From behind the veil: Evaluating allocation rules by ex-ante properties

Patrick Harless*
University of Rochester

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Abstract

We study rules for allocating objects. Departing from standard analysis, we evaluate rules according to their performance at an ex-ante stage, before individuals learn their preferences. Introducing an appropriate notion of ex-ante efficiency, we search for rules that are both efficient and provide incentive for individuals to truthfully report their eventual preferences. Our main results characterize the priority (or "serial dictatorship") rules by ex-ante efficiency and either strategy-proofness or Bayesian incentive compatibility on natural preference domains. Allowing indifferences identifies the extended priority rules. For domains on which utilities correspond to ordinal preference rank, the implications of our incentive requirements diverge: ex-ante efficiency and strategy-proofness continue to characterize the priority rules, but many additional rules, including rules which maximize utilitarian welfare, are Bayesian incentive compatible.

When truly behind the veil, agents and objects are indistinguishable, which we model as symmetric problems. Remarkably, all rules in a large family achieve the same utilitarian welfare. Moreover, rules adapting the top trading cycles algorithm are Lorenz maximal and priority rules are Lorenz minimal within this family. Allowing the size of the economy to grow, we find that average welfare under each rule approaches that of a utilitarian rule. To further compare rules, we introduce solidarity properties and consider an interim participation constraint. These considerations distinguish methods of randomizing over families of rules.

*Address: Department of Economics, University of Rochester, Rochester, NY 14627, USA; e-mail: patrick.harless@rochester.edu. I thank Paulo Barelli, Srihari Govindan, Romans Pancs, and participants at the 2015 Fall Midwest Economic Theory Meeting for helpful discussion. I am particularly grateful to Vikram Manjunath and William Thomson for detailed comments.

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Contents

1 Introduction

Markets and prices commonly allocate resources, but transaction costs, practicality, and ethical objections sometimes require allocation without recourse to money. Instead, we observe centralized assignment of resources from school seats, offices, and dorm rooms to work schedules and tasks, even organs for transplantation. This is particularly common when each agent receives one object, the problem we study. While the absence of money increases the challenge, it does not preclude accounting for preferences, and attractive allocation methods will aim to improve the welfare of the agents.

As typically modeled, an allocation problem includes a set of agents with preferences over available objects and then searches for rules with desirable properties. Following the implicit timeline, agents form preferences before the designer chooses a rule. In fact, though, the timing is often reversed: The designer chooses a rule and then agents' learn their preferences. In the school context, for example, students attend fairs and open houses, make visits, and research programs long after the assignment procedure has been announced. Moreover, even when the implied timing is accurate for the agents, it is not for the designer who must choose a rule before learning the agents' preferences. Consequently, at the planning stage, evaluations of rules that assume preferences are known may prove misleading.[1]

With the goal of improving the welfare of the agents in mind, we examine two questions: Which rules are efficient when viewed from the ex-ante perspective? How do rules compare from *behind the veil of ignorance*? To isolate the role of incomplete information, we abstract from the process by which preferences form and assume that all ex-ante uncertainty resolves before agents report their preferences. That is, agents have sufficient time, access to information, etc. to learn all of the relevant information about the objects.

1.1 Efficiency from the ex-ante perspective

To address our first question, we introduce an appropriate notion of ex-ante efficiency based on agents' expected utilities. Modeling a problem as probability distribution over von Neumann-Morgenstern utility profiles, we study rules which recommend allocations for each realizable profile. Recognizing that ex-post randomization may be desirable, we also include rules which select distributions over allocations. We consider a rule *ex-ante efficient* at a problem if its expected utility profile is not Pareto dominated by that of an alternative rule.[2] This implies the ex-post requirement

[1] It may also be desirable to choose the rule before agents learn their preferences. For example, debates surrounding walk-zone and other priorities are contentious because students already know whether they live near a desirable school (Abdulkadiroğlu et al., 2006; Dur et al., 2014). Evaluating the merits of these priorities without such knowledge – from behind the veil of ignorance (Harsanyi, 1953; Rawls, 1972) – might promote consensus.

[2] We also consider an efficiency notion defined by restricting dominance to deterministic rules, the appropriate condition when ex-post randomization is infeasible. For many problems, the notions are actually equivalent, and a more general equivalence follows if we define efficiency for collections of problems rather each problem separately.

that a rule select allocations which are Pareto undominated. While *ex-ante efficiency* strengthens its ex-post counterpart, often significantly, it is correspondingly weaker than and implied by the requirement that a rule select allocations which maximize the sum of the agents' utilities.[3] As well as intrinsically desirable, efficiency acts as a source of stability, eliminating the most obvious objections to an allocation.[4] Just as *ex-post efficiency* ensures that no group of agents prefer to exchange assignments, *ex-ante efficiency* guarantees that no group prefer a conditional contract prescribing exchange under some circumstances.

To measure welfare, we require information about preferences. As this information is private, we seek rules which provide incentives to report preferences truthfully. The standard requirement, *strategy-proofness*, insists that truthful reporting be a dominant strategy for each agent. This strong guarantee has become the sine qua non and proved decisive in many debates about school assignment procedures (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2006). With incomplete information, weaker conditions may also be appropriate. In particular, we also study *Bayesian incentive compatibility*, the more modest requirement that agents prefer to truthfully report their preferences given their beliefs about others' preferences. If agents in fact know the preferences of all other agents, then the requirements are the same, though *Bayesian incentive compatibility* is satisfied much more generally with incomplete information.

Ensuring truthful reports frees us to focus on our central concern, the welfare of the agents. Our conclusions are somewhat disappointing, identifying the priority rules as the only *ex-ante efficient* rules satisfying our incentive requirements. Each priority rule simulates sequential choice according a single, fixed priority order over the agents, thereby systematically favoring some agents over others to the maximum extent possible. More precisely, at each problem with strict preferences satisfying a mild richness condition,[5] *ex-ante efficiency* and *strategy-proofness* characterize the priority rules (Theorem 1(i)). Even more surprising, relaxing our incentive constraint provides essentially no additional flexibility: Subject to a slightly stronger richness condition, only the priority rules are *ex-ante efficient* and *Bayesian incentive compatible* (Theorem 1(ii)). To accommodate indifferences, we extend the definition of priority rules and find that the same requirements characterize the extended priority rules (Theorem 1(iii)).

Although our richness conditions require only minimal diversity of potential preferences, we also study their implications for problems defined on restricted domains of preferences. To investigate, we consider domains in which all realizable utility vectors are permutations of the same set of values. This implies, for example, that no two utility vectors correspond to the same ranking over

[3] This stronger requirement is appropriate when transfer payments are feasible and preferences are quasi-linear.

[4] When other considerations override efficiency, agents may object. For example, after identifying an improving exchanging of school seats, two families sued the Amsterdam school district (Ashlagi and Nikzad, 2015). Recognizing the tension with *strategy-proof* assignment, the court ultimately rejected the suit.

[5] Essentially, we require that there be some cardinal diversity at the top of preferences. We describe specific richness conditions in detail subsequently and also provide examples when they are violated.

objects. Utilities in queuing problems may exhibit this type of structure. These "scoring" domains[6] may also arise from an observe's attempt to give cardinal meaning to ordinal rankings. Even here, only the priority rules are *ex-ante efficient* and *strategy-proof* (Theorem 2). As priority rules treat agents very differently, these axioms are incompatible with symmetry axioms, whether considered ex-ante or ex-post (Corollary 6). In contrast, *Bayesian incentive compatible* is now satisfied by many *ex-ante efficient* rules (Proposition 3). Among these are not only symmetric rules, but also rules which maximize the sum of the agents' utilities.

Our results extend readily to domains which allow structured indifferences as when some objects are available in multiple copies. Consequently, the full force of our results apply to the school assignment problem (Corollary 4). As a second application, we derive implications for a model of learning as in Bade (2015) (Corollary 3).[7] Finally, testing the tightness of our characterizations, we explore domains violating our richness conditions and find cases where additional rules satisfy our properties.

1.2 Choosing rules from behind the veil

Turning to our second question, we take seriously choosing rules from *behind the veil of ignorance* and study problems in which agents and objects are ex-ante indistinguishable. Beyond embodying our normative motivation, these problems capture many economic environments including those in which auctions are commonly studied. Following a formulation from the school assignment literature, we model utilities as composed of common value and private value components and call problems defined this way symmetric problems.

Since symmetric problems satisfy our richness conditions, the priority rules remain the only *ex-ante efficient* and *strategy-proof* rules (Remark 5). On the other hand, broadening our point of view, we find that other rules nevertheless compare favorably with the priority rules. First, we show that all *ex-post efficient*, *strategy-proof*, and *non-bossy* rules yield the same utilitarian welfare in each symmetric problem (Proposition 4). In other words, moving between rules in this class amounts to transferring expected utility among the agents. If the real efficiency measure is what we may achieve with money, then all of these rules stand on equal footing. Exploring further, we introduce Lorenz comparisons which refine and generalize the "maximin" criterion. Although the incompleteness of the Lorenz order typically makes comparisons inconclusive, our setting permits unusually definitive conclusions within the class of *ex-post efficient*, *strategy-proof*, and *non-bossy* rules. At one extreme, the expected utility profiles of priority rules are Lorenz dominated by all other rules in the class (Proposition 5(ii)). At the other extreme, the Lorenz maximal rules constitute a familiar family, namely those rules which adapt Gale's top trading cycles (TTC) algorithm and simulate trade from hypothetical endowments (Proposition 5(i)). This strong egalitarian conclusion implies that,

[6] The term is meant to suggest a parallel to *scoring rules* in social choice which are defined similarly (Young, 1975).
[7] We describe the relationship between results in greater detail when discussing related literature.

subject to incentive constraints, TTC rules[8] maximize ex-ante social welfare according to every measure exhibiting aversion to interim inequality.

Allowing the size of the economy to vary, we find that all rules in this family further achieve the maximum average utilitarian welfare in the limit (Proposition 6). In other words, efficiency losses due to incentive requirements and the absence of money essentially vanish. Somewhat tempering this optimistic conclusion, we observe that it may be overturned under different distribution of common values varies as the number of objects grows (Remark 6). Refining this point, we show that greater variability of the private values typically enhances welfare under each rule, whereas greater variability of the common value component diminishes welfare (Proposition 7). Whether the overall conclusion is positive or negative depends on specifics of the problem.

As a related concern, we inquire about the solidarity properties of rules as the environment changes. From the ex-ante perspective, we ask whether agents can expect to be better off when problems are augmented by additional agents and objects. This is true for all agents under TTC rules, whereas some agents may be worse off in an augmented problem when a priority rule is applied (Proposition 8(iii)). On the other hand, ex-post comparisons are ambiguous. While it is always possible for agents to be better off ex-post in larger problems (Proposition 8(i)), most agents can also be worse off. By introducing new agents at the bottom of a priority order, priority rules may offer a strong welfare guarantee to at least some agents, but even this is out of the reach of TTC rules (Proposition 8(ii)).

Further extending the model, we introduce outside options which provide opportunities for agents to drop out and thereby lead to interim participation constraints. Our comparisons are mixed, but generally suggest that TTC rules encourage greater participation by agents (Corollary9). Comparing welfare of individual agents, we find that a majority of agents are always better off ex-ante under a given priority rule than in each TTC rule. On the other hand, in some environments, nearly one half of agents are better off when assignments are made independent of preferences than under any priority rule. In contrast, all agents are ex-ante better off under each TTC rule than under preference-independent allocation (Proposition 9). We also derive comparisons across lotteries over either priority orders or and endowments, finding differences in the information revealed by each type of randomization. Although the lotteries are equivalent in expectation, an inequality averse interim welfare measure will favor randomization over endowments (Corollary 11). Taken together, these extensions help to explain when priority or TTC rules are likely to be desirable in practice.

Finally, we move outside the symmetric environment and investigate the extent to which conclusions generalize. First, specific types of correlation may in fact lead to opposite conclusions. Utilitarian and maximin comparisons may favor either TTC rules or priority rules, and priority rules may ex-ante Pareto dominate TTC rules. Furthermore, *strategy-proof* and *bossy* rules some-

[8]All of these rules apply the TTC algorithm from Shapley and Scarf (1974) and are distinguished by the allocation used in place of endowments.

times achieve higher social welfare than is possible among *non-bossy* rules (Propositions 10, 11, and 12). These conclusions suggest that restriction to *non-bossy* rules may sometimes be undesirable and underscore the standard admonition that choice among rules must be sensitive to details of the environment. As an application, we compare the desirability of a risky versus a safe endowment under TTC in a two-agent problem. Comparing expected utilities of the agents, we observe an endogenous risk preference induced by the rule (Example 8).

Taken as a whole, the ex-ante perspective yields both encouraging and discouraging conclusions. On one hand, only the priority rules are robustly efficient and incentive compatible. For general environments, *ex-ante efficiency* provides a compelling argument for priority rules despite their asymmetry. Nevertheless, in some special cases, incomplete information sufficiently relaxes incentive requirements to make possible *utilitarian-optimal* allocation. Moreover, in symmetric environments, many rules are undominated within the class of *strategy-proof* and *non-bossy* rules. As these rules in fact achieve the same utilitarian welfare as priority rules, selection based on *ex-ante efficiency* is now less persuasive and additional equity concerns more salient. Comparison from *behind the veil*, then, strongly favors the top trading cycles rules.

1.3 Related literature

Closest to our work is Bade (2015) who also studies *ex-ante efficient* and *strategy-proof* in a model with endogenous learning. Her Theorem 2 shows that, among *strategy-proof* and *non-bossy* rules, only priority rules are *ex-ante efficient* when agents learn their preferences through costly information acquisition. This result applies on the unrestricted domain of finite problems with strict preferences and arbitrary learning costs. Applied to her model, our Theorem 1(i) generalizes this result to characterize the priority rules on narrow domains of preferences, without imposing *non-bossiness*, and even when learning is free (Corollary 3).[9]

While at least the priority rules are *ex-ante efficient* and *strategy-proof* with costly learning, no rules satisfy these properties when agents receive interdependent signals of their values (Che et al., 2015). However, the negative conclusion relies critically on lingering uncertainty at the reporting stage. In contrast, because agents in our model learn their own values precisely before the reporting stage, correlation does not lead to the same impossibility. The incompatibility we do observe with symmetry (Corollary 6) parallels several impossibility results (Zhou, 1990; Bogomolnaia and Moulin, 2001; Kasajima, 2013).[10]

More generally, our work falls with in a growing literature studying object allocation rules. In the complete information setting with strict preferences, we now have a full description of all *ex-post efficient*, *strategy-proof*, and *non-bossy* rules (Pápai, 2000a; Pycia and Ünver, 2014; Bade,

[9] Also, our Theorem 1(ii) and Theorem 1(iii) have no parallels in her work.

[10] The Myerson-Satterthwaite Theorem (Myerson and Satterthwaite, 1983) can also be adapted to our setting where it implies that no rule is *utilitarian-optimal* and *strategy-proof*.

2014b). Within this family, only the priority rules are also symmetric with respect to the objects (Svensson, 1999).[11] Generalizing the model to accommodate indifferences requires extending the definitions of the rules.[12]

Within the ordinal framework, refinements of *ex-post efficiency* have been proposed based on stochastic dominance (Bogomolnaia and Moulin, 2001), applicable when assignments are lotteries, as well as rank (Featherstone, 2013; Harless, 2012). Recent work has also begun to also account for preference intensities (Manea, 2008; Featherstone and Niederle, 2013; Abdulkadiroğlu et al., 2011; Hafalir and Miralles, 2015).[13] With the goal of comparing equilibrium outcomes under leading school assignment rules, this analysis evaluates welfare at the interim stage.[14] For the special case of common ordinal preferences, Troyan (2012) makes similar comparisons at the ex-ante stage.

Our normative analysis on the symmetric domain is philosophically inspired (Harsanyi, 1953; Rawls, 1972).[15] Closest to our formal results are the Lorenz comparisons of Harless and Manjunath (2015) and the asymptotic analysis of Che and Tercieux (2015), each of which studies object allocation. Harless and Manjunath (2015) study a model with learning and show that equilibrium utility profiles under top trading cycles rules Lorenz dominate those under priority rules. Che and Tercieux (2015) show that all *ex-post efficient* rules are asymptotically payoff equivalent, although with somewhat different modeling assumptions from ours.

The remainder of the paper is organized as follows. In Section 2, we introduce the object assignment model with incomplete information and describe the properties we study. We introduce our richness conditions, derive implications of our axioms, and present our characterization theorems in Section 3. In Section 4, we introduce the symmetric domain and turn to welfare analysis from *behind the veil*. Section 5 concludes and the appendix collects omitted proofs.

[11] See Bu (2014) for a related characterization. Top trading cycles rules have also been singled out within the larger family (Ma, 1994; Morrill, 2013; Anno, 2015).

[12] Svensson (1994) defines the extension of the priority rules which we follow. Bogomolnaia et al. (2005) characterize the slightly larger family of "bi-polar" priority rules in this setting, remarkably without imposing *ex-post efficiency*. Alcalde-Unzu and Molis (2011), Jaramillo and Manjunath (2014), and Ehlers (2014) propose extensions of the top trading cycles rules and study indifferences more generally.

[13] Formally, these papers study a slightly different model augmented by object-specific priorities.

[14] Although sometimes called ex-ante, agents' have already learned their own preferences, so this line of analysis corresponds to the interim stage in our model. We define ex-ante following Myerson and Holmström (1983) and Jackson and Sonnenschein (2007).

[15] Similarly motivated, Nehring (2004) studies an allocation setting from the interim stage. Although agents know their own preferences, he argues that rules are appropriately evaluated from behind a *veil of public ignorance* which makes comparisons as if from an ex-ante stage with a common prior.

2 Model

2.1 Introducing incomplete information

As in a standard allocation problem, there are finite sets of objects $A \equiv \{a_1, \ldots, a_n\}$ and agents $N \equiv \{1, \ldots, n\}$ so $|A| = |N| = n$.[16] Each agent must receive one object, so an **allocation** is a bijection $\omega\colon N \to A$. For each $i \in N$, we call x_i agent i's **assignment at $\boldsymbol{\omega}$**. The set of feasible allocations is $X \equiv \{\omega \in A^N : i \neq j \Rightarrow \omega_i \neq \omega_j\}$. We write ω_{-i} for $(\omega_1, \ldots, u_{i-1}, \omega_{i+1}, \ldots, \omega_n)$. To extend this notation to ω_{-ij}, ω_{-ijk}, and so on, we adopt the convention of reading components in the order of removal so that $(\alpha, \beta, \omega_{-ij})$ means $\omega_i = \alpha$ and $\omega_j = \beta$. The same notation applies to all vector quantities.

Each agent $i \in N$ has a **type** $u_i \in \mathbb{R}^A$ which specifies the value of each object and we write $u \equiv (u_i)_{i \in N}$. When there is no risk of confusion, we sometimes write $u_i(\omega)$ instead of $u_i(\omega_i)$ for the utility of agent i with type u_i at allocation ω. With a quasi-linear environment in mind, we interpret these values according to a common scale in terms of an un-modelled monetary equivalent. In particular, we consider affine transformations to be distinct types.[17] To measure the distance between types, we use the "max" norm: For each pair $u_0, u_0' \in \mathbb{R}^A$, $d(u_0, u_0') \equiv \max_{a \in A} |u_0(a) - u_0'(a)|$. For each $\varepsilon \in \mathbb{R}_{++}$ and $u_0 \in \mathcal{U}$, the open ball of radius ε around u_0 is $B_\varepsilon(u_0) \equiv \{u_0' \in \mathbb{R}^A : d(u_0, u_0') < \varepsilon\}$. Each type induces an (ordinal) preference ranking over A. For each $u_0 \in \mathbb{R}^A$, $R(u_0)$ is the binary relation on A such that for each $a, b \in A$, $a\ R(u_0)\ b \Leftrightarrow u_a \geq u_b$. We distinguish the set of types $\mathcal{U}$ consistent with strict preferences: $\mathcal{U} \equiv \{u_0 \in \mathbb{R}^A : \forall\, a, b \in A,\ u_0(a) \neq u_0(b)\}$. Let Π be the set of strict orders over N, $\mathcal{P}$ be the set of strict orders over A, and $\mathcal{R}$ be the reflexive closure of $\mathcal{P}$. For each $P_0 \in \mathcal{P}$, let $\mathcal{U}(P_0) \equiv \{u_0 \in \mathcal{U} : \forall\, a, b, \in A,\ a\ P\ b \Leftrightarrow u_0(a) > u_0(b)\}$, the set of types consistent with P_0.

With N and A fixed, a **problem** is specified by a probability measure μ over $\mathbb{R}^{A \times N}$. Let Σ be the Borel σ-algebra on $\mathbb{R}^{A \times N}$ and Σ_0 be the Borel σ-algebra on $\mathbb{R}^A$. For each $D \in \Sigma$, $\mu(D)$ is the probability that $u \in D$. For each $i \in N$, μ_i is the **marginal distribution for agent $\boldsymbol{i}$**. The **support of $\boldsymbol{\mu}$** is[18] $\operatorname{supp}(\mu) \equiv \overline{\{V \in \Sigma : \mu(V) > 0\}}$ and for each $i \in N$, the support of μ_i is $\operatorname{supp}(\mu_i) \equiv \overline{\{V \in \Sigma_0 : \mu_i(V) > 0\}}$.

In our model, agents report their types, so reported profiles may differ from the truth. To avoid interpretive difficulties with *strategy-proofness* which arise when a deviation leads to a profile that does not occur, we restrict attention to problems with a product structure. A problem μ is **rectangular** if $\operatorname{supp}(\mu) = \prod_N \operatorname{supp}(\mu_i)$ and $\bar{\mathcal{D}}$ is the collection of **rectangular problems**.[19] A

[16] Given a set X, $|X|$ is the cardinality of X; $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$ denote the real and non-negative real, and positive real numbers respectively and $\mathbb{R}^X$ denotes the $|X|$ dimensional Euclidean space.

[17] While we adopt this interpretation for comparability with transferable utility settings, many of our results can also be interpreted in terms of relative utilitarianism (Dhillon and Mertens, 1999).

[18] For $V \in \Sigma$ or $V \in \Sigma_0$, $\bar{V}$ denotes the closure of V.

[19] Rectangularity is consistent with almost arbitrary correlation. By comparison, if types are independent across agents, then $\mu = \prod_N \mu_i$.

priori, we place no further restrictions on μ, capturing settings with finite and continuum type spaces and further allowing type spaces and marginal distributions to differ across agents. Also, by allowing indifference and correlation, we accommodate objects available in multiple copies. This will allow us to extend our results to the school assignment problem.

Of course, we are also interested in subclasses of problems satisfying additional properties. First, we distinguish problems in which agents' types are drawn from a common set: For each $V \subseteq \mathbb{R}^A$, let $\mathcal{D}(V) \subseteq \mathcal{D}$ consist of those problems such that for each $i \in N$, $\mu_i(V) = 1$. Also important are: **strict problems** ($\mathcal{D}$) in which indifferences occur with zero probability; **finite problems** ($\bar{\mathcal{D}}^*$) and **strict finite problems** ($\mathcal{D}^*$) whose supports contain a finite number of types[20]; and **continuous problems** ($\bar{\mathcal{D}}^c$) and **strict continuous problems** ($\mathcal{D}^c$) whose measures are non-atomic. Formally,[21]

$$\begin{aligned}
\mathcal{D} &\equiv \{\mu \in \bar{\mathcal{D}} : \forall\, i \in N,\ \mu_i(\mathbb{R}^A \backslash \mathcal{U}) = 0\}, \\
\bar{\mathcal{D}}^* &\equiv \{\mu \in \bar{\mathcal{D}} : \forall\, i \in N,\ \text{supp}(\mu_i) \text{ is finite }\}, \\
\mathcal{D}^* &\equiv \{\mu \in \mathcal{D} : \forall\, i \in N,\ \text{supp}(\mu_i) \text{ is finite }\}, \\
\bar{\mathcal{D}}^c &\equiv \{\mu \in \mathcal{D} : \forall\, i \in N,\ \text{supp}(\mu_i) \text{ is absolutely continuous }\}, \text{ and} \\
\mathcal{D}^c &\equiv \{\mu \in \mathcal{D} : \forall\, i \in N,\ \text{supp}(\mu_i) \text{ is absolutely continuous }\}.
\end{aligned}$$

We will also consider further sub-domains satisfying additional properties, such as symmetry across agents or objects or independence across objects.

Our objective is to compare allocation methods, or rules, which select allocations for profiles of types. We distinguish rules according to whether they permit randomization ex-post. A **deterministic rule** is a measurable function $F\colon \mathbb{R}^{A\times N} \to X$ and a **probabilistic rule** is[22] a measurable function $F\colon \mathbb{R}^{A\times N} \to \Delta(X)$. By the Birkhoff–von-Neuman Theorem, each probabilistic assignment can be decomposed as a lottery over deterministic assignments (Birkhoff, 1946; von Neumann, 1953). Let $\mathcal{F}^*$ and $\mathcal{F}$ denote the classes of deterministic and probabilistic rules respectively.

A well-studied class of rules are insensitive to differences in preference intensities and respond only to the agents' rankings over objects. Formally, a rule F is **ordinal** if for each $u \in \mathbb{R}^{A\times N}$, each $i \in N$, and each $u'_i \in \mathcal{U}(P(u_i))$, $F(u'_i, u_{-i}) = F(u)$. Given μ, we consider F and $\hat{F}$ **equivalent at $\boldsymbol{\mu}$** if they differ on at most a negligible set: There is $V \subseteq \mathbb{R}^A_+$ such that $\mu(V) = 0$ and for each $u \in \mathcal{U}\backslash V$, $F(u) = \hat{F}(u)$. For each $i \in N$, the **expected utility of agent $\boldsymbol{i}$ under $\boldsymbol{F}$** is $U_i(F,\mu) \equiv \int u_i(F(u))\, d\mu$ and the **expected utility profile of $\boldsymbol{F}$** is $U(F,\mu) \equiv (U_i(F,\mu))_{i\in N}$. The sum $U^*(F,\mu) \equiv \sum_N U_i(F,\mu)$ is the **utilitarian welfare under $\boldsymbol{F}$**. When there is no confusion about the measure, we suppress it in the notation and write $U_i(F)$, $U(F)$, and $U^*(F)$.

[20]In particular, $\mathcal{D}^*$ corresponds to the sub-domain of Bade (2015) with zero investigation cost.

[21]Absolute continuity is with respect to the Lebesgue measure.

[22]We denote the simplex of lotteries over X by $\Delta(X)$.

2.2 Properties

2.2.1 Efficiency

We first describe efficiency. Here, we consider three requirements. Our first requirements apply at a given profile. Given $u \in \mathbb{R}^{A\times N}$, an allocation $\omega \in X$ is **ex-post efficient at $\boldsymbol{u}$** if it is Pareto undominated: For each $\omega' \in X$, if there is $i \in N$ such that $u_i(\omega_i) < u_i(\omega'_i)$, then there is $j \in N$ such that $u_j(\omega_j) > u_j(\omega'_j)$. It is **utilitarian-optimal at $\boldsymbol{u}$** if it maximizes the sum of the agents' utilities: $\sum_N u_i(\omega_i) = \max_{\omega \in X} \sum_N u_i(\omega_i)$. A rule F is **ex-post efficient** or **utilitarian-optimal** if it always selects allocations with these properties.[23]

Immediate from the definition, utilitarian welfare is the same at each *utilitarian-optimal* allocation. When transfers are possible, *utilitarian optimality* is a component of *ex-post efficiency* and achieved by auction mechanisms (Demange et al., 1986). However, in our setting, *utilitarian optimality* is not based on Pareto comparisons. Whereas *ex-post efficiency* guarantees that objects cannot be reallocated to make all agents better off, *utilitarian optimality* further ensures that objects are allocated to their highest valued uses. Consequently, the set of *utilitarian-optimal* allocations at a given profile is generally a small subset of those which are *ex-post efficient.*

We aim to evaluate rules before agents learn their types. At this stage, efficiency is properly measured in terms of expected utility. Given $\mu \in \mathcal{D}$, a rule F is **ex-ante efficient at $\boldsymbol{\mu}$** if it is ex-ante Pareto undominated: For each $F' \in \mathcal{F}$, if there is $i \in N$ such that $U_i(F,\mu) < U_i(F',\mu)$, then there is $j \in N$ such that $U_j(F,\mu) > U_j(F',\mu)$. Extending this idea to a domain of problems $D \subseteq \mathcal{D}$, F is **ex-ante efficient on $\boldsymbol{D}$** if for each $\mu \in D$, F is *ex-ante efficient* at μ. Our definition requires that a rule be undominated among all probabilistic rules. In some cases, we may wish to restrict dominance to deterministic rules. We say that F is **ex-ante efficient* at $\boldsymbol{\mu}$** or **on $\boldsymbol{D}$** if it satisfies the previous definitions for each $F' \in \mathcal{F}^*$. Comparing notions, *utilitarian-optimal* implies *ex-ante efficiency*, which in turn implies *ex-post efficiency.*[24]

2.2.2 Incentives

As agents' types are private information, our central requirement is incentive compatibility. To ensure that agents never benefit by misreporting their preferences, the most common requirement is *strategy-proofness*, which ensures that truth-telling is a dominant strategy for each agent. Formally,

Strategy-proofness: For each $u \in \mathcal{U}^N$, each $i \in N$, and each $u'_i \in \mathcal{U}$, $u_i \cdot F_i(u) \geq u_i \cdot F_i(u'_i, u_{-i})$.

With incomplete information, as in our model, it is also natural to consider a weaker requirement. We suppose that agents have correct beliefs so, given $\mu \in \mathcal{D}$, μ is also the agents' common

[23] For probabilistic rules, we require that each allocation in the support of the lottery allocation be *ex-post efficient* or *utilitarian-optimal.*

[24] *Ex-ante efficiency** on $\mathcal{D}^*$ corresponds to Pareto optimality in Bade (2015) without learning costs.

prior. Our requirement is now that, after updating his beliefs according to Bayes rule, no agent obtain higher expected utility by misreporting. To state the property formally, we introduce additional notation. For each $F \in \mathcal{F}$, each $\mu \in \mathcal{D}$, each $i \in N$, and each pair $u_0, u_0' \in \mathcal{U}$, let $U_i(u_0', u_0, F) \equiv \int u_0(F_i(u_0', u_{-i}))\, d(\mu|u_0)$. That is, $U_i(u_0', u_0, F)$ is the expected utility under F of agent i with type u_0 when reporting u_0'.

Bayesian incentive compatibility (BIC) at $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}$: For each $i \in N$ and each pair $u_0, u_0' \in \mathcal{U}$, $U_i(u_0, u_0, F) \geq U_i(u_0', u_0, F)$.

Since *BIC* applies only to expected utilities, *strategy-proofness* immediately implies *BIC*.

We also mention an invariance property that is often imposed together with *strategy-proofness*. It says that when an agent changes his own report, either his own assignment changes, or the entire allocation remains the same.

Non-bossiness: For each $u_N \in \mathcal{U}^N$, each $i \in N$, and each $u_i' \in \mathcal{U}$, if $F_i(u) = F_i(u_i', u_{-i})$, then $F(u) = F(u_i', u_{-i})$.

Non-bossiness ensures a kind of robustness, rendering the allocation invariant to changes in preferences that are may appear irrelevant. It also helps to connect incentive properties. In particular, when preferences are strict, the combination of *strategy-proofness* and *non-bossiness* ensures that no group of agents can profitably manipulate. Nevertheless, *non-bossiness* does limit flexibility,[25] so we will not impose it a priori.

2.2.3 Additional properties

We conclude with desirable equity properties. When agents have equal claims on the objects, it is natural to ask that they be treated equally, which we define in terms of welfare. Our first requirement applies to individual profiles and asks that agents with the same type obtain the same utility. Formally, given $D \subseteq \mathcal{D}$, F is **ex-post symmetric on $\boldsymbol{D}$** if for each $u \in \mathbb{R}^{A \times N}$ and each pair $i, j \in N$, if $u_i = u_j$, then $u_i(F_i(u)) = u_j(F_j(u))$.

Incomplete information permits a second notion of symmetry. From the ex-ante perspective, we ask that agents with the same distribution of types obtain the same expected utility. Formally, given $D \subseteq \mathcal{D}$, F is **ex-ante symmetric on $\boldsymbol{D}$** if for each $\mu \in D$ and each pair $i, j \in N$, if $\mu_i = \mu_j$, then $U_i(F, \mu) = U_j(F, \mu)$.

The relative strength of these properties depends on the domain. For example, on the continuous domain, *ex-post symmetry* applies to a set of profiles with measure zero. In contrast, *ex-ante symmetry* may still be demanding, particularly when all agents' types are dr a common distribution. On the other hand, on each domain in which at least two agents' types spaces overlap,

[25] In particular, a *strategy-proof* and *non-bossy* deterministic rule must be ordinal (Ehlers et al., 2014).

no deterministic rule is *ex-post symmetric*. In contrast, a variety of domains admit deterministic *ex-ante symmetric* rules.[26]

In addition to axiomatic analysis, we are interested in comparing rules by their expected utility profiles. Given a list $x_1, \ldots, x_l$, for each $k = 1, \ldots, l$, let $x^{(k,l)}$ be the kth smallest value so

$$\min\{x_1, \ldots, x_l\} = x^{(1,l)} \leq x^{(2,l)} \leq \cdots \leq x^{(l-1,l)} \leq x^{(l,l)} = \max\{x_1, \ldots, x_l\}.$$

When the x_i are random variables, $x^{(k,l)}$ is their kth **order statistic**. Comparing rules, F **Lorenz dominates** F' if for each $k = 1, \ldots, n$, $\sum_{i=1}^{k} U_i^{(i,n)}(F) \geq \sum_{i=1}^{k} U_i^{(i,n)}(F')$ and F **strictly Lorenz dominates** F' if additionally at least one inequality is strict. That is, F is more favorable than F' to its least well off agents. Lorenz dominance defines an incomplete order, so rules need not be Lorenz comparable.

