(t) - 2h(t+\tau).$$

[54] As in the confidential setting, it is easily seen that $\lambda_0^* < 0$ is a necessary second-order condition.

Similarly, if instead $H(t) = \int_0^\infty h(j)e^{-|j+\tau-t|}\,\mathrm{d}j$ then if $t > \tau$,

$$H(t) = \int_0^{t-\tau} h(j)e^{(j+\tau-t)}\,\mathrm{d}j + \int_{t-\tau}^{\infty} h(j)e^{-(j+\tau-t)}\,\mathrm{d}j,$$

and for every t,

$$H''(t) = H(t) - 2h(t-\tau).$$

Finally, if $H(t) = \int_0^\infty e^{-r\tau}\left[h(t+\tau) + h(t-\tau)\right]\mathrm{d}\tau$, then

$$H'(t) = e^{-rt}h(0) + \int_0^\infty e^{-r\tau}\left[h'(t+\tau) + h'(t-\tau)\right]\mathrm{d}\tau,$$

and

$$H''(t) = -re^{-rt}h(0) + e^{-rt}h'(0) + \int_0^\infty e^{-r\tau}\left[h''(t+\tau) + h''(t-\tau)\right]\mathrm{d}\tau.$$

We can now compute $p_k \coloneqq L_k - L_k''$ as

$$\begin{aligned}
p_k(t) = L_k(t) - L_k''(t) = \alpha_k U_0(1-r^2)e^{-rt} \\
&+ 2\lambda_0\sigma_k^2[u_k(t) - u_k''(t)] + 2\lambda_0\gamma^2\beta_k U(t) \\
&+ \lambda_0(1-r)\sigma_k^2 \int_0^\infty e^{-r\tau}[u_k(t+\tau) + u_k(t-\tau)]\,\mathrm{d}\tau \\
&- \lambda_0(1-r)\sigma_k^2 \int_0^\infty e^{-r\tau}[u_k''(t+\tau) + u_k''(t-\tau)]\,\mathrm{d}\tau \\
&- \lambda_0(1-r)\sigma_k^2\left[-re^{-rt}u_k(0) + u_k'(0)e^{-rt}\right] \\
&+ \lambda_0(1-r)\gamma^2\beta_k \int_0^\infty e^{-r\tau}U(t-\tau)\,\mathrm{d}\tau \\
&+ \lambda_0(1-r)\gamma^2\beta_k \int_0^\infty e^{-r\tau}U(t+\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Next, we let

$$\begin{aligned}
J_k(t) &= \int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right]\mathrm{d}\tau, \text{ and} \\
J(t) &= \sum_{k=1}^{K} \beta_k J_k(t).
\end{aligned}$$

We observe that $J_k'' = -2ru_k + r^2 J_k$. Plugging J_k in the expression for p_k:

$$\begin{aligned} p_k(t) = \alpha_k U_0(1-r^2)e^{-rt} + 2\lambda_0\sigma_k^2\left[u_k(t) - u_k''(t)\right] + 2\lambda_0\gamma^2\beta_k U(t) \\ + 2r\lambda_0(1-r)\sigma_k^2 u_k(t) + \lambda_0(1-r)(1-r^2)\sigma_k^2 J_k(t) + \lambda_0(1-r)\gamma^2\beta_k J(t). \end{aligned}$$

After differentiation, we get

$$\begin{aligned} p_k''(t) = r^2\alpha U_0(1-r^2)e^{-rt} + 2\lambda_0\sigma_k^2\left[u_k''(t) - u_k''''(t)\right] + 2\lambda_0\gamma^2\beta U''(t) \\ + 2r\lambda_0(1-r)\sigma_k^2 u_k''(t) + \lambda_0(1-r)(1-r^2)\sigma_k^2\left[-2ru_k(t) + r^2 J_k(t)\right] \\ + \lambda_0(1-r)\gamma^2\beta_k\left[-2rU(t) + r^2 J(t)\right]. \end{aligned}$$

Finally, we let $q_k \coloneqq p_k'' - r^2 p_k$. We have

$$\begin{aligned} q_k(t) = 2\lambda_0\sigma_k^2\left[u_k''(t) - u_k''''(t)\right] - r^2 2\lambda_0\sigma_k^2\left[u_k(t) - u_k''(t)\right] \\ + 2\lambda_0\gamma^2\beta_k U''(t) - 2r^2\lambda_0\gamma^2\beta_k U(t) \\ + 2r\lambda_0(1-r)\sigma_k^2 u_k''(t) - 2r^3\lambda_0(1-r)\sigma_k^2 u_k(t) \\ - 2r\lambda_0(1-r)(1-r^2)\sigma_k^2 u_k(t) \\ - 2r\lambda_0(1-r)\gamma^2\beta_k U(t). \end{aligned}$$

We must have $q_k(t) = 0$ for all k and all t. In particular, and since $\lambda_0 \neq 0$,

$$\frac{1}{2\lambda_0}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}q_k(t) = 0,$$

hence

$$\begin{aligned} U'' - U'''' - r^2(U - U'') + \gamma^2 m_\beta U'' - r^2\gamma^2 m_\beta U \\ + r(1-r)U'' - r(1-r)U - r(1-r)\gamma^2 m_\beta U = 0. \end{aligned}$$

The characteristic polynomial associated with this homogeneous linear differential equation has roots $\pm\sqrt{1+\gamma^2 m_\beta} = \pm\kappa$ and $\pm\sqrt{r}$. As we have assumed that the solution to the optimization problem is admissible, it follows that U must be bounded, and we discard the positive roots. Thus, U must have the form

$$U(t) = C_1 e^{-\sqrt{r}t} + C_2 e^{-\kappa t}, \tag{60}$$

for some constants C_1 and C_2.

Next, pick an arbitrary pair (i,j) with $i \neq j$, and define $\zeta_{ij}(t) \coloneqq \beta_i\sigma_j^2 u_j(t) -$

$\beta_j \sigma_i^2 u_i(t)$. That $(\beta_i q_j(t) - \beta_j q_i(t))/(2\lambda_0) = 0$ yields, after simplification, the following differential equation for ζ_{ij}:

$$\zeta_{ij}'' - \zeta_{ij}'''' - r^2(\zeta_{ij} - \zeta_{ij}'') + r(1-r)(\zeta_{ij}'' - \zeta_{ij}) = 0.$$

The characteristic polynomial associated with this homogeneous linear differential equation has roots ± 1 and $\pm\sqrt{r}$. As ζ_{ij} must be bounded, we get that ζ_{ij} has the form

$$\zeta_{ij}(t) = C_1' e^{-\sqrt{r}t} + C_2' e^{-t}, \tag{61}$$

for some constants C_1' and C_2'.

Putting together (60) and (61), we get that

$$u_k(t) = D_{1,k} e^{-\sqrt{r}t} + D_{2,k} e^{-\kappa t} + D_{3,k} e^{-t}, \tag{62}$$

for some constants $D_{1,k}$, $D_{2,k}$, $D_{3,k}$.

Determination of the constants. As in the proof of Theorem 3.1, we have established that the solution belongs to a family of functions that are sums of some given scaled time exponentials. We now solve for the constant factors $D_{1,k}$, $D_{2,k}$, $D_{3,k}$, $k \geq 1$.

We first note that, since the term e^{-t} vanishes in Equation (60) that gives the general form of the function U, the equality

$$\sum_{k=1}^{K} \beta_k D_{3,k} = 0 \tag{63}$$

obtains.

Using (63), we plug (62) in the equation for $L_k(t)$ and get that

$$L_k(t) = L_{1,k} e^{-rt} + L_{2,k} e^{-\kappa t} + L_{3,k} e^{-t},$$

where $L_{1,k}$, $L_{2,k}$ and $L_{3,k}$ are scalar factors that will be expressed as a function of the primitives of the model and the constants $D_{1,k}$, $D_{2,k}$, $D_{3,k}$. Note that the exponential $e^{-\sqrt{r}t}$, which appears in the general form of $u_k(t)$ given in (62), vanishes after simplification, while instead an exponential e^{-rt} appears, which is not present in $u_k(t)$.

We observe that

$$L_{2,k} = \frac{2\sigma_k^2 \lambda_0 (r-\kappa^2)}{(r-\kappa)(\kappa+r)} D_{2,k} + \frac{2\gamma^2 \lambda_0 \beta_k \left(r-\kappa^2\right)}{(\kappa-1)(\kappa+1)(\kappa-r)(\kappa+r)} \sum_{i=1}^{K} \beta_i D_{2,i}$$
$$= \frac{2\lambda_0 \sigma_k^2 \left(r-\kappa^2\right) D_{2,k}}{(r-\kappa)(\kappa+r)} + \frac{2\lambda_0 \beta_k \left(\kappa^2 - r\right)}{m_\beta (r-\kappa)(\kappa+r)} \sum_{i=1}^{K} \beta_i D_{2,i},$$

using that $\gamma^2 = (\kappa^2-1)/m_\beta$. That $L_{2,k}=0$ for all k implies

$$D_{2,k} = a \frac{\beta_k}{\sigma_k^2}, \tag{64}$$

for some constant a. It can be seen that if $a=0$, then $D_{1,k} = D_{2,k} = D_{3,k} = 0$ for all k, in which case $u_k = 0$ and the variance normalization constraint is violated. Hence, in the remainder of the proof, we assume $a \neq 0$. (As it turns out, as ratings yield the same market belief up to a scalar, the precise value of a is irrelevant, as long as it is non-zero.) In particular,

$$\sum_{k=1}^{K} \alpha_k D_{2,k} = a m_{\alpha\beta},$$

and

$$\sum_{k=1}^{K} \beta_k D_{2,k} = a m_\beta.$$

Using (63), (64), and $\gamma^2 = (\kappa^2-1)/m_\beta$, we get

$$\begin{aligned} L_{3,k} = & \frac{\left(\kappa^2-1\right)\lambda_0 \beta_k}{\left(\sqrt{r}-1\right)(r+1) m_\beta} \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\beta_k}{r+\sqrt{r}} \sum_{i=1}^{K} \alpha_i D_{1,i} \\ & + \frac{\beta_k}{r+1} \sum_{i=1}^{K} \alpha_i D_{3,i} + \frac{2\lambda_0 \sigma_k^2}{r+1} D_{3,k} + \frac{a \beta_k m_{\alpha\beta}}{\kappa+r} + \frac{a(\kappa+1)\lambda_0 \beta_k}{r+1}. \end{aligned} \tag{65}$$

As $L_{3,k}=0$ for all k, we can multiply (65) by β_k/σ_k^2, sum over k, and use (63) to get that $D_{3,k}=0$. In addition, after plugging $D_{3,k}=0$, the term $L_{3,k}$ simplifies to

$$\frac{\left(\kappa^2-1\right)\lambda_0 \beta_k}{\left(\sqrt{r}-1\right)(r+1) m_\beta} \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\beta_k}{r+\sqrt{r}} \sum_{i=1}^{K} \alpha_i D_{1,i} + \frac{a \beta_k m_{\alpha\beta}}{\kappa+r} + \frac{a(\kappa+1)\lambda_0 \beta_k}{r+1} = 0, \tag{66}$$

which we will use to determine λ_0.

Finally, given $D_{2,k} = a\beta_k/\sigma_k^2$ and $D_{3,k} = 0$, and using that $\gamma^2 = (\kappa^2 - 1)/m_\beta$, the remaining constant $L_{1,k}$ simplifies to

$$\begin{aligned} L_{1,k} = &\left(\frac{\alpha_k}{\sqrt{r}+1} - \frac{(\kappa^2-1)\,\lambda_0\beta_k}{(\sqrt{r}-1)\,\sqrt{r}(r+1)m_\beta} \right) \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\lambda_0\,(\sqrt{r}+1)\,\sigma_k^2}{\sqrt{r}} D_{1,k} \\ &+ \frac{a\alpha_k m_\beta}{\kappa+1} + \frac{a\lambda_0\beta_k(\kappa+r)}{r+1}. \end{aligned} \tag{67}$$

As $L_{1,k} = 0$ must hold for every k, we multiply (67) by β_k/σ_k^2, sum over k, and we get an equation that the term $\sum_i \beta_i D_{1,i}$ must satisfy:

$$\begin{aligned} &\left(\frac{m_{\alpha\beta}}{\sqrt{r}+1} - \frac{(\kappa^2-1)\,\lambda_0 m_\beta}{(\sqrt{r}-1)\,\sqrt{r}(r+1)m_\beta} \right) \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\lambda_0\,(\sqrt{r}+1)}{\sqrt{r}} \sum_{i=1}^{K} \beta_i D_{1,i} \\ &\qquad + \frac{a m_{\alpha\beta} m_\beta}{\kappa+1} + \frac{a\lambda_0 m_\beta(\kappa+r)}{r+1} = 0. \end{aligned} \tag{68}$$

As $m_{\alpha\beta} \geq 0$ and $m_\beta > 0$,

$$\frac{a m_{\alpha\beta} m_\beta}{\kappa+1} + \frac{a\lambda_0 m_\beta(\kappa+r)}{r+1} \neq 0, \tag{69}$$

which implies that the factor of $\sum_i \beta_i D_{1,i}$ is non-zero. Similarly, if we multiply (67) by α_k/σ_k^2 and sum over k, we get an equation that the term $\sum_i \alpha_i D_{1,i}$ must satisfy:

$$\begin{aligned} &\left(\frac{m_{\alpha}}{\sqrt{r}+1} - \frac{(\kappa^2-1)\,\lambda_0 m_{\alpha\beta}}{(\sqrt{r}-1)\,\sqrt{r}(r+1)m_\beta} \right) \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\lambda_0\,(\sqrt{r}+1)}{\sqrt{r}} \sum_{i=1}^{K} \alpha_i D_{1,i} \\ &\qquad + \frac{a m_{\alpha} m_\beta}{\kappa+1} + \frac{a\lambda_0 m_{\alpha\beta}(\kappa+r)}{r+1} = 0. \end{aligned} \tag{70}$$

Now we can solve for $\sum_i \alpha_i D_{1,i}$ and $\sum_i \beta_i D_{1,i}$, using (68) and (70). Plugging in the solutions in (66), we get a rational expression in λ_0, whose denominator is

$$\begin{aligned} &(\kappa+1)(r+1)^2 \left(\sqrt{r}-1\right) \sqrt{r} \left(\sqrt{r}+1\right)^2 m_{\alpha\beta}(\kappa+r) \\ &\qquad + (\kappa+1)\lambda_0(r+1) \left(\sqrt{r}+1\right)^3 (r-\kappa)(\kappa+r)^2, \end{aligned}$$

and whose numerator is

$$
\begin{aligned}
- a(r+1)^2 \left(\sqrt{r}-\kappa\right) \left(m_\alpha m_\beta (\kappa+r)^2 - (\kappa+1) m_{\alpha\beta}^2 (\kappa+2r-1)\right) \\
+ 4a(\kappa+1)\lambda_0 (r+1)\sqrt{r} \left(\sqrt{r}+1\right)^2 m_{\alpha\beta} \left(\sqrt{r}-\kappa\right)(\kappa+r) \\
+ a(\kappa+1)^2 \lambda_0^2 \left(\sqrt{r}+1\right)^4 \left(\sqrt{r}-\kappa\right)(\kappa+r)^2.
\end{aligned}
$$

We observe that the numerator, which must equal zero, yields a quadratic equation in λ_0,

$$
a\left(\sqrt{r}-\kappa\right)\left(A\lambda_0^2 + B\lambda_0 + C\right) = 0, \tag{71}
$$

where:

$$
\begin{aligned}
A &= (\kappa+1)^2 (\kappa+r)^2 \left(\sqrt{r}+1\right)^4, \\
B &= 4(\kappa+1)(\kappa+r)\left(\sqrt{r}+1\right)^2 \sqrt{r}(r+1) m_{\alpha\beta}, \\
C &= -(r+1)^2 \left(m_\alpha m_\beta (\kappa+r)^2 - (\kappa+1) m_{\alpha\beta}^2 (\kappa+2r-1)\right), \\
&= -(r+1)^2 \left((\kappa+r)^2 \left(m_\alpha m_\beta - m_{\alpha\beta}^2\right) + (1-r)^2 m_{\alpha\beta}^2\right).
\end{aligned}
$$

Next, we have that $A > 0$, and also that $C < 0$, which owes to the Cauchy-Schwarz inequality $m_\alpha m_\beta \geq m_{\alpha\beta}^2$ and to $\kappa > 1$. Hence, there are two real roots of (71), one negative, and one positive. As $B > 0$ and we have established that $\lambda_0 < 0$, it follows that

$$
\lambda_0 = \frac{-B + \sqrt{B^2 - 4AC}}{2A},
$$

which, after simplification, reduces to

$$
\lambda_0 = -\frac{(r+1)\left(\sqrt{\Delta} + 2\sqrt{r} m_{\alpha\beta}\right)}{(\kappa+1)\left(\sqrt{r}+1\right)^2 (\kappa+r)},
$$

with $\Delta = (r+\kappa)^2 (m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2$.

Finally, (67) and (68) yield a linear equation that determines $D_{1,k}$:

$$
\begin{aligned}
D_{1,k} = &- \frac{a\sqrt{r}(r+1) m_\beta \left(\sqrt{r}-\kappa\right)(\kappa+r)}{(\kappa+1)\left(\left(r^2-1\right)\sqrt{r} m_{\alpha\beta} + \lambda_0 \left(\sqrt{r}+1\right)^2 \left(r^2-\kappa^2\right)\right)} \frac{\alpha_k}{\sigma_k^2} \\
&- \frac{a(r+1)\sqrt{r} m_{\alpha\beta} \left(\kappa + r - \sqrt{r} - 1\right) + a\lambda_0 \left(r - \sqrt{r}\right)\left(\sqrt{r}+1\right)^2 (\kappa+r)}{\left(\left(r^2-1\right)\sqrt{r} m_{\alpha\beta} + \lambda_0 \left(\sqrt{r}+1\right)^2 \left(r^2-\kappa^2\right)\right)} \frac{\beta_k}{\sigma_k^2}.
\end{aligned}
$$

Letting $\lambda = (\kappa - 1)\sqrt{r}(1 + r)m_{\alpha\beta} + (\kappa - r)\sqrt{\Delta}$, we can make further simplifications, and express the solution in a form similar to that of the confidential case. We have

$$u_k(t) = ad_k \frac{\sqrt{r}}{\lambda} e^{-\sqrt{r}t} + a\frac{\beta_k}{\sigma_k^2} e^{-\kappa t},$$

with

$$d_k = \frac{\kappa - \sqrt{r}}{\kappa - r} c_k + \lambda \frac{\sqrt{r} - 1}{\kappa - r} \frac{\beta_k}{\sigma_k^2},$$

and, as in the confidential setting,

$$c_k = (\kappa^2 - r^2)m_\beta \frac{\alpha_k}{\sigma_k^2} + (1 - \kappa^2)m_{\alpha\beta} \frac{\beta_k}{\sigma_k^2}. \tag{72}$$

As in the case of confidential ratings, because a rating process induces the same effort level up to scaling, all constant multipliers a yield the same effort level. We can use, for example, $a = 1$ in the preceding expressions.

If

$$a = -\frac{(\kappa - 1)\left((\kappa - 1)m_{\alpha\beta}(r + 1)\sqrt{r} + \sqrt{\Delta}(\kappa - r)\right)}{\sqrt{\Delta} m_\beta \left(\sqrt{r} + 1\right)\left(\sqrt{r} - \kappa\right)},$$

then it can be verified that the corresponding rating process satisfies the conditions of Lemma A.4, so is a market belief for a public information structure.

B.2.2 Part II: Verification

As for the case of confidential ratings described in Section B.1.2, we verify that the candidate rating of Section B.2.1 is optimal among all ratings. We use the same auxiliary setting described in Section B.1.2, with the same variables and notation, except for the principal's instantaneous payoff function H.

To define the principal's payoff, we introduce an additional state variable, Λ, with initial value $\Lambda_0 = 0$, and which evolves according to

$$\mathrm{d}\Lambda_t = -r\Lambda_t \, \mathrm{d}t + Y_t \, \mathrm{d}t. \tag{73}$$

Instead of using H as in equation (54), we let

$$H_t = c'(A_t) - \phi_1 Y_t (Y_t - \nu_t) - \phi_2 Y_t \left(\frac{Y_t}{1 + r} - \Lambda_t \right),$$

where

$$\phi_1 := \frac{2\sqrt{\Delta}}{(1+r)(\kappa-1)(\kappa+r)},$$
$$\phi_2 := \frac{\sqrt{\Delta}(r-1)}{(\kappa-1)(\kappa+r)},$$

and Δ is as defined in Section 3.2: $\Delta = (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2$.

Compared to the case of the confidential exclusive setting described in Section B.1.2, we now require two penalty terms to ensure that the principal's payoff (in the auxiliary setting) and the intermediary's objective (in the original setting) are comparable. As in the confidential exclusive case, the term $\phi_1 Y_t(Y_t - \nu_t)$ can be interpreted as a Lagrangian penalty term that guarantees that the optimal transfer for the principal is close to a market belief (in the sense of the original setting). The second term, $\phi_2 Y_t\left(\frac{Y_t}{1+r} - \Lambda_t\right)$, is new. It captures the public constraint: together with the first term, it ensures that the transfer for the principal is close to a market belief derived from a *public information structure*. Indeed, recall that any public market belief μ satisfies $\mathbf{Cov}[\mu_t, \mu_{t+\tau}] = \mathbf{Var}[\mu_t]e^{-\tau}$, by Lemma 2.8. If

$$\Lambda_t = \int_0^t e^{-r(t-s)}\mu_s \, \mathrm{d}s,$$

it is immediate that Λ satisfies (73) for $Y = \mu$ and, as

$$\mathbf{E}\left[\mu_t \Lambda_t\right] = \int_0^t e^{-r(t-s)} \, \mathbf{Cov}\left[\mu_s, \mu_t\right] \mathrm{d}s = \frac{\mathbf{Var}\left[\mu_t\right]}{1+r}\left(1 - e^{-(1+r)t}\right).$$

Thus,

$$\mathbf{E}\left[\mu_t\left(\frac{\mu_t}{1+r} - \Lambda_t\right)\right] = \frac{\mathbf{Var}\left[\mu_t\right]}{1+r}e^{-(1+r)t}.$$

As opposed to the first penalty term, this expectation does not vanish for finite values of t, because $\Lambda_0 = 0$ (more generally, as long as Λ_0 is set independently of the contract, the above expectation cannot be zero for every market belief). However, it converges exponentially to zero as t grows, and this turns out to be sufficient for our purposes. The specific values for ϕ_1 and ϕ_2 are carefully selected using the conjectured optimal rating derived from the Euler-Lagrange necessary conditions in Section B.2.1.

The principal's problem is then an optimal control problem with three natural state variables: the agent's estimate of his ability, ν, the state associated with the public constraint, Λ, and the agent's continuation transfer, J. We have the following equations for the evolution of the state variables:

$$\begin{aligned}
\mathrm{d}\nu_t &= -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\\
\mathrm{d}J_t &= (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\frac{\xi_\beta}{m_\beta}\frac{\kappa-1}{1+r}\frac{\beta_k}{\sigma_k^2} + C_k\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],\\
\mathrm{d}\Lambda_t &= -r\Lambda_t\,\mathrm{d}t + Y_t\,\mathrm{d}t,
\end{aligned}$$

with $\xi_\beta := \sum_k \beta_k C_k$ and $C_k := \int_{\tau\geq 0} e^{-r\tau}u_k(\tau)\,\mathrm{d}\tau$.

As in the confidential exclusive setting considered in Section B.1, the principal's problem can be restated as follows: the principal seeks to find a stationary linear contract (A, Y), along with processes $\widehat{C}_k$, $k = 1, \ldots, K$, such that, for all t, the principal maximizes

$$\mathbf{E}\left[\int_t^\infty \rho e^{-\rho(s-t)}\left(c'(A_s) - \phi_1 Y_s(Y_s - \nu_s) - \phi_2 Y_s\left(\frac{Y_s}{1+r} - \Lambda_s\right)\right)\mathrm{d}s \,\middle|\, \mathcal{G}_t\right]$$

subject to:

1. Incentive compatibility: $c'(A_t) = \widehat{\xi}_\alpha$, where $\widehat{\xi}_\alpha := \sum_k \alpha_k \widehat{C}_k$.

