Dynamic College Admissions Problem*

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Abstract

We study a dynamic two-sided many-to-one matching market in the context of universities and students. It is a generalization of the college admissions problem. Our main solution concept, dynamic stability, involves backward induction and maximin. A dynamically stable matching always exists, and a weaker version of rural hospitals theorem also holds. The set of dynamically stable matchings does not form a lattice with respect to universities' preferences, but there does exist a university-optimal stable matching. The generalizations of stability, group stability, and weak core no longer coincide with each other.

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1 Introduction

In the United States, a typical community college offers two-year college education with almost no barrier to entry. After that, students who want a bachelor degree will transfer to a four-year university to continue their education. In a report of the National Center of Education Statistics by Radford et al. (2010), an estimate of 26.6% of the students started college in 2003-2004 and obtained a Bachelor's degree by spring 2009 are transfer students.

We generalize the college admissions problem into a two-period context. In the college admissions problem, each year, a cohort of high school seniors enter the market and apply to a bunch of universities. A university will identify a group of acceptable students based on its criteria and give them offers. A matching outcome then finalizes after each student accepts an offer (if there is any). We put in a new element: community college.[1] Students can go to a community college if they are not matched with a university in the first period. In the second period, students who want to transfer, either from a university or a college, will re-enter the matching market. At the same time, universities that have a transfer quota will enter the market as well.

Community college indeed serves as an institution that stores agents who prefer to match with the other side of the market in a later period. This kind of institution has different representations in other dynamic markets. For example, in a dynamic daycare problem, a child can stay home for one period before he goes to daycare; in a dynamic marriage market, a man can stay alone for one period before he matches with some woman.[2]

In a college admissions problem, agents on both sides have preferences over the other side, while in a school choice problem, students have preferences over school but schools rank students by priorities. In our model, schools will continue to have preferences over students, but there are also priorities involved. The first type of priority can also be found in the dynamic daycare problem of Kennes et al. (2014). In period-2, a student whom was admitted by some university in the first period is an incumbent to the seat his is possessing. Note that this priority is not predetermined exogenously as in a school choice problem since a university will not admit some unacceptable student and grant him this priority. Another type of priority is new: in period-2, a seat that was unmatched in last period or abandoned by a student that was admitted last period perishes, that is, the highest priority for that seat is self-matching.[3] When universities consider who to admit in the first period, these

[1] To avoid confusion, we use "university" for four-year university and use "college" for two-year community college.

[2] See Kennes et al. (2014) and Kadam and Kotowski (2016), respectively.

[3] In other dynamic matching markets, a similar situation happens when the object being allocated is non-storable or the matching opportunity expired. For example, see Lauermann and Nöldeke (2014); Doval (2016).

priorities together might make them prefer students who will stay for two periods.

This paper provides three solution concepts: dynamically stability, dynamically group stability, and dynamic weak core, where each of them is a generalization of stability, group stability, and the weak core in the college admissions problem, respectively. These three solution concepts will degenerate to their conventional counterpart if the dynamic model degenerates to the college admissions problem, in which they will coincide with each other. They resemble three Nash equilibria in a finitely repeated non-cooperative game, where a stage game Nash equilibrium must be played in the last period. Even though they do not coincide in the dynamic model, all of them require a static stable matching in the last period.

This paper lies between two strands of literature. One is the classic matching theory. After the original many-to-one matching problem has been introduced by Gale and Shapley (1962), it has been studied extensively in the literature; for example, see Roth (1985); Roth and Sotomayor (1989, 1992); Roth (1986, 1984).

Another is the growing dynamic matching literature. Doval (2016) introduces a stability concept, which is also called dynamic stability, in dynamic matching markets where matching opportunities arrive over time, matching is one-to-one, and irreversible. In those environments, she shows that dynamically stable matchings always exist in one-sided markets but not always in the allocation of objects with priorities and two-sided markets.[4] Even though the dynamic stability in her paper also incorporates a backward induction notion and requires a stable matching in the second period, it is more similar to our dynamic group stability in two aspects. First, it is not pairwise and requires no group of agents can find in profitable to deviate. Second, it does not allow agents to form agreements intertemporally, which is also similar to the dynamic group stability here. In contrast, our dynamic stability will allow agents to form binding agreements intertemporally. Kurino (2014) studies dynamic house allocation with priorities using an overlapping generations model. Damiano and Lam (2005) and Kurino (2009) study dynamic two-sided one-to-one matching market as well, but they use different solution concepts other than dynamic stability. The closest work is Kadam and Kotowski (2016), in which they study a dynamic marriage market with ordinal preference forms. They introduce a stability concept, which is also called dynamic stability. The major difference between this paper and Kadam and Kotowski (2016) is preference structure. Universities in our model adopt the preference structure in the college admissions problem, but it is unclear how to extend their preference into a many-to-one model. The existence of dynamically stable matchings is not guaranteed in their paper without preference restrictions, while the built in preference structures here guarantee the existence of dynamically stable

[4]Example for one-sided market: deceased-donor organ allocation. Example for allocation of objects with priorities: public housing. Examples for one-sided market: marriage market, adoption market.

Table 1: Relationship to Existing Dynamic Matching Literature

	One-sided	Priority	Two-sided
Dynamic one-to-one	Doval (2016)	Kurino (2014), Doval (2016), etc.	Kadam and Kotowski (2016), Doval (2016), etc.
Dynamic many-to-one	-	Pereyra (2013), Kennes et al. (2014), etc.	This paper

matchings.

The existing papers of dynamic many-to-one matching typically adopt an overlapping generations approach and focus on school choice problems with priorities to certain students or teachers; for example, see Dur (2012); Pereyra (2013); Kennes et al. (2014).

This paper makes the two contributions. First, to the best of our knowledge, this paper is the first one that provides a framework to study dynamic *two-sided* many-to-one matching markets. Second, it shows many results from the college admissions problem continue to hold in a dynamic context.

2 The Models

2.1 The Benchmark: College Admissions Problem

We briefly review the original college admissions problem in Gale and Shapley (1962). It will serve as a benchmark in this paper. There is a set of students $S = \{s_1, s_2, ..., s_{|S|}\}$ and a set of universities $U = \{u_1, u_2, ..., u_{|U|}\}$. Each student can match with at most one university or remain unmatched (being unmatched is denoted by $\emptyset$). For each university u, it can match with at most q_u students, where q_u is a positive integer. A *matching* μ is a mapping satisfies

1. $\forall s \in S$, $\mu(s) \in U \cup \{\emptyset\}$ and $|\mu(s)| \leq 1$;

2. $\forall u \in U, \mu(u) \subseteq S$ and $|\mu(u)| \leq q_u$;

3. $\mu(s) = u$ if and only if s is in $\mu(u)$.

Each student s has a strict preference relation $\rhd_s$ over $U \cup \{\emptyset\}$, and a university is *acceptable* to student s if $u \rhd_s \emptyset$. Each university u has a strict preference relation$\rhd_u$ over $S \cup \{\emptyset\}$, and a student is *acceptable* to university u if $s \rhd_u \emptyset$. We use $\unrhd_l$ to mean $\rhd_l$ or $=$. A matching μ is *individually rational* if for each student s $\mu(s) \unrhd_s \emptyset$ and for each university u $s \rhd_u \emptyset$ for all $s \in \mu(u)$. A matching μ is *stable* if it is individually rational and there do not exist a university u and a student s such that $u \rhd_s \mu(s)$ and $s \rhd_u \sigma$ for some $\sigma \in \mu(u)$. Many of the results for the college admissions problem are derived under an assumption that a university u's preference over sets of students, $\succ_u$, is responsive to $\rhd_u$. We will define responsiveness in section 2.3.

If all $q_u = 1$, the model degenerates to the marriage problem.

2.2 A Two-period Model

Now we extend the benchmark model to *dynamic college admissions problem.* There is a set of students $S = \{s_1, s_2, ..., s_{|S|}\}$ and a set of universities $U = \{u_1, u_2, ..., u_{|U|}\}$. Each student can match with at most one university at a time or remain unmatched. For each university u, it can match with at most q_u students, where q_u is a positive integer. It can be viewed as the maximum number of students that can graduate from u at the end of the second period. To be more specific, $q_u = (q_u^1, q_u^2)$ is an intertemporal capacity constraint, in the sense that it comprises the quota of university u for freshmen in the first period, q_u^1, and the quota for transfer students in the second period, q_u^2, where each q_u^t is a non-negative integer.[5][6] There is an anonymous community college c which has $|S|$ seats to enroll all students in the market in the first period, i.e. $q_c = |S|$.

A **period-1 matching** μ^1 is a mapping satisfies

1. $\forall s \in S$, $\mu^1(s) \in U \cup \{c\} \cup \{\emptyset\}$ and $|\mu^1(s)| \leq 1$;
2. $\forall u \in U, \mu^1(u) \subseteq S$ and $|\mu^1(u)| \leq q_u^1$;
3. $\mu^1(s) = u$ if and only if s is in $\mu^1(u)$.

There are four extra requirements that connect a period-2 matching it to its corresponding period-1 matching. First, a student cannot take a period-1 seat and a period-2 seat from the same university. Second, a university can not unilaterally drop or replace a student whom it admitted in the first period. Third, a university can not use an empty period-1 seat to admit a transfer student. Otherwise, q_u^1 and q_u^2 will not longer be exogenous since a university might strategically keep a period-1 seat empty to enroll an extra transfer student.[7] Forth, if a student transfers, then the occupied seat vanishes. The intuition is that an university is not informed until it realizes someone is transferred in the second period, so it does not have sufficient time and thus can not reuse the seat immediately in the second period. For

[5] We use superscript to indicate time.

[6] So we allow a university to admit student in only one period, that is, it can admit no transfer student or only transfer students.

[7] Dur et al. (2015) study a one-period two-sided many-to-one exchange model with endogenous quotas.

instance, a Economics PhD program that experiences attrition in the first-year is unlikely to go out and search some first-year student from another program to fill the vacancy. These four restriction correspond to requirement 4 to 7 below, respectively. Let u_β be some period-1 seat of university u in the second period. We denote a binary relation over $S \cup \emptyset$ that generates the priority ranking for u_β given μ^1 by $\gtrdot_{u_\beta}^{\mu^1}$. In another word, if given μ^1, student s has a high priority than s' for u_β , then we write $s \gtrdot_{u_\beta}^{\mu^1} s'$.

A **period-2 matching** μ^2 is a mapping satisfies

1. $\forall s \in S, \mu^2(s) \in U \cup \{\emptyset\}$ and $|\mu^2(s)| \leq 1$;
2. $\forall u \in U, \mu^2(u) \subseteq S$ and $|\mu^2(u) \setminus \mu^1(u)| \leq q_u^2$;
3. $\mu^2(s) = u$ if and only if s is in $\mu^2(u)$;
4. if $\mu^1(s) = u$ then $s \notin \mu^2(u) \setminus \mu^1(u)$;
5. if $\mu^1(s) = u$ then $\exists u_\beta$ such that $s \gtrdot_{u_\beta}^{\mu^1} \sigma$ for all $\sigma \in S \cup \emptyset$;
6. if $\nexists s \in S$ such that $\mu^1(s) = u_\beta$ then $\emptyset \gtrdot_{u_\beta}^{\mu^1} s'$ for all $s' \in S$;
7. if $\mu^1(s) = u$ and $\mu^2(s) \neq u$ then $\exists u_\beta$ such that $\emptyset \gtrdot_{u_\beta}^{\mu^1} s'$ for all $s' \in S$.

A **matching** μ is a mapping such that $\forall l \in S \cup U$, $\mu(l) = (\mu^1(l), \mu^2(l))$. A matching specifies which student goes to which university in both periods. For a set of agents $L \subseteq S \cup U$, let $\mu(L)$ be the set of agents that match with some member of L under μ. Let $\mathcal{M}^{\mathrm{t}}$ be the set of all period-t matchings, and let $\mathcal{M}$ be the set of all matchings.

2.2.1 Preferences

To make preferences simple, we make two innocuous assumptions. First, we assume a student in the market plans to graduate from a university at the end; second, a student who does not go to school in the first period will not be accepted in the second period.[8] No result will change without these assumptions.[9] A student may match with two different schools in period-1 and period-2. We call such sequence of matches a plan; (σ^1, σ^2) is a **plan** for a student where he is matched with $\sigma^1 \in U \cup \{c\} \cup \{\emptyset\}$ in the first period and with $\sigma^2 \in U$ in the

[8] We consider the first period matching as general education or as credential/experience in other labor markets.