2.3 Rules

Central to our analysis is the family of priority rules. A priority rule begins with an order over agents and uses their reported types to simulate sequential choice. When types are strict, the agent with the highest priority receives his most preferred object, the agent with the second highest priority receives his most preferred object among those that remain, and so on. Formally,

Priority rule associated with $\prec \in \Pi^N$, $F^{\prec}$: For each $u \in \mathcal{U}^N$ and each $i \in N$,

$$F_i^{\prec}(u) \equiv \operatorname{argmax} \left\{ u_i(a) : a \in A \backslash \left(\bigcup_{j \prec i} F_j^{\prec}(u) \right) \right\}.$$

To accommodate indifferences, we must generalize our definition, although the idea is essentially the same: The highest priority agent identifies those allocations at which he receives one of his most preferred objects. The second highest priority agent refines this set by identifying those allocations at which he receives one of his most preferred objects among those he may receive within this set of allocations. Subsequent agents further refine the set in turn. Formally, for each $\prec \in \Pi^N$ and $u \in \mathbb{R}^{A \times N}$, let $X^{\prec}(u, 0) \equiv X$ and for each $i = 1, \ldots, n$, let

$$X^{\prec}(u, i) \equiv \Big\{ \omega \in X^{\prec}(u, i-1) : \omega_i \in \operatorname{argmax}\{u_i(\omega'_i) : \omega' \in X^{\prec}(u, i-1)\} \Big\}.$$

With these sets in hand, we define a priority correspondence as well as rules which select from it.

[26] On some domains, *ex-ante symmetry* may be infeasible. In such cases, it a more appropriate measure of equity might compare agents' expected utility under a rule to the maximum or minimum expected utility they could receive under any rule. We will be interested in *ex-ante efficiency* primarily when agents share a common type space where these perspectives are equivalent.

Extended priority correspondence associated with $\prec \in \Pi^N$, $\overline{F^{\prec}}$: For each $u \in \mathbb{R}^{AN}$, $\overline{F^{\prec}}(u) \equiv X^{\prec}(u, n)$.

An **extended priority rule** is a selection from an extended priority correspondence. When types are strict, there is a single extended priority rule and it coincides with the priority rule defined by the same order. Although $\overline{F^{\prec}}$ is technically a correspondence, all of the allocations selected for given profile are welfare equivalent. That is, $\overline{F^{\prec}}$ is **essentially single valued**: For each $u \in \mathbb{R}^{AN}$, each $i \in N$, and each pair $\omega, \omega' \in \overline{F^{\prec}}(u)$, $u_i(\omega_i) = u_i(\omega'_i)$. Since all extended priority rules defined by the same order are welfare equivalent, we abuse notation slightly and use $F^{\prec}$ to refer to these rules.

A second family of rules also stand out, those based on Gale's top trading cycles algorithm (Shapley and Scarf, 1974), which are defined for strict preferences.[27] Given $\omega \in X$ representing endowments, we call agent i the "owner" of ω_i. In the first round of the algorithm, each agent points at the owner of his most preferred object. This forms at least one cycle. Agents in a cycle are assigned the objects to which they are pointing and these agents and objects are removed. Each agent without an assignment now points at the owner of his most preferred object among those which remain. Again at least one cycle forms, and the process continues until each agent has been assigned an object. Applying this algorithm defines a rule: For each $\omega \in X$, the **top trading cycles (TTC) rule associated with ω, F^{ω}**, defined by the top trading cycles algorithm.

The priority rules and TTC rules belong to larger families of rules which have been characterized by desirable efficiency and incentive properties in the standard model. In particular, "trading and braiding" algorithms describe rules the large family of *ex-post efficient*, *strategy-proof*, and *non-bossy* rules (Pesendorfer, 1979; Pycia and Ünver, 2014; Bade, 2014a). While some of our results apply to this family, we will not study other specific members and so omit their formal definitions.

As a benchmark, we introduce rules which maximize the sum of the agents' utilities. In general, there may be several allocations which maximize this sum, so we first introduce the correspondence which identifies all such allocations. Formally,

Utilitarian-optimal correspondence, $\overline{W}$: For each $u \in \mathbb{R}^{AN}$, $\overline{W}(u) \equiv \text{argmax}_{\omega \in X} \sum_N u_i(\omega_i)$.

Given a tie-breaking procedure, a **utilitarian rule**, is a selection from the utilitarian correspondence which breaks ties according to W. Agents are not indifferent among tie-breaking procedures. However, allowing for ex-post randomization, but we can recover symmetry by randomization. We take as our leading example the utilitarian rule F^W which randomizes uniformly over $\overline{W}$. In a

[27] Abdulkadiroğlu and Sönmez (2003) extend TTC to situations with multiple copies of objects and otherwise strict types. Recent work has introduced further refinements to accommodate indifferences more generally (Alcalde-Unzu and Molis (2011); Saban and Seturaman (2013); Jaramillo and Manjunath (2014); Ehlers (2014)). Doing so without jeopardising desirable properties TTC satisfies when types are strict is a delicate process. As we will study TTC rules on domains where indifferences occur with probability zero, we do not introduce these definitions.

quasi-linear model with transfers, utilitarian rules may be implemented via ascending price auctions (Shapley and Shubik, 1971; Demange et al., 1986), so the utilitarian rules help us to compare the outcomes of our rules with those we may expect to see when money is available.

3 Efficiency and incentives: Characterizing priority rules

Our first goal is to identify rules with desirable ex-ante properties and, in particular, understand the implications of *ex-ante efficiency*. Our results, a variety of characterizations, show that when efficiency is a central goal and preferences must be elicited from agents, incomplete information about preferences requires allocating objects according to priority.

3.1 Efficiency

In contrast with the extensively studied incentive properties, little is known about *ex-ante efficiency*. Our first goal is therefore to derive implications of *ex-ante efficiency*. With a clear understanding of *ex-ante efficiency*, we will be in position to study the interaction of incentive and efficiency properties on rich domains.

3.1.1 Distinguishing rules by ex-ante efficiency

We first note for reference that priority rules, as well as their extensions, are *ex-ante efficient* generally.

Lemma 1. *For each problem, each extended priority rule is* ex-ante efficient *and* strategy-proof*.*

The rules we have introduced distinguish our efficiency conditions. All utilitarian rules are *utilitarian-optimal* and therefore *ex-ante efficient* and *ex-post efficient* as well. Priority rules are *ex-ante efficient* and *ex-post efficient*, but generally fall short of *utilitarian-optimal*. The TTC rules are of course *ex-post efficient*, but generally violate the stronger requirements. Given the generally desirable ex-post properties of the TTC rules, their failure to satisfy *ex-ante efficiency* is both disappointing and surprising. To better understand both the property and the rules, we illustrate a typical violation which points to the pervasiveness of the inefficiency.

Example 1. TTC rules violate ex-ante efficiency. Let $N \equiv \{1,2,3\}$, $A \equiv \{a,b,c\}$, $\omega \equiv (a,b,c)$, and $V \subseteq \mathbb{R}^A$ consist of the six types in the table.

	u^{ab}	u^{ac}	u^{ba}	u^{bc}	u^{ca}	u^{cb}
a	2	2	1	0	1	0
b	1	0	2	2	0	1
c	0	1	0	1	2	2

Let $\mu \in \mathcal{D}(V)$ be the uniform measure over V^N so that for each $u \in V^N$, $\mu(u) = p_0 \equiv \frac{1}{(3!)^3} = \frac{1}{216}$. Under F^ω, $F^\omega(u^{ab}, u^{ac}, u^{ac}) = F^\omega(u^{bc}, u^{ba}, u^{bc}) = (a, b, c)$. These assignments permit an ex-ante improvement. Let F' modify F^ω so that $F'(u^{ab}, u^{ac}, u^{ac}) = F'(u^{bc}, u^{ba}, u^{bc}) = (b, a, c)$. Comparing expected utilities,

$$U_1(F', \mu) - U_1(F^\omega, \mu) = p_0 \cdot (1 - 2) + p_0 \cdot (2 - 0) = p_0 > 0 \text{ and}$$
$$U_2(F', \mu) - U_2(F^\omega, \mu) = p_0 \cdot (2 - 0) + p_0 \cdot (1 - 2) = p_0 > 0.$$

Since agent 3 receives the same assignments under F' and F^ω, F' ex-ante Pareto dominates F^ω.

As Example 1 shows, TTC rules typically leave opportunities for agents to arbitrage an ex-ante improvement by agreeing to exchange objects when their preference are relatively more intense. The modification takes advantage of one such case, though there are many others. Moreover, the environment is symmetric with respect to agents and objects, showing that inefficiency is not due to peculiar type structures or unusual profiles.

In Section 4, we will show that TTC rules actually possess desirable properties in symmetric problems. In particular, they are ex-ante Pareto undominated among *ex-post efficient*, *strategy-proof*, and *non-bossy* rules. However, for general problems, such guarantee is no longer possible. In some problems, performance in terms of *ex-ante efficiency* is sufficiently poor that a TTC rule is ex-ante Pareto dominated by a priority rule.

Example 2. Priority rules may ex-ante Pareto dominate TTC rules. Let $N \equiv \{1, 2, 3, 4\}$, $A \equiv \{a, b, c, d\}$, $\prec \equiv (1, 2, 3, 4)$, and $\omega \equiv (a, b, c, d)$. Let $\hat{u}, \hat{u}' \in \mathcal{U}^N$ be as specified in the table.

	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$	$\hat{u}_4$	$\hat{u}'_1$	$\hat{u}'_2$	$\hat{u}'_3$	$\hat{u}'_4$
a	2	3	3	2	2	3	2	3
b	1	2	0	3	1	1	3	1
c	3	1	2	1	0	0	0	2
d	0	0	1	0	3	2	1	0

Let $\mu \in \mathcal{D}^*$ be such that $\mu(\hat{u}) = \mu(\hat{u}') = \frac{1}{2}$. The assignments under F^ω and $F^\prec$ are

$$F^\omega(\hat{u}) = (c, b, a, d), \qquad F^\omega(\hat{u}') = (d, b, c, a), \text{ and}$$
$$F^\prec(\hat{u}) = (c, a, d, b), \qquad F^\prec(\hat{u}') = (d, a, b, c).$$

Comparing expected utilities,

$$U(F^\omega, \mu) = \tfrac{1}{2} \cdot (3, 2, 3, 0) + \tfrac{1}{2} \cdot (3, 1, 0, 3) = \tfrac{1}{2} \cdot (6, 3, 3, 3) \text{ and}$$
$$U(F^\prec, \mu) = \tfrac{1}{2} \cdot (3, 3, 1, 3) + \tfrac{1}{2} \cdot (3, 3, 3, 2) = \tfrac{1}{2} \cdot (6, 6, 4, 5).$$

Then agent 1 is equally well off under each rule and agents 2, 3, and 4 are better off under $F^{\prec}$. That is, $F^{\prec}$ ex-ante Pareto dominates F^{ω} at μ.

Although TTC rules are not *ex-ante efficient*, the priority and utilitarian rules are not the only rules to satisfy the property. In fact, a diverse family of hybrid rules are also *ex-ante efficient*. These rules mix the utilitarian and prioritarian motives of the utilitarian rules and priority rules which appear as the extreme members. We illustrate this family with two agents on the continuous domain.

Example 3. Ex-ante efficient rules for two agents on the continuous domain. Let $N \equiv \{1,2\}$, $A \equiv \{a,b\}$, and $\mu \in \mathcal{D}^c$. Let $\delta \in \mathbb{R}$ and define F^{δ} for each $u \in \mathcal{U}$ by

$$F^{\delta}(u) \equiv \begin{cases} (a,b) & \text{if } u_1(a)+u_2(b) \geq u_1(b)+u_2(a)+\delta \\ (b,a) & \text{if } u_1(b)+u_2(a) \geq u_1(a)+u_2(b)+\delta \\ (a,b) & \text{if } \big|(u_1(a)+u_2(b))-(u_1(b)+u_2(a))\big| < \delta \text{ and } u_1(a) > u_1(b) \\ (b,a) & \text{if } \big|(u_1(a)+u_2(b))-(u_1(b)+u_2(a))\big| < \delta \text{ and } u_1(a) > u_1(b) \end{cases}.$$

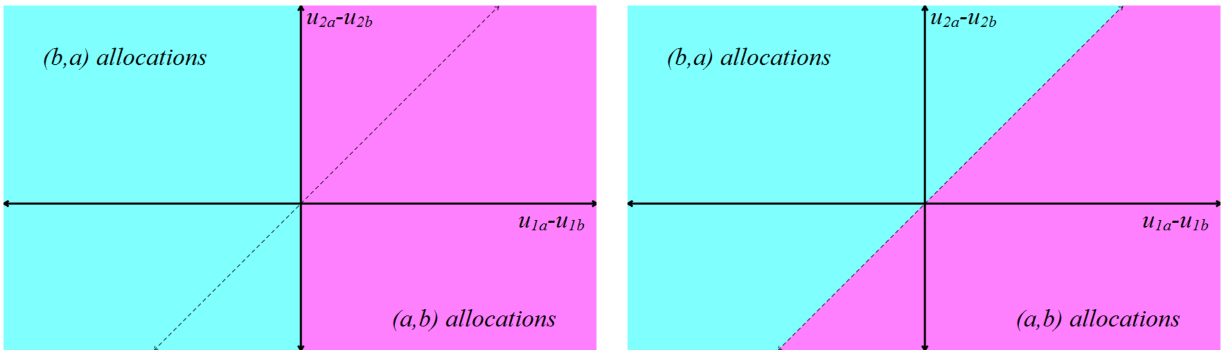
In words, F^{δ} applies the utilitarian rule when the potential utilitarian gains are large as measured by δ and otherwise follows the priority rule $F^{\prec}$. Following F^{δ}, no ex-ante renegotiation benefits both agents.

By carefully choosing the the profiles at which to favor the utilitarian objective over priority, F^{δ} achieves *ex-ante efficiency*. To verify this, consider a utilitarian improvement over F^{δ} which includes all possible ex-ante Pareto improvements. This reverses the assignments at some profiles determined in favor of agent 1 according to priority. However, compensating agent 1 for ceding priority at some of these profiles requires reversing some of the utilitarian assignments favoring agent 2. By construction, these reversals entail a decrease in utilitarian welfare that more than offsets the increase from the previous reversals. Altogether, F^{δ} is *ex-ante efficient*.

3.1.2 Implications of ex-ante efficiency

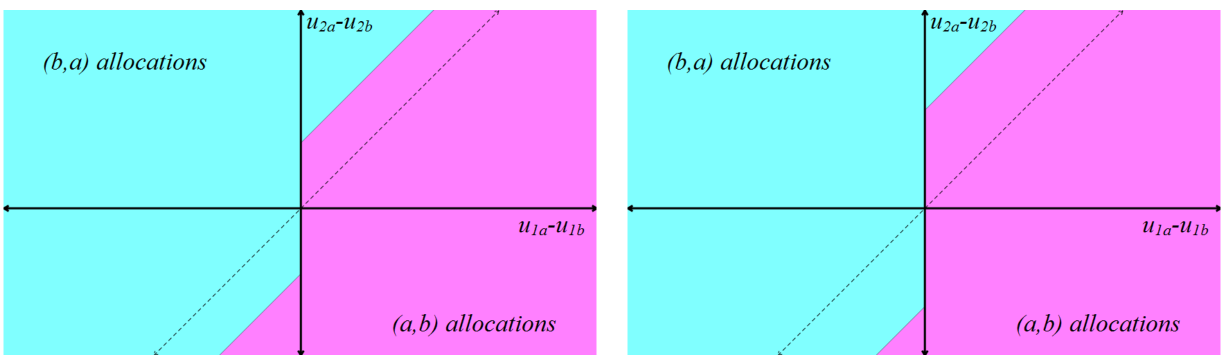
The structure of the two-agent rules in Example 3 reveal some general properties of *ex-ante efficient* rules. As with those rules, all *ex-ante efficient* rules reflect some mixing of utilitarian and prioritarian objectives. Intuitively, if conflicts of preferences will sometimes be resolved in favor of one agent and sometimes in favor of another agent, then these situations should be aligned so that each agent is favored when he benefits the most.

Some additional terminology will facilitate formal comparisons. Consider two agents $i, j \in N$, a profile $u \in \mathbb{R}^{A \times N}$, and an ex-post efficient allocation $\omega \in X$. We say that agent i is **favored**



(a) Allocations under a priority rule. **(b)** Allocations under a utilitarian rule.

Figure 1: Canonical ex-ante efficient rules for two agents. The figure represents profiles by the utility differences for each agent. The region to the right of the y-axis indicates profiles at which agent 1 prefers a to b and the region to the left indicates the opposite preference. For agent 2, the corresponding regions are above and below the x-axis. Greater distance from the origin reflects more intense preferences in terms of the perceived utility difference between objects.



(a) Allocations under F^δ for large δ. **(b)** Allocations under F^δ for small δ.

Figure 2: Additional ex-ante efficient rules for two agents. When δ is large, F^δ is close to a priority rule. As δ decreases, F^δ moves closer to a utilitarian rule. In each case, translations by δ of the 45° line partition the sets of profiles at which F^δ allocations (a, b) or (b, a).

over agent j at ω if agent j prefers agent i's assignment to his own: $u_j(\omega_i) > u_j(\omega_j)$.[28] Since ω is ex-post efficient, at most one of the agents is favored at ω. Now suppose that $u_i(\omega_i) > u_i(\omega_j)$ and $u_j(\omega_i) > u_j(\omega_j)$ so both agents prefer ω_i to ω_j. We say that agent i has **more intense preference** than agent j at ω if he has a larger utility difference for the preferred object: $u_i(\omega_i) - u_i(\omega_j) > u_j(\omega_i) - u_j(\omega_j)$. We define **less intense preference** by the reverse condition. For a more nuanced comparison, we may quantify the difference in preference intensity by choosing $\delta \in \mathbb{R}_+$ so that $u_i(\omega_i) - u_i(\omega_j) + \delta = u_j(\omega_i) - u_j(\omega_j)$. If this condition is satisfied, we say that agent i has **$\boldsymbol{\delta}$-more intense preference** than agent j at ω.

For a deterministic rule, *ex-ante efficiency* requires that if at one profile the rule favors one agent over another agent with more intense preference, then the rule also favors the first agent over

[28] This condition is sometimes introduced as saying that agent j "envies" agent i. We prefer a more neutral language that emphasizes a feature of the rule.

the second agent whenever those agents' preferences are equally intense. More generally, if an agent is favored over an agent at one profile, then he is also favored at each profile at which the difference in preference intensity between the two agents is larger. Lemma 2, proved in the appendix, states this conclusion generally.

Lemma 2. *Let $\mu \in \bar{\mathcal{D}}$ and $F \in \mathcal{F}^*$ be* ex-ante efficient *at μ. For each pair $i, j \in N$ and each $\delta \in \mathbb{R}_+$, if*

$$\mu\left(\left\{u \in \mathbb{R}^{A\times N} : \begin{array}{c} u_i(F_i(u)) > u_i(F_j(u)),\, u_j(F_i(u)) > u_j(F_j(u)),\ \textit{and} \\ u_i(F_i(u)) - u_i(F_j(u)) \leq u_j(F_i(u)) - u_j(F_j(u)) + \delta \end{array}\right\}\right) > 0$$

then for each $\delta' \in \mathbb{R}$ such that $\delta' < \delta$,

$$\mu\left(\left\{u \in \mathbb{R}^{A\times N} : \begin{array}{c} u_i(F_i(u)) < u_i(F_j(u)),\, u_j(F_i(u)) < u_j(F_j(u)),\ \textit{and} \\ u_i(F_j(u)) - u_i(F_i(u)) \geq u_j(F_j(u)) - u_j(F_i(u)) + \delta' \end{array}\right\}\right) = 0.$$

For finite problems, the conditions simply compare profiles in the support of the problem. The special case $0 = \delta' < \delta$ identifies a useful fact: If an agent is favored over another agent with relatively more intense preference, then he is favored over that agent whenever reporting the same type. Reversing the implications, we deduce a utilitarian property: If two agents are each sometimes favored when they have equally intense preferences, then the objects they receive are always assigned between then to maximize the sum of their utilities.

Generalizing to probabilistic rules, the implications are similar. In fact, *ex-ante efficiency* severely restrict opportunities to randomize ex-post. First, there are now more allocations to consider: The conclusions of Lemma 2 apply when the conditions are met for any allocation in the support. Second, randomizing over allocations which cycle objects among some agents means favoring each of those agents over the others. In the language of Lemma 2, this identifies a single value δ and limits randomization among those agents to cases in which the same agent's preferences are δ-more intense.

As an important special case, consider randomizing the assignments of agents reporting the same type. If this occurs with positive probability, then Lemma 2 requires that objects always be allocated among these agents to maximize the sum of their utilities. As a second special case, consider a continuous problem. Here, the preference intensity conditions are met on a measure zero set of profiles meaning that *ex-ante efficient* rules are essentially deterministic. For reference, we summarize these conclusion in Remark 1.

Remark 1. (i) If an *ex-ante efficient* rule randomizes the assignments of agents reporting the same types with positive probability, then the rule always allocates objects among these agents to maximize their utilitarian welfare.

(ii) For each continuous problem, an *ex-ante efficient* rule randomizes assignments for at most a measure zero set of profiles.

Broadly speaking, randomization by an *ex-ante efficient* rule is meaningful only at profiles which occur with positive probability, namely atoms of μ. Nevertheless, as well will see, randomization is consistent with *ex-ante efficiency* on notable finite domains.

3.1.3 Comparing ex-ante efficiency and ex-ante efficiency*

Our definition of *ex-ante efficiency* allows us to search for ex-ante Pareto improvements among the full class of probabilistic rules. In some cases, we may be interested in domination restricted to deterministic rules. For continuous problems, this entails no difference: A rule is *ex-ante efficient* at a continuous problem if and only if it is *ex-ante efficient** at that problem. This is true moreover for each problem whose support contains an open set in $\mathbb{R}^{A\times N}$. For finite problems, the two notions may differ. In the appendix, we provide an example of a problem at which a two-agent TTC rule is *ex-ante efficient** but not *ex-ante efficient* (Example 9). The intuition behind the example is that, although an ex-ante Pareto improvement is possible with ex-post randomization, the types space and measure are too coarse to implement the same improvement deterministically. This type of example relies crucially on the coarseness of type space and even this is generally ameliorated in larger problems which typically assign lower probabilities to each individual profile.

While *ex-ante efficiency* and *ex-ante efficiency** may differ, they can be formally related by expanding the notion of *ex-ante efficiency** apply to a domain of problems. Most important for translating our results, *ex-ante efficiency** on a domain carries the same implications regarding favoring agents according to preference intensity described in Lemma 2.

Remark 2. For each $V \subseteq \mathbb{R}^A$ and each $F \in \mathcal{F}^*$ which is *ex-ante efficient** on $\mathcal{D}(V)$, the conclusions of Lemma 2 apply.

Intuitively, if a rule admits an ex-ante Pareto improvement is possible via ex-post randomization at a given problem, then there will be another problem at which it admits a similar improvement without ex-post randomization, this randomization now absorbed by the probabilities assigned to different profiles. While we emphasize *ex-ante efficiency*, Remark 2 ensures that our conclusions also apply when the notion is *ex-ante efficiency** on a domain of problems.

3.2 Two-agent problems

Although a variety of rules satisfy either *ex-ante efficiency* or *strategy-proofness*, we will see that their combination quickly narrows the range of possibilities. To build intuition, we begin with two-agent problems on finite domains of strict preferences. The conclusion is stark: Essentially any amount of cardinal diversity among potential types, only the priority rules satisfy our requirements.

Let $N \equiv \{1, 2\}$ and $A \equiv \{a, b\}$. To formalize our richness condition, we say that types $u_0, u'_0 \in \mathcal{U}$ are **non-collinear** if $u_0(a) - u_0(b) \neq u'_0(a) - u'_0(b)$. Non-collinear types reflect different intensity of preference. A set $V \subseteq \mathcal{U}$ is **rich** if it contains at least three non-collinear types and a problem $\mu \in \mathcal{D}^*$ is **rich** if there is a rich type space $V \subseteq \mathcal{U}$ such that $V^N \subseteq \text{supp}(\mu)$.[29] With even this minimal diversity, *strategy-proofness* and *ex-ante efficiency* single out the priority rules. Moreover, relaxing the incentive requirement to *Bayesian incentive compatibility* does not expand the range of possibilities. This is surprising because many rules which are not *strategy-proof* nevertheless meet this requirement.

Proposition 1. *For each rich two-agent problem $\mu \in \mathcal{D}^*$, a rule is* ex-ante efficient *and* strategy-proof *or* BIC *and* ex-ante efficient *at μ if and only if it is an extended priority rule.*

We present here a proof when imposing *strategy-proofness.* The arguments in this simple case elucidate the interplay between our axioms, providing the essential intuition for our general results. Extending the argument to *BIC* builds on these ideas, establishing that a *BIC* and *ex-ante efficient* rule is in fact *strategy-proof.* Since this argument is somewhat more elaborate, we relegate it to the appendix.

Proof. With two agents, the rules which are *ex-post efficient* and *strategy-proof* comprise the two priority rules, the two TTC rules, and randomizations among them. To uniquely identify the priority rules, we first argue that the TTC rules either coincide with the priority rules or violate *ex-ante efficiency.* Second, we argue that non-trivial randomization over the priority rules violates *ex-ante efficiency.* Let $\mu \in \mathcal{D}^*$ be a rich problem and $V \subseteq \mathcal{U}$ be rich set of common types in its support.

TTC rules. First, if for each pair $u_0, u'_0 \in V$, $P(u_0) = P(u'_0)$ so that all types induce the same ordinal preferences, then there is no distinction between TTC and priority rules: Being endowed with the preferred object is equivalent to having higher priority.

Suppose instead that V includes types which induce distinct ordinal preferences. Then there are non-collinear types $u^1, u^2, u^3 \in V$ such that $u^1(a) > u^1(b)$ and $u^3(b) > u^3(a)$. Relabeling objects if necessary, suppose that $u^2(a) > u^2(b)$. For each $k = 1, \ldots, 3$, there are $\alpha_k, \beta_k \in \mathbb{R}$ such that $u^k = (\alpha_k, \beta_k)$ so $\alpha_1 > \beta_1$, $\alpha_2 > \beta_2$, and $\alpha_3 < \beta_3$. Also, since the types are non-collinear, we may label the types so that $0 < \alpha_1 - \beta_1 < \alpha_2 - \beta_2$ so that u^2 represents a more intense preference for a.

Then $F^\omega(u^1, u^2) - F^\omega(u^3, u^3) = (a, b)$. By modifying these allocations, we construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u^1, u^2)$ and $p_2 \equiv \mu(u^3, u^3)$ so that $p_1 > 0$ and p_2 by our richness condition. Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < \min\left\{\frac{1}{\alpha_1 - \beta_1}, \frac{1}{(\beta_3 - \alpha_3)}\right\}$, $q_1 \equiv \frac{\varepsilon \cdot p_2}{\alpha_1 - \beta_1}$, and $q_2 \equiv \frac{\varepsilon \cdot p_1}{\beta_3 - \alpha_3}$ so

[29] Although we define richness of a problem in terms of a common set of types, this is for convenience only and is not required for the result.

$q_1, q_2 \in (0,1)$. Now define F for each $u \in \mathcal{U}^N$ by

$$F(u) \equiv \begin{cases} (b,a) & \text{with probability } q_1 \text{ if } u = (u^1,u^2) \\ (b,a) & \text{with probability } q_2 \text{ if } u = (u^3,u^3) \\ F^\omega(u) & \text{otherwise} \end{cases}.$$

Intuitively, F modifies F^ω to better match the agents intensity of preference. Under F, agent 2 sometimes receives a when both agents prefer a but agent 2 values it relatively more highly than does agent 1, which increases utilitarian welfare. As ex-ante compensation, F assigns b to agent 1 when both agents prefer b. Comparing expected utilities for the two agents,

$$\begin{aligned} U_1(F,\mu) - U_1(F^\omega,\mu) &= p_1 \cdot q_1 \cdot \big[\beta_1 - \alpha_1\big] + p_2 \cdot q_2 \cdot \big[\beta_3 - \alpha_3\big] \\ &= -\varepsilon p_1 p_2 + \varepsilon p_2 p_1 \\ &= 0 \\ U_2(F,\mu) - U_1(F^\omega,\mu) &= p_1 \cdot q_1 \cdot \big[\alpha_2 - \beta_2\big] + p_2 \cdot q_2 \cdot \big[\alpha_3 - \beta_3\big] \\ &= \frac{\varepsilon p_1 p_2}{(\alpha_2 - \beta_2)(\alpha_1 - \beta_1)} - \varepsilon p_2 p_1 \\ &> -\varepsilon p_1 p_2 + \varepsilon p_2 p_1 \\ &= 0. \end{aligned}$$

Since agent 1's expected utility is the same under F and F^ω and agent 2's expected is higher under F, F ex-ante Pareto dominates F^ω at μ.

Random priority rules. Let $\prec, \prec' \in \Pi^N$ be such that $1 \prec 2$ and $2 \prec' 1$. Let $\delta \in (0,1)$ and $F^\delta \equiv \delta \cdot F^{\prec} + (1-\delta) \cdot F^{\prec'}$. Now $F^{\prec}(u^1,u^2) = F^{\prec}(u^2,u^1) = (a,b)$ and $F^{\prec'}(u^1,u^2) = F^{\prec'}(u^2,u^1) = (b,a)$ so F^δ follows these allocations with probability δ and $1-\delta$ respectively. As with F^ω, these allocations leave room for ex-ante Pareto improvement. Let $p_1' \equiv \mu(u^1,u^2)$ and $p_2' \equiv \mu(u^2,u^1)$, and $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < \min\{\delta p_1', \delta p_2'\}$. Now let $q_1' \equiv \delta - \frac{\varepsilon}{p_1'}$, $q_2' \equiv \delta + \frac{\varepsilon}{p_2'}$, and define F' for each $u \in \mathcal{U}^N$ by

$$F'(u) \equiv \begin{cases} (a,b) & \text{if } u = (u^1,u^2) \text{ with probability } q_1 \\ (b,a) & \text{if } u = (u^2,u^1) \text{ with probability } q_2 \\ F^\delta(u) & \text{otherwise} \end{cases}.$$

Comparing expected utilities for the two agents,

$$
\begin{aligned}
U_1(F', \mu) - U_1(F^\delta, \mu) &= p'_1 \cdot \big[(q'_1\alpha_1 + (1 - q'_1)\beta_1) - (\delta\alpha_1 + (1 - \delta)\beta_1)\big] + p'_2 \cdot \big[(q'_2\alpha_2 + (1 - q'_2)\beta_2) - (\delta\alpha_2 + (1 - \delta)
\\
&= -\tfrac{\varepsilon p'_1}{p'_1} \cdot \big[\alpha_1 - \beta_1\big] + \tfrac{\varepsilon p'_2}{p'_2} \cdot \big[\alpha_2 - \beta_2\big] \\
&= \varepsilon\big[(\alpha_2 - \beta_2) - (\alpha_1 - \beta_1)\big] \\
&> 0 U_2(F', \mu) - U_2(F^\delta, \mu) \\
&= \tfrac{\varepsilon p'_1}{p'_1} \cdot \big[\alpha_2 - \beta_2\big] - \tfrac{\varepsilon p'_2}{p'_2} \cdot \big[\alpha_1 - \beta_1\big] \\
&= \varepsilon\big[(\alpha_2 - \beta_2) - (\alpha_1 - \beta_1)\big] \\
&> 0.
\end{aligned}
$$

Each agent's expected utility is higher under F' than F^δ, so F' ex-ante Pareto dominates F^δ at μ. □

The proof of Proposition 1 shows how inefficiency arises from cardinal diversity. Intuitively, incomplete information creates a kind of currency at the ex-ante stage and, via probabilities, agents are able to transfer utility. With a *strategy-proof* rule, even a minimal amount of cardinal diversity creates room for improvement in a utilitarian sense. Consequently, as long as conflicts are resolved differently in some states, the agents can increase their expected utilities by accounting for cardinal preferences. Of course, these desirable ex-ante improvements are not incentive compatible ex-post and so inconsistent with *strategy-proofness.*

The arguments in proof of Proposition 1 extended to $\bar{D}^*$ to imply that each *ex-ante efficient* and *strategy-proof* rule is welfare-equivalent to a priority rule. The same is true when *strategy-proofness* is replaced by *BIC*. Thus, we characterize the extended priority rules.

Corollary 1. *For each rich two-agent problem $\mu \in \bar{\mathcal{D}}^*$, a rule is* ex-ante efficient *and* strategy-proof *or* BIC *and* ex-ante efficient *at μ if and only if it is an extended priority rule.*

If we instead relax our efficiency condition to *ex-post efficiency*, TTC rules become admissible, as well as all rules randomizing over TTC and priority rules. This collection is quite large and includes rules which are *ex-ante symmetric* or *ex-post symmetric*. Nevertheless, all but the priority rules leave room for improvement: ex-ante, agents unanimously[30] prefer to change the rule.

3.3 Problems with arbitrary populations

Two-agent problems are somewhat special, most notably because all rules are trivially *non-bossy*. Even when this property is imposed, a large family of "hierarchical exchange", "trading cycles",

[30] Although we construct weak ex-ante Pareto improvements in the proof of Proposition 1, we can make both agents better off by slightly modifying the probabilities.

or "trading and braiding" rules are *ex-post efficient* and *strategy-proof* (Pápai, 2000b; Pycia and Ünver, 2014; Bade, 2014b). Including arbitrary randomizations among these rules leads to a diverse collection satisfying these properties. Yet here again, a small amount of incomplete information dramatically changes our perspective with only the priority rules satisfying *ex-ante efficiency*.

3.3.1 Richness conditions

Our results apply on domains with at least some diversity of types so that incomplete information is meaningful. In this section, we study problems on domains satisfying two richness requirements: (i) diversity of top preferences and (ii) diversity or preference intensities. Diversity of top preferences simply means that each agent may rank each object first. While natural, this does limit the strength of correlation among agents' preferences. For example, a setting in which agents receive a common signal about which object lacks this diversity.

Diversity of preference intensity concerns perceived utility differences. For two-agent problems, non-collinearity reflects this diversity. Generalizing to many objects, we ask that there be types which rank objects in the same way but differ according to their intensity of preference for the top object relative to the remaining objects. For example, proportional types are related in this way, as are types which differ only by the utilities they assign to the top object. Since we impose this requirement for only one object, it is satisfied by every continuous problem. In contrast, a purely ordinal domain in which each type corresponds to a unique ranking of the objects does not satisfy this condition.

We call a set of types rich if it meets our two requirements and call a problem rich if it places positive probability on profiles formed by the product of a rich set. Implicitly, we suppose that a single rich set of types is common among the agents. This seems natural, but the assumption is for convenience only. In fact, our results require no amount of overlap among the type spaces. It suffices that each agent's type space include types meeting these criteria and that profiles consisting of their products occur with positive probability.