2. The evolution of the agent's belief ν, given by

$$\mathrm{d}\nu_t = -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$

3. The evolution of the state Λ, given by

$$\mathrm{d}\Lambda_t = -r\Lambda_t\,\mathrm{d}t + Y_t\,\mathrm{d}t.$$

4. The evolution of the agent's continuation transfer J, given by

$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\widehat{\xi}_{\beta,t}\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],$$

where $\widehat{\xi}_\beta := \sum_k \beta_k \widehat{C}_k$.

5. The transversality conditions, given by

$$\lim_{\tau \to +\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau} \mid \mathcal{G}_t] = 0, \text{ and}$$
$$\lim_{\tau \to +\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau}^2 \mid \mathcal{G}_t] = 0.$$

We use dynamic programming to solve the principal's problem. The principal maximizes

$$\mathbf{E}\left[\int_t^{\infty} \rho e^{-\rho(s-t)} \left(\widehat{\xi}_{\alpha,s} - \phi_1 Y_s (Y_s - \nu_s) - \phi_2 Y_s \left(\frac{Y_s}{1+r} - \Lambda_s\right)\right) \mathrm{d}s \,\middle|\, \mathcal{G}_t\right],$$

for every t, subject to the evolution of the different state variables and the transversality conditions. As before, we solve the principal's problem without imposing the restriction that transfer processes be stationary linear, and verify *ex post* that the optimal transfer in this relaxed problem is indeed stationary linear.

Assume the principal's value function V, as a function of the three states J, ν and Λ, is $\mathcal{C}^2(\mathbf{R}^3)$. By standard arguments, an application of Itô's Lemma yields the Hamilton-Jacobi-Bellman (HJB) equation for V:

$$\begin{aligned} \rho V = \sup_{y, c_1, \ldots, c_K} & \rho \widehat{\xi}_\alpha - \rho \phi_1 y (y - \nu) - \rho \phi_2 y \left(\frac{y}{1+r} - \Lambda\right) \\ & - \nu V_\nu + (rJ - y) V_J + (-r\Lambda + y) V_\Lambda \\ & + \frac{\widehat{\xi}_\beta}{m_\beta} \frac{(\kappa - 1)(\kappa + r)}{1+r} V_{\nu J} + \frac{(\kappa-1)^2}{2 m_\beta} V_{\nu\nu} \\ & + \frac{1}{2} \sum_{k=1}^{K} \left(\frac{\widehat{\xi}_\beta}{m_\beta} \frac{\kappa - 1}{1+r} \frac{\beta_k}{\sigma_k} + \sigma_k c_k\right)^2 V_{JJ}, \end{aligned} \tag{74}$$

where, as before, to shorten notation, we have used the subscript notation for the (partial) derivatives of V, and have abused notation by using $\widehat{\xi}_\alpha$ and $\widehat{\xi}_\beta$ to denote $\sum_k \alpha_k c_k$ and $\sum_k \beta_k c_k$, respectively.

We conjecture a quadratic value function V of the form

$$\begin{aligned} V(J, \nu, \Lambda) = & a_0 + a_1 \nu + a_2 J + a_3 \Lambda \\ & + a_4 \nu J + a_5 \nu \Lambda + a_6 J \Lambda \\ & + a_7 \nu^2 + a_8 J^2 + a_9 \Lambda^2. \end{aligned} \tag{75}$$

We plug (75) into the dynamic programming equation (74) and solve for the optimal control variables $y, c_1, \ldots, c_K$.

The equation is quadratic in $(y, c_1, \ldots, c_K)$. The second-order conditions are

$$\phi_1 + \frac{\phi_2}{1+r} > 0, \text{ and} \tag{76}$$

$$a_8 < 0. \tag{77}$$

That condition (76) is satisfied is immediate by the definition of ϕ_1 and ϕ_2. Assuming momentarily that (77) holds, the first-order conditions yield as maximizers

$$\begin{aligned} y(J,\Lambda,\nu) &= \frac{(a_6 - 2a_8)\,(r+1)}{2\rho\left((r+1)\phi_1 + \phi_2\right)} J + \frac{(r+1)\left(-a_6 + 2a_9 + \rho\phi_2\right)}{2\rho\left((r+1)\phi_1 + \phi_2\right)} \Lambda \\ &\quad + \frac{(r+1)\left(-a_4 + a_5 + \rho\phi_1\right)}{2\rho\left((r+1)\phi_1 + \phi_2\right)} \nu + \frac{(a_3 - a_2)\,(r+1)}{2\rho\left((r+1)\phi_1 + \phi_2\right)}, \end{aligned} \tag{78}$$

$$c_k(J,\Lambda,\nu) = \frac{(\kappa - 1)\left(m_{\alpha\beta}\rho(\kappa + 2r + 1) - a_4(r+1)(\kappa + r)\right)}{2a_8 m_\beta (\kappa + r)^2} \cdot \frac{\beta_k}{\sigma_k^2} - \frac{\rho}{2a_8} \cdot \frac{\alpha_k}{\sigma_k^2}. \tag{79}$$

Note that y is affine in the three state variables, and every c_k is constant.

Define

$$\bar{\rho} = \sqrt{(\rho+2)(\rho+2r)}.$$

We then plug the optimal controls in (74) to identify the coefficients $a_0, \ldots, a_9$. Contrary to the confidential exclusive case, the system is linear-quadratic. There are two sets of coefficients that satisfy the equality (74) and the second-order conditions. However, only one set of coefficients yields a state J that satisfies the transversality condition. Keeping that set of coefficients, we get:

$$a_1 = a_2 = a_3 = 0,$$

$$a_4 = \frac{\sqrt{\Delta}\rho(2r - \rho)(\rho + r + 1)\left(\rho(\rho+2) + (r - 1 - \rho)\bar{\rho}\right)}{(\kappa - 1)(\rho+2)(r-1)r(r+1)^2(\kappa + r)},$$

$$a_5 = \frac{\sqrt{\Delta}\rho^2\left((\rho+2)(r-1-\rho)(\rho+2r) + (\rho - r + 1)(\rho + r + 1)\bar{\rho}\right)}{(\kappa - 1)(\rho+2)(r-1)r(r+1)^2(\kappa + r)},$$

$$a_6 = \frac{\sqrt{\Delta}\rho(2r - \rho)\left(2r^2 - (\rho+2)r + \rho\left(\bar{\rho} - \rho - 1\right)\right)}{4(\kappa - 1)r^2(\kappa + r)},$$

$$a_7 = \frac{2\sqrt{\Delta}\rho(\rho - r + 1)^2(\rho + r + 1)^2\left(\rho - \bar{\rho} + r + 1\right)}{(\kappa - 1)(\rho+2)^2(r-1)^2(r+1)^4(\kappa + r)},$$

$$a_8 = -\frac{\sqrt{\Delta}\rho(2r-\rho)\left(\rho^2+\rho-\rho r+2r(r+1)\right)}{8(\kappa-1)r^2(\kappa+r)} - \frac{\sqrt{\Delta}\rho\bar{\rho}(\rho-2r)^2}{8(\kappa-1)r^2(\kappa+r)},$$

$$a_9 = \frac{\sqrt{\Delta}\rho^3\left(\rho-\bar{\rho}+r+1\right)}{8(\kappa-1)r^2(\kappa+r)}.$$

The expression for the coefficient a_0 is lengthy and does not impact the calculations that follow. Therefore, it is omitted. Note that, if $\rho < r$, the coefficient a_8 is negative, hence (77) is satisfied, and the maximizers are determined by the first-order condition.

After inserting the coefficients $a_1, \ldots, a_9$ into (78) and (79), we obtain the optimal processes $\widehat{C}_k$, which are constant (and whose expression is lengthy and omitted), as well as the optimal transfer at time t, Y_t, as a linear function of the state variables J_t, ν_t, Λ_t:

$$Y_t = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cdot \begin{bmatrix} J_t \\ \Lambda_t \\ \nu_t \end{bmatrix}, \tag{80}$$

with

$$b_1 = \frac{(2r-\rho)\left(\bar{\rho}-\rho+2r\right)}{4r},$$
$$b_2 = \frac{\rho\left(\rho-\bar{\rho}+2r\right)}{4r},$$
$$b_3 = \frac{\left((\rho+1)^2-r^2\right)\left(\bar{\rho}-\rho-2\right)}{(\rho+2)(r-1)(r+1)^2}.$$

We insert the expression of the optimal control Y_t back into the equations that determine the evolution of the state variables. Doing so yields a linear three-dimensional stochastic differential equation for the state variables, namely

$$\mathrm{d}\begin{bmatrix} J_t \\ \Lambda_t \\ \nu_t \end{bmatrix} = M \begin{bmatrix} J_t \\ \Lambda_t \\ \nu_t \end{bmatrix} + \sum_{k=1}^{K} \begin{bmatrix} \widehat{\xi}_{\beta,t} \dfrac{\gamma^2}{(1+\kappa)(1+r)} \dfrac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t} \\ 0 \\ \dfrac{\kappa-1}{m_\beta} \dfrac{\beta_k}{\sigma_k^2} \end{bmatrix} \left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right],$$

where

$$M = \begin{bmatrix} r - b_1 & -b_2 & -\widehat{\xi}_\beta \dfrac{\kappa + r}{1 + r} - b_3 \\ b_1 & -r + b_2 & b_3 \\ 0 & 0 & -\kappa \end{bmatrix}.$$

The matrix M has three eigenvalues,

$$\begin{aligned}
\delta_f &= \frac{1}{4}\left(3\rho - \bar{\rho} - \sqrt{2}\sqrt{\rho(\rho + \bar{\rho} + 1) + 2r^2 - r(\rho + 2\bar{\rho} - 2)} - 2r\right), \\
\delta_g &= \frac{1}{4}\left(3\rho - \bar{\rho} + \sqrt{2}\sqrt{\rho(\rho + \bar{\rho} + 1) + 2r^2 - r(\rho + 2\bar{\rho} - 2)} - 2r\right), \\
\delta_h &= -\kappa.
\end{aligned}$$

Note that, as $\rho \to 0$, it holds that $\delta_f \to -\sqrt{r}$, and $\delta_g \to -r$. Hence, if ρ is close enough to zero (*i.e.*, $\rho < \rho_0$, for some $\rho_0 > 0$), the eigenvalues of the matrix M are all distinct and negative. We can write

$$\begin{bmatrix} J_t \\ \Lambda_t \\ \nu_t \end{bmatrix} = \sum_{k=1}^{K} \int_{s \le t} \left(\mathbf{f}_k e^{\delta_f (t-s)} + \mathbf{g}_k e^{\delta_g (t-s)} + \mathbf{h}_k e^{\delta_h (t-s)}\right) [\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t],$$

where $\mathbf{f}_k, \mathbf{g}_k$ and $\mathbf{h}_k$ are three-dimensional vectors that can be expressed in closed form as a function of the parameters of the model (the expressions for $\rho > 0$ are lengthy and omitted). From (80), we get

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t - s) [\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t],$$

with

$$u_k(\tau) = F_k e^{\delta_f \tau} + G_k e^{\delta_g \tau} + H_k e^{\delta_h \tau},$$

for some constants $F_k, G_k, H_k, k = 1, \ldots, K$ that depend on the parameters of the model, and, in particular, on ρ. As $\rho \to 0$, we can simplify these constants as

$$
\begin{aligned}
F_k \to\ & \frac{(\kappa-1)m_\beta\left(\sqrt{r}-\kappa\right)(\kappa+r)}{m_\beta\left(\sqrt{r}+1\right)\sqrt{\frac{\Delta}{r}}\left(\sqrt{r}-\kappa\right)}\frac{\alpha_k}{\sigma_k^2} \\
& + \frac{(\kappa-1)\left(\sqrt{\Delta}+(\kappa-1)m_{\alpha\beta}\left(\kappa+r-\sqrt{r}+1\right)-\sqrt{\Delta r}\right)}{m_\beta\left(\sqrt{r}+1\right)\sqrt{\frac{\Delta}{r}}\left(\sqrt{r}-\kappa\right)}\frac{\beta_k}{\sigma_k^2}, \\
G_k \to\ & 0, \\
H_k \to\ & -\frac{(\kappa-1)\left((\kappa-1)m_{\alpha\beta}(r+1)\sqrt{r}+\sqrt{\Delta}(\kappa-r)\right)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)\left(\sqrt{r}-\kappa\right)}\frac{\beta_k}{\sigma_k^2}.
\end{aligned}
$$

Also, as $\rho \to 0$,

$$
\widehat{C}_k \to \frac{m_\beta(\kappa+r)^2}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)^2(\kappa+r)}\frac{\alpha_k}{\sigma_k^2} + \frac{(\kappa-1)\left(2\sqrt{\Delta r}-(\kappa-1)m_{\alpha\beta}(\kappa+2r+1)\right)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)^2(\kappa+r)}\frac{\beta_k}{\sigma_k^2}.
$$

Thus,

$$
\widehat{\xi}_\alpha(=c'(A)) \to \frac{(\kappa-1)\left(m_\alpha m_\beta(\kappa+r)^2-(\kappa-1)m_{\alpha\beta}^2(\kappa+2r+1)+2m_{\alpha\beta}\sqrt{\Delta r}\right)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)^2(\kappa+r)},
$$

and so, after simplification,

$$
c'(A_t) \to \frac{(\kappa-1)\left(1-\left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right)^2\right)\left(2m_{\alpha\beta}+\sqrt{\frac{\Delta}{r}}\right)}{4m_\beta(\kappa+r)}.
$$

Back to the original model. We conclude the verification and make the connection between the auxiliary model and the original model. The procedure is analogous to the confidential case explained in the last part of Section B.1.2.

Let (A^*, Y^*) be the incentive-compatible contract defined by

$$
c'(A_t^*) = \frac{(\kappa-1)}{4m_\beta(\kappa+r)}\left(1-\left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right)^2\right)\left(2m_{\alpha\beta}+\sqrt{\frac{\Delta}{r}}\right),
$$

and

$$Y_t^* = -\frac{(\kappa-1)\Big((\kappa-1)m_{\alpha\beta}(r+1)\sqrt{r}+\sqrt{\Delta}(\kappa-r)\Big)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)\left(\sqrt{r}-\kappa\right)} \cdot \sum_{k=1}^{K}\int_{s\le t} u_k^p(t-s)\left[\mathrm{d}S_{k,s}-\alpha_k A_s^*\,\mathrm{d}s\right].$$

Here, Y_t^* is the market belief of the conjectured optimal rating of the original setting, and A_t^* is the conjectured optimal action. Let $\widehat{\mathcal{F}}$ be a public information structure, generated by some rating process, which induces a constant action process $\widehat{A}$. Let $\widehat{Y} := \mathbf{E}[\theta_t \mid \widehat{\mathcal{F}}_t]$, and observe that $(\widehat{A},\widehat{Y})$ is an incentive-compatible stationary linear contract. We show that $c'(A^*) \ge c'(\widehat{A})$. For $\rho < \rho_0$, let $(A^{(\rho)}, Y^{(\rho)})$ be the optimal incentive-compatible stationary linear contract defined above.

Let V^* be the principal's expected payoff under contract (A^*, Y^*), $\widehat{V}$ be her expected payoff under $(\widehat{A},\widehat{Y})$, and $V^{(\rho)}$ be her expected payoff $(A^{(\rho)}, Y^{(\rho)})$.

For every public exclusive information structure $\mathcal{F}$ generated by some rating process, the equilibrium market belief of the original setting, $\mu_t = \mathbf{E}[\theta_t \mid \mathcal{F}_t]$, satisfies $\mathbf{Cov}[\mu_t, \nu_t] = \mathbf{Var}[\mu_t]$, as well as $\mathbf{Cov}[\mu_t, \mu_{t+\tau}] = \mathbf{Var}[\mu_t]e^{-\tau}$ for $\tau > 0$, by Lemma 2.8. Thus, under the contract (μ, A), where A is the equilibrium action, the state variable Λ is expressed as

$$\Lambda_t = \int_0^t e^{-r(t-s)}\mu_s\,\mathrm{d}s,$$

and the principal's payoff is

$$\int_0^\infty e^{-\rho t}\left(c'(A)-\phi_1\mu_t(\mu_t-\nu_t)-\phi_2\mu_t\left(\frac{\mu_t}{1+r}-\Lambda_t\right)\right)\mathrm{d}t = \frac{c'(A_t)}{\rho} - \phi_2\frac{\mathbf{Var}\left[\mu_t\right]}{(1+r)(1+r+\rho)}.$$

Hence, as $\rho \to 0$, $\rho V^* \to c'(A^*)$, and $\rho\widehat{V} \to c'(\widehat{A})$. For every ρ small enough, $V^{(\rho)} \ge \widehat{V}$ must hold, because $(A^{(\rho)}, Y^{(\rho)})$ is optimal. However, as $\rho \to 0$, $c'(A^{(\rho)}) \to c'(A^*)$, and the linear filter of $Y^{(\rho)}$ converges pointwise to the linear filter of Y^*. In particular, $\mathbf{Cov}[Y^{(\rho)}, \nu_t] - \mathbf{Var}[Y^{(\rho)}] \to 0$, and, for every $\tau > 0$, $\mathbf{Cov}[Y_t^{(\rho)}, Y_{t+\tau}^{(\rho)}] - \mathbf{Var}[Y_t^{(\rho)}]e^{-\tau} \to 0$. Together, these two limits imply that, as $\rho \to 0$,

$$\rho V^{(\rho)} - \rho V^* \to 0.$$

Thus, $\rho V^{(\rho)} \to c'(A^*)$, implying that $c'(A^*) \geq c'(\widehat{A})$.

B.3 Proof of Lemma 3.4 and Lemma 3.5

Fix a confidential or a public information structure $\mathcal{F}$.

Given a rating process Y that is proportional to a market belief induced by $\mathcal{F}$ and with linear filter $\{u_k\}_k$, the equilibrium marginal cost is given by

$$c'(A) = \frac{\mathbf{Cov}[Y_t, \theta_t]}{\mathbf{Var}[Y_t]} \sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(\tau) e^{-r\tau}\, \mathrm{d}\tau,$$

and the equilibrium market belief induced by the information structure is

$$\mu_t = \mathbf{E}[\theta_t \mid \mathcal{F}_t] = \mathbf{E}[\theta_t \mid Y_t] = \frac{\mathbf{Cov}[Y_t, \theta_t]}{\mathbf{Var}[Y_t]} Y_t.$$

This follows from the projection formulas for jointly normal random variables, with

$$\mathbf{Cov}[Y_t, \theta_t] = \frac{\gamma^2}{2} \sum_{k=1}^{K} \beta_k \int_0^\infty u_k(\tau) e^{-\tau}\, \mathrm{d}\tau,$$

and

$$\mathbf{Var}[Y_t] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(\tau)^2\, \mathrm{d}\tau + \sum_{k=1}^{K} \sum_{k'=1}^{K} \int_0^\infty \int_0^\infty u_k(s) u_{k'}(s') e^{-|s-s'|}\, \mathrm{d}s\, \mathrm{d}s'.$$

Thus,

$$\mathbf{Var}[\mu_t] = \frac{\mathbf{Cov}[Y_t, \theta_t]^2}{\mathbf{Var}[Y_t]}.$$

The expressions $c'(A^c)$ and $c'(A^p)$ given in the statement of Lemma 3.4, as well as the expressions $\mathbf{Var}[\mu^c]$ and $\mathbf{Var}[\mu^p]$ given in the statement of Lemma 3.5, follow by plugging in the expressions of the linear filters for the optimal rating as described in Theorem 3.1 and Theorem 3.2. The calculations are lengthy and omitted, a detailed proof being available upon request.

Appendix C: Proofs of Section 4

C.1 Proof of Proposition 4.1

1. If Y is a belief for a confidential information structure $\mathcal{F}$, with non-exclusive signals $S_1, \ldots, S_{K_0}$, then $Y_t = \mu_t$, where, by definition, $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{F}_t] = \mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}, \mu_t]$, where the second equality follows from the law of iterated expectations, using that $\mathcal{F}_t$ includes all information about $\{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}$. Conversely, if $Y_t = \mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}, Y_t]$, then Y is the belief μ for the confidential information structure induced by ratings Y and the signals $S_1, \ldots, S_{K_0}$.

2. If Y is a belief for a public information structure structure $\mathcal{F}$, with non-exclusive signals $S_1, \ldots, S_{K_0}$, then $Y_t = \mu_t$, where, by definition, $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{F}_t] = \mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}, \{\mu_s\}_{s \leq t}]$, using that $\mathcal{F}$ is a filtration and includes all information about $\{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}$. Conversely, if $Y_t = \mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}, \{Y_s\}_{s \leq t}]$, then Y is the belief μ for the public information structure that is the filtration generated by Y and the signals $S_1, \ldots, S_{K_0}$.

C.2 Proof of Lemma 4.2

Let Y be a rating process associated with a confidential or public information structure with non-exclusive signals $S_1, \ldots, S_{K_0}$. At a given time t, conditionally on Y_t, the random variable θ_t is independent from all past non-exclusive signals $S_{k,s}$, $s \leq t$. That is, $\mathbf{Cov}[\theta_t, S_{k,s} \mid Y_t] = 0$ for every $k = 1, \ldots, K_0$, as the market belief Y_t carries by assumption all relevant information about θ_t that is already contained in the non-exclusive signals. Hence, for every $k = 1, \ldots, K_0$, every t and every $\tau \geq 0$,

$$\mathbf{Cov}[\theta_{t+\tau}, S_{k,t} \mid Y_{t+\tau}] = 0. \tag{81}$$

By the projection formulas for jointly normal random variables,

$$\mathbf{Cov}[\theta_{t+\tau}, S_{k,t} \mid Y_{t+\tau}] = \mathbf{Cov}[\theta_{t+\tau}, S_{k,t}] - \frac{\mathbf{Cov}[\theta_{t+\tau}, Y_{t+\tau}]\,\mathbf{Cov}[S_{k,t}, Y_{t+\tau}]}{\mathbf{Var}[Y_{t+\tau}]}.$$

By Lemma 2.7, $\mathbf{Cov}[\theta_{t+\tau}, Y_{t+\tau} = \mathbf{Var}[Y_{t+\tau}]$. Thus,

$$\mathbf{Cov}[\theta_{t+\tau}, \mu_{t+\tau}] = \mathbf{Cov}[\theta_{t+\tau}, S_{k,t}]\ \forall k = 1, \ldots, K_0, \forall t, \forall \tau \geq 0. \tag{82}$$

Conversely, suppose Y is a rating process that is a market belief associated a confidential structure and satisfies (82). Reversing the argument above, the equality (81) must hold. As all random variables involved are jointly normal, given any time t, such equality implies that conditionally on Y_t, the random variable θ_t is independent from all past non-exclusive signals $S_{k,s}$, $s \leq t$, $k = 1, \ldots, k_{K_0}$. Hence, by Proposition 2.6,

$$\mathbf{E}^* \left[\theta_t \mid \{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}, Y_t\right] = \mathbf{E}^*[\theta_t \mid Y_t] = Y_t,$$

and it follows from Proposition 4.1 that Y is a market belief for a confidential information structure with non-exclusive signals $S_1, \ldots, S_{K_0}$. If instead Y is a market belief associated a public structure, then

$$\mathbf{E}^* \left[\theta_t \mid \{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}, \{Y_s\}_{s \leq t}\right] = \mathbf{E}^*[\theta_t \mid Y_t] = Y_t,$$

and by Proposition 4.1, Y is a market belief for a public information structure with non-exclusive signals $S_1, \ldots, S_{K_0}$.

C.3 Proof of Theorem 4.3

In this section, we prove Theorem 4.3. As in the two exclusive settings, the proof proceeds in two parts. In the first part, we provide a candidate optimal rating by deriving first-order conditions using a variational argument. In the second part, we verify the optimality of the candidate.