[9] Let s in a student's plan represents self-matching. Without the first assumption, we need to add a requirement $uu \succ_s us$ in the definitions of binding agreement, no active student condition, period-1 individual rationality, individual rationality in the related marriage market since we have us plans in students' preferences; without the second assumption, we need to remove the requirement $\mu^1(s) \neq \emptyset$ in the definition of period-2 blocking since we have su plans in students' preferences.

second period. We sometimes write $\sigma^1\sigma^2 \equiv (\sigma^1, \sigma^2)$ if no confusion arises. Each student s has a *strict* preference relation $\succ_s$ over the set of plans. As usual, for any two plans (σ^1, σ^2), $(\sigma^1, \sigma^2)'$, we write $(\sigma^1, \sigma^2) \succeq_l (\sigma^1, \sigma^2)'$ if $(\sigma^1, \sigma^2) \succ_l (\sigma^1, \sigma^2)'$ or $(\sigma^1, \sigma^2) = (\sigma^1, \sigma^2)'$. Plan $\sigma^1\sigma^2$ is **acceptable** to student s if $\sigma^1\sigma^2 \succ_s \emptyset$ and **unacceptable** if $\emptyset \succ_s \sigma^1\sigma^2$. [10]

Each university u has a *strict* preference relation $\succ_u$ over 2^S, all possible subsets of students. Note that university u only cares about its final outcome, $\mu^2(u)$, which is a subset of S.[11] In another word, a university prefers the outcome of $\mu(u)$ over the outcome of $\mu'(u)$ when $\mu^2(u) \succ_u \mu'^2(u)$. To be consistent with the literature, when defining or referring to some conventional definition, we sometimes, by an abuse of notation, write $\mu(u) \succ_u \mu'(u)$ instead of $\mu^2(u) \succ_u \mu'^2(u)$. As usual, for any two subsets of students S', S'', we write $S' \succeq_u S''$ if $S' \succ_u S''$ or $S' = S''$. Student s is acceptable to university u if $s \succ_u \emptyset$.[12] Preference relation $\succ_u$ is *responsive with capacity* q_u if $\forall S' \subseteq S, \forall s' \in S'$ and $\forall s \notin S'$

1. $S' \setminus s' \cup s \succ_u S' \iff s \succ_u s'$;

2. $S' \cup s \succ_u S' \iff |S'| < q_u$ and $s \succ_u \emptyset$;

3. $|S'| > q_u \implies \emptyset \succ_u S'$.[13]

Following the literature, we assume the preference of each university is responsive with its capacity throughout the paper. By extracting all singleton sets from $\succ_u$, we will have its ranking over individual students. We denote this ranking by $\rhd_u$, and this is the university's preference over individual students in the benchmark model. We say $\succ_u$ is responsive to $\rhd_u$. Note that there maybe more than one $\succ_u$ that are responsive to a $\rhd_u$, but only one $\rhd_u$ can be extracted from a $\succ_u$.

Let $\succ = (\succ_l)_{l \in S \cup U}$ be the preference profile of all students and universities. For convenience, we often only write the acceptable plans (students) to denote a preference relation. For example,

$$\succ_s : u_1u_1, u_2u_1$$

means to student s, u_1u_1 is the most preferred plan, u_2u_1 follows the second, and no other plan is acceptable.

$(S, U, \succ)$ is a dynamic college admissions problem.

[10]Here $\emptyset$ should be $(\emptyset, \emptyset)$, but by the two assumptions we made, we can abbreviate $\sigma^1\emptyset$ and $\emptyset\sigma^2$ to $\emptyset$.

[11]In Roth and Sotomayor (1989), an outcome for a university is a set of incoming students, but it will be natural to say an outcome for a university is a set of graduating students here.

[12]We use s_i to denote the singleton set $\{s_i\}$ if no confusion arises.

[13]The original definition can be found in Roth (1985), and this version is modified from Kamada and Kojima (2015).

2.2.2 Dynamic Stability

We define our first solution concept, dynamic stability, recursively by using the idea of backward induction. For a student s, $u \succ_s^{\mu^1} u' \iff (\mu^1(s), u_j) \succ_s (\mu^1(s), u')$, where $\succ_s^{\mu^1}$ is the conditional preference of student s given μ^1. A period-2 matching μ^2 is **period-2 individually rational** if for each student $s \in S$, $\mu^2(s) \succeq_s^{\mu^1} \emptyset$, and if for each university $u \in U$, $\forall s \in \mu^2(u), s \succ_u \emptyset$. In word, μ^2 is period-2 individually rational if, conditioning on the first period matching μ^1, there is no student matched with some unacceptable plan, and there is no university matched with someone unacceptable. A period-2 matching μ^2 is **period-2 blocked** if there exists student s with $\mu^1(s) \neq \emptyset$ and university u such that $u \succ_s^{\mu^1} \mu^2(s)$ and $s \succ_u \sigma$ for some $\sigma \in \mu^2(u) \setminus \mu^1(u)$. A matching is **period-2 stable** if it is period-2 individually rational and it is *not* period-2 blocked.

We present two examples to shed light on period-1 concepts.

Example 1. Consider the following market. $S = \{s_1, s_2\}$, $U = \{u_1, u_2\}$,

$\rhd_{u_1}$: s_1 $\quad q_{u_1} = (0, 1)$,
$\rhd_{u_2}$: s_1, s_2 $q_{u_2} = (1, 0)$,
$\succ_{s_1}$: $u_2 u_1$,
$\succ_{s_2}$: $u_2 u_2$.

The set of all matchings $\mathcal{M} = \{\emptyset, \mu, \hat{\mu}\}$, where $\emptyset$ is the empty matching,

$$\mu = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ \emptyset & s_1 & \emptyset & s_2 \\ s_1 & \emptyset & \emptyset & s_2 \end{pmatrix} \begin{matrix} \\ \}\mu^1 \\ \}\mu^2 \end{matrix} \quad \textit{and}\ \hat{\mu} = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ \emptyset & s_2 & \emptyset & s_1 \\ \emptyset & s_2 & \emptyset & s_1 \end{pmatrix}.$$

The matrix μ is read as $\mu(u_1) = (\emptyset, \{s_1\})$, $\mu(u_2) = (\{s_1\}, \emptyset)$, $\mu(s_1) = u_2 u_1$, and $\mu(s_2) = (\emptyset, \emptyset)$. All three matchings in $\mathcal{M}$ are period-2 stable. Let's look through μ. The final outcome of u_2 is the empty set, so it rather has s_2 as its final outcome, but s_2 would not be willing to match with it in period-2 if he was not admitted in period-1. Anticipating this, u_2 would not extend offer to s_1 in the first period. Instead, it would admit s_2. Observe that $s_1 \succ_{u_2} s_2$, this implies when a university is contemplating to admit some student, its own preference is not the only determinant but also the student's preference.

We now draw connections between period-1 and period-2 matchings. In period-1, a university and a student can form an agreement on a seat in the second period.[14] An **agreement** $a(s, u)$ is a contract between s and u signed in period-1 such that university u holds a seat for s in period-2. A student s can sign at most one agreement, and u can sign at most q_u^2 agreements. But such an agreement is not credible if a university reneges when

[14] One example is "transfer agreement guarantee," which is generally offered by state universities to transfer students from in-state community colleges .

there are better alternatives to the university in the second period, so a student may hang back because of the possibility of being unmatched.

Example 2. Consider the following market. $S = \{s_1, s_2\}$, $U = \{u_1, u_2\}$,

$\rhd_{u_1}$: s_1, s_2 $q_{u_1} = (0, 1)$,

$\rhd_{u_2}$: s_1 $\quad q_{u_2} = (1, 0)$,

$\succ_{s_1}$: u_2u_1, cu_1,

$\succ_{s_2}$: cu_1.

Suppose, in a matching μ, s_2 signs an agreement with u_1 and $\mu^1(u_2) = \{s_1\}$. Then at the beginning of period-2, s_1 will bump s_2 out of u_1. Anticipating this, s_2 would not sign the agreement in the first period. This conservatism arises from complementarity. In example 2, when s_2 goes to c in the first period, going to u_1 in the second period is a complement to him because conditional preferences of a student are history dependent. Indeed, going to u_1 in the second period is a perfect complement to s_2 in this case since there is no other cu plan that is deemed to be acceptable to him. Moreover, since a university would lose a seat whenever a student transfers, it would rather admit someone who would not transfer, as shown in example 1. This can also be explained by complementarity: in period-1, a university considers the presences of a student in both periods as perfect complements.

We say an agreement $a(s, u)$ is **credible** if $s \succ_u \emptyset$ and the ranking of s in $\rhd_u$, $r_u(s)$, is less than or equal to q_u^2. It is easy to see a university has no incentive to deviate from any credible agreement. We say an agreement $a(s, u)$ is **binding** under μ^1 if it is credible and $u \succ_s^{\mu^1} u'$ for all $u' \neq u$; let k_u^2 be the remaining quota for transfer students in u if some positive number of binding agreements have been signed. We denote the set of students who sign a binding agreement under μ^1 by $S_b^{\mu^1} \subseteq S$.

While period-2 concepts are relatively normative, period-1 concepts rely on the idea of maximin. An agent will take an action if the worst potential outcome is better than the current outcome regardless others' action, and thus this action must only rely on agents' characteristics. In a dynamic market, it is necessary for an agent to anticipate the future development of the market, and such anticipation must ground on the contemporaneous market condition. If an agent is contemplating to match or to block the current matching, he must conjecture all possible subsequent evolutions induced by his action. This type of elaboration will be burdensome once the size of market gets large. The inability to track market evolution might make a university hesitate to admit a student that might transfer or might make a student hesitate to pursue a better plan. In this sense, the agents in our model are risk averse. An agent in general cannot be insulated from prophetic reasoning in a dynamic setting, so we want to have a stability concept that at least minimizes the prophetic counterfactual reasoning on market evolution.

Now we turn to a closer look at the conditional preferences of students. In a Nash equilibrium of a finitely repeated non-cooperative game, a stage game Nash equilibrium must be played in the last period. Following the same intuition, when an agent decides to participate in a period-1 matching, he does not anticipate any period-2 matching to arise, but a stable one, which is from a subset of stable period-2 matchings given the contemporaneous period-1 matching. We now construct this subset and then use it to form a student's conditional preference.

We say a student is **active** (given μ^1) in period-2 if (i) $s \notin S_b^{\mu^1}$; (ii) $\mu^1(s) \neq \emptyset$ and $\exists u \in U, u \neq \mu^1(s)$ such that $u \succ_s^{\mu^1} \mu^1(s)$ for some u with $q_u^2 > 0$ and $s \succ_u \emptyset$.[15] We define the *no active student condition* for a university u: $\forall s \in \mu^1(u)$, $u \succ_s^{\mu^1} u'$ for all $u' \neq u$ with $q_{u'}^2 > 0$ and $s \succ_{u'} \emptyset$. Let $S_a^{\mu^1} \subseteq S$ denote the group of active students given μ^1. We say a university is active in period-2 if $k_u^2 > 0$. Let $U_a^{\mu^1} \subseteq U$ denote the group of active universities given μ^1. Let $\succ^{\mu^1} = (\succ_l^{\mu^1})_{l \in S_a^{\mu^1} \cup U_a^{\mu^1}}$ denote the preference profile that consists the conditional preferences of all students in $S_a^{\mu^1}$ and the preferences of all universities in $U_a^{\mu^1}$. Then $A^{\mu^1} \equiv (S_a^{\mu^1}, U_a^{\mu^1}, \succ^{\mu^1})$ is actually a college admissions problem. Let m^{μ^1} be a stable outcome of this problem, and let M^{μ^1} be the set of stable outcomes. It is well-known that the set of stable outcome is nonempty.

Returning to period-2 matching. The set of students are now being partitioned into two groups, $S_a^{\mu^1}$ and $S \setminus S_a^{\mu^1}$, where $S \setminus S_a^{\mu^1}$ is the group of inactive students (given μ^1). Let $S_{inactive}^{\mu^1} \equiv S \setminus S_a^{\mu^1}$; note that $S_b^{\mu^1} \subseteq S_{inactive}^{\mu^1}$. If an active student is not matched with a university at a stable matching of A^{μ^1}, then in period-2, he either continues to match with his old university or be unmatched if he is matched with the community college without signing a credible agreement. Formally, $\forall s \in S_a^{\mu^1}$,

1. if $m^{\mu^1}(s) \neq \emptyset$ then $\mu^2(s) = m^{\mu^1}(s)$;
2. if $m^{\mu^1}(s) = \emptyset$ and $\mu^1(s) \neq c$, then $\mu^2(s) = \mu^1(s)$;
3. if $m^{\mu^1}(s) = \emptyset$ and $\mu^1(s) = c$, then $\mu^2(s) = \emptyset$.

Also, if $s \in S_b^{\mu^1}$, then $\mu^2(s) = u$, where u is the university that signs a binding agreement with s. Lastly, if $s \in S_{inactive}^{\mu^1} \setminus S_b^{\mu^1}$, then $\mu^2(s) = \mu^1(s)$. These five cases fully specify $\mu^2(s)$ for all students given any μ^1. Note that a student who was able to sign a credible agreement will not be unmatched in A^{μ^1} since he can at least match with the university that signs a credible agreement with him, i.e., he is in case 1. Recall that, for a university, if a student transfers, then the period-1 seat vanishes in the second period. Let $S_u^{m^{\mu^1}}$ be the set of students who are active in u and are able to matched with some university at a stable matching of A^{μ^1}, that

[15] If $\mu^1(s) = c$, (ii) is trivially satisfied.

is, $\forall s \in S_u^{m^{\mu^1}}$, $s \in \mu^1(u)$ and $m^{\mu^1}(s) \neq \emptyset$. Let $\mu^1_{m^{\mu^1}}(u) = \mu^1(u) \setminus S_u^{m^{\mu^1}}$ be the set of students that are either not active or active but are not matched with some university at a stable matching of A^{μ^1}. For each university u, $\mu^2(u) = \mu^1_{m^{\mu^1}}(u) \cup m^{\mu^1}(u) \cup S_b^{\mu^1}(u)$, where $S_b^{\mu^1}(u)$ is the set of students that sign a binding agreement with u under μ^1. We say $\tilde{\mu}^2_{\mu^1} \in \mathcal{M}^2$ is an **achievable** period-2 stable matching given the first period matching μ^1 if $\forall l \in S \cup U$, $\mu^2(l)$ is constructed as above. Fixing any μ^1, we will have $A^{\mu^1}$and thus a set of stable outcomes M^{μ^1}. Then we can construct a $\tilde{\mu}^2_{\mu^1}$ for each $m^{\mu^1} \in M^{\mu^1}$ to obtain a set of stable period-2 matchings give μ^1; denote it $\mathcal{M}^2_{\mu_1} \subseteq \mathcal{M}^2$.

Since A^{μ^1} is fully characterized by μ^1, a student can deduce $\mathcal{M}^2_{\mu^1}$ by following the rules above. In another word, he knows the set of stable period-2 matchings given μ^1.