We now formalize our requirements. For each $a \in A$, let $V^a \equiv \{u_0 \in \mathbb{R}^A : a = \text{argmax}_{c \in A} u(c)\}$, the set of types which rank a (uniquely) first. For each $a \in A$ and each $V_0 \subseteq V^a$, let

$$V^a(V_0) \equiv \left\{u'_0 \in V^a : \forall\, u_0 \in V_0\, \forall\, c \in A\backslash\{a\},\, u_0(a) - u_0(c) < u'_0(a) - u'_0(c)\right\}.$$

That is, $V^a(V_0)$ consists of those types with greater intensity of preference for a than all of the types in V_0. Slightly refining this set, also let

$$V^a_*(V_0) \equiv \left\{u'_0 \in V^a(V_0) : \exists\, u_0 \in V_0\, \exists\, \delta \in \mathbb{R}_{++}\, \forall\, c, c' \in A,\, u_0(c) < u_0(c') \Rightarrow u'_0(c) + \delta < u'_0(c')\right\}.$$

Then $V^a_*(V_0)$ consists of types which increase the spread of all values specified by some types in V_0.

For example, a type in $V_*^a(V_0)$ may scale a type in V_0. As a consequence, paired types correspond to the same ordinal ranking. A set $V \equiv \bigcup_{k=1}^{n+2} V^k \subseteq \mathbb{R}^A$ is **rich** if it satisfies:

1. Diversity of top preference: For each $k \in \{1, \ldots, n\}$, $\emptyset \neq V^k \subseteq V^{a_k}$.

2. Diversity of preference intensity: There is $a \in A$ such that $\emptyset \neq V^{n+2} \subseteq V^a(V^{n+1})$.

Strengthening richness, a set $V \equiv \bigcup_{k=1}^{2n} V^k \subseteq \mathbb{R}^A$ is **rich*** if for each $k \in \{1, \ldots, n\}$, $\emptyset \neq V^{n+k} \subseteq V_*^{a_k}(V^k) \subseteq V^{a_k}$. This defines diversity of preference intensity according to the refined comparison and extends the requirement to all objects. Finally, $\mu \in \bar{\mathcal{D}}$ is **rich** if there is a rich set $V \equiv \bigcup_{k=1}^{n+2} V^k \subseteq \mathbb{R}^A$ such that for each $l \in \{1, \ldots, n+2\}^N$, $\mu(\prod_N V^{l_i}) > 0$. It is **rich*** if there is a rich* set $V \equiv \bigcup_{k=1}^{2n} V^k \subseteq \mathbb{R}^A$ such that for each $l \in \{1, \ldots, 2n\}^N$, $\mu(\prod_N V^{l_i}) > 0$. While a rich problem need not be rich*, the conditions are equivalent among continuous problems. More generally, both conditions are satisfied whenever the sets $\{V^1, \ldots, V^n\}$ are open, so the difference is meaningful primarily among finite problems.

In the definition of a rich set, V^{n+1} and V^{n+2} may be subsets of one of the V^k. Consequently, a rich set may consist of as few as $n+1$ types and a finite problem may place probability only on profiles consisting of these types. For example, these types could be constructed from a single type differing only by moving one object to the top, an extremely minimal amount of diversity. For a continuous problem, a collection of $n+1$ arbitrarily small open sets suffices for richness.

3.3.2 Characterizations for rich problems

Returning to the implications of our axioms, we find that efficiency and incentive properties continue characterize the priority and extended priority rules.

Theorem 1. *(i) For each rich strict problem, a rule is* ex-ante efficient *and* strategy-proof *if and only if it is a priority rule.*
(ii) For each rich strict problem, a rule is* ex-ante efficient *and* BIC *if and only if it is a priority rule.*
(iii) For each rich or rich problem, the same axioms characterize the extended priority rules.*

Theorem 1 is proved in the appendix. The essence of the argument is to determine the extent to which some agents are favored over others. *Ex-ante efficiency* says that once an agent is favored, this agent must also be favored whenever his relative preference is at least as intense. On the other hand, *strategy-proofness* says that a small amount of favoritism – even breaking a tie at profiles of identical preferences – spreads to all cases. Under *strategy-proofness*, "partial" favoritism is not sustainable. We show by example (Example 10 in the appendix) that the characterization by *ex-ante efficiency* and *BIC* may fail at problems that are rich but not rich*. Again, however, the difference is narrow, essentially restricted to very narrow finite problems. We investigate a more substantive domain differentiating *strategy-proofness* and *BIC* in Section 3.6.

Surprisingly, our characterization does not invoke standard robustness requirements. We allow rules to condition on types rather than only ordinal reports, yet those meeting our requirements do not. Moreover, we allow *bossy* rules. This permits, for example, resolving a conflict between two agents differently depending on a portion of the report of a third agent irrelevant to his own assignment. This is consistent with *strategy-proofness* and, particularly when preferences are correlated, may be desirable from utilitarian perspective. Nevertheless, it introduces opportunities for ex-ante Pareto improvements and so conflicts with *ex-ante efficiency.*

Similarly, allowing ex-post randomization does not expand the range of admissible rules. Although randomization softens the incentive requirements – for example, preventing spreading from breaking ties – *ex-ante efficiency* applies even more forcefully to random assignments. In these cases, as described in Remark 1, randomizing among allocations means simultaneously "favoring" all agents in a group over all other agents in the group. This severely restricts the situations in which randomization is relevant. Moreover, if relevant at all, randomization even on this narrow range is inconsistent with the axioms.

3.4 Applications to related models

Our Theorem 1 carries implications for models of endogenous learning as well as the school assignment problem.

3.4.1 Endogenous learning

In an environment with costly learning, Bade (2015) characterizes the priority rules by *strategy-proof*, *non-bossiness*, and *Pareto optimality.* Her efficiency requirement is ex-ante like ours, but much stronger, additionally requiring efficiency for all admissible learning cost functions. Our notion of *ex-ante efficient** on $\mathcal{D}^*$ corresponds to her *Pareto optimality* when learning is free. By Remark 2, we have an immediate corollary of Theorem 1.

Corollary 2. *(i) A rule is* ex-ante efficient* *and* strategy-proof *on $\mathcal{D}^*$ if and only if it is a priority rule.*
(ii) A rule is ex-ante efficient* *and* strategy-proof *on $\bar{\mathcal{D}}^*$ if and only if it is an extended priority rule.*

Turning to Bade (2015)'s model, Corollary 2 implies a generalization of her Theorem 2.

Corollary 3. *In the Bade (2015) model of endogenous learning, a rule is* strategy-proof *and* Pareto optimal *if and only if it is a priority rule even when learning is costless.*

Corollary 3 provides a point of reference for understanding the difference between uncertainty and endogenous learning. Corollary 3 strengthen's Bade (2015)'s Theorem 2 in two ways. First, it does not invoke *non-bossiness.* Second, it restricts the range of *Pareto optimality* to trivial cost functions. Compared to Bade (2015)'s results for costly learning, there remains an important point of contrast: With incomplete information alone or when learning is costless, no priority rule ex-ante Pareto dominates a TTC rule in any problem. On the other hand, Bade (2015) demonstrates learning technologies for which a priority rule may dominate a TTC rule. Summarizing, *incomplete information* suffices for uniqueness whereas *costly learning* is required for dominance.

3.4.2 Replicated objects and school assignment

Within our model, it is easy accommodate multiple copies of object. This allows us to model "school choice" problems which associate each object with a capacity (Abdulkadiroğlu and Sönmez, 2003). To see how, let $\mathcal{A} \equiv \{A_1, \ldots, A_K\}$ be a partition of A, and

$$V^{\mathcal{A}} \equiv \Big\{u_0 \in \mathbb{R} : \forall\, k \in \mathbb{N}\, \forall\, a, b \in A_k,\; u_0(a) = u_0(b)\Big\}.$$

Then for each $\mu \in \overline{\mathcal{D}(V^{\mathcal{A}})}$, all agents assign the same value to all objects of the same type. For example, A_k represents a school with $|A_k|$ indistinguishable seats.

Problems in the domain $\mathcal{D}(V^{\mathcal{A}})$, violate our richness conditions. However, because no agent distinguishes among replicas of an object, we can rephrase our richness requirement for this setting in terms of object types. Given a partition $\mathcal{A}$ of A, for each $k \in \{1, \ldots, K\}$, let

$$V^k \equiv \big\{u_0 \in V^{\mathcal{A}} : \forall\, k \in \mathbb{N}\, \forall\, a \in A_k\, \forall c \in A \backslash A_k,\; u_0(a) > u_0(c)\Big\}.$$

For each $k \in \{1, \ldots, K\}$ and each $V_0 \subseteq V^k$, let

$$V^k(V_0) \equiv \Big\{u_0' \in V^k : \forall\, u_0 \in V_0\, \forall\, a \in A_k\, \forall\, c \in A \backslash A_k,\; u_0(a) - u_0(c) < u_0'(a) - u_0'(c)\Big\}.$$

Slightly refining this set as in the original model, let

$$V_*^k(V_0) \equiv \Big\{u_0' \in V^k(V_0) : \exists\, u_0 \in V_0\, \exists\, \delta \in \mathbb{R}_{++}\, \forall\, c, c' \in A,\; u_0(c) < u_0(c') \Rightarrow u_0'(c) + \delta < u_0'(c')\Big\}.$$

A set $V_0 \equiv \bigcup_{k=1}^{K+2} V_0^k \subseteq \mathbb{R}^A$ is **partition-rich** with respect to $\mathcal{A}$ if:

1. For each $k \in \{1, \ldots, K\}$, $V_0^k \subseteq V^k$.

2. There is $k \in \{1, \ldots, K\}$ such that $V^{K+2} \subseteq V^k(V^{K+1})$.

A problem $\mu \in \overline{\mathcal{D}(V^{\mathcal{A}})}$ is **partition-rich** with respect to $\mathcal{A}$ if there is a rich set $V_0 \equiv \bigcup_{k=1}^{K+2} V_0^k \subseteq \mathbb{R}^A$ such that for each $l \in \{1, \ldots, K+2\}^N$, $\mu(\prod_N V_0^{l_i}) > 0$. If additionally $V^{K+2} \subseteq V_*^k(V^{K+1})$, we say that V and μ are **partition-rich***. With these adapted richness definitions, our results again apply.

Corollary 4. *Let $\mathcal{A}$ be a partition of A.*
(i) For each problem $\mu \in \mathcal{D}(V^{\mathcal{A}})$ which is partition-rich with respect to $\mathcal{A}$, a rule is ex-ante efficient *and* strategy-proof *if and only if it is a priority rule.*
(ii) For each problem $\mu \in \mathcal{D}(V^{\mathcal{A}})$ which is partition-rich with respect to $\mathcal{A}$, a rule is* ex-ante efficient *and* BIC *if and only if it is a priority rule.*
(iii) For each $\mu \in \overline{\mathcal{D}(V^{\mathcal{A}})}$ which is partition-rich or partition-rich with respect to $\mathcal{A}$, the same axioms characterize the extended priority rules.*

In a typical school choice problem, objects are augmented by priority orders. Enriching the model in this way, our results say that the only object-priority orders compatible with *ex-ante efficiency* and *strategy-proofness* are essentially trivial: Object-priorities must be assignment-equivalent to a single, uniform priority order.[31]

3.5 Role of richness in the characterizations

The thrust of our results is that even a little bit of uncertainty about preferences creates inefficiencies with all of these rules but the priority rules. While our richness conditions are very mild, some cardinal diversity is generally required to uniquely identify the priority rules. In this section, we elaborate on these requirements and possibilities when they are violated.

3.5.1 Diversity of top preferences

We first consider restricting the set of objects appearing at the top while retaining diversity of preference intensity. To gain intuition, first consider the extreme of complete information. In this case, *ex-ante efficient* is equivalent to *ex-post efficiency.* Lifting results from the standard framework, it appears at first glance that the large families of "hierarchical exchange", "trading cycles", or "trading-and-braiding" rules are now available (Pesendorfer, 1979; Pycia and Ünver, 2014; Bade, 2014b). However, this is misleading: On this narrow domain, each such rule is indistinguishable from a priority rule.[32] In other words, our characterization continues to apply.

Next suppose that each agent's type space is restricted to a particular cone in $\mathbb{R}^{A\times N}$. Then the ordinal rankings are completely determined, though intensities may be quite varied. Assuming that there is more than one ex-post efficient allocation so that not all *ex-post efficient* rules coincide, we might consider ex-post randomization or using the report one agent to differently resolve conflicts among other agents. Both possibilities are consistent with incentives. However, neither procedure is able to perfectly align the resolution of these conflicts with their preference intensities. Consequently, a agent will sometimes be favored when his preference intensity is relatively low yet

[31] It is possible that object-priorities differ in trivial ways. For example, given an object with capacity k, reordering the first k agents does not change the allocation at any profile.

[32] As Abdulkadiroğlu and Sönmez (1998) show, for each *ex-post efficient* allocation, there is a priority order which induces it. All rules selecting a given allocation will appear identical to this priority rule.

disfavored at other times when his preference intensity is relatively high. This violates *ex-ante efficiency.* Thus, we return to the priority rules.

While these examples are suggestive, we cannot entirely dispense with top preference diversity. For example, suppose that all types rank the same two objects at the bottom. Moreover, suppose that the utility difference between these objects is the same. With this special structure, modifying a priority rule to randomize the assignments of the two agents with lowest priority is consistent with *strategy-proofness* and *ex-ante efficiency.* Although some non-priority rules now meet our requirements, provided there is one object satisfying diversity of preference intensity, *strategy-proofness* and *ex-ante efficiency* imply that one agent always receives his most preferred object. Proposition 2 summarizes.

Proposition 2. *For each problem $\mu \in \bar{D}$ satisfying diversity of preference intensity and each rule that is* ex-ante efficient *and* strategy-proof *at μ, there is an agent who always receives one of his most preferred objects.*

As the number of objects which may be ranked first increases, we move closer to the priority rules. More generally, given one object satisfying diversity of preference intensity and k additional objects satisfying top preference diversity, each *ex-ante efficient* and *strategy-proof* rule agrees with a priority rule for agents with priorities 1 through $k + 1$.

3.5.2 Diversity of preference intensity

Broadly speaking, relaxing our first richness condition leads to few new possibilities. As we will see, diversity of preference intensity is more significant. Without this diversity, the demands of *ex-ante efficiency* are much less stringent. We begin illustrative examples here, reserving a complete analysis more ordinal domains for the next section.

First consider a two-agent problem and let $u^a, u^b \in \mathbb{R}^A$ be such that $u^a(a) = u^b(b) > u^a(b) = u^b(a)$. Then $V \equiv \{u^a, u^b\}$ is ordinally diverse, though lacks diversity of preference intensity. On this domain, each TTC rule is *ex-ante efficient*, as are all probabilistic rules randomizing among the allocations under TTC and priority rules. Remark 3, verified in the appendix, formalizes these observations.

Remark 3. With two agents and two objects, for each pair $u_0, u_0' \in \mathbb{R}^A$ and each $\mu \in \bar{\mathcal{D}}^*(\{u_0, u_0'\})$, all *ex-post efficient* rules are *ex-ante efficient* and *strategy-proof* at μ.

The key observation is that on this domain, TTC rules in fact maximize utilitarian welfare because *ex-post efficiency*, *ex-ante efficiency*, and *utilitarian optimality* coincide. Unfortunately, this positive result for two-agent problems does not generalize. As we have seen in Example 1, once there are even three agents, TTC rules violate *ex-ante efficiency* even on domains with no diversity of preference intensity.

While the combination of *ex-ante efficiency* and *strategy-proofness* is limiting, we are nevertheless able to identify a rather large domain which admits *ex-ante efficient* and *strategy-proof* rules that are not priority rules. A set is $V \subseteq \mathbb{R}^A$ is **top-collinear** if there are $\varepsilon \in \mathbb{R}_{++}$ and $\tau \colon A \to A$ such that for each $a \in A$, $\tau(a) \neq a$ and

$$V \equiv \left\{ u_0 \in \mathcal{R}^A : \exists\, a \in A, \begin{array}{r} a = \text{argmax}_{c\in A} u_0(c), \\ \tau(a) = \text{argmax}_{c\in A\setminus\{a\}}, \\ \text{and } u_0(a) - u_0(\tau(a)) = \varepsilon \end{array} \right\}.$$

That is, V consists of types which "pair" the top two object. For all types with the same top-ranked object, the second-ranked object is also the same; moreover, for all types the utility difference between the top two objects is constant.

We identify rules which are *ex-ante efficient* and *strategy-proof* on $\mathcal{D}(V)$ by modifying priority rules following Bogomolnaia et al. (2005). The modified rules allow conflicts between the two highest priority agents to be resolved differently for different objects.

Bi-polar priority rule with respect to $\prec \in \Pi^N$ and $B \subseteq A$, $F^{\prec,B}$: For each $u \in \mathcal{U}^N$, labeling agents so that $\prec = (1, 2, 3, \ldots, n)$ and letting $\prec' \equiv (2, 1, 3, \ldots, n)$,

$$F^{\prec,B} \equiv \begin{cases} F^{\prec'}(u) & \text{if } \text{argmax}_{c\in A} u_1(c) = \text{argmax}_{c\in A} u_2(c) \in B \\ F^{\prec}(u) & \text{otherwise} \end{cases}.$$

Thinking in terms of endowments, $F^{\prec,B}$ modifies $F^{\prec}$ by initially endowing agent 2 with B. Allowing for indifferences, we define an **extended bi-polar priority rule** similarly from the corresponding extended priority rules. Similarly, we can randomize over the bi-polar rules defined by the same order but different sets of objecs. The **randomized bi-polar priority rule** with respect to $\prec$ randomizes over the allocations recommended by rules among $\{F^{\prec,B} : B \subseteq A\}$. Tailored to the structure of a top-collinear domain, randomized bi-polar priority rules are *ex-ante efficient* and *strategy-proof* on $\mathcal{D}(V)$.

Remark 4. For each top-collinear set $V \subseteq \mathcal{R}^A$ and each $\mu \in \mathcal{D}^*(V)$, each randomized bi-polar priority rules is *ex-ante efficient* and *strategy-proof* at μ.

We verify *ex-ante efficiency* in the appendix. A top-collinear domain includes a continuum of types as well as types corresponding to most ordinal rankings. As such domains are large in a meaningful sense, they are important examples illustrating the limits of our characterizations. To better understand our axioms, we turn next to ordinal domains which also lack diversity of preference intensity.

3.6 Ordinal and scoring domains

In many common settings, agents report preference rankings rather than utilities, so complete information about types may be unavailable. Still interested in ex-ante expected utility, we take the perspective of an observer who must then infer this information. For example, even believing that the underlying distribution is diverse, we may assign utilities representing the average over compatible types. This leads us to investigate a class of ordinal domains in which each type corresponds to a unique ranking over objects.

Among ordinal domains, we emphasize a subclass with uniform structure. On these domains, utilities are completely determined by ordinal ranks, similar to "scoring rules" familiar from social choice (Young, 1975). Relative to general ordinal domains, these "scoring domains" possess two additional features. First, agents share a common underlying type space, so we draw the same inferences about utility from a given report regardless of which agent submits it. Second, types treat objects symmetrically. For example, an agent receives the same utility whenever assigned his most preferred object, regardless of which object it turns out to be. Scoring domains reflect the perspective an observer who assigns a common value to each rank to indicate preference intensity.[33]

Because utilities on ordinal domains are highly structured, *ex-ante efficiency* has much less force. Nevertheless, subject again to a mild richness requirement, *strategy-proofness* and *ex-ante efficient* continue to characterize the priority rules. Our richness requirement is familiar from social choice as the "free triple" property.[34] In lieu of our diversity of preference intensity assumption, we suppose that at least one triple of objects appears in all possible orders at the top of the ranking for each agent. On the other hand, relaxing our incentive requirement to *BIC* now greatly expands the range of rules for many problems. On scoring domains in particular, we are often able to achieve even *utilitarian optimality* and *ex-ante symmetry*. We begin with some definitions and then turn to the implications of our incentive requirements.

3.6.1 Preliminaries

We first formalize the domains we study. A set $V \subseteq \mathbb{R}^A$ is **ordinal** if each type induces a distinct ranking over objects: For each pair $u_0, u_0' \in V$, if $P(u_0) = P(u_0')$, then $u_0 = u_0'$. An ordinal set includes at most $n!$ strict types and is necessarily finite. A problem $\mu \in \bar{D}^*$ is **ordinal** if for each $i \in N$, there is an ordinal set $V_i \subseteq \mathbb{R}^A$ such that $\mu(\prod_N V_i) = 1$. Within this class, we distinguish sets of types on which all utility information can be inferred from ordinal ranks. Let $S \equiv \Big\{(s_1, \ldots, s_n) \in \mathbb{R} : s_1 > s_2 > \cdots > s_n\Big\}$ and for each $s \in S$, let $V^s \equiv \Big\{u_0 \in \mathcal{U} : \{u_1, \ldots, u_n\} = \{s_1, \ldots, s_n\}\Big\}$. We call S the set of **scoring vectors** and $\mathcal{D}(V^s)$ the **scoring domain** with respect

[33]Alternatively, we can view scoring domains as normalizations. In this case, we interpret *utilitarian-optimal* in terms of *relative utilitarianism* (Dhillon and Mertens, 1999).

[34]For example, see Blau (1957); Sen (1966), and Duggan (2015).

to s. In the special case of constant differences, i.e. $s_1 - s_2 = s_2 - s_3 = \cdots = s_{n-1} - s_n$, we call $\mathcal{D}(V^s)$ a **Borda domain**.

Next we formalize our richness condition. Given $V \subseteq \mathbb{R}^A$, a triple $a, b, c \in A$ is **free at the top** in V if for each labeling $\{\alpha, \beta, \gamma\} = \{a, b, c\}$, there is $u_0 \in V$ such that for each $d \in A \backslash \{a, b, c\}$, $u_0(a) > u_0(b) > u_0(c) > u_0(d)$. A set $V \subseteq \mathbb{R}^A$ is **ordinally diverse** if it satisfies:

1. Diversity of top preference: For each $a \in A$, there is $u_0 \in V$ such that $a = \text{argmax}_{c \in A} u_0(c)$.

2. Free triple property: There is a triple $a, b, c \in A$ which is free at the top in V.

Finally, $\mu \in \bar{\mathcal{D}}^*$ is **ordinally rich** if there is an ordinally rich set $V \subseteq \mathbb{R}^A$ and for each $i \in N$ an ordinal set $V_i \subseteq \mathbb{R}^A$ such that $V \subseteq V_i$ and $\text{supp}(\mu) = \prod_N V_i$.

3.6.2 Characterization with strategy-proofness

With two object, a scoring domain consists of exactly two types. By Remark 3, we achieve *ex-ante efficiency* and *strategy-proofness* simply by ensuring *ex-post efficiency*. TTC rules as well as many probabilistic rules therefore satisfy these properties. Unfortunately, once there are as few as three objects, these additional possibilities evaporate.

Theorem 2. *Let $|A| \geq 3$. For each ordinally rich problem $\mu \in \mathcal{D}^*$, a rule is* ex-ante efficient *and* strategy-proof *at μ if and only if it is a priority rule.*

As an important special case, this results applies to each scoring domain. These domains will provide our leading point of contrast between the requirements of *strategy-proofness* and *BIC*.

Corollary 5. *Let $|A| \geq 3$. For each $s \in S$ and each ordinally rich problem $\mu \in \mathcal{D}(V^s)$, a rule is* ex-ante efficient *and* strategy-proof *at μ if and only if it is a priority rule.*

The proofs appear in the appendix. Our proof in fact begins with scoring domain where the constant difference property of a scoring domain reduces the opportunities for potential ex-ante Pareto improvements. The conclusions then extend straightforwardly. To prove the result on scoring domains, we begin with three-agent problems where the three triple property ensures support on the full domain. To extend the result problems with many agents and objects, we employ a calibration strategy, using the guaranteed free triple to obtain partial orders over each triple of agents. We conclude by showing that these orders coincide, thereby defining a priority rule.

While we relegate the full proof of Theorem 2 to the appendix, we develop intuition by considering bi-polar priority rules. We have argued that bi-polar priority rules are *ex-ante efficient* on top-collinear domains. On scoring domains, it is impossible to reassign objects to make either of the two top priority agents better off without making the other worse off. The inefficiency we observe with these rules is qualitatively different from the inefficiency we saw among TTC rules in Example 1.

Example 4. Bi-polar priority rules may violate ex-ante efficiency on scoring domains. Let $N \equiv \{1,2,3\}$, $A \equiv \{a,b,c\}$, $s \equiv S$, and $\mu \in \mathcal{D}(V^s)$ be the uniform measure so that for each $u \in (V^s)^N$, $\mu(u) = p_0 \equiv \frac{1}{(3!)^3} = \frac{1}{216}$. For each pair $\alpha, \beta \in \{a,b,c\}$, we write $u^{\alpha\beta}$ for the unique type that ranks α first and β second so $u^{\alpha\beta}(\alpha) = s_1$ and $u^{\alpha\beta}(\beta) = s_2$. Let $\prec \equiv (1,2,3)$, $B \equiv \{a\}$, and consider $F^{\prec,B}$. Up to relabeling, $F^{\prec,B}$ is the only non-trivial bi-polar priority rule with three objects.

Following the bi-polar priority rule, $F^{\prec,B}(u^{ab}, u^{ac}, u^{ba}) = (b,a,c)$ and $F^{\prec,B}(u^{ca}, u^{cb}, u^{bc}) = (c,b,a)$. Although it is impossible to make either agent 1 or agent 2 better off without making the other worse off, these assignments leave room to improve the welfare of agent 3. Let F' modify $F^{\prec,B}$ so that $F'(u^{ab}, u^{ac}, u^{ba}) = (a,c,b)$ and $F'(u^{ca}, u^{cb}, u^{bc}) = (a,c,b)$. Comparing expected utilities,

$$U_1(F', \mu) - U_1(F^{\prec,B}, \mu) = p_0 \cdot (s_1 - s_2) + p_0 \cdot (s_2 - s_1) = 0,$$
$$U_2(F', \mu) - U_2(F^{\prec,B}, \mu) = p_0 \cdot (s_1 - s_2) + p_0 \cdot (s_2 - s_1) = 0, \text{ and}$$
$$U_3(F', \mu) - U_3(F^{\prec,B}, \mu) = p_0 \cdot (s_1 - s_3) + p_0 \cdot (s_1 - s_3) > 0.$$

Therefore, F' ex-ante Pareto dominates $F^{\prec,B}$. If μ is not the uniform measure but still has full support on $(V^s)^N$, we obtain a similar improvement with ex-post randomization.

As a final implication of our characterization, we observe a general incompatibility with symmetry.

Corollary 6. *Let $|A| \geq 3$. For each ordinally rich problem $\mu \in \mathcal{D}^*$, no rule is* ex-ante efficient *and* strategy-proof *and either* ex-ante symmetric *or* ex-post symmetric *at μ.*

The result follows immediately from our characterization since priority rules are neither *ex-ante symmetric* nor *ex-post symmetric*. Most important is the narrowness of the underlying domain which allows us to compare the conclusion to similar impossibilities. Corollary 6 can be viewed as an extension of the impossibilities derived by Zhou (1990) and Bogomolnaia and Moulin (2001). Compared with Zhou (1990), the narrowness of the Borda domain makes our efficiency notion much weaker. Compared with Bogomolnaia and Moulin (2001), the Borda domain makes our notion of efficiency stronger, but significantly weakens *strategy-proofness*.

3.6.3 Additional possibilities with BIC

Theorem 2 characterizes the priority rules, even on ordinal domains. On contrast with domains exhibiting diversity of preference intensity, we obtain additional positive results by relaxing our incentive requirement to *BIC*. In fact, as we show for scoring domains, we are often able to achieve *utilitarian optimality* and even *ex-ante symmetry* in addition to *BIC*.

Proposition 3. *On each scoring domain, there exists an open set of problems including the uniform measure on which* ex-ante symmetry*,* utilitarian optimality*, and* BIC *are compatible.*

Proposition 3 does not restrict ordinal diversity and so the positive conclusion applies even for problems in which all profiles of ordinal rankings occur. Proposition 3 is also true if *ex-ante symmetry* is replaced by *ex-post symmetry.* The symmetry axioms are compatible for the uniform problem, where *ex-post symmetry* in fact implies *ex-ante symmetry*, and more generally whenever types are independent and identically distributed. However, these requirements may be incompatible, even for nearby problems with correlation or heterogeneous distributions.

When we interpret a score vector as representing average utilities, Proposition 3 suggests that the efficiency loss may limited even on a rich domain, provided a problem is well-approximated by the uniform problem on a scoring domain. In such cases, we may recommend a rule with desirable properties on scoring domain even though it will leave agents with opportunities to renegotiation ex-ante.

4 Social welfare behind the veil: Arguments for Top Trading Cycles

Modeling choice of rules from *behind the veil of ignorance*, we turn to a class problems in which agents and objects are ex-ante symmetric.[35] We first formally define the domain and then turn to evaluation of rules. For most of the analysis, we emphasize the well-known family of *ex-post efficient*, *strategy-proof*, and *non-bossy* rules.

Our first results draw general conclusions about the performance of rules in symmetric problems. Although the priority rules continue to be distinguished by *ex-ante efficient* and *strategy-proof* (Remark 5), the argument in their favor is not conclusive. We find that many rules perform equally well in terms of *utilitarian optimality* (Proposition 4) and essentially all are more egalitarian with TTC rules standing out (Proposition 5). Our next results show that all of these rules approximate *utilitarian optimality* in large problems (Proposition 6) and that TTC rules generally have the most favorable solidarity properties as the size of the economy grows (Proposition 8).

Extending symmetric problems, we introduce interim outside options and find that TTC rules generally meet a stronger participation constraint than do priority rules (Corollary9). We also compare welfare of individual agents across rules (Proposition 9) and compare common proposals to randomize over either priority orders or endowments (Corollary 11). Finally, we consider problems in which agents and object are no longer ex-ante symmetric. Depending on the details of the problem, some comparisons may reverse with different rules being well suited to take advantage of particular correlations among preferences. All proofs appear in the appendix.

[35]While our motivation is normative, as discussed in the introduction, symmetric problems also describe many important economic environments including most auction settings.

4.1 Symmetric problems

We begin by describing the domain. We model values of objects as decomposable into a "common" value and a "private" value. Thinking of agents and objects as ex-ante symmetric, we imagine that all common values are drawn from the same distribution and that all private values are drawn from the same distribution. We further suppose that these draws are independent across agents and objects, so correlation enters only through the common values. To ensure diversity of preference intensity and also to avoid complications due to indifferences, we assume that private values are drawn from a continuous distribution. Finally, so that we may analyze limits as the size of the economy grows, we restrict attention to bounded distributions.

To formalize these ideas, let G and H be bounded cdfs with G also continuous and $\mathbf{E}_G[x] = \mathbf{E}_H[x] = 0$.[36] For each $a \in A$, the **common value** of a is a random variable v_a distributed according to H. For each $i \in N$ and each $a \in A$, the **private value** of a to agent i is a random variable ε_{ia} distributed according to G. Altogether, agent i value for a is $v_{ia} \equiv v_a + \varepsilon_{ia}$ which defines the type $v_i \equiv (v_{ia})_{a \in A}$. Since G is continuous, $Pr(v_i \in \mathcal{U}) = 1$. Let $\tau_G \equiv \sup(\text{supp}(\text{G}))$ and $\tau_H \equiv \sup \text{supp}(H)$. Also, for each collection of random variables $\{x_1, \ldots, x_l\}$ distributed according to G, let $\nu_G^{(k,l)} \equiv \mathbf{E}[x^{k,l)}]$, so $\nu^{(k,l)}$ is the expected value of their kth order statistic.

Distributions G and H describe the **symmetric problem** $\mu_{G,H}$, the measure over $\mathbb{R}^{A \times N}$ defined by independently drawing the $(\varepsilon_{ia})_{i \in N, a \in A}$ and $(v_a)_{a \in A}$ according to G and H respectively. When H is degenerate, values are independent. In this case, $\mu_{G,H}$ is an **independent symmetric problem** which we denote by μ_G. The other extreme, a degenerate distribution G, corresponds to common values. By our assumption that G is non-atomic, this case is not a symmetric problem. Moreover, each symmetric problem is rich and so Theorem 1 applies.

Remark 5. For each symmetric problem, a rule is *ex-ante efficient* and *strategy-proof* if and only if it is a priority rule.

4.1.1 Equity comparisons and the Lorenz order

While only the priority rules are *ex-ante efficient* at symmetric problems, a large family of rules achieve the same utilitarian welfare.

Proposition 4. *For each symmetric problem, all* ex-post efficient*,* strategy-proof*, and* non-bossy *rules achieve the same utilitarian welfare.*

In particular, Proposition 4 implies that no *ex-post efficient*, *strategy-proof*, and *non-bossy* rule is ex-ante Pareto dominated at a symmetric problem. Instead, moving from one rule to another amounts to transferring expected utility among the agents.

[36]Since we will assume that values are iid, normalizing the distributions to have mean zero is without loss of generality.

The proof of Proposition 4 builds on equivalence results concerning the distributions of allocations when randomizing the roles of agents in different rules (Abdulkadiroğlu and Sönmez, 1998; Lee and Sethuraman, 2011; Carroll, 2014; Bade, 2014c). These results compare, for example, randomizing over priority orders and randomizing over endowments and show that each method induces the same lottery over final allocations.[37] However, symmetry is essential. In fact, the conclusions are not true more generally. As we have seen in Example 2, a priority rule may in fact ex-ante Pareto dominate a TTC rule. Furthermore, no similar comparisons are possible among ex-post utilities at a given profile. Consequently, the conclusions do not hold when there is very little uncertainty about preferences. We return to these points subsequently.

Based on Proposition 4, we can provide simple formulas for agents' expected welfare under priority and TTC rules in independent symmetric problems.

Corollary 7. *Let μ_G be an independent symmetric problem. (i) Under each priority rule, the expected utility of the agent with kth highest priority is $\nu_G^{(n+1-k,n+1-k)}$. (ii) Under each TTC rule, each agent's expected utility is $\sum_{k=1}^{n} \frac{1}{n} \nu_G^{(k,k)}$.*

The formulas in Corollary 7 can be interpreted in terms of "option sets". A priority rule, for example, provides the agent with kth highest priority a choice among $n-k+1$ objects. The formula for expected utility under a TTC rule tells us that the size of an agent's option set is equally likely to be each possible value in $\{1, \ldots, n\}$. At one extreme, with probability $\frac{1}{n}$, all other agents will "point" at a given agent and he will be able to choose his most preferred among all of the objects. At the other extreme, also with probability $\frac{1}{n}$, all other agents will trade among themselves and never "point" at the given agent, leaving him with no choice but to consume his endowment.

Observing that many rules achieve the same utilitarian welfare, we inspect the expected utility profiles they induce for a more nuanced comparison. We find that TTC rules are the most egalitarian and the priority rules the least egalitarian in the class. Within the class of symmetric problems, we are able to make strong comparisons according the Lorenz order.