C.3.1 Part I: First-Order Conditions

We continue to use the following shorthand notation:

$$\begin{aligned}
U(t) &:= \sum_{k=1}^{K} \beta_k u_k(t), \\
V(t) &:= \sum_{k=1}^{K} \alpha_k u_k(t), \\
U_0 &:= \int_0^\infty U(t) e^{-t} \, \mathrm{d}t, \\
V_0 &:= \int_0^\infty V(t) e^{-rt} \, \mathrm{d}t.
\end{aligned}$$

We seek to maximize the equilibrium marginal cost $c'(A)$, where A is the stationary equilibrium action of the agent, among all confidential information structures with non-exclusive signals $S_1, \ldots, S_{K_0}$ that are generated by some rating process Y that satisfies the variance normalization $\mathbf{Var}[Y_t] = 1$ and that is proportional to the market belief. Recall that such a rating process takes the form

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \left[\mathrm{d}S_{k,s} - \alpha_k A_s \, \mathrm{d}s \right],$$

where $\mathbf{u} = \{u_k\}_k$ is the associated linear filter.

Proposition C.1, accounting for the variance normalization, yields the constraints that Y must satisfy to be proportional to the market belief: for every non-exclusive signal S_k,

$$\mathbf{Cov}[\theta_t, Y_t] \, \mathbf{Cov}[S_{k,t}, Y_{t+\tau}] = \mathbf{Cov}[S_{k,t}, \theta_{t+\tau}], \qquad \forall t, \forall \tau \ge 0.$$

An equivalent constraint, more convenient to work with, is as follows: for every non-exclusive signal S_k,

$$\mathbf{Cov}[\theta_t, Y_t] \, \mathbf{Cov}[(S_{k,t+\tau} - S_t), Y_{t+\tau}] = \mathbf{Cov}[(S_{k,t+\tau} - S_{k,t}), \theta_{t+\tau}], \qquad \forall t, \forall \tau \ge 0.$$

Lemma 2.12 continues to hold and maximizing the equilibrium marginal cost is maximizing

$$\frac{\gamma^2}{2} \left[\int_0^\infty U(t) e^{-t} \, \mathrm{d}t \right] \left[\int_0^\infty V(t) e^{-rt} \, \mathrm{d}t \right].$$

Recall from the proof for the corresponding result in the confidential exclusive setting that the variance normalization constraint can be expressed in terms of the linear filter as

$$\sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)^2 \, \mathrm{d}s + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i) U(j) e^{-|j-i|} \, \mathrm{d}i \, \mathrm{d}j = 1.$$

As for the constraints associated with the non-exclusive signals, we have:

$$\mathbf{Cov}[\theta_t, Y_t] = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \, \mathbf{Cov}[\theta_t, \theta_s] \, \mathrm{d}s = \frac{\gamma^2}{2} \int_0^\infty U(s) e^{-s} \, \mathrm{d}s,$$

$$
\begin{aligned}
\mathbf{Cov}[(S_{k,t+\tau} - S_{k,t}), \theta_{t+\tau}] &= \int_0^\tau \beta_k \, \mathbf{Cov}[\theta_{t+\tau}, \theta_{t+\tau-s}] \, \mathrm{d}s \\
&= \frac{\beta_k \gamma^2}{2} \int_0^\tau e^{-s} \, \mathrm{d}s = \frac{\beta_k \gamma^2}{2}(1 - e^{-\tau}),
\end{aligned}
$$

and using Itô's isometry,

$$
\begin{aligned}
&\mathbf{Cov}[(S_{k,t+\tau} - S_t), Y_{t+\tau}] \\
&\qquad = \frac{\gamma^2}{2} \left[\int_0^\tau u_k(s) \sigma_k^2 \, \mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_{i=0}^\tau \int_{j=0}^\infty U(j) e^{-|i-j|} \, \mathrm{d}i \, \mathrm{d}j \right].
\end{aligned}
$$

Thus, the constraints that the linear filter $\{u_k\}_k$ must satisfy are:

$$
G(\mathbf{u}) = 1,
$$

and for all $\tau \geq 0$, and all $k = 1, \ldots, K_0$,

$$
H_k(\mathbf{u}, \tau) = \beta_k (1 - e^{-\tau}),
$$

where we define

$$
\begin{aligned}
G(\mathbf{u}) &= \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i) U(j) e^{-|j-i|} \, \mathrm{d}i \, \mathrm{d}j + \sum_{i=1}^K \int_0^\infty \sigma_i^2 u_i(t)^2 \, \mathrm{d}t, \\
H_k(\mathbf{u}, \tau) &= \left[\int_0^\infty U(t) e^{-t} \, \mathrm{d}t \right] \left[\sigma_k^2 \int_0^\tau u_k(s) \, \mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_{i=0}^\tau \int_{j=0}^\infty U(j) e^{-|i-j|} \, \mathrm{d}i \, \mathrm{d}j \right].
\end{aligned}
$$

As in the exclusive public setting, it is difficult to solve the optimization problem directly, given the continuum of constraints. We solve a relaxed optimization problem with $1 + 2K_0$ constraints: one constraint is associated to the variance normalization, to which we append two constraints for every non-exclusive signal. We thus maximize $F(\mathbf{u})$, defined as in the exclusive setting, namely

$$
F(\mathbf{u}) \coloneqq \left[\int_0^\infty U(t) e^{-t} \, \mathrm{d}t \right] \left[\int_0^\infty V(t) e^{-rt} \, \mathrm{d}t \right],
$$

subject to

$$G(\mathbf{u}) = 1,$$
$$\int_0^\infty e^{-r\tau} H_k(\mathbf{u}, \tau)\, \mathrm{d}\tau = \frac{\beta_k}{r(1+r)}, \qquad \forall k = 1, \dots, K_0,$$
$$\int_0^\infty e^{-\hat{\kappa}\tau} H_k(\mathbf{u}, \tau)\, \mathrm{d}\tau = \frac{\beta_k}{\hat{\kappa}(1+\hat{\kappa})}, \qquad \forall k = 1, \dots, K_0.$$

Assume there exists a solution $\mathbf{u}^* = \{u_k^*\}_k$ to the relaxed problem, where $\mathbf{u}^*$ is four times differentiable, integrable, and square-integrable. It will be shown that the solution of this relaxed optimization problem satisfies the original (continuum of) constraints.

To solve the relaxed problem, we consider the Lagrangian

$$\begin{aligned} L\left(\mathbf{u}, \lambda_0, \{\lambda_{r,k}\}_{k \le K_0}, \{\lambda_{\kappa,k}\}_{k \le K_0}\right) \coloneqq{}& F(\mathbf{u}) + \lambda_0 G(\mathbf{u}) \\ &+ \sum_{i=1}^{K_0} \lambda_{r,i} \int_0^\infty e^{-r\tau} H_i(\mathbf{u}, \tau)\, \mathrm{d}\tau \\ &+ \sum_{i=1}^{K_0} \lambda_{\kappa,i} \int_0^\infty e^{-\hat{\kappa}\tau} H_i(\mathbf{u}, \tau)\, \mathrm{d}\tau. \end{aligned}$$

Assume there exist $\lambda_0^*, \lambda_{r,1}^*, \dots, \lambda_{r,K_0}^*, \lambda_{\kappa,1}^*, \dots, \lambda_{\kappa,K_0}^*$ such that $\mathbf{u}^*$ maximizes $\mathbf{u} \mapsto L\left(\mathbf{u}, \lambda_0^*, \{\lambda_{r,k}^*\}_{k \le K_0}, \{\lambda_{\kappa,k}^*\}_{k \le K_0}\right)$. Assume $\lambda_0^* < 0$.[55] We shall define these constants in such a way that there is a unique solution to the unconstrained maximization problem (up to a scalar factor), that, in addition, solves the constraints of the relaxed optimization problem.

In the sequel, we drop the star notation for simplicity. Throughout, let

$$z_k(\tau) = \sigma_k^2 \int_0^\tau u_k(s)\, \mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_{i=0}^\tau \int_{j=0}^\infty U(j) e^{-|i-j|}\, \mathrm{d}i\, \mathrm{d}j.$$

We apply Proposition D.1 to get first-order conditions: for all k and all t, $L_k(t) = 0$, where L_k is defined as follows.

[55] The inequality is a Legendre second-order condition.

If k indexes an exclusive signal, then

$$
\begin{aligned}
L_k(t) \coloneqq{} & U_0\alpha_k e^{-rt} + V_0\beta_k e^{-t} \\
& + \lambda_0 \left(2\sigma_k^2 u_k(t) + \gamma^2\beta_k \int_0^\infty U(j)e^{-|j-t|}\,\mathrm{d}j \right) \\
& + \beta_k e^{-t} \sum_{i=1}^{K_0} \lambda_{r,i} \int_0^\infty e^{-r\tau} z_i(\tau)\,\mathrm{d}\tau \\
& + \beta_k e^{-t} \sum_{i=1}^{K_0} \lambda_{\kappa,i} \int_0^\infty e^{-\hat{\kappa}\tau} z_i(\tau)\,\mathrm{d}\tau \\
& + \frac{U_0\beta_k\gamma^2}{2} \int_0^\infty \int_0^\tau e^{-r\tau} e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}\tau \sum_{i=1}^{K_0} \lambda_{r,i} \\
& + \frac{U_0\beta_k\gamma^2}{2} \int_0^\infty \int_0^\tau e^{-\hat{\kappa}\tau} e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}\tau \sum_{i=1}^{K_0} \lambda_{\kappa,i}.
\end{aligned}
$$

If, instead, k indexes a non-exclusive signal, then we must include two additional terms:

$$
\begin{aligned}
L_k(t) \coloneqq{} & U_0\alpha_k e^{-rt} + V_0\beta_k e^{-t} \\
& + \lambda_0 \left(2\sigma_k^2 u_k(t) + \gamma^2\beta_k \int_0^\infty U(j)e^{-|j-t|}\,\mathrm{d}j \right) \\
& + \beta_k e^{-t} \sum_{i=1}^{K_0} \lambda_{r,i} \int_0^\infty e^{-r\tau} z_i(\tau)\,\mathrm{d}\tau \\
& + \beta_k e^{-t} \sum_{i=1}^{K_0} \lambda_{\kappa,i} \int_0^\infty e^{-\hat{\kappa}\tau} z_i(\tau)\,\mathrm{d}\tau \\
& + \frac{U_0\beta_k\gamma^2}{2} \int_0^\infty \int_0^\tau e^{-r\tau} e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}\tau \sum_{i=1}^{K_0} \lambda_{r,i} \\
& + \frac{U_0\beta_k\gamma^2}{2} \int_0^\infty \int_0^\tau e^{-\hat{\kappa}\tau} e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}\tau \sum_{i=1}^{K_0} \lambda_{\kappa,i} \\
& + \lambda_{r,k} U_0 \sigma_k^2 \int_t^\infty e^{-r\tau}\,\mathrm{d}\tau \\
& + \lambda_{\kappa,k} U_0 \sigma_k^2 \int_t^\infty e^{-\hat{\kappa}\tau}\,\mathrm{d}\tau,
\end{aligned}
$$

where we note that

$$\int_t^\infty e^{-r\tau}\,\mathrm{d}\tau = \frac{e^{-rt}}{r},$$
$$\int_t^\infty e^{-\hat{\kappa}\tau}\,\mathrm{d}\tau = \frac{e^{-\hat{\kappa}t}}{\hat{\kappa}},$$
$$\int_0^\infty \int_0^\tau e^{-r\tau} e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d} = \left[\frac{2e^{-rt}}{r(1-r^2)} - \frac{e^{-t}}{r(1-r)}\right],$$
$$\int_0^\infty \int_0^\tau e^{-\hat{\kappa}\tau} e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d} = \left[\frac{2e^{-\hat{\kappa}t}}{\hat{\kappa}(1-\hat{\kappa}^2)} - \frac{e^{-t}}{\hat{\kappa}(1-\hat{\kappa})}\right].$$

We first obtain conditions on u_k when k indexes an exclusive signal. In the same fashion as in the public exclusive case, we obtain

$$\begin{aligned} L_k(t) - L_k''(t) = {} & \alpha_k U_0 (1-r^2) e^{-rt} \\ & + 2\lambda_0 \sigma_k^2 \left[u_k(t) - u_k''(t)\right] + 2\lambda\gamma^2 \beta_k U(t) \\ & + U_0 \beta_k \frac{2e^{-rt}}{r} \sum_{i=1}^{K_0} \lambda_{r,i} \frac{\beta_i \gamma^2}{2} \\ & + U_0 \beta_k \frac{2e^{-\hat{\kappa}t}}{\hat{\kappa}} \sum_{i=1}^{K_0} \lambda_{\kappa,i} \frac{\beta_i \gamma^2}{2}. \end{aligned} \tag{83}$$

Now, let k denote the index of a non-exclusive signal. We then obtain

$$\begin{aligned} L_k(t) - L_k''(t) = {} & \alpha_k U_0 (1-r^2) e^{-rt} \\ & + 2\lambda_0 \sigma_k^2 \left[u_k(t) - u_k''(t)\right] + 2\lambda_0\gamma^2 \beta_k U(t) \\ & + U_0 \beta_k \frac{2e^{-rt}}{r} \sum_{i=1}^{K_0} \lambda_{r,i} \frac{\beta_i \gamma^2}{2} \\ & + U_0 \beta_k \frac{2e^{-\hat{\kappa}t}}{\hat{\kappa}} \sum_{i=1}^{K_0} \lambda_{\kappa,i} \frac{\beta_i \gamma^2}{2} \\ & + (1-r^2)\lambda_{r,k} U_0 \sigma_k^2 \frac{e^{-rt}}{r} \\ & + (1-\hat{\kappa}^2)\lambda_{\kappa,k} U_0 \sigma_k^2 \frac{e^{-\hat{\kappa}t}}{\hat{\kappa}}. \end{aligned} \tag{84}$$

The equality $L_k - L_k'' = 0$ must hold for every k. Multiplying (83) by $\frac{\beta_k}{2\sigma_k^2}$, and summing over the exclusive signal index k, multiplying (84) by $\frac{\beta_k}{2\sigma_k^2}$ and summing over the non-exclusive signal index k, and aggregating those two summations, we get that

$$2\lambda_0\left[U(t) - U''(t)\right] + 2\lambda_0\gamma^2 m_\beta U(t)$$

is equal to a sum of two exponentials with rates $-r$ and $-\hat{\kappa}$.

As by assumption, $\lambda_0 \neq 0$, the characteristic polynomial associated with this linear differential equation has the familiar roots $\pm\sqrt{1+\gamma^2 m_\beta} = \pm\kappa$. As U is bounded, it implies that U, expressed as a sum of a particular solution to the above ODE and a solution to the homogenous ODE, can be written as the sum of three exponentials:

$$U(t) = C_1 e^{-rt} + C_2 e^{-\kappa t} + C_3 e^{-\hat{\kappa} t}. \tag{85}$$

Plugging back (85) into (83) and (84) and equating to zero, we obtain K additional differential equations, one for every u_k. Accounting for the fact that u_k must remain bounded, the general solution of these equations yields that u_k is the sum of four exponentials, that is,

$$u_k(t) = D_{1,k}e^{-rt} + D_{2,k}e^{-\kappa t} + D_{3,k}e^{-\hat{\kappa} t} + D_{4,k}e^{-t}, \tag{86}$$

for some constants $D_{1,k}, D_{2,k}, D_{3,k}, D_{4,k}$.

Determination of the constants. We plug the general form of u_k obtained in (86) in the expression for $L_k(t)$. We get that, for both exclusive and non-exclusive signal indices k, L_k can be written in the form of a sum of four exponential terms

$$L_k = L_{1,k}e^{-rt} + L_{2,k}e^{-\kappa t} + L_{3,k}e^{-\hat{\kappa} t} + L_{4,k}e^{-t},$$

where, as in the exclusive cases, the constant factors $L_{1,k}$, $L_{2,k}$, $L_{3,k}$, and $L_{4,k}$ depend on the primitives of the model and the constants $D_{1,k}$, $D_{2,k}$, $D_{3,k}$ and $D_{4,k}$. Asserting that $L_k = 0$ is equivalent to $L_{1,k} = L_{2,k} = L_{3,k} = L_{4,k} = 0$.

Let

$$A_i = \sum_{k=1}^{K} \alpha_k D_{i,k}, \qquad B_i = \sum_{k=1}^{K} \beta_k D_{i,k},$$

$$B_r = \sum_{k=1}^{K_0} \beta_k \lambda_{r,k}, \qquad B_\kappa = \sum_{k=1}^{K_0} \beta_k \lambda_{\kappa,k},$$

and let

$$\xi_{r,i} = \int_0^\infty e^{-r\tau} z_i(\tau)\,\mathrm{d}\tau,$$
$$\xi_{\kappa,i} = \int_0^\infty e^{-\kappa\tau} z_i(\tau)\,\mathrm{d}\tau.$$

For a non-exclusive signal index k, we have the following:

$$L_{1,k} = \frac{\left(\left(r^2-1\right)\sigma_k^2\left(2\lambda_0 r D_{1,k} + U_0\lambda_{r,k}\right) + \gamma^2\left(-\beta_k\right)\left(2B_1\lambda_0 r + B_r U_0\right) + r\left(r^2-1\right)U_0\alpha_k\right)}{r\left(r^2-1\right)},$$
$$L_{2,k} = \frac{2\lambda_0\left(\left(\kappa^2-1\right)\sigma_k^2 D_{2,k} - B_2\gamma^2\beta_k\right)}{\kappa^2-1},$$
$$L_{3,k} = \frac{\left(\left(\hat\kappa^2-1\right)\sigma_k^2\left(2\hat\kappa\lambda_0\left(D_{3,k}+D_{4,k}\right) + U_0\lambda_{\kappa,k}\right) - \gamma^2\beta_k\left(2B_3\hat\kappa\lambda_0 + B_\kappa U_0\right)\right)}{\hat\kappa\left(\hat\kappa^2-1\right)},$$
$$L_{4,k} = \frac{1}{2}\beta_k\left(\gamma^2\lambda_0\left(2\left(\frac{B_1}{r-1} + \frac{B_2}{\kappa-1} + \frac{B_3}{\hat\kappa-1}\right) + 2B_4 t + B_4\right)\right),$$
$$+ \frac{1}{2}\beta_k\left(\gamma^2 U_0\left(\frac{B_\kappa}{(\hat\kappa-1)\hat\kappa} + \frac{B_r}{(r-1)r}\right) + 2\sum_{i=1}^{K_0}\lambda_{\kappa,i}\xi_{\kappa,i} + 2\sum_{i=1}^{K_0}\lambda_{r,i}\xi_{r,i} + 2V_0\right).$$

Instead, for an exclusive signal index k, we have the following:

$$L_{1,k} = \frac{\left(2\lambda_0 r\left(r^2-1\right)\sigma_k^2 D_{1,k} + \gamma^2\left(-\beta_k\right)\left(2B_1\lambda_0 r + B_r U_0\right) + r\left(r^2-1\right)U_0\alpha_k\right)}{r\left(r^2-1\right)},$$
$$L_{2,k} = \frac{2\lambda_0\left(\left(\kappa^2-1\right)\sigma_k^2 D_{2,k} - B_2\gamma^2\beta_k\right)}{\kappa^2-1},$$
$$L_{3,k} = \frac{\left(2\hat\kappa\left(\hat\kappa^2-1\right)\lambda_0\sigma_k^2\left(D_{3,k}+D_{4,k}\right) - \gamma^2\beta_k\left(2B_3\hat\kappa\lambda_0 + B_\kappa U_0\right)\right)}{\hat\kappa\left(\hat\kappa^2-1\right)},$$
$$L_{4,k} = \frac{1}{2}\beta_k\left(\gamma^2\lambda_0\left(2\left(\frac{B_1}{r-1} + \frac{B_2}{\kappa-1} + \frac{B_3}{\hat\kappa-1}\right) + 2B_4 t + B_4\right)\right)$$
$$+ \frac{1}{2}\beta_k\left(\gamma^2 U_0\left(\frac{B_\kappa}{(\hat\kappa-1)\hat\kappa} + \frac{B_r}{(r-1)r}\right) + 2\sum_{i=1}^{K_0}\lambda_{\kappa,i}\xi_{\kappa,i} + 2\sum_{i=1}^{K_0}\lambda_{r,i}\xi_{r,i} + 2V_0\right).$$

Similarly, H_k can be written in the form of a sum of four exponential terms and a constant:

$$H_k = H_{1,k}e^{-rt} + H_{2,k}e^{-\kappa t} + H_{3,k}e^{-\hat\kappa t} + H_{4,k}e^{-t} + H_{5,k},$$

where it can be shown that

$$
\begin{aligned}
H_{1,k} &= \frac{U_0\left(B_1\gamma^2\beta_k - \left(r^2-1\right)\sigma_k^2 D_{1,k}\right)}{r\left(r^2-1\right)}, \\
H_{2,k} &= \frac{U_0\left(B_2\gamma^2\beta_k - \left(\kappa^2-1\right)\sigma_k^2 D_{2,k}\right)}{\kappa\left(\kappa^2-1\right)}, \\
H_{3,k} &= \frac{U_0\left(B_3\gamma^2\beta_k - \left(\hat{\kappa}^2-1\right)\sigma_k^2\left(D_{3,k}+D_{4,k}\right)\right)}{\hat{\kappa}\left(\hat{\kappa}^2-1\right)}, \\
H_{4,k} &= \frac{1}{4}\gamma^2 U_0\beta_k\left(-\frac{2B_1}{r-1} - \frac{2B_2}{\kappa-1} - \frac{2B_3}{\hat{\kappa}-1} - B_4(2t+3)\right), \\
H_{5,k} &= \frac{1}{4}U_0\left(\frac{4\sigma_k^2\left(\kappa\hat{\kappa}D_{1,k} + r\left(\kappa\left(D_{3,k}+D_{4,k}\right)+\hat{\kappa}D_{2,k}\right)\right)}{\kappa\hat{\kappa}r}\right) \\
&\quad + \frac{1}{4}U_0\left(\gamma^2\beta_k\left(2\left(\frac{B_1(r+2)}{r(r+1)} + \frac{B_2(\kappa+2)}{\kappa(\kappa+1)} + \frac{B_3(\hat{\kappa}+2)}{\hat{\kappa}(\hat{\kappa}+1)}\right) + 3B_4\right)\right).
\end{aligned}
$$

Using that $L_{2,k} = 0$ for all k, we immediately get $D_{2,k} = a\beta_k/\sigma_k^2$ for some scalar a, and thus $B_2 = am_\beta$. Using $L_{4,k} = 0$ for all k, we get that $D_{4,k}$ is proportional to β_k/σ_k^2. As the term in e^{-t} vanishes in (85), we get $B_4 = 0$, which in turn implies $D_{4,k} = 0$ for all k. Using that $H_{3,k} = 0$ for every non-exclusive signal index k, we infer that $D_{3,k}$ is also proportional to β_k/σ_k^2 for every non-exclusive signal index k. Summing the term $H_{3,k}$ over these indices, and equating to zero, we get

$$
B_3 = \sum_{k=1}^{K_0} \beta_k D_{3,k}.
$$

Next, for an exclusive signal index k, $L_{3,k} = 0$ implies that $D_{3,k}$ is proportional to β_k/σ_k^2 for the exclusive signal indices as well. Further,

$$
\sum_{k=K_0+1}^{K} \beta_k D_{3,k} = 0
$$

implies that $D_{3,k} = 0$ for exclusive signal indices k.

Similarly, $H_{1,k} = 0$ implies that

$$
D_{1,k} = \frac{\beta_k}{\sigma_k^2}\frac{\gamma^2}{\left(r^2-1\right)}B_1.
$$

As $L_{3,k} = 0$, we get

$$B_3 = -\frac{B_\kappa U_0}{2\hat{\kappa}\lambda_0}. \tag{87}$$

Then, plugging this value of B_3 into the equation $L_{3,k} = 0$ for non-exclusive signal indices k, we get that, for some ν,

$$\lambda_{\kappa,k} = \nu\frac{\beta_k}{\sigma_k^2}.$$

Furthermore, $B_\kappa = \nu m_\beta^n$.