A period-1 matching μ^1 is **period-1 individually rational** if for each student $s \in S$, $(\mu^1(s), \tilde{\mu}^2_{\mu^1}(s)) \succeq_s \emptyset$, and if for each university $u \in U$, $\forall s \in \mu^1(u)$, $s \succ_u \emptyset$ and $u \succ_s^{\mu^1} u'$ for all $u' \neq u$ with $q^2_{u'} > 0$ and $s \succ_{u'} \emptyset$.[16] In word, μ^1 is period-1 individually rational if no student is matched with some unacceptable plan under any achievable period-2 stable matching $\tilde{\mu}^2_{\mu^1}$, and no university is matched with someone either unacceptable or who will become an active student after matched with u under μ^1. By matching with an inactive student, a university can pledge itself this seat will not vanish at any matching. A period-1 matching μ^1 is **period-1 blocked** if there exists student s and university u such that (i) $(c, u) \succ_s (\mu^1(s), \tilde{\mu}^2_{\mu^1}(s))$ and $0 < r_u(s) \leq q^2_u$; or (ii) $(u, u) \succ_s (\mu^1(s), \tilde{\mu}^2_{\mu^1}(s))$ and $s \succ_u \sigma$ for some $\sigma \in \mu^1(u)$. In (i), the matching is blocked by a credible agreement $a(s, u)$; because of period-1 individual rationality, it is infeasible for a student s to block with u with a credible agreement to achieve some plan (u', u) with $u' \neq u$, and the only feasible path is (c, u). Note that although a credible agreement $a(s, u)$ can period-1 block μ^1, s is not in $\mu^1(u)$. In (ii), (u, u) is the worst plan s can pledge himself at any matching if he block μ^1 with u. Note that a specific μ^2 can be characterized by a bunch of agreements (not necessarily credible) given the current μ^1.

A matching μ is **dynamically individually rational** if it is period-t individually rational for all t, and it is **dynamically stable** if it is dynamically individually rational and *not* period-t blocked by any pair. At a dynamically stable matching: (i) no student would be forced to match with some unacceptable plan in the second period because of period-1 individual rationality; (ii) there is no active student in universities, a student either transfers to a university from the community college or stay in the same university in both periods.

In example 1, μ is period-1 individually irrational for u_2, $\emptyset$ is period-1 blocked by (u_2, s_2),

[16] It is also well-known that under strict preferences, the set of students and positions filled in a college admissions problem is the same at every stable matching by Theorem 9 of Roth (1984). So by checking whether a student is matched in one stable matching of A^{μ^1}, we can assert whether his period-1 individual rationality is satisfied.

and $\hat{\mu}$ is the unique dynamically stable matching. In example 2, μ is period-1 blocked by the credible agreement $a(s_1, u_1)$.

When $q_u^1 = 0$ for all u, the model degenerates to the college admissions problem, and even the solution concept, dynamic stability, would coincide with the benchmark stability. To see that, note that when $q_u^1 = 0$ for all u, only plans of the form cu will be meaningful. Also, if (s, u) period-1 block a matching via a credible agreement here, then (s, u) can also period-2 block the matching. Without loss of generality, we exclude the possibility of agreement here, so every student that has acceptable cu plan will be an active student. We denote the period-1 matching with only community college on the university side by μ^c. Given μ^c, we construct a period-2 matching μ^2 by following the rules above. Let m^{μ^c} be a stable outcome of $A^{\mu^c} \equiv (S_a^{\mu^c}, U_a^{\mu^c}, \succ^{\mu^c})$. For a student $s \in S_a^{\mu^c}$, if $m^{\mu^c}(s) \neq \emptyset$ then $\mu^2(s) = m^{\mu^c}(s)$ and $\mu(s) = (c, m^{\mu^c}(s))$; if $m^{\mu^c}(s) = \emptyset$ then $\mu(s) = (\emptyset, \emptyset)$ because of period-1 individual rationality. For a university $u \in U_a^{\mu^c}$, $\mu^2(u) = \mu^1_{m^{\mu^1}}(u) \cup m^{\mu^1}(u) \cup S_b^{\mu^1}(u) = m^{\mu^1}(u) = m^{\mu^c}(u)$. By construction, $\mu = (\mu^c, m^{\mu^c})$ is always period-1 individually rational and never period-1 blocked; μ^2 satisfies period-2 individual rationality if and only if m^{μ^c} is individually rational, and μ^2 is period-2 blocked if and only if m^{μ^c} is blocked. Hence, μ is dynamically stable if and only if m^{μ^c} is stable.

3 Properties of Dynamic Stability

3.1 Existence of Dynamically Stable Matchings

To show the existence of dynamically stable matching, we define the **student-proposing plan and agreement deferred acceptance algorithm (S-PA-DA):**

1. Begin with the empty matching, namely, a matching μ where $\mu(l) = (\emptyset, \emptyset)$ for all l.

2. Step 1: (a) For each student s, if uu is his most preferred plan, he proposes a two-period plan to u; if cu is his most preferred plan, he proposes an agreement to u; if uu' with $u \neq u'$ is his most preferred plan, he goes to the next most preferred plan. (b) Subject to q_u^1, each university u keeps the most preferred students who are proposing plan if they are acceptable and never become active. Also, subject to q_u^2, it keeps most preferred students who are proposing agreement if they are acceptable. Then it rejects all other students.

3. Step $n \geq 2$: (a) For each student s who was rejected in Step *(n-1)*, if uu is his next most preferred plan, he proposes a two-period plan to u; if cu is his next most preferred plan, he proposes an agreement to u; if uu' with $u \neq u'$ is his next most preferred plan,

he goes to the next next most preferred plan. (b) Subject to q_u^1, each university u keeps the most preferred students who are proposing plan if they are acceptable and never become active. Also, subject to q_u^2, it keeps most preferred students who are proposing agreement if they are acceptable. Then it rejects all other students.

4. Termination: Stops when no more application is made. Students who matched with a university via an agreement go to the community college in the first period.

This algorithm terminates in finite steps. We call the S-PA-DA matching μ_S.
Lemma 1: μ_S is dynamically stable.
Theorem 1: There exists a dynamically stable matching.

Theorem 1 is a direct result of Lemma 1. Following Roth and Sotomayor (1989), we construct the related marriage market for $(S, U, \succ)$ to further study the properties of dynamic stability.

3.2 The Related Marriage Market

Consider a college admission problem with transfer $(S, U, \succ)$ with student $S = \{s_1, s_2, ..., s_{|S|}\}$ and university $U = \{u_1, u_2, ..., u_{|U|}\}$ having quotas $(q_{u_1}^1, q_{u_1}^2), (q_{u_2}^1, q_{u_2}^2), ..., (q_{u_{|U|}}^1, q_{u_{|U|}}^2)$. The preferences of students are given by $\succ_{s_1}, \succ_{s_2}, ..., \succ_{s_{|S|}}$, and the preferences of universities on all possible subsets of students are given by $\succ_{u_1}, \succ_{u_2}, ..., \succ_{u_{|U|}}$. Also, we have the extracted rankings over individual students, $\rhd_{u_1}, \rhd_{u_2}, ..., \rhd_{u_{|U|}}$.

In the related marriage market of benchmark model, a university is divided into q_u pieces. Here, we first divide a university into two: one that does not admit transfer student and one that only admit transfer student; and then divide them into q_u^1 and q_u^2 pieces, respectively. We now replace u by q_u^1 period-1 positions of u and q_u^2 period-2 positions of u; denote them $u_{j,1}^1, u_{j,2}^1, ...u_{j,q_j^1}^1$ and $u_{j,1}^2, u_{j,2}^2, ...u_{j,q_j^2}^2$, respectively, where the subscripts $j \in \{1, 2, ..., |U|\}$. Let $\beta \leq q_j^1$ and $\gamma \leq q_j^2$. From now on we will keep calling our anonymous university u in the two-period market and call it u_j in the related marriage market; when we refer to an anonymous period-1 or period-2 position of some university, we will call it p. Denote the set of period-1 positions by U^1 , the set of period-2 positions by U^2, and the set of all positions by $\bar{U}$.[17] Now each position has a quota of 1, so we do not need to consider preferences over sets of students but only extracted rankings over individual students. Each student's preference is modified by replacing u by $u_{j,1}^1, u_{j,2}^1, ...u_{j,q_j^1}^1$ or $u_{j,1}^2, u_{j,2}^2, ...u_{j,q_j^2}^2$ depending on which period it appears in a plan. However, when we come to modify a university's preference, we need to consider separately for different periods. For a period-2 position, $u_{j,\gamma}^2$, it has "preference" $\rhd_u$,

[17] We use bar to indicate related marriage market.

but in a marriage market, we need both sides of the market to have a symmetric preference form. This means each position should have a preference over plans, so for each element s in the ranking $\rhd_u$, we extend it to us, where u in a position's plan represents self-matching. Denote this new "preference over plans" by $\succ_{u_j^2}$. For a period-1 position, $u_{j,\beta}^1$, we extend each element s in $\rhd_u$ to ss, and we denote this new "preference over plans" by $\succ_{u_j^1}$. Let $\bar{\succ} = (\succ_l)_{l \in S \cup \bar{U}}$ be the preference profile of all students and university positions.

Remark 1: In Kadam and Kotowski (2016), $\succ_u$ exhibits *strong inertia* if for s, s', $s \neq s'$, then $ss \succ_u ss'$ and $ss \succ s's$. $\succ_{u_j^1}$ satisfies this strong inertia requirement since there are only ss plans in it. [18]

The related marriage market of $(S, U, \succ)$ is described by $(S, \bar{U}, \bar{\succ})$. Let $\bar{\mathcal{M}}$ be the set of matchings in $(S, \bar{U}, \bar{\succ})$. Let $\bar{\mu} \in \bar{\mathcal{M}}$ be the matching that matches student in $\mu^t(u)$ to the ordered positions of a university according to $r_u(s)$. That is, if s is u's most preferred student in $\mu^1(u)$, then $\bar{\mu}^1(s) = u_{j,1}^1$; if s is u's second most preferred student in $\mu^2(u)$, then $\bar{\mu}^2(s) = u_{j,2}^2$ and so on. We call $\bar{\mu}$ the corresponding matching of μ; by defining $\bar{\mu}$ as above, there is a natural bijection between $\mathcal{M}$ and $\bar{\mathcal{M}}$.

A matching $\bar{\mu}$ is **individually rational** if (i) $\forall s \in S$, $\bar{\mu}(s) \succeq_s \emptyset$; (ii) $\forall u_{j,\beta}^1 \in \bar{U}$, $\bar{\mu}(u_{j,\beta}^1) \succeq_{u_j^1} \emptyset$ and if $\bar{\mu}(u_{j,\beta}^1) = ss$, $u \succ_s^u u'$ for all $u' \neq u$ with $q_{u'}^2 > 0$ and $s \succ_{u'} \emptyset$; (iii) $\forall u_{j,\gamma}^2 \in \bar{U}$, $\bar{\mu}(u_{j,\gamma}^2) \succ_{u_j^2} \emptyset$. A pair $(s, u_{j,\beta}^1)$ or $(s, u_{j,\gamma}^2)$ can **block** the matching $\bar{\mu}$ if

1. $u_{j,\beta}^1 u_{j,\beta}^1 \succ_s \bar{\mu}(s)$ and $ss \succ_{u_j^1} \bar{\mu}(u_{j,\beta}^1)$; or

2. $cu_{j,\gamma}^2 \succ_s \bar{\mu}^1(s)$ and $u_{j,\gamma}^2 s \succ_{u_j^2} \bar{\mu}(u_{j,\gamma}^2)$.

A matching is **stable** if it is individually rational and cannot be blocked by any pair.

Lemma 2: A matching μ is dynamically stable if and only if its corresponding matching $\bar{\mu}$ is stable.

We define the **university-proposing plan and agreement deferred acceptance algorithm (U-PA-DA)** in the related marriage market:

1. Begin with the empty matching, namely, a matching $\bar{\mu}$ where $\bar{\mu}(l) = \emptyset$ for all l.

2. Step 1: Each position (of universities) proposes to its most preferred student according to its individual rationality. Each student keeps his most preferred plan and rejects all others.

[18] Requirement 5 of period-2 implies ss plans exist. Requirement 6 and 7 of a period-2 matching veto us plans and ss' plans, respectively. su plans are vetoed since a university u only cares about its final outcome, $\mu^2(u)$.

3. Step $n \geq 2$: (a) Each position that was rejected in Step *(n-1)* then proposes to its next most preferred student according to its individual rationality. Each student keeps his most preferred plan and rejects all others.

4. Termination: Stops when no more proposal is made.

We denote this matching by $\bar{\mu}_{\bar{U}}$.
Lemma 3: $\bar{\mu}_{\bar{U}}$ is stable.

3.3 Positive Results of the Benchmark Model

As shown in Roth (1985), since the marriage problem is a special case of the college admissions problem, many positive results for the marriage problems do not generalize to the college admissions problem. We showed at the end of section 2, if $q_u^1 = 0$ for all u, the two-period model degenerates to the college admissions problem, so it inherits *all* negative results of the college admissions problem. In this section, we try to extend the relevant positive results in Gale and Shapley (1962); Roth (1984, 1985, 1986); Roth and Sotomayor (1989, 1992). Note that some of these results are built on other results from the marriage market and our related marriage market differs from the marriage market because of the two-period nature. Therefore, we need to prove all relevant results again in the two-period context. Surprisingly, when a positive result continue to hold in the two-period context, its proof only requires slight modification from the original. We continue to assume preferences are *strict*, and univerisities' preferences satisfy responsiveness.

3.3.1 Positive Results That Continue to Hold

A stable matching is **student-optimal** if each student prefers his assignment in that matching to his assignment in all other stable matchings. A **position-optimal** stable matching and a **university-optimal** stable matching are defined analogously. A student and a position (or a university) are said to be *achievable* to each other if they are matched in some (dynamically) stable matching. It is well-known that, in the benchmark model, optimal stable matchings exist if preferences are strict. This will continue to hold in the dynamic model.
Theorem 2: (i) μ_S is student-optimal; (ii) $\bar{\mu}_{\bar{U}}$ is position-optimal.