Proposition 5. *For each symmetric problem, among* ex-post efficient*,* strategy-proof*, and* non-bossy *rules, (i) all TTC rules Lorenz dominate all non-TTC rules and (ii) all priority rules are Lorenz dominated by all non-priority rules.*

Our Lorenz comparisons imply that the TTC rules are best and priority rules worst in this class according to the Rawlsian maximin criterion (Rawls, 1972). Proposition 5 shows that the priority rules lie at one extreme, maximally favoring some agents over others, and yielding the least egalitarian expected utility profiles. In contrast, because of the ex-ante symmetry among agents and objects, each TTC rule equally divides the utilitarian welfare, thereby achieving the

[37] To define agents' roles in rules other than priority or TTC rules would require a full description the "trading-and-braiding" rules (Pycia and Ünver, 2014; Bade, 2014b). As the conditions are somewhat elaborate and tangential to our analysis, we refer to Bade (2014c) for a formal description of the randomization process.

most egalitarian expected utility profile. Our Lorenz comparisons echo findings by Harless and Manjunath (2015) in a model of strategic learning.

4.1.2 Comparing social welfare across problems

While Proposition 4 provides a general utilitarian equivalence among rules, it provides little information by which to judge the welfare they achieve. We next investigate how common value compares to natural benchmarks. Here again our results are encouraging: In large economies, average welfare under each *ex-post efficient*, *strategy-proof*, and *non-bossy* rule approaches the maximum feasible average (Proposition 6). In other words, in large populations, application of one of these rules entails essentially no efficiency loss. Just as larger populations improve efficiency, we find that greater variability of private values has similar effect while greater variability of common values exerts a countervailing effect (Proposition 7). These results help us to predict when the efficiency cost of *strategy-proofness* and absence of money is likely to be mild or severe.

For comparison, we establish two benchmarks. An "ideal" allocation would assign to each agent the object he values most highly. Of course, when values are correlated, this is generally impossible. From a social perspective, the common values are always realized and so make the same contribution to welfare regardless of how objects are assigned. Instead, *utilitarian optimality* depends on private values, so maximizing welfare essentially means assigning to each agent an object for which he has a high private value. This sets an upper bound on the welfare gains we may achieve through optimal assignment, our optimistic benchmark. At the other extreme, we consider assigning objects without regard for preferences. With preference-independent allocation, expected utilitarian welfare is simply the sum of the common values. So that we may investigate welfare as the size of the population changes, we index each symmetric problem by the size of its population and write $\mu^{(n)}_{G,H}$. To evaluate rules, we measure the increase in expected utility per agent that each rule obtains over preference-independent allocation. Given our normalization $\mathbf{E}_G[x] = \mathbf{E}_H[x] = 0$, preference-independent allocation yields an expected utilitarian welfare of 0 for each agent. Thus, our measure is simply the average expected utility. Our optimistic benchmark, the **maximum average surplus** is $\nu^{(n,n)}$. In general, the maximum average surplus is infeasible, so the benchmark is more optimistic than striving for *utilitarian optimality.* Nevertheless, all of our rules approach this welfare.

Proposition 6. *For each sequence of symmetric problems, the average expected utility under each* ex-post efficient*,* strategy-proof*, and* non-bossy *rule approaches the maximum average surplus.*[38]

When values are uncorrelated, Proposition 6 follows directly from the formulas derived in Corollary 7. In this case, it is in fact possible to assign a vast majority of agents their most preferred

[38]In fact, the conclusion applies to each *ex-post efficient* rule.

objects in expectation. Somewhat more surprisingly, each of our rules also approaches the maximum when values include a common component. Our efficiency results suggest that we may appeal to other criteria when comparing *ex-post efficient*, *strategy-proof*, and *non-bossy* rules, at least in large economies.

While Proposition 6 is encouraging, we make the strong assumption that the distribution of common values is the same for all populations. In applications, this may be overly optimistic. For example, increasing the number of school seats may necessitate larger class sizes or hiring inexperienced or less qualified teachers. The opposite scenario in which the new objects are on average better than the original may also be relevant. For example, seats at a newly built school may be particularly desirable. More generally, moving from a small to large economy, we may expect the objects originally present to systematically differ from the new objects.

To formalize these ideas, we say that a sequence of distributions (H^k) is **deteriorating** if there are $\varepsilon, L \in \mathbb{R}_{++}$ such that for each $k \in \mathbb{N}$, $\text{supp}(H^k) \subseteq [-L - k\varepsilon, L - k\varepsilon]$. The sequence is **improving** if instead $\text{supp}(H^k) \subseteq [-L + k\varepsilon, L + k\varepsilon]$. Even allowing ε to be arbitrarily small and L to be arbitrarily large, we see that either condition is sufficient to undermine the conclusion of Proposition 6.

Remark 6. If the distribution of common values in a sequence of symmetric problems is improving or deteriorating, then the average expected utility under each *ex-post efficient*, *strategy-proof*, and *non-bossy* rule is bounded away from the maximum average surplus.

In addition to the effect of population size, it is important to understand the effects of different distributions of values within a given population. Just as some rules may stand out with large populations, others may be particularly well suited to situations with high or low variability in values. Generally speaking, greater variability among private values increases the utilitarian surplus. In contrast, since common values are realized regardless of the allocation, greater variability among common values does not change the utilitarian surplus. Common values nevertheless figure prominently under *strategy-proofness*. The effect of greater variability is to reduce the number of reversals in ordinal preferences, reducing the surplus realized by *strategy-proof* rules. To formalize these ideas, we compare scaled distributions. Given a cdf H and $\delta \in \mathbb{R}_+$, let $\delta \cdot H$ be the distribution such that for each $x \in \mathbb{R}$, $\delta \cdot H(x) = H(\delta x)$.

Proposition 7. *For each symmetric problem $\mu_{G,H}$ and $\delta \in \mathbb{R}_{++}$,*

1. *The utilitarian surplus at $\mu_{\delta \cdot G,H}$ is increasing in δ and the utilitarian surplus at $\mu_{G,\delta \cdot H}$ is constant in δ.*

2. *The fraction of the utilitarian surplus at $\mu_{G,\delta \cdot H}$ realized by each* ex-post efficient*,* strategy-proof*, and* non-bossy *rule is non-increasing in δ.*

When the common value component is strong relative to the private value component, agents rank objects more similarly and fewer mutually beneficial trades are possible beginning from a random assignment. This reduces the realized surplus. For a continuous distribution of common values, scaling reduces the probability of a preference reversal due to private values. This probability goes to zero in the limit, meaning that all agents' ordinal rankings coincide. The disappointing implication, summarized in Corollary 8, is that scaling drives the surplus to zero.

Corollary 8. *For each symmetric problem $\mu_{G,H}$ with continuous cdf H, $\lim_{\delta\to\infty} U^*(F^{\prec}, \mu_{G,\delta\cdot H}) = 0$.*

When the distribution is atomic, some surplus survives scaling. Intuitively, scaling the distribution creates distinct "bins" at the atoms of the distribution and limits trades among objects whose common values fall in the same bin. This suggests that it is not variance per se that decreases surplus. For example, consider cdfs G, H, and $\hat{H}$ such that $\text{supp}(G) \subseteq [-1, 1]$ and H and $\hat{H}$ place equal probabilities on $\{-3, 0, 3\}$ and $\{-4, 4\}$ respectively. Then $var(H) = 6 < 16var(\hat{H})$. However, since H creates more "bins" than does $\hat{H}$, $U(F^{\prec}, \mu_{G,H}) < U(F^{\prec}, \mu_{G,\hat{H}})$. In fact, for each $\delta \in \mathbb{R}_{++}$, $U(F^{\prec}, \mu_{G,H}) < U(F^{\prec}, \mu_{G,\delta\cdot\hat{H}})$, so surplus is higher at $\mu_{G,\delta\cdot\hat{H}}$ even as the variance of the distribution of common values becomes arbitrarily large.

As it is scaling on one distribution relative to the other that drives the result, another interpretation is possible. Suppose that values have the form $v_{ia} = \alpha v_a + (1-\alpha)\varepsilon_{ia}$ where $\alpha \in [0, 1]$. Scaling either G or H is then equivalent to varying α to place more or less weight on private values relative to common values. When α is close to 1, Corollary 8 shows that the cost of *strategy-proofness* is high: *Strategy-proof* rules achieve no higher welfare than preference-independent allocation.

4.1.3 Solidarity properties of rules

Related to our limit results, we compare the welfare of agents before and after augmenting a problem with new objects and agents. Depending on the environment and the allocation rule, the arrival of new agents and objects may have important welfare consequences for the original agents. For example, a school district expecting an increase in the student population may open a new school or an expanding firm may hire new workers and make available new shift assignments. An organization seeing to attract new members may make concessions to ensure the new members expect to be reasonably well off. On the other hand, support of the original members may require guarantees that recruitment will not make them worse off. Similarly, if new agents simply increase competition for the most desirable school seats or tasks, the original agents may be worse off. More generally, we may imagine merging problems as when a district combines assignment procedures for public and charter schools or two firms consolidate operations. Welfare comparisons before and after the merge for both groups are then relevant.

We restrict our formal analysis to comparisons among priority and TTC rules. In particular, we assume that agents and objects arrive in pairs and that the values of all objects follow the same distributions. Consistent with our previous results, agents expect to be better off in larger problems (Proposition 8). Priority rules with carefully aligned orders offer a stronger guarantee to some agents, ensuring that they are in fact at least as well of in the augmented problem as in the original problem for each realization of types. Less encouraging, TTC rules provide such a guarantee to at most one agent. On the other hand, under TTC rules, all agents are better off in expectation in the augmented problem whereas priority rules guarantee this only for agents with higher priority than all additional agents.

Proposition 8. *Let $N \equiv N_1 \cup N_2$ and $A \equiv A_1 \cup A_2$ be partitions of N and A such that $|N_1| = |A_1|$ and $|N_2| = |A_2|$. Also let $\mu_{G,H}$ be a symmetric problem for N, $\prec \in \Pi^N$, and $\omega \in X$ with $\{\omega_i : i \in N_1\} = A_1$ and $\{\omega_i : i \in N_2\} = A_2$. Then for each $i \in N_1$:*

(i) There is $u \in supp(\mu_{G,H})$ such that $u_i(F_i^{\prec}|_{N_1}(u|_{N_1})) < u_i(F_i^{\prec}(u))$ and $u_i(F_i^{\omega}|_{N_1}(u|_{N_1})) < u_i(F_i^{\omega}(u))$.

(ii) There is $u \in supp(\mu_{G,H})$ such that $u_i(F_i^{\prec}|_{N_1}(u|_{N_1})) > u_i(F_i^{\prec}(u))$ if and only if there is $j \in N_2$ such that $j \prec i$; and there is $u \in supp(\mu_{G,H})$ such that $u_i(F_i^{\omega}|_{N_1}(u|_{N_1})) > u_i(F_i^{\omega}(u))$ if and only if $|N_1| > 1$.

(iii) If for each $j \in N_2$, $i \prec j$, then $U_i(F^{\prec}, \mu) > U_i(F^{\prec}|_{N_1}, \mu|_{N_1})$; and $U_i(F^{\omega}, \mu) > U_i(F^{\omega}|_{N_1}, \mu|_{N_1})$.

The common value component makes the welfare comparisons for $F^{\prec}$ ambiguous. For an independent symmetric problem μ_G, we can strengthen Proposition 8(iii) to: $U_i(F^{\prec}, \mu) \geq U_i(F^{\prec}|_{N_1}, \mu|_{N_1})$ with strict inequality if there is $j \in N_2$ such that $i \prec j$. That is, in expectation, all agents are at least as well off in the augmented problem as in the original problem and better off if at least one of the additional agents has lower priority in the augmented problem.

Our results suggest an expanding organization faces a tension between providing guarantees to existing agents and to joining agents. If appeasing current members is most important, then priority rules stand out: Adding new members to the bottom of the priority order leaves all original agents at least as well off for each realization of types and better off in expectation. On the other hand, if the common value component is large, then the new agents may be worse off than under random assignment. In contrast, TTC rules facilitate recruitment: All agents are better off in expectation in the augmented problem. Of course, the risk is that some original agents may be disgruntled once types are realized. When these concerns are paramount, we may expect to see allocation by priorities, as with commonly observed procedures based on seniority.

4.2 Extensions of the symmetric domain

We adapt the framework of symmetric problems to incorporate additional real-world considerations and further compare priority and TTC rules. First, we consider outside options which may arrive in the interim. We next look more closely at the distribution of welfare under these rules. Finally, we compare randomization over either of the two families.

4.2.1 Outside options

In addition to the objects available for assignment, agents may have outside options against which they compare participation in the assignment procedure. For example, in addition to public schools, families may consider independent schools, religious schools, or even home schooling. Similarly, students applying for housing also consider off-campus options and employees expecting onerous assignments may quit.

To model this, we imagine that each agent also draws the value of an outside option. We allow the distribution of this value to differ from the distribution governing the objects available for assignment, a special case being a degenerate distribution representing an option common to all agents. We further imagine that outside options are fleeting: Outside options are available after the rule is announced, but must be chosen or discarded before agents learn their types. This timing allows us to compare the effects of outside options on different rules. While options that remain available after agents learn their types or that are available only before the rule is announced are also interesting, they affect rules similarly in expectation and so do not help us to distinguish among rules.

Formally, let H_0 be a cdf representing the distribution of outside option values and c_0 be drawn according to H_0. Given $\mu_{G,H}$ and $i \in N$, the **interim participation constraint** for F at μ of agent i with outside option c_0 is $U_i(F,\mu) \geq c_0$. Our first result is for independent symmetric problems.

Corollary 9. *Let μ_G be an independent symmetric problem, $\prec \in \Pi^N$, and $\omega \in X$. Let $k \in N$ be the agent with kth priority under $F^{\prec}$, endowment a_k under F^{ω}, and outside option $c_0 \in \mathbb{R}$. Then agent k's interim participation constrains for $F^{\prec}$ and F^{ω} are $\nu^{(n-k+1,n-k+1)} \geq c_0$ and $\frac{1}{n}\sum_{l=1}^{n} \nu^{(l,l)}$.*

Corollary 9 follows immediately from the formulas in Corollary 7. In particular, the threshold is the same for all agents under F^{ω}. Under $F^{\prec}$, the constraint becomes more stringent for lower priority agents. Example 5 illustrates.

Example 5. Interim participation constraint in an independent symmetric problem. Let $G, H_0 \sim Unif[-1,1]$ and consider the independent symmetric problem μ_G with outside options distributed according to H_0. For each $k \in \mathbb{N}$, $\nu^{(k,k)} = \frac{k}{k+1}$.

Let $k \in N$ be the agent with kth priority under $F^{\prec}$. In expectation, agent k participates with probability $\frac{n-k+1}{n-k+2}$. These constraints are most stringent for those agents with lowest priority. The lowest priority agent, for example, participates with probability $\frac{1}{2}$. Now consider F^{ω}. In expectation, all agents participate with the same probability. More precisely, each agent participates with probability $\frac{1}{n}\sum_{k=1}^{n}\frac{k}{k-1}$. The expected number of participants and dropouts is the same under either rule.

Suppose instead H_0 is degenerate at $\alpha \in \mathbb{R}$ representing a common outside option with known value. If $\alpha < 0$ so that the outside option is less desirable than the expected value of an object available for assignment, then all agents participate under either rule. Once $\alpha > 0$, the agent with lowest priority under $F^{\prec}$ drops out. In contrast, provided $\alpha < \sum_{k=1}^{n}\nu^{(k,k)}$, all agents continue to participate under F^{ω}. As the value of α increases, additional agents with low priority drop out under $F^{\prec}$. For α above the threshold $\sum_{k=1}^{n}\nu^{(k,k)}$, all agents drop out under F^{ω}. The highest priority agent under $F^{\prec}$ continues to participate until $\alpha > \nu^{(n,n)}$.

While rules are partly distinguished even in independent symmetric problems, the presence of common values has a much larger effect on priority rules. Intuitively, under F^{ω}, each agent expects to receive an object with the mean common value. Expected utility is lower with a strong common value component because the expected gains from trade are smaller, but the common value does not enter the computation directly. In contrast, under $F^{\prec}$, agents with high priority are more likely to receive objects with high common values and agents with low priority are more likely to receive objects with low common values. Thus, the common value directly enters the agents' expected utility calculations. Example 6 shows how these considerations may influence agents' decisions to participate.

Example 6. Interim participation constrain in a symmetric problem. Let $G \sim Unif[-1,1]$ and H_0 be degenerate at 0 representing a common outside option whose value is equal to the expected value of an object available for assignment. Even if the outside option has unknown value, this is the correct model if, like the objects available for assignment, its precise value will be learned only after it is no longer available. As in the previous example, all agents participate under either rule at μ_G.[39]

Now let $M \in \mathbb{R}_+$ and H be a binary distribution such that the values $\{-M, M\}$ are equally likely and consider $\mu_{G,H}$. Under F^{ω}, each agent expects an endowment with high or low common value with equal probability and expects gains from trade with positive probability. Therefore, for each $i \in N$, $U_i(F^{\omega}, \mu_{G,H}) > 0$ and all agents prefer to participate. Notably, this is independent of M.

Now consider $F^{\prec}$. For each $k \in N$ with $k > \frac{n}{2}$, the agent with kth highest priority expects to receive an object with common value $-M$ with probability greater than $\frac{1}{2}$. In particular, since the

[39]The lowest priority agent under $F^{\prec}$ is in fact indifferent, so we suppose that he participates as well.

lowest priority agent was indifferent between participating and dropping out at μ_G, he now prefers to drop out. Moreover, as M increases, more assignments are determined by common values so for each $k \in N$ with $k > \frac{n}{2}$, the probability of receiving an object with common value $-M$ is increasing in M. For sufficiently large M, all such agents drop out.

4.2.2 Further welfare comparisons

The differential effects we observe with an interim participation constraint reflect an important difference in how TTC and priority rules distribute expected utilities, particularly the chances of bad outcomes. On one hand, TTC rules provide all agents with the same expected utility; expected bad outcomes are distributed equally among all agents. In contrast, priority rules concentrate the expected bad outcomes on the agents with the lowest priority who receive almost no benefit from the presence of many objects.

When values are independent, this means that a majority of the population prefer a given priority rule to each TTC rule. However, when values include a large common component, the comparison is more nuanced. In fact, depending on the distribution of common values, it is possible to obtain a near consensus in favor of either a priority or a TTC rule. Moreover, it is possible only a bare majority prefer their expected assignments under a priority rule to preference-independent allocation. In contrast, all agents prefer a TTC rule to preference-independent allocation. Proposition 9 summarizes these observations.

Proposition 9. *For each priority rule and each TTC rule,*

(i) For each independent symmetric problem, a majority of agents obtain higher expected utility under the priority rule than under the TTC.

(ii) There exist symmetric problems at which an arbitrarily large or arbitrarily small proportion of the agents obtain higher expected utility under the priority rule than under the TTC rule.

(iii) For each symmetric problem, all agents obtain higher expected utility under the TTC rule than under preference-independent allocation.

(iv) There exist symmetric problems at which nearly one-half of the agents prefer preference-independent allocation to the priority rule.

According to Proposition 9, at each independent symmetric problem, at least one-half of agents prefer a given priority rule to every TTC rule. In fact, the fraction of agents who prefer the priority rule is typically much larger. The reason is simply that, given G, the sequence $(\nu_G^{(k,k)})$ of order statistics converges quickly. Their average, which corresponds to expected utility under a TTC rule, is therefore heavily weighted by the low values appearing early in the sequence. While this effect is present at each symmetric problem, it may be overwhelmed by the influence of common values.

This leads to the stark conclusion in (iv): Although the purpose of allocation rules is to improve welfare, nearly one half of agents expect no gains compared to preference-independent allocation. This situation is impossible under a TTC rule because of its equitable distribution of expected utility. These considerations may loom large when agents face interim participation constraints as considered in the previous section.

Related to these results is a fundamental difference in the information revealed by an agent's position in a priority or TTC rule. On one hand, an agent's position in a priority order provides information about his expected utility. Given symmetry, the agent's position provides no information about his eventual assignment. On the other hand, again due to symmetry, an agent's endowment provides no information about his expected utility. Even so, because of the structure of a TTC rule, the agent is more likely to keep his endowment than to receive any other individual object. A TTC rule therefore provides partial information about the *final allocation* but no information about the *final utility profile* whereas a priority rule provides partial information about the *final utility profile* but no information about the *final allocation*. These differences may be particularly salient when considering randomization over either priority orders or endowment profiles. To illustrate, we consider a typical independent symmetric problem.

Example 7. Rank distributions induced by TTC and priority rules in independent symmetric problems. Let μ_G be an independent symmetric problem. We compute the probabilities that an agent receives the object he ultimately ranks first, second, or third, as well as the probability that he receives each particular object. The tables provide these probabilities for two- and three-agent problems under $F^{\prec}$ and F^{ω} where $1 \prec 2 \prec 3$ and $\omega \equiv (a, b, c)$.

	$F^{\prec}$		F^{ω}	
	1	2	1	2
1st rank	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
2nd rank	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

	$F^{\prec}$		F^{ω}	
	1	2	1	2
a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$
b	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$

	$F^{\prec}$			F^{ω}		
	1	2	3	1	2	3
1st rank	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
2nd rank	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
3rd rank	0	0	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

	$F^{\prec}$			F^{ω}		
	1	2	3	1	2	3
a	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{22}{36}$	$\frac{7}{36}$	$\frac{7}{36}$
b	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{36}$	$\frac{22}{36}$	$\frac{7}{36}$
c	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{22}{36}$

These numerical results generalize. Corollary 10 provides general formulas for the probability that a given agent receives his r-th most preferred object under a priority rule or a TTC rule. To simplify notation, we extend the binomial coefficient by setting it equal to zero whenever undefined.[40]

[40]That is, for each pair $n, k \in \mathbb{N} \cup \{0\}$, let $\binom{n}{k} \equiv \frac{n!}{(n-k)!k!}$ if $k \leq n$ and $\binom{n}{k} \equiv 0$ if $k > n$.

Corollary 10. *Let G define an independent symmetric problem. For (i) Under each priority rule, the probability that the assignment of the agent with k-th highest priority receives his r-th most preferred object is $\binom{n-r}{k-r} \div \binom{n}{k-1}$. (ii) Under each TTC rule, the probability that an agent receives his rth most preferred object is $\frac{1}{n}\sum_{k=1}^{n}\binom{n-r}{k-r} \div \binom{n}{k-1}$.*

Interestingly, the probability that an agent receives his most preferred object under a TTC rule converges quickly to $\frac{1}{2}$. Explicitly,

$$\frac{1}{n}\sum_{k=1}^{n}\binom{n-1}{k-1} \div \binom{n}{k-1} = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n} = \frac{n+1}{2n}.$$

Then $\lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$.

Combined with Propositions 4 and 5, we are able to compare rank distributions across randomizations of TTC or priority rules, as well as other *ex-post efficient*, *strategy-proof*, and *non-bossy* rules. Each method of randomization induces a distinct two stage lottery. On one hand, there appears to be no difference as all methods of randomization achieve the same expected utilities. However, an inequality-averse society may not simply compare the expected utilities under the induced two-stage lotteries. Formally, inequality-aversion means applying a concave function to aggregate interim utilities.[41] Such a society will have strong preference for randomizing over endowments rather than priority orders.

Corollary 11. *Let F be an* ex-post efficient, strategy-proof, *and* non-bossy *rule and $\bar{F}$ be rule defined by uniformly randomizing over the roles of the agents in F. For each symmetric problem,*

(i) The utilitarian welfare and same expected rank distribution under $\bar{F}$ are the same as randomization over the priority rules.

(ii) All agents obtain the same expected utility under $\bar{F}$.

(iii) If F is not a priority rule, then according the each concave interim social aggregator, $\bar{F}$ achieves higher social welfare than each priority rule.

(iv) If F is not a TTC rule, then according the each concave interim social aggregator, $\bar{F}$ achieves lower social welfare than each TTC rule.

The comparisons in Corollary 11(ii) and (iii) derive from the differences in how the various methods of randomization reveal information. Just as interim participation constraints lead us to prefer randomization over endowments, so too does interim inequality aversion.

[41]Another interpretation is possible from the perspective of an agent in terms of a preference for the timing of the resolution of uncertainty. For example, agents who prefer "one shot" resolution will favor TTC (Dillenberger, 2010). See also Ergin and Gul (2009); Ergin and Saver (2015), and Grant et al. (2009).

4.3 Heterogeneous problems

We are motivated by fairness considerations to choose rules before preferences are learned. This leads naturally to the symmetric domain. Incomplete information remains relevant, however, even with partial information about preferences. We next consider the extent to which our welfare comparisons apply when agents and objects are no longer symmetric.

4.3.1 Correlated preferences

For insight into environments with correlated preferences, we begin with facts about ex-post utilities under TTC and priority rules.

Proposition 10. *Let $\mu \in \bar{D}$ be a problem with at least four agents whose support on a scoring domain. For each pair of priority and TTC rules, there exist profiles at which either rule utilitarian and maximin dominates the other rule.*

Proposition 10 applies to each symmetric problem. More generally, ex-post comparison shows that for problems with little uncertainty about preferences, utilitarian and maximin criteria may favor rule in either class.

On the symmetric domain, no pair of priority rules or TTC rules can be Pareto ranked. We have already seen that priority rules may ex-ante Pareto dominate TTC rules (Example 2). In fact, one TTC rule may dominate another, even with as few as two agents. This is notable because priority rules may not ex-ante Pareto dominate TTC rules with only two agents.

Proposition 11. *(i) With two or more agents, there exist problems at which one TTC rule ex-ante Pareto dominates another TTC rule.*
(ii) With three or more agents, there exist problems at which a priority rule ex-ante Pareto dominates the rules randomizing over priority rules or TTC rules.

The disappointing conclusions of Proposition 11 do not apply to priority rules, as they are always *ex-ante efficient*. The examples verifying Proposition 11 depend on particular correlations among preferences. On the other hand, correlation may also be advantageous. With recourse to *bossy* rules, we sometimes achieve improvements in social welfare which are impossible among rules satisfying the common invariance requirement.

Proposition 12. *With three or more agents, there exist problems at which a* strategy-proof *and* bossy *rule:*

(i) Achieves higher utilitarian welfare than all priority and TTC rules.

(ii) Lorenz dominates all priority and TTC rules.

(iii) ex-ante Pareto dominates the rules randomizing over priority rules or TTC rules.

Proposition 12 shows that focus on *non-bossy* rules, as is common, may forego desirable flexibility.[42] In the example considered in the proof, we construct a rule which takes advantage of negative correlation among agents' values. As the result suggests, welfare can sometimes be improved by carefully tailoring a rule to account for correlations among preferences. Of course, it is crucial that the direction of the correlation be known; if the structure of the problem is misspecified, then the same modifications may yield the opposite conclusions. Although our general results are somewhat disappointing, they further underscore the relevance of the ex-ante perspective.

4.3.2 Risky versus safe endowments

Agents in our model are risk-neutral expected utility maximizers. Nevertheless, with incomplete information, a kind of endogenous risk preference may emerge. To illustrate, we consider a simple problem with two agents and a TTC rule. We ask whether an agent would prefer to be endowed with a "risky" or "safe" object in the sense of having a more or less variable value. To do so, we minimally modify an independent symmetric problem to allow the distributions to private values to depend on the objects.

For the remaining discussion, let $N \equiv \{1, 2\}$, $A \equiv \{a, b\}$, and $\omega \equiv (a, b)$. Now consider two independent symmetric problems μ_{G_a} and μ_{G_b}. The **extended independent symmetric problem** with respect to G_a and G_b, μ_{G_a,G_b}, is a modified symmetric problem in which for each $i \in N$, $\varepsilon_{ia} \sim G_a$ and $\varepsilon_{ib} \sim G_b$ are independent random variables.

To decide which object is more desirable as an endowment, we compare the agent's expected utilities under F^ω. For each μ_{G_a,G_b}, let $p_a \equiv \mu\big(\{u \in \mathbb{R}^{A\times N} : u_1(a) > u_1(b)\}\big)$, $\alpha^H \equiv \mathbf{E}_{\mu_{G_a,G_b}}[\varepsilon_{1a}|\varepsilon_{1a} > \varepsilon_{1b}]$, $\alpha^L \equiv \mathbf{E}_{\mu_{G_a,G_b}}[\varepsilon_{1a}|\varepsilon_{1a} < \varepsilon_{1b}]$, $\beta^H \equiv \mathbf{E}_{\mu_{G_a,G_b}}[\varepsilon_{1b}|\varepsilon_{1a} < \varepsilon_{1b}]$, $\beta^L \equiv \mathbf{E}_{\mu_{G_a,G_b}}[\varepsilon_{1b}|\varepsilon_{1a} > \varepsilon_{1b}]$. By independence, these values also apply to agent 2. Moreover, because F^ω is ordinal, the agents' expected utilities depend only on these values. We are now able to compare expected utilities.

Lemma 3. *For each extended independent symmetric problem μ_{G_a,G_b},*

$$U_1(F^\omega, \mu_{G_a,G_b}) = p_a\alpha^H + p_a(1-p_a)\beta^H + (1-p_a)^2\alpha^L, \text{ and}$$
$$U_2(F^\omega, \mu_{G_a,G_b}) = (1-p_a)\beta^H + p_a(1-p_a)\alpha^H + (p_a)^2\beta^L.$$

Moreover, $U_1(F^\omega, \mu_{G_a,G_b}) \geq U_2(F^\omega, \mu_{G_a,G_b})$ if and only if $p_a \geq \frac{1}{2}$.

According to Lemma 3, the object more likely to be preferred ex-post is more desirable as an endowment ex-ante. Given that the expected values of objects are the same, the difference in distributions affects agents' expected utilities only through this probability. We apply Lemma 3 to compare distributions in terms of riskiness.

[42] Although we state the result for TTC and priority rules because they are our benchmarks, the conclusion is more general, as the example in the proof makes essential use of correlation that can only be identified by cardinal information.

Example 8. Comparing risky and safe endowments. First, to represent a "safe" object with known value, let G_0 be degenerate at 0. To represent a risky object, let $\gamma_L, \gamma_H \in \mathbb{R}_{++}$ and G_1 be a binary distribution which takes values $-\gamma_L$ and γ_H with probabilities $p \equiv \frac{\gamma_H}{\gamma_L+\gamma_H}$ and $1-p = \frac{\gamma_L}{\gamma_L+\gamma_H}$ respectively. Then $\mathbf{E}_{G_1}[x] = 0$, so μ_{G_0,G_1} is an extended independent symmetric problem with $a^L = a^H = 0$, $b^L = -\gamma_L$, $b^H = \gamma_H$, and $p_a = 1-p$. Comparing endowments, the safe object is more desirable whenever agent 1 obtains higher expected utility. This is the case whenever $p_a = \frac{\gamma_L}{\gamma_L+\gamma_H} > \frac{1}{2}$, or equivalently, $\gamma_H > \gamma_L$.

Endogenous risk preference. TTC induces an endogenous risk preference. To interpret the conditions comparing expected utilities in terms of riskiness, first suppose that $\gamma_L = \gamma_H$ so $p_a = \frac{1}{2}$. In this case, G_1 is symmetric and the objects are equally desirable. This accords with risk neutrality. Next suppose that $\gamma_L > \gamma_H$ so $p_a > \frac{1}{2}$. The safe object is more desirable in this case. On the other hand, consider the final case $\gamma_L < \gamma_H$ so $p_a < \frac{1}{2}$. Now the risky object is more desirable as an endowment. Intuitively, when $\gamma_L > \gamma_H$, the risky object represents a lottery that usually loses, but sometimes pays off with a large reward. On the other hand, when $\gamma_L < \gamma_H$, the risky object represents a lottery that usually wins, but sometimes leads to a very low value. Interestingly, a single risk averse agent would have precisely the opposite preference between these lotteries. This shows how the potential to trade provides insurance for the agent endowed with the risky object, something particularly valuable when there is a small probability of a very low value.

Social welfare and risk. We can also compare the effect of riskiness on social welfare. In μ_{G_0,G_1}, the agents' utilities simplify to

$$U_1(F^\omega, \mu_{G_a,G_b}) = \frac{\gamma_L(\gamma_H)^2}{(\gamma_L+\gamma_H)^2} \text{ and } \quad U_2(F^\omega, \mu_{G_a,G_b}) = \frac{\gamma_L(\gamma_H)^2 + (\gamma_H)^3 - (\gamma_L)^3}{(\gamma_L+\gamma_H)^2}.$$

Inspecting the formulas, agent 1's expected utility is increasing in both γ_L and γ_H while agent 2's expected utility is increasing in γ_L and decreasing in γ_H. Social welfare at μ_{G_0,G_1} depends on the variability of the values of b, the source of all gains from trade, and is also increasing in both γ_L and γ_H.

We consider binary distributions in Example 8 because are easy to interpret in terms of riskiness and allow us to directly control the relevant parameters. Of course, through Lemma 3, the same conclusions apply to other distributions corresponding to the same values of b^L, b^H, and p_a.

5 Conclusion

We extended the object assignment problem to incorporate incomplete information about preferences. Comparing rules at the ex-ante stage, we identified the priority rules as the only *strategy-proof* or *Bayesian incentive compatible* rules which meet our strengthened notion of efficiency in most problems in which incomplete information is meaningful. Even more, we saw that priority rules may ex-ante Pareto dominate TTC rules. For robust efficiency when even the underlying distribution of preferences is uncertain, then, leaves essentially no alternatives to priority rules. The narrowness of our results in comparison with those in the complete information model underscore the importance and relevance of the ex-ante perspective.

The ex-ante perspective also allowed us to compare rules from *behind the veil* when agents and objects as ex-ante symmetric. In this environment, we found that a large family are never ex-ante Pareto dominated by the priority rules and in fact achieve the same utilitarian welfare as priority rules. Moreover, all of these rules approach the average welfare of an utilitarian-optimal allocation in large economies, muting concerns about efficiency. Turning to equity criteria, we identified TTC rules as the Lorenz dominant members of this family. As a benchmark, all agents prefer TTC to allocation without regard for preferences, but this is not true for priority rules. Positing an interim outside option, we found that TTC rules generally induce greater participation than priority rules. Allowing for randomization over priority orders or endowments, our results continued to favor the endowments approach, at least whenever interim equality is desirable. On equity grounds, then, our findings lend strong support for choosing top trading cycles rules over alternatives, even in comparison to randomization over priority orders.