Finally, $\sum_{k=1}^{K_0} L_{1,k} = 0$ implies that

$$B_r = -\frac{r\left(r^2-1\right)m_{\alpha\beta}^n}{-\gamma^2 m_\beta^n + r^2 - 1},$$

and plugging B_r back into the equation $L_{1,k} = 0$ gives

$$\lambda_{r,i} = -\frac{\gamma^2 r\beta_k m_{\alpha\beta}^n}{\sigma_k^2\left(r^2-\hat{\kappa}^2\right)} - \frac{r\alpha_k}{\sigma_k^2}.$$

Then, computing $\sum_{k=1}^{K} L_{1,k}$ and equating to zero, we get

$$B_1 = -\frac{\left(r^2-1\right)U_0\left(m_\alpha\left(r^2-\hat{\kappa}^2\right) + m_{\alpha\beta}^n\left(\kappa^2-r^2\right)\right)}{2\lambda_0(r-\kappa)(\kappa+r)\left(r^2-\hat{\kappa}^2\right)},$$

and

$$D_{1,k} = -\frac{U_0\left(\gamma^2\beta_k m_\alpha + \alpha_k\left(r^2-\kappa^2\right)\right)}{2\lambda_0\sigma_k^2\left(r^2-\kappa^2\right)}.$$

Note that we have

$$U_0 = \frac{am_\beta}{\kappa+1} + \frac{B_1}{r+1} + \frac{B_3}{\hat{\kappa}+1},$$
$$V_0 = \frac{am_{\alpha\beta}}{\kappa+r} + \frac{A_1}{2r} + \frac{A_3}{\hat{\kappa}+r}.$$

At this stage, beside a, only two unknown variables remain: λ_0 and ν.

We plug the values of the variables obtained thus far to express the values of $L_{4,k}$, $H_{5,k}$, and G. This yields two quadratic equations in λ_0 and ν, obtained by setting

$L_{4,k}$ to zero and set $H_{5,k}$ equal to G (which is equal to 1).[56]

Solving for the quadratic system of equations, we obtain after simplification

$$\lambda_0 = \frac{\sqrt{r}\left((\kappa+1)m^n_{\alpha\beta}(\kappa+r) - (\hat{\kappa}+1)m_{\alpha\beta}(\hat{\kappa}+r)\right)}{(\kappa+1)(\hat{\kappa}+1)\sqrt{r}(\kappa+r)(\hat{\kappa}+r)} - \frac{\sqrt{\Delta}(\kappa\hat{\kappa}-1)}{(\kappa+1)(\hat{\kappa}+1)\sqrt{r}},$$

and

$$\nu = \frac{\gamma^2\hat{\kappa}}{\hat{\kappa}+1}\left(\frac{(r+1)m^n_{\alpha\beta}}{(r-\hat{\kappa})(\hat{\kappa}+r)} + \frac{\sqrt{\Delta}}{\sqrt{r}}\right),$$

where ν is chosen to be the unique negative root of the quadratic equation that results (the other root is the unique positive root and is associated to a minimum of the objective function), and where

$$\Delta := \frac{(\kappa+1)(\hat{\kappa}+1)}{2(\kappa-\hat{\kappa})}\left[\frac{m^e_\alpha m^e_\beta}{\kappa^2-\hat{\kappa}^2} + \frac{(1+2r+\hat{\kappa})(m^n_{\alpha\beta})^2}{(r+\hat{\kappa})^2(\hat{\kappa}+1)} - \frac{(1+2r+\kappa)m^2_{\alpha\beta}}{(r+\kappa)^2(\kappa+1)}\right].$$

(We prove below that $\Delta \geq 0$, so the square root is well-defined.)

Plugging back these expressions into the variables obtained so far, and defining

$$\widetilde{\lambda} = (\kappa-1)\left(\sqrt{r}(1+r)m_{\alpha\beta} + (\kappa^2-r^2)\sqrt{\Delta}\right),$$

we obtain after simplification, for a given factor a, the following variables.

For $k = 1, \ldots, K_0$:

$$D_{1,k} = -\frac{a\left(\kappa^2-1\right)\sqrt{r}\left(m_{\alpha\beta}\left(r^2-\hat{\kappa}^2\right) + m^n_{\alpha\beta}\left(\kappa^2-r^2\right)\right)}{\widetilde{\lambda}\left(r^2-\hat{\kappa}^2\right)}\frac{\beta_k}{\sigma_k^2},$$

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2},$$

$$D_{3,k} = a\left[\frac{(\kappa+1)}{(\hat{\kappa}+1)} - \frac{\left(\kappa^2-1\right)\sqrt{r}(r+1)\left(m_{\alpha\beta}\left(r^2-\hat{\kappa}^2\right) + m^n_{\alpha\beta}\left(r^2-\kappa^2\right)\right)}{(\hat{\kappa}+1)\widetilde{\lambda}\left(r^2-\hat{\kappa}^2\right)}\right]\frac{\beta_k}{\sigma_k^2},$$

[56] The expressions of $L_{4,k}$, $H_{5,k}$, and G are lengthy and therefore omitted. The details of the derivation and the two equations are available upon request.

and for $k = K_0 + 1, \ldots, K$:

$$\begin{aligned}
D_{1,k} &= -\frac{a\left(\kappa^2 - 1\right) m_{\alpha\beta}\sqrt{r}}{\widetilde{\lambda}}\frac{\beta_k}{\sigma_k^2} - \frac{a\left(\kappa^2 - 1\right)\left(r^2 - \kappa^2\right)\sqrt{r}}{\gamma^2\widetilde{\lambda}}\frac{\alpha_k}{\sigma_k^2},\\
D_{2,k} &= a\frac{\beta_k}{\sigma_k^2},\\
D_{3,k} &= 0.
\end{aligned}$$

It can be verified that the rating process Y defined by such linear filter $\{u_k\}_k$ satisfies the initial set of constraints for every a, except for the normalization constraint; but as rating policies yield the same market belief when multiplied by a non-scalar, the exact value of a does not need to be determined.

To conclude, we show that $\Delta \geq 0$, or equivalently that $\delta \geq 0$, with

$$\delta := \frac{(\kappa+1)(r+\kappa)^2}{\kappa^2 - \hat{\kappa}^2} m_\alpha^e m_\beta^e - (1 + 2r + \kappa) m_{\alpha\beta}^2 + \frac{(\kappa+1)(r+\kappa)^2(1+2r+\hat{\kappa})}{(r+\hat{\kappa})^2(\hat{\kappa}+1)}(m_{\alpha\beta}^n)^2.$$

We can express δ as

$$\delta = \frac{(\kappa+1)(r+\kappa)^2}{\kappa^2 - \hat{\kappa}^2}(m_\alpha^e m_\beta^e - (m_{\alpha\beta}^e)^2) + f((m_{\alpha\beta}^n)^2, (m_{\alpha\beta}^e)^2),$$

where the first term is nonnegative, by the Cauchy-Schwarz inequality, and the second term is the quadratic form

$$f((m_{\alpha\beta}^n)^2, (m_{\alpha\beta}^e)^2) = \begin{pmatrix}(m_{\alpha\beta}^n)^2 & (m_{\alpha\beta}^e)^2\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}(m_{\alpha\beta}^n)^2\\ (m_{\alpha\beta}^e)^2\end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= \frac{(\kappa - \hat{\kappa})\left(2r^3 + (1+\kappa)(1+\hat{\kappa})(\kappa+\hat{\kappa}) + 4r^2(1+\kappa+\hat{\kappa}) + 2r(1+\kappa+\hat{\kappa})^2\right)}{(1+\hat{\kappa})(r+\hat{\kappa})^2},\\
a_{12} &= a_{21} = -(1 + 2r + \kappa),\\
a_{22} &= \frac{r(r + (2+r)\kappa) + (1+2r+\kappa)\hat{\kappa}^2}{\kappa^2 - \hat{\kappa}^2}.
\end{aligned}$$

To prove that this quadratic form is positive semidefinite, we need to check that the three principal minors are nonnegative. Clearly $a_{11} \geq 0$ and $a_{22} \geq 0$, and finally

$$a_{11}a_{22} - a_{12}a_{21} = \frac{2r(1+r)^2(1+\kappa)(r+\kappa)^2}{(1+\hat{\kappa})(r+\hat{\kappa})^2(\kappa+\hat{\kappa})} \geq 0,$$

as required.

C.3.2 Part II: Verification

We now verify that the candidate rating introduced in Section C.3.1 is optimal among all ratings. We continue to use the auxiliary setting described in Part I of Section B.1 in Appendix B, with the same variables and notation. However, we must re-define the principal's instantaneous payoff function, H.

To do so, we first introduce K_0 extra state variables, $\Lambda^r_1, \ldots, \Lambda^r_{K_0}$, with initial value

$$\Lambda^r_{k,0} = \frac{1}{r}\int_{s\leq 0} e^{-r(t-s)}\,\mathrm{d}S_{k,s},$$

and which evolve according to

$$\mathrm{d}\Lambda^r_{k,t} = -r\Lambda^r_{k,t}\,\mathrm{d}t + \frac{1}{r}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right]. \tag{88}$$

We also introduce K_0 additional state variables, $\Lambda^\kappa_1, \ldots, \Lambda^\kappa_{K_0}$, with initial value

$$\Lambda^\kappa_{k,0} = \frac{1}{\hat{\kappa}}\int_{s\leq 0} e^{-\hat{\kappa}(t-s)}\,\mathrm{d}S_{k,s},$$

which evolve as

$$\mathrm{d}\Lambda^\kappa_{k,t} = -\hat{\kappa}\Lambda^\kappa_{k,t}\,\mathrm{d}t + \frac{1}{\hat{\kappa}}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right]. \tag{89}$$

Instead of using H as in equation (54), we re-define H as follows:

$$H_t := c'(A_t) - \phi_1 Y_t(Y_t - \nu_t) + \sum_{k=1}^{K_0}\phi_{2,k}\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)} - Y_t\Lambda^r_{k,t}\right) - \sum_{k=1}^{K_0}\phi_{3,k}\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta\hat{\kappa}(1+\hat{\kappa})} - Y_t\Lambda^\kappa_{k,t}\right),$$

where

$$\phi_1 := \sqrt{\Delta/r},$$
$$\phi_{2,k} := r\frac{\alpha_k}{\sigma_k^2} + \frac{rm_{\alpha\beta}^n(\kappa^2-1)}{m_\beta(r^2-\hat{\kappa}^2)}\frac{\beta_k}{\sigma_k^2},$$
$$\phi_{3,k} := \frac{\hat{\kappa}(\kappa^2-1)\left(m_{\alpha\beta}^n\sqrt{r}(1+r)+\sqrt{\Delta}(r^2-\hat{\kappa}^2)\right)}{m_\beta\sqrt{r}(1+\hat{\kappa})(r^2-\hat{\kappa}^2)}\frac{\beta_k}{\sigma_k^2}.$$

Recall that Δ is defined in the main text, Section 4.

Compared to the case of confidential exclusive ratings, we now include $1+2K_0$ penalty terms. As before, these penalty terms ensure that the principal's payoff in this auxiliary setting and the intermediary's objective in the original setting are comparable. The (by now usual) term $\phi_1 Y_t(Y_t-\nu_t)$ is a penalty term that ensures that the optimal transfer of the principal remains close to a market belief of the original setting. To ensure that the optimal transfer is close to a market belief that incorporates all relevant information of the non-exclusive signals, we add an additional penalty term for every non-exclusive signal S_k:

$$\phi_{2,k}\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)} - Y_t\Lambda_{k,t}^r\right) - \phi_{3,k}\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta\hat{\kappa}(1+\hat{\kappa})} - Y_t\Lambda_{k,t}^\kappa\right).$$

To grasp the intuition behind this term, recall that, by Lemma 4.2, for every non-exclusive signal S_k, any market belief that includes all relevant information about S_k satisfies

$$\mathbf{Cov}\left[\theta_t, S_{k,t-\tau}\right] = \mathbf{Cov}\left[\mu_t, S_{k,t-\tau}\right],\ \forall\tau\geq 0, \forall t,$$

or equivalently,

$$\mathbf{Cov}\left[\theta_t, S_{k,t}-S_{k,t-\tau}\right] = \mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right],\ \forall\tau\geq 0, \forall t. \tag{90}$$

Note that

$$\mathbf{Cov}\left[\theta_t, S_{k,t}-S_{k,t-\tau}\right] = \frac{\beta_k\gamma^2}{2}\left(1-e^{-\tau}\right) = \frac{\beta_k(\kappa^2-1)}{2m_\beta}\left(1-e^{-\tau}\right).$$

Hence,

$$\int_0^\infty e^{-r\tau}\,\mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau = \int_0^\infty e^{-r\tau}\,\mathbf{Cov}\left[\theta_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau$$
$$= \frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)},$$

and

$$\int_0^\infty e^{-\hat{\kappa}\tau}\,\mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau = \int_0^\infty e^{-\hat{\kappa}\tau}\,\mathbf{Cov}\left[\theta_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau$$
$$= \frac{\beta_k(\kappa^2-1)}{2m_\beta \hat{\kappa}(1+\hat{\kappa})}.$$

Next, remark that

$$\begin{aligned}
&\int_0^\infty e^{-r\tau}\,\mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau \\
&\qquad = \mathbf{E}\left[\mu_t \int_0^\infty e^{-r\tau}\left(S_{k,t} - \textstyle\int_{s\le t}\alpha_k A_s\,\mathrm{d}s - S_{k,t-\tau} + \int_{s\le t-\tau}\alpha_k A_s\,\mathrm{d}s\right)\mathrm{d}\tau\right] \\
&\qquad = \mathbf{E}\left[\mu_t \frac{1}{r}\int_{s\le t} e^{-r(t-s)}\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right]\right] \\
&\qquad = \mathbf{E}\left[\mu_t \Lambda^r_{k,t}\right],
\end{aligned}$$

where the second equality is obtained by change of variables and integration by part. Similarly,

$$\int_0^\infty e^{-\hat{\kappa}\tau}\,\mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau = \mathbf{E}\left[\mu_t \Lambda^\kappa_{k,t}\right].$$

So, if μ is a market belief that incorporates the information of non-exclusive signal S_k, we have

$$\mathbf{E}\left[\frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)} - \mu_t\Lambda^r_{k,t}\right] = 0, \quad \text{and} \quad \mathbf{E}\left[\frac{\beta_k(\kappa^2-1)}{2m_\beta \hat{\kappa}(1+\hat{\kappa})} - \mu_t\Lambda^\kappa_{k,t}\right] = 0.$$

If μ was a market belief, but did not incorporate the information of non-exclusive signal S_k, then these expectations would, in general, be non-zero. The factors $\phi_{2,k}$ and $\phi_{3,k}$ are chosen so as to induce the principal to choose, as optimal transfer, a market belief that does incorporate such non-exclusive information.[57]

In the remainder of this proof, we will also use the following notation:

$$\Phi_{2,\alpha} = \sum_{k=1}^{K_0} \alpha_k \phi_{2,k} = r m_\alpha^n + \frac{r(m_{\alpha\beta}^n)^2(\kappa^2-1)}{m_\beta(r^2-\hat{\kappa}^2)},$$
$$\Phi_{2,\beta} = \sum_{k=1}^{K_0} \beta_k \phi_{2,k} = r m_{\alpha\beta}^n + \frac{r m_{\alpha\beta}^n m_\beta^n(\kappa^2-1)}{m_\beta(r^2-\hat{\kappa}^2)},$$
$$\Phi_{3,\alpha} = \sum_{k=1}^{K_0} \alpha_k \phi_{3,k} = \frac{\hat{\kappa}(\kappa^2-1)m_{\alpha\beta}^n\left(m_{\alpha\beta}^n\sqrt{r}(1+r)+\sqrt{\Delta}(r^2-\hat{\kappa}^2)\right)}{m_\beta\sqrt{r}(1+\hat{\kappa})(r^2-\hat{\kappa}^2)},$$
$$\Phi_{3,\beta} = \sum_{k=1}^{K_0} \beta_k \phi_{3,k} = \frac{\hat{\kappa}(\kappa^2-1)m_{\beta}^n\left(m_{\alpha\beta}^n\sqrt{r}(1+r)+\sqrt{\Delta}(r^2-\hat{\kappa}^2)\right)}{m_\beta\sqrt{r}(1+\hat{\kappa})(r^2-\hat{\kappa}^2)}.$$

The principal's problem is then an optimal control problem with the following natural state variables: the agent's estimate of his ability, ν, the agent's continuation transfer J, and the states associated with the non-exclusive constraints. We have the following equations for the evolution of the state variables:

$$\mathrm{d}\nu_t = -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],$$
$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\frac{\xi_\beta}{m_\beta}\frac{\kappa-1}{1+r}\frac{\beta_k}{\sigma_k^2} + C_k\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],$$
$$\mathrm{d}\Lambda_{k,t}^r = -r\Lambda_{k,t}^r\,\mathrm{d}t + \frac{1}{r}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\ \forall k = 1,\ldots,K_0,$$
$$\mathrm{d}\Lambda_{k,t}^\kappa = -\hat{\kappa}\Lambda_{k,t}^\kappa\,\mathrm{d}t + \frac{1}{\hat{\kappa}}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\ \forall k = 1,\ldots,K_0.$$

Recall that, as in the exclusive cases, $\xi_\beta \coloneqq \sum_k \beta_k C_k$ and $C_k \coloneqq \int_{\tau\geq 0} e^{-r\tau}u_k(\tau)\,\mathrm{d}\tau$.

[57] In other words, the factor

$$\phi_{2,k}e^{-r\tau} - \phi_{3,k}e^{-\hat{\kappa}\tau}$$

corresponds to an infinitesimal Lagrangian multiplier associated with the constraint of non-exclusivity

$$\mathbf{Cov}\left[\theta_t, S_{k,t} - S_{k,t-\tau}\right] = \mathbf{Cov}\left[\mu_t, S_{k,t} - S_{k,t-\tau}\right].$$

As in the confidential exclusive setting considered in Section B.1 of Appendix B, the principal's problem can be restated as follows: the principal seeks to find a stationary linear contract (A, Y), along with processes $\widehat{C}_k$, $k = 1, \ldots, K$, such that, for all t, the principal maximizes

$$\mathbf{E}\left[\int_t^{\infty} \rho e^{-\rho(s-t)} H_s \,\mathrm{d}s \;\middle|\; \mathcal{G}_t\right]$$

subject to:

1. Incentive compatibility: $c'(A_t) = \widehat{\xi}_\alpha := \sum_k \alpha_k \widehat{C}_k$.
2. The evolution of the agent's belief ν,
$$\mathrm{d}\nu_t = -\kappa\nu_t \,\mathrm{d}t + \frac{\kappa - 1}{m_\beta} \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} \left[\mathrm{d}S_{k,s} - \alpha_k A_s \,\mathrm{d}s\right].$$
3. The evolution of the agent's continuation transfer J,
$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K} \left(\widehat{\xi}_{\beta,t} \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right) \left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\,\mathrm{d}t\right],$$
with $\widehat{\xi}_\beta := \sum_k \beta_k \widehat{C}_k$.
4. The evolution of the states Λ_k^r and Λ_k^κ, for $k = 1, \ldots, K_0$,
$$\mathrm{d}\Lambda_{k,t}^r = -r\Lambda_{k,t}^r \,\mathrm{d}t + \frac{1}{r}\left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right], \ \forall k = 1, \ldots, K_0,$$
$$\mathrm{d}\Lambda_{k,t}^\kappa = -\hat{\kappa}\Lambda_{k,t}^\kappa \,\mathrm{d}t + \frac{1}{\hat{\kappa}}\left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right], \ \forall k = 1, \ldots, K_0.$$
5. The following transversality conditions
$$\lim_{\tau \to +\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau} \mid \mathcal{G}_t] = 0, \text{ and}$$
$$\lim_{\tau \to +\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau}^2 \mid \mathcal{G}_t] = 0.$$

We use dynamic programming to solve the principal's problem. The principal maximizes the expected value of

$$\int_t^\infty \rho e^{-\rho(s-t)}\Bigg(\widehat{\xi}_{\alpha,s} - \phi_1 Y_t(Y_t-\nu_t) + \sum_{k=1}^{K_0}\phi_{2,k}Y_t\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)} - \Lambda^r_{k,t}\right) \\ - \sum_{k=1}^{K_0}\phi_{3,k}Y_t\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta\hat{\kappa}(1+\hat{\kappa})} - \Lambda^{\hat{\kappa}}_{k,t}\right)\Bigg),$$

conditional on $\mathcal{G}_t$, for every t, subject to the evolution of the different state variables and the transversality conditions. As in the proof of the exclusive cases, we solve the principal's problem without imposing the restriction that transfer processes be stationary linear, and verify that the optimal transfer in this relaxed problem is indeed stationary linear.

Assume the principal's value function V is jointly twice continuously differentiable, as a function of all the state variables. The Hamilton-Jacobi-Bellman (HJB) equation for V reads

$$\begin{aligned}\rho V = \sup_{y,c_1,\ldots,c_K} &\rho\widehat{\xi}_\alpha - \rho\phi_1 y(y-\nu) - \rho y\sum_{k=1}^{K_0}\left(\phi_{2,k}\Lambda^r_{k,t} - \phi_{3,k}\Lambda^\kappa_{k,t}\right) \\ &+ V^J(rJ-y) - V^\nu\nu + \sum_{k=1}^{K_0}V^r_k\left(-r\Lambda^r_k + \frac{\beta_k}{r}\nu\right) + \sum_{k=1}^{K_0}V^\kappa_k\left(-\hat{\kappa}\Lambda^\kappa_k + \frac{\beta_k}{\hat{\kappa}}\nu\right) \\ &+ \frac{V^{JJ}}{2}\left(\frac{\widehat{\xi}^2_\beta}{m_\beta}\frac{(\kappa-1)(1+2r+\kappa)}{(1+r)^2} + \sum_{k=1}^{K}\sigma_k^2c_k^2\right) + \frac{V^{J\nu}\xi_\beta(\kappa-1)(\kappa+r)}{m_\beta(1+r)} \\ &+ \frac{V^{\nu\nu}(\kappa-1)^2}{2m_\beta} + \sum_{k=1}^{K_0}\left(\frac{V^{Jr}_k}{r} + \frac{V^{J\kappa}_k}{\hat{\kappa}}\right)\left(\frac{\xi_\beta(\kappa-1)\beta_k}{m_\beta(1+r)} + \sigma_k^2c_k\right) \\ &+ \sum_{k=1}^{K_0}\left(\frac{V^{\nu r}_k}{r} + \frac{V^{\nu\kappa}_k}{\hat{\kappa}}\right)\frac{(\kappa-1)\beta_k}{m_\beta} + \sum_{k=1}^{K_0}\left(\frac{V^{rr}_{kk}}{2r^2} + \frac{V^{r\kappa}_{kk}}{r\hat{\kappa}} + \frac{V^{\kappa\kappa}_{kk}}{2\hat{\kappa}^2}\right)\sigma_k^2.\end{aligned} \tag{91}$$

To shorten notation, we have used the following superscript/subscript notation for the (partial) derivatives of V. We use superscripts to denote the variables $(\nu, J, \Lambda^r, \Lambda^\kappa)$, and subscripts to denote the index of the variables Λ^r and Λ^κ. For example,

$$V^{JJ} := \frac{\partial^2 V}{\partial J^2},\ V^{Jr}_k := \frac{\partial^2 V}{\partial J\partial\Lambda^r_k},\quad \text{and}\quad V^{r\kappa}_{kj} := \frac{\partial^2 V}{\partial\Lambda^r_k\partial\Lambda^\kappa_j}.$$

We have also abused notation by using $\widehat{\xi}_\alpha$ and $\widehat{\xi}_\beta$ to denote $\sum_k \alpha_k c_k$ and $\sum_k \beta_k c_k$, respectively.