Let $\mu >_S \mu'$ denote that all students like μ at least as well as μ' with at least one student strictly prefer, that is, $\mu(s) \succeq_s \mu'(s)$ for all s, and $\mu(s) \succ_s \mu'(s)$ for at least one s. $>_{\bar{U}}$ and $>_U$ are defined analogously for positions and universities.
Theorem 3: If $\bar{\mu}$ and $\bar{\mu}'$ are stable matchings, then $\bar{\mu} >_S \bar{\mu}'$ if and only if $\bar{\mu}' >_{\bar{U}} \bar{\mu}$.

So any stable matching that is better for all student is worse for all positions, and vice versa. Following from Theorem 2 and Theorem 3, we have the following corollary.

Corollary 1: (i) μ_U is the worst dynamically stable matching for students; (ii)$\bar{\mu}_S$ is the worst stable matching for positions.

Theorem 4: μ_U is university-optimal, and μ_S is the worst dynamically stable matching for universities.

Though Theorem 4 is closely related to Theorem 2 and 3, it requires some extra arguments because $>_{\bar{U}}$ and $>_U$ are two different partial orders.

Lemma 4 (Decomposition Lemma): Let $\bar{\mu}$ and $\bar{\mu}'$ be stable matching in $(S, \bar{U}, \bar{\succ})$. Let $S^{\bar{\mu}}$ $(\bar{U}^{\bar{\mu}})$ be the sets of students (positions) who prefer $\bar{\mu}$ to $\bar{\mu}'$, and let $S^{\bar{\mu}'}$ $(\bar{U}^{\bar{\mu}'})$ be those who prefer $\bar{\mu}'$. Then $\bar{\mu}$ and $\bar{\mu}'$ map $S^{\bar{\mu}}$ onto $\bar{U}^{\bar{\mu}'}$and $S^{\bar{\mu}'}$ onto $\bar{U}^{\bar{\mu}}$.

If μ and μ' are matchings we define

$$\lambda(u) \quad = \mu \vee_U \mu' = \begin{cases} \mu(u) & if\; \mu(u) \succ_u \mu'(u) \\ \mu'(u) & otherwise \end{cases}, \lambda(s) = \mu \wedge_S \mu' = \begin{cases} \mu(s) & if\; \mu'(s) \succ_s \mu(s) \\ \mu(s) & otherwise \end{cases};$$

similarly,

$$v(u) \quad = \mu \wedge_U \mu' = \begin{cases} \mu(u) & if\; \mu'(u) \succ_u \mu(u) \\ \mu'(u) & otherwise \end{cases}, v(s) = \mu \vee_S \mu' = \begin{cases} \mu(s) & if\; \mu(s) \succ_s \mu'(s) \\ \mu(s) & otherwise \end{cases}.$$

Again, we write $\mu(u) \succ_u \mu'(u)$ instead of $\mu^2(u) \succ_u \mu'^2(u)$ to keep consistency. If λ is a matching, then λ is the least upper bound for $\{\mu, \mu'\}$ under $>_U$ and the greatest lower bound for $\{\mu, \mu'\}$ under $>_S$. In the related marriage market, $\bar{\lambda}(p) = \bar{\mu} \vee_{\bar{U}} \bar{\mu}'$, $\bar{\lambda}(s) = \bar{\mu} \wedge_S \bar{\mu}'$, $\bar{v}(p) = \bar{\mu} \wedge_{\bar{U}} \bar{\mu}'$, and $\bar{v}(s) = \bar{\mu} \vee_S \bar{\mu}'$ are defined analogously. If $\bar{\lambda}$ is a matching, then $\bar{\lambda}$ will be the least upper bound for $\{\bar{\mu}, \bar{\mu}'\}$ under $>_{\bar{U}}$ and the greatest lower bound for $\{\bar{\mu}, \bar{\mu}'\}$ under $>_S$.

Theorem 5 (Lattice Theorem): If $\bar{\mu}$ and $\bar{\mu}'$ are stable matchings, then $\bar{\lambda}$ and $\bar{v}$ are stable matchings.

Corollary 2 below follows from Theorem 3 together with Theorem 5.

Corollary 2: The set of stable matchings forms a lattice under $>_{\bar{U}}$ and $>_S$, and the lattice under $>_{\bar{U}}$ is the the dual to the lattice under $>_S$.

Theorem 6 (Weak Pareto Optimality for the Students): There is no dynamically individually rational matching μ (dynamically stable or not) such that $\mu \succ_s \mu_S$ for all $s \in S$.

This result will also hold for $\bar{\mu}_{\bar{U}}$ in the related marriage market, but the counterpart for μ_U has been shown false in Roth (1985).

Lemma 5 (Blocking Lemma): Let μ be any dynamically individually rational matching under $\succ$ and let S' be all students who prefer μ to μ_S. If S' is nonempty, there is a pair (s,p) that blocks μ such that $s \in S \setminus S'$ and $p \in \mu(S')$.

Again, this result will hold for $\bar{\mu}_{\bar{U}}$ in the related marriage market. But unlike Theorem 6, this result also holds for μ_U. When some university prefers another individual rational matching to the university-optimal matching μ_U, by responsiveness, it is equivalent to say some of its positions prefer another individual rational matching to the position-optimal matching $\mu_{\bar{U}}$ in the related marriage market.

Theorem 7: The set of students and positions that are matched is the same for all stable matchings.

The following two results are the extended versions of their original counterpart.

Lemma 6: Let μ and μ' be dynamically stable matchings in $(S, U, \succ)$ and let $\bar{\mu}$ and $\bar{\mu}'$ be their corresponding stable matching in $(S, \bar{U}, \bar{\succ})$, respectively. If $\mu^1(u) \neq \mu'^1(u)$ for some u such that $\bar{\mu}(u^1_{j,i}) \succ_{u^1_j} \bar{\mu}'(u^1_{j,i})$ for some $u^1_{j,i}$, then $\bar{\mu}(u^1_{j,\beta}) \succeq_{u^1_j} \bar{\mu}'(u^1_{j,\beta})$ for all positions $u^1_{j,\beta}$ of u. If $\mu^2(u) \setminus \mu^1(u) \neq \mu'(u) \setminus \mu'^1(u)$ for some u such that $\bar{\mu}(u^2_{j,i}) \succ_{u^2_j} \bar{\mu}'(u^2_{j,i})$ for some $u^2_{j,i}$, then $\bar{\mu}(u^2_{j,\gamma}) \succeq_{u^1_j} \bar{\mu}'(u^2_{j,\gamma})$ for all positions $u^2_{j,\gamma}$ of u.

Theorem 8 (Rural Hospitals Theorem): Any university that does not fill its quota in some period at some dynamically stable matching will match with the same set of students at every dynamically stable matching for that period.

3.3.2 Positive Results That Fail to Hold

A mechanism φ is a function that maps a preference profile to a matching. $\varphi(\succ)$ denotes the matching outcome of φ given the preference profile $\succ$, and $\varphi_l(\succ)$ denotes the outcomes for $l \in S \cup U$. A mechanism φ is said to be *stable* if it yields a (dynamically) stable matching for each preference profile. A mechanism φ is said to be **strategy-proof** if there does not exist a preference profile $\succ$, an agent $l \in S \cup U$ with $\hat{\succ}_l$ such that $\varphi_l(\hat{\succ}_l, \succ_{-l}) \succ_l \varphi(\succ)$. A mechanism φ is said to be **strategy-proof for students** if there does not exist a preference profile $\succ$, an student $s \in S$ with $\hat{\succ}_s$ such that $\varphi_s(\hat{\succ}_s, \succ_{-s}) \succ_s \varphi(\succ)$. We define the **S-PA-DA mechanism** to be a mechanism that use S-PA-DA algorithm to produces a matching for each input. A well-known result in Roth (1982) says there does not exist a stable matching mechanism that is strategy-proof. Roth (1985) shows the following Theorem.

Theorem 1*: In a college admission problem, a stable mechanism that yields the student-optimal stable matching is strategy-proof for students.

This result does not hold in the two-period context.

Proposition 3: There does not exist a stable matching mechanism that is strategy-proof for students.

Proof: In example 1, the dominant strategy for s_1 is to state $\hat{\succ}_{s_1}$: u_2u_2 instead of his true preference $\succ_{s_1}$: u_2u_1. The only dynamically stable matching given $(\hat{\succ}_{s_1}, \succ_{-s_1})$ matches s_1 to u_2u_2, but s_1 can then transfer to u_1. □

The main results of Roth and Sotomayor (1989) rely on Lemma 6 (its single period version); nevertheless, in Roth and Sotomayor (1992), they indicate that some further results of the college admission model also depend critically on Lemma 6. Although Lemma 6 still holds, it is not strong enough to support those results in the two-period environment. We now review those theorems and provide counterexamples in the two-period context.

Theorem 2*: In a college admissions problem, if μ and μ' are two stable matchings, and u is a university with $q_u = k$ such that $\mu(u) \neq \mu'(u)$ and $\mu(u) = \{s_1, s_2, ..., s_k\}$ and $\mu'(u) = \{s'_1, s'_2, ..., s'_k\}$, where the students are listed in order of $\rhd_u$, that is, $\forall i$, $s_i \succ_u s_{i+1}$ and $s'_i \succ s'_{i+1}$. If i is any index such that $s_i \succ_u s'_i$, then $s_\gamma \succeq_u s'_\gamma$ for all $\gamma \in \{1, 2, ..., k\}$ and $\mu(u) \succ_u \mu'(u)$.

Theorem 2* is an alternative statement for Theorem 3 of Roth and Sotomayor (1989).[19] This result is for the college admissions problem while Lemma 6 is for its related marriage market. They use the following example to illustrate the result: consider a university u with quota 2 and preference over individual students $\rhd_u$: s_1, s_2, s_3, s_4. Let $\mu(u) = \{s_1, s_4\}$ and $\mu'(u) = \{s_2, s_3\}$. Then they can not both be stable because $s_1 \succ_u s_2$ but $s_4 \not\succeq_u s_3$ and $s_3 \succ_u s_4$ but $s_2 \not\succeq_u s_1$.

Proposition 4: The result of Theorem 2* does not hold in the dynamic college admissions problem.

Proof: We prove it by an example.

Example 3. Consider the following market. $S = \{s_1, s_2, s_3, s_4\}$, $U = \{u_1, u_2\}$,

$\rhd_{u_1}$: s_1, s_2, s_3, s_4 $q_{u_1} = (1, 1)$,

$\rhd_{u_2}$: s_4, s_3, s_2, s_1 $q_{u_2} = (1, 1)$,

$\succ_{s_1}$: u_2u_2, u_1u_1,

$\succ_{s_2}$: u_1u_1, u_2u_2,

$\succ_{s_3}$: cu_2, cu_1,

$\succ_{s_4}$: cu_1, cu_2.

[19] We take this alternative statement from Roth and Sotomayor (1992). In Roth and Sotomayor (1989), preferences over individual students are strict, but unlike our setting, preferences over groups of students are not assumed to be strict. The original theorem states that if universities and students have strict preference over individuals, then universities have strict preferences over those groups of students that they may be assigned at stable matchings. Then this implies if μ and μ' are two different stable matchings, then there does not exist responsive preference that is indifferent between $\mu(u)$ and $\mu'(u)$.

In this market,

$$\mu = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ s_1 & s_2 & s_3, s_4 & \emptyset \\ s_1, s_4 & s_2, s_3 & \emptyset & \emptyset \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ s_2 & s_1 & s_3, s_4 & \emptyset \\ s_2, s_3 & s_1, s_4 & \emptyset & \emptyset \end{pmatrix},$$

so both $\mu^2(u_1) = \{s_1, s_4\}$ and $\mu'^2(u_1) = \{s_2, s_3\}$ are now stable outcome for u_1. However, $s_1 \succ_{u_1} s_2$ but $s_4 \not\succeq s_3$ and $s_3 \succ_{u_1} s_4$ but $s_2 \not\succeq_u s_1$, which contradicts to the conclusion of Theorem 2*. □

Theorem 3*: In a college admissions problem, if μ and μ' are both stable matchings and $\mu(u) \succ_u \mu'(u)$ for some u, then $s \succ_u s'$ for all $s \in \mu(u)$ and $s' \in \mu'(u) \setminus \mu(u)$.

In words, u prefers every student in $\mu(u)$ to every student in $\mu'(u)$ but not in $\mu(u)$. This is Theorem 4 of Roth and Sotomayor (1989). They use the following example to illustrate the result: consider a university u with quota 2 and preference $\rhd_u$: s_1, s_2, s_3, s_4. Let $\mu(u) = \{s_1, s_3\}$ and $\mu'(u) = \{s_2, s_4\}$. Note than we cannot use Theorem 2* to determine whether they are both stable or not. Obviously, $\mu(u) \succ_u \mu'(u)$ by responsiveness and transitivity. By Theorem 3*, they can not both be stable since $s_3 \not\succ_u s_2$. Corollary 1* follows immediately from Theorem 3* and responsiveness.

Corollary 1*: In a college admissions problem, if $\succ_u$ and $\succ'_u$ are responsive to $\rhd_u$, then for every pair of stable matchings μ and μ', $\mu(u) \succ_u \mu'(u)$ if and only if $\mu(u) \succ'_u \mu'(u)$.

Proposition 5: The results of Theorem 3* and Corollary 1* do not hold in the dynamic college admissions problem.

Proof: Consider Example 3 again. In that market,

$$\mu_S = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ s_2 & s_1 & s_3, s_4 & \emptyset \\ s_2, s_4 & s_1, s_3 & \emptyset & \emptyset \end{pmatrix} \text{ and } \mu_U = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ s_1 & s_2 & s_3, s_4 & \emptyset \\ s_1, s_3 & s_2, s_4 & \emptyset & \emptyset \end{pmatrix},$$

so both $\mu_S^2(u_1) = \{s_2, s_4\}$ and $\mu_U^2(u_1) = \{s_1, s_3\}$ are now stable outcome for u_1. However, $s_3 \not\succ_{u_1} s_2$, which contradicts to the conclusion of Theorem 3*.