Our framework leaves open many avenues for future research. We adopted the perspective of a social planner or outside observer, omitting the process by which a rule is selected. An important extension, then will formalize and compare social choice procedures at the ex-ante stage. Further recognizing that the true "original position" is largely hypothetical, a related extension would consider choosing a rule with partial knowledge of preferences. Capturing many informational settings observed in practice, this would extend our analysis of correlated environments. where agents have some information about their preferences face legitimate uncertainty about their preferences, rules are rarely chosen before agents learned any information about their eventual preferences. Allowing information to arrive in stages, perhaps with interim opportunities to revisit the choice of rule, would also yield insights, perhaps suggesting conditions under which it is optimal to announce and commit to a rule ex-ante or to delay choice until agents learn their preferences. Most ambitiously, the task of endogenizing the learning process remains almost entirely open, very recent early work testifying to both the difficulty and import of the general problem.

A Proofs from Section 3

We begin with additional notation to describe Bayesian incentives. For each $\mu \in \bar{\mathcal{D}}$, each $i \in N$, each $a \in A$, and each $u_0 \in \mathbb{R}^A$, let

$$x_i(u_0, a) \equiv \mu_{-i}\left(\{\hat{u} \in \mathcal{U}^N : F_i(u_0, u_{-i}) = a\}\right).$$

Then $x_i(u_0, a)$ is the probability with which agent i receives a under F when reporting u_0 at μ. Also let $x_i(u_0) \equiv \big(x_i(u_0, a)\big)_{a \in A}$, agent i expected assignment in this case.

A.1 Efficiency

Lemma 1 *For each problem, each extended priority rule is* ex-ante efficient *and* strategy-proof*.*

Proof. Let $\prec \in \Pi$ and label the agents so that $1 \prec 2 \prec \cdots \prec n$. Suppose that F is such that for each $i \in N$, $U_i(F) \geq U_i(F^{\prec})$. We show that $F = F^{\prec}$. First consider agent 1. By definition, $F^{\prec}$ maximizes agent 1's expected utility. Now $U_1(F) \geq U_1(F^{\prec})$ implies that for each $u \in \mathbb{R}^{A \times N}$, $u_1(F_1(u)) = u_1(F_1^{\prec}(u))$. If $u_1 \in \mathcal{U}$, then agent 1's assignment is unique. Now consider agent 2. Again by definition, $F^{\prec}$ maximizes agent 2's expected utility conditional on agent 1's utility guarantee. Therefore, $U_2(F) \geq U_2(F^{\prec})$ implies that for each $u \in \mathbb{R}^{A \times N}$, $u_2(F_2(u)) = u_2(F_2^{\prec}(u))$. Again, if $u_2 \in \mathcal{U}$, then agent 2's assignment is unique. Considering each remaining agent in turn, for each $u \in \mathbb{R}^{A \times N}$ and each $i \in N$, $u_i(F(u)) = u_i(F^{\prec}(u))$. Therefore, F is itself an extended priority rule. □

Lemma 2 *Let $\mu \in \bar{\mathcal{D}}$ and $F \in \mathcal{F}^*$ be* ex-ante efficient *at μ. For each pair $i, j \in N$ and each $\delta \in \mathbb{R}_+$, if*

$$\mu\left(\left\{u \in \mathbb{R}^{A \times N} : \begin{array}{c} u_i(F_i(u)) > u_i(F_j(u)),\, u_j(F_i(u)) > u_j(F_j(u)), \text{ and} \\ u_i(F_i(u)) - u_i(F_j(u)) \leq u_j(F_i(u)) - u_j(F_j(u)) + \delta \end{array}\right\}\right) > 0$$

then for each $\delta' \in \mathbb{R}$ such that $\delta' < \delta$,

$$\mu\left(\left\{u \in \mathbb{R}^{A \times N} : \begin{array}{c} u_i(F_i(u)) < u_i(F_j(u)),\, u_j(F_i(u)) < u_j(F_j(u)), \text{ and} \\ u_i(F_j(u)) - u_i(F_i(u)) \geq u_j(F_j(u)) - u_j(F_i(u)) + \delta' \end{array}\right\}\right) = 0.$$

Proof. Let $\mu \in \bar{\mathcal{D}}$, $F \in \mathcal{F}^*$ be *ex-ante efficient* at μ, $i, j \in N$, $\delta \in \mathbb{R}_+$ and $\delta' \in \mathbb{R}$ with $\delta' < \delta$,

$$V \equiv \left\{u \in \mathbb{R}^{A \times N} : \begin{array}{c} u_i(F_i(u)) > u_i(F_j(u)),\, u_j(F_i(u)) > u_j(F_j(u)), \text{ and} \\ u_i(F_i(u)) - u_i(F_j(u)) \leq u_j(F_i(u)) - u_j(F_j(u)) + \delta \end{array}\right\}, \text{ and}$$

$$V' \equiv \left\{u \in \mathbb{R}^{A \times N} : \begin{array}{c} u_i(F_i(u)) < u_i(F_j(u)),\, u_j(F_i(u)) < u_j(F_j(u)), \text{ and} \\ u_i(F_j(u)) - u_i(F_i(u)) \geq u_j(F_j(u)) - u_j(F_i(u)) + \delta' \end{array}\right\}.$$

Suppose by way of contradiction that $\mu(V) > 0$ and $\mu(V') > 0$. Let $p_1 \equiv \mu(V)$, $p_2 \equiv \mu(V')$, and for each $k \in \{i, j\}$,

$$\varepsilon_k \equiv \mathbf{E}_\mu\big[u_k(F_i(u)) - u_k(F_j(u)) \big| u \in V\big] \text{ and}$$
$$\varepsilon'_k \equiv \mathbf{E}_\mu\big[u_k(F_j(u)) - u_k(F_i(u)) \big| u \in V'\big].$$

Then $\varepsilon_i, \varepsilon'_i, \varepsilon_j, \varepsilon'_j > 0$, $\varepsilon_i \leq \varepsilon_j + \delta$, and $0\varepsilon'_j \leq \varepsilon'_i - \delta'$. Also,

$$\frac{\varepsilon'_j}{\varepsilon_j} \leq \frac{\varepsilon'_i - \delta'}{\varepsilon_i - \delta} < \frac{\varepsilon'_i - \delta}{\varepsilon_i - \delta} \leq \frac{\varepsilon'_i}{\varepsilon_i}$$

so $\varepsilon'_j \varepsilon_i < \varepsilon'_i \varepsilon_j$.

We now construct an ex-ante Pareto improvement. Let $\varepsilon_0 \in \mathbb{R}_{++}$ be such that $\varepsilon_0 < \min\left\{\frac{1}{\varepsilon_i}, \frac{1}{\varepsilon'_i}\right\}$, $q_1 \equiv p_2 \varepsilon_0 \varepsilon'_i$, and $q_2 \equiv p_1 \varepsilon_0 \varepsilon_i$. Now for each $u \in \mathbb{R}^{A \times N}$, let

$$F'_{ij}(u) \equiv \begin{cases} (F_j(u), F_i(u)) & \text{with probability } q_1 \text{ if } u \in V \\ (F_j(u), F_i(u)) & \text{with probability } q_2 \text{ if } u \in V' \\ (F_i(u), F_j(u)) & \text{otherwise} \end{cases}$$

and $F'_{-ij}(u) = F_{-ij}(u)$. That is, F' modifies F by reversing the assignments of agents i and j on $V \cup V'$ with some probability. Comparing expected utilities,

$$U_i(F', \mu) - U_i(F, \mu) = p_1 \cdot q_1 \cdot (-\varepsilon_i) + p_2 \cdot q_2 \cdot (\varepsilon'_i) = -p_1 p_2 \varepsilon_0 \varepsilon'_i \varepsilon_i + p_2 p_1 \varepsilon_0 \varepsilon_i \varepsilon'_i = 0$$
$$U_j(F', \mu) - U_j(F, \mu) = p_1 \cdot q_1 \cdot (\varepsilon_j) + p_2 \cdot q_2 \cdot (-\varepsilon'_j) = p_1 p_2 \varepsilon_0 \varepsilon'_i \varepsilon_j - p_2 p_1 \varepsilon_0 \varepsilon_i \varepsilon'_j = p_1 p_2 \varepsilon_0 (\varepsilon'_i \varepsilon_j - \varepsilon_i \varepsilon'_j) > 0$$

For each $k \in N \backslash \{i, j\}$, $U_k(F', \mu) = U_k(F, \mu)$, so F' is an ex-ante Pareto improvement over F and F is not *ex-ante efficient* at μ. □

Example 9. There exist finite problems at which a TTC rule is ex-ante efficient* and not ex-ante efficient. We illustrate with two agents. Let $N \equiv \{1, 2\}$, $A \equiv \{a, b\}$, $\omega \equiv (a, b)$, and $V \equiv \{u^1, u^2, u^3, u^4\}$ be as specified in the table.

	u^1	u^2	u^3	u^4
a	20	0	20	0
b	1	1	3	3

Let $\mu \in \mathcal{D}(V)$ be the uniform measure so that for each $u \in V^N$, $\mu(u) = \frac{1}{4^2} = \frac{1}{16}$. Then μ is rich. The expected value of a is 10 whereas the expected value of b is 2, so a is ex-ante more desirable than b. In contrast, the objects are equally likely to be more desirable ex-post.

Because μ is rich, F^ω is not *ex-ante efficient* at μ. Specifically, there is room to improve utilitarian welfare by reversing allocations at (u^3, u^1) and (u^4, u^2). In these cases, $F^\omega(u^3, u^1) = (a, b)$ and $F^\omega(u^4, u^2) = (a, b)$, which yield utilities $(20, 1)$ and $(0, 1)$ respectively. Reversing the these allocations yields utilities $(3, 20)$ and $(3, 0)$. An ex-ante Pareto improvement is possible by, for example, modifying F^ω to allocate (a, b) at (u^3, u^1) with probability $\frac{1}{6}$ and allocate (a, b) at (u^4, u^2)

with probability 1. Since $\frac{1}{6}(3-20)+1\cdot(3-0)=\frac{1}{6}>0$ and $\frac{1}{6}(20-1)+1\cdot(0-1)=\frac{13}{6}>0$, this makes both agents better off.

We argue that F^{ω} is *ex-ante efficient** at μ. First, to make agent 2 better off, we must allocate (b,a) at some profile in which both agents prefer a. We do so at the lowest cost to agent 1 by reversing the allocation at (u^3,u^1). Now the maximum compensation to agent 1 relative to F^{ω} is to assign b to agent 1 whenever he prefers b to a. There are four such profiles: (u^2,u^2), (u^2,u^4), (u^4,u^2), and (u^4,u^4). Since all states are equally like, it suffices to compare utility sums. Relative to F^{ω}, reversing all of these allocations increases agent 1's expected utility by $\frac{1}{16}(1-0)+\frac{1}{16}(1-0)+\frac{1}{16}(3-0)+\frac{1}{16}(3-0)=\frac{1}{2}$. However, the change at (u^3,u^1) decreases agent 1's utility by $\frac{1}{16}(20-3)=\frac{17}{16}>\frac{1}{2}$, so agent 1 is worse off. Thus, F^{ab} is *ex-ante efficient** at μ.

A.2 Two-agent problems

Proposition 1 *For each rich two-agent problem $\mu\in\mathcal{D}^*$, a rule is* BIC *and* ex-ante efficient *if and only if it is a priority rule.*

Proof. Let $\mu\in\mathcal{D}^*$ be a rich problem with rich set $V\subseteq\mathcal{U}$ in its common support and let F be *BIC* and *ex-ante efficient* at μ. To minimize notation, suppose further that F is deterministic. If in fact F randomizes at the profiles considered, the same arguments apply after adjusting the probabilities chosen when constructing ex-ante Pareto improvements.

Relabeling the objects if necessary, since V is rich, there are $u_0,u_0'\in V$ such that $u_0(a)-u_0(b)>u_0'(a)-u_0'(b)>0$. Label the agents so that $F(u_0,u_0)=(a,b)$ and let $\delta\equiv u_0(a)-u_0(b)$, and $\delta'\equiv u_0'(a)-u_0'(b)$ so $\delta>\delta'>0$.

Step 1: Agent 1 has priority for *a*. We first argue that $F_1(u_0',u_0)=(a,b)$. Suppose by way of contradiction that $F(u_0',u_0)=(b,a)$. By *BIC*, there is $u_0''\in\mathcal{U}$ with $\mu(u_0,u_0'')>0$ such that $F(u_0,u_0'')=(b,a)$ and $F(u_0',u_0'')=(a,b)$. Then by *ex-ante efficiency*, $u_0''(a)>u_0''(b)$. Let $p_1\equiv\mu(u_0,u_0'')$, $p_2\equiv\mu(u_0',u_0'')$, and $\delta''\equiv u_0''(a)-u_0''(b)$.

We construct an ex-ante Pareto improvement. For each $u\in\mathcal{U}^N$, define F' by

$$F'(u)\equiv\begin{cases}(a,b) & \text{with probability } p_2 \text{ if } u=(u_0,u_0'')\\ (b,a) & \text{with probability } p_1 \text{ if } u=(u_0',u_0'')\\ F(u) & \text{otherwise}\end{cases}.$$

Comparing the agents' expected utilities,

$$U_1(F',\mu)-U_1(F,\mu)=p_1\cdot p_2\cdot(\delta)+p_2\cdot p_1\cdot(-\delta')>0 \text{ and}$$
$$U_2(F',\mu)-U_2(F,\mu)=p_1\cdot p_2\cdot(-\delta'')+p_2\cdot p_1\cdot(\delta'')=0.$$

Then F' ex-ante Pareto dominates F at μ. Instead, $F_1(u_0', u_0) = (a, b)$.

Finally, let $u \in \mathcal{U}^N$ be such that $u_1(a) > u_1(b)$. Then u_1 is non-collinear with at least one of u_0 and u_0', so by repeating the previous arguments, $F(u) = (a, b)$.

Step 2: Agent 1 has priority for *b*. Let $\hat{u} \in V$ be such that $\hat{u}_1(b) > \hat{u}_1(a)$ and suppose by way of contradiction that $F(\hat{u}) = (a, b)$. By *ex-ante efficiency*, $\hat{u}_2(b) > \hat{u}_2(a)$. If $\hat{u}_1$ and $\hat{u}_2$ are non-collinear, then by repeating the arguments in Step 1 for agent 2, $F(\hat{u}_1, \hat{u}_1) = (b, a)$. Therefore, passing to this profile if necessary, we may assume that $\hat{u}_1$ and $\hat{u}_2$ are collinear. Let $\hat{p}_1 \equiv \mu(u_0', u_0)$, $\hat{p}_2 \equiv \mu(\hat{u}_1, \hat{u}_2)$, and $\hat{\delta} \equiv \hat{u}_1(b) - \hat{u}_1(a) = \hat{u}_2(b) - \hat{u}_2(a)$.

We construct an ex-ante Pareto improvement. Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < \min\left\{\frac{1}{\delta'}, \frac{1}{\hat{\delta}}\right\}$ and for each $u \in \mathcal{U}^N$, define $\hat{F}$ by

$$\hat{F}(\hat{u}) \equiv \begin{cases} (b, a) & \text{with probability } \frac{\hat{p}_2 \varepsilon}{\delta'} \text{ if } u = (u_0', u_0) \\ (b, a) & \text{with probability } \frac{\hat{p}_1 \varepsilon}{\hat{\delta}} \text{ if } u = (\hat{u}_1, \hat{u}_2) \\ F(u) & \text{otherwise} \end{cases} \cdot$$

Comparing the agents' expected utilities,

$$U_1(\hat{F}, \mu) - U_1(F, \mu) = \hat{p}_1 \cdot \tfrac{\hat{p}_2 \varepsilon}{\delta'} \cdot (-\delta') + \hat{p}_2 \cdot \tfrac{\hat{p}_1 \varepsilon}{\hat{\delta}} \cdot (\hat{\delta}) = 0 \text{ and}$$
$$U_2(\hat{F}, \mu) - U_2(F, \mu) = \hat{p}_1 \cdot \tfrac{\hat{p}_2 \varepsilon}{\delta'} \cdot (\delta) + \hat{p}_2 \cdot \tfrac{\hat{p}_1 \varepsilon}{\hat{\delta}} \cdot (-\hat{\delta}) > 0 \text{ and}$$

Then $\hat{F}$ ex-ante Pareto dominates F at μ. Therefore, $F(\hat{u}) = (b, a)$. Combined with the conclusion from Step 1, F is a priority rule. □

A.3 Problems with arbitrary populations

We prove the components of Theorem 1 separately.

Theorem 1(i) *For each rich strict problem, a rule is* ex-ante efficient *and* strategy-proof *if and only if it is a priority rule.*

Proof. Let $\mu \in \mathcal{D}$ be rich with rich set $V \equiv \bigcup_{k=1}^{n+2} V^k \subseteq \mathbb{R}^A$ and let $a \in A$ be the object such that $V^{n+1} \subseteq V^a$. Let F be *ex-ante efficient* and *strategy-proof* at μ.

Case 1: The problem places positive probability on each profile formed by a finite rich set. This case includes, for example, all rich problems in $\mathcal{D}^*$. Suppose that there is a set $\hat{V} = \{u^1, \ldots, u^{n+2}\}$ such that for each $k = 1, \ldots, n+2$, $u^k \in V^k$, and for each $l \in \{1, \ldots, n+2\}^N$, $\mu(\prod_N u^{l_i}) > 0$. For notational convenience, label these types so that for each $b \in A$, if $b = a_k$, then

$u^b \equiv u^k$. Also label $u^a \equiv u^{n+1}$ and $\hat{u}^a \equiv u^{n+2}$.[43] In particular, u^a represents less intense preference for a than does $\hat{u}^a$. We argue that F is a priority rule.

Step 1: There is an agent who always receives his most preferred object. Consider the profile $(\hat{u}^a, \hat{u}^a, \ldots, \hat{u}^a)$. Label as 1 the agent who receives a so that $F_1(\hat{u}^a, \hat{u}^a, \ldots, \hat{u}^a) = a$. By *strategy-proofness*, $F_1(u^a, \hat{u}^a, \ldots, \hat{u}^a) = a$. Then by Lemma 2, agent 1 never prefers another agent's assignment to his own when both report the same type. We claim that agent 1 always receives his most preferred object.

Suppose by way of contradiction that there is $u' \in \mathcal{U}^N$ such that $b \equiv \text{argmax}_{c \in A} u'_1(c)$ and $F_1(u') \neq b$. By *strategy-proofness*, $F_1(u^b, u'_{-1}) \neq b$. Let $i \in N$ be such that $F_i(u^b, u'_{-1}) = b$. By *strategy-proofness*, $F_i(u^b, u^b, u'_{-1i}) = b$. But agent 1 prefers agent i's assignment when both report the same type. This contradicts Lemma 2. Instead, $F_1(u') = b$.

Step 2: There is an agent who always receives his most preferred object among those not assigned to agent 1. Let $b \in A \backslash \{a\}$ and consider the profile $(u^b, \hat{u}^a, \ldots, \hat{u}^a)$. By Step 1, $F_1(u^b, \hat{u}^a, \ldots, \hat{u}^a) = b$. Label as 2 the agents who receives a so that $F_2(u^b, \hat{u}^a, \ldots, \hat{u}^a) = a$. By *strategy-proofness*, $F_2(u^b, u^a, \ldots, \hat{u}^a) = a$. Then by Lemma 2, agent 2 never prefers the assignment of an agent among $N \backslash \{1, 2\}$ to his own when both report the same type. We claim that, conditional on agent 1's report, agent 2 always receives his most preferred object among those not assigned to agent 1.

Let $u_1 \in \mathcal{U}$, $b \equiv \text{argmax}_{c \in A} u_1(c)$, and suppose by way of contradiction that there is $u' \in \mathcal{U}^N$ such that $c \equiv \text{argmax}_{\tilde{c} \in a \backslash \{b\}} u_2(\tilde{c})$, and $F_2(u') \neq c$. By *strategy-proofness*, $F_1(u^c, u'_{-2}) \neq c$. Let $i \in N$ be such that $F_i(u^c, u'_{-2}) = b$. By Step 1, $F_1(u') = F_1(u^c, u'_{-2}) = b$, so $i \neq 1$. By *strategy-proofness*, $F_i(u^c, u^c, u'_{-2i}) = c$. But agent 2 prefers agent i's assignment when both report the same type. This contradicts Lemma 2. Instead, $F_2(u') = c$.

Step 3: F is a priority rule. We calibrate the remainder of the priority order in similar fashion. Let $u_1, u_2 \in \mathcal{U}$, $b_1 \equiv \text{argmax}_{c \in A} u_1(c)$, $b_2 \equiv \text{argmax}_{c \in A \backslash \{b_1\}} u_2(c)$. Consider the profile $(u^{b_1}, u^{b_2}, \hat{u}^a, \ldots, \hat{u}^a)$ and label as 3 the agents who receives a so that $F_3(u^{b_1}, u^{b_2}, \hat{u}^a, \ldots, \hat{u}^a) = a$. By Steps 1 and 2, $F_1(u^{b_1}, u^{b_2}, \hat{u}^a, \ldots, \hat{u}^a) = b_1$ and $F_2(u^{b_1}, u^{b_2}, \hat{u}^a, \ldots, \hat{u}^a) = b_2$ so $3 \in N \backslash \{1, 2\}$. Repeating the argument in Step 2, agent 3 always receives his most preferred object among those those assigned to agents 1 and 2. Continuing in this fashion, for each $k \in \{1, \ldots, n\}$, there is an agent, called agent k, who always receives his most preferred object among those not assigned to one of the agents $\{1, \ldots, k-1\}$. Let $\prec \in \Pi^N$ be such that $1 \prec 2 \cdots \prec n$. Then $F = F^{\prec}$ and F is a priority rule.

Case 2: The problem is general. We show how to adapt the arguments from Case 1. Without loss of generality, label the $V^1, \ldots, V^n$ so that $V^n \subseteq V^a$.

[43] Although we have now defied u^a twice, it is without loss of generality to suppose that the two types with this label are the same.

First, let $N_1 \subseteq N$ be the set of agents assigned a with positive probability at profiles among $\prod_N V^{n+2}$. Let $1 \in N_1$ and $\bar{V}^1 \equiv \{u \in \prod_N V^{n+2} : F_1(u) = a\}$. By *strategy-proofness*, for each $\bar{u} \in \bar{V}^1$ and each $u_1 \in V^{n+1} \cup V^{n+2}$, $F_1(u_1, \bar{u}_{-1}) = a$. By richness, $\mu(V^{n+1} \times \bar{V}^1_{-1}) > 0$. Now at each profile among $(V^{n+1} \times \bar{V}^1_{-1})$, agent 1 has less intense preference for a than each of the other agents. Therefore, Lemma 2 applies. Arguing as in Step 1 of Case 1, agent 1 always receives his most preferred object.

Second, let $N_2 \subseteq N$ be the set of agents assigned a with positive probability at profiles among $V^1 \times \prod_{N\backslash\{1\}} V^{n+2}$. By our labeling, $V^1 \cap V^a = \emptyset$ and so by the previous argument $1 \notin N_2$. Let $2 \in N_2$ and $\bar{V}^2 \equiv \{u \in V^1 \times \prod_{N\backslash\{1\}} V^{n+2} : F_2(u) = a\}$. By *strategy-proofness*, for each $\bar{u} \in \bar{V}^2$ and each $u_2 \in V^{n+1} \cup V^{n+2}$, $F_1(u_1, \bar{u}_{-1}) = a$. By richness, $\mu(V^{n+1} \times \bar{V}^2_{-2}) > 0$. Now at each profile among $(V^{n+1} \times \bar{V}_{-1})$, agent 2 has less intense preference for a than each the agents among $N\backslash\{1\}$. Therefore, Lemma 2 applies. Arguing as in Step 2 of Case 1, agent 2 always receives his most preferred object among those not assigned to agent 1.

Continuing in this fashion, we identify for each $k = 1, \dots, n$ an agent, called agent k, whose receives a with positive probability at profiles among $\prod_{\{1,\dots,k-1\}} V^l \times \prod_{N\backslash\{1,\dots,k-1\}} V^{n+2}$. By *strategy-proofness*, agent k also receives a when instead reporting a type among $V^{n+1} \cup V^{n+2}$. By richness, the set of such profiles occurs with positive probability. Since agent k has less intense preference for a at each such profile than each agent among $N\backslash\{1, \dots, k-1\}$. Therefore, Lemma 2 applies. Arguing as in Step 3 of Case 1, agent'k always receives his most preferred object among those not assigned among agents $\{1, \dots, k-1\}$. Let $\prec \in \Pi^N$ be such that $1 \prec 2 \prec \cdots \prec n$. Then $F = F^{\prec}$ except possibly on a set of measure zero and so is a priority rule. □

Theorem 1(ii) *For each rich* strict problem, a rule is* ex-ante efficient *and* BIC *if and only if it is a priority rule.*

Proof. Let $\mu \in \mathcal{D}$ be rich with rich* set $V \equiv \bigcup_{k=1}^{2n} V^k \subseteq \mathbb{R}^A$ and let F be *ex-ante efficient* and *strategy-proof* at μ. We argue that F is equivalent to a priority rule. Since differences up to a set of measure zero are irrelevant, we will assume inconsequential redefinitions and sometimes say "always" without the caveat "except possibly on a set of measure zero".

For each $a \in A$ and each $i \in N$, let

$$\bar{V}^a_i \equiv \{u \in (V^a)^N : F_i(u) = a\}, \text{ and}$$
$$N^a \equiv \{j \in N : \mu(\bar{V}^a_j) > 0\}.$$

That is, $\bar{V}^a_i$ consists of those profiles at which a is each agent's most preferred object and agent i receives a. Then N^a consists of those agents who are favored for receiving a at a set of profiles with positive measure. By minimal richness, for each $a \in A$, $\mu((V^a)^N) > 0$.

Step 1: For each object, there is an agent who always receives that object whenever

it is his most preferred object. Let $a \in A$ and label sets so that $V^1 \subseteq V^a$. If V^1 contains an open set, then we may replace it with that set. Also, since redefinition on a measure zero set is inconsequential, we may suppose that either V^1 is open or V^1 is finite. We consider these cases in turn.

Case 1: V^1 is open. Suppose by way of contradiction that $|N^a| > 1$. For each $i \in N^a$ and $j \in N \backslash N^a$, agent i is favored over agent j when reporting less intense preference. Therefore, by Lemma 2, the agents among N^a are always favored over agents among $N \backslash N^a$ whenever reporting preferences at least as intense. For each pair $i, j \in N^a$, since $(V^1)^N \subseteq (V^a)^N$ is open, Lemma 2 implies that there is a unique $\delta_{ij} \in \mathbb{R}$ such that agent i is δ_{ij}-favored over agent j. Then *ex-ante efficient* dictates a unique assignment among N^a Also, there is at least one $i \in N$ such that for each $j \in N^a$, $\delta_{ij} > 0$. Label one such agent as 1.

Let $u_0 \in V^1$ and let $u_0' \in V^{n+1} \cap V_*^a(u_0)$ be a paired type. Relabel the objects according to the common order $P(u_0) = P(u_0')$ so that $a_1 = a$ and $u_0(a_1) > u_0(a_2) \geq u_0(a_3) \geq \cdots \geq u_0(a_n)$. We compare $x_1(u_0)$ and $x_1(u_0')$, agent 1's expected assignments when reporting u_0 and u_0' respectively. By assumption, $x_1(u_0, a) < 1$. Since u_0' represents more intense preference for a_1 than does u_0, so by Lemma 2, at each profile such that agent 1 receives a_1 when reporting u_0, he also receives a_1 when reporting u_0'. In fact, because V^1 and V^{n+1} contain open sets on which a_1 is assigned among N^{a_1} according to the δ criterion, he receives a_1 at a positive measure of additional profiles. Altogether, $x_1(u_0', a_1) > x_1(u_0, a_1)$. Moreover, for each $k = 1, \ldots, n-1$, u_0' represents more intense preference for $\{a_1, \ldots, a_k\}$ than does u_0'. Therefore,

$$\sum_{h=1}^{l} x_k(u_0, a_h) \leq \sum_{h=1}^{l} x_k\big(u_0^{\delta}, a_h\big) \leq \sum_{h=1}^{l} x_k\big(u_0^{\delta'}, h\big).$$

That is, $x_1(u_0')$ strictly first-order stochastically dominates $x_1(u_0)$ according to $P(u_0)$. Agent 1 therefore has a profitable deviation. Since this applies to each $u_0 \in V^1$, agent 1 has a profitable deviation at a non-trivial set of types which violates *BIC*. Instead, $|N^a| = 1$. This implies that agent 1 always receives a whenever it is his most preferred object.

Case 2: V^1 is finite. Suppose by way of contradiction that $|N^a| > 1$. We may suppose further that V^1 and V^{n+1} are singletons so $V^1 \equiv \{u^1\}$ and $V^{n+1} \equiv \{\hat{u}^1\}$ where $\hat{u}^1 \in V_*^a(u_0)$. Relabel the objects according to the common order $P(u_0) = P(u_0')$ so that $a_1 = a$ and $u_0(a_1) > u_0(a_2) \geq u_0(a_3) \geq \cdots \geq u_0(a_n)$. As in Case 1, for each $i \in N^a$ and $j \in N \backslash N^a$, agent i is favored over agent j when reporting less intense preference. Therefore, by Lemma 2, the agents among N^a are always favored over agents among $N \backslash N^a$ whenever reporting preferences at least as intense. For each $k = 1, \ldots, n-1$, u_0' represents more intense preference for $\{a_1, \ldots, a_k\}$ than does u_0'. Therefore, for each $i \in N^a$, at each profile in which agent i receives an object among $\{a_1, \ldots, a_k\}$ when reporting

u_0, he also receives an object among $\{a_1, \ldots, a_k\}$ when reporting u_0'. That is,

$$\sum_{h=1}^{l} x_k(u_0, a_h) \leq \sum_{h=1}^{l} x_k(u_0^{\delta}, a_h) \leq \sum_{h=1}^{l} x_k(u_0^{\delta'}, h)$$

so $x_i(u_0')$ weakly first-order stochastically dominates $x_i(u_0)$. Since these profiles occur with positive probability, *BIC* requires that $x_i(u_0') = x_i(u_0)$. Let $i \in N^a$. Since $x_i(u_0') < 1$, there exist profiles at which an agent j receives a while agent j reports u_0 and agent i reports u_0'. Since u_0 represents less intense preference for a than does u_0', this implies that agent j is favored over agent i and so $j \in N^a$. By the same argument, there is $k \in N^a$ such that agent k is favored over agent j and similarly for all agents among N^a. Since the favored agents are also among N^a, this is a contradiction. Instead, $|N^a| = 1$. This implies that agent 1 always receives a whenever it is his most preferred object.

Step 2: F is a priority order. We first argue that in fact a single agent always receives his most preferred object and then apply the same arguments to calibrate the remainder of a priority order.

Substep 2.1: There is an agent who always receives his most preferred object. By Step 1, for each $a_k \in A$, there is $i_k \in N$ who receives a_k whenever it is his most preferred object. In particular, agent i_k receives a_k at each profile $u \in V^k \times (V^{n+k})^{N\backslash\{i_k\}}$. Since types among V^{n+k} represent more intense preference for a_k than do types among V^k, by Lemma 2, agent i_k is favored over all other agents whenever reporting preferences at least as intense. Let $a_k, a_l \in A$ and $i_k, i_l \in N$ be the corresponding agents. If $i_k \neq i_l$, then each is favored over the other, which is a contradiction. Instead, $i_k = i_l$. That is, there is a single agent who always receives his most preferred object. Label this agent as 1.

Substep 2.2: For each $k = 1, \ldots, n$, there is an agent, called agent k, who always receives his most preferred object among those not assigned to agents $\{1, \ldots, k-1\}$. We next argue that there is an agent who always receives his most preferred object among those not assigned to agent 1. For each pair $a, b \in A$ and $i \in N\backslash\{1\}$, let

$$\begin{aligned} \bar{V}_i^{ab} &\equiv \{u \in V^a \times (V^b)^{N\backslash\{1\}} : F_i(u) = b\}, \text{ and} \\ N^{ab} &\equiv \{j \in N : \mu(\bar{V}_j^{ab}) > 0\}. \end{aligned}$$

At each profile among $V^a \times (V^b)^{N\backslash\{1\}}$, a is the most preferred object for agent 1 and b is the most preferred object of all other agents. Thus, $\bar{V}_i^{ab}$ consists of those profiles at which F favors agent i among $N\backslash\{1\}$ for receiving b and N^{ab} consists of those agents who are favored for receiving b at a set of profiles with positive measure. By Step 1, agent 1 receives a at each such profile in V^{ab}.

Let $a \in A$ and consider profiles in which $u_1 \in V^a$. By the arguments in Step 1, for each $b \in A\backslash\{a\}$, there is an agent who receives b at all such profiles in which it is his most preferred object among $A\backslash\{a\}$. By the arguments in Subset 2.1, there is a single such agent. Label this agent as i_a. Similarly, for each $b \in A$, there is an among $N\backslash\{1\}$ who always receives his most preferred object at all profiles in which $u_1 \in V^b$. Label this agent as i_b. Let $a, b \in A$. By Lemma 2, agent i_a is favored over all agents among $N\backslash\{1\}$ whenever reporting preferences at least as intense. Similarly, agent i_b is favored over all agents among $N\backslash\{1\}$ whenever reporting preferences at least as intense. Therefore, $i_a = i_b$. That is, there is an agent who always receives his most preferred object among those not assigned to agent 1. Label this agent as 2.

Continuing in this fashion, we identify a single third agent who always receives his most preferred object among those not assigned to agent 1 or agent 2 and, more generally, a single agent k who always receives his most preferred object among those not assigned to agents $\{1, \ldots, k-1\}$. Let $\prec \in \Pi^N$ be such that $1 \prec 2 \prec \cdots \prec n$. Then $F = F^{\prec}$ and F is a priority rule. □

Theorem 1(iii) *For each rich problem, a rule is* ex-ante efficient *and* strategy-proof *if and only if it is an extended priority rule. For rich* problem, a rule is* ex-ante efficient *and* BIC *if and only if it is an extended priority rule.*

Proof. The arguments constructing the priority order in the proof of Theorem 1(i) also apply to each rich problem in $\bar{\mathcal{D}}$. Therefore, for each rich $\mu \in \bar{\mathcal{D}}$, each rule that is *ex-ante efficient* and *strategy-proof* at μ is welfare-equivalent to a priority rule. That is, the rule is an extended priority rule.