We conjecture a quadratic value function V of the form

$$\begin{aligned}V(J,\nu,\Lambda^r,\Lambda^\kappa) &= a_0 + a^J J + a^\nu \nu + a^{J\nu} J\nu + a^{JJ} J^2 + a^{\nu\nu}\nu^2 \\ &\quad + \sum_{k=1}^{K_0} \left(a_k^r \Lambda_k^r + a_k^{Jr} J \Lambda_k^r + a_k^{\nu r} \nu \Lambda_k^r + a_k^\kappa \Lambda_k^\kappa + a_k^{J\kappa} J \Lambda_k^\kappa + a_k^{\nu\kappa} \nu \Lambda_k^\kappa \right) \\ &\quad + \sum_{k,j=1}^{K_0} a_{kj}^{r\kappa} \Lambda_k^r \Lambda_j^\kappa + \sum_{1\le k\le j\le K_0} \left(a_{kj}^{rr} \Lambda_k^r \Lambda_j^r + a_{kj}^{\kappa\kappa} \Lambda_k^\kappa \Lambda_j^\kappa \right).\end{aligned} \tag{92}$$

After we substitute (92) into the HJB equation (91), we solve for the optimal control variables $y, c_1, \ldots, c_K$. The right-hand side of the resulting equation is a sum of two quadratic functions, one in y, the other in $(c_1, \ldots, c_K)$. These quadratic functions are strictly concave when the following second-order conditions are satisfied:

$$\phi_1 > 0 \quad \text{and} \quad a^{JJ} < 0.$$

It is immediate that the first inequality is satisfied by the definition of ϕ_1. Let us assume momentarily that the second inequality holds. The first-order conditions then yield the value of the optimal control variables, which permits to identify the constant factors of quadratic value function. We get:

$$0 = a^J = a^\nu = a_k^\kappa = a_k^r, \quad \forall k = 1, \ldots, K_0.$$

and

$$\begin{aligned}a^{JJ} &= -\rho(2r-\rho)\phi_1, \\ a^{J\nu} &= \frac{(2r-\rho)\rho\left(2r^2\hat{\kappa}(r+\hat{\kappa})\phi_1 - \hat{\kappa}(r+\hat{\kappa})\Phi_{2,\beta} + 2r^2\Phi_{3,\beta}\right)}{2r^2(1+r)\hat{\kappa}(r+\hat{\kappa})}, \\ a^{\nu\nu} &= \rho(r-1-\rho) \\ &\quad \times \left((r-1-\rho)\phi_1^2 + d_2^{\nu\nu}\phi_1\Phi_{2,\beta} + d_3^{\nu\nu}\phi_1\Phi_{3,\beta} + d_{22}^{\nu\nu}\Phi_{2,\beta}^2 + d_{23}^{\nu\nu}\Phi_{2,\beta}\Phi_{3,\beta} + d_{33}^{\nu\nu}\Phi_{3,\beta}^2\right) \\ &\quad \times \left(4(1+r)^2(2+\rho)\phi_1\right)^{-1},\end{aligned}$$

where

$$
\begin{aligned}
d_2^{\nu\nu} &= \frac{-2r(1+r)+\rho(2+\rho)}{r^2(1+r+\rho)}, \\
d_3^{\nu\nu} &= \frac{2\left(r^2-\hat{\kappa}(1+\rho)-\rho(2+\rho)+r(3+\hat{\kappa}+\rho)\right)}{\hat{\kappa}(r+\hat{\kappa})(1+\hat{\kappa}+\rho)}, \\
d_{22}^{\nu\nu} &= \frac{8r^3(r+1)-4r^2(r+3)\rho+2\left(r+1-2r^2\right)\rho^2+(r+3)\rho^3+\rho^4}{4r^4(r-1-\rho)(r+1+\rho)(2r+\rho)}, \\
d_{23}^{\nu\nu} &= \Big(-\rho(1+\rho)(2+\rho)(\hat{\kappa}+\rho)(\hat{\kappa}+1+\rho)-2r^5-4r^4(\hat{\kappa}+2+\rho) \\
&\quad -2r^3\left(\hat{\kappa}^2+\hat{\kappa}(4+\rho)-\rho^2+\rho+3\right) \\
&\quad +r^2\left(2\hat{\kappa}^2\rho+\hat{\kappa}(-4+\rho(5\rho+6))+\rho(1+\rho)(8+3\rho)\right) \\
&\quad +r\left(\hat{\kappa}^2(2+\rho(4+\rho))+\hat{\kappa}\rho(6+\rho(6+\rho))+\rho(1+\rho)(2+\rho)\right)\Big) \\
&\quad \times\left(\hat{\kappa}r^2(\hat{\kappa}+1+\rho)(\hat{\kappa}+r)(r-1-\rho)(r+1+\rho)(\hat{\kappa}+r+\rho)\right)^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
d_{33}^{\nu\nu} = &\left(2\hat{\kappa}^2(1+\rho)^2+\hat{\kappa}\rho(2+\rho)(2+3\rho)+\rho^2(1+\rho)(2+\rho)+2r^4+4r^3(\hat{\kappa}+1)\right. \\
&\left.+2r^2(\hat{\kappa}(\hat{\kappa}+4)+1)-2r\left(2\hat{\kappa}^2(1+\rho)+\hat{\kappa}(2+\rho(8+3\rho))+\rho(1+\rho)(2+\rho)\right)\right) \\
&\times\left(\hat{\kappa}^2(\hat{\kappa}+1+\rho)(2\hat{\kappa}+\rho)(\hat{\kappa}+r)^2(r-1-\rho)\right)^{-1}.
\end{aligned}
$$

Since $\phi_1 > 0$, we have $a^{JJ} < 0$ as long as $\rho \in (0, 2r)$. Thus, the second-order conditions are satisfied. For $k = 1, \ldots, K_0$,

$$
\begin{aligned}
a_k^{Jr} &= -\frac{\rho(2r-\rho)\phi_{2,k}}{2r}, \\
a_k^{J\kappa} &= \frac{(2r-\rho)\rho\phi_{3,k}}{r+\hat{\kappa}}, \\
a_k^{\nu r} &= \frac{\rho^2\left((r-1-\rho)\phi_1+d_2^{\nu r}\Phi_{2,\beta}+d_3^{\nu r}\Phi_{3,\beta}\right)\phi_{2,k}}{4r(1+r)(1+r+\rho)\phi_1}, \\
a_k^{\nu\kappa} &= \frac{\rho(r-\hat{\kappa}-\rho)\left((r-1-\rho)\phi_1+d_2^{\nu\kappa}\Phi_{2,\beta}+d_3^{\nu\kappa}\Phi_{3,\beta}\right)\phi_{3,k}}{2(1+r)(r+\hat{\kappa})(1+\hat{\kappa}+\rho)\phi_1},
\end{aligned}
$$

where

$$
\begin{aligned}
d_2^{\nu r} &= \frac{-4r^2+\rho+r\rho+\rho^2}{2r^2(2r+\rho)}, \\
d_2^{\nu r} &= \frac{3r^2+r(1+\hat{\kappa})-(1+\rho)(\hat{\kappa}+\rho)}{\hat{\kappa}(r+\hat{\kappa})(r+\hat{\kappa}+\rho)}, \\
d_2^{\nu\kappa} &= \frac{-2r^2-2r\hat{\kappa}+\rho(1+\hat{\kappa}+\rho)}{2r^2(r+\hat{\kappa}+\rho)}, \\
d_2^{\nu\kappa} &= \frac{r^2-\rho(1+\rho)+r(1+3\hat{\kappa}+\rho)-\hat{\kappa}(1+2\rho)}{\hat{\kappa}(r+\hat{\kappa})(2\hat{\kappa}+\rho)}.
\end{aligned}
$$

Finally, for $1 \le k, j \le K_0$,

$$
a_{kj}^{r\kappa} = \frac{(r-\hat{\kappa}-\rho)\rho^2\phi_{2,k}\phi_{3,j}}{4r(r+\hat{\kappa})(r+\hat{\kappa}+\rho)\phi_1},
$$

and for $1 \le k \le j \le K_0$,

$$
\begin{aligned}
a_{kj}^{rr} &= \begin{cases} \frac{\rho^3\phi_{2,k}\phi_{2,j}}{8r^2(2r+\rho)\phi_1} & \text{if } k<j, \\ \frac{\rho^3\phi_{2,k}^2}{16r^2(2r+\rho)\phi_1} & \text{if } k=j, \end{cases} \\
a_{kj}^{\kappa\kappa} &= \begin{cases} \frac{\rho(-r+\hat{\kappa}+\rho)^2\phi_{3,k}\phi_{3,j}}{2(r+\hat{\kappa})^2(2\hat{\kappa}+\rho)\phi_1} & \text{if } k<j, \\ \frac{\rho(-r+\hat{\kappa}+\rho)^2\phi_{3,k}^2}{4(r+\hat{\kappa})^2(2\hat{\kappa}+\rho)\phi_1} & \text{if } k=j. \end{cases}
\end{aligned}
$$

The constant term a_0 is unwieldy and irrelevant for the sequel. Therefore, it is omitted.

In turn, the controls are expressed as follows. If $k > K_0$, *i.e.*, if S_k is an exclusive signal, then

$$
c_k(J,\nu,\Lambda) = \frac{\alpha_k}{2(2r-\rho)\phi_1\sigma_k^2} - \frac{(\kappa-1)\left(m_{\alpha\beta}(1+2r+\kappa)-(r+\kappa)(2r-\rho)\phi_1\right)\beta_k}{2m_\beta(r+\kappa)^2(2r-\rho)\phi_1\sigma_k^2},
$$

while if $k \le K_0$,

$$
c_k(J,\nu,\Lambda) = \frac{\alpha_k}{2(2r-\rho)\phi_1\sigma_k^2} - \frac{(\kappa-1)\left(m_{\alpha\beta}(1+2r+\kappa)-(r+\kappa)(2r-\rho)\phi_1\right)\beta_k}{2m_\beta(r+\kappa)^2(2r-\rho)\phi_1\sigma_k^2} - \frac{\hat{\kappa}(r+\hat{\kappa})\phi_{2,k}-2r^2\phi_{3,k}}{4r^2\hat{\kappa}(r+\hat{\kappa})\phi_1},
$$

and finally,

$$y(J, \nu, \Lambda) = \begin{bmatrix} b^J \\ b^\nu \\ \mathbf{b}^r \\ \mathbf{b}^\kappa \end{bmatrix} \cdot \begin{bmatrix} J \\ \nu \\ \mathbf{\Lambda}^r \\ \mathbf{\Lambda}^\kappa \end{bmatrix},$$

where

$$\begin{aligned} b^J &= 2r - \rho, \\ b^\nu &= \frac{1 - r + \rho}{2 + 2r} + \frac{(2r - \rho)\Phi_{2,\beta}}{4r^2(r+1)\phi_1} - \frac{(2r-\rho)\Phi_{3,\beta}}{2(r+1)\hat{\kappa}(r+\hat{\kappa})}, \\ \mathbf{b}^r &= \frac{\rho}{4r\phi_1}\boldsymbol{\phi}_2, \\ \mathbf{b}^\kappa &= -\frac{(r - \hat{\kappa} - \rho)}{2(r+\hat{\kappa})\phi_1}\boldsymbol{\phi}_3. \end{aligned}$$

In the above equations, $\boldsymbol{\phi}_2 \coloneqq (\phi_{2,1}, \ldots, \phi_{2,K_0})$ and $\boldsymbol{\phi}_3 \coloneqq (\phi_{3,1}, \ldots, \phi_{3,K_0})$.

Plugging these controls back into the equations of evolution of the state variables, we obtain a $(2 + 2K_0)$-dimensional stochastic differential equation:

$$\mathrm{d} \begin{bmatrix} J_t \\ \nu_t \\ \mathbf{\Lambda}_t^r \\ \mathbf{\Lambda}_t^\kappa \end{bmatrix} = M \begin{bmatrix} J_t \\ \nu_t \\ \mathbf{\Lambda}_t^r \\ \mathbf{\Lambda}_t^\kappa \end{bmatrix} + \sum_{k=1}^{K} \begin{bmatrix} \frac{\widehat{\xi}_{\beta,t}(\kappa-1)\beta_k}{m_\beta(1+r)\sigma_k^2} + \widehat{C}_{k,t} \\ \frac{(\kappa-1)\beta_k}{m_\beta \sigma_k^2} \\ \mathbf{e}_k / r \\ \mathbf{e}_k / \hat{\kappa} \end{bmatrix} \left[\mathrm{d}S_{k,t} - \alpha_k A_t \, \mathrm{d}t\right],$$

where

$$M = \begin{bmatrix} -(r-\rho) & \frac{\widehat{\xi}_\beta(\kappa+r)}{1+r} - b^\nu & -(\mathbf{b}^r)^\intercal & -(\mathbf{b}^\kappa)^\intercal \\ 0 & -\kappa & 0 & 0 \\ 0 & 0 & -r\mathbf{I}_{K_0} & 0 \\ 0 & 0 & 0 & -\hat{\kappa}\mathbf{I}_{K_0} \end{bmatrix}.$$

In these equations, $\mathbf{I}_{K_0}$ denotes the $K_0 \times K_0$ identity matrix, and $\mathbf{e}_k \in \mathbf{R}^{K_0}$ is the vector with one in the k-th component and zeros elsewhere. The matrix M has four eigenvalues, $-r, -\hat{\kappa}, -\kappa, -(r-\rho)$, which are distinct and negative for $\rho \in (0, r)$.

Therefore, we can write

$$\begin{bmatrix} J_t \\ \nu_t \\ \mathbf{\Lambda}^r \\ \mathbf{\Lambda}^\kappa \end{bmatrix} = \sum_{k=1}^{K} \int_{s \le t} \left(\mathbf{f}_k^r e^{-r(t-s)} + \mathbf{f}_k^\kappa e^{-\kappa(t-s)} + \mathbf{f}_k^{\hat{\kappa}} e^{-\hat{\kappa}(t-s)} + \mathbf{f}_k^\rho e^{-(r-\rho)(t-s)} \right) \left[\mathrm{d}S_{k,t} - \alpha_k A_t \, \mathrm{d}t \right],$$

for some $(2+2K_0)$-vectors $\mathbf{f}^r, \mathbf{f}^\kappa, \mathbf{f}^{\hat{\kappa}}, \mathbf{f}^\rho$ that can be expressed in closed form as a function of the parameters of the model (as in the public exclusive case, the expressions for $\rho > 0$ are lengthy and thus omitted). It follows that

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \left[\mathrm{d}S_{k,t} - \alpha_k A_t \, \mathrm{d}t \right],$$

with

$$u_k(\tau) = F_k^r e^{-r\tau} + F_k^\kappa e^{-\kappa\tau} + F_k^{\hat{\kappa}} e^{-\hat{\kappa}\tau} + F_k^\rho e^{-(r-\rho)\tau}.$$

In the limit as $\rho \to 0$, each factor converges, and the first and last exponential become a single exponential with rate $-r$ and factor $F^r + F^\rho$. For non-exclusive signals S_k, $k \le K_0$ we have

$$\begin{aligned} F^r + F^\rho &\to -\frac{a(\kappa^2-1)\sqrt{r}\left(m_{\alpha\beta}(r^2-\hat{\kappa}^2) + m^n_{\alpha\beta}(\kappa^2-r^2)\right)}{\tilde{\lambda}(r^2-\hat{\kappa}^2)} \frac{\beta_k}{\sigma_k^2}, \\ F^\kappa &\to a\frac{\beta_k}{\sigma_k^2}, \\ F^{\hat{\kappa}} &\to a\left[\frac{\kappa+1}{\hat{\kappa}+1} - \frac{(\kappa^2-1)\sqrt{r}(r+1)(m_{\alpha\beta}\left(r^2-\hat{\kappa}^2\right) + m^n_{\alpha\beta}(r^2-\kappa^2))}{(\hat{\kappa}+1)\tilde{\lambda}(r^2-\hat{\kappa}^2)}\right] \frac{\beta_k}{\sigma_k^2}, \end{aligned}$$

while for the exclusive signal S_k, $k > K_0$,

$$
\begin{aligned}
F^r + F^\rho &\to -\frac{a(\kappa^2-1)m_{\alpha\beta}\sqrt{r}}{\tilde{\lambda}}\frac{\beta_k}{\sigma_k^2} - \frac{am_\beta(r^2-\kappa^2)\sqrt{r}}{\tilde{\lambda}}\frac{\alpha_k}{\sigma_k^2},\\
F^\kappa &\to a\frac{\beta_k}{\sigma_k^2},\\
F^{\hat{\kappa}} &\to 0,
\end{aligned}
$$

where the scaling factor is

$$
a = -\frac{(\kappa-1)\left(m_{\alpha\beta}(1+r)-(r^2-\kappa^2)\phi_1\right)}{2m_\beta(r^2-\kappa^2)\phi_1}.
$$

The limit of $\widehat{\xi}_\alpha$ as $\rho \to 0$ is

$$
c'(A) = \frac{(\kappa-1)\left(m^n_{\alpha\beta}(\kappa+1)(r+\kappa)+m_{\alpha\beta}(\hat{\kappa}+1)(r+\hat{\kappa})\right)}{2m_\beta(r+\kappa)(1+\hat{\kappa})(r+\hat{\kappa})} + \frac{(\kappa-1)(\kappa-\hat{\kappa})}{2m_\beta(\hat{\kappa}+1)}\sqrt{\frac{\Delta}{r}}.
$$

We observe that, as $\rho \to 0$, the value obtained for $c'(A)$ corresponds to the conjectured optimum of the original model, and the optimal transfer Y corresponds to the conjectured optimal market belief of the optimal rating of the original model, presented at the end of Part I.

Back to the original model. We can now conclude the verification, in a way very similar to the confidential exclusive setting explained in Section B.1.2 of the Appendix B; we will therefore skip the details.

Let (A^*, Y^*) be the incentive-compatible contract defined by Y^* to be the market belief of the conjectured optimal rating of the original setting, defined by the linear filter described at the end of Part I, and A^*_t to be the associated conjectured optimal action.

Let $\widehat{\mathcal{F}}$ be a confidential information structure with non-exclusive signals S_k, $k \le K_0$, associated with market belief $\widehat{Y}$ and stationary action $\widehat{A}$; $(\widehat{A}, \widehat{Y})$ is then a well-defined incentive-compatible stationary linear contract. We want to show $c'(A^*) \ge c'(\widehat{A})$.

Let $(A^{(\rho)}, Y^{(\rho)})$ be the optimal incentive-compatible stationary linear contract defined as the optimal solution above, as a function of the discount rate of the principal ρ, with $V^{(\rho)}$ the corresponding principal's expected payoff. Under both (A^*, Y^*) and $(\widehat{A}, \widehat{Y})$, the expectation of the penalty term in the principal's payoff vanishes. Thus, the principal's expected payoff for contract (A^*, Y^*) is $V^* := c'(A^*)/\rho$, while the principal's expected payoff for contract $(\widehat{A}, \widehat{Y})$ is $\widehat{V} := c'(\widehat{A})/\rho$. Then, for every $\rho \in (0, r)$, the inequalities $\rho V^{(\rho)} \geq \rho \widehat{V} = c'(\widehat{A})$ must hold. However, as $\rho \to 0$, $c'(A^{(\rho)}) \to c'(A^*)$, and the linear filter of $Y^{(\rho)}$ converges pointwise to the linear filter of Y^*, as shown by the limits above. Thus, the expectation of the penalty term of the principal's payoff converges to zero, which in turn implies that $\rho V^{(\rho)} \to c'(A^*)$. Hence, $c'(A^*) \geq c'(\widehat{A})$.

C.4 Proof of Theorem 4.4

As for the other cases, the proof proceeds in two parts.

C.4.1 Part I: First-Order Conditions

Given a linear filter $\{u_k\}_k$, we recall the following shorthand notation:

$$U(t) := \sum_{k=1}^{K} \beta_k u_k(t),$$
$$V(t) := \sum_{k=1}^{K} \alpha_k u_k(t),$$
$$U_0 := \int_0^\infty U(t) e^{-t} \, \mathrm{d}t,$$
$$V_0 := \int_0^\infty V(t) e^{-rt} \, \mathrm{d}t.$$

We maximize $c'(A)$, with A the stationary equilibrium action of the agent, among all public information structures with non-exclusive signals $S_1, \ldots, S_{K_0}$, that are generated by some rating process Y that satisfies the variance normalization $\mathbf{Var}[Y_t] = 1$ and that is proportional to the market belief. We express such a rating process Y via its linear filter $\mathbf{u} = \{u_k\}_k$,

$$Y_t = \sum_{k=1}^{K} \int_0^\infty u_k(t-s) \left[\mathrm{d}S_{k,s} - \alpha_k a^* \, \mathrm{d}s\right].$$

Proposition C.1 and Proposition 2.6 yield the constraints that Y must satisfy to be proportional to the market belief.

1. The constraints of non-exclusivity: for every non-exclusive signal S_k,

$$\mathbf{Cov}[\theta_t, Y_t]\,\mathbf{Cov}[S_{k,t}, Y_{t+\tau}] = \mathbf{Cov}[S_{k,t}, \theta_{t+\tau}], \qquad \forall t, \forall \tau \geq 0,$$

or equivalently,

$$\mathbf{Cov}[\theta_t, Y_t]\,\mathbf{Cov}[(S_{k,t+\tau} - S_t), Y_{t+\tau}] = \mathbf{Cov}[(S_{k,t+\tau} - S_{k,t}), \theta_{t+\tau}], \qquad \forall t, \forall \tau \geq 0.$$

2. The constraints of publicness together with the variance normalization:

$$\mathbf{Cov}\left[Y_t, Y_{t+\tau}\right] = e^{-\tau}, \qquad \forall r, \forall \tau \geq 0.$$

Recall from the proofs of Theorem 4.3 and Theorem 3.2 that the constraints can be expressed in terms of the linear filter of the rating process as follows:

$$G(\mathbf{u}, \tau) = e^{-\tau} \qquad \text{and} \qquad H_k(\mathbf{u}, \tau) = \beta_k(1 - e^{-\tau}) \qquad \forall \tau \geq 0, \forall k = 1, \ldots, K_0,$$

with

$$G(\mathbf{u}, \tau) = \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j+\tau-i|}\,\mathrm{d}i\,\mathrm{d}j + \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)u_k(s+\tau)\,\mathrm{d}s,$$

$$H_k(\mathbf{u}, \tau) = \left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right]\left[\sigma_k^2 \int_0^\tau u_k(s)\,\mathrm{d}s + \frac{\beta_k\gamma^2}{2}\int_{i=0}^{\tau}\int_{j=0}^{\infty} U(j)e^{-|i-j|}\,\mathrm{d}i\,\mathrm{d}j\right].$$

We solve a relaxed optimization problem with $2 + K_0$ constraints: one constraint is associated to the variance normalization, one constraint associated with the constraints of public ratings, and one constraint associated to every non-exclusive signal. Specifically, we maximize $F(\mathbf{u})$, defined as

$$F(\mathbf{u}) = \left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right]\left[\int_0^\infty V(t)e^{-rt}\,\mathrm{d}t\right],$$

and equal to a constant factor of $c'(A)$, subject to

$$
\begin{aligned}
G(\mathbf{u},0) &= 1 \\
\int_0^\infty e^{-r\tau} G(\mathbf{u},\tau)\,\mathrm{d}\tau &= \frac{1}{1+r}, \\
\int_0^\infty e^{-r\tau} H_k(\mathbf{u},\tau)\,\mathrm{d}\tau &= \frac{\beta_k}{r(1+r)}, \qquad \forall k = 1,\dots,K_0.
\end{aligned}
$$

Assume there exists a solution $\mathbf{u}^* = \{u_k^*\}_k$ to the above problem that is four times continuously differentiable, integrable, and square-integrable, and that in addition satisfies the continuum of constraints of the original problem.