Now consider

$$\succ_{u_1}: ..., \{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4$$

and

$$\succ'_{u_1}: ..., \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4.$$

Note that both preferences are responsive to their extracted ranking $\rhd_{u_1}$, so both of them can be u_1's preference, which contradicts to the conclusion of Corollary 1*. □

The following three theorems and their corollaries can be found in Roth and Sotomayor (1992).

Theorem 4*: In a college admissions problem, if μ and μ' are stable matchings, then $\mu >_S \mu'$ if and only if $\mu' >_U \mu$.

Theorem 4* is a parallel result to Theorem 3 in the related marriage market.

Proposition 6: The result of Theorem 4* does not hold in the dynamic college admissions problem.

Proof: Consider μ and μ' in Example 3 again with

$$\succ_{u_1}: ..., \{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4$$

and

$$\succ_{u_2}: ..., \{s_3, s_4\}, \{s_2, s_4\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_1, s_3\}, \{s_1, s_2\}, s_4, s_3, s_2, s_1.$$

Now $\mu >_U \mu'$ but $\mu' \not>_S \mu$ since s_3 and s_4 prefer μ over μ', which contradicts to the conclusion of Theorem 4*. □

Theorem 5*: In a college admissions problem, if μ and μ' are stable matchings, then λ and v are stable matchings.

Corollary 2*: In a college admissions problem, the set of stable matchings forms a lattice under $>_U$ and $>_S$, and the lattice under $>_U$ is the the dual to the lattice under $>_S$.

Theorem 5* is parallel to Theorem 3 in the related marriage market. Corollary 2* follows from Theorem 4* and Theorem 5*.

Proposition 7: The results of Theorem 5* and Corollary 2* do not hold in the dynamic college admissions problem.

Proof: Consider μ and μ' in Example 3 again with

$$\succ_{u_1}: ..., \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4.$$

Notice that $\lambda^2(u_1) = \mu'^2(u_1) = \{s_2, s_3\}$ and $\lambda(s_1) = \mu(s_1) = u_1u_1$, so λ is not a matching. Similarly, $v^2(u_1) = \mu^2(u_1) = \{s_1, s_4\}$ and $v(s_1) = \mu'(s_1) = u_2u_2$, so v is not a matching, which contradicts to the conclusions of Theorem 5*.

So Corollary 2* does not hold too. □

Despite the failure of Corollary 2*, Corollary 2 together with Lemma 2 ensure the set of stable matchings forms a lattice under $>_S$ in the two-period market. In Example 3,

$\mu_U = \mu \wedge_S \mu'$ and $\mu_S = \mu \vee_S \mu'$, but if

$$\succ_{u_1}: ..., \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4$$

as in the proof of Proposition 7, and u_2 to have the preference in the proof of Proposition 6

$$\succ_{u_2}: ..., \{s_3, s_4\}, \{s_2, s_4\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_1, s_3\}, \{s_1, s_2\}, s_4, s_3, s_2, s_1$$

, then $\mu \vee_U \mu'$ ($\neq \mu_U$) is not a matching since both of them want $\{s_2, s_3\}$. Of course, if one of them prefers $\{s_1, s_4\}$ to $\{s_2, s_3\}$, then the set of dynamically stable matchings can be a lattice under $>_U$, too. Here, since $>_U$ and $>_{\bar{U}}$ are different partial orders, when a university points to a preferable stable matching, its period-1 positions point to one stable matching, while its period-2 positions point to another stable matching. This does not happen in the college admissions problem since $>_{\bar{U}}$ always agrees with $>_U$ when there is only one period.

Theorem 6*: In a college admissions problem, if μ and μ' are stable matchings and $u = \mu(s)$ or $u = \mu'(s)$, then if $\mu(u) \succ_u \mu'(u)$ then $\mu'(s) \succeq_s \mu(s)$ (and if $\mu'(s) \succ_s \mu(s)$ then $\mu(u) \succeq_u \mu'(u)$).

Theorem 6* is an analogue of Decomposition Lemma (Lemma 4).

Proposition 8: The result of Theorem 6* does not hold in the dynamic college admissions problem.

Proof: Consider μ and μ' in Example 3 again with

$$\succ_{u_1}: ..., \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4.$$

$\{s_2, s_3\} = \mu'^2(u_1) \succ_{u_1} \mu^2(u_1) = \{s_1, s_4\}$ and $\mu'(s_2) = u_1 u_1 \succ_{s_2} \mu(s_2) = u_2 u_2$, which contradicts to the conclusion of Theorem 6*. □

Even the two-period model reduces to the benchmark model when $q_u^1 = 0$ for all u, Lemma 6 delivers less power in the more general case. In particular, in the single period market, Lemma 6 enables us to rank the students in a university's outcome from 1 to q_u, but in the two-period market, we can no longer do it. Theorem 2* to 6* concern two stable matchings. When we consider two dynamically stable outcomes for a university, $\mu(u)$ and $\mu'(u)$, one outcome can be from a matching consists of a superior period-1 matching and an inferior period-2 matching, while another outcome comes from a matching consists of an inferior period-1 matching and a superior period-2 matching, that is, $\mu^1(u) \succ_u \mu'^1(u)$ and $\mu'^2(u) \setminus \mu'^1(u) \succ_u \mu^2(u) \setminus \mu^1(u)$.

$q_u^1 = 0$ for all u is a strong condition; indeed, Theorem 2* to 6* and their corollaries will hold in the two-period model under a slightly weaker condition: when $q_u^1 = 0$ or $q_u^2 = 0$ for all u. Under such condition, the power of Lemma 6 is restored. The reader can easily verify them by modifying their original proofs together with Lemma 6.

Now we consider two other solution concepts for the benchmark model, the core and group stability. A matching μ is **dominated** by another matching μ' via a coalition $B \subseteq S \cup U$ if (i) $\forall l \in B$, $i \in \mu'(l)$ implies $i \in B$, and (ii) $\forall l \in B, \mu'(l) \succ_l \mu(l)$. The set of matchings that are not dominated by any other matching is the **core**. A matching μ is **Pareto dominated** by μ' if (i) $\forall l \in S \cup U$, $\mu'(l) \succeq_l \mu(l)$, and (ii)$\exists l \in S \cup U$ such that $\mu'(l) \succ_l \mu(l)$. A matching μ is **Pareto efficient** if it is not Pareto dominated by any other matching, and we call the set of matchings that are not Pareto dominated by any other matching the **Pareto set**. A matching μ is **weakly dominated** by another matching μ' via a coalition $B \subseteq S \cup U$ if (i) $\forall l \in B$, $i \in \mu'(l)$ implies $i \in B$, (ii) $\forall l \in B$, $\mu'(l) \succeq_l \mu(l)$, and (iii) $\exists l \in B$ such that $\mu'(l) \succ_l \mu(l)$. The set of matchings that are not weakly dominated by any other matching is the **weak core**. When a matching is not weakly dominated by another matching, it is not Pareto dominated by any other matching and not dominated by any other matching, that is, the weak core is inside the core and also inside the Pareto set; yet, there is no logical relationship between the core and Pareto set.

In a basic one-to-one matching (marriage) problem, the set of stable matchings is equivalent to the core, and under strict preferences, the core is equivalent to the weak core, so in this case, the matchings in the core are Pareto efficient.[20] In a basic many-to-one matching (college admissions) problem, the set of stable matchings is equivalent to the weak core, while the core might contain some unstable outcomes.[21] That implies the set of stable matchings is still inside the Pareto set, but the core might not. The following result can be found in Proposition 5.36 of Roth and Sotomayor (1992).

Proposition 1*: In a college admissions problem, a matching is in the weak core if and only if it is stable.

Roth and Sotomayor (1989) defines another notion for multilateral blocking, group stability, that does not require condition (i) in the definition of core or weak core, which requires the coalition members only match with other members within the coalition after a deviation. A **group deviation** from μ is a group $B \subseteq S \cup U$ and a matching μ' such that (i) $\forall s \in B$, $\mu'(s) \in B$, (ii) $\forall u \in B$, $\sigma \in \mu'(u) \implies \sigma \in B \cup \mu(u)$, and (iii) $\forall l \in B$, $\mu'(l) \succ_l \mu(l)$.[22] A matching μ is **group stable** if it is immune to any group deviation from μ.

Proposition 2*: In a college admissions problem, a matching is group stable if and only if it is stable.

[20] The first equivalence can be found in Theorem 3.3 of Roth and Sotomayor (1992).
[21] Roth and Sotomayor (1992) provide an example for unstable outcomes in the core.
[22] Konishi and Ünver (2006) defines a more general definition for many-to-many matching problems.

This result is Proposition 1 of Roth and Sotomayor (1989); it says in a college admissions problem, the set of group stable matchings coincides with the set of stable matchings. There is no logical relationship between the weak core and the set of group stable matchings. But in a college admissions problem, they coincide with each other via proposition 1* and 2*.

In the literature of dynamic matching, the core equivalent is sometimes called the "dynamic core" (Kadam and Kotowski (2016)) or the "recursive core" (Becker and Chakrabarti (1995); Damiano and Lam (2005)). A matching μ is **period-1 dominated** by another matching $\mu' = (\mu_B^1, \mu_B^2)$ via a coalition $B \subseteq S \cup U$ if (i) $\forall l \in B$, $i \in \mu_B^t(l)$ implies $i \in B$ and $l \in \mu_B^t(i)$, and (ii)$\forall l \in B$, $\mu'(l) \succ_l \mu(l)$. A matching μ is **period-2 dominated** by another matching $\mu' = (\mu^1, \mu_B^2)$ via a coalition $B \subseteq S \cup U$ if (i) $\forall s \in B$, $\mu_B^2(s) = u$ implies $u \in B$ and $s \in \mu_B^2(u) \setminus \mu^1(u)$, (ii) $\forall u \in B$, $s \in \mu_B^2(u) \setminus \mu^1(u)$ implies $s \in B$ and $\mu_B^2(s) = u$, and (iii) $\forall l \in B, \mu'(l) \succ_l \mu(l)$. The set of matchings that are not period-t dominated by any other matching is the **dynamic core**. The dynamic core, unlike the core, specifies that a student might go to different universities in different periods. If a coalition is formed in period-1, a student can go to different universities within the coalition in different periods.[23] If a coalition is formed in period-2, a student can go to a university in the coalition that might be different from his period-1 school. A coalition forms in period-2 can not alter the period-1 matching.

Recall that in a many-to-one matching problem, the core might contain some unstable outcomes, and this implies the dynamic core might contain some unstable outcomes too since the dynamic core is equivalent to the core when $q_u^1 = 0$ for all u.[24]

Now we define the weak core equivalent. A matching μ is **weakly period-1 dominated** by another matching $\mu' = (\mu_B^1, \mu_B^2)$ via a coalition $B \subseteq S \cup U$ if (i) $\forall l \in B$, $i \in \mu_B^t(l)$ implies $i \in B$ and $l \in \mu_B^t(i)$, (ii) $\forall l \in B$, $\mu'(l) \succeq_l \mu(l)$, and (iii) $\exists l \in B$ such that $\mu'(l) \succ_l \mu(l)$. A matching μ is **weakly period-2 dominated** by another matching $\mu' = (\mu^1, \mu_B^2)$ via a coalition $B \subseteq S \cup U$ if (i) $\forall s \in B$, $\mu_B^2(s) = u$ implies $u \in B$ and $s \in \mu_B^2(u) \setminus \mu^1(u)$, (ii) $\forall u \in B$, $s \in \mu_B^2(u) \setminus \mu^1(u)$ implies $s \in B$ and $\mu_B^2(s) = u$, (iii) $\forall l \in B$, $\mu'(l) \succeq_l \mu(l)$, and (iv) $\exists l \in B$ such that $\mu'(l) \succ_l \mu(l)$. The set of matchings that are not weakly period-t dominated by any other matching is the **dynamic weak core**. Note that μ_B^2 must be period-2 stable since Proposition 1* holds for $A^{\mu_B^1}$ and A^{μ^1}. It is also easy to see the dynamic weak core is inside the Pareto set.

Proposition 9: The dynamic weak core might not coincide with the set of dynamically stable matchings.

[23] He can go to c in the first period as well.

[24] In a dynamic marriage market, Kadam and Kotowski (2016) also show the dynamic core might not be equivalent to the set of dynamically stable matchings.

Proof: We prove it by an example.

Example 4. Consider the following market. $S = \{s_1\}$, $U = \{u_1, u_2\}$,

$\succ_{u_1}$: s_1 $q_{u_1} = (0, 1)$,

$\succ_{u_2}$: s_1 $q_{u_2} = (1, 0)$,

$\succ_{s_1}$: $u_2 u_1$.

The unique dynamically stable matching is the empty matching, and

$$\mu = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ \emptyset & s_1 & \emptyset & \emptyset \\ s_1 & \emptyset & \emptyset & \emptyset \end{pmatrix}$$

is the only matching in the dynamic weak core. The empty matching is weakly period-1 dominated by μ via the grand coalition $B = S \cup U$ with $\mu(s_1) = u_2 u_1 \succ_{s_1} \emptyset$, $\mu^2(u_1) = s_1 \succ_u \emptyset$, and $\mu^2(u_2) = \emptyset \succeq_{u_2} \emptyset$. □

Example 4 shows that there might be some dynamically individually irrational matching in the dynamic weak core. Also, a dynamically stable matching might be Pareto dominated by another matching. In another word, a dynamically stable matching is not necessarily Pareto efficient.