Similarly, the arguments constructing the priority order in the proof of Theorem 1(ii) also apply to each rich* problem in $\bar{\mathcal{D}}$. Therefore, for each rich* $\mu \in \bar{\mathcal{D}}$, each rule that is *ex-ante efficient* and *BIC* at μ is welfare-equivalent to a priority rule. That is, the rule is an extended priority rule. □

A.4 Role of richness in the characterizations

Proposition 2 *For each problem $\mu \in \bar{D}$ satisfying diversity of preference intensity and each rule that is* ex-ante efficient *and* strategy-proof *at μ, there is an agent who always receives one of his most preferred objects.*

Proof. Let $\mu \in \bar{D}$ satisfy diversity of preference intensity and F be *ex-ante efficient* and *strategy-proof* at μ. Let $a \in A$ be an object for which diversity of preference intensity is satisfied. The arguments in Step 1 of the proof of Theorem 1 only use diversity of preference intensity and so continue to apply. Therefore, there is an agent 1 who always receives a whenever it is his most preferred object. Similarly, for each object $b \in A$ which agent 1 may rank first, the arguments from Step 2 also apply. Therefore, agent 1 always receives his most preferred object. □

Remark 3 *With two agents and two objects, for each pair $u_0, u_0' \in \mathbb{R}^A$ and each $\mu \in \bar{\mathcal{D}}^*(\{u_0, u_0'\})$, all* ex-post efficient *rules are* ex-ante efficient *and* strategy-proof *at μ.*

Proof. Let $N \equiv \{1,2\}$, $A \equiv \{a,b\}$, $V \equiv \{u_0, u_0'\} \subseteq \mathcal{U}$, $\mu \in \mathcal{D}^*(V)$, and F be *ex-post efficient* and *strategy-proof* at μ. If either $u_0(a) \geq u_0(b)$ and $u_0'(a) \geq u_0'(b)$ or $u_0(b) \geq u_0(a)$ and $u_0'(b) \geq u_0'(a)$, then each *ex-post efficient* and *strategy-proof* rule is constant on V^N, so *ex-ante efficiency* is satisfied trivially. Suppose instead that $u_0(a) > u_0(b)$ and $u_0'(a) < u_0'(b)$, relabeling if necessary. By *ex-post efficiency*, $F(u_0, u_0') = (a,b)$ and $F(u_0', u_0) = (b,a)$. Let $q_1, q_2 \in [0,1]$ be such that

$$F(u) = \begin{cases} (a,b) & \text{with probability } q_1 \text{ if } u = (u_0, u_0) \\ (a,b) & \text{with probability } q_2 \text{ if } u = (u_0' u_0') \end{cases}.$$

Strategy-proofness. Each agent receives his most preferred object at (u_0, u_0') and (u_0', u_0), so neither has a profitable deviation. Consider (u_0, u_0). If either agent deviates to u_0', he receives b with probability 1. Since $0 \leq q_1$ and $0 \leq 1 - q_1$, neither agent has a profitable deviation at (u_0, u_0). Next consider (u_0', u_0'). If either agent deviates to u_0, he receives a with probability 1. Since $0 \leq q_2$ and $0 \leq 1 - q_2$, neither agent has a profitable deviation at (u_0', u_0'). Thus, F is *strategy-proof.*

Ex-ante efficiency. To make agent 1 better off, we must either increase q_1 or decrease q_2. Similarly, to make agent 2 better off, we must either decrease q_1 or increase q_2. However, since both agents have the same types at the corresponding profiles, the gain for one agent is exactly offset by the loss for the other agent. Thus, ex-ante, neither agent can be made better off without making the other agent worse off and F is *ex-ante efficient* at μ. $\square$

Remark 4 *For each top-collinear set $V \subseteq \mathcal{R}^A$ and each $\mu \in \mathcal{D}^*(V)$, each randomized bi-polar priority rules is* ex-ante efficient *and* strategy-proof *at μ.*

Proof. Let $V \subseteq \mathbb{R}^A$ be a top-collinear set defined by τ and ε, $\mu \in \mathcal{D}(V)$, $\prec \in \Pi^N$, and $B \subseteq A$. We show that $F^{\prec,B}$ is *ex-ante efficient* at μ and then extend the result to randomized bi-polar priority rules with respect to $\prec$. Label the agents so that $\prec = (1,2,3,\ldots,n)$ and let $\prec' \equiv (2,1,3,\ldots,n)$.

We look for a possible ex-ante Pareto improvement. First consider making agent 1 better off. The only possibility is to resolve some of the conflicts over B in favor of agent 1 rather than agent 2. But then to make agent 2 at least as well off, we must resolve some of the conflicts over $A \backslash B$ in favor of agent 2 rather than agent 1. The difference in ex-post utility is ε for each agent in each case. Therefore, making agent 1 better off requires that changes of the first type occur with higher probability while making agent 2 at least as well off requires that changes of the second type occur with at least as high probability. These conditions are incompatible. As the reverse argument applies to agent 2, no other rule makes either agent 1 or agent 2 better off without making the other worse off.

Therefore, an ex-ante Pareto improvement cannot make agent 1 or agent 2 better off. Instead, each of these agents must receive the same expected utility as under F. In this case, from the definitions of V and F, these agents collectively receive the same objects in each state. Therefore, conditional on agents 1 and 2 receiving the same expected utility, agent 3 cannot be made better off. As the remainder of the rule follows a priority order, each agent's utility is maximized conditional on the utilities of those agents with higher priority. Therefore, the same argument applies and no ex-ante Pareto improvement is possible.

Finally consider randomization over $\{F^{\prec,B} : B \subseteq A\}$. Let $u \in V$, $\equiv \text{argmax}_{c\in A} u_1(c)$, and $b \equiv \text{argmax}_{c\in A} u_2(c)$. If $a \neq b$, then for each $B \subseteq A$, $F^{\prec,B}_{12}(u) = (a, b)$. Since all rules in the family follow the same priority order for the remaining agents, for each $B, B' \subseteq A$, $F^{\prec,B}_{-12}(u) = F^{\prec,B'}_{-12}(u)$. If $a = b$, then for each $B \subseteq A$, $F^{\prec,B}_{12}(u) \in \{(a, \tau(a)), (\tau(a), a)\}$. Since all rules in the family follow the same priority order for the remaining agents, for each $B, B' \subseteq A$, $F^{\prec,B}_{-12}(u) = F^{\prec,B'}_{-12}(u)$. Thus, all agents $N\backslash\{1, 2\}$ receive the same assignments at each profile under each rule in the family. By the previous arguments, none of these agents can be made better off without making another agent worse off. Similarly, as argued previously, neither agents 1 nor agent 2 can be made better off without making the other worse off. Altogether, no ex-ante Pareto improvement is possible. □

Example 10. A non-priority rule which is ex-ante efficient and BIC at a rich problem. Let $N \equiv \{1, 2, 3, 4\}$, $A \equiv \{a, b, c, d\}$, and $V \equiv \{u^L, u^H, u^b, u^c, u^d\}$ as specified in the table.

	u^L	u^a	u^b	u^c	u^d
a	20	30	0	0	0
b	18	8	30	4	4
c	12	4	4	30	8
d	0	0	8	8	30

Let F be the rule which randomizes uniformly over the set of utilitarian allocations at each profile $u \in V^N$. Then F is *ex-ante efficient* and even *utilitarian-optimal* on $\mathcal{D}(V)$. We show that F is also *BIC* at some problems on this domain. To simplify computations, we consider a non-rectangular problem and therefore not rich. However, all incentive constraints will hold with slack, so F is also *BIC* at nearby rectangular problems which are rich. Let μ_0 be a measure on V such that $\mu_0(u^L) = \frac{1}{15}$, $\mu(u^a) = \frac{3}{5}$, $\mu_0(b) = \frac{1}{3}$, and $\mu_0(c) = \mu(d) = 0$. Let $\mu \in \mathcal{D}(V)$ be such that for each $u \in V^N$, $\mu(u) = \prod_N \mu_0(u_i)$.

We argue that F is *BIC* at μ. First, for each $u_0 \in V\backslash\{u^L\}$, the utility difference between the first and second ranked object is $30 - 8 = 22$. Since largest utility difference between the second and fourth ranked object is $18 - 0 = 18$, by *utilitarian optimality*, each object is assigned to an agent who ranks it first whenever such an agent is present. Also, since a is ranked last by u^b, u^c, and u^d, agents with these types never receive a when there is an agent whose type is u^L or u^a.

We verify that no type has a profitable deviation by computing expected assignments. By independence, the computations are the same for agents of each type. First, since $\mu_0(u^c) = \mu_0(u^d)$, each agent with either type receives his most preferred object with probability 1. That is, $x_i(u^c) = (0, 0, 1, 0)$ and $x_i(u^d) = (0, 0, 0, 1)$.

To compute the remaining expected assignments, we distinguish profiles according to the number of agents who report u^L, u^a, or u^b.

num. u^L	num. of u^a	Prob/50625	u^L assign.	u^a assign.	u^b assign.
0	0	625	$\emptyset$	$\emptyset$	$\{a, b, c, d\}$
0	1	4500	$\emptyset$	$\{a\}$	$\{b, c, d\}$
0	2	12150	$\emptyset$	$\{a, c\}$	$\{b, d\}$
0	3	14580	$\emptyset$	$\{a, c, d\}$	$\{b\}$
0	4	6561	$\emptyset$	$\{a, b, c, d\}$	$\emptyset$
1	0	500	$\{a\}$	$\emptyset$	$\{b, c, d\}$
1	1	2700	$\{c\}$	$\{a\}$	$\{b, d\}$
1	2	4860	$\{c\}$	$\{a, c\}$	$\{b\}$
1	3	2916	$\{b\}$	$\{a, c, d\}$	$\emptyset$
2	0	150	$\{a, c\}$	$\emptyset$	$\{b, d\}$
2	1	540	$\{c, d\}$	$\{a\}$	$\{b\}$
2	2	486	$\{b, c\}$	$\{a, d\}$	$\emptyset$
3	0	20	$\{a, c, d\}$	$\emptyset$	$\{b\}$
3	1	36	$\{b, c, d\}$	$\{a\}$	$\emptyset$
4	0	1	$\{a, b, c, d\}$	$\emptyset$	$\emptyset$

Since F randomizes uniformly over *utilitarian-optimal* allocations, the expected economy-wide allocations by type as fractions out of 607500 are:

	a	b	c	d
u^L	8052	41268	105516	7164
u^a	591948	78732	434484	352836
u^b	7500	487500	67500	247500

The expected allocations for agent i when reporting each type are then approximately:

$$x_i(u^L) \approx \big(.050, .255, .651, .044\big)$$
$$x_i(u^a) \approx \big(.406, .054, .298, .242\big)$$
$$x_i(u^b) \approx \big(.009, .602, .083, .306\big)$$
$$x_i(u^c) \approx \big(.406, .054, .298, .242\big)$$
$$x_i(u^d) \approx \big(.009, .602, .083, .306\big)$$

Finally, the expected utility for each these types when reporting each other type are:

	$x_i(u^L)$	$x_i(u^a)$	$x_i(u^b)$	$x_i(u^c)$	$x_i(u^c)$
u^L	13.4	12.7	12.0	12.0	0
u^a	6.1	13.8	5.4	4.0	0
u^b	10.6	4.7	20.8	4.0	8.0

As seen from the table, an agent of each type has a strict incentive to report truthfully. Thus, F is *BIC*.

A.5 Ordinal and scoring domains

We prove Theorem 2 in five lemmas. We begin with three-agent problems. We first show that the axioms are incompatible with ex-post randomization. Next, we verify the characterization on scoring domains and then on general ordinal domains. Finally, we extend these results to larger populations, again considering scoring domains separately.

For easy reference, we introduce notation to distinguish types. We write u^a for a type which ranks a first, u^{ab} for a type which ranks a first and b second, and u^{abc} for a type which ranks a first, b second, and c third. For example, when $|A| = 3$, the domain V^s consists of six types: $V^s = \{u^{ab}, u^{ac}, u^{ba}, u^{bc}, u^{ca}, u^{cb}\}$. In this case, the types are completely determined by specifying the objects ranked first and second. More generally, several types may be represented by u^a, etc., though we will assume that u^a is fixed once mentioned.

Lemma 4. *Let $|A| = 3$. For each $s \in S$ and each $\mu \in \mathcal{D}(V^s)$ with full support, each rule that is* ex-ante efficient *and* strategy-proof *at μ is deterministic.*

Proof. Let $N \equiv \{1, 2, 3\}$, $A \equiv \{a, b, c\}$, $s \in S$, $\mu \in \mathcal{D}(V^s)$ have full support, and F be *ex-ante efficient* and *strategy-proof* at μ. Suppose by way of contradiction that there are $i \in N$, $a, b \in A$, and $u \in (V^s)^N$ such that $\mu(u) > 0$ and $\{a, b\} \subseteq \text{supp}(F_i(u))$. Without loss of generality, we may label the objects so that $u_i(a) > u_i(b)$ and either $u_i(b) > u_i(c)$ or $c \notin \text{supp}(F_i(u))$. Let $N^a \equiv \{k \in N : a \in \text{supp}(F_k(u'))\}$. By feasibility, $|N^a| \geq 2$. Let $j \in N^a \backslash \{i\}$ and label the third agent k. By *ex-post efficiency*, $u_j(a) > u_j(b)$.

Case 1: $N^a = \{i, j, k\}$. Then $u_i = u_j = u_k = u^{ab}$. By Remark 1, F allocates objects among $\{i, j, k\}$ to maximize their utilitarian welfare at each profile. In particular, $F(u^{ab}, u^{ac}, u^{ba}) = (a, c, b)$ and $F(u^{ac}, u^{ab}, u^{ba}) = (c, a, b)$. But now *strategy-proofness* implies $F(u^{ac}, u^{ac}, u^{ba}) = (a, c, b)$ and $F(u^{ac}, u^{ac}, u^{ba}) = (c, a, b)$, a contradiction.

Case 2: $N^a = \{i, j\}$. Then $F_k(u) = c$. By *strategy-proofness*, $F(u^{ab}, u_j, u_k) = F(u)$ and $F(u^{ab}, u^{ab}, u_k) = F(u)$. By Remark 1, F allocates objects among $\{i, j\}$ to maximize their utilitarian welfare at each profile. Now consider $u' \equiv (u^{ab}, u^{ab}, u^{ab})$. If $F_k(u')$ is not deterministic, then by Remark 1, F allocates objects among $\{i, j, k\}$ to maximize their utilitarian welfare at each profile and the argument in Case 1 applies. Suppose instead that $F_k(u')$ is deterministic.

Subcase 2.1: $F_k(u') = a$. Then $F(u^{ba}, u^{ab}, u^{ab}) = (b, c, a)$ and $F(u^{ab}, u^{ba}, u^{ab}) = (c, b, a)$. But now *strategy-proofness* implies $F(u^{ba}, u^{ba}, u^{ab}) = (a, c, b)$ and $F(u^{ba}, u^{ba}, u^{ab}) = (c, b, a)$, a contradiction.

Subcase 2.2: $F_k(u') = b$. By *strategy-proofness*, $F_k(u^{ab}, u^{ab}, u^{ba}) = b$. Then $F(u^{ac}, u^{ab}, u^{ba}) = (c, a, b)$ and $F(u^{ab}, u^{ac}, u^{ba}) = (a, c, b)$. But now *strategy-proofness* implies $F(u^{ac}, u^{ac}, u^{ba}) = (a, c, b)$ and $F(u^{ac}, u^{ac}, u^{ba}) = (a, c, b)$, a contradiction.

Subcase 2.3: $F_k(u') = c$. Consider $F(u^{ac}, u^{ab}, u^{ab})$. If $a \in \text{supp}(F_k(u^{ac}, u^{ab}, u^{ab}))$, then by Lemma 2, F allocates objects among $\{i, j, k\}$ to maximize their utilitarian welfare at each profile. The argument in Case 1 then applies, so suppose instead that $a \notin \text{supp}(F_k(u^{ac}, u^{ab}, u^{ab}))$. Similarly, $a \notin \text{supp}(F_k(u^{ab}, u^{ac}, u^{ab}))$. By *ex-post efficiency*, $\text{supp}(F_i(u^{ac}, u^{ab}, u^{ab})) \neq \{a\}$. If $\text{supp}(F_j(u^{ac}, u^{ab}, u^{ab})) = \{a\}$, then by repeated application of *strategy-proofness*,

$$a = F_j(u^{ac}, u^{ab}, u^{ab}) = F_j(u^{ac}, u^{ac}, u^{ab}) = F_j(u^{ab}, u^{ac}, u^{ab}) = F_j(u^{ab}, u^{ab}, u^{ab}).$$

This contradicts $a \in \text{supp}(F_i(u^{acb}, u^{abc}, u^{abc}))$. Instead,

$$\text{supp}(F(u^{ab}, u^{ac}, u^{ab})) = \big\{(a, b, c), (c, a, b)\big\} \text{ and } \text{supp}(F(u^{ab}, u^{ac}, u^{ab})) \quad = \big\{(a, b, c), (c, a, b)\big\}.$$

Let $p_1 \equiv \mu(u^{ab}, u^{ac}, u^{ab})$ and $p_2 \equiv \mu(u^{ac}, u^{ab}, u^{ab})$ so $p_1, p_2 \in (0, 1)$. Let $q_1, q_2 \in (0, 1)$ be such that

$$F(u) = \begin{cases} (a, b, c) & \text{with probability } q_1 \text{ if } u = (u^{ac}, u^{ab}, u^{ab}) \\ (b, a, c) & \text{with probability } q_2 \text{ if } u = (u^{ab}, u^{ac}, u^{ab}) \end{cases}.$$

We construct an ex-ante Pareto improvement. Let F' modify F at (u^{ac}, u^{ab}, u^{ab}) and (u^{ab}, u^{ac}, u^{ab}) so that

$$F'(u) = \begin{cases} (a, b, c) & \text{with probability } q_1 - q_1 q_2 p_2 \text{ if } u = (u^{ac}, u^{ab}, u^{ab}) \\ (b, a, c) & \text{with probability } q_2 - q_1 q_2 p_1 \text{ if } u = (u^{ab}, u^{ac}, u^{ab}) \end{cases}.$$

Comparing expected utilities,

$$\begin{aligned} U_i(F', \mu) - U_i(F, \mu) &= p_1(-q_1 q_2 p_2)(s_1 - s_2) + p_2(-q_1 q_2 p_1)(s_2 - s_1), \\ U_j(F', \mu) - U_j(F, \mu) &= p_1(-q_1 q_2 p_2)(s_2 - s_1) + p_2(-q_1 q_2 p_1)(s_1 - s_2), \text{ and} \\ U_k(F', \mu) - U_k(F, \mu) &= p_1(-q_1 q_2 p_2)(s_3 - s_2) + p_2(-q_1 q_2 p_1)(s_3 - s_2) > 0. \end{aligned}$$

Therefore, F' ex-ante Pareto dominates F at μ. Altogether, $\{a, b\} \not\subseteq \text{supp}(F_i(u))$. Instead, F is deterministic. □

Lemma 5. *Let $|A| = 3$. For each $s \in S$ and each $\mu \in \mathcal{D}(V^s)$ with full support, a rule is* ex-ante efficient *and* strategy-proof *at μ if and only if it is a priority rule.*

Proof. Let $N \equiv \{1,2,3\}$, $A \equiv \{a,b,c\}$, $s \in S$, $\mu \in \mathcal{D}(V^s)$ have full support, and F be *ex-ante efficient* and *strategy-proof* at μ. By Lemma 4, we may suppose F is deterministic. Label the agents so that $F(u^{ab}, u^{ab}, u^{ab}) = (a,b,c)$. We argue that $F = F^{\prec}$ where $1 \prec 2 \prec 3$.

Step 1: Agent 1 receives *a* whenever all agents rank *a* first. We divide the eight profiles in which all agents rank a first into two groups according to whether agent 2 reports u^{ab} or u^{ac} and consider each set of profiles in turn.

Case 1.1: $u_2 = u^{ab}$. By *strategy-proofness*, $F_1(u^{ac}, u^{ab}, u^{ab}) = a$, $F_3(u^{ab}, u^{ab}, u^{ac}) = c$, and $F_3(u^{ac}, u^{ab}, u^{ac}) \neq a$. Thus, agent 3 does not receive a at any of these profiles. Suppose by way of contradiction that $F_2(u^{ab}, u^{ab}, u^{ac}) = a$ so $F_1(u^{ab}, u^{ab}, u^{ac}) = b$. By *strategy-proofness*, $F_1(u^{ac}, u^{ab}, u^{ac}) \neq a$ so $F_2(u^{ac}, u^{ab}, u^{ac}) = a$. Together, $F(u^{ac}, u^{ab}, u^{ab}) \in \{(a,b,c),(a,c,b)\}$ and $F(u^{ac}, u^{ab}, u^{ac}) \in \{(c,a,b),(b,a,c)\}$.

We construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u^{ac}, u^{ab}, u^{ab})$ and $p_2 \equiv \mu(u^{ac}, u^{ab}, u^{ac})$. Let F' modify F so that for each $u \in (V^s)^N$,

$$F'(u) \equiv \begin{cases} (c,a,b) & \text{with probability } p_2 \text{ if } u = (u^{ac}, u^{ab}, u^{ab}) \\ (a,b,c) & \text{with probability } p_1 \text{ if } u = (u^{ac}, u^{ab}, u^{ac}) \\ F(u) & \text{otherwise} \end{cases}.$$

Comparing expected utilities,

$$\begin{aligned} U_1(F',\mu) - U_1(F,\mu) &\geq p_1p_2(s_2 - s_1) + p_2p_1(s_1 - s_2) = 0, \\ U_2(F',\mu) - U_2(F,\mu) &\geq p_1p_2(s_1 - s_2) + p_2p_1(s_2 - s_1) = 0, \text{ and} \\ U_3(F',\mu) - U_3(F,\mu) &\geq p_1p_2(s_2 - s_2) + p_2p_1(s_2 - s_2) = 0, \end{aligned}$$

so each agent is at least as well off under F' as under F. If $F(u^{ac}, u^{ab}, u^{ab}) = (a,b,c)$, then in fact $U_3(F',\mu) - U_3(F,\mu) \geq p_1p_2(s_2 - s_3) + p_2p_1(s_2 - s_2) > 0$. If instead $F(u^{ac}, u^{ab}, u^{ab}) = (a,c,b)$, then $U_2(F',\mu) - U_2(F,\mu) = p_1p_2(s_1 - s_3) + p_2p_1(s_2 - s_1) > 0$. In each case, at least one agent is better off and F' ex-ante Pareto dominates F at μ. Instead, $F_1(u^{ab}, u^{ab}, u^{ac}) = a$ and so by *strategy-proofness*, $F_1(u^{ac}, u^{ab}, u^{ac}) = a$ as well. Summarizing, agent 1 receives a at each of the four profiles in which agent 2 reports u^{ab}.

Case 1.2: $u_2 = u^{ac}$. By Substep 1.1, agent 2 does not receive a when reporting u^{ab} while the other agents report u^{ab} or u^{ac}. Therefore, by *strategy-proofness*, agent 2 also does not receive a when reporting u^{ac} while the other agents report u^{ab} or u^{ac}. Suppose by way of contradiction that agent 3 receives a at one of the remaining profiles in which all agents rank a first. Then repeating the arguments from Substep 1.1, agent 3 receives a at each of these profiles.

Now consider (u^{ac}, u^{ac}, u^{ab}). If $F_1(u^{ac}, u^{ac}, u^{ab}) = b$, then by Lemma 2, agent 3 never prefers agent 1's assignment to his own when both report the same type. However, $F_3(u^{ab}, u^{ab}, u^{ab}) = c$, so instead $F(u^{ac}, u^{ac}, u^{ab}) = (c, b, a)$. Lemma 2 now implies that agent 3 never prefers agent 2's assignment to his own when both report the same type. However, $F_3(u^{ab}, u^{ab}, u^{ab}) = c$, so this is a contradiction. Instead, agent 1 receives $\boldsymbol{a}$ whenever all agents rank a first.

Step 2: Agent 1 receives $\boldsymbol{a}$ whenever he ranks $\boldsymbol{a}$ first. Let $u \in (V^s)^N$ with $u_1 \in \{u^{ab}, u^{ac}\}$ and suppose by way of contradiction that $F_1(u) \neq a$. By *strategy-proofness*, $F_1(u^{ab}, u_{-1}) \neq a$. Let $i \in N$ be such that $F_i(u^{ab}, u_{-1}) = a$ and label the third agent j. By *strategy-proofness*, $F_i(u^{ab}, u^{ab}, u_j) = F_i(u^{ab}, u^{ac}, u_j) = a$. Then also $F_1(u^{ac}, u^{ab}, u_j) \neq a$ and $F_1(u^{ac}, u^{ac}, u_j) \neq a$. By Step 1, $u_j \in \{u^{ba}, u^{bc}, u^{ba}, u^{bc}\}$. We distinguish two cases according to u_j.

Case 2.1: $u_j \in \{u^{ba}, u^{bc}\}$. Consider (u^{ab}, u^{ac}, u_j). By *strategy-proofness*, $F_1(u^{ab}, u^{ac}, u_j) \neq a$. If $F_j(u^{ab}, u^{ac}, u_j) = a$, then by *strategy-proofness*, $F_j(u^{ab}, u^{ac}, u^{ab}) = a$. However, $F_1(u^{ab}, u^{ac}, u^{ab}) = a$, so instead $F_i(u^{ab}, u^{ac}, u_j) = a$. If $F_1(u^{ab}, u^{ac}, u_j) = c$, then by Lemma 2, agent 3 never prefers agent 1's assignment to his own when both report the same type. However, $F_3(u^{ab}, u^{ab}, u^{ab}) = c$, so instead $F(u^{ab}, u^{ac}, u_j) = (b, a, c)$. By *strategy-proofness* together with Step 1, $F(u^{ab}, u^{ac}, u^{ac}) = (a, b, c)$.

We construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u^{ab}, u^{ac}, u_j)$ and $p_2 \equiv \mu(u^{ab}, u^{ac}, u^{ac})$. Let F' modify F so that for each $u \in (V^s)^N$,

$$F'(u) \equiv \begin{cases} (a, c, b) & \text{with probability } p_2 \text{ if } u = (u^{ab}, u^{ac}, u_j) \\ (b, a, c) & \text{with probability } p_1 \text{ if } u = (u^{ab}, u^{ac}, u^{ac}) \\ F(u) & \text{otherwise} \end{cases}.$$

Comparing expected utilities,

$$\begin{aligned} U_1(F', \mu) - U_1(F, \mu) &= p_1 p_2 (s_1 - s_2) + p_2 p_1 (s_2 - s_1) = 0, \\ U_i(F', \mu) - U_i(F, \mu) &= p_1 p_2 (s_2 - s_1) + p_2 p_1 (s_1 - s_3) > 0, \text{ and} \\ U_j(F', \mu) - U_j(F, \mu) &\geq p_1 p_2 (s_1 - s_2) + p_2 p_1 (s_2 - s_2) > 0, \end{aligned}$$

Therefore, F' ex-ante Pareto dominates F at μ.

Case 2.2: $u_j \in \{u^{ca}, u^{cb}\}$. Consider (u^{ac}, u^{ab}, u_j). By *strategy-proofness*, $F_1(u^{ac}, u^{ab}, u_j) \neq a$. If $F_j(u^{ac}, u^{ab}, u_j) = a$, then by *strategy-proofness*, $F_j(u^{ac}, u^{ab}, u^{ab}) = a$. However, $F_1(u^{ac}, u^{ac}, u^{ab}) = a$, so instead $F_i(u^{ac}, u^{ab}, u_j) = a$. If $F_1(u^{ac}, u^{ab}, u_j) = b$, then by Lemma 2, agent 3 never prefers agent 1's assignment to his own when both report the same type. However, $F_3(u^{ab}, u^{ab}, u^{ab}) = c$, so instead $F(u^{ac}, u^{ab}, u_j) = (c, a, b)$. By *strategy-proofness* together with Step 1, $F(u^{ac}, u^{ab}, u^{ab}) = (a, c, b)$.

We construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u^{ac}, u^{ab}, u_j)$ and $p_2 \equiv \mu(u^{ac}, u^{ab}, u^{ab})$.

Let F' modify F so that for each $u \in (V^s)^N$,

$$F'(u) \equiv \begin{cases} (a,b,c) & \text{with probability } p_2 \text{ if } u = (u^{ac}, u^{ab}, u_j) \\ (c,a,b) & \text{with probability } p_1 \text{ if } u = (u^{ac}, u^{ab}, u^{ab}) \\ F(u) & \text{otherwise} \end{cases} .$$

Comparing expected utilities,

$$\begin{aligned} U_1(F',\mu) - U_1(F,\mu) &= p_1p_2(s_1 - s_2) + p_2p_1(s_2 - s_1) = 0, \\ U_i(F',\mu) - U_i(F,\mu) &= p_1p_2(s_2 - s_1) + p_2p_1(s_1 - s_3) > 0, \text{ and} \\ U_j(F',\mu) - U_j(F,\mu) &\geq p_1p_2(s_1 - s_2) + p_2p_1(s_2 - s_2) > 0. \end{aligned}$$

Therefore, F' ex-ante Pareto dominates F at μ. Instead, $F_1(u) = a$.

Step 3: Agent 1 always receives his most preferred object. First consider b. By Steps 1 and 2, there is an agent $i \in N$ who always receives b whenever he ranks b first. Suppose by way of contradiction that $i \neq 1$ and label the third agent j. Let $u \equiv (u_1, u_i, u_j) \equiv (u^{ab}, u^{ac}, u^{ac})$ and $u' \equiv (u'_1, u'_i, u'_j) \equiv (u^{ba}, u^{bc}, u^{ba})$. Then $F_1(u) = a$ and $F_i(u') = b$. Moreover, by *strategy-proofness*, $F(u) = (a,c,b)$ and $F(u') = (a,b,c)$.

We construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u)$ and $p_2 \equiv \mu(u')$. Let F' modify F so that for each $\hat{u} \in (V^s)^N$,

$$F'(\hat{u}) \equiv \begin{cases} (b,a,c) & \text{with probability } p_2 \text{ if } \hat{u} = u \\ (b,c,a) & \text{with probability } p_1 \text{ if } \hat{u} = u' \\ F(\hat{u}) & \text{otherwise} \end{cases} .$$

Comparing expected utilities,

$$\begin{aligned} U_1(F',\mu) - U_1(F,\mu) &= p_1p_2(s_2 - s_1) + p_2p_1(s_1 - s_2) = 0, \\ U_i(F',\mu) - U_i(F,\mu) &= p_1p_2(s_1 - s_2) + p_2p_1(s_2 - s_1) = 0, \text{ and} \\ U_j(F',\mu) - U_j(F,\mu) &= p_1p_2(s_2 - s_3) + p_2p_1(s_2 - s_3) > 0. \end{aligned}$$

Therefore, F' ex-ante Pareto dominates F at μ. Instead, $i = 1$.

Finally, consider c. By Steps 1 and 2, there is an agent $j \in N$ who always receives c whenever he ranks c first. Repeating the previous argument, $j = 1$ and so agent 1 always receives his most preferred object.

Step 4: Conditional on agent 1's assignment, agent 2 always receives his most pre-

ferred object. Let $u \in (V^s)^N$ and suppose by way of contradiction that $u_2(F_3(u)) > u_2(F_2(u))$. Relabeling objects if necessary, suppose that $F_2(u) = b$ and $F_3(u) = a$. By *ex-post efficiency* and *strategy-proofness*,

$$F(u_1, u^{ab}, u^{ab}) = F(u_1, u^{ac}, u^{ab}) = F(u_1, u^{ab}, u^{ac}) = F(u_1, u^{ac}, u^{ac}) = (c, b, a).$$

Then by Lemma 2, agent 3 never prefers agent 2's assignment to his own when both report the same type. However, $F_3(u^{ab}, u^{ab}, u^{ab}) = c$, so this is a contradiction. Instead, $u_2(F_3(u)) < u_2(F_2(u))$. Altogether, $F = F^{\prec}$. □

Lemma 6. *Let $|A| = 3$. For each ordinally rich set $V \subseteq \mathbb{R}^A$ and each $\mu \in \mathcal{D}^*$ with full support on V^N, a rule is* ex-ante efficient *and* strategy-proof *at μ if and only if it is a priority rule.*

Proof. Let $N \equiv \{1, 2, 3\}$, $A \equiv \{a, b, c\}$, $V \subseteq \mathbb{R}^A$ be ordinally rich, $\mu \in \mathcal{D}^*$ have full support on V^N, and F be *ex-ante efficient* and *strategy-proof* at μ. To apply the arguments from the proof of Lemma 5, it suffices to show that the ex-ante Pareto improvements constructed for a scoring domain are also an ex-ante Pareto improvements for V; all conclusions drawn from *strategy-proofness* translate directly. Let $\{u^{ab}, u^{ac}, u^{ba}, u^{bc}, u^{ca}, u^{cb}\} \subseteq V$ represent the types guaranteed by ordinal diversity.

To apply Step 1, consider the ex-ante Pareto improvement from Case 1.1. In this case, $F(u^{ac}, u^{ab}, u^{ab}) \in \{(a, b, c), (a, c, b)\}$ and $F(u^{ac}, u^{ab}, u^{ac}) \in \{(c, a, b), (b, a, c)\}$ while $F'(u^{ac}, u^{ab}, u^{ab}) = (c, a, b)$ and $F(u^{ac}, u^{ab}, u^{ac}) = (b, a, c)$. Agents 1 and 2 have the same types in each profile, so the comparisons of their expected utilities are the same as in the lemma. Agent 3 is at least as well off at each profile individually. Since $u^{ab}(b) > u^{ab}(c)$ and $u^{ac}(c) > u^{ac}(b)$, the comparison of his expected utilities is also the same same as in the lemma. The conclusion from Step 1 of Lemma 5 now applies: agent 1 receives a whenever all agents rank a first.

To apply Step 2, label the agents $\{1, i, j\}$ as there and in each profile list the types so that agent 1 is first, agent i is second, and agent j is third. First consider the ex-ante Pareto improvement from Case 2.1 where $u_j \in \{u^{ba}, u^{bc}$. In each profile, we list the types so that agent 1 is first, agent i is second, and agent j is third. In this case, $F(u^{ab}, u^{ac}, u^{ac}) = (b, a, c)$ and $F(u^{ab}, u^{ac}, u^{ac}_j) = (a, b, c)$ while $F'(u^{ab}, u^{ac}, u_j) = (a, c, b)$ and $F'(u^{ab}, u^{ac}, u^{ac}) = (b, a, c)$. Agents 1 and i have the same types in each profile, so the comparisons of their expected utilities are the same as in the lemma. Since $u_j(b) > u_j(c)$ and $u^{ac}(c) = u^{ac}(c)$, agent j is better off.