Let

$$
\begin{aligned}
L\left(\mathbf{u},\lambda_0,\lambda_1,\{\lambda_{r,k}\}_{k\le K_0}\right) = F(\mathbf{u}) + \lambda_0 G(\mathbf{u},0) + \lambda_1 \int_0^\infty e^{-r\tau} G(\mathbf{u},\tau)\,\mathrm{d}\tau \\
+ \sum_{i=1}^{K_0} \lambda_{r,i} \int_0^\infty e^{-r\tau} H_i(\mathbf{u},\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Assume there exist $\lambda_0^*, \lambda_1^*, \lambda_{r,1}^*, \dots, \lambda_{r,K_0}^*$ such that $\mathbf{u}^*$ maximizes

$$
\mathbf{u} \mapsto L\left(\mathbf{u},\lambda_0^*,\lambda_1^*,\{\lambda_{r,k}^*\}_{k\le K_0}\right).
$$

Assume $\lambda_1^*/\lambda_0^* > -r$. We will choose these constants in such a way that there is a unique solution to the unconstrained maximization problem (up to a scalar factor), that in addition solves the constraints of the original problem.

In the sequel, we drop the star notation for simplicity. Throughout, let

$$
z_k(\tau) = \sigma_k^2 \int_0^\tau u_k(s)\,\mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_{i=0}^{\tau} \int_{j=0}^{\infty} U(j) e^{-|i-j|}\,\mathrm{d}i\,\mathrm{d}j.
$$

We apply Proposition D.1 to get first-order conditions: for all k and all t, $L_k(t) = 0$, where L_k is defined as follows.

If k indexes an exclusive signal, then

$$\begin{aligned}
L_k(t) \coloneqq{} & U_0\alpha_k e^{-rt} + V_0\beta_k e^{-t} \\
& + \lambda_0 \left(2\sigma_k^2 u_k(t) + \gamma^2\beta_k \int_0^\infty U(j) e^{-|j-t|}\,\mathrm{d}j \right) \\
& + \lambda_1\sigma_k^2 \int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right]\mathrm{d}\tau \\
& + \lambda_1\beta_k\frac{\gamma^2}{2}\int_0^\infty e^{-r\tau}\int_0^\infty U(j)e^{-|j+\tau-t|}\,\mathrm{d}j\,\mathrm{d}\tau \\
& + \lambda_1\beta_k\frac{\gamma^2}{2}\int_0^\infty e^{-r\tau}\int_0^\infty U(j)e^{-|t+\tau-j|}\,\mathrm{d}j\,\mathrm{d}\tau \\
& + \beta_k e^{-t}\sum_{i=1}^{K_0}\lambda_{r,i}\int_0^\infty e^{-r\tau} z_i(\tau)\,\mathrm{d}\tau \\
& + U_0\beta_k\left[\frac{2e^{-rt}}{r(1-r^2)} - \frac{e^{-t}}{r(1-r)}\right]\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2}.
\end{aligned}$$

If k indexes a non-exclusive signal, then

$$\begin{aligned}
L_k(t) \coloneqq{} & U_0\alpha_k e^{-rt} + V_0\beta_k e^{-t} \\
& + \lambda_0 \left(2\sigma_k^2 u_k(t) + \gamma^2\beta_k \int_0^\infty U(j) e^{-|j-t|}\,\mathrm{d}j \right) \\
& + \lambda_1\sigma_k^2 \int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right]\mathrm{d}\tau \\
& + \lambda_1\beta_k\frac{\gamma^2}{2}\int_0^\infty e^{-r\tau}\int_0^\infty U(j)e^{-|j+\tau-t|}\,\mathrm{d}j\,\mathrm{d}\tau \\
& + \lambda_1\beta_k\frac{\gamma^2}{2}\int_0^\infty e^{-r\tau}\int_0^\infty U(j)e^{-|t+\tau-j|}\,\mathrm{d}j\,\mathrm{d}\tau \\
& + \beta_k e^{-t}\sum_{i=1}^{K_0}\lambda_{r,i}\int_0^\infty e^{-r\tau} z_i(\tau)\,\mathrm{d}\tau \\
& + U_0\beta_k\left[\frac{2e^{-rt}}{r(1-r^2)} - \frac{e^{-t}}{r(1-r)}\right]\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2} \\
& + \lambda_{r,k}U_0\sigma_k^2\frac{e^{-rt}}{r}.
\end{aligned}$$

We first obtain conditions on u_k when k indexes an exclusive signal. In a similar fashion as in the case of public exclusive information structures, we obtain that

$$\begin{aligned}
L_k(t) - L_k''(t) = {} & \alpha_k U_0(1-r^2)e^{-rt} \\
& + 2\lambda_0\sigma_k^2[u_k(t) - u_k''(t)] + 2\lambda_0\gamma^2\beta_k U(t) \\
& + \lambda_1\sigma_k^2 \int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right] \mathrm{d}\tau \\
& - \lambda_1\sigma_k^2 \int_0^\infty e^{-r\tau}\left[u_k''(t+\tau) + u_k''(t-\tau)\right] \mathrm{d}\tau \\
& - \lambda_1\sigma_k^2\left[-re^{-rt}u_k(0) + u_k'(0)e^{-rt}\right] \\
& + \lambda_1\gamma^2\beta_k \int_0^\infty e^{-r\tau}U(t+\tau)\,\mathrm{d}\tau \\
& + U_0\beta_k \frac{2e^{-rt}}{r}\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2}.
\end{aligned}$$

Let

$$p_k(t) = L_k(t) - L_k''(t),$$

and let

$$\begin{aligned}
J_k(t) &= \int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right] \mathrm{d}\tau, \\
J(t) &= \sum_{i=1}^{K}\beta_i J_i(t).
\end{aligned}$$

Observe that

$$J_k''(t) = -2ru_k(t) + r^2 J_k(t),$$

and, inserting J_k in p_k:

$$\begin{aligned}
p_k(t) = {} & \alpha_k U_0(1-r^2)e^{-rt} + 2\lambda_0\sigma_k^2(u_k(t) - u_k''(t)) + 2\lambda_0\gamma^2\beta_k U(t) \\
& + 2r\lambda_1\sigma_k^2 u_k(t) + \lambda_1(1-r^2)\sigma_k^2 J_k(t) + \lambda_1\gamma^2\beta_k J(t) + U_0\beta_k\frac{2e^{-rt}}{r}\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2}.
\end{aligned}$$

After differentiation, we get

$$
\begin{aligned}
p_k''(t) &= r^2\alpha U_0(1-r^2)e^{-rt} + 2\lambda_0\sigma_k^2\left[u_k''(t) - u_k''''(t)\right] + 2\lambda_0\gamma^2\beta U''(t) \\
&\quad + 2r\lambda_1\sigma_k^2 u_k''(t) + \lambda_1(1-r^2)\sigma_k^2\left[-2ru_k(t) + r^2 J_k(t)\right] \\
&\quad + \lambda_1\gamma^2\beta_k\left[-2rU(t) + r^2 J(t)\right] + r^2 U_0\beta_k\frac{2e^{-rt}}{r}\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2}.
\end{aligned}
$$

Finally, let

$$q_k(t) = p_k''(t) - r^2 p_k(t).$$

We have

$$
\begin{aligned}
q_k(t) &= 2\lambda_0\sigma_k^2\left[u_k''(t) - u_k''''(t)\right] - r^2 2\lambda_0\sigma_k^2\left[u_k(t) - u_k''(t)\right] \\
&\quad + 2\lambda_0\gamma^2\beta_k U''(t) - 2r^2\lambda_0\gamma^2\beta_k U(t) \\
&\quad + 2r\lambda_1\sigma_k^2 u_k''(t) - 2r^3\lambda_1\sigma_k^2 u_k(t) \\
&\quad - 2r\lambda_1\sigma_k^2(1-r^2)u_k(t) \\
&\quad - 2r\lambda_1\gamma^2\beta_k U(t).
\end{aligned}
\tag{93}
$$

We must have $q_k(t) = 0$ for all k and all t, and this defines a differential equation that u_k must satisfy.

Now, let k denote the index of a non-exclusive signal. We have:

$$
\begin{aligned}
L_k(t) - L_k''(t) &= \alpha_k U_0(1-r^2)e^{-rt} + 2\lambda_0\sigma_k^2\left[u_k(t) - u_k''(t)\right] + 2\lambda_0\gamma^2\beta_k U(t) \\
&\quad + \lambda_1\sigma_k^2\int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right]\mathrm{d}\tau \\
&\quad - \lambda_1\sigma_k^2\int_0^\infty e^{-r\tau}\left[u_k''(t+\tau) + u_k''(t-\tau)\right]\mathrm{d}\tau \\
&\quad - \lambda_1\sigma_k^2\left(-re^{-rt}u_k(0) + u_k'(0)e^{-rt}\right) + \lambda_1\gamma^2\beta_k\int_0^\infty e^{-r\tau}U(t+\tau)\,\mathrm{d}\tau \\
&\quad + U_0\beta_k\frac{2e^{-rt}}{r}\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2} + (1-r^2)\lambda_{r,k}U_0\sigma_k^2\frac{e^{-rt}}{r}.
\end{aligned}
$$

Compared to the case of exclusive signals, we note the presence of one additional term in the expression for $L_k(t) - L_k''(t)$. However, this term being a constant factor of e^{-rt}, the chain of transformations we used for the case of exclusive signals continues to yield the same equation (93). Hence, the differential equation obtained for u_k, by setting $q_k = 0$ when k denotes a non-exclusive signal, is the same as when k denotes an exclusive signal.

Thus, multiplying (93) by $\frac{\beta_k}{2\sigma_k^2}$ and summing over k's, we get

$$\sum_{k=1}^{K}\frac{\beta_k}{2\sigma_k^2}q_k(t)=\lambda_0(U''(t)-U''''(t))-r^2\lambda_0(U(t)-U''(t))+\lambda_0\gamma^2 m_\beta U''(t)$$
$$-\lambda_0 r^2\gamma^2 m_\beta U(t)+r\lambda_1 U''(t)-r\lambda_1 U(t)-r\lambda_1\gamma^2 m_\beta U(t).$$

As U is bounded, we can discard the positive roots. We conclude that U has the form

$$U(t)=C_1 e^{-\sqrt{r(r+\lambda_1/\lambda_0)}t}+C_2 e^{-\kappa t}, \tag{94}$$

for some constants C_1 and C_2.

Since $\sum_k \frac{\beta_k}{2\sigma_k^2}q_k=0$, U is solution of a homogeneous linear differential equation, whose characteristic polynomial has roots $\pm\sqrt{1+\gamma^2 m_\beta}=\pm\kappa$ and $\pm\sqrt{r(r+\lambda_1/\lambda_0)}$ (recall our assumption that $\lambda_1/\lambda_0>-r$).

Next, let us fix an arbitrary pair (i,j) with $i\neq j$, define $\zeta_{ij}(t)\coloneqq 2(\beta_i\sigma_i^2 u_j(t)-\beta_j\sigma_j^2 u_i(t))$. We have

$$\beta_i p_j(t)-\beta_j p_i(t)=\lambda_0(\zeta_{ij}''(t)-\zeta_{ij}''''(t))-r^2\lambda_0(\zeta_{ij}(t)-\zeta_{ij}''(t))+r\lambda_1\zeta_{ij}(t)''-r\lambda_1\zeta_{ij}(t),$$

and since $\beta_i p_j-\beta_j p_i=0$ must hold, we obtain a homogeneous linear differential equation that ζ_{ij} must satisfy. The roots of the characteristic polynomial are ± 1 and $\pm\sqrt{r(r+\lambda_1/\lambda_0)}$. As ζ_{ij} is bounded, we can discard the positive roots. We conclude that ζ_{ij} has the form

$$\zeta_{ij}(t)=C_1' e^{-\sqrt{r(r+\lambda_1/\lambda_0)}t}+C_2 e^{-\kappa t}+C_2' e^{-t}, \tag{95}$$

for some constants C_1' and C_2'.

Putting together (94) and (95), it holds that

$$u_k(t)=D_{1,k}e^{-\sqrt{r(r+\lambda_1/\lambda_0)}t}+D_{2,k}e^{-\kappa t}+D_{3,k}e^{-t}, \tag{96}$$

for some constants $D_{1,k}$, $D_{2,k}$ and $D_{3,k}$. We can anticipate that $D_{3,k}=0$, because U does not include a term e^{-t}.

Determination of the constants. Let

$$\hat{r} = \sqrt{r\left(r + \frac{\lambda_1}{\lambda_0}\right)}.$$

We plug the general form of u_k from (96) into the expression for L_k. For both exclusive and non-exclusive signals S_k, L_k can be written in the form of a sum of three exponential terms, namely

$$L_k = L_{1,k}e^{-\hat{r}t} + L_{2,k}e^{-\kappa t} + L_{3,k}e^{-t} + L_{4,k}e^{-rt},$$

where the constant factors $L_{1,k}$, $L_{2,k}$, and $L_{3,k}$ depend on the primitives of the model, as well as on the constants $D_{1,k}$, $D_{2,k}$ and $D_{3,k}$. Asserting that $L_k = 0$ is equivalent to $L_{1,k} = L_{2,k} = L_{3,k} = 0$.

We solve for the constant factors $D_{1,k}$, $D_{2,k}$ and $D_{3,k}$, $k = 1, \ldots, K$, as well as the variables $\lambda_0, \lambda_1, \lambda_{r,k}$, using both the first-order condition that L_k, and also the constraints $G_1(\mathbf{u}, \tau) = e^{-\tau}$ and $H_k(\mathbf{u}, \tau) = \beta_k(1 - e^{-\tau})$.

First, we note that for both exclusive and non-exclusive signals S_k, the term $L_{4,k}$ is

$$\frac{2\left(\lambda_0\left(r^2-\kappa^2\right)+\lambda_1 r\right)\left(\gamma^2\beta_k\sum_{i=1^K}\beta_i D_{2,i}-\left(\kappa^2-1\right)\sigma_k^2 D_{2,k}\right)}{(\kappa-1)(\kappa+1)(\kappa-r)(\kappa+r)}.$$

By assumption, we have

$$\left(\lambda_0\left(r^2-\kappa^2\right)+\lambda_1 r\right) \neq 0,$$

which, together with $L_{4,k} = 0$, implies that

$$\gamma^2\beta_k\sum_{i=1^K}\beta_i D_{2,i}-\left(\kappa^2-1\right)\sigma_k^2 D_{2,k} = 0,$$

so that, for every k,

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2}. \tag{97}$$

Second, we have, for both exclusive and non-exclusive signals S_k,

$$L_{3,k} = \frac{\gamma^2 U_0 \beta_k}{2(r-1)r} \sum_{i=1}^{K_0} \beta_i \lambda_{2,i} + \beta_k \sum_{i=1}^{K_0} \zeta_i \lambda_{2,i}$$
$$+ \frac{2\sigma_k^2 \left(\lambda_0 \left(r^2-1\right) + \lambda_1 r\right)}{r^2-1} D_{3,k} + \frac{\gamma^2 \beta_k \left(\lambda_0 \left(r^2-1\right) + \lambda_1 r\right)}{(\tau-1)\left(r^2-1\right)} \sum_{i=1}^{K} \beta_i D_{1,k}$$
$$+ \frac{\gamma^2 \beta_k \left(\lambda_0 \left(r^2-1\right) + \lambda_1 r\right)}{(\kappa-1)\left(r^2-1\right) \sum_{i=1}^{K} \beta_i D_{2,k}} + V_0 \beta_k,$$

for some expression ζ_i (whose expression is lengthy and therefore omitted). Thus, as

$$\lambda_0 \left(r^2-1\right) + \lambda_1 r \neq 0,$$

solving for $D_{3,k}$ in the equation $L_{3,k} = 0$ yields that $D_{3,k}$ is proportional to β_k/σ_k^2. Moreover, the term e^{-t} vanishes in (94), which yields the equality

$$\sum_{k=1}^{K} \beta_k D_{3,k} = 0,$$

which, in turn, implies $D_{3,k} = 0$.

We have thus identified two factors. Therefore, we may already write

$$u_k(t) = D_{1,k} e^{-\tau t} + a \frac{\beta_k}{\sigma_k^2} e^{-\kappa t}.$$

Using this simplified expression for u_k, we also get

$$\zeta_k = \frac{\gamma^2 \beta_k (\tau + r + 2)}{2(\tau+1)r(r+1)(\tau+r)} \sum_{i=1}^{K_0} \beta_i D_{1,i} + \frac{\sigma_k^2}{r(\tau+r)} D_{1,k}$$
$$+ \frac{a\beta_k \left(\gamma^2 m_\beta (\kappa + r + 2) + 2(\kappa+1)(r+1)\right)}{2(\kappa+1)r(r+1)(\kappa+r)}.$$

Given this simplification in the expression for u_k, we now get to the remaining equations that characterize the unique solution to the problem. These are the equations labeled (a)–(e) below. The numerator of $L_{4,k}$ for an exclusive signal S_k

must be zero, which yields the equation

$$
\begin{aligned}
0 = &-\gamma^2 U_0 \beta_k (r-\tau)(r-\kappa) \sum_{i=1}^{K_0} \beta_i \lambda_{2,i} - \lambda_1 r \left(r^2-1\right) \sigma_k^2 (r-\kappa) D_{1,k} \\
&+ r(r-\tau)\left(a\lambda_1\beta_k\left(\gamma^2 m_\beta - r^2 + 1\right) + \left(r^2-1\right) U_0 \alpha_k (r-\kappa)\right) \\
&+ \gamma^2 \lambda_1 r \beta_k (r-\kappa).
\end{aligned} \tag{98a}
$$

The numerator of $L_{4,k}$ for a non-exclusive signal S_k must be zero, which yields the equation

$$
\begin{aligned}
0 = &-\gamma^2 U_0 \beta_k (r-\tau)(r-\kappa) \left(\sum_{i=1}^{K_0} \beta_i \lambda_{2,i}\right) \\
&+ (r-\tau)\left(\left(r^2-1\right) U_0 \sigma_k^2 (r-\kappa)\lambda_{2,k} + a\lambda_1 r \beta_k \left(\gamma^2 m_\beta - r^2 + 1\right) + r\left(r^2-1\right) U_0 \alpha_k (r-\kappa)\right) \\
&- \lambda_1 r \left(r^2-1\right) \sigma_k^2 (r-\kappa) D_{1,k} + \gamma^2 \lambda_1 r \beta_k (r-\kappa) \sum_{i=1}^{K} \beta_i D_{1,k}.
\end{aligned} \tag{98b}
$$

Because $L_{3,k} = 0$, we must have

$$
\begin{aligned}
0 = &\frac{\gamma^2 U_0 \beta_k}{2(r-1)r} \sum_{i=1}^{K_0} \beta_i \lambda_{2,i} + \beta_k \sum_{i=1}^{K_0} \zeta_i \lambda_{2,i} \\
&+ \frac{\beta_k \left(a\gamma^2 m_\beta \left(\lambda_0 \left(r^2-1\right) + \lambda_1 r\right) + (\kappa-1)\left(r^2-1\right) V_0\right)}{(\kappa-1)\left(r^2-1\right)} \\
&+ \frac{\gamma^2 \beta_k \left(\lambda_0\left(r^2-1\right) + \lambda_1 r\right)}{(\tau-1)\left(r^2-1\right)} \sum_{i=1}^{K} \beta_i D_{1,k}.
\end{aligned} \tag{98c}
$$

The public constraint, *i.e.*, the equation $G_1(\mathbf{u}, \tau) = e^{-\tau}$, also yields an equation that involves a sum of three exponential terms, and can be written

$$
G_1 e^{-\hat{r}\tau} + G_2 e^{-\tau} = e^{-\tau},
$$

where G_1 and G_2 and constant factors obtained when plugging the expression obtained for u_k above into the definition of the function $G(\mathbf{u}, \tau)$, and simplifying, using $\kappa^2 = 1 + \gamma^2 m_\beta$.

We must have $G_1 = 0$, which yields the equation

$$0 = \frac{a\left(\tau^2 + \gamma^2(-m_\beta) - 1\right)}{(\tau^2 - 1)(\tau + \kappa)} \sum_{k=1}^{K} \beta_i D_{1,i} + \frac{\gamma^2}{2\tau - 2\tau^3} \left(\sum_{k=1}^{K} \beta_i D_{1,i} \right)^2 + \frac{1}{2\tau} \sum_{k=1}^{K} \sigma_i^2 D_{1,i}^2. \tag{98d}$$

We then incorporate the constraint of non-exclusivity, $H_k(\mathbf{u}, \tau) = \beta_k(1 - e^{-\tau})$, for every non-exclusive signal S_k. After plugging the expression obtained for u_k just obtained, the term $G_{2,k}(\mathbf{u}, \tau)$ is expressed as a sum of three exponentials and a constant term:

$$H_k(\mathbf{u}, \tau) = H_{k,1} e^{-\hat{r}\tau} + H_{k,2} e^{-\tau} + H_{k,3}.$$

Because $H_{k,2} = 0$, it holds that, for every non-exclusive signal index k,

$$D_{1,k} = \frac{\gamma^2 \beta_k}{(\tau^2 - 1)\sigma_k^2} \sum_{k=1}^{K} \beta_i D_{1,i}. \tag{98e}$$

Finally, we must also have $G_2 = 1$, which yields

$$\frac{1}{2}\gamma^2 U_0 \left(-\frac{\sum_{k=1}^{K} \beta_i D_{1,i}}{\tau - 1} - \frac{a m_\beta}{\kappa - 1} \right) + 1 = 0, \tag{98f}$$

where we note that

$$U_0 = \frac{a m_\beta}{\kappa + 1} + \frac{1}{\tau + 1} \sum_{k=1}^{K} \beta_i D_{1,i}.$$

However, we will not use (98f). As will be verified, equations (98a)–(98e) yield a unique candidate $\mathbf{u}$, up to the scalar constant a which is pinned down by (98f), and this candidate satisfies the first-order condition, and the public and non-exclusivity constraints.

Throughout, we use the notation

$$\rho_\beta := \frac{m_\beta^n}{m_\beta}.$$

We describe $(c_k^n, c_k^e, \lambda_{r,k}, \lambda_0)$ as a function of λ_1 and $\hat{r}$; then, λ_1 as a function of $\hat{r}$; finally, we define $\hat{r}$ as a one of the positive roots of some polynomial.

First, we briefly sketch how to solve the system, then state the solution.

1. First, we solve (98a), (98b) for c_k, taking the sum $\sum_{k=0}^{K} c_k \beta_k$ as a parameter.

We multiply each by β_k, and add them up, giving us an affine equation for $\sum_{k=0}^{K} c_k\beta_k$, which we solve for.

2. Using the solution for each c_k, we obtain an expression for each of them, which is a function of λ_1, λ_0 and $\sum_{k=0}^{K_0} \beta_k\lambda_{r,k}$.

3. We plug the formula for c_k, $k \leq K_0$ in (98c), which yields an equation for each $\lambda_{r,k}$ (as a function of λ_1, λ_0 and $\sum_{k=0}^{K_0} \beta_k\lambda_{r,k}$).

4. Taking $\sum_{k=0}^{K_0} \beta_k\lambda_{r,k}$ as a parameter, we solve for each $\lambda_{r,k}$, multiply by β_k, add them up, which gives us an affine equation for $\sum_{k=0}^{K_0} \beta_k\lambda_{r,k}$, which we solve for.

5. We plug back into each of our formulas for c_k and $\lambda_{r,k}$, and obtain a solution for each of them, as a function of λ_0, λ_1 and $\hat{r}$ only. Using that $\lambda_0 = \frac{r\lambda_1}{\hat{r}^2-r^2}$, we may eliminate λ_0 altogether, and obtain the formulas for c_k and $\lambda_{r,k}$ given below.