In a period-1 core deviation, the coalition members only match within the coalition, so there is no uncertainty. But when a group is contemplating a group deviation in the first period, the group needs to consider the outsiders who are dropped by its members. Those dropped agents will trigger evolutions in both periods, and this will make it very demanding to form a group deviation in the first period. A **period-1 group deviation** from μ is a group $B \subseteq S \cup U$ and some matching $\tilde{\mu} = (\tilde{\mu}_B^1, \tilde{\mu}_{\tilde{\mu}_B^1}^2)$ such that (i) $\forall s \in B$, $\tilde{\mu}_B^1(s) \in B$, (ii) $\forall u \in B$, $\sigma \in \tilde{\mu}_B^1(u) \implies \sigma \in B \cup \mu^1(u)$, and (iii) $\forall l \in B$, $\tilde{\mu}(l) \succ_l \mu(l)$.

A student in a core deviation might match with c in the first period. In contrast, condition (i) here actually says a student in B will match with some university in B in the first period. The intuition is that if a student wants to sign a credible agreement with some $u \in B$, he does not need to join the group; if a student wants to sign an incredible agreement with some $u \in B$, his presence will not affect the evolution in the first period and since the definition explicitly requires the period-2 matching must be period-2 stable, he is not guaranteed to match with u, which will potentially violate condition (iii).

A **period-2 group deviation** from μ is a group $B \subseteq S \cup U$ and a matching $\mu' = (\mu^1, \mu_B^2)$ such that (i) $\forall s \in B$, $\mu_B^2(s) \in B$, (ii) $\forall u \in B$, $\sigma \in \mu_B^2(u) \implies \sigma \in B \cup \mu^2(u) \setminus \mu^1(u)$, and (iii) $\forall l \in B$, $\mu'(l) \succ_l \mu(l)$. Again, note that μ_B^2 must be period-2 stable since Proposition 2* holds for $A^{\tilde{\mu}_B^1}$ and A^{μ^1}. A matching μ is **dynamically group stable** if it is immune to any period-t group deviation from μ.[25]

Proposition 10: The set of dynamically group stable matchings might not coincide with the set of dynamically stable matchings.

Proof: We prove it by an example.

Example 5. $S = \{s_1, s_2\}$, $U = \{u_1\}$,

$\rhd_{u_1}$: s_1, s_2 $q_{u_1} = (1, 1)$,

$\succ_{s_1}$: $u_1 u_1, c u_1$,

$\succ_{s_2}$: $u_1 u_1$.

It is easy to check that there is a unique dynamically stable matching

$$\mu = \begin{pmatrix} u_1 & c & \emptyset \\ s_1 & \emptyset & s_2 \\ s_1 & \emptyset & s_2 \end{pmatrix}.$$

Yet, the unique dynamically group stable matching is

$$\mu' = \begin{pmatrix} u_1 & c & \emptyset \\ s_2 & s_1 & \emptyset \\ s_1, s_2 & \emptyset & \emptyset \end{pmatrix}.$$

There is a period-1 group deviation from μ with $B = \{u_1, s_2\}$ and μ'. Note that μ'^2 is the unique period-2 stable matching given μ'^1, while μ'^1 is not period-1 stable. □

Note that in example 4, the empty matching is in the set of dynamic group stable matchings since condition (iii) of period-1 group deviation is never satisfied for u_2 ($\forall \mu \in \mathcal{M}$, $\mu^2(u_2) = \emptyset$), and in example 5, μ is in the dynamic weak core, so these shows there is no logical relationship between the dynamic weak core and the set of dynamic group stable matchings.

There are some similar results in different contexts. In many-to-many matching problems, the set of pairwise-stable matchings is not Pareto efficient (and thus no in the weak core) and no long equivalent to the set of group stable matchings.[26] In a dynamic one-to-one matching problem, Kadam and Kotowski (2016) show the dynamic core might not be equivalent to the set of dynamically stable matchings.

[25] This is a different concept to the dynamic group stability in Kurino (2009), which is defined in a dynamic marriage market.

[26] These results can be found in Proposition 5.23 of Roth and Sotomayor (1992). In Example 2.6 of Blair (1988), the first result is shown with substitutable preferences.

3.4 Results That Have No Parallel in the Benchmark Model

The following two results try to recondition Theorem 1* and Proposition 1* that we vetoed in the last section. It turns out if the no active student condition is not a concern, then they will hold in the two-period model.

Theorem 9: If $\nexists \succ_s$ that contains uu' such that $u \neq u'$, then S-PA-DA is strategy-proof for students.

As we shown in the proof of Proposition 3, whenever there is a uu' plan in a student's preference, it is a dominant strategy to hide it. The major reason is that a uu' plan is never an equilibrium plan for the solution concept dynamic stability, so without any preference restriction, Theorem 9 and Proposition 11 below can not hold. In contrast, uu' plan can be an equilibrium plan in a matching within the dynamic weak core as we show in Example 4. Indeed, uu' can be an equilibrium plan in a dynamic group stable matching by a similar example.

Example 6. Consider the following market. $S = \{s_1, s_2\}$, $U = \{u_1, u_2\}$,

$\rhd_{u_1}$: s_1, s_2 $q_{u_1} = (1, 1)$,

$\rhd_{u_2}$: s_1, s_2 $q_{u_2} = (1, 1)$,

$\succ_{s_1}$: $u_2 u_1$,

$\succ_{s_1}$: $u_1 u_2$.

Again, the unique dynamically stable matching is the empty matching, and

$$\mu = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ s_2 & s_1 & \emptyset & \emptyset \\ s_1 & s_2 & \emptyset & \emptyset \end{pmatrix}$$

is in the unique dynamically group stable matching. There is a period-1 group deviation from the empty matching with $B = \{u_1, u_2, s_1, s_2\}$ and μ.

Proposition 11: If $\nexists \succ_s$ that contains uu' such that $u \neq u'$, then a matching is in the dynamic weak core if and only if it is dynamically stable.

This proposition also says under the this preference restriction, the dynamically stable matchings are Pareto efficient. When a coalition is formed in period-1, the coalition members always match with its members in both periods, so they are exempted from market evolutions. This is true since by definition they can enforce some period-2 stable μ_B^2 from the first period. Of course, there might be more than one such μ_B^2 in the second period as long as all conditions of weakly period-1 domination are respected. However, a period-1 group deviation will induce the market to evolve twice, one in each period, so when we try to recondition Proposition 2*, the two layers of uncertainty make it very difficult to tackle.

We show that the sets of achievable students for period-1 and period-2 positions of a university do not need to be disjoint.

Claim 1: Suppose μ is a dynamically stable matching and $s \in \mu^1(u)$ for some u, there could be another dynamically stable matching μ' such that $s \in \mu'^2(u) \setminus \mu'^1(u)$.

Proof: We prove it by an example.

Example 7. Consider the following market. $S = \{s_1, s_2, s_3\}$, $U = \{u_1, u_2\}$,

$\rhd_{u_1}$: s_1, s_2, s_3 $q_{u_1} = (1, 1)$,

$\rhd_{u_2}$: s_3, s_1 $q_{u_2} = (1, 0)$,

$\succ_{s_1}$: u_2u_2, u_1u_1,

$\succ_{s_2}$: u_1u_1, cu_1,

$\succ_{s_3}$: cu_1, u_2u_2.

In this market,

$$\mu_S = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ s_2 & s_1 & s_3 & \emptyset \\ s_2, s_3 & s_1 & \emptyset & \emptyset \end{pmatrix} \textit{ and } \mu_U = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ s_1 & s_3 & s_2 & \emptyset \\ s_1, s_2 & s_3 & \emptyset & \emptyset \end{pmatrix},$$

so s_2 is an achievable student for both period-1 and period-2 positions of u. □

Theorem 10: If μ and μ' are dynamically stable matchings, then $\mu^2(u) = \mu'^2(u)$ if and only if $\mu(u) = \mu'(u)$.

That is, there do not exists two period-1 matchings such that they would give u the same final outcome at two different dynamically stable matchings. The intuition behind this theorem is very simple. Consider a university u_1 with 1 position in both periods and preference $\succ_u$: s_1. If $\mu^2(u) = \{s_1\}$ and $\mu'^2(u) = \{s_1\}$ but $\mu^1(u_1) = \emptyset \neq s_1 = \mu'^1(u_1)$, then they can not be both stable because s_1 would have strict preference over plans cu_1 and u_1u_1.

Theorem 11: Suppose μ and μ' are dynamically stable matchings with $s \in \mu^1(u)$ and $s' \in \mu^2(u) \setminus \mu^1(u)$ for some u. If $\mu(s) \succ_s \mu'(s)$ and $\mu(s') \succ_{s'} \mu'(s')$ then $\mu'(u) \succ_u \mu(u)$. If $\mu(u) \succ_u \mu'(u)$, then $\mu'(s) \succeq_s \mu(s)$ or $\mu'(s') \succeq_{s'} \mu(s')$.

Theorem 11 says there is no Pareto improvement between two dynamically stable matchings. It delivers a message similar to the Decomposition Lemma, yet it is not comparable with Theorem 6* since it concerns two students while Theorem 6* only concerns one.

4 Conclusion

In this paper, we propose a dynamic two-sided many-to-one matching model that is generalized from the college admissions problem. We show that many of the results from the college admission problem carry over to our model, yet some of them fail because of the two-period nature. Our solution concepts are all natural generalizations of their single period counterpart. However, even the two-period model is an extension of the many-to-one matching model, it is also similar to the many-to-many model in the sense that each student might match with two schools, one for each period. Dynamic stability requires at a dynamically stable matching, each student only matches with one university. But neither dynamically group stability nor dynamic weak core requires this.

Multilateral blocking is rarely observed in admissions markets. While both dynamic weak core and dynamic group stability require large scale of prophetic reasoning, they might work in a small market, but it will be difficult to track evolutions in a large market. Dynamic stability is more conservative and prophetic reasoning minimizing. Yet it does not exhibit all fine properties of the single period stability, it remains the best option among the three solution concepts to approach the problem with moderate or large market size given its tractability.

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A Omitted Proofs

Proof of Lemma 1: μ_S is dynamically individually rational since at each step of the algorithm, no university keeps an unacceptable or active student, and no student proposes an unacceptable plan; also, $(\mu_S^1(s), \mu_S^2(s)) = (\mu_S^1(s), \tilde{\mu}^2_{\mu_S^1}(s)) \succeq_s \emptyset$ for all students since there is no active student.

μ_S is not period-1 blocked by any pair with a two-period plan. Suppose $uu \succ_s \mu_S(s)$ for some u and s. This means that s applied to u with a two-period plan and was rejected at some step n of S-PA-DA. Denote a tentative matching of u at step n by $\mu_S^n(u)$. $\mu_S^n(u)$ only improves weakly as n goes up, i.e., $\mu_S^{n+1}(u) \succeq_u \mu_S^n(u)$, so u will not block with s.

μ_S is not period-1 blocked by any credible agreement. We show a stronger result that μ_S is not period-1 blocked by any agreement. Suppose $cu \succ_s \mu_S(s)$ for some u and s. This means that s applied to u with an agreement and was rejected at some step n of S-PA-DA. Again, $\mu_S^n(u)$ only improves weakly as n goes up, so u will not block with s.

μ_S is not period-2 blocked by any pair. Suppose $u \succ_s^{\mu_S^1} \mu_S^2(s)$ for some u and s. Notice that $\mu_S^1(s) = c$; otherwise, say $\mu_S^1(s) = u'$. But u' will not admit s in the first period since μ_s is dynamically individually rational. Now we go back to the case above. $\square$

Proof of Lemma 2: ($\Longrightarrow$) Suppose $\bar{\mu}$ is unstable.

Suppose $\bar{\mu}$ is not individually rational for some agent. If $\exists s$ such that $\emptyset \succ_s (\bar{\mu}^1(s), \bar{\mu}^2(s))$, then this implies $\emptyset \succ_s (\mu^1(s), \tilde{\mu}^2_{\mu^1}(s)) = (\mu^1(s), \mu^2(s))$. If $\exists u^1_{j,\beta}$ such that $\emptyset \succ_{u_j^1} \bar{\mu}(u^1_{j,\beta})$, then this implies $\emptyset \succ_u s$ for some $s \in \mu^1(u)$; if there is some active student in $\bar{\mu}$, then there must be some in μ. If $\exists u^2_{j,\gamma}$ such that $\emptyset \succ_{u_j^2} \bar{\mu}(u^2_{j,\gamma})$, then this implies $\emptyset \succ_u s$ for some $s \in \mu^2(u)$. Period-1 individual rationality fails in the first three cases, and period-2 individual rationality fails in the last case. Hence, μ is not dynamically individually rational.

Suppose $\bar{\mu}$ is blocked by a pair. If $\exists (s, u^1_{j,\beta})$ such that $u^1_{j,\beta}u^1_{j,\beta} \succ_s \bar{\mu}(s)$ and $s \succ_{u_j^1} \bar{\mu}(u^1_{j,\beta})$, then this implies $(u,u) \succ_s (\mu^1(s), \tilde{\mu}^2_{\mu^1}(s)) = (\mu^1(s), \mu^2(s))$ and $s \succ_u \sigma$ for some $\sigma \in \mu^1(u)$. If $\exists (s, u^2_{j,\gamma})$ such that $cu^2_{j,\gamma} \succ_s \bar{\mu}^1(s)$ and $u^2_{j,\gamma}s \succ_{u_j^2} \bar{\mu}(u^2_{j,\gamma})$, then this implies either $(c,u) \succ_s (\mu^1(s), \tilde{\mu}^2_{\mu^1}(s)) = (\mu^1(s), \mu^2(s))$ with $0 < r_u(s) \leq q_u^2$ or $u \succ_s^{\mu^1} \mu^2(s)$ and $s \succ_u \sigma$ for some $\sigma \in \mu^2(u) \setminus \mu^1(u)$. In the first two cases, μ is period-1 blocked, and it is period-2 blocked in the last case.

So, μ is dynamically unstable.

($\Longleftarrow$) Suppose μ is dynamically unstable.