Next consider the ex-ante Pareto improvement from Case 2.2 where $u_j \in \{u^{ca}, u^{cb}$. In this case, $F(u^{ac}, u^{ab}, u_j) = (c, a, b)$ and $F(u^{ac}, u^{ab}, u^{ac}) = (a, c, b)$ while $F'(u^{ac}, u^{ab}, u_j) = (a, b, c)$ and $F'(u^{ac}, u^{ab}, u^{ac}) = (c, a, b)$. Agents 1 and i have the same types in each profile, so the comparisons of their expected utilities are the same as in the lemma. Since $u_j(c) > u_j(b)$ and $u^{ac}(b) = u^{ac}(b)$, agent j is better off. The conclusion from Step 1 of Lemma 5 now applies: agent 1 receives a

whenever he ranks a first.

Repeating the previous arguments, for each $a_i \in A$, there is $i \in N$ such that agent i always receives a_i whenever he ranks a_i first. Let $a, b \in A$. First suppose that the preference intensities differ across types ranking a and b first. That is, $u^{ab}(a) - u^{ab}(b) \neq u^{ba}(b) - u^{ba}(a)$. Then by Lemma 2, the agent favored for a is the same as the agent favored for b.

Suppose instead that preference intensities are the same so that $u^{ab}(a) - u^{ab}(b) = u^{ba}(b) - u^{ba}(a)$ and consider the ex-ante Pareto improvement in Step 3. In this case, $F(a^{ab}, u^{ac}, u^{ac}) = (a, c, b)$ and $F(a^{ba}, u^{bc}, u^{ba}) = (a, b, c)$ while $F(a^{ab}, u^{ac}, u^{ac}) = (b, a, c)$ and $F(a^{ba}, u^{bc}, u^{ba}) = (b, c, a)$. By assumption, $u^{ab}(a) - u^{ab}(b) = u^{ba}(b) - u^{ba}(a)$, so the comparisons of their expected utilities for agents 1 and 2 are the same as in the lemma. Since $u^{ac}(c) > u^{ac}(b)$ and $u^{ba}(a) > u^{ba}(c)$, agent 3 is better off. The conclusion from Step 3 of Lemma 5 now applies: agent 1 always receives his most preferred object.

To conclude, the argument from Step 4 relies only on Lemma 2 and so applies directly. Therefore, F is a priority rule. □

Lemma 7. *Let $|A| \geq 3$. For each $s \in S$ and each ordinally rich problem $\mu \in \mathcal{D}(V^s)$, a rule is* ex-ante efficient *and* strategy-proof *at μ if and only if it is a priority rule.*

Proof. Let $s \in S$, $\mu \in \mathcal{D}(V^s)$ be ordinally rich, and F be *ex-ante efficient* and *strategy-proof* at μ. Our proof makes frequent use of Lemma 5. To show that F is a priority rule, we first calibrate a conditional priority structure from problems with three agents and three objects. Next, we argue that the structure represents a single priority order. Before continuing, we identify subsets of types which will play an important role of the proof. Let $s \in S$ and for each triple $a, b, c \in A$ and each triple $i, j, k \in N$, let

$$V^{+}(a,b,c,i,j,k) \equiv \Big\{u_0 \in V^s : \forall d \in A \backslash \{a,b,c\},\, u_0(d) < \min\{u_0(a), u_0(b), u_0(c)\}\Big\}$$

$$V^{-}(a,b,c,i,j,k) \equiv \left\{ u_0 \in V^s : \begin{array}{c} \forall\, h, h' \in N \backslash \{i,j,k\} \quad \exists\, d, d' \in A \backslash \{a,b,c\}, \\ d = \text{argmax}_{d'' \in A} u_h(d''), \\ d' = \text{argmax}_{d'' \in A} u_{h'}(d''), \\ \text{and } d = d' \Leftrightarrow h = h' \end{array} \right\}.$$

That is, $V^{+}(a,b,c,i,j,k)$ consists of those partial profiles at which agents $\{i,j,k\}$ rank $\{a,b,c\}$ at the top. Similarly, $V^{-}(a,b,c,i,j,k)$ consists of partial profiles excluding agents $\{i,j,k\}$ at which each of the remaining agents rank distinct objects among $A \backslash \{a,b,c\}$ at the top. For each $u \in V^{+}(a,b,c,i,j,k) \times V^{-}(a,b,c,i,j,k)$, *ex-post efficiency* requires that $\{a,b,c\}$ be assigned among $\{i,j,k\}$.

Step 1: Calibrating conditional priorities for triples of agents. Let $a, b, c \in A$ be a triple that is free at the top in the support of μ. By ordinal richness, there is at least one such triple. Let

$i, j, k \in N$, and $u_{-ijk} \in V^-(a,b,c,i,j,k)$. Again by ordinal richness, $V^+(a,b,c,i,j,k) \times u_{-ijk} \subseteq$ supp(μ). For each $u_{ijk} \in V^+(a,b,c,i,j,k)$, F assigns $\{a,b,c\}$ among $\{i,j,k\}$ at (u_{ijk}, u_{-ijk}), so $(V^+(a,b,c,i,j,k) \times u_{-ijk}$ corresponds to a three-agent problem with agents $\{i,j,k\}$ and objects $\{a,b,c\}$. By Lemma 5, there is an order $\prec (abc, u_{-ijk})$ over $\{i,j,k\}$ such that the restriction of F to $\{i,j,k\}$ corresponds to $F^{\prec(abc,u_{-ijk})}$ on $V^+(a,b,c,i,j,k) \times u_{-ijk}$.

Step 2: Comparing conditional priorities for a given triple of agents. Let $i,j,k \in N$, $a,b,c \in A$, and $x,y,z \in A$ be triples (possibly with $\{a,b,c\} \cap \{x,y,z\} \neq \emptyset$) for which $\prec (abc, u_{-ijk})$ and $\prec (xyz, u_{-ijk})$ are defined. Let $u_{-ijk} \in V^-(a,b,c,i,j,k)$, $u'_{-ijk} \in V^-(x,y,z,i,j,k)$, $\prec \equiv \prec (abc, u_{-ijk})$, and $\prec' \equiv \prec (xyz, u'_{-ijk})$. Label the agents so that $i \prec j \prec k$.

Substep 2.1: $i \prec' k$ and $j \prec' k$. By definition of $\prec$, $F_{ijk}(u^{acb}, u^{abc}, u^{abc}, u_{-ijk}) = (a,b,c)$ and $F_{ijk}(u^{bac}, u^{acb}, u^{abc}, u_{-ijk}) = (a,b,c)$. Now

$$u^{acb}(a) - u^{acb}(c) = s_1 - s_2 < s_1 - s_3 = u^{abc}(a) - u^{abc}(c).$$

Therefore, by Lemma 2, the assignment at $(u^{acb}, u^{abc}, u^{abc}, u_{-ijk})$ implies that agent i never prefers agent k's assignment to his own when both agents report the same type. Similarly, the assignment at $(u^{bac}, u^{acb}, u^{abc}, u_{-ijk})$ implies that agent j never prefers agent k's assignment to his own when both agents report the same type. Together, these conditions imply that $F_{ijk}(u^{xyz}, u^{xyz}, u^{xyz}, u'_{-ijk}) \in \{(x,y,z),(y,x,z)\}$. Then by definition of $\prec'$, $i \prec' k$ and $j \prec' k$.

Substep 2.2: $i \prec' j$. Suppose by way of contradiction that $j \prec' i \prec' k$. Then $F_{ijk}(u^{xyz}, u^{xzy}, u^{yxz}, u'_{-ijk}) = (y,x,z)$. Also, by definition of $\prec$, $F_{ijk}(u^{acb}, u^{abc}, u^{bac}, u_{-ijk}) = (a,b,c)$. We construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u^{xyz}, u^{xzy}, u^{yxz}, u'_{-ijk})$ and $p_2 \equiv \mu(u^{acb}, u^{abc}, u^{bac}, u_{-ijk})$ so $p_1 > 0$ and $p_2 > 0$. Let F' modify F by adjusting the assignments of $\{i,j,k\}$ under F so that for each $u \in (V^s)^N$,

$$F'_{ijk}(u) \equiv \begin{cases} (x,z,y) & \text{with probability } p_2 \text{ if } u = (u^{xyz}, u^{xzy}, u^{yxz}, u'_{-ijk}) \\ (c,a,b) & \text{with probability } p_1 \text{ if } u = (u^{acb}, u^{abc}, u^{bac}, u_{-ijk}) \end{cases}.$$

Comparing expected utilities,

$$\begin{aligned} U_i(F',\mu) - U_i(F,\mu) &= p_1p_2(s_1 - s_2) + p_2p_1(s_2 - s_1) = 0, \\ U_j(F',\mu) - U_j(F,\mu) &= p_1p_2(s_2 - s_1) + p_2p_1(s_1 - s_2) = 0, \text{ and} \\ U_k(F',\mu) - U_k(F,\mu) &= p_1p_2(s_1 - s_3) + p_2p_1(s_1 - s_3) > 0. \end{aligned}$$

Therefore, F' ex-ante Pareto dominates F at μ. Instead, $i \prec' j \prec' k$. Altogether, the orders are independent of the sets of objects and reports of other agents. For each triple $i,j,k \in N$, let $\prec (ijk)$ denote the common order.

Step 3: Comparing priorities across triples of agents. Let $i, j, k, l \in N$, $\prec \equiv \prec(ijk)$, $\prec' \equiv \prec(ijl)$, $\prec'' \equiv \prec(ikl)$, and $\prec''' \equiv \prec(jkl)$. Label the agents so that $i \prec j$ and suppose by way of contradiction that $j \prec' i$. Let $a, b, c \in A$ and $u_{-ijk} \in V^-(a, b, c, i, j, k)$. Also let $u_k = u_l$ so that $u_{-ijl} \in V^-(a, b, c, i, j, l)$.

Substep 3.1: $j \prec k$. Suppose by way of contradiction that $k \prec j$. Then $F_{ijk}(u^{abc}, u^{acb}, u^{cab}, u_{-ijk}) = (a, b, c)$. Also, since $j \prec' i$, $F_{ijl}(u^{abc}, u^{abc}, u^{cab}, u_{-ijl}) = (b, a, c)$. We construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u^{abc}, u^{acb}, u^{cab}, u_{-ijk})$ and $p_2 \equiv \mu(u^{abc}, u^{abc}, u^{cab}, u_{-ijl})$ so $p_1 > 0$ and $p_2 > 0$. Let F' modify F by adjusting the assignments of $\{i, j, k\}$ under F so that for each $u \in (V^s)^N$,

$$F'_{ijk}(u) \equiv \begin{cases} (b, a, c) & \text{with probability } p_2 \text{ if } u = (u^{abc}, u^{acb}, u^{cab}, u_{-ijk}) \\ (a, b, c) & \text{with probability } p_1 \text{ if } u = (u^{abc}, u^{abc}, u^{cab}, u_{-ijl}) \end{cases}.$$

Comparing expected utilities,

$$\begin{aligned} U_i(F', \mu) - U_i(F, \mu) &= p_1 p_2 (s_2 - s_1) + p_2 p_1 (s_1 - s_2) = 0 \text{ and} \\ U_j(F', \mu) - U_j(F, \mu) &= p_1 p_2 (s_1 - s_3) + p_2 p_1 (s_2 - s_1) > 0. \end{aligned}$$

Since the assignments of all other agents are unchanged, F' ex-ante Pareto dominates F at μ. Instead, $i \prec j \prec k$.

Substep 3.2: $i \prec' l$. Suppose by way of contradiction that $l \prec' i$. Then $F_{ijl}(u^{acb}, u^{abc}, u^{cab}, u_{-ijl}) = (b, a, c)$. Also, since $i \prec j$, $F_{ijk}(u^{abc}, u^{abc}, u^{cab}, u_{-ijk}) = (a, b, c)$. We construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u^{acb}, u^{abc}, u^{cab}, u_{-ijl})$ and $p_2 \equiv \mu(u^{abc}, u^{abc}, u^{cab}, u_{-ijk})$ so $p_1 > 0$ and $p_2 > 0$. Let F' modify F by adjusting the assignments of $\{i, j, k\}$ under F so that for each $u \in (V^s)^N$,

$$F'_{ijk}(u) \equiv \begin{cases} (a, b, c) & \text{with probability } p_2 \text{ if } u = (u^{acb}, u^{abc}, u^{cab}, u_{-ijl}) \\ (b, a, c) & \text{with probability } p_1 \text{ if } u = (u^{abc}, u^{abc}, u^{cab}, u_{-ijk}) \end{cases}.$$

Comparing expected utilities,

$$\begin{aligned} U_i(F', \mu) - U_i(F, \mu) &= p_1 p_2 (s_1 - s_3) + p_2 p_1 (s_2 - s_1) > 0 \text{ and} \\ U_j(F', \mu) - U_j(F, \mu) &= p_1 p_2 (s_2 - s_1) + p_2 p_1 (s_1 - s_2) = 0. \end{aligned}$$

Since the assignments of all other agents are unchanged, F' ex-ante Pareto dominates F at μ. Instead, $j \prec' i \prec' l$.

Substep 3.3: $i \prec j \prec k$ and $j \prec' i \prec' l$ are contradictory. By definition of $\prec$ and $\prec'$,

$F_{ijk}(u^{acb}, u^{abc}, u^{bac}, u_{-ijk}) = (a, b, c)$ and $F_{ijl}(u^{abc}, u^{acb}, u^{bac}, u_{-ijl}) = (b, a, c)$. We construct an ex-ante Pareto improvement. Let $p_1 \equiv \mu(u^{acb}, u^{abc}, u^{cab}, u_{-ijk})$ and $p_2 \equiv \mu(u^{abc}, u^{acb}, u^{bac}, u_{-ijl})$ so $p_1 > 0$ and $p_2 > 0$. Let F' modify F by adjusting the assignments of $\{i, j, k\}$ under F so that for each $u \in (V^s)^N$,

$$F'_{ijk}(u) \equiv \begin{cases} (c, b, a) & \text{with probability } p_2 \text{ if } u = (u^{acb}, u^{abc}, u^{cab}, u_{-ijk}) \\ (a, c, b) & \text{with probability } p_1 \text{ if } u = (u^{abc}, u^{acb}, u^{bac}, u_{-ijl}) \end{cases}.$$

Comparing expected utilities,

$$\begin{aligned} U_i(F', \mu) - U_i(F, \mu) &= p_1 p_2 (s_2 - s_1) + p_2 p_1 (s_1 - s_2) = 0, \\ U_j(F', \mu) - U_j(F, \mu) &= p_1 p_2 (s_1 - s_2) + p_2 p_1 (s_2 - s_1) = 0, \\ U_k(F', \mu) - U_k(F, \mu) &= p_1 p_2 (s_1 - s_3) > 0, \text{ and} \\ U_l(F', \mu) - U_l(F, \mu) &= p_2 p_1 (s_1 - s_3) > 0. \end{aligned}$$

Therefore, F' ex-ante Pareto dominates F at μ. Instead, $i \prec' j$. Altogether, agents i and j are ordered the same way in each order over a triple of agents that includes them both. Since this is true for each pair of agents, the orders over triples agree with a single order over N. Let $\prec$ denote the common order.

Step 4: Extending agent 1's priority to all profiles. Let $u \in \text{supp}(\mu)$ and $a \in A$ be such that $u_1(a) = s_1$ and suppose by way of contradiction that $F_1(u) = b \neq a$. Let $i \in N$ be such that $F_i(u) = a$. Let $c \in A \backslash \{a, b\}$, $j \in N \backslash \{1, i\}$, and $u'_{-1ij} \in V^-(a, b, c, i, j, k)$. Also let $u' \equiv (u^{abc}, u_{-i})$, $u'' \equiv (u^{abc}, u^{acb}, u^{acb}, u'_{-1ij})$, $p_1 \equiv \mu(u)$, $p_2 \equiv \mu(u')$, and $p_3 \equiv \mu(u'')$. By assumption, $p_1 > 0$, and by ordinal richness, $p_2 > 0$ and $p_3 > 0$.

By *strategy-proofness*, $F_i(u') = a$, and by Step 3, $F_1(u'') = a$. If $F_i(u'') = b$, then by Lemma 2, agent 1 never prefers agent i's assignment to his own when both agents report the same type. This contradicts $F_i(u^{abc}, u^{abc}, u_{-1i}) = a$, so instead $F_{1ij}(u^{abc}, u^{acb}, u^{acb}, u'_{-1ij}) = (a, c, b)$. We construct an ex-ante Pareto improvement by modifying F at u, u', and u''. We first exchange the assignments at u and u', adjusting probabilities so that all agents $N \backslash \{1, i, j\}$ are indifferent. Next, we reassign the objects received by $\{1, i, j\}$ at at u' and u''. Let F' modify F so that for each $\hat{u} \in (V^s)^N$,

$$F'(\hat{u}) \equiv \begin{cases} F(u') & \text{with probability } p_2 p_3 \text{ if } \hat{u} = u \\ F(u) & \text{with probability } p_1 p_3 \text{ if } \hat{u} = u' \\ F(\hat{u}) & \text{otherwise} \end{cases}.$$

Next let $\hat{F}$ further modify F' at u' and u'' so that

$$\hat{F}_{1i}(u') = (a,b) \text{ with probability } (p_1p_3)p_3$$
$$\hat{F}_{1ij}(u'') = (b,a,c) \text{ with probability } (p_1p_3)p_2$$

with all remaining assignments unchanged. This modification is feasible given the probabilistic assignments under F' at u' and u''. Comparing expected utilities,

$$U_1(\hat{F},\mu) - U_1(F,\mu) \geq p_2(p_1p_3)p_3 \cdot (s_1 - s_2) + p_3(p_1p_3)p_2 \cdot (s_2 - s_1) = 0,$$
$$U_i(\hat{F},\mu) - U_i(F,\mu) = p_2(p_1p_3)p_3 \cdot (s_2 - s_1) + p_3(p_1p_3)p_2 \cdot (s_2 - s_1) = 0, \text{ and}$$
$$U_j(\hat{F},\mu) - U_j(F,\mu) = p_3(p_1p_3)p_2 \cdot (s_2 - s_3) > 0.$$

For each $k \in N\backslash\{1,i,j\}$, agent k's assignments change only by reversing his assignments at u and u'. Since $\mu(u) = \mu(u')$ and $u_k = u'_k$, $U_k(\hat{F},\mu) = U_k(F,\mu)$. Therefore, $\hat{F}$ ex-ante Pareto dominates F at μ.

Instead, $F_1(u) = a$. Since this is true for each object, agent 1 always receives his most preferred object.

Step 5: Extending priorities to all agents. We first draw a general implication of the priority order.

Substep 5.1: Among agents reporting identical types, objects are assigned according to priority in $\prec$. Let $i,j \in N\backslash\{1\}$ be such that $i \prec j$, $a,b,c \in A$, and $u_{-1ij} \in V^-(a,b,c,i,j,k)$. By Step 3, $F_{1ij}(u^{cab}, u^{abc}, u^{acb}, u_{-1ij}) = (c,a,b)$. Then by Lemma 2, agent i never prefers agent j's assignment to his own when both agents report the same type. Therefore, whenever a subset of agents report the same type, F favors them according to their priority in $\prec$.

Substep 5.2: All objects are assigned according to priority in $\prec$. Suppose by way of contradiction that $F \neq F^{\prec}$ and let $i \in N$ be the agent with highest priority in $\prec$ whose assignments under F and $F^{\prec}$ differ. By Step 4, $i > 1$. Then there are $j \in N$ such that $i \prec j$, $u \in (V^s)^N$, and $a,b \in A$ such that $u_i(a) > u_i(b)$ and $F_{ij}(u) = (b,a)$. By *strategy-proofness*, $F_i(u^{abc}, u_{-i}) = b$ and $F_j(u^{abc}, u_{-j}) = a$. By Substep 5.1, since $i \prec j$, $F_j(u^{abc}, u^{abc}, u_{-ij}) \neq a$. By *strategy-proofness*, since $F_i(u^{abc}, u_{-j}) \neq a$, $F_i(u^{abc}, u^{abc}, u_{-ij}) \neq a$.

Instead, there is $k \in N\backslash\{i,j\}$ such that $F_k(u^{abc}, u^{abc}, u_{-ij}) = a$. By *strategy-proofness*, $F_k(u^{abc}, u^{abc}, u^{abc}, u_{-ijk}) = a$. Then by Substep 5.1, $k \prec i$. Now $F_j(u^{abc}, u_{-j}) = a$, so $F_k(u^{abc}, u_{-j}) \neq a$. In particular, $F_k(u^{abc}, u_{-j}) \neq F_k(u^{abc}, u^{abc}, u_{-ij})$. Since the only agent whose type changes between profiles has lower priority than agent k, agent k's assignments under F and $F^{\prec}$ also differ. This contradicts the choice of agent i as the highest priority agent whose assignments differ. Instead, $F = F^{\prec}$. □

Lemma 8. *Let $|A| \geq 3$. For each ordinally rich problem $\mu \in \mathcal{D}^*$, a rule is* ex-ante efficient *and* strategy-proof *at μ if and only if it is a priority rule.*

Proof. Let $\mu \in \mathcal{D}^*$ be ordinally rich with corresponding ordinally rich set $V \subseteq \mathbb{R}^A$ and let F be *ex-ante efficient* and *strategy-proof* at μ. Let $a, b, c \in A$ be a triple which is free at the top in V. We show that the arguments from the proof of Lemma 7 continue to apply when V is a subset of a scoring domain. The arguments from the proof of Lemma 6 then extend the result to the general case.

Following Step 1, we calibrate conditional priorities for each triple of agents with respect to $\{a, b, c\}$. This is possible because each object appears at the top for at least one type. For each triple that is free at the top, Steps 2 and 3 continue to imply that the conditional priorities calibrated with respect to these objects coincide. Let $\prec \in \Pi^N$ be this common order. By Lemma 2, among agents reporting identical types, objects are assigned according to priority in $\prec$.

To see that agent 1 always receives his most preferred object (Step 4), suppose by way of contradiction that there is $u \in V^N$ and $a', b' \in A$ such that $u_1 = u^{a'}$ and $F_1(u) = b'$. Then there is $i \in N \backslash \{1\}$ such that $F_i(u) = a'$. By *strategy-proofness*, $F_i(u^{a'}, u_{-i}) = a'$. But then $u_1 = u_i$ and agent i is favored over agent 1 when reporting identical types, a contradiction. Instead, agent 1 always receives his most preferred object.

Finally, we show that all assignments are made according to $\prec$ (Step 5). Suppose by way of contradiction that $F \neq F^{\prec}$ and let $i \in N$ be the agent with highest priority in $\prec$ whose assignments under F and $F^{\prec}$ differ.

Then $i > 1$ and there are $j \in N$ such that $i \prec j$, $u \in V^N$, and $a', b' \in A$ such that $u_i(a') > u_i(b')$ and $F_{ij}(u) = (b', a')$. By choice of agent i, all agents with higher priority under $\prec$ than agent i receive objects they prefer to a'. By assumption, there is $u^{a'} \in V$ with a' at the top. By *strategy-proofness*, $F_i(u^{a'}, u_{-i}) \neq a'$. Then there is $j' \in N$ (possibly $j' = j$) such that $F_{j'}(u^{a'}, u_{-i}) = a'$. By *strategy-proofness*, $F_{j'}(u^{a'}, u^{a'}, u_{-ij'}) = a'$. Again by choice of agent i, all agents with higher priority under $\prec$ receive the same objects at u and $(u^{a'}, u_{-i})$ so $i \prec j'$. But agent j' is favored over agent i at $(u^{a'}, u^{a'}, u_{-ij'})$ when reporting identical types, a contradiction. Instead, $F = F^{\prec}$. □

Proposition 3 *On each scoring domain, there exists an open set of problems including the uniform measure on which* ex-ante symmetry*,* utilitarian optimality*, and* BIC *are compatible.*

Proof. We construct a rule which is *ex-post symmetric* and *utilitarian-optimal* for each problem on a scoring domain. At the uniform problem, it is also *ex-ante symmetric*. Moreover, when agents know only their own types, each agent has a strict incentive to report truthfully. Therefore, the rule satisfies *BIC* on an open set of problems including the uniform measure. Also, rules formed from the original rule by slightly perturbing the ex-post randomization probabilities are also *BIC* on an open set of problems including the uniform measure. By continuity of expected utilities, for each problem in an open set including the uniform measure, this collection of perturbed rules includes rules which are also *ex-ante symmetric* at the chosen problem. Let $s \in S$ and $\mu \in \mathcal{D}(V^s)$ be the uniform measure. Let F be the rule which randomizes uniformly over the set of utilitarian allocations at each profile. By definition, F is *utilitarian-optimal* and *ex-post symmetric* at μ. Since μ is uniform over a common type space, *ex-post symmetry* implies that the expected utility calculations are the same for all agents. Thus, F is *ex-ante symmetric* at μ.

It remains to show that each agent has a strict interim incentive to report truthfully. Because μ is uniform, for each pair $i, j \in N$ and each $u_0 \in V^s$, $x_i(u_0) = x_j(u_0)$ and for each $a \in A$, $x_i(u_0, a) > 0$. Moreover, permuting the ranking of u_0 permutes the values of $x_i(u_0)$ correspondingly. Then by *utilitarian-optimal*, for each $i \in N$ and each pair $u_0, u_0' \in V^s$, $x_i(u_0)$ strictly first-order stochastically dominates $x_i(u_0')$ from the perspective of u_0. Therefore, $U_i(u_0, u_0, F) > U_i(u_0', u_0, F)$. □

B Proofs from Section 4

B.1 Symmetric problems

B.1.1 Equity comparisons and the Lorenz order

Proposition 4 *For each symmetric problem, all* ex-post efficient*,* strategy-proof*, and* non-bossy *rules achieve the same utilitarian welfare.*

Proof. Let $\mu_{G,H}$ be a symmetric problem and let F be a rule satisfying the axioms. We argue that the utilitarian welfare under F is the same as under a priority rule. To do so, we consider randomizations over all priority orders and roles under F. For each $\pi \in \Pi^N$, let $\pi(F)$ be the rule formed by permuting the roles of the agents in F. For each $P_0 \in \mathcal{P}$ also let $\pi(P_0)$ be the ordinal ranking formed by permuting the objects in P_0.

Let $P_0 \in \mathcal{P}$, and label the objects so that $a_1 \; P_0 \; a_2 \; P_0 \; \cdots \; P_0 \; a_n$. Let $i \in N$ and for each $k = 1, \ldots, n$, let $s_k \equiv \mathbf{E}_{\mu_{G,H}}[a_k | v_i \in \mathcal{U}(P_0)]$. By independence, $(s_k)_{k=1}^n$ depends on agent i's type. By symmetry across objects, for each $\pi \in \Pi^N$, $Pr(v_i \in \mathcal{U}(\pi(P_0))) = \frac{1}{n!}$ and $\pi(s_k) = \mathbf{E}_{\mu_{G,H}}[a_k | v_i \in \mathcal{U}(\pi(P_0))]$. That is, each ordinal ranking occurs with the same probability and the expected values of objects depend only on their ordinal ranks. Finally, by symmetry across agents, these values are the same for all agents.

Now consider the sum of utilitarian welfare over all permutations of F, $\sum_{\pi \in \Pi^N} \sum_N U_k(\pi(F), \mu_{G,H})$. By symmetry across agents, for each $P \in \mathcal{P}^N$, P and $(P_{\pi(i)})_{i \in N}$ are equally likely. Therefore, for each $i \in N$,

$$\sum_{\pi \in \Pi^N} U_i(\pi(F), \mu_{G,H}) = \frac{1}{n} \sum_{\pi \in \Pi^N} \sum_N U_k(\pi(F), \mu_{G,H}).$$

Similarly by symmetry across agents and objects, for each $\pi \in \Pi^N$,

$$\sum_N U_k(\pi(F), \mu_{G,H}) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \sum_N U_k(\pi(F, \mu_{G,H})).$$

That is, all rules formed by permuting the roles of agents in F achieve the same utilitarian welfare. It remains to relate these to the priority rules.

We argue that $\sum_{\pi \in \Pi^N} \sum_N U_k(\pi(F), \mu_{G,H}) = \sum_{\pi \in \Pi^N} \sum_N U_k(F^{\prec}, \mu_{G,H})$. Now for each $\pi \in \Pi^N$, $\pi(F)$ and $F^{\prec}$ are ordinal and the expected values of objects depend only on their ordinal ranks. Therefore, it suffices to show that for each $P \in \mathcal{P}^N$, allocations occur with the same frequency within either family. As shown by Lee and Sethuraman (2011); Carroll (2014); Bade (2014c), for each $P \in \mathcal{P}^N$, $i \in N$, and $a \in A$,

$$\left|\pi \in \Pi^N : \pi(F)_i(P) = a\right| = \left| \prec \in \Pi^N : F_i^{\prec}(P) = a\right|.$$

That is, the relative frequencies of assignments are the same when randomizing over roles in F as when randomizing over roles in $F^{\prec}$. Therefore, $\sum_{\pi \in \Pi^N} \sum_N U_k(\pi(F), \mu_{G,H}) = \sum_{\pi \in \Pi^N} \sum_N U_k(F^{\prec}, \mu_{G,H})$. Since all rules in the same family achieve the same utilitarian welfare, this implies $\sum_N U_k(F, \mu_{G,H}) = \sum_N U_k(F^{\prec}, \mu_{G,H})$. □

Corollary 7 *Let μ_G be an independent symmetric problem. (i) Under each priority rule, the expected utility of the agent with kth highest priority is $\nu_G^{(n+1-k,n+1-k)}$. (ii) Under each TTC rule, each agent's expected utility is $\sum_{k=1}^n \frac{1}{n} \nu_G^{(k,k)}$.*

Proof. Let $\mu_{G,H}$ be an independent symmetric problem, $\prec \in \Pi^N$, and $\omega \in X$.

(i) First consider agent 1, the agent with highest priority under $\prec$. For each realization u, agent 1 receives his most preferred among the n objects and obtains utility $\max\{u_1(a) : a \in A\}$. Ex-ante, these values are independent random variables distributed according to G. Therefore, agent 1's expected utility is $\nu_G^{(n,n)}$.

(ii) Because the objects are symmetric, each agent obtains the same expected utility regardless of his endowment. Therefore, for each $i \in N$, $U_i(F^{\omega}, \mu_G) = \frac{1}{n} \sum_N U_k(F^{\omega}, \mu_G)$. By Proposition 4, $\sum_N U_k(F^{\omega}, \mu_G) = \sum_N U_k(F^{\prec}, \mu_G)$, and by (i), $\sum_N U_k(F^{\prec}, \mu_G) = \sum_{k=1}^n \nu_G^{(k,k)}$. Combining, for each $i \in N$, $U_i(F^{\omega}, \mu_G) = \frac{1}{n} \sum_{k=1}^n \nu_G^{(k,k)}$. □

Proposition 5 *For each symmetric problem, among* ex-post efficient*,* strategy-proof*, and* non-bossy *rules, (i) all TTC rules Lorenz dominate all non-TTC rules and (ii) all priority rules are Lorenz dominated by all non-priority rules.*

Proof. Let μ_G be an independent symmetric problem and F be a rule satisfying the axioms.

(i) Let $y \equiv U(F, \mu)$. Let $\omega \in X$ and $\hat{y} \equiv U(F^\omega, \mu)$. Then for each $i = 1, \ldots, n$, $\hat{y}_i = \frac{1}{n}\sum_{k=1}^n \nu^{(k,n)}$. By Proposition 4, $\sum_N y_i = \sum_N \hat{y}_i$, so $\sum_{i=1}^n \tilde{y}^{(i,n)} = \sum_{i=1}^n y^{(i,n)}$. Therefore, $\hat{y}$ Lorenz dominates y.

(iii) Let $\prec \in \Pi^N$ and $\tilde{y} \equiv U(F^\prec, \mu)$. Labeling the agents so that $1 \prec 2 \prec \cdots \prec n$, for each $i = 1, \ldots, n$, $\tilde{y}^{(n-i+1,n)} = \tilde{y}_i = \nu^{(n-i+1,n)}$. By construction, for each realization of preferences, agent 1 receives his most preferred object under $F^\prec$. By symmetry across agents, this implies $\tilde{y}^{(n,n)} \geq y^{(n,n)}$. Similarly, conditional on agent 1 receiving his most preferred object, agent 2 receives his most preferred object among those which remain under $F^\prec$. Again by symmetry across agents, this implies $\tilde{y}^{(n,n)} + \tilde{y}^{(n-1,n)} \geq y^{(n,n)} + y^{(n-1,n)}$. Continuing in this fashion, for each $k = 1, \ldots, n$, $\sum_{i=n-k+1}^n \tilde{y}^{(i,n)} \geq \sum_{i=n-k+1}^n y^{(i,n)}$. By Proposition 4, $\sum_N y_i = \sum_N \tilde{y}_i$, so $\sum_{i=1}^n \tilde{y}^{(i,n)} = \sum_{i=1}^n y^{(i,n)}$. The previous inequalities then imply that for each $k = 1, \ldots, n$, $\sum_{i=1}^n \tilde{y}^{(i,n)} \leq \sum_{i=1}^n y^{(i,n)}$. Therefore, y Lorenz dominates $\tilde{y}$. □

B.1.2 Comparing social welfare across problems

We begin with a simple fact about the limits of order statistics.

Lemma 9. *Let G be a cdf with bounded support. Then $\lim_{n\to\infty} \nu^{(n,n)} = \tau_G$.*

Proof. Let G be a cdf with bounded support. Let $\varepsilon \in \mathbb{R}_{++}$ and $p_\varepsilon \equiv G(\tau_G - \frac{\varepsilon}{2})$ so $p_\varepsilon < 1$. There are $M, N \in \mathbb{N}$ such that $-M < \inf \operatorname{supp}(G)$ and for each $n \in \mathbb{N}$ with $n \geq N$, $(p_\varepsilon)^n < \frac{\varepsilon}{2}(\tau_G + \frac{\varepsilon}{2} + M)$. Then for $n \geq N$,

$$\begin{aligned}
\nu^{(n,n)} &= \mathbf{E}_G\big[\max\{x_1, \ldots, x_n\}\big] \\
&> (p_\varepsilon)^n \cdot (-M) + \big(1 - (p_\varepsilon)^n\big)\big(\tau_G - \tfrac{\varepsilon}{2}\big) \\
&= (p_\varepsilon)^n\big(\tau_G - \tfrac{\varepsilon}{2} - M\big) + \big(\tau_G - \tfrac{\varepsilon}{2}\big) \\
&\geq -\tfrac{\varepsilon}{2} + \tau_G - \tfrac{\varepsilon}{2} \\
&= \tau_G + \varepsilon.
\end{aligned}$$

By the definition of τ_G, $\nu^{(n,n)} \leq \tau_G$, so $\lim_{n\to\infty} \nu^{(n,n)} = \tau_G$. □

Proposition 6 *For each sequence of symmetric problems, the average expected utility under each* ex-post efficient*,* strategy-proof*, and* non-bossy *rule approaches the maximum average surplus.*

Proof. Let $\big(\mu_{G,H}^{(n)}\big)_{n=1}^\infty$ be a sequence of symmetric problems.