We then turn to (98d) and (98e). Substituting for c_k and $\lambda_{r,k}$, (98e) becomes a quadratic expression for λ_1 (and independent of λ_0), while (98d) is quadratic in λ_1 and affine in λ_0. Using that $\lambda_0 = \frac{r\lambda_1}{\hat{r}^2-r^2}$, this becomes another expression that is quadratic in λ_1 and independent of λ_0. We may eliminate the quadratic term by taking the weighted average of these two quadratic expressions, thereby obtaining an equation that is affine in λ_1. Solving it gives us λ_1 as a function of $\hat{r}$ only, given below. Using that $\lambda_0 = \frac{r\lambda_1}{\hat{r}^2-r^2}$, the formula for λ_0 follows. Plugging these two formulas into either quadratic expression gives us a condition that $\hat{r}$ must satisfy, which turns out to be precisely $P(\hat{r})P^*(\hat{r}) = 0$, with P as defined below, and P^* a product of polynomials of degree no larger than two that admit no real roots.

The above procedure yields the solution

$$c_k^n = \frac{C_{1,k}}{R_k}, \quad c_k^e = \frac{C_{2,k}}{R_k},$$

where k is the indexed of a non-exclusive or an exclusive signal, and $C_{1,k}, C_{2,k}$ and R_k are given by

$$\begin{aligned}
C_{1,k} = {} & \left(\kappa^2 - 1\right) \beta_k(\hat{r} - r) \left(m_{\alpha\beta} \left(r^2 - 1 - \left(\kappa^2 - 1\right) \rho_\beta\right)\right. \\
& \left. + (\kappa + r) \left((\kappa + 1)\lambda_1 \left(\rho_\beta - 1\right) + m_{\alpha\beta}^n(\kappa - r)\right)\right), \\
C_{2,k} = {} & (r - \hat{r}) \left((\kappa\right. \\
& +1)\beta_k \left((\hat{r} - 1) \left((r + 1)m_{\alpha\beta}(\hat{r} - \kappa + 1 - r) + (\kappa + r) \left((\hat{r} + 1)\lambda_1 + m_{\alpha\beta}^n(r - \hat{r})\right)\right)\right. \\
& \left. + \left(\kappa^2 - 1\right) \rho_\beta \left((\kappa - 1)m_{\alpha\beta} - \lambda_1(\kappa + r)\right)\right) \\
& \left. + (\hat{r} - \kappa)\alpha_k m_\beta(\kappa + r) \left((\kappa + 1)\rho_\beta(\hat{r} + \kappa - r - 1) - (\hat{r} - 1)(r + 1)\right)\right),
\end{aligned}$$

$$R_k = (\kappa+1)\sigma_k^2 \left((1-\hat{r})m_{\alpha\beta}(\hat{r}-r)\left(r^2-1-\left(\kappa^2-1\right)\rho_\beta\right)\right.$$
$$\left. + (r-\kappa)(\kappa+r)\left(\lambda_1\left(\left(\kappa^2-1\right)\rho_\beta - \hat{r}^2+1\right) + (\hat{r}-1)m^n_{\alpha\beta}(\hat{r}-r)\right)\right).$$

The Lagrangian coefficients $\lambda_{r,k}$ are given by

$$\lambda_{r,k} = \frac{L_{2,k}}{R'_k},$$

where

$$L_{2,k} = r\left((\kappa+1)\beta_k\left((r+1)m_{\alpha\beta}(\hat{r}-r) + (\kappa+r)\left(\lambda_1(\hat{r}+\kappa) + m^n_{\alpha\beta}(r+1-\hat{r}-\kappa)\right)\right)\right.$$
$$\left. + \alpha_k m_\beta(\kappa+r)\left((\kappa+1)\rho_\beta(\hat{r}+\kappa-r-1) + (1-\hat{r})(r+1)\right)\right),$$
$$R'_k = \sigma_k^2 m_\beta(\kappa+r)\left((\hat{r}-1)(r+1) - (\kappa+1)\rho_\beta(\hat{r}+\kappa-r-1)\right).$$

Finally,

$$\lambda_0 = \frac{r\lambda_1}{\hat{r}^2-r^2}.$$

It is time to define λ_1 given $\hat{r}$, namely

$$\lambda_1 = \frac{rQ_1Q_2}{(\hat{r}+r)(r+\kappa)(1+\kappa)\delta_1},$$

where

$$Q_1 = (\hat{r}+1)\left(\hat{r}^2-r\right) - (\kappa-1)\rho_\beta\left(r(\hat{r}+\kappa+1)-\hat{r}^2\right),$$

and

$$Q_2 = \left(\kappa^2-1\right)^2(1-\rho_\beta)\,m^2_{\alpha\beta}\left((\kappa+1)\rho_\beta(2\hat{r}+\kappa-1) + (\hat{r}-1)^2\right)$$
$$+ 2\left(\kappa^2-1\right)m_{\alpha\beta}m^e_{\alpha\beta}\left(r^2-\kappa^2\right)\left((\kappa+1)\rho_\beta(2\hat{r}+\kappa-1) + (\hat{r}-1)^2\right)$$
$$+ (\kappa+r)^2\left(m_\beta m^e_\alpha\left((\kappa+1)\rho_\beta(r+1-\hat{r}-\kappa) + (\hat{r}-1)(r+1)\right)^2\right.$$
$$\left. + (\kappa+1)(m^e_{\alpha\beta})^2\left((\kappa+1)\rho_\beta(r+1-\hat{r}-\kappa)^2 + (1-\hat{r})^2(\kappa-2r-1)\right)\right),$$

and

$$\delta_1 = d_0 + d_1\hat{r} + d_2\hat{r}^2 + d_3\hat{r}^3 + d_4\hat{r}^4 + d_5\hat{r}^5 + d_6\hat{r}^6,$$

with

$$d_0 = -r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)\left((\kappa-1)(\rho_\beta-1)\,m_{\alpha\beta}\left(2(\kappa+1)\right.\right.$$
$$\left. + (\kappa+1)\rho_\beta\left(-\left(-2\kappa^2+r(r+2)+2\right)\right) + r(\kappa+r+2)\right)$$
$$\left. + m^e_{\alpha\beta}(\kappa+r)\left((\kappa+1)\rho_\beta\left(2\kappa^2-2\kappa(r+1)+r\right) + \kappa(r+2)-r\right)\right),$$

$$
\begin{aligned}
d_1 = r^2 \left(\left(\kappa^2-1\right)\rho_\beta+1\right) \big((\kappa-1)\left(\rho_\beta-1\right) m_{\alpha\beta} \left((r+1)(\kappa+r)-4\kappa(\kappa+1)\rho_\beta\right) \\
+ m^e_{\alpha\beta}(\kappa+r)\left(\rho_\beta(-\kappa(4\kappa+1)+3\kappa r+r+1)+(\kappa-1)(r+1)\right)\big),
\end{aligned}
$$

$$
\begin{aligned}
d_2 = r \big((\kappa-1)\left(\rho_\beta-1\right) m_{\alpha\beta} \big(\kappa \\
+(\kappa+1)\rho_\beta \big(5\kappa-2\left(\kappa^2-1\right)\rho_\beta(-2\kappa+2r+1)+\kappa^2(r+1)+\kappa r(r+2)-r(r(r+3)+9)-4\big) \\
+ r(3\kappa+r(\kappa+r+3)+5)+2\big) + m^e_{\alpha\beta}(\kappa+r)\big(\kappa \\
+ \rho_\beta \big(\kappa(\kappa(\kappa+5)-2)+\left(\kappa^2-1\right)\rho_\beta\left(\kappa(4\kappa-1)+2r^2-(7\kappa+1)r-1\right)-(\kappa-3)r^2+ \\
\left(\kappa\left(\kappa^2+\kappa-10\right)-2\right)r-2\big)+(\kappa-1)r^2+3\kappa r+r+1\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
d_3 = m^e_{\alpha\beta}(\kappa+r)\big(\rho_\beta\big(2(\kappa-1)+\kappa^2\left(-4r^2+r+1\right)+\kappa r\left(r^2+r+6\right)+\kappa^3(3r-1) \\
- r(r(r+5)+2)\big)+(\kappa-1)(-r-1)\left(r^2+1\right) \\
+ (\kappa-1)(\kappa+1)\rho_\beta^2(\kappa+r(5\kappa-4r-1)-1)\big) \\
- (\kappa-1)\left(\rho_\beta-1\right) m_{\alpha\beta}\big((r+1)\left(r^2+1\right)(\kappa+r) \\
- (\kappa+1)\rho_\beta\left(\kappa-\kappa r^2+4\left(\kappa^2-1\right)r\rho_\beta+\kappa^2(3r-1)+(r+5)r\right)\big),
\end{aligned}
$$

$$
\begin{aligned}
d_4 = -m^e_{\alpha\beta}(\kappa+r)\big(-\kappa \\
+\rho_\beta\big(\kappa\left(\kappa^2+\kappa+2\right)+(3\kappa-1)r^2+(\kappa-1)(\kappa+1)\rho_\beta(\kappa-2r-1)-\left(4\kappa^2+\kappa+5\right)r-2\big) \\
+ r(\kappa+\kappa r+r+3)+1\big)-(\kappa-1)\left(\rho_\beta-1\right) m_{\alpha\beta}\big(-\kappa \\
+(\kappa+1)\rho_\beta\left(\left(\kappa^2-1\right)\rho_\beta+\kappa^2+\kappa-2r^2-3(\kappa+1)r+1\right)+r(\kappa+r(\kappa+r+3)+3)\big),
\end{aligned}
$$

$$
\begin{aligned}
d_5 = (\kappa-1)\left(\rho_\beta-1\right) m_{\alpha\beta}\left((r+1)(\kappa+r)-4\kappa(\kappa+1)\rho_\beta\right) \\
+ m^e_{\alpha\beta}(\kappa+r)\left(\rho_\beta(-\kappa(4\kappa+1)+3\kappa r+r+1)+(\kappa-1)(r+1)\right),
\end{aligned}
$$

$$
d_6 = m^e_{\alpha\beta}(\kappa+r)\left(-(\kappa+1)\rho_\beta-\kappa+2r+1\right)-(\kappa-1)\left(\rho_\beta-1\right) m_{\alpha\beta}\left((\kappa+1)\rho_\beta+\kappa-r\right).
$$

Finally, we are left with pinning down $\hat{r}$. Namely, $\hat{r}$ must be one of the roots (if any) of the polynomial P, defined as

$$
P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6,
$$

with

$$
\begin{aligned}
a_0 = r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)\big((\kappa+r)^2\left(m_\beta m^e_\alpha\left(\left(\kappa^2-1\right)\rho_\beta+1\right)+\left(\kappa^2-1\right)(m^e_{\alpha\beta})^2\right) \\
- 2(\kappa-1)\kappa m_{\alpha\beta}m^e_{\alpha\beta}(\kappa+r+1)(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)^2\big),
\end{aligned}
$$

$$
\begin{aligned}
a_1 = 2r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)\big((\kappa+r)^2\left(m_\beta m^e_\alpha\left((\kappa-1)\rho_\beta+1\right)+(\kappa-1)(m^e_{\alpha\beta})^2\right) \\
+ (1-\kappa)m_{\alpha\beta}m^e_{\alpha\beta}(2\kappa+r+1)(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_2 = r\big(2(\kappa-1)m_{\alpha\beta}m^e_{\alpha\beta}(\kappa+r)\big((1-\kappa)\rho_\beta\left(r^2-2\kappa(\kappa+1)+r\right)+\kappa\left(r^2+r+2\right)+\kappa^2(r+2) \\
- r(r+1)\big)+(\kappa+r)^2\big(m_\beta m^e_\alpha\left((\kappa-1)\rho_\beta+1\right)\left((\kappa-1)\rho_\beta(r-2\kappa-2)+r-2\right) \\
+ (\kappa-1)(m^e_{\alpha\beta})^2\left((\kappa-1)\rho_\beta(r-2\kappa-2)-\kappa(r+2)+r-2\right)\big) \\
+ (\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)\left(2\left(\kappa^2-1\right)\rho_\beta+r^2+\kappa(r+2)+r+2\right)\big),
\end{aligned}
$$

$$
\begin{aligned}
a_3 = r \Big(2(\kappa+r)^2 \Big(m_\beta m_\alpha^e \left((1-\kappa)\rho_\beta \left((\kappa-1)\rho_\beta+\kappa+3\right)-2\right) \\
- (m_{\alpha\beta}^e)^2 \left((\kappa-1)^2\rho_\beta+\kappa^2-1\right)\Big) \\
+ 4(\kappa-1)m_{\alpha\beta}m_{\alpha\beta}^e(\kappa+r)\left(\left(\kappa^2-1\right)\rho_\beta+\kappa(\kappa+r+1)+1\right) \\
+ 2(\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2\left(\left(\kappa^2-1\right)\rho_\beta+\kappa(\kappa+2)+r^2+2(\kappa+1)r+2\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
a_4 = -(\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2\left(\left(\kappa^2-1\right)\rho_\beta+2\kappa+1\right)+(\kappa \\
+r)^2\left((\kappa-1)(m_{\alpha\beta}^e)^2\left((\kappa-1)\rho_\beta+2\right)-m_\beta m_\alpha^e\left((\kappa-1)\rho_\beta+1\right)\left(\rho_\beta(1-\kappa)+2r-1\right)\right) \\
+ 2(\kappa-1)m_{\alpha\beta}m_{\alpha\beta}^e(\kappa+r)\left((\kappa-1)\rho_\beta(r-\kappa)+r-2\kappa\right),
\end{aligned}
$$

$$
\begin{aligned}
a_5 = 2(\kappa+r)^2\left(m_\beta m_\alpha^e\left((\kappa-1)\rho_\beta+1\right)+(\kappa-1)(m_{\alpha\beta}^e)^2\right) \\
- 2(\kappa-1)m_{\alpha\beta}m_{\alpha\beta}^e(2\kappa+r+1)(\kappa+r)-2(\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2(\kappa+r+1),
\end{aligned}
$$

$$
a_6 = m_\beta m_\alpha^e(\kappa+r)^2-2(\kappa-1)m_{\alpha\beta}m_{\alpha\beta}^e(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2.
$$

C.4.2 Part II: Verification

To be completed.

C.5 Missing Formulas for Theorem 4.4

The missing formulas for Theorem 4.4 are

$$
d^n := -\frac{\kappa-1}{\delta-1}-\frac{\Lambda_1 R_\beta(\delta+\kappa-r-1)}{z(\delta-1)\left(\delta+\kappa-R_\beta\right)},
$$

$$
c^e := \frac{(\delta-r)\left(m_{\alpha\beta}R_\beta+z\right)}{(r-\kappa)z}, \quad d^e := \frac{(\delta-r)(\kappa+r)m_\beta R_\beta}{\left(\kappa^2-1\right)z},
$$

where

$$\Lambda_1 := \frac{\lambda_1(\kappa + r)\left((1-\delta^2)\, m_\beta + (\kappa^2-1)\, m_\beta^n\right)}{(\delta-1)m_\beta(r-\delta)},$$

$$R_\beta := \frac{(\kappa-1)\left((\delta-1)(r+1)m_\beta + (\kappa+1)m_\beta^n(r+1-\delta-\kappa)\right)}{(\delta-1)m_\beta(r-\delta)},$$

$$z := \frac{m_{\alpha\beta}\left((r^2-1)\, m_\beta - (\kappa^2-1)\, m_\beta^n\right)}{(\delta-\kappa)m_\beta} + \frac{(r^2-\kappa^2)\left((\kappa^2-1)\,\lambda_1 m_\beta^n - (\delta-1)m_\beta\left((\delta+1)\lambda_1 + m_{\alpha\beta}^n(r-\delta)\right)\right)}{(\delta-1)(\delta-\kappa)m_\beta(r-\delta)},$$

in terms of λ_1 and δ.

The parameter λ_1 is a function of δ, and we accordingly write $\lambda_1(\delta)$ when convenient. It holds that

$$\lambda_1 = \frac{(r-\delta)\left((\kappa-1)\sigma_\beta\left(r(\delta+\kappa+1)-\delta^2\right) + (\delta+\kappa)\left(\delta^2-\kappa r\right)\right)(A_1+A_2)}{(1-\kappa)\sigma_\beta D_1 + \sigma_{\alpha\beta}(\kappa+r)D_2},$$

where

$$A_1 := \left(\kappa^2-1\right) m_{\alpha\beta}^2\left((\delta+\kappa)^2 - (\kappa+1)\sigma_\beta(2\delta+\kappa-1)\right)\left(\left(\kappa^2-1\right)\sigma_\beta + 2\sigma_{\alpha\beta}\left(r^2-\kappa^2\right)\right),$$
$$A_2 := (\kappa+r)^2\left(x^2\sigma_\alpha m_\alpha m_\beta - (\kappa+1)\sigma_{\alpha\beta}^2 m_{\alpha\beta}^2((\delta-1)(\delta+r)(r-\kappa) + x(\delta+\kappa-r-1))\right),$$

with

$$x := (\kappa+1)\sigma_\beta(\delta+\kappa-r-1) + (\delta+\kappa)(r-\kappa).$$

The expressions for D_1 and D_2 are somewhat unwieldy. It holds that

$$\begin{aligned}
D_1 := {}& (\kappa-1)(\kappa+1)^2\sigma_\beta^2\left(\delta^4 - r^4 - 2r^3 + 2r^2\left(2\delta^2+2\delta\kappa+\kappa^2-1\right) - 2\delta^2 r(2\delta+2\kappa-1)\right)\\
&- (\kappa+1)\sigma_\beta(\delta+\kappa)\Big(\delta^3\left(\delta^2+3\delta\kappa+\kappa-1\right) + r^4(\delta-2\kappa+1)\\
&\quad + r^3(\delta(3-\kappa)-3\kappa+1) + r^2\left(\delta^2(3\kappa-1) - 2\delta^3 + \delta\left(4\kappa^2-\kappa+1\right) + 4\kappa\left(\kappa^2-1\right)\right)\\
&\quad + \delta^2 r\left(3(1+\kappa)(1-\delta) - 8\kappa^2\right)\Big)\\
&+ (\delta+\kappa)^2\Big(\delta^3(2\delta\kappa+\delta+\kappa) + r^4(\delta-\kappa) + (\delta+1)r^3(\delta-\kappa)\\
&\quad + r^2\left(\delta^2(\kappa+1) - \delta^3 - \delta\kappa + 2\kappa^2(\kappa+1)\right) - \delta^2 r\left(\delta^2-\delta\kappa+\delta+\kappa(4\kappa+3)\right)\Big),
\end{aligned}$$

and

$$D_2 := \left(\kappa^2 - 1\right) \sigma_\beta^2 \Big((\delta-1)\delta^3(\kappa-1) - r^3 \left(2(\delta+\kappa)^2 + (1-\delta)(\kappa-1)\right)$$
$$+ r^2 \left(4\delta^3 + \delta^2(7\kappa+1) + \delta\left(4\kappa^2+\kappa-1\right) + 2\kappa\left(\kappa^2-1\right)\right)$$
$$+ \delta^2 r \left(\delta - 2\delta^2 - 5\delta\kappa - 4\kappa^2 + \kappa + 1\right) \Big)$$
$$+ \sigma_\beta(\delta+\kappa) \Big(\delta^3 \left(\delta\left(1-3\kappa^2\right) - \delta^2(\kappa+1) + (\kappa-1)\kappa\right)$$
$$+ r^3 \left(\delta^2(\kappa-1) + \delta\left(3\kappa^2-1\right) + \kappa\left(4\kappa^2+\kappa-3\right)\right)$$
$$+ r^2 \left(\delta^3(1-3\kappa) + \delta^2\left(3-9\kappa^2\right) - \delta\kappa\left(4\kappa^2+\kappa-1\right) - 4\kappa^2\left(\kappa^2-1\right)\right)$$
$$+ \delta^2 r \left(\delta^2(3\kappa+1) + 5\delta\kappa^2 + \delta + \kappa\left(8\kappa^2-\kappa-5\right)\right) \Big) - 2(\delta+\kappa)^2(r-\kappa)\left(\delta^2-\kappa r\right)^2.$$

Finally, regarding δ, consider the polynomial

$$\tilde{P}(z) := b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + z^6,$$

with

$$b_0 := \zeta\left(\zeta + \psi g_{\alpha\beta}\right),$$
$$b_1 := \zeta\left(2\eta_\beta + g_{\alpha\beta}\right),$$
$$b_2 := \frac{1}{2}\left(-2\eta_\beta\left(2\zeta - \eta_\beta\right) - g_{\alpha\beta}\left((4\psi-1)\eta_\beta + \psi\right) - |g_{\alpha\beta}|\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta}\right),$$
$$b_3 := -2\left(\eta_\beta^2 + \zeta\right) - g_{\alpha\beta}\left(\eta_\beta + \psi\right) - |g_{\alpha\beta}|\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta},$$
$$b_4 := \frac{1}{2}\left(2\left(\eta_\beta - 2\right)\eta_\beta + g_{\alpha\beta}\left(\eta_\beta + \psi\right) - |g_{\alpha\beta}|\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta}\right),$$
$$b_5 := 2\eta_\beta + g_{\alpha\beta},$$

where $\sigma_\beta := 1 - m_\beta^n/m_\beta$, $\sigma_\alpha := 1 - m_\alpha^n/m_\alpha$, $\sigma_{\alpha\beta} := 1 - m_{\alpha\beta}^n/m_{\alpha\beta}$ and

$$\eta_\beta := \frac{\kappa(1-\sigma_\beta) + \sigma_\beta}{r}, \quad \zeta := \frac{\kappa^2(1-\sigma_\beta) + \sigma_\beta}{r^2},$$
$$g_{\alpha\beta} := \frac{2(\kappa-1)(r+1)^2\chi(\chi+1)m_{\alpha\beta}^2}{r\left(\sigma_\alpha m_\alpha m_\beta(\kappa+r)^2 + (\kappa-1)m_{\alpha\beta}^2\left(2(r+1)\chi - (\kappa-1)\sigma_\beta\right)\right)},$$
$$\psi := \frac{(\kappa-1)\sigma_\beta + \chi(\kappa(\chi+2)+\chi)}{2r\chi(\chi+1)}, \quad \chi := \frac{(\kappa-1)\sigma_\beta - \sigma_{\alpha\beta}(\kappa+r)}{r+1}.$$

In the supplementary Appendix, we show

Lemma C.1 *The polynomial $\tilde{P}$ is irreducible and admits no solutions in terms of radicals. It has exactly two positive distinct roots $\tilde{\delta}_-, \tilde{\delta}_+$. Let $\delta_- = r\tilde{\delta}_-, \delta_+ = r\tilde{\delta}_+$. It holds that either $(\delta_-^2 - r)\lambda_1(\delta_-) < 0$ or $(\delta_+^2 - r)\lambda_1(\delta_+) < 0$, but not both. The parameter δ is equal to δ_- if $(\delta_-^2 - r)\lambda_1(\delta_-) < 0$, and to δ_+ otherwise.*

C.6 Proof of Lemma 4.5

The Representation Lemma (Lemma 2.11) continues to hold. Every rating process is described by its linear filter, which is essentially unique. Hence, as in the baseline model, maximizing an objective over rating processes is equivalent to maximizing the objective over linear filters.

For a rating process Y with linear filter $\{u_k\}_k$, let

$$V_\ell(t) = \sum_{k=1}^{K} \alpha_{k,\ell} u_k(t),$$

for $\ell = 1, \ldots, L$. The extension of Equation (7) of Section 2 to the multi-action setting is immediate: the equilibrium actions $A_1, \ldots, A_L$ are pinned down by the first-order conditions

$$c'(A_\ell) = \frac{\mathbf{Cov}\left[\theta_t, Y_t\right]}{\mathbf{Var}\left[Y_t\right]} \int_0^\infty e^{-rt} V_\ell(t)\,\mathrm{d}t, \qquad \forall \ell = 1, \ldots, L. \tag{99}$$

As in the baseline model, the equilibrium action is constant.