Suppose μ is not dynamically individually rational. If $\exists s$ such that $\emptyset \succ_s (\mu^1(s), \mu^2(s))$, then this implies $\emptyset \succ_s (\bar{\mu}^1(s), \bar{\mu}^2(s))$ for some s. If $\emptyset \succ_u s$ for some $s \in \mu^1(u)$, then this implies $\exists u^1_{j,\beta}$ such that $\emptyset \succ_{u_j^1} \bar{\mu}(u^1_{j,\beta})$; if there is some active student in μ, then there must be some in $\bar{\mu}$. If $\emptyset \succ_u s$ for some $s \in \mu^2(u)$, then this implies $\exists u^2_{j,\gamma}$ such that $\emptyset \succ_{u_j^2} \bar{\mu}(u^2_{j,\gamma})$.

Hence, $\bar{\mu}$ is not individually rational.

Suppose μ is blocked by a pair. If μ is period-1 blocked by a plan, i.e., $\exists s$ and u such that $(u,u) \succ_s (\mu^1(s), \tilde{\mu}^2_{\mu^1}(s)) = (\mu^1(s), \mu^2(s))$ and $s \succ_u \sigma$ for some $\sigma \in \mu^1(u)$, then this implies $\exists(s, u^1_{j,\beta})$ such that $u^1_{j,\beta}u^1_{j,\beta} \succ_s \bar{\mu}(s)$ and $s \succ_{u^1_j} \bar{\mu}(u^1_{j,\beta})$. If μ is period-1 blocked by a credible agreement, i.e., $\exists s$ and u such that $(c,u) \succ_s (\mu^1(s), \tilde{\mu}^2_{\mu^1}(s)) = (\mu^1(s), \mu^2(s))$ with $0 < r_u(s) \leq q^2_u$, then this implies $\exists(s, u^2_{j,\gamma})$ such that $cu^2_{j,\gamma} \succ_s \bar{\mu}^1(s)$ and $u^2_{j,\gamma}s \succ_{u^2_j} \bar{\mu}(u^2_{j,\gamma})$. If μ is period-2 blocked, i.e., $\exists s$ and u such that $u \succ_s^{\mu^1} \mu^2(s)$ and $s \succ_u \sigma$ for some $\sigma \in \mu^2(u) \backslash \mu^1(u)$, and this implies $\exists(s, u^2_{j,\gamma})$ such that $cu^2_{j,\gamma} \succ_s \bar{\mu}^1(s)$ and $u^2_{j,\gamma}s \succ_{u^2_j} \bar{\mu}(u^2_{j,\gamma})$.

So, $\bar{\mu}$ is unstable. $\square$

Proof of Lemma 3: $\bar{\mu}_{\bar{U}}$ is individually rational since at each step of the algorithm, no position proposes an unacceptable or active student, and no student keeps an unacceptable plan.

$\bar{\mu}_{\bar{U}}$ is not blocked by any pair. Suppose $ss \succ_{u^1_j} \bar{\mu}_{\bar{U}}(u^1_{j,\beta})$ for some $u^1_{j,\beta}$ and s. This means that $u^1_{j,\beta}$ proposed to s and was rejected at some step n of U-PA-DA. Denote a tentative matching of s at step n by $\bar{\mu}^n_{\bar{U}}(s)$. $\bar{\mu}^n_{\bar{U}}(s)$ only improves weakly as n goes up, i.e. $\bar{\mu}^{n+1}_{\bar{U}}(s) \succeq_s \bar{\mu}^n_{\bar{U}}(s)$, so s will not block with $u^1_{j,\beta}$. Suppose $u^2_{j,\gamma}s \succ_{u^2_j} \bar{\mu}_{\bar{U}}(u^2_{j,\gamma})$ for some $u^2_{j,\gamma}$ and s. This means that $u^2_{j,\gamma}$ proposed to s and was rejected at some step n of U-PA-DA. Again, $\bar{\mu}^n_{\bar{U}}(s)$ only improves weakly as n goes up, so s will not block with $u^2_{j,\gamma}$. $\square$

Proof of Theorem 2: We prove (ii) here, and (i) can be proved in a symmetric way. We will show that in one of the deferred acceptance algorithms, no one on the proposing side is rejected by an achievable agent.

The proof is by inductions. Assume that up to a given step in the algorithm no position has been rejected by a student who is achievable for it so far. At this step, suppose s rejects p.

If p is unacceptable, then p is unachievable for s.

If s rejects p because he is proposed a preferable plan p'. We must show p is not achievable for s. We know p' prefers s to any student except those who have previously rejected it (so those students by assumption are unachievable for it). For contradiction, consider a hypothetical matching μ that matches p to s and everyone else to an achievable match such that if p is achievable for s, then μ is one of the stable matching. But p' prefers s to his match at μ, so p' and s will form a blocking pair. Hence, there is no stable matching that matches s and p, so p is unachievable for s. $\square$

Proof of Theorem 3: Suppose it is not true such that $\bar{\mu} >_S \bar{\mu}'$ and $\bar{\mu} >_{\bar{U}} \bar{\mu}'$. Then $\exists s$ such that $\bar{\mu}(s) \succ_s \bar{\mu}'(s)$, where$\bar{\mu}(s) \succ_s \emptyset$ by strict preferences. Then the position $p = \bar{\mu}(s)$ must match with someone else. Since p also has strict preferences and $\bar{\mu} >_{\bar{U}} \bar{\mu}'$. p and s will form a blocking pair at $\bar{\mu}'$. So $\bar{\mu}'$ is not stable, a contradiction. $\square$

Proof of Theorem 4: Let μ be some stable matching other than μ_U. By Theorem 2 (ii), $\bar{\mu}_{\bar{U}}$ is position-optimal, that is, $\forall p \in \bar{U}$, $\bar{\mu}_{\bar{U}}(p) \succeq_p \mu(p)$ and $\exists p \in \bar{U}$ such that $\bar{\mu}_{\bar{U}}(p) \succ_p \mu(p)$. By responsiveness, $\forall u \in U$, $\mu^1_U(u) \succeq_u \mu^1(u)$ and $\mu^2_U(u) \setminus \mu^1_U(u) \succeq_u \mu'^2(u) \setminus \mu'^1(u)$, and $\exists u \in U$ such that $\mu^1_U(u) \succ_u \mu^1(u)$ or $\mu^2_U(u) \setminus \mu^1_U(u) \succ_u \mu^2(u) \setminus \mu^1(u)$. By construction, $\mu^1_U(u)$ and $\mu^2_U(u) \setminus \mu^1_U(u)$ are disjoint subsets of S, so are $\mu^1(u)$ and $\mu^2(u) \setminus \mu^1_U(u)$. Then by responsiveness again, $\forall u \in U$, $\mu_U(u) \succeq_u \mu(u)$ and $\exists u \in U$, $\mu_U(u) \succ_u \mu(u)$. That is, $\mu_U >_U \mu$.

The second part follows from a symmetric argument together with Corollary 1, which says $\bar{\mu}_S$ is the worst stable matching for positions. □

Proof of Lemma 4: Suppose for some s, $p = \bar{\mu}(s) \succ_s \bar{\mu}'(s) = p'$. Then since μ' is stable, this implies $\bar{\mu}'(p) \succ_p \bar{\mu}(p) = s$. Hence, $\bar{\mu}(S^{\bar{\mu}})$ is contained in $\bar{U}^{\bar{\mu}'}$, so $|S^{\bar{\mu}}| \le |\bar{U}^{\bar{\mu}'}|$. Symmetrically, $\bar{\mu}(\bar{U}^{\bar{\mu}'})$ is contained in $S^{\bar{\mu}}$, so $|S^{\bar{\mu}}| \ge |\bar{U}^{\bar{\mu}'}|$. Since $\bar{\mu}$and $\bar{\mu}'$ are one-to-one and $S^{\bar{\mu}}$ and $\bar{U}^{\bar{\mu}'}$ are finite, both $\bar{\mu}$ and $\bar{\mu}'$ are onto. □

Proof of Theorem 5: By definition, $\bar{\lambda}(l) = \bar{\mu}'(l)\ \forall l \in S^{\bar{\mu}'} \cup \bar{U}^{\bar{\mu}}$ and $\bar{\lambda}(l) = \bar{\mu}(l)\ \forall l \in S^{\bar{\mu}} \cup \bar{U}^{\bar{\mu}'}$. By Lemma 4, λ is therefore a matching. Suppose (s, p) blocks $\bar{\lambda}$. Then $s \succ_p \bar{\lambda}(p)$ and $p \succ_s \bar{\lambda}(s)$. Hence (s, p) blocks $\bar{\mu}$ if $\bar{\lambda}(p) = \bar{\mu}(p)$ or $\bar{\mu}$' if $\bar{\lambda}(p) = \bar{\mu}'(p)$. In either case we have a contradiction.

By a symmetric argument, $\bar{v}$ is also a stable matching. □

Proof of Theorem 6: Using the S-PA-DA in the related marriage market. Let $\bar{\mu}$ and $\bar{\mu}_S$ be the corresponding matchings of μ and μ_S in $(S, \bar{U}, \bar{\succ})$, respectively. The matching $\bar{\mu}$ would match every student s to some position that rejected him in the algorithm because a preferable student s' proposed to it (even s satisfies its individual rationality). So all those positions $\bar{\mu}(S)$ have been matched under $\bar{\mu}_S$, this implies $\bar{\mu}_S(\bar{\mu}(S)) = S$. Hence, all students would have been matched under $\bar{\mu}_S$ and $\bar{\mu}_S(S) = \bar{\mu}(S)$. But since all students are matched under $\bar{\mu}_S$, any position which gets a proposal in the last step of the algorithm at which proposals were issued has not rejected any acceptable student, i.e. the algorithm stops as soon as every position in $\bar{\mu}_S(S)$ has an acceptable proposal. So such a position must be unmatched at μ since every student prefers $\bar{\mu}$ to $\bar{\mu}_S$, which contradicts $\bar{\mu}_S(S) = \bar{\mu}(S)$. □

Proof of Lemma 5: Let $\bar{\mu}$ and $\bar{\mu}_S$ be the corresponding matchings of μ and μ_S in $(S, \bar{U}, \bar{\succ})$, respectively.

Case 1: $\bar{\mu}(S') \neq \bar{\mu}_S(S')$. Pick a p in $\bar{\mu}(S') \setminus \bar{\mu}_S(S')$, that is, $\bar{\mu}(p) = s' \in S'$ and $\bar{\mu}_S(p) = s \notin S'$. For $\bar{\mu}_S$ to be stable, $s \succ_p s'$. But $s \notin S'$, so he does not prefer $\bar{\mu}$ to $\bar{\mu}_S$ and by strictness of preferences, we have $\bar{\mu}_S(s) = p \succ_s \bar{\mu}(s)$. So (s, p) blocks μ.

Case 2: $\bar{\mu}(S') = \bar{\mu}_S(S') = \bar{U}'$. Let p be the last position of university in $\bar{U}'$ to receives a proposal from a student in S' that satisfies its individual rationality in the S-PA-DA. Under $\bar{\mu}$, since $p \in \bar{U}'$, it matches with some student in S'. Let us denote him s'. We show that s'

is rejected before p gets this last proposal. Since $p \succ_{s'} \bar{\mu}_S(s')$, s' must proposed to p and get rejected in S-PA-DA and then ends up matching with $\bar{\mu}_S(s')$. But if p only rejects s' when it gets its last proposal from a student in S' that satisfies its individual rationality, then the proposal of s' to $\bar{\mu}_S(s')$ would have came later, which contradicts how we pick p. So p rejects s' before it got its last proposal from a student in S' that satisfies its individual rationality, that implies it holds a proposal from some other student. Let us denote him s. So this s would prefers p to $\bar{\mu}_S(s)$ since he proposed to $\bar{\mu}_S(s)$ after p rejected him. Since s satisfies p's individual rationality but later proposed to $\bar{\mu}_S(s)$ and p is the last position of university in $\bar{U}'$ to receives a proposal from a student in S' that satisfies its individual rationality, $s \notin S'$. That implies $p \succ_s \bar{\mu}_S(s) \succ_s \bar{\mu}(s).p$ rejects s' because of s. So (s, p) block $\bar{\mu}$. □

Proof of Theorem 7: $|\bar{\mu}_S(\bar{U})| = |\bar{\mu}_S(S)| \geq |\bar{\mu}_{\bar{U}}(S)| = |\bar{\mu}_{\bar{U}}(\bar{U})| \geq |\bar{\mu}_S(\bar{U})|$. □

Proof of Lemma 6: We show that $\bar{\mu}(u^1_{j,\beta}) \succ_{u^1_j} \bar{\mu}'(u^1_{j,\beta})$ for all $\beta > i$. Suppose not. $\exists\beta$ such that $\bar{\mu}(u^1_{j,\beta}) \succ_{u^1_j} \bar{\mu}'(u^1_{j,\beta})$ and $\bar{\mu}'(u^1_{j,\beta+1}) \succeq_{u^1_j} \bar{\mu}(u^1_{j,\beta+1})$. Note that $u^1_{j,\beta}$ is matched under $\bar{\mu}$ (we cannot have $\emptyset = \bar{\mu}(u^1_{j,\beta}) \succ_{u^1_j} \bar{\mu}'(u^1_{j,\beta})$), so Theorem 7 implies $\bar{\mu}'(u^1_{j,\beta}) \in S$. Let $s' \equiv \bar{\mu}'(u^1_{j,\beta})$. By Lemma 4, $u^1_{j,\beta} \equiv \bar{\mu}'(s') \succ_s \bar{\mu}(s')$, and since $s' \equiv \bar{\mu}'(u^1_{j,\beta}) \succ_{u^1_j} \bar{\mu}'(u^1_{j,\beta+1}) \succeq_{u^1_j} \bar{\mu}(u^1_{j,\beta+1})$, $s' \neq \bar{\mu}(u^1_{j,\beta+1})$. Since by construction $u^1_{j,\beta+1}$ is the next most preferred plan in the preference of s', $u^1_{j,\beta+1} \succ_s \bar{\mu}(s')$. So s' and $u^1_{j,\beta+1}$ will block $\bar{\mu}'$, which contradicts the stability of μ. Since i is arbitrary, the first part follows.