Case 1: $\operatorname{supp}(H) = \{0\}$. Then each $\mu_{G,H}^{(n)}$ is an independent symmetric problem and the formulas for $U^*\big(F^\prec, \mu_{G,H}^{(n)}\big)$ and $U^*\big(F^\omega, \mu_{G,H}^{(n)}\big)$ computed in Corollary 7 apply. By Lemma 9, $\lim_{n\to\infty} \nu^{(n,n)} = \tau_G$. Therefore, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \nu^{(k,k)} = \tau_G$ as well, so $\lim_{n\to\infty} \frac{1}{n} U^*\big(F^\prec, \mu_{G,H}^{(n)}\big) = \lim_{n\to\infty} \frac{1}{n} U^*\big(F^\omega, \mu_{G,H}^{(n)}\big) = \tau_G$.

Case 2: *H* is discrete. For each $v_0 \in \text{supp}(H)$, let $p_0 \equiv H(v_0) - \sup\{H(v) : v < v_0\}$ and $K_0(n) \equiv \{a \in A^{(n)} : v_a = v_0\}$. Since $p_0 > 0$, $\lim_{n\to\infty} \mathbf{E}_H[\,|K_0(n)|\,] = \infty$.

To establish a lower bound on the common utilitarian welfare $U^*(F^{\prec}, \mu^{(n)}_{G,H}) = U^*(F^{\omega}, \mu^{(n)}_{G,H})$, consider a restricted TTC rule. Beginning from arbitrary endowments, for each $v_0 \in \text{supp}(H)$, apply the corresponding TTC rule to reassign objects $K_0(n)$ among those agents endowed with these objects. Trades across these sets permitted by the unrestricted TTC rule reflect private values which are large in expectation. Therefore, the unrestricted TTC rule achieves expected welfare at least as large as the restricted TTC rule.

By the previous computation, $\lim_{n\to\infty} \mathbf{E}_H[\,|K_0(n)|\,] = \infty$, so the conclusion from Case 1 applies individually to each subproblem. Therefore, $\lim_{n\to\infty} \frac{1}{n}\, U^*(F^{\prec}, \mu^{(n)}_{G,H}) = \lim_{n\to\infty} \frac{1}{n}\, U^*(F^{\omega}, \mu^{(n)}_{G,H}) = \tau_G$.

Case 3: *H* has general form. We approximate H by a discrete distribution. Let $M, K \in \mathbb{N}$ be such that $\text{supp}(H) \subseteq [-M, M]$, $\varepsilon \in \mathbb{R}_{++}$, and $K \geq \frac{2M}{\varepsilon}$. Let $s^0 \equiv -M$ and for each $k \in \mathbb{N}$, let $s^k \equiv -M + k \cdot \frac{\varepsilon}{M}$ and $S^k \equiv [s^{k-1}, s^k)$. Then $\text{supp}(H) \subseteq \bigcup_{k=0}^{K} S^k$. Since H has at most a finite number of atoms, we may suppose that ε is sufficiently small that no s^k is an atom. Finally, let $p_k \equiv H(s^k) - H(s^{k-1})$ and $S^k(n) \equiv \{a \in A^{(n)} : v_a \in S^k\}$.

Let $k \in \{0, \ldots, K\}$. If $p_k = 0$, then $\mathbf{E}_H[\,|S^k(n)|\,] = 0$ and S^k has no effect on expected utilities, so suppose instead that $p_k > 0$. Then $\lim_{n\to\infty} \mathbf{E}_H[\,|S^k(n)|\,] = \infty$. To establish a lower bound on the common utilitarian welfare $U^*(F^{\prec}, \mu^{(n)}_{G,H}) = U^*(F^{\omega}, \mu^{(n)}_{G,H})$, consider again a restricted TTC rule. Beginning from arbitrary endowments, for each $S^k(n)$ such that $S^k(n) \cap \text{supp}(H) \neq \emptyset$, apply the corresponding TTC rule to reassign objects $S^k(n)$ among those agents endowed with these objects. Trades across these sets permitted by the unrestricted TTC rule reflect private values which are large in expectation. Therefore, the unrestricted TTC rule achieves expected welfare at least as large as the restricted TTC rule.

First suppose that all common value realizations in S^k are in fact s^{k-1}. Under this assumption, since $\lim_{n\to\infty} \mathbf{E}_H[\,|K_0(n)|\,] = \infty$, the arguments from Case 1 apply: The average utilitarian welfare in each $S^k(n)$ approaches τ_G. Compared to this approximation, the expected values of the private values of the assigned objects under the restricted TTC rule differ by at most $s^k - s^{k-1} = \varepsilon$. Therefore, the average utilitarian welfare under the restricted TTC rule in the subproblem defined by $S^k(n)$ approaches $\tau_G - \varepsilon$. Since this is true for each $\varepsilon \in \mathbb{R}_{++}$, the average utilitarian welfare under the restricted TTC rule in each subproblem also approaches τ_G. Therefore, $\lim_{n\to\infty} \frac{1}{n}\, U^*(F^{\prec}, \mu^{(n)}_{G,H}) = \lim_{n\to\infty} \frac{1}{n}\, U^*(F^{\omega}, \mu^{(n)}_{G,H}) = \tau_G$. □

Remark 6 *If the distribution of common values in a sequence of symmetric problems is improving or deteriorating, then the average expected utility under each* ex-post efficient*,* strategy-proof*, and* non-bossy *rule is bounded away from the maximum average surplus.*

Proof. We consider a deteriorating sequence of common value distributions; the argument for an improving sequence is similar. We argue that average expected utility under $F^{\prec}$ is bounded away from the maximum average surplus under $F^{\prec}$. Again, the argument for other *ex-post efficient*, *strategy-proof*, and *non-bossy* rules is similar.

Let $\mu_{G,H}$ be a symmetric problem. Let $\varepsilon, L \in \mathbb{R}_{++}$ and for each $k \in \mathbb{N}$, let H^k be such that $\text{supp}(H^k \subseteq [-L - k\varepsilon, L + k\varepsilon]$. Also let $K, M \in \mathbb{R}_{++}$ be such that $\text{supp}(G) \subseteq [-M, M]$ and $K \geq \frac{2(L+M)}{\varepsilon}$. Then for each pair $k, k' \in \mathbb{N}$ with $k - k' \geq K$, the value of v_k is greater than the value of $v_{k'}$ for all realizations of common and private values.

Under $F^{\prec}$, an agent with kth highest priority always receives an object among $\{a_l : k - K \leq l \leq k_K\}$. The expected value of the private value component of this agent's assignment is then at most $\nu^{(2K,2K)}$. The average expected utility under each priority rule is therefore bounded above by $\nu^{(2K,2K)}$ for all populations. Consequently, the maximum average surplus is not reached even in the limit. □

Proposition 7 *For each symmetric problem $\mu_{G,H}$ and $\delta \in \mathbb{R}_{++}$,*

1. *The utilitarian surplus at $\mu_{\delta \cdot G,H}$ is increasing in δ and the utilitarian surplus at $\mu_{G,\delta \cdot H}$ is constant in δ.*
2. *The fraction of the utilitarian surplus at $\mu_{G,\delta \cdot H}$ realized by each* ex-post efficient*,* strategy-proof*, and* non-bossy *rule is non-inceasing in δ.*

Proof. Let $(\mu_{G,H}$ be a symmetric problem and $\delta \in \mathbb{R}_{++}$ with $\delta > 1$.

(1) We compare the utilitarian surplus at paired profiles. Let $v \in \mathbb{R}^A$ be a vector of common values and $\varepsilon \in \mathbb{R}^{A \times N}$ be a vector of private values. Let $u \equiv v + \varepsilon$ and $\hat{u} \equiv v + \delta \cdot \varepsilon$. The utilitarian surplus at $\hat{u}$ is equal to δ times the utilitarian surplus at u. Since this is true for each $u \in \mathbb{R}^{A \times N}$ and these profiles are paired under $\mu_{G,H}$ and $\mu_{\delta \cdot G,H}$, the utilitarian surplus at $\mu_{\delta \cdot G,H}$ is equal to δ times the utilitarian surplus at $\mu_{G,H}$. Since the utilitarian surplus at $\mu_{G,H}$ is positive, the utilitarian surplus at $\mu_{\delta \cdot G,H}$ is greater.

From the definition, the utilitarian surplus depends only on the private values and is independent of the common values. Therefore, the utilitarian surplus at $\mu_{G,\delta \cdot H}$ is the same as the utilitarian surplus at $\mu_{G,H}$.

(2) We verify the result for priority rules. The conclusion extends to TTC rules and other *ex-post efficient*, *strategy-proof*, and *non-bossy* rules by Proposition 4. We prove a slightly stronger statement comparing the utilitarian surplus at paired profiles with fixed common values which differ only by the common value assigned to the object with the highest common value. Let $v \in \mathbb{R}^A$ be a vector of common values. Abusing notations slightly, we write $\mu_{G,v}$ for the problem in which H is replaced by a distribution which assigns probability 1 to v.

Case 1: $|A| = 2$. Then $v = (v_a, v_b)$. Label the objects so that $v_a \geq v_b$ and let $hatv \equiv (v_b, v_b)$. Now compare the utilitarian welfare at $\mu_{G,v}$ and $\mu_{G,\hat{v}}$. If $v_a = v_b$, the problems are identical, so suppose instead that $v_a > v_b$ ad let $\varepsilon_1 \in \mathbb{R}^A$. When common values are $\hat{v}$, agent 1 receives a under $F^{\prec}$ if $\varepsilon_{1a} < \varepsilon_{1b}$ and b if $\varepsilon_{1a} > \varepsilon_{1b}$. When common values are v, agent 1 receives a under $F^{\prec}$ if $\varepsilon_{1a} + v_a - v_b > \varepsilon_{1b}$ and b if $\varepsilon_{1a} + v_a - v_b > \varepsilon_{1b}$. In particular, private value of agent 1's assignment is when common values are v. Since this is true for each $\varepsilon_1 \in \mathbb{R}^A$, the expected value of agent 1's private value at his assignment is smaller when common values are v. Since agent 2 receives the remaining object in each case, the expected value of agent 2's private value at his assignment is the same for either set of common values. Overall, the sum of expected sum of the private values is smaller when common values are v. Therefore, the surplus under $F^{\prec}$ is smaller at $\mu_{G,v}$ than at $\mu_{G,\hat{v}}$.

Case 2: $|A| > 2$. We argue by induction. For the induction hypothesis, we suppose that for each vector of common values in each problem with $|N| - 1$ agents, increasing the common value of one of the objects with the highest common value decreases the surplus under $F^{\prec}$.

Label the objects so that $v_{a_1} \geq v_{a_2} \geq \cdots \geq v_{a_n}$ and let $hatv \equiv (v_{a_2}, v_{a_2}, v_{a_3}, \ldots, v_{a_n})$. Now compare the utilitarian welfare at $\mu_{G,v}$ and $\mu_{G,\hat{v}}$. If $v_a = v_b$, the problems are identical, so suppose instead that $v_a > v_b$ ad let $\varepsilon_1 \in \mathbb{R}^A$. If the difference in common values changes agent 1's assignment under $F^{\prec}$, then agent 1 receives a_1. In particular, private value of agent 1's assignment is when common values are v. Since this is true for each $\varepsilon_1 \in \mathbb{R}^A$, the expected value of agent 1's private value at his assignment is smaller when common values are v.

In those cases in which agent 1 receives a_1, the assignments of the remaining agents are the same when common values are v or $\hat{v}$ and so the expected values of their private values are the same as well. In those cases in which agent 1 does not receive a_1, he receives the same object when common values are v or $\hat{v}$. Then by the induction hypothesis, the surplus under $F^{\prec}$ obtained by the remaining agents in the reduced problem is smaller when common values are v. Altogether, the surplus under $F^{\prec}$ in the original problem is smaller when common values are v and the surplus under $F^{\prec}$ is smaller at $\mu_{G,v}$ than at $\mu_{G,\hat{v}}$.

Returning to $\mu_{G,\delta\cdot H}$, for each $v \in \mathbb{R}^A$, we obtain $\delta \cdot v$ from v by a sequence of transformations which increase the common values some objects by the same amount. For example, to move from $(0,0,0,0)$ to (v_a, v_b, v_c, v_d), we consider in turn $(0,0,0,0)$, $(v_a - v_b, 0, 0, 0)$, $(v_a - v_c, v_b - v_c, 0, 0)$, $(v_a - v_d, v_b - v_d, v_c - v_d, 0)$, and (v_a, v_b, v_c, v_d). By the previous arguments, each step decreases the surplus under $F^{\prec}$. Therefore, the surplus under $F^{\prec}$ is decreasing in δ. □

B.1.3 Population changes

Proposition 8 *Let $N \equiv N_1 \cup N_2$ and $A \equiv A_1 \cup A_2$ be partitions of N and A such that $|N_1| = |A_1|$ and $|N_2| = |A_2$. Also let $\mu_{G,H}$ be a symmetric problem for N, $\prec \in \Pi^N$, and $\omega \in X$ with $\{\omega_i : i \in N_1\} = A_1$ and $\{\omega_i : i \in N_2\} = A_2$. Then for each $i \in N_1$*

1. *There is $u \in supp(\mu_{G,H})$ such that $u_i(F_i^{\prec}|_{N_1}(u|_{N_1})) < u_i(F_i^{\prec}(u))$ and $u_i(F_i^{\omega}|_{N_1}(u|_{N_1})) <$*

$u_i(F_i^\omega(u))$.

2. *There is $u \in supp(\mu_{G,H})$ such that $u_i(F_i^\prec|_{N_1}(u|_{N_1})) > u_i(F_i^\prec(u))$ if and only if there is $j \in N_2$ such that $j \prec i$; and there is $u \in supp(\mu_{G,H})$ such that $u_i(F_i^\omega|_{N_1}(u|_{N_1})) > u_i(F_i^\omega(u))$ if and only if $|N_1| > 1$.*

3. *If for each $j \in N_2$, $i \prec j$, then $U_i(F^\prec, \mu) > U_i(F^\prec|_{N_1}, \mu|_{N_1})$; and $U_i(F^\omega, \mu) > U_i(F^\omega|_{N_1}, \mu|_{N_1})$.*

Proof. Let N, N_1, N_2, A, A_1, A_2, $\mu_{G,H}$, $\prec$, and ω be as specified in the statement. Let $i \in N_1$.

(i) By definition of $\mu_{G,H}$, there are $b \in A_2$ and $u \in \text{supp}(\mu_{G,H})$ such that for each $j \in N\backslash\{i\}$ and each $b' \in A\backslash\{b\}$, $u_i(b) > u_i(b')$ and $u_j(b) < u_j(b)$. Then $F_i^\prec(u) = F_i^\omega(u) = b$. Since $b \notin A_1$, $F^\prec|_{N_1}u(|_{N_1}) \neq b$ and $F^\omega|_{N_1}u(|_{N_1}) \neq b$. Therefore, $u_i(F_i^\prec|_{N_1}(u|_{N_1})) < u_i(F_i^\prec(u))$ and $u_i(F_i^\omega|_{N_1}(u|_{N_1})) < u_i(F_i^\omega(u))$.

(ii) **Priority rule**. If for each $j \in N_2$, $i \prec j$, then agent i's assignment under $F^\prec|_{N_1}$ is always available when his priority is reached under $F^\prec$ and so he is at least as well off under $F^\prec$. Suppose instead that there is $j \in N_2$ such that $j \prec i$. By definition of $\mu_{G,H}$, there are $b \in A_1$ and $u \in \text{supp}(\mu_{G,H})$ such that for each $k \in N\backslash\{i,j\}$ and each $b' \in A\backslash\{b\}$, $u_i(b) > u_i(b')$, $u_j(b) > u_j(b')$, and $u_k(b) < u_k(b)$. Then $F_i^\prec|_{N_1}(u|_{N_1}) = F_j^\prec(u) = b \neq F_i^\prec(u)$. Therefore, $u_i(F_i^\prec|_{N_1}(u|_{N_1})) > u_i(F_i^\prec(u))$.

TTC rule. If $|N_1| = 1$, then agent i always receives a_i under $F^\omega|_{N_1}$. Under F^ω, agent i always receives a_i or a more preferred object. Therefore, for each $u \in \text{supp}(\mu_{G,H})$, $u_i(F_i^\omega|_{N_1}(u|_{N_1})) \leq u_i(F_i^\omega(u))$. Suppose instead that $|N_1| > 1$. Let $j \in N_1\backslash\{i\}$ and $k \in N_2$. By definition of $\mu_{G,H}$, there is $u \in \text{supp}(\mu_{G,H})$ such that for each $l \in N\backslash\{i,j,k\}$ and each $b \in A\backslash\{\omega_i, \omega_j, \omega_k\}$, $u_i(\omega_j) > u_i(\omega_k) > u_i(\omega_i) > u_i(b)$, $u_j(\omega_k) > u_j(\omega_i) > u_j(\omega_j) > u_j(b)$, and $u_k(\omega_j) > u_k(\omega_k) > u_k(\omega_i) > u_k(b)$. Then $F_i^\omega|_{N_1}(u|_{N_1}) = F_k^\omega(u) = \omega_j \neq F_i^\omega(u)$. Therefore, $u_i(F_i^\omega|_{N_1}(u|_{N_1})) > u_i(F_i^\omega(u))$.

(iii) **Priority rule**. Suppose that for each $j \in N_2$. By (2) agent i is at least as well off under $F^\prec$ as under $F^\prec|_{N_1}$ at each profile. By (1) and the definition of $\mu_{G,H}$, agent i is better of at a positive measure of profiles. Therefore, $U_i(F^\prec, \mu) > U_i(F^\prec|_{N_1}, \mu|_{N_1})$.

TTC rule. The expected utilitarian welfare in the combined problem is higher than the sum of the expected utilitarian welfares in the reduced problems. Since each agent obtains the same expected utility, this implies that $U_i(F^\omega, \mu) > U_i(F^\omega|_{N_1}, \mu|_{N_1})$. □

B.2 Extensions of the symmetric domain

Proposition 9 *For each priority rule and each TTC rule,* *(i) For each independent symmetric problem, a majority of agents obtain higher expected utility under the priority rule than under the TTC.*

(ii) There exist symmetric problems at which an arbitrarily large or arbitrarily small proportion of the agents obtain higher expected utility under the priority rule than under the TTC rule.

(iii) For each symmetric problem, all agents obtain higher expected utility under the TTC rule than under preference-independent allocation.

(iv) There exist symmetric problems at which nearly one-half of the agents prefer preference-independent allocation to the priority rule.

Proof. (i) Let μ_G be an independent symmetric problem, $\prec \in \Pi^N$, and $\omega \in X$. If $n = 1$, then $F^\prec$ and F^ω are equivalent and if $n = 2$, then $\big|\{i \in N : U_i(F^\prec, \mu_G) \geq U_i(F^\omega, \mu_G)\}\big| = \big|\{1\}\big| = 1 = \frac{n}{2}$. Suppose now that $n \geq 3$.

By Corollary 7, for each $k \in N$, $U_k(F^\prec, \mu_G) = \nu_G^{(n+1-k,n+1-k)}$ and $U_k(F^\omega, \mu_G) = \sum_{i=1}^n \frac{1}{n} \nu_G^{(i,i)}$. By the definition of order statistics, the sequence $\{\nu_G^{(k,k)}\}_{k\in\mathbb{N}}$ is increasing and bounded with strictly decreasing differences: For each $k \in \mathbb{N}$, $0 < \nu_G^{(k+2,k+2)} - \nu_G^{(k+1,k+1)} < \nu_G^{(k+1,k+1)} - \nu_G^{(k,k)}$. Therefore, $\nu_G^{(2,2)} > \frac{1}{3}\big(\nu_G^{(1,1)} + \nu_G^{(2,2)} + \nu_G^{(3,3)}\big)$, so if $n = 3$, $\big|\{i \in N : U_i(F^\prec, \mu_G) \geq U_i(F^\omega, \mu_G)\}\big| = \big|\{1, 2\}\big| = 2 > \frac{n}{2}$. More generally, for each $k \in \mathbb{N}$ with $k \geq \frac{n}{2}$, $\nu_G^{(k,k)} > \frac{1}{n}\sum_{i=1}^n \nu_G^{(i,i)}$. Therefore,

$$\big|\{i \in N : U_i(F^\prec, \mu_G) \geq U_i(F^\omega, \mu_G)\}\big| \geq \big|\{1, \ldots, k\}\big| > \frac{n}{2}.$$

(ii) We provide a one parameter family of examples in which the proportion of agents whose expected utility is higher under one rule can be made arbitrarily close to either 0 or 1. Let $M \in \mathbb{N}$, $p \in (0, 1)$, and $\mu_{G,H}$ be a symmetric problem such that $H(-M) = H\big(\frac{2p-1}{1-p}M\big) = p$. In such a problem, the realizations of the common values divide objects into groups, either very good or very bad. The likelihood and relative magnitude of these realizations depends on whether p is close to 0 or close to 1.

Given a priority rule, those agents who expect objects with very good common values prefer the priority rule to each TTC rule while those agents who expect objects with very bad common values prefer each TTC rule to the priority rule. When p is close to 0, nearly all agents expect objects with very good common values and prefer the priority rule. On the other hand, when p is close to 0, nearly all agents expect objects with very bad common values and prefer each TTC rule. By varying p, the proportion of agents who prefer the priority rule can be made arbitrarily large or small.

(iii) Given a symmetric problem, for each TTC rule, all agents obtain the same expected utility. Moreover, this common expected utility is positive. In contrast, each agent's expected utility under preference-independent allocation is zero. Therefore, each agent obtains higher expected utility under the TTC rule.

(iv) We provide a sequence of problems in which one-half of the population prefers preference-independent allocation in the limit. Let $\mu_{G,H}$ be a symmetric problem such that $H(-2) = H(2) = \frac{1}{2}$. For example, H may be a binary distribution under which the values $\{-2, 2\}$ are equally likely. Let $K \in \mathbb{N}$ and N^K be a population with K agents. Under random assignment, each agent's ex-ante expected utility is 0. Now consider $F^{\prec}$ and let $k \in N$ be such that $k \geq \frac{K}{2}$. Then each agent $i \in N^K$ with priority lower than k, $U_k(F^{\prec}, \mu_{G,H}) \leq -2 + 1 = -1$ so these agents are worse off than under random assignment.

Whereas $F^{\prec}$ partitions the population into groups with high and low expected utilities, all agents obtain the same expected utilities under each TTC rule. Although bad outcomes occur with the same probability under each TTC rule as under each priority rule, they remain evenly distributed across the population rather than concentrated among a particular subpopulation. In particular, all agents prefer each TTC rule to random assignment. □

Corollary 10 *Let G define an independent symmetric problem. For (i) Under each priority rule, the probability that the assignment of the agent with kth highest priority receives his rth most preferred object is $\binom{n-r}{k-r} \div \binom{n}{k-1}$. (ii) Under each TTC rule, the probability that an agent receives his rth most preferred object is $\frac{1}{n}\sum_{k=1}^{n} \binom{n-r}{k-r} \div \binom{n}{k-1}$.*

Proof. (i) Let $k, r \in \mathbb{N}$, $\prec \in \Pi^N$, and label agents according to their priority under $\prec$. Under $F^{\prec}$, agent k always receives one of his k most preferred objects. For each $r > k$, agent k receives his rth most preferred object with probability 0, confirming the formula.

Now suppose $r \leq k$. The $k-1$ agents with higher priority than agent k choose $k-1$ of the n objects. By independence and symmetry, the $\binom{n}{k-1}$ different combinations of objects are equally likely. Agent k is able to choose his most preferred object if it is not among the $k-1$ selected by higher priority agents. This object remains in $\binom{n-1}{k-1}$ of the $\binom{n}{k-1}$ combinations. The probability that agent k receives his most preferred object is then $\binom{n-1}{k-1} \div \binom{n}{k-1} = \frac{n-k-1}{n}$, confirming the formula.

In general, agent k receives his rth most preferred object when the $k-1$ agents with higher priority select all of agent k's $r-1$ most preferred objects and the remaining $(k-1)-(r-1) = k-r$ objects from among agent k's $n-r$ least preferred objects. This occurs in $\binom{r-1}{r-1}\binom{n-r}{k-r} = \binom{n-r}{k-r}$ combinations. Therefore, the probability that agent k receives his rth most preferred object is $\binom{n-r}{k-r} \div \binom{n}{k-1}$, confirming the formula.

(ii) By the formulas from Corollary 7, randomization over the roles of the agents in each TTC rule yields the same expected rank distribution as randomization over the priority rules. Randomizing over the roles of the agents in a given TTC rule is equivalent to randomizing over the family of TTC rules. Moreover, for each independent symmetric problem, each TTC rule yields the same expected rank distribution. Therefore, the expected rank distribution for each TTC rule is the average of the rank distributions over all priority rules: In expectation, each agent receives his rth most preferred object with probability $\frac{1}{n}\sum_{k=1}^{n}\binom{n-r}{k-r}\div\binom{n}{k-1}$. □

B.3 Heterogeneous problems

Proposition 10 *Let $\mu \in \bar{D}$ be a problem with at least four agents whose support on a scoring domain. For each pair of priority and TTC rules, there exist profiles at which either rule utilitarian and maximin dominates the other rule.*

Proof. We provide an example with four agents and objects. Similar constructions may be embedded in larger problems. Lett and $N \equiv \{1,2,3,4\}$, $A \equiv \{a,b,c,d\}$, and $s \in S$. Let $\hat{u}, \hat{u}' \in V^s$ be as specified in the ordinal rankings in the table.

$P(\hat{u}_1)$	$P(\hat{u}_2)$	$P(\hat{u}_3)$	$P(\hat{u}_4)$
b	b	c	d
a	c	b	b
c	a	d	c
d	d	a	a

At $\hat{u}$, there are four *ex-post efficient* allocations:

$$x \equiv (b,c,d,a), \quad x' \equiv (b,c,a,d), \quad x'' \equiv (b,a,c,d), \text{ and } \quad x''' \equiv (a,b,c,d).$$

The corresponding utilities are

$$\begin{aligned} U(x,\hat{u}') &= (s_1,s_2,s_1,s_4), \qquad U(x',\hat{u}') = (s_2,s_1,s_1,s_1), \text{ and} \\ U(x'',\hat{u}') &= (s_1,s_1,s_3,s_1), \qquad U(x''',\hat{u}') = (s_1,s_2,s_3,s_4), \end{aligned}$$

Comparing, each of allocation among $\{x', x'', x'''\}$ utilitarian and (extended) maximin dominates x.

TTC and priority rules. By relabeling agents and objects, it suffices to make each comparison for one pair of rules. First consider $\omega \equiv (a,b,c,d)$ and $\prec \equiv (1,2,3,4)$. Then $F^{\omega}(\hat{u}) = x'''$ and $F^{\prec}(\hat{u}) = x$, so F^{ω} utilitarian and maximin dominates $F^{\prec}$ at $\hat{u}$. Now consider $\hat{\omega} \equiv (b,c,d,a)$ and $\hat{\prec} \equiv (2,3,4,1)$. Then $F^{\hat{\omega}}(\hat{u}) = x$ and $F^{\hat{\prec}}(\hat{u}) = x'''$ so $F^{\hat{\prec}}$ utilitarian and maximin dominates $F^{\hat{\omega}}$ at $\hat{u}$.

Two priority rules. Again let $\prec \equiv (1,2,3,4)$ and $\prec' \in \Pi^N \backslash \{\prec\}$. If agent 1 has first priority, then $F^{\prec'}(\hat{u}) \in \{x'', x'''\}$; if agent 2 has first priority, then $F^{\prec'}(\hat{u}) = x'''$; if agent 3 has first priority, then $F^{\prec'}(\hat{u}) \in \{x'', x'''\}$; and if agent 4 has highest priority, then $F^{\prec'}(\hat{u}) \in \{x', x'', x'''\}$. In each case, $F^{\prec'}(\hat{u}') \neq x$, so $F^{\prec'}$ utilitarian and maximin dominates $F^{\prec}$ at $\hat{u}$.

Two TTC rules. Let $\omega' \equiv (b, c, d, a)$ and let $\omega'' \in X \backslash \{\omega'\}$. If $\omega''_1 = b$, then $F^{\omega''}(\hat{u}) \in \{x'', x'''\}$; if $\omega''_2 = b$, then $F^{\omega''}(\hat{u}) = x'''$; if $\omega''_3 = b$, then $F^{\omega''}(\hat{u}) \in \{x'', x'''\}$; and if $\omega''_4 = b$, then $F^{\omega''}(\hat{u}) \in \{x', x'', x'''\}$. In each case, $F^{\omega''}(\hat{u}') \neq x$, so $F^{\omega''}$, utilitarian and maximin dominates $F^{\omega'}$ at $\hat{u}$. □

Proposition 11 *(i) With two or more agents, there exist problems at which one TTC rule ex-ante Pareto dominates another TTC rule.*
(ii) With three or more agents, there exist problems at which a priority rule ex-ante Pareto dominates the rules randomizing over priority rules or TTC rules.

Proof. To simplify computations, our examples are not rectangular problems. However, as the strict inequalities make clear, the conclusions continue to apply in problems which place small probability on the excluded profiles.

(i) We provide an example with two agents which may be embedded in larger problems.
Let and $N \equiv \{1, 2\}$, $A \equiv \{a, b\}$, and $V \equiv \{u^1, u^2, \hat{u}^1, \hat{u}^2\}$ be as defined below.

	u^1	u^2	$\hat{u}^1$	$\hat{u}^2$
a	3	2	1	0
b	0	1	2	3

Let μ such that $\mu(u^1, u^2) = \mu(\hat{u}^1, \hat{u}^2) = \frac{1}{2}$. To interpret, we imagine that agent 1 has a slight intrinsic preference for a while agent 2 has a slight intrinsic preference for b. However, these predispositions are overshadowed by a common signal about the quality of the objects: In the first case, the agents receive a common signal favorable to a; in the first case, the agents receive a common signal favorable to b.

Since the agents' top references are identical at each realized profile, each agent keeps his endowment under either TTC rule, so $U(F^{ab}, \mu) = (2, 2)$ and $U(F^{ba}, \mu) = (1, 1)$. In this case, F^{ab} ex-ante Pareto dominates F^{ba}. The idea is that, ex-ante, the rule which "matches" the agents' endowments to their intrinsic preferences performs better.

By comparison, the expected utilities under the two priority rules are $U(F^{12}, \mu) = \left(\frac{5}{2}, \frac{1}{2}\right)$ and $U(F^{21}, \mu) = \left(\frac{1}{2}, \frac{5}{2}\right)$. Thus, averaging over the TTC rules or the priority rules yields the same common value $\left(\frac{3}{2}, \frac{3}{2}\right)$. Moreover, no ex-ante Pareto comparison is possible between either TTC rule and either Priority rule.

(ii) Example 2 applies. Moreover, the example also illustrates that ex-ante Pareto domination of one TTC rule by another TTC rule is possible even on scoring domains. Continuing with the definitions in Example 2, let $\omega' \equiv (b, a, d, c)$. Then $F^{\omega'}(\hat{u}) = F^{\prec}(\hat{u})$ and $F^{\omega'}(\hat{u}') = F^{\prec}(\hat{u}')$. Therefore, $U(F^{\omega'}, \mu) = U(F^{\prec}, \mu)$ and $F^{\omega'}$ ex-ante Pareto dominates F^{ω}. □

Proposition 12 *With three or more agents, there exist problems at which a* strategy-proof *and* bossy *rule:*

(i) Achieves higher utilitarian welfare than all priority and TTC rules.

(ii) Lorenz dominates all priority and TTC rules.

(iii) ex-ante Pareto dominates the rules randomizing over priority rules or TTC rules.

Proof. We provide an example with three agents and objects. Similar constructions may be embedded in larger problems. Let $N \equiv \{1, 2, 3\}$, $A \equiv \{a, b, c\}$, and $V \equiv \{u^1, u^2, u^3, u^4\}$ be as defined below.

	u^1	u^2	u^3	u^4
a	2	2	1	0
b	1	0	2	2
c	0	1	0	1

Let μ such that $\mu(u^1, u^3, u^4) = \mu(u^2, u^4, u^3) = \frac{1}{2}$. That is, agent 1 always ranks a first while agents 2 and 3 always rank b first. Moreover, the agents rankings of the remaining objects are correlated. This captures a situation in which agents have relatively fixed preferences but observe a signal about c which makes it appear more favorable to agents 1 and 2 and less favorable to agent 3. The expected utility profiles for the six priority and six TTC rules are

	U_1	U_2	U_3
F^{123}	2	2	$\frac{1}{2}$
F^{132}	2	$\frac{1}{2}$	2
F^{213}	2	2	$\frac{1}{2}$
F^{231}	1	2	1
F^{312}	2	$\frac{1}{2}$	2
F^{321}	1	1	2

	U_1	U_2	U_3
F^{abc}	2	2	$\frac{1}{2}$
F^{acb}	2	$\frac{1}{2}$	2
F^{bac}	2	2	$\frac{1}{2}$
F^{bca}	2	$\frac{1}{2}$	2
F^{cab}	1	1	2
F^{cba}	1	2	1

The individual rules do not yield the same utilitarian welfare. However, each agent's expected utility averaged over all priority rules is the same as his expected utility averaged over all TTC rules. In each case, the average is $\left(\frac{5}{3}, \frac{4}{3}, \frac{4}{3}\right)$. All twelve rules are *ex-ante efficient* with respect to μ.

We define a *bossy* and *strategy-proof* rule which yields higher utilitarian welfare. The idea is to take advantage of the negative correlation between the preferences of agents 1 and 3. Let F' modify $F^{\prec}$ so that agent 1 continues to have first priority. If agent 1 reports u^1, then agent 2 has second highest priority; if agent 1 reports u^2, then agent 3 has second highest priority. In each case, agent 1 receives a and the agent with second priority receives b. The expected utility profile for this rule is $U(F', \mu) = \left(2, \frac{3}{2}, \frac{3}{2}\right)$ so the utilitarian welfare under F' is greater than under any priority or TTC rule. Moreover, comparing utility profiles, F' Lorenz dominates all priority and TTC rules. Furthermore, F' ex-ante Pareto dominates randomization over either class of rules. $\square$

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