The intermediary seeks to maximize the discounted expected output, which is equivalent to maximizing the drift of the output process,

$$\sum_{\ell=1}^{L} \alpha_{1,\ell} A_\ell,$$

over linear filters $\{u_k\}_k$ that define the rating process Y by

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s)\,\mathrm{d}S_{k,s},$$

subject to (99). With quadratic costs, using without loss a normalized factor $c = 1$, this optimization problem reduces to maximizing

$$\frac{\mathbf{Cov}\left[\theta_t, Y_t\right]}{\mathbf{Var}\left[Y_t\right]} \sum_{\ell=1}^{L} \int_0^\infty e^{-rt} V_\ell(t)\,\mathrm{d}t. \tag{100}$$

Let

$$\alpha_k = \frac{\sum_{\ell=1}^{L} \alpha_{0,\ell}\alpha_{k,\ell}}{\sum_{\ell=1}^{L} \alpha_{0,\ell}}.$$

Maximizing (100) is the same as maximizing

$$\frac{\mathbf{Cov}\left[\theta_t, \widetilde{Y}_t\right]}{\mathbf{Var}\left[\widetilde{Y}_t\right]} \int_0^\infty e^{-rt} \widetilde{V}(t)\,\mathrm{d}t, \tag{101}$$

over $\{u_k\}_k$, where

$$\widetilde{Y}_t := \sum_{k=1}^{K} \int_{s \le t} u_k(t-s)\,\mathrm{d}\widetilde{S}_{k,s},$$

and

$$\widetilde{V}(t) := \sum_{k=1}^{K} \alpha_k u_k(t).$$

Note that the objective (101) is the equilibrium marginal cost of the fictitious setting, under the confidential information structure generated by the rating process $\widetilde{Y}$.

Thus, the linear filter of the optimal confidential rating for the original setting is the same as the linear filter of the optimal confidential rating for the fictitious setting.

C.7 Proof of Lemma 4.6

Here, we prove that the optimal exponential smoothing system dominates any moving window system under confidential (exclusive) ratings, in the context of two signals, $S_1 = X$ (the output) and $S_2 = S$.[58] For simplicity, given that the parameters of the output process $\alpha_1 = \beta_1$ are normalized to 1, we simply write α for α_2, β for β_2, σ for σ_1 and ϵ for σ_2.

Recall that a moving window system is defined, up to an additive constant, by a

[58]We suspect that the result generalizes to more signals, but have not investigated this claim.

two-parameter rating process

$$Y_t = \int_{t-\tau}^{t} c\,\mathrm{d}X_j + (1-c)\,\mathrm{d}S_j,$$

at time t, which is the market information $\mathcal{F}_t$. The parameters are $\tau > 0$, the size of the moving window, and c, the relative weight put on the output.

Straightforward calculations yield that, in the stationary equilibrium, equilibrium effort a^{mw}, if positive, satisfies

$$c'(a^{\mathrm{mw}}) = \gamma^2(\beta(1-c)+c)(c+(1-c)\alpha)\frac{1-e^{-r\tau}}{r}\rho^{\mathrm{mw}}, \tag{102}$$

with

$$\rho^{\mathrm{mw}} := \frac{(1-e^{-\tau})}{2\left(c^2\tau\sigma^2 + \gamma^2\left(\tau + e^{-\tau} - 1\right)\left(\beta - \beta c + c\right)^2 + (c-1)^2\tau\epsilon^2\right)}.$$

Instead, an exponential smoothing system is defined (up to an additive constant) by a two-parameter rating process

$$Y_t = \int_{-\infty}^{t} e^{-\lambda(t-j)}(c\,\mathrm{d}X_j + (1-c)\,\mathrm{d}S_j),$$

at time t, which is the market information $\mathcal{F}_t$. Here, the parameters are $\lambda > 0$, the coefficient of smoothing, and c, as before, the relative weight put on the output.

It is readily verified that, in the stationary equilibrium, equilibrium effort a^{es}, if positive, satisfies

$$c'(a^{\mathrm{es}}) = \gamma^2(\beta(1-c)+c)(c+(1-c)\alpha)\frac{\rho^{\mathrm{es}}}{r+\lambda}, \tag{103}$$

with

$$\rho^{\mathrm{es}} := \frac{\lambda}{(\lambda+1)\left(c^2\sigma^2 + \frac{\gamma^2(\beta-\beta c+c)^2}{\lambda+1} + (c-1)^2\epsilon^2\right)}.$$

Our objective is to prove that the maximum over (c,λ) of the right-hand side of (103) exceeds the maximum over (c,τ) of the right-hand side of (102).

We will prove this pointwise in c. First, note that if $(\beta(1-c)+c)(c+(1-c)\alpha) \leq 0$, there is nothing to show. So we are left with showing that

$$\sup_{\lambda>0} \min_{\tau\geq 0} f(c,\tau,\lambda) \geq 0,$$

where

$$f(c,\tau,\lambda) \coloneqq \frac{\rho^{\text{es}}}{r+\lambda} - \frac{1-e^{-r\tau}}{r}\rho^{\text{mw}}.$$

Let

$$g = \frac{\gamma^2(\beta(1-c)+c)^2}{c^2\sigma^2+(c-1)^2\epsilon^2},$$

which is clearly nonnegative. Computing f, we obtain

$$f(c,\tau,\lambda) = \frac{2\lambda r\left(e^\tau(\tau+(\tau-1)g)+g\right)e^{\tau r} - \left(e^\tau-1\right)\left(g+\lambda+1\right)\left(e^{\tau r}-1\right)\left(\lambda+r\right)}{2r\left(e^\tau(\tau+(\tau-1)g)+g\right)\left(g+\lambda+1\right)e^{\tau r}(\lambda+r)\left(c^2\sigma^2+(c-1)^2\epsilon^2\right)}.$$

The elementary inequality

$$e^\tau(\tau-1)+1 \geq 0$$

implies that

$$e^\tau(\tau+(\tau-1)g)+g \geq 0,$$

and so the denominator of f is positive. We note that the numerator is quadratic in λ, concave, with

$$f(c,\tau,0) = \left(1-e^\tau\right)(g+1)r\left(e^{\tau r}-1\right) \leq 0,$$

and it is readily verified that also $\partial f(c,\tau,0)/\partial\lambda \geq 0$. So it suffices to show the discriminant $\widetilde{\tau}$ is positive. A simple calculation shows that this discriminant is quadratic and convex in g, as

$$\frac{\mathrm{d}^2\widetilde{\tau}}{\mathrm{d}g^2} = 2\left(e^\tau + e^{\tau r}\left(e^\tau(2(\tau-1)r-1)+2r+1\right)-1\right)^2 \geq 0.$$

Evaluating the minimum of $\widetilde{\tau}$ with respect to g, we get

$$\min_g \widetilde{\tau} = (x-1)^3 x^r \left(x^r-1\right)^2 \left((1-x)\left((r+1)x^r+r-1\right)+rx\left(x^r+1\right)\ln x\right),$$

where $x \coloneqq e^\tau \geq 1$. Hence it remains to argue that

$$(1-x)\left((r+1)x^r+r-1\right)+rx\left(x^r+1\right)\ln x \geq 0,$$

or equivalently,

$$\frac{(1-x)\left((r+1)x^r+r-1\right)}{x\left(x^r+1\right)}+r\ln x \geq 0.$$

This is clearly true for $x=1$, and the derivative of the left-hand side with respect to x is

$$\frac{r(x-1)\left(x^{2r}+1\right)+1-x^{2r}}{x^2\left(x^r+1\right)^2},$$

which is positive because

$$\begin{aligned}
r(x-1)\left(x^{2r}+1\right)+1-x^{2r} &= r(y^{\frac{1}{2r}}-1)\left(y+1\right)+1-y \\
&\geq r\left(1+\frac{1}{2r}(y-1)-1\right)(y+1)+1-y \\
&= \frac{1}{2}(y-1)^2,
\end{aligned}$$

where $y \coloneqq x^{2r}$, and we use Bernoulli's inequality in the second line.

Appendix D: Ancillary Mathematical Results

D.1 Euler-Lagrange First-Order Conditions

In this section, we derive the first-order conditions for the particular type of control problems considered in this paper.

Let N, M, K, L, be positive integers. For $\ell = 1, \ldots, L$, let $F^\ell : \mathbf{R}_+^N \to \mathbf{R}$, and $G^\ell : \mathbf{R}^{K\times M} \to \mathbf{R}$, where every G^ℓ can be written

$$G^\ell((y_{1,1}; \ldots; y_{K,1}), \ldots, (y_{1,M}; \ldots; y_{K,M})) = y_{k,i} y_{k',i'},$$

for some k, k', i, i'. In other words, letting

$$F(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_M) = \sum_{\ell=1}^{L} F^\ell(\mathbf{x}) G^\ell(\mathbf{y}_1, \ldots, \mathbf{y}_M),$$

we have that $F(\mathbf{x}, \cdot)$ is a quadratic form, and $F^\ell(\mathbf{x})$ are the coefficients.

For every $i = 1, \ldots, M$, let $\phi_i : \mathbf{R}_+^N \to \mathbf{R}_+$ be a (possibly shifted) projection, in the following sense: $\phi_i((x_1; \ldots; x_N)) = x_j + \delta$ for some j and some $\delta \geq 0$. Let $\mathcal{U}$ be the space of measurable functions $\mathbf{u} : \mathbf{R}_+ \to \mathbf{R}^K$ that are continuous, integrable and square integrable.

Define $G^\ell_{k,i}((y_{1,1}; \ldots; y_{K,1}), \ldots, (y_{1,M}; \ldots; y_{K,M}))$ as

$$\frac{\partial G((y_{1,1}; \ldots; y_{K,1}), \ldots, (y_{1,M}; \ldots; y_{K,M}))}{\partial y_{k,i}},$$

and let

$$F_{k,i}(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_M) = \sum_{\ell=1}^{L} F^\ell(\mathbf{x}) G^\ell_{k,i}(\mathbf{y}_1, \ldots, \mathbf{y}_M).$$

We consider the problem of maximizing

$$\int_{\mathbf{R}_+^N} F(\mathbf{x}, \mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x})))\, \mathrm{d}\mathbf{x}, \tag{104}$$

over control functions $\mathbf{u} \in \mathcal{U}$.

We make the following assumptions:

1. For every ℓ, every $u \in \mathcal{U}$, $\mathbf{x} \mapsto F^\ell(\mathbf{x}) G^\ell(\mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x})))$ is integrable on $\mathbf{R}_+^N$.

2. For every ℓ, i, k, $\mathbf{x} \mapsto F^{\ell}(\mathbf{x}) G^{\ell}_{k,i}(\mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x})))$ is integrable on $\mathbf{R}^N_+ \cap \{\phi_i = t\}$ for every t.

3. The map

$$t \mapsto \int_{\mathbf{R}^N_+ \cap \{\phi_i = t\}} F^{\ell}(\mathbf{x}) G^{\ell}_{k,i}(\mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x}))) \, \mathrm{d}\mathbf{x}$$

is piecewise continuous, where the integral is taken with respect to the Lebesgue measure on $\mathbf{R}^N_+ \cap \{\phi_i = t\}$.

Compared to standard problems of calculus of variations (see, for example, Burns (2014), Chapter 3), this optimization problem involves delayed terms and integrals over a domain whose dimension is unrelated to the dimension of the control. The classical Euler-Lagrange equations do not hold. However, the argument can be adapted to yield the following first-order condition.

Proposition D.1 *Assume the control function* $\mathbf{u}^* \in \mathcal{U}$ *maximizes* (104)*. Then, for every* k *and every* t*,*

$$\sum_{i=1}^{M} \int_{\mathbf{R}^N_+ \cap \{\phi_i = t\}} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x}))) \, \mathrm{d}\mathbf{x} = 0.$$

Proof. For a control function $\mathbf{u} \in \mathcal{U}$, let

$$J(\mathbf{u}) := \int_{\mathbf{R}^N_+} F(\mathbf{x}, \mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x}))) \, \mathrm{d}\mathbf{x},$$

and assume $J(\mathbf{u})$ is maximized for $\mathbf{u} = \mathbf{u}^*$.

The proof relies on classical variational arguments. Fix k and let $\mathbf{v} : \mathbf{R}_+ \to \mathbf{R}^K$, where we write $\mathbf{v} = (v_1, \ldots, v_K)$ and where $v_{k'} = 0$ for $k' \neq k$, and assume v_k is continuous with bounded support. Let $j(\epsilon) = J(\mathbf{u}^* + \epsilon \mathbf{v})$. Differentiating under the integral sign (see, for example, Theorem 6.28 of Klenke (2014)), we get

$$j'(0) = \int_{\mathbf{R}^N_+} \sum_{i=1}^{M} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x}))) v_k(\phi_i(\mathbf{x})) \, \mathrm{d}\mathbf{x}.$$

We observe that j is maximized at $\epsilon = 0$, and so $j'(0) = 0$.

Suppose by contradiction that, for some t,

$$\sum_{i=1}^{M} \int_{\mathbf{R}_+^N \cap \{\phi_i = t\}} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x}))) \, \mathrm{d}\mathbf{x}$$

is nonzero—for example, positive. The sum is piecewise continuous with respect to t, and so by continuity,

$$\sum_{i=1}^{M} \int_{\mathbf{R}_+^N \cap \{\phi_i = t'\}} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x}))) \, \mathrm{d}\mathbf{x}$$

is positive for t' on an interval to the left or the right of t. Let I_t be such an interval, and let v_k be a function that is zero outside of I_t and that is positive inside I_t. Then

$$\begin{aligned} 0 &< \int_{t' \in I_t} \sum_{i=1}^{M} \int_{\mathbf{R}_+^N \cap \{\phi_i = t'\}} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x}))) v_k(t') \, \mathrm{d}\mathbf{x} \, \mathrm{d}t' \\ &= \sum_{i=1}^{M} \int_{\mathbf{R}_+^N} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x}))) v_k(\phi_i(\mathbf{x})) \, \mathrm{d}\mathbf{x}, \end{aligned}$$

which contradicts $j'(0) = 0$. ■

D.2 Deterministic Ratings Must be Gaussian: Lemma D.2

Gaussian processes have the remarkable property that their variances and covariances are non-random. The goal of this section is to provide a partial converse, showing how this property implies that a process must be Gaussian. For simplicity, we focus on the case of a single signal ("X").

Throughout, fix a probability space $(\Omega, \mathcal{F}, P)$ with two independent standard Brownian motion Z and W. Let θ be the Ornstein-Uhlenbeck process defined by

$$\theta_t = \gamma \int_0^t e^{-(t-s)} \, \mathrm{d}Z_s,$$

and let X be the process defined by

$$X_t = W_t + \int_0^t \theta_s \, \mathrm{d}t.$$

Let $\mathcal{G}$ be the natural augmented filtration generated by X. The following lemma is the main result of this section.

Lemma D.2 *Let Y be a progressively measurable process on $\mathcal{G}$ such that:*

1. *For every $T > t + \tau > t$, $\mathbf{Cov}[Y_T, X_{t+\tau} \mid \mathcal{G}_t]$ is a (non-random) function of (t, T, τ), differentiable in τ, and whose derivative is uniformly Lipschitz continuous in t.*

2. *For every $T > t$, $\mathbf{Cov}[Y_T, \theta_t \mid \mathcal{G}_t]$ is a (non-random) function of (t, T).*

3. *For every t, $\mathbf{E}[Y_t^2] < \infty$ and $E[Y_t] = 0$.*

Then, for every t, there exists a square-integrable function $s \mapsto u(t, s)$ such that

$$Y_t = \int_0^t u(t,s)\,\mathrm{d}X_s.$$

Note that this implies that Y is a Gaussian process as well. The proof of Lemma D.2 makes use of the following lemma.

Lemma D.3 *Fix $T > 0$ and let ξ be a square-integrable predictable process adapted to $\mathcal{G}$ defined over $[0, T]$. Suppose that there exists a function $f(s, t)$ defined on $\{(s,t) \in [0,T]^2 : s \leq t\}$, continuously differentiable in t, that satisfies the uniform Lipschitz continuity condition*

$$\left| \frac{\partial f}{\partial t}(x,z) - \frac{\partial f}{\partial t}(y,z) \right| \leq M|x-y|,$$

for some M and all triples (x, y, z), and such that for every $s < t$,

$$\int_s^t \mathbf{E}[\xi_x \mid \mathcal{G}_s]\,\mathrm{d}x = f(s,t).$$

Then,

$$\mathbf{E}\left[\int_0^T (\xi_t - \partial f(t,t)/\partial t)^2\,\mathrm{d}x\right] = 0.$$

Proof of Lemma D.3. Let us consider a sequence $\{\xi^k\}_k$ of simple processes (as defined, for example, in Karatzas and Shreve (1991), Chapter 3), such that

$$\mathbf{E}\left[\int_0^T (\xi_x^k - \xi_x)^2\,\mathrm{d}x\right] \to 0.$$

That ξ is square-integrable and progressively measurable guarantees existence of such a sequence.

For every integer $k \geq 0$, let $\{(s_i^k, t_i^k]\}_i$ be a subdivision of the interval $(0, T]$ such that

$$\lim_{k \to +\infty} \max_i |t_i^k - s_i^k| = 0,$$

and

$$\xi_x^k = \mathbf{E}\left[\xi_x^k \mid \mathcal{G}_{s_i^k}\right].$$

The Cauchy-Schwarz inequality implies

$$\begin{aligned}
\frac{1}{5}\mathbf{E}\left[\int_0^T \left(\xi_x - \frac{\partial f}{\partial t}(x,x)\right)^2 \mathrm{d}x\right] &= \frac{1}{5}\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x - \frac{\partial f}{\partial t}(x,x)\right)^2 \mathrm{d}x\right] \\
&\leq \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x - \xi_x^k\right)^2 \mathrm{d}x\right] \\
&\quad + \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x^k - \mathbf{E}\left[\xi_x^k \mid \mathcal{G}_{s_i^k}\right]\right)^2 \mathrm{d}x\right] \\
&\quad + \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x^k \mid \mathcal{G}_{s_i^k}\right] - \mathbf{E}\left[\xi_x \mid \mathcal{G}_{s_i^k}\right]\right)^2 \mathrm{d}x\right] \\
&\quad + \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x \mid \mathcal{G}_{s_i^k}\right] - \frac{\partial f}{\partial t}(s_i^k, x)\right)^2 \mathrm{d}x\right] \\
&\quad + \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\frac{\partial f}{\partial t}(s_i^k, x) - \frac{\partial f}{\partial t}(x,x)\right)^2 \mathrm{d}x\right].
\end{aligned}$$

Let us show that each term converges to zero. First, by choice of the sequence of simple processes, we immediately get

$$\lim_{k \to +\infty} \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x - \xi_x^k\right)^2 \mathrm{d}x\right] = 0.$$

Also, by choice of the subdivisions, we immediately get

$$\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x^k - \mathbf{E}\left[\xi_x^k \mid \mathcal{G}_{s_i^k}\right]\right)^2 \mathrm{d}x\right] = 0.$$

Then, the hypothesis that, for every $s < t$, we have

$$\int_s^t \mathbf{E}[\xi_x \mid \mathcal{G}_s] \, \mathrm{d}x = f(s,t)$$

implies that $t \mapsto f(s,t)$ is absolutely continuous when $t \geq s$, and thus for almost every $\omega \in \Omega$ and almost every $t \geq s$,

$$\mathbf{E}[\xi_t \mid \mathcal{G}_s] \, \mathrm{d}x = \frac{\partial f}{\partial t}(s,t).$$

In turn, this last equality implies

$$\lim_{k \to +\infty} \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x \,\middle|\, \mathcal{G}_{s_i^k}\right] - \frac{\partial f}{\partial t}(s_i^k, x)\right)^2 \mathrm{d}x\right] = 0.$$

Next, we observe that by Jensen's inequality,

$$\left(\mathbf{E}\left[\xi_x - \xi_x^k \,\middle|\, \mathcal{G}_{s_i^k}\right]\right)^2 \leq \mathbf{E}\left[\left(\xi_x - \xi_x^k\right)^2 \,\middle|\, \mathcal{G}_{s_i^k}\right],$$

which after integration implies

$$\int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x \,\middle|\, \mathcal{G}_{s_i^k}\right] - \mathbf{E}\left[\xi_x^k \,\middle|\, \mathcal{G}_{s_i^k}\right]\right)^2 \mathrm{d}x \leq \int_{s_i^k}^{t_i^k} \mathbf{E}\left[\left(\xi_x - \xi_x^k\right)^2 \,\middle|\, \mathcal{G}_{s_i^k}\right] \mathrm{d}x,$$

and so, by the law of iterated expectations,

$$\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x \,\middle|\, \mathcal{G}_{s_i^k}\right] - \mathbf{E}\left[\xi_x^k \,\middle|\, \mathcal{G}_{s_i^k}\right]\right)^2 \mathrm{d}x\right] \leq \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x - \xi_x^k\right)^2 \mathrm{d}x\right].$$

Finally,

$$\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\frac{\partial f}{\partial t}(s_i^k, x) - \frac{\partial f}{\partial t}(x,x)\right)^2 \mathrm{d}x\right] \leq M \sum_i |t_i^k - s_i^k|^3 \to 0.$$

Hence, each term converges to zero, which naturally implies

$$\mathbf{E}\left[\int_0^T \left(\xi_x - \frac{\partial f}{\partial t}(x,x)\right)^2 \mathrm{d}x\right] = 0,$$

and concludes the proof. ■

We now turn to the proof of Lemma D.2.

Proof of Lemma D.2. Let $T > 0$. In most of the proof, we work on the finite horizon $[0, T]$.

Let the process L be defined by $L_t = -\int_0^t \theta_s \, \mathrm{d}W_s$, and H be the Doléans-Dade exponential of L, *i.e.*,

$$H_t = \exp\left(L_t - \frac{1}{2}\int_0^t \theta_s^2 \, \mathrm{d}s\right).$$

The Novikov condition is satisfied on $[0, T]$ (*e.g.*, Corollary 3.5.13 of Karatzas and Shreve, 1991). Therefore, H is a martingale density process for some probability measure Q equivalent to P on $\mathcal{F}_T$. By the Girsanov-Cameron-Martin Theorem (*e.g.*, Theorem 3.5.1 of Karatzas and Shreve, 1991),

$$W_t + \int_0^t \theta_s \, \mathrm{d}s \qquad (= X_t)$$

is a standard Brownian motion on $\mathcal{F}$, for $t \in [0, T]$, with respect to Q.

We observe that Y_T is a square-integrable random variable measurable with respect to $\mathcal{G}_T$. In addition, under Q, X is a standard Brownian motion. Thus, invoking the Martingale Representation Theorem under Q (*e.g.*, see Karatzas and Shreve, Section 3.4), there exists a square-integrable predictable process $\xi = \{\xi_t\}_{t \in [0,T]}$ adapted to $\mathcal{G}$ such that, almost surely with respect to Q,

$$Y_T = \mathbf{E}^Q[Y_T] + \int_0^T \xi_t \, \mathrm{d}X_t, \tag{105}$$

where $\mathbf{E}^Q$ denotes the expectation operator under Q. Since P and Q are equivalent probability measures, we immediately have that (105) also holds almost surely under P.

Then, using $\mathbf{E}[Y_T] = 0$ and $\mathbf{E}[\mathrm{d}X_t] = 0$, we get

$$Y_T = \int_0^T \xi_t \, \mathrm{d}X_t$$

(in the almost sure sense).

Assumptions (1) and (2) of the lemma statement imply that

$$\begin{aligned}\mathbf{Cov}\left[Y_T, \int_t^{t+\tau} (\mathrm{d}X_x - \theta_x \, \mathrm{d}x) \,\middle|\, \mathcal{G}_t\right] &= \mathbf{Cov}\left[Y_T, \int_t^{t+\tau} \mathrm{d}W_x \,\middle|\, \mathcal{G}_t\right] \\ &= \int_t^{t+\tau} \mathbf{E}\left[\xi_x \mid \mathcal{G}_t\right] \mathrm{d}x,\end{aligned}$$

where the equality is obtained using Itô's isometry, and uses the fact that ξ is adapted to $\mathcal{G}$ and that W remains a standard Brownian motion in the filtration $\mathcal{G}$. We conclude by application of Lemma D.3. ■

Additional References

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