The second part can be proved by the same argument. □

Proof of Theorem 8: Suppose not. Then $\exists u^1_{j,i}$ such that $\bar{\mu}(u^1_{j,i}) \succ_{u^1_j} \bar{\mu}'(u^1_{j,i})$, then from the proof of Lemma 6 we know that $\bar{\mu}(u^1_{j,\beta}) \succ_{u^1_j} \bar{\mu}'(u^1_{j,\beta})$ for all $\beta > i$. But we know by Theorem 7, if a period-1 position is unmatched in some stable matching, then it is unmatched under all stable matchings. That means the $\bar{\mu}(u^1_{j,q^1_u}) = \bar{\mu}'(u^1_{j,q^1_u}) = \emptyset$ for the last period-1 position in u, and this contradicts $\bar{\mu}(u^1_{j,q^1_u}) \succ_{u^1_j} \bar{\mu}'(u^1_{j,q^1_u})$. So such $u^1_{j,i}$ does not exists, that is, $\bar{\mu}(u^1_{j,\beta}) = \bar{\mu}'(u^1_{j,\beta})$ for all β.

The same result can be proved for period-2 by the same argument. □

Proof of Theorem 9: Note that $\nexists \succ_s$ that contains uu' such that $u \neq u'$, so a student will not misstate any uu' preference as uu. Suppose a student s can misstate his preference as $\hat{\succ}_s$, along with whatever preferences everyone else is reporting, we have a new preference profile $\hat{\succ}$ such that $\hat{\mu}_S(s) \succ_s \mu_S(s)$, where μ_S is the student-optimal matching under the true preference. Since S-PA-DA is used, $\hat{\mu}_S$ will be individually rational, apply the Lemma 5 to the true preference$\succ$, $\hat{S}$ the set of students who prefer $\hat{\mu}_S$ to μ_S is not empty (s will be in $\hat{S}$ since $\hat{\mu}_S(s) \succ_s \mu_S(s)$, and there maybe more students) , a pair (s', p) blocks $\hat{\mu}_S$ under the true preference such that $s' \in S \setminus \hat{S}$ and $p \in \hat{\mu}_S(\hat{S})$. □

Proof of Proposition 11: Note that $\nexists \succ_s$ that contains uu' such that $u \neq u'$, so a university does not need to care its no active student condition that embeds in its period-1 individual rationality.

($\Longrightarrow$) If μ is dynamically unstable via a single agent because μ is not dynamically individually rational, then it is clearly not in the dynamic weak core since it is weakly period-1 dominated via a coalition $B = \{s\}$ by any matching μ' with $\mu'(s) = \emptyset$ or $B = u \cup \mu^2(u) \setminus s$ for some s that $\emptyset \succ_u s$ by any matching μ' with $\mu'^2(u) = \mu^2(u) \setminus s$.

If μ is dynamically unstable via a pair of student and university (s, u) such that μ is period-1 blocked, then it is weakly period-1 dominated via a coalition $B = u \cup s \cup \mu^2(u) \setminus \sigma$ by any matching μ' with $\mu'(s) = uu$, $\mu'^1(u) = s \cup \mu^1(u) \setminus \sigma$, and $\mu'^2(u) \setminus \mu'^1(u) = \mu^2(u) \setminus \mu^1(u)$ or μ' with $\mu'(s) = cu$, $\mu'^1(u) = \mu^1(u)$, and $\mu'^2(u) \setminus \mu^1(u) = s \cup \mu^2(u) \setminus \mu^1(u) \setminus \sigma$. Similarly, if μ is μ is period-2 blocked, then it is weakly period-2 dominated via coalition $B = u \cup s \cup \mu^2(u) \setminus \mu^1(u) \setminus \sigma$ by any matching μ' with $\mu'^2(s) = u$ and $\mu'^2(u) \setminus \mu^1(u) = s \cup \mu^2(u) \setminus \mu^1(u) \setminus \sigma$.

($\Longleftarrow$) Suppose μ is not in the dynamic weak core. Then μ is period-1 or period-2 weakly dominated by some matching μ' via some coalition B, and hence some student or university prefer μ' to μ. If μ is not dynamically individually rational, then it is not dynamically stable.

Suppose μ satisfies dynamically individual rationality.

Suppose also μ is period-2 weakly dominated by some matching $\mu' = (\mu^1, \mu^2_B)$ via some coalition B.

Suppose $\mu^2_B(u) \setminus \mu^1(u) \succ_u \mu^2(u) \setminus \mu^1(u)$ for some $u \in B$. There must exists $s \in \mu^2_B(u) \setminus \mu^1(u) \setminus \mu^2(u)$ and $\sigma \in \mu^2(u) \setminus \mu^1(u) \setminus \mu^2_B(u)$ such that $s \succ_u \sigma$; otherwise, $\sigma \succeq_u s$ $\forall s \in \mu^2_B(u) \setminus \mu^1(u) \setminus \mu^2(u)$ implies $\mu^2(u) \setminus \mu^1(u) \succ_u \mu^2_B(u) \setminus \mu^1(u)$ by responsiveness. By definition, $s \in \mu^2_B(u) \setminus \mu^1(u)$ implies $s \in B$, so $u = \mu'(s) \succeq_s \mu(s)$; furthermore, $s \notin \mu^2(u) \setminus \mu^1(u)$ and $s \in \mu^2_B(u) \setminus \mu^1(u)$, so $\mu'(s) = (\mu^1(s), \mu^2_B(s)) \neq (\mu^1(s), \mu^2(s)) = \mu(s)$. Therefore, $\mu'(s) \succ_s \mu(s)$. Hence, μ it is period-2 blocked by (s, u).

Suppose some $s \in B$ with $\mu^2_B(s) = u$ with $\mu'(s) \succ_s \mu(s)$, this implies $u \in B$, so $\mu^2_B(u) \setminus \mu^1(u) \succeq_u \mu^2(u) \setminus \mu^1(u)$. $\mu^2_B(s) = u$ with $\mu'(s) \succ_s \mu(s)$ also implies $\mu^2(s) \neq u$ (a student can not switch between cu and uu given μ^1), so $\mu^2_B(u) \setminus \mu^1(u) \neq \mu^2(u) \setminus \mu^1(u)$. Hence, $\mu^2_B(u) \setminus \mu^1(u) \succ_u \mu^2(u) \setminus \mu^1(u)$. This implies that there is a student $s' \in \mu^2_B(u) \setminus \mu^1(u) \setminus \mu^2(u)$ (possibly different from s) and $\sigma \in \mu^2(u) \setminus \mu^1(u) \setminus \mu^2_B(u)$ such that $s' \succ_u \sigma$. Then μ is blocked by (s', u).

Suppose also μ is period-1 weakly dominated by some matching $\mu' = (\mu^1_B, \mu^2_B)$ via some coalition B.

Suppose $\mu^2_B(u) \succ_u \mu^2(u)$ for some $u \in B$. There must exists $s \in \mu^2_B(u) \setminus \mu^2(u)$ and $\sigma \in \mu^2(u) \setminus \mu^2_B(u)$ such that $s \succ_u \sigma$; otherwise, $\sigma \succeq_u s$ $\forall s \in \mu^2_B(u) \setminus \mu^2(u)$ implies $\mu^2(u) \succ_u \mu^2_B(u)$ by responsiveness. By definition, $s \in \mu^2_B(u)$ implies $s \in B$, so $u = \mu'(s) \succeq_s \mu(s)$; furthermore, $s \notin \mu^2(u)$ and $s \in \mu^2_B(u)$, so $\mu'(s) \neq \mu(s)$. Therefore, $\mu'(s) \succ_s \mu(s)$. Hence, μ it is period-2 blocked by (s, u).

Suppose some $s \in B$ with $\mu_B^2(s) = u$ with $\mu'(s) \succ_s \mu(s)$, this implies $u \in B$, so $\mu_B^2(u) \succeq_u \mu^2(u)$.

If $\mu_B^2(u) \neq \mu^2(u)$, then $\mu_B^2(u) \succ_u \mu^2(u)$. This in turn implies there is a student $s' \in \mu_B^2(u) \setminus \mu^2(u)$ (possibly different from s) and $\sigma \in \mu^2(u) \setminus \mu_B^2(u)$ such that $s' \succ_u \sigma$. Then μ is blocked by (s', u).

If $\mu_B^2(u) = \mu^2(u)$, then it means s switches between cu and uu. Without loss of generality, assume $uu \succ_s cu$. So $\mu'(s) = uu$, then either there is an unmatched period-1 seat under μ, or there is no unmatched period-1 seat under μ but there exists some student s' such that $\mu'(s') = cu \succ_{s'} uu = \mu(s')$ and he switches his seat with s (note that $s' \in B$, so $\mu'(s') \succeq_{s'} \mu(s')$, since there is no more period-1 seat for s under μ, given $\mu_B^2(u) = \mu^2(u)$, s' has to move to cu under μ', this implies $\mu'(s') = cu \succ_{s'} uu = \mu(s')$). In either case, μ is blocked by (s, u). □

Proof of Theorem 10: Suppose u fills all its quota under μ and μ' in both periods; otherwise by Theorem 7, $\mu^1(u) = \mu'^1(u)$ or $\mu^2(u) \setminus \mu^1(u) = \mu'^2(u) \setminus \mu'^1(u)$.

($\Longleftarrow$) This direction is by definition of μ.

($\Longrightarrow$) If $\mu(u) \neq \mu'(u)$ but $\mu^2(u) = \mu'^2(u)$. Then (without loss of generality) $\mu^1(u) \succ_u \mu'^1(u)$ and $\mu^2(u) \setminus \mu^1(u) \succ_u \mu'^2(u) \setminus \mu'^1(u)$. $\exists s \in \mu^1(u)$ and $s = \bar{\mu}'(u_{j,\gamma}^2) \in \mu'^2(u) \setminus \mu'^1(u)$. $\exists s' \in \mu'^1(u)$ and $s' \in \mu^2(u) \setminus \mu^1(u)$. Since preferences are strict, without loss of generality, suppose $s \rhd_u s'$ and $cu \succ_s uu$. Then $u_{j,\gamma}^2$ and s will block μ, a contradiction. □

Proof of Theorem 11: Suppose u fills all its quota under μ and μ' in both periods for the first part; otherwise, by Theorem 7, $\mu^1(u) = \mu'^1(u)$ or $\mu^2(u) \setminus \mu^1(u) = \mu'^2(u) \setminus \mu'^1(u)$. This will imply $\mu(s) = \mu'(s)$ or $\mu'(s) = \mu(s')$. For the second part, u has to fill all its period-1 positions or period-2 positions under μ and μ'; otherwise, by Theorem 7, $\mu(u) = \mu'(u)$.

Let $\bar{\mu}$ and $\bar{\mu}'$ be the corresponding matchings of μ and μ' in $(S, \bar{U}, \bar{\succ})$, respectively. $\exists u_{j,\beta}^1, u_{j,\gamma}^2$ such that $\bar{\mu}(u_{j,\beta}^1) = s$ and $\bar{\mu}'(u_{j,\gamma}^2) = s'$.

Suppose $\mu(s) \succ_s \mu'(s)$ and $\mu(s') \succ_{s'} \mu'(s')$. Since μ' is stable, $\exists s'' = \bar{\mu}'(u_{j,\beta}^1) \succ_u s$ and $\exists s''' = \bar{\mu}(u_{j,\gamma}^2) \succ_u s'$. By Lemma 6, $\bar{\mu}(u_{j,i}^1) \succeq_{u_j^1} \bar{\mu}'(u_{j,i}^1)$ for all positions $u_{j,i}^1$ of u with $u_{j,\beta}^1$ strictly prefers, and $\bar{\mu}(u_{j,i}^2) \succeq_{u_j^1} \bar{\mu}'(u_{j,i}^2)$ for all positions $u_{j,\gamma}^2$ of u with with $u_{j,\gamma}^2$ strictly prefers. So by responsiveness, $\mu'^1(u) \succ_u \mu^1(u)$ and $\mu'^2(u) \setminus \mu'^1(u) \succ_u \mu^2(u) \setminus \mu^1(u)$. By responsiveness again, $\mu'(u) \succ_u \mu(u)$.

Suppose $\mu(u) \succ_u \mu'(u)$, then either $\mu^1(u) \succ_u \mu'^1(u)$ or $\mu^2(u) \setminus \mu^1(u) \succ_u \mu'^2(u) \setminus \mu'^1(u)$. Define $\bar{U}^{\bar{\mu}} = \{p \in \bar{U} | \bar{\mu}(p) \succ_p \bar{\mu}'(p)\}$ and $S^{\bar{\mu}'} = \{s \in S | \bar{\mu}'(s) \succ_s \bar{\mu}(s)\}$.

If $\mu^1(u) \succ_u \mu'^1(u)$, then by Lemma 6, $\bar{\mu}(u_{j,i}^1) \succeq_u \bar{\mu}'(u_{j,i}^1)$ for all period-1 position of u, so either $\bar{\mu}(u_{j,i}^1) = \bar{\mu}'(u_{j,i}^1)$ or $u_{j,i}^1 \in \bar{U}^{\bar{\mu}}$. Note that $\bar{\mu}(u_{j,\beta}^1) = s$, together with Lemma 4, $\bar{\mu}(s) = \bar{\mu}'(s)$ or $s \in S^{\bar{\mu}'}$. That is, $\mu'(s) \succeq_s \mu(s)$ (note that it is possible that s is just moving to a "worse" period-1 position at u).

If $\mu^2(u) \setminus \mu^1(u) \succ_u \mu'^2(u) \setminus \mu'^1(u)$, by the same argument, $\mu'(s') \succeq_s \mu(s')$. $\